Complex Wavefunctions

We saw in the sinusoidal solutions section that an advantage of sinusoidal waveforms is that they are linear and orthogonal so they can be added together to construct other solutions. However, sinusoidal waveforms are not solutions to wave equations of some systems we observe in real life, for example the thermal diffusion equation or the Schrodinger Equation for a free quantum particle. In these cases, a complex exponential solution is more convenient.

Consider two solutions to a wave equation:

$$\psi_1 = \cos(kx - \omega t) \quad \psi_2 = \sin(kx - \omega t)$$

These can be added to make another solution:

$\psi = a\psi_1 + b\psi_2$ where a and b are arbitrary constants

Suppose that b is complex, $b = ia$:

$\psi = a\cos(kx - \omega t) + ia\sin(kx - \omega t)$ which is better written:

$\underline{\underline{\psi = a\exp(i(kx - \omega t))}}$ as a complex exponential

On an Argand Diagram, this would look like a vector of length a that rotates anticlockwise with angular frequency ω as the position coordinate x is advanced. We can confirm this can be a solution to general wave equation $\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial x^2}$:

$\psi = ae^{i(kx - \omega t)}$ so: $\frac{\partial^2 \psi}{\partial t^2} = -a\omega^2 e^{i(kx - \omega t)}$ and $\frac{\partial^2 \psi}{\partial x^2} = -ak^2 e^{i(kx - \omega t)}$

Subbing into wave eqn:

$a\omega^2 e^{i(kx - \omega t)} = v^2 k^2 a e^{i(kx - \omega t)} \Rightarrow v = \pm \frac{\omega}{k}$ as we expect.

So a complex exponential function is theoretically a solution to a wave equation

But what does it mean for the amplitude (arbitrary constant) of a sinusoid (that makes up the solution) to be complex? How can a physical quantity be complex?
Well, here is where we say that the complex solution is just an intermediary step that we use. Physical properties must always correspond to real expressions but sometimes it is easier to do wave analysis on complex exponentials and convert them to real quantities later.

So how do we convert to real numbers? Sinusoidal waves can be formed from superpositions or manipulations of complex exponentials. eg.

$$\cos(kx - \omega t) = \frac{1}{2}\left\{ e^{i(kx-\omega t)} + e^{-i(kx-\omega t)} \right\}$$

or

$$\cos(kx - \omega t) = \frac{1}{2}\left\{ e^{i(kx-\omega t)} + \left(e^{i(kx-\omega t)}\right)^{*} \right\}$$

$$= Re\left\{ e^{\pm i(kx-\omega t)} \right\}$$

Dispersion in Dissipative Systems

One example where it is much easier to use complex exponentials is in systems in which energy is lost over time, for example through friction. Let's go back to the long string example but this time we need to add an extra term for friction, which we take to have magnitude proportional to the transverse velocity and is in the opposite direction to the transverse velocity.

$$M \frac{\partial^2 \psi}{\partial t^2} = W \frac{\partial^2 \psi}{\partial x^2} - \gamma \frac{\partial \psi}{\partial t}$$

Clearly a sine wave canot satisfy this as a sine way is perfectly periodic with no decay.

So let's try a complex exponential:

$\psi(x,t) = a e^{i(kx-\omega t)}$

$$\frac{\partial^2 \psi}{\partial x^2} = -ak^2 e^{i(kx-\omega t)} \qquad \frac{\partial \psi}{\partial t} = -ia\omega e^{i(kx-\omega t)} \qquad \frac{\partial^2 \psi}{\partial t^2} = -a\omega^2 e^{i(kx-\omega t)}$$

subbing into the wave equation and dividing by $ae^{i(kx-\omega t)}$:

$$M(-\omega^2) = W(-k^2) - \gamma(-i\omega) \Rightarrow M\omega^2 = Wk^2 - i\gamma\omega$$

$$\Rightarrow k^2 = \frac{M}{W}\omega^2 + i\omega\frac{\gamma}{W} = \omega^2 \frac{M}{W}\left[1 + \frac{i\gamma}{M\omega}\right]$$

$$\therefore \ k = \pm\sqrt{\frac{M}{W}}\,\omega\sqrt{1 + i\frac{\gamma}{M\omega}}$$

if we now use binomial expansion on the second square root term:

$$k \approx \pm\sqrt{\frac{M}{W}}\,\omega\left(1 + \frac{i\gamma}{2M\omega}\right)$$

This is the dispersion relation so we can write the solution to the wave equation as:

$$\psi(x,t) = a e^{i(\pm\sqrt{\frac{M}{W}}\omega x - \omega t) \pm i\sqrt{\frac{M}{W}}\frac{\gamma}{2M\omega}x}$$

$$= a\exp\left[i\left(\pm\sqrt{\frac{M}{W}}\omega x - \omega t\right)\right]\exp\left[\sqrt{\frac{M}{W}}\frac{\gamma}{2M\omega}x\right]$$

This probably won't come up in this form in the exam so don't worry! It's just proof that complex exponentials are useful!