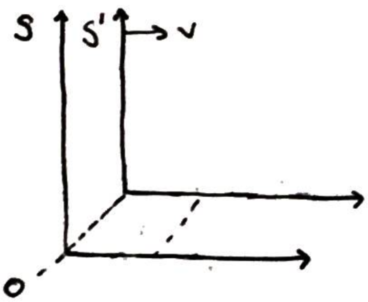
Relativity

Prior to Einstein's theory of relativity, accomplished scientists like Newton and Galileo believed in absolute Time and absolute Space. i.e that there was an absolute frame of reference that all other frames could be related to, by using tools like Galilean Velocity Addition and Galilean Transformations.



Here, events in the S' frame look in S like:

$$x = x' + vt' \qquad x' = x - vt$$
$$y = y' \qquad y' = y$$
$$z = z' \qquad z' = z$$
$$t = t' \qquad t' = t$$

The velocity u' of an object in S' looks in S like:

$$u = \frac{dx}{dt} = \frac{dx}{dt'} \quad \text{but } x = x' + vt'$$

$$\therefore u = \frac{d}{dt'}(x' + vt') \qquad \therefore \underline{\underline{u = u' + v}}$$

Note

The acceleration remains unchanged $a = a'$ so therefore any measured force also remains unchanged.

This all feels like common sense and does work for slow moving frames. However, experimental analysis of high speed frames shows that this does not work!!!

A note on reference frames

This is a set of coordinates defining a plane moving at constant velocity. In this frame, all Newton's Laws will hold and it will be experimentally impossible to deduce whether the frame is moving without looking outside it.

Einstein's Postulates

Einstein aimed to fix this with his theory of special relativity. Instead of starting with the assumption that there existed an absolute reference frame and that $t = t'$ he started with 2 postulates:

> Postulate 1: The laws of physics are the same in all inertial reference frames.

This means that the laws of electromagnetism must also be true in all inertial frames, thus Maxwell's equations hold true in all frames. However, we can get the speed of light in a vacuum from Maxwell's equations: $c = 1/\sqrt{\varepsilon_0 \mu_0}$ implying:

> Postulate 2: The speed of light in a vacuum is the same in all inertial reference frames

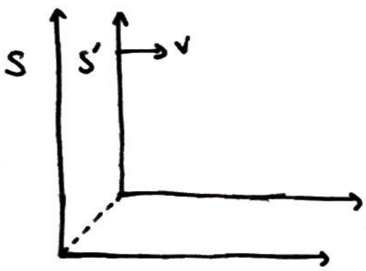
This clearly contradicts Galilean Relativity since light travels at speed c for moving and stationary people.

Michelson-Morley Experiment

This was experimentally proven by the Michelson-Morley experiment. Prior to this experiment, it was thought that lightwaves needed a medium to propagate called the luminiferous ether. If this was the case, moving through the ether would result in light moving at different speeds. The Michelson-Morley experiment showed this was not the case and light was a universal constant.

Lorentz Transformations

We know that Galilean Transformations are wrong so how do time and space in one reference frame map to the other reference frame



we know that since there is no motion in y and z:

$y' = y \quad z' = z$

and $t \neq t'$ since this was an assumption of Galilean Transformations.

So $x' = f(x,t)$ and $t' = g(x,t)$

So how do we start finding the functions f and g ?

We know that objects moving at constant velocity in S will move at constant velocity in S', this is the principle of inertia.

If we choose $x=0$ and $x'=0$ to coincide at $t=0$, we can say that the functions f and g are linear.

So, an object in S' at $x'=0$ moves along the path $x=vt$ in S. So:

$$x' = \gamma_v (x - vt)$$

$$x = \gamma_{-v} (x + vt)$$

where γ_v and γ_{-v} are some quantity that applies the function f.

Let's assume $\gamma_v = \gamma_{-v}$. Note, if we enforce $t = t'$, we get $\gamma = 1$ and retrieve Galilean Transformation.

So how do we find γ?

