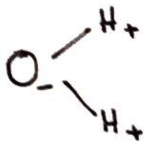
Magnetic Force and Field

One way to think about magnets is as electric dipoles. This is how magnets seem to interact with each other: like two oppositely charged particles a fixed distance away from each other.

Electric dipoles can be observed in nature in molecules of H_2O. The molecule has an uneven distribution of charge and so can be considered a dipole:



$\Rightarrow \quad -q \overset{d}{\text{———}} +q$

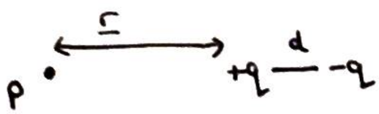
A dipole moment can be calculated from this:

$$\boxed{\text{dipole moment} = q d \hat{\underline{r}}}$$

where $\hat{\underline{r}}$ is a vector from $-$ to $+$.

In an $\underline{E}$ field, the dipole will align with the field due to a torque that causes the dipole to spin to alignment. In non-uniform fields, dipoles experience a net force which can be calculated:

The field originates at point P and is parallel to $\underline{r}$.



$$|\underline{E}| = \frac{q}{4\pi\varepsilon_0 |\underline{r}|^2} - \frac{q}{4\pi\varepsilon_0 (|\underline{r}|+d)^2}$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{|\underline{r}|^2} - \frac{1}{|\underline{r}|^2\left(1+\frac{d}{|\underline{r}|}\right)^2}\right)$$

we can use binomial expansion to approximate $\left(1+\frac{d}{|\underline{r}|}\right)^2$ to $1-2\frac{d}{|\underline{r}|}$

$$\therefore \approx \underline{\underline{\frac{q}{4\pi\varepsilon_0}\,\frac{2d}{|\underline{r}|^3}}}$$

Back to Magnets

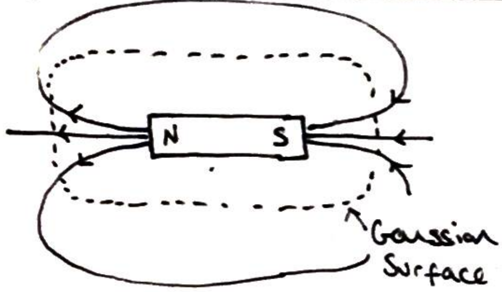
Magnets attract and repel like electric dipoles. They will also align themselves in the magnetic field of a larger magnet. So:

If we could cut magnets in half into "magnetic monopoles", they would interact like electric charges and we could use all the same equations as before.

But magnetic monopoles do not exist!!!

Because of this, we can state one important fact:

$$\boxed{\text{Magnetic Gauss' Law:} \quad \int_{\substack{\text{closed}\\ \text{surface}}} \underline{B} \cdot d\underline{A} = 0}$$



Since magnets are always dipoles, any field lines that leave a surface will also reenter so the flux through the closed surface is 0.

So if there are no magnetic monopoles, what is the source of magnetic fields? Moving electric charges generate magnetic fields!

So, since we can't define a magnetic field by its effect on a monopole, we define magnetic fields by their effects on electric charges.

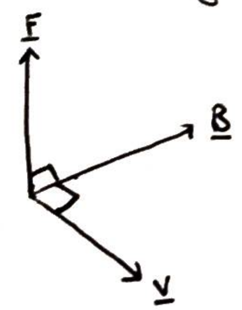
Magnetic Force on an Electric Charge

The magnetic field $\underline{B}$ is a vector field filling space that generates the force on charge q.

$$\boxed{\underline{F} = q\,\underline{v} \times \underline{B}}$$

This is similar to the equation for an E field : $\underline{F} = q\,\underline{E}$

Note that the cross product means that the direction of the force will perpendicular to both the magnetic field $\underline{B}$ and direction of movement of the charge $\underline{v}$.



We can use Fleming's Left Hand Rule to work out the direction of the field. Note, in this rule, the second finger is direction of conventional current, which may not necessarily be the direction of motion of charge, depending on the charge.

The force can also be written:

$$\boxed{\begin{aligned} |\underline{F}| &= q\,|\underline{v}||\underline{B}|\sin\theta \\ \underline{F} &= q\,|\underline{v}||\underline{B}|\sin\theta\,\hat{\underline{n}} \end{aligned}}$$

where $\hat{\underline{n}}$ is the normal vector to $\underline{v}$ and $\underline{B}$.

Note : The $\underline{B}$ field has units $kg s^{-1} C^{-1}$ or T for Tesla.

Some common fields are :

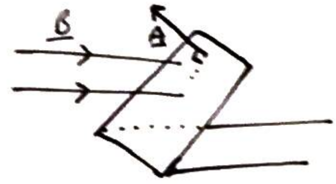
Earth's $\underline{B}$ field - $30\mu T$
Fridge magnet - $5mT$
MMR machine - $3T$

Magnetic Flux

As we did for an $\underline{E}$ field, we can define a quantity magnetic flux as the number of magnetic field lines passing through a unit area.

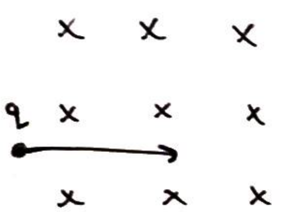
$$\boxed{\Phi_B = \int_{\text{surface}} \underline{B} \cdot d\underline{A}}$$

where the magnetic flux has units webers Wb



Generally, due to various widereaching fields like the Earth's magnetic field, there is always some flux through objects. However, the flux through a closed surface is always 0.

Example: consider a charge moving perpendicular to a plane of magnetic field



The magnetic field is going into the page. The force will always be perpendicular to both the direction of motion and the field, so the charge will undergo circular motion.

So, equating the force to the force of circular motion:

$$q|\underline{v}||\underline{B}|\sin\theta = \frac{m|\underline{v}|^2}{|\underline{r}|} \quad \text{but } \theta = 90° \text{ so } \sin\theta = 1$$

$$\therefore q|\underline{v}||\underline{B}| = \frac{m|\underline{v}|^2}{|\underline{r}|} \Rightarrow |\underline{r}| = \frac{m|\underline{v}|}{q|\underline{B}|}$$

This is the radius of rotation.

Note, if the charge has a component of $\underline{v}$ in the $\underline{B}$ direction, this component will remain unaffected since $\sin 0 = 0$ and the particle undergoes a helix motion:  in the direction of the field.

Cyclotrons

From the previous example, we saw that the magnitude of the radius of motion for a charge undergoing circular motion in a $\underline{B}$ field is given by:

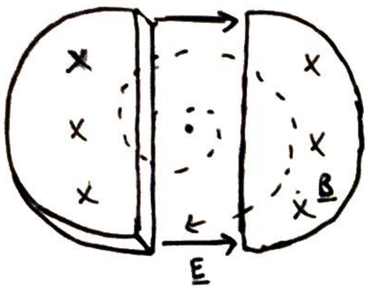
$$|\underline{r}| = \frac{m|\underline{v}|}{q|\underline{B}|}$$

so the number of revolutions per unit time, the Period, is given by:

$$T = \frac{2\pi|\underline{r}|}{|\underline{v}|} = \frac{2\pi m}{q|\underline{B}|}$$

Note that this is independent of $|\underline{v}|$.

Since the period of the circular motion is independent of the velocity, it is possible to build simple particle accelerators: cyclotrons



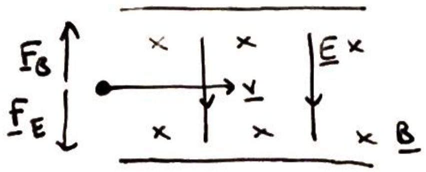
A radioactive source will generate α or β particles. The perpendicular $\underline{B}$ field causes circular motion in the particles. Everytime the particles leave one of the D's, they are accelerated through a potential difference in the gap. This potential is reversed every half period so the acceleration happens every time. After each acceleration, the radius of motion increases but the period remains the same, so the potential can be reversed at the same rate.

Eventually, the particle will reach a maximum velocity when the radius is the same as the radius of the D's.

$$|\underline{v}_{max}| = \frac{|\underline{r}_0| q |\underline{B}|}{m}$$

Perpendicular $\underline{E}$ and $\underline{B}$ fields

Consider perpendicular $\underline{E}$ and $\underline{B}$ fields and a charge q with $\underline{v}$ perpendicular to both.



There is a force upwards due to the $\underline{B}$ field and a force downwards due to the $\underline{E}$ field.

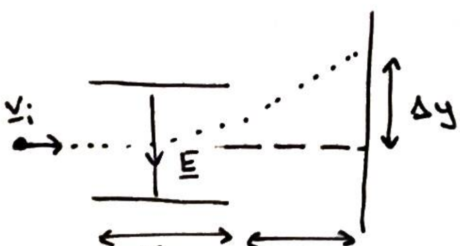
Since the forces are both in the vertical plane but in opposite directions, it is possible for them to be equal. In this case:

$$q\,|\underline{v}|\,|\underline{B}| = q\,|\underline{E}|$$

$$\therefore\ |\underline{v}| = \frac{|\underline{E}|}{|\underline{B}|}$$

This acts as a velocity selector. Any charges with higher or lower velocity will veer off and hit the plates.

JJ Thompson in 1897 used such a velocity selector to calculate the charge : mass ratio of an electron:



The beam will move at constant $\underline{v}$ in x direction. It will also accelerate at a constant rate in y from stationary. This $a = qE/m$ by Newton's 2nd Law.

$$\Delta y = \underbrace{\tfrac{1}{2}at_1^2}_{\text{acceleration between plates}} + \underbrace{v_y t_2}_{\text{motion between plates and screen}} \Rightarrow \tfrac{1}{2}a\left(\frac{x_1}{v_i}\right)^2 + \underbrace{at_1\frac{x_2}{v_i}}_{\frac{a x_1 x_2}{v_i^2}}$$

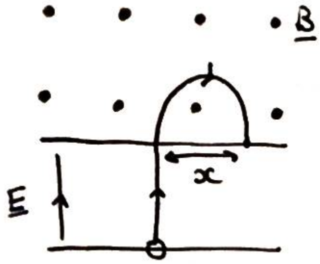
$$= a\left(\frac{1}{2}\frac{x_1^2}{v_i^2} + \frac{x_1 x_2}{v_i^2}\right)$$

$$\Delta y = \frac{q}{m}\frac{E x_1}{v_i^2}\left(\tfrac{1}{2}x_1 + x_2\right)$$

Since he could measure everything else, he could calculate $\frac{q}{m}$.

Mass Spectrometer

This allows for the separation of a beam of ions into a spectrum divided by $\frac{q}{m}$ values.



A substance is vaporised and ionised and these ions are accelerated as a beam through the electric field. When the accelerated beam reaches the magnetic field, the ions are deflected in a semi circle and hit the plates.

The velocity of the ions when entering the magnetic field is given by:

$$\frac{1}{2}mv^2 = qV$$

$$\therefore \quad v = \sqrt{\frac{2qV}{m}}$$ where V is the potential difference.

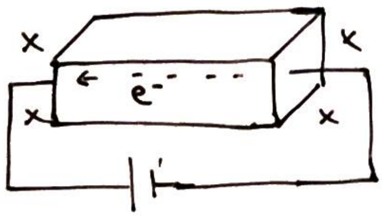
Position x depends on the radius of the circular motion and is given by:

$$x = 2|\underline{r}| = 2 \times \frac{m|\underline{v}|}{q|\underline{B}|} = \frac{2m}{q|\underline{B}|}\sqrt{\frac{2qV}{m}} = \sqrt{\frac{4 \times 2 \times m^2 \times q \times V}{q^2 \times |\underline{B}|^2 \times m}}$$

$$\therefore \quad x = \sqrt{\frac{8mV}{q|\underline{B}|^2}}$$

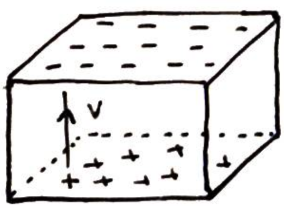
so different $\frac{m}{q}$ values will have different x values, allowing for the separation of a mixed compound into its constituents.

Hall Effect



If we take a block of conducting material and run a current through it, when the material is brought into a magnetic field, the electrons in the material will experience a force.

In this case, with the $\underline{B}$ field into the page, the electrons will experience a force upwards. This will cause a net number of electrons to accumulate at the top of the material. Once a significant amount of electrons have accumulated, the force of repulsion will balance the magnetic force. So no more electrons will accumulate at the top of the material.



Due to the net difference in charge, there is a potential difference in the material. This is called a "Hall Voltage," V_H caused by electric field $\underline{E}_H$ from bottom to top.

In the equilibrium position:

$$q|\underline{E}_H| = q|\underline{v}||\underline{B}| \Rightarrow |\underline{E}_H| = |\underline{v}||\underline{B}|$$

but current $I = nq|\underline{v}|A$, $\therefore |\underline{E}_H| = \dfrac{|\underline{B}| I}{nqA}$ where n is number of electrons

where $A = \text{height} \times \text{width} \Rightarrow h \times w$

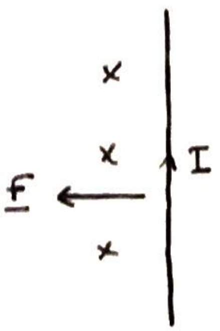
and $|\underline{E}_H| = \dfrac{\text{Potential difference}}{\text{height}}$

$$\therefore \frac{V_H}{\text{height}} = \frac{|\underline{B}| I}{nq} \times \frac{1}{\text{height} \times \text{width}} \Rightarrow V_H = \frac{|\underline{B}| I}{nqt}$$ where t is width

$$\boxed{\text{Hall Voltage } V_H = \frac{|\underline{B}| I}{nqt}}$$

$\frac{1}{nq}$ is called the Hall Coefficient and reveals the nature and density of charge carriers.

Wire in a $\underline{B}$ field



The charges moving through the wire in $\underline{B}$ experience a force as shown.

$$\underline{F} = q\,\underline{v} \times \underline{B} \,.\, (\text{number of charges})$$

The number of charges is given by: $n\,\underline{A} \cdot \underline{L}$
where A is the cross sectional area, L is the length of wire

$$\therefore \quad \underline{F} = q\,\underline{v} \times \underline{B} \cdot (n\,\underline{A} \cdot \underline{L})$$

but $\quad I = q n \underline{A} \cdot \underline{L}$

so $\quad \boxed{\underline{F} = I(\underline{L} \times \underline{B})} \Rightarrow \boxed{|\underline{F}| = |\underline{B}|\, I\, |\underline{L}| \sin\theta}$