Fourier Series and Transforms

As we saw when we discussed linearity previously, it is possible to make a new solution by superposing other solutions. This is the principle of fourier synthesis and analysis.

The principle of fourier synthesis

Any wave function may be reproduced by simply adding together sinusoidal components with appropriate amplitudes c_n and phases φ_n that is, we can write:

$$\psi(t) = \sum_n c_n \cos(\omega_n t + \varphi_n)$$

$$\psi(t) \boxed{= \sum_n a_n \cos(\omega_n t) + b_n \sin(\omega_n t)}$$

where $\boxed{a_n = c_n \cos\varphi_n}$ and $\boxed{b_n = -c_n \sin\varphi_n}$

writing a wave function in this form is called a Fourier series

We can also write this as a Fourier Integral:

$$\boxed{\psi(t) = \int_{-\infty}^{\infty} a(\omega)\cos(\omega t) + b(\omega)\sin(\omega t)\, d\omega}$$

where

$$\boxed{\begin{aligned} a(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(t)\cos(\omega t)\, dt \\ b(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(t)\sin(\omega t)\, dt \end{aligned}}$$

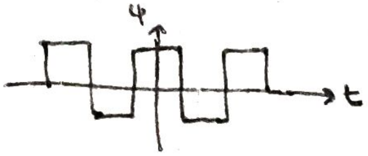
Fourier analysis is breaking down a waveform into single-frequency components and Fourier synthesis is the reverse process of constructing a waveform from single-frequency components.

The underlying mathematics depends on the orthogonality of sine waves.

Turning waveforms into series and integrals and vice versa is known as Fourier Transforms

Square Wave Example

Consider a square wave symmetric about $t=0$ and oscilates between $\pm a_0$ with a period T:



From this, we can intuitively see that the sine wave contributions must be 0 since it is symmetric about the y-axis, which a sine function is not.

So from the fourier series: $\psi(t) = \sum_n a_n \cos(\omega_n t) + b_n \sin(\omega_n t)$

we can say $b_n = 0$ so $\psi(t) = \sum_n a_n \cos(\omega_n t)$

what is ω_n? The wave is periodic with period T so the basic sets (components) must also be periodic with an integer multiple of ω_n so:

$$\omega = 2\pi f = \frac{2\pi}{T} \quad \text{so} \quad \omega_n = \frac{2\pi n}{T}$$

This gives $\psi(t) = \sum_n a_n \cos\left(\frac{2\pi n}{T} t\right)$ Now we need to work out a_n

we can do this by multiplying by an arbitrary cosine wave of the same period and integrating w.r.t. time:

$$\int_{-T/2}^{T/2} \psi(t) \cos\left(\frac{2\pi M}{T} t\right) dt = \int_{-T/2}^{T/2} \sum_n a_n \cos\left(\frac{2\pi n}{T} t\right) \times \cos\left(\frac{2\pi M}{T} t\right) dt$$

but $\cos A \cos B = \frac{1}{2}\left[\cos[A+B] + \cos[A-B]\right]$

$$\therefore \int_{-T/2}^{T/2} \psi(t) \cos\left(\frac{2\pi M}{T} t\right) dt = \int_{-T/2}^{T/2} \frac{a_m}{2}\left\{\cos\left(\frac{2\pi(M+n)}{T} t\right) + \cos\left(\frac{2\pi(M-n)}{T} t\right)\right\} dt$$

if $M \neq n$, then the integrand is 0

$$\text{if } M = n: \int_{-T/2}^{T/2} \psi(t) \cos\left(\frac{2\pi M}{T} t\right) dt = \int_{-T/2}^{T/2} \frac{a_M}{2}\left\{\cos\left(\frac{4\pi M}{T} t\right) + 1\right\} dt$$

$$= \frac{a_M}{2}\left[t\right]_{-T/2}^{T/2} = \frac{a_M T}{2}$$

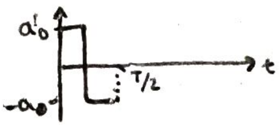
rearranging: $$a_M = \frac{2}{T} \int_{-T/2}^{T/2} \psi(t) \cos\left(\frac{2\pi M}{T} t\right) dt$$

$$a_m = \frac{2}{T}\int_{-T/2}^{T/2} \psi(t)\cos\left(\frac{2\pi m}{T}t\right) dt$$

if our waveform was not symmetric, we would have found a corresponding $b_m = \frac{2}{T}\int_{-T/2}^{T/2} \psi(t)\sin\left(\frac{2\pi m}{T}t\right) dt$

since our waveform is symmetric, we can simplify the integral by integrating between 0 and $T/2$ and doubling the result so:

$$a_m = \frac{4}{T}\int_{0}^{T/2} \psi(t)\cos\left(\frac{2\pi m}{T}t\right) dt$$



$$= \frac{4}{T}\left\{\int_0^{T/4} a_0\cos\left(\frac{2\pi m}{T}t\right) dt + \int_{T/4}^{T/2} -a_0\cos\left(\frac{2\pi m}{T}t\right) dt\right\}$$

$$= \frac{4}{T}\cdot\frac{T}{2\pi m}\, a_0 \left\{\left[\sin\left(\frac{2\pi m}{T}t\right)\right]_0^{T/4} - \left[\sin\left(\frac{2\pi m}{T}t\right)\right]_{T/4}^{T/2}\right\}$$

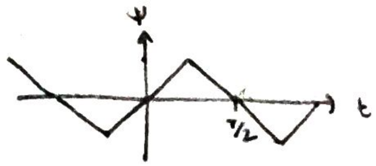
$$= \frac{2a_0}{\pi m}\left\{\sin\left(\frac{\pi m}{2}\right) - \