Let's use the 2nd Postulate. A light ray $x = ct$ should map onto $x' = ct'$ with c unchanged.

$$\therefore \quad ct' = \gamma(ct - vt) \qquad ct = \gamma(ct' + vt')$$

$$ct' = \gamma(c-v)t \qquad ct = \gamma(c+v)t'$$

$$t = \frac{ct'}{\gamma(c-v)} \quad \text{sub into} \uparrow$$

$$\frac{c^2 t'}{\gamma(c-v)} = \gamma(c+v)t' \Rightarrow \gamma^2 = \frac{c^2}{(c+v)(c-v)}$$

$$\therefore \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

To get the temporal transformation:

$$x = \gamma(x' + vt') \qquad x' = \gamma(x - vt)$$

$$x = \gamma x' + \gamma vt' \qquad x' = \gamma x - \gamma vt$$

subbing x' into x:

$$x = \gamma^2 x - \gamma^2 vt + \gamma vt'$$

$$\gamma vt' = x + \gamma^2 vt - \gamma^2 x$$

$$= x(1 - \gamma^2) + \gamma^2 vt$$

Note

$$1 - \gamma^2 = \frac{1 - v^2/c^2}{1 - v^2/c^2} - \frac{1}{1 - v^2/c^2}$$

$$= -v^2/c^2 \, [\gamma]^2$$

$$\therefore \gamma vt' = \gamma^2 vt - \gamma^2 x \frac{v^2}{c^2}$$

$$t' = \gamma t - \gamma x \frac{v}{c^2}$$

$$\therefore \quad t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

Similarly

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

Final Lorentz Transformations

This all gives us the true transformation equations:

$$\boxed{t = \gamma\left(t' + \frac{vx'}{c^2}\right)} \qquad \boxed{t' = \gamma\left(t - \frac{vx}{c^2}\right)}$$

$$\boxed{x = \gamma(x' + vt')} \qquad \boxed{x' = \gamma(x - vt)}$$

$$\boxed{y = y' \quad z = z'}$$

$$\boxed{\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}}$$

Note that the inverse transformation is always just to change the primes to nonprimes and vice-versa and then swap the sign!

Note that for all non-zero speeds $\gamma > 1$.

For $v \ll c$, $\gamma \gtrsim 1$ and the Galilean Transformation is obtained.

Time Dilation

From the Lorentz transformation of time we get an interesting result. If we let:

$t_1' = \gamma(t_1 - vx_1/c^2)$ and $t_2' = \gamma(t_2 - vx_2/c^2)$

$$t' = t_1' - t_2' = \gamma(t_1 - t_2) = \gamma t_p$$

$$\therefore \boxed{t = \gamma t_p}$$

Since γ is > 1 t is longer than t_p. This implies that time is larger in the stationary observor's reference frame. "Moving clocks run slow"

Length Contraction

This is a similar phenomena to time dilation and results in measured length to be shorter in the stationary observer's reference frame.

If we let:

$x_1' = \gamma(x_1 - vt_1)$ and $x_2' = \gamma(x_2 - vt_1)$

Note that in the second equation we let $t_2 = t_1$ since when measuring the length of a moving object from a stationary reference frame, the positions of both ends of the object have to be measured at the same time. It doesn't matter in the object's frame.

Subtracting x_1' from x_2':

$$x_2' - x_1' = \gamma(x_2 - x_1)$$

$$\therefore \quad \boxed{L_p = \gamma L}$$

Length is contracted in the stationary observer's frame.

A note on "proper" length and time

"Proper" Length and Time is the length and duration of an event in the reference frame in which the event is stationary. i.e. if you are holding your breath on a spaceship, the proper time is the time you measure from the spaceship. If you are measuring the length of the spaceship, proper length is the length you measure if you move alongside the ship.

Simultaneity –

Two separated events that are simultaneous in one inertial frame are not simultaneous in another inertial frame.

This can be proven by Lorentz transformations. We can show that if $t_1 = t_2 = 0$, i.e. two events are simultaneous in one inertial frame, then $t_1' \neq t_2' \neq 0$ i.e they are not simultaneous in another frame.
Two events cannot be simultaneous in both frames:

Event 1 (x_1, t_1) in S seen as (x_1', t_1') in S':
$x_1' = \gamma(x_1 - vt_1)$ $\quad t_1' = \gamma(t_1 - vx_1/c^2)$

Event 2 (x_2, t_2) in S seen as (x_2', t_2') in S':
$x_2' = \gamma(x_2 - vt_2)$ $\quad t_2' = \gamma(t_2 - vx_2/c^2)$

if $t_1 = t_2 = 0$, then:

$t_1' = -\gamma v x_1/c^2$ and $t_2' = -\gamma v x_2/c^2$
which are not equal. So the two events are not simultaneous in S' !

Velocity Addition

So we know that Galilean Velocity Addition is wrong, so how do velocities add? we can differentiate the Lorentz transformation equations.

$$t = \gamma(t' + vx'/c^2) \Rightarrow dt = \gamma(dt' + vdx'/c^2)$$

$$x = \gamma(x' + vt') \Rightarrow dx = \gamma(dx' + vdt')$$

$$y = y' \Rightarrow dy = dy'$$

$$z = z' \Rightarrow dz = dz'$$

$$\frac{dy}{dt} = \frac{dy'}{\gamma(dt' + vdx'/c^2)}$$

dividing top and bottom by dt':

$$\frac{dy}{dt} = \frac{dy'/dt'}{\gamma\left(1 + \frac{vdx'}{c^2 dt'}\right)}$$

Similarly:

$$\therefore \boxed{u_y = \frac{u'_y}{\gamma\left(1 + \frac{vu_x'}{c^2}\right)}} \qquad \boxed{u_z = \frac{u'_z}{\gamma\left(1 + \frac{vu_x'}{c^2}\right)}}$$

$$\boxed{u_x = \frac{v + u_x'}{1 + \frac{vu_x'}{c^2}}}$$

Note that there is no gamma factor in the transformation in the x direction

This is a very surprising result since even if the frame is only moving in the x direction, the transformation in the y and z direction are affected!

Relativistic Doppler Shifts

When a source emitting electromagnetic radiation moves towards an observer, the observer receives light with shorter wavelength and higher frequencies. This is called blueshift.

When the source moves away from the observer, the observer receives light with longer wavelengths and lower frequencies. This is called redshift.

We must also take into account relativistic effects when trying to work out exactly how much radiation has been red/blue shifted.

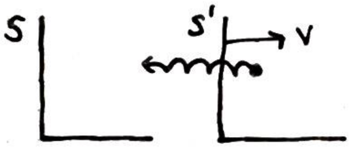
The formula for the classical doppler effect is given by:

$$\boxed{f' = f\left(\frac{c + v_0}{c - v_s}\right)}$$

where v_0 (velocity of observor) is positive for observor motion toward source and v_s (velocity of source) is positive for source motion toward observer.

Also note that in this formula, observor motion toward source is not the same as source motion toward observer.

For the relativistic doppler effect:



Here, a source in S' is moving away from an observer in S with velocity v.
In S', in a time interval t', N pulses are emitted. $\therefore \; f' = \frac{N}{t'}$

This means there are N waves inside a distance ct' (the distance the wave travelled in t')

$\therefore \; \lambda' = ct'/N$

But the observer in S will see N pulses in a distance $ct + vt$ (note that this is $t \neq t'$). So the wavelength observed by the observer is $\lambda = \dfrac{ct + vt}{N}$.

So observed frequency is given by $c = f\lambda$, $f = \dfrac{c}{\lambda}$

$$f = \frac{Nc}{(c+v)t} \times \frac{t'}{t'} = \frac{N}{t'} \times \frac{t'}{t} \times \frac{c}{c+v}$$

$$= f' \times \frac{t'}{t} \times \frac{1}{1 + v/c}$$

But, since $t = \gamma t_p$ and here, $t_p = t'$: $\dfrac{t'}{t} = \dfrac{1}{\lambda}$

$$\therefore f = f' \times \frac{1}{\lambda} \times \frac{1}{1 + v/c} = f' \times \sqrt{1 - v^2/c^2} \times \frac{1}{1 + v/c}$$

$$= f' \times \sqrt{(1 + v/c)(1 - v/c)} \times \frac{1}{\sqrt{(1 + v/c)^2}}$$

$$\therefore \boxed{f = f' \sqrt{\frac{1 - v/c}{1 + v/c}}}$$

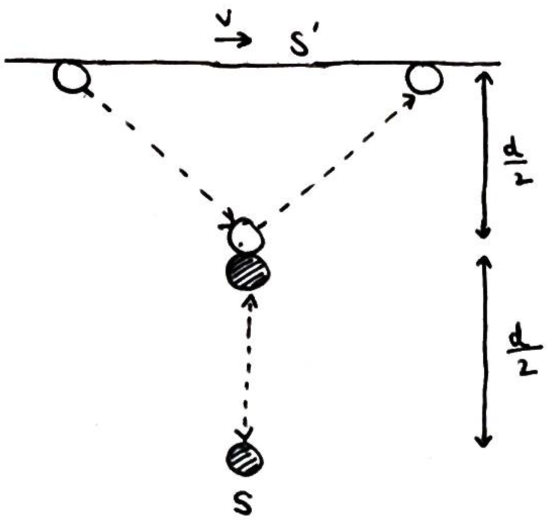
This is the formula for the relativistic doppler effect.

Relativistic Energy and Momentum

Can we still use our Newtonian formulae for things like momentum, kinetic energy and force now that we know about relativity?

Classically, if we apply a constant force to an object we will continuously accelerate it, eventually to a velocity greater than c. This is not possible! So we will have to think differently.

Thought Experiment



Two identical balls are thrown, one from a moving train. The balls are thrown in the y direction perpendicular to the track with the velocity u measured in their own reference frames. The collisions are completely elastic so the balls return to their starting points with the velocity −u measured in their own reference frames.

Although in S frame the coloured ball was thrown with velocity u, in S' frame we must use relativistic velocity addition.

$$u_y' = \frac{u_y}{\gamma\left(1-\frac{u_x v}{c^2}\right)} = \frac{u}{\gamma\left(1-\frac{0 \cdot v}{c^2}\right)} = \frac{u}{\gamma}$$

In S frame, this is the velocity of the clear ball.

So each observer thinks the other has thrown the ball slower than they have. So how did the balls get back with the same velocity? Momentum does not appear to have been conserved! So we need new momentum equations!

S sees coloured balls change in momentum as $2mu$ and clear balls change in momentum as $2m'\frac{u}{\gamma}$. So to conserve momentum:

$$2mu = 2m'\frac{u}{\gamma}$$

$$\therefore \boxed{m' = \gamma m} \quad \text{and} \quad \boxed{p = \gamma m v}$$

Note

The apparent increase in mass has been confirmed experimentally!

Relativistic Energy

Now that we have a better formula for momentum, we can get a better formula for force which is defined by Newton's Second Law as the rate of change of linear momentum.

$$\therefore \quad F = \frac{dp}{dt} = \frac{d}{dt}(\gamma m v)$$

We can use this to work out the work done in moving a particle from rest ($x=0, t=0, v=0$) to some future point ($x=x$, $t=t$, $v=v$). The work done is the kinetic energy of the particle. Let's omit vectors for convenience.

$$\therefore \quad K = \int_0^x F \cdot dx = \int_0^t F \cdot \frac{dx}{dt}\, dt = \int_0^t F \cdot v\, dt$$

but $F = \frac{dp}{dt}$ so $K = \int_0^t v \cdot \frac{dp}{dt}\, dt = \int_0^p v \cdot dp$

but $\frac{d}{dt}(pv) = v\frac{dp}{dt} + p\frac{dv}{dt} \Rightarrow d(pv) = v\,dp + p\,dv$

$$\therefore v\,dp = d(pv) - p\,dv$$

$$K = \int_0^v d(pv) - \int_0^v p\,dv = \Big[pv\Big]_0^v - \int_0^v p\,dv \qquad \text{but } p = \gamma m v$$

$$K = \left[\frac{mv^2}{\sqrt{1-\frac{v^2}{c^2}}}\right]_0^v - m\underbrace{\int_0^v \frac{v}{\sqrt{1-\frac{v^2}{c^2}}}\,dv}_{\text{how do we integrate this?}}$$

We can write it as $mc\int \frac{v}{\sqrt{c^2-v^2}}\,dv$ which is a known standard integral with the solution

$$-mc\left[\sqrt{c^2-v^2}\right]$$

This gives us:

$$K = \left[\frac{mv^2}{\sqrt{1 - v^2/c^2}}\right]_0^v + mc\left[\sqrt{c^2 - v^2}\right]_0^v$$

$$= \left[\frac{mv^2}{\sqrt{1 - v^2/c^2}}\right]_0^v + mc^2\left[\sqrt{1 - v^2/c^2}\right]_0^v$$

$$= mc^2\left[\frac{v^2/c^2}{\sqrt{1 - v^2/c^2}} + \sqrt{1 - v^2/c^2}\right]_0^v$$

$$= mc^2\left\{\frac{v^2/c^2}{\sqrt{1 - v^2/c^2}} + \sqrt{1 - v^2/c^2}\right\} - mc^2$$

$$= \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2$$

$$\therefore \boxed{K = \gamma mc^2 - mc^2}$$

Note that there are two parts to this equation

$$\boxed{K = \text{Total Energy} - \text{Rest Mass Energy}}$$

Note that at low speeds, this reduces to the Newtonian $\frac{1}{2}mv^2$. As the velocity v approaches the speed of light, the kinetic energy becomes infinite for any finite m. Thus, it requires an infinite amount of work to accelerate a particle to the speed of light which is in practice impossible for finite masses.

Relativistic Mass

We saw from the derivation of momentum that there is an apparent change in mass for moving particles $m' = \gamma m$. When the object is at rest, this mass is called the rest mass m_0.

The rest mass energy is the energy a particle has simply by virtue of existence.

$K = \gamma mc^2 - mc^2$

If at rest, $K = 0$

We define a total relativistic energy to be $E = K +$ rest mass

$\therefore \boxed{E_0 = mc^2}$

All of this means we need only worry about two conservation laws in relativistic calculations:

> Conservation of total relativistic momentum.
> Conservation of total relativistic energy.

Energy Momentum Relations

Relativistic Momentum ($P = \gamma mv$) and Relativistic Energy ($E = \gamma mc^2$) are related by:

$$\boxed{p = \frac{E}{c^2} v}$$

This also applies for particles travelling at the speed of light (only possible for massless particles). But velocity v isn't a very useful quantity for particles travelling near c. Then, we can use:

$$\boxed{E^2 - p^2c^2 = m^2c^4}$$

The formula $\boxed{E^2 - p^2c^2 = m^2c^4}$ is an example of an invariant.

This means the formula is true regardless of the reference frame we use to measure the quantities.

i.e, in another frame we would find $E'^2 - p'^2c^2 = m^2c^4$!

For a particle at rest, we get $E = mc^2$ and for massless particles we get $E = pc$ like on the previous page.

Binding Energy

A good example of the mass-energy interchange $E = mc^2$ is that the mass of a nucleus is always less than the sum of the masses of its constituents. The mass deficit is the mass-equivalent of the energy released when the nucleons bind.

A note on units

Energies are typically quoted in electron volts (eV). 1 eV is equivalent to the energy gained by an electron falling through a potential of 1 Volt.

$1\,eV = 1.602 \times 10^{-19}\,J$

Masses are typically quoted in eV/c^2

Momentums are typically quoted in eV/c

All this is just to make the c's cancel out in the various equations in relativity.