

# Investigating Cross-Sectional Size Distribution of Randomly Distributed 3D Spheres

Sai Pandian, ID: 29899923

**Abstract—Text**

## I. INTRODUCTION

**I**N the world of Material Science and Crystallography, it is often the case that a material is identified and classified using the distribution of its constituent particle sizes. However, direct observation of the particles so as to construct a distribution is often impossible. Using a modern microscope or laser imaging techniques, it is possible to observe particles in a cross-section of material. But the observed radii of particles in this cross-section do not necessarily correspond directly to the radii of the particles in the material, since the plane of the cross-section is not necessarily oriented such that the observed particles lie exactly on the plane. Thus, it becomes difficult to relate the observed distribution of particle sizes in the plane to the real distribution of particle sizes in the material.

Mathematically, determining the distribution of spherical particles from planar sections is a well known problem called [Wicksell's Corpuscle Problem](#), named after S. D. Wicksell, who solved the problem in 1925. With this solution, it is possible to determine the original distribution of spherical particles. Thus, it is possible to determine the distribution of particle sizes in a solid from its observed cross-section, allowing us to identify and classify materials better.

In this paper we aim to show that it is possible to reproduce theoretical results on cross-sectional distributions using numerical simulation, and it is thus possible to solve Wicksell's Corpuscle Problem in this manner.

## II. METHOD

Consider many spheres of different radii distributed randomly in space. For the purposes of the numerical simulation, we will consider a cube of length 20. The spheres were not allowed to overlap, which we ensured by placing the spheres such that if one sphere has vector position  $\vec{a}$  and radius  $r_1$  and the other has vector position  $\vec{b}$  and radius  $r_2$ , the spheres must have positions such that:

$$|\vec{a} - \vec{b}| > (r_1 + r_2)$$

Random coordinates for a sphere are drawn from a uniform distribution and checked against every existing sphere until a set of coordinates is drawn that ensures no overlap, at which point the sphere is placed in space and the process is repeated until all the necessary spheres have been placed.

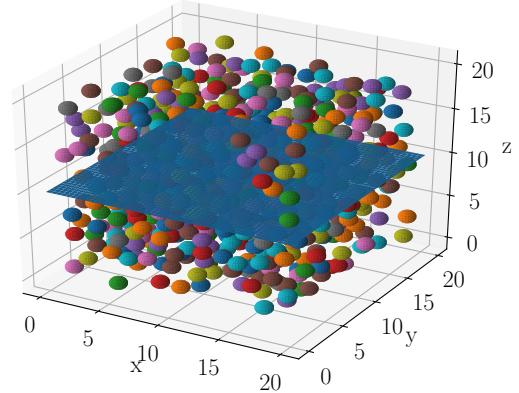


Fig. 1. 750 spheres of radius 0.7 distributed randomly in a cube of length 20, such that no sphere is touching or overlapping another sphere. A randomly oriented plane passes through the box and “cuts” through the spheres.

We then placed a plane into the cube, with a random orientation. The plane will “cut” through many of the spheres in the cube. This is shown in Figure 1. Here we see 750 spheres, each with radius 0.7 distributed randomly in a cube of length 20. The plane is shown in blue, and can be seen intersecting many of the spheres in the cube.

In order to obtain the planar cross-section, we employed the spherical cap method. If we consider a sphere with centre  $\vec{c}_0 = (x_0, y_0, z_0)$  and radius  $R$ , and a plane with equation  $Ax + By + Cz = D$  such that  $\vec{n} = (A, B, C)$  is the normal vector to the plane, then the distance from the centre of the circle to the closest point on the plane  $p_0$  is given by  $\rho$ :

$$\rho = \frac{(\vec{c}_0 - \vec{p}_0) \cdot \vec{n}}{|\vec{n}|} = \frac{Ax_0 + By_0 + Cz_0 - D}{\sqrt{A^2 + B^2 + C^2}}$$

If  $|\rho| < R$ , then there is an intersection of the plane and the sphere. The radius of the circle  $r$  in the planar cross-section is given by:

$$r = \sqrt{R^2 - \rho^2}$$

And we then binned this radius to produce a histogram of the distribution of planar circle radii. It is also possible to recover the centre  $\vec{c}$  of the circle visible on the plane using:

$$\vec{c} = \vec{c}_0 + \rho \frac{\vec{n}}{|\vec{n}|} = (x_0, y_0, z_0) + \rho \frac{(A, B, C)}{\sqrt{A^2 + B^2 + C^2}}$$

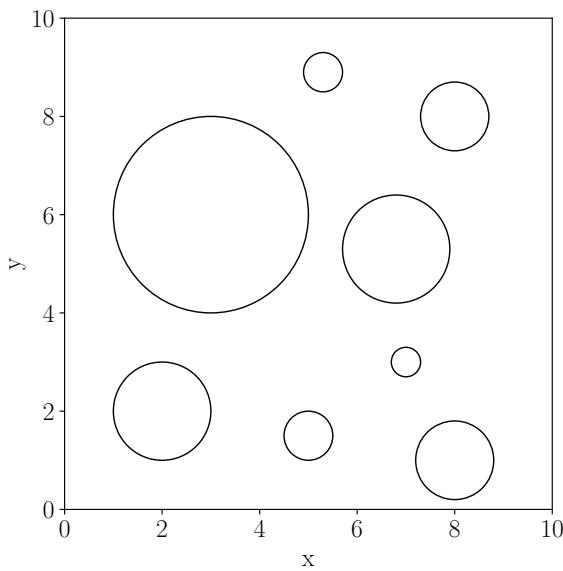


Fig. 2. Planar cross-section constructed using mock data. Spheres cut by the plane appear as circles on the plane, with radius dependent on the relative position of the spheres to the plane. This distribution of circles can be modelled.

Figure 2 presents a planar cross-section generated using this method and mock data. As we see, there are a range of circular radii observed since even though the spheres all had the same radius, they were at different distances from the plane, and were thus cut at different points. The many observed circular radii are binned to produce a distribution.

If  $f(R)dR$  is the number of spheres per unit volume with radii in interval  $[R, R + dR]$ , then we can define a distribution  $\phi(x)dx$  which represents the number of circles per unit area on the plane with radii in interval  $[x, x + dx]$ . This distribution is given by:

$$\phi(x) = \int_x^\infty \frac{xf(R)}{\sqrt{R^2 - x^2}} \quad (1)$$

We can compute this integral to obtain a theoretical planar distribution for different starting spherical radius distributions.

We first considered the case in which all the spheres had the same radius  $R_0$ . In this case, the spherical radius distribution is simply a Dirac delta function centred on the radius  $R_0$ .

If we compute the integral in Equation 1, replacing  $f(R)$  with  $\delta(R - R_0)$  we find:

$$\phi(x) = \int_x^\infty \frac{x\delta(R - R_0)}{\sqrt{R^2 - x^2}} = \frac{xH(R_0 - x)}{\sqrt{R_0^2 - x^2}} \quad (2)$$

where  $H(R_0 - x)$  is the heaviside function, which “turns on” the distribution only for  $x < R_0$  as it is not possible to have a circular radius  $x$  larger than the spherical radius. So we aim to see our experimental data reproduce this distribution. We also investigated how changing the value of  $R_0$  affects the experimental distribution.

We next considered the case in which each sphere has a radius  $R$  uniformly distributed in a range  $R_1 < R < R_2$ . In this case, we modify  $f(R)$  in Equation 1 to be:

$$f(R) = \begin{cases} 1, & \text{if } R_1 < R < R_2 \\ 0, & \text{Otherwise} \end{cases}$$

In this case the integral in Equation 1 does not need to be computed with the limits shown, but instead with limits between  $R_1$  and  $R_2$  since  $f(R)$  is 0 otherwise:

$$\begin{aligned} \phi(x) &= \int_{R_1}^{R_2} \frac{xf(R)}{\sqrt{R^2 - x^2}} \\ &= x \ln \left( \sqrt{R_2^2 - x^2} + R_2 \right) - x \ln \left( \sqrt{R_1^2 - x^2} + R_1 \right) \end{aligned} \quad (3)$$

Since this distribution produces complex numbers for  $x < R_1$ , we only consider the real part of this distribution, which appears to produce a peak at  $x = R_1$ .

### III. RESULTS

Text

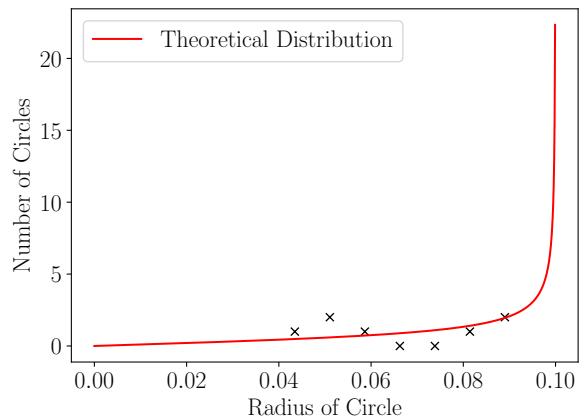


Fig. 3. Distribution of circle radii on plane for 750 spheres of radius 0.1 each. There are relatively few points since the spheres are small, so there is relatively small chance that they will be cut by the plane. So only a few circles are visible on the plane. The theoretical distribution is shown, with the experimental data matching this distribution, showing a slow increase in number of circles as radius increases.

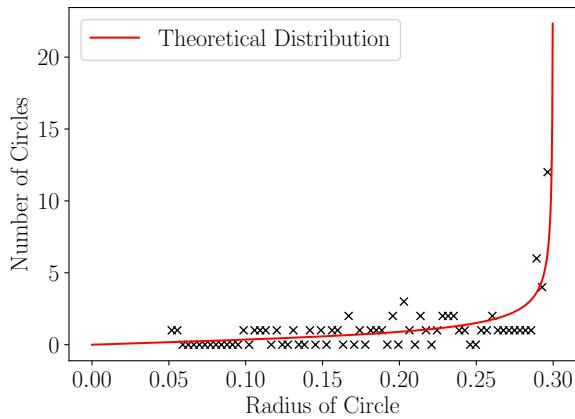


Fig. 4. Distribution of circle radii on plane for 750 spheres of radius 0.3 each. Compared to Figure 3, there are many more experimental data points, since it is more likely that the plane will intersect these larger spheres. The experimental data more clearly resembles the theoretical distribution, showing a strong increase in number of circles as radius increases towards the maximum of 0.3.

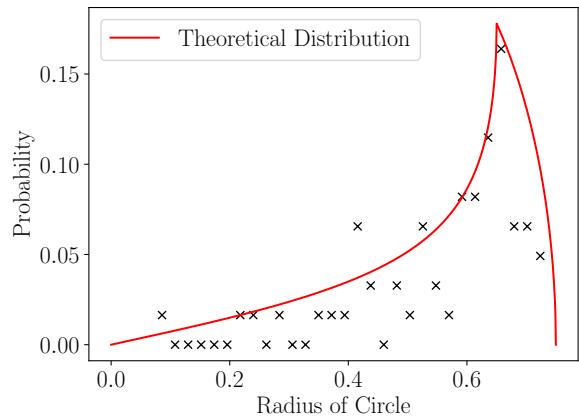


Fig. 6. Distribution of circle radii on plane for 750 spheres of radius  $R$  where  $0.65 < R < 0.7$ . The theoretical distribution is now different, although the experimental data still resembles this new distribution, showing a maximum around  $R = 0.7$  and a sharp decrease after this. This is likely because there are more spheres with radii such that an overlap between 0 and 0.7 is possible, and relatively few spheres with radii such that a larger overlap is possible.

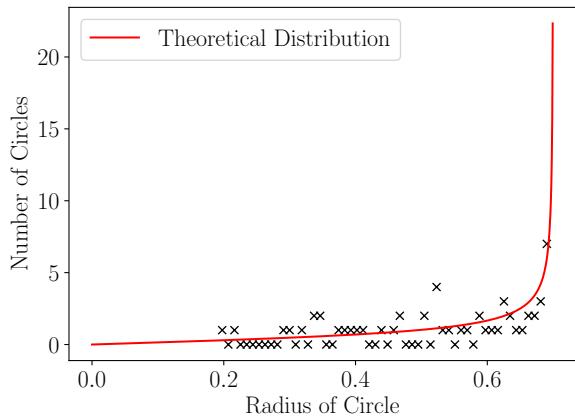


Fig. 5. Distribution of circle radii on plane for 750 spheres of radius 0.7 each. We do not observe much difference in how well the experimental data resembles the theoretical distribution between this figure and Figure 4. With larger sized spheres, the simulation becomes harder to run as it is more difficult to randomly place spheres in a finite space without overlap.

#### IV. DISCUSSION AND SUMMARY

Text

#### APPENDIX A APPENDIX NAME

Text

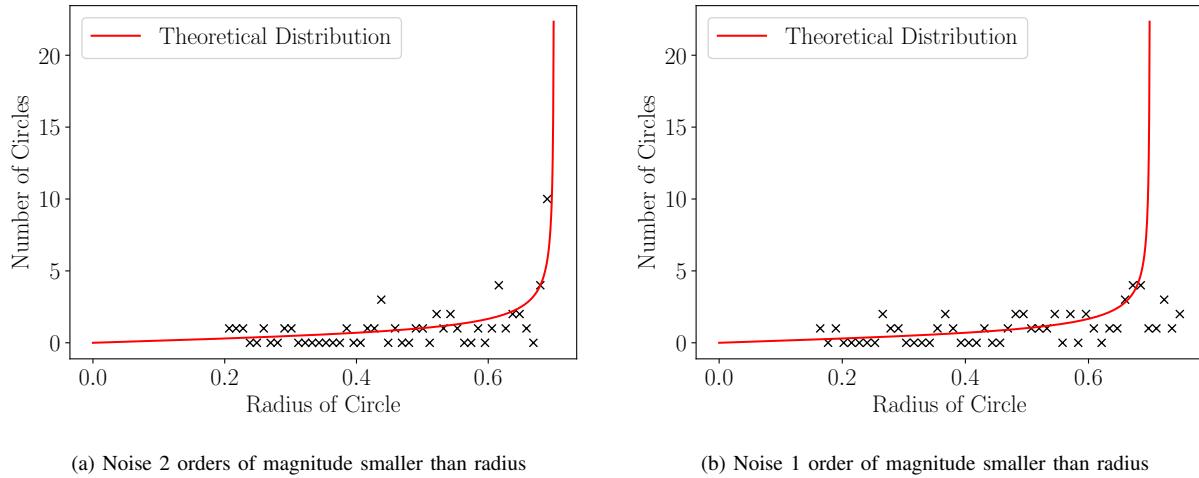


Fig. 7. Distribution of circle radii on plane for 750 balls of radius 0.7 each. Each recorded radius has a random noise added to it in order to determine how sensitive the data is to experimental error. Figure 7a shows the distribution for an added noise 2 orders of magnitude smaller than the recorded radius, while Figure 7b is the same for noise 1 order of magnitude smaller. As we can see, the experimental data very resembles the theoretical distribution when noise of 2 orders of magnitude smaller are added. But this is not true when noise of 1 order of magnitude smaller is added, implying the data is very sensitive to experimental data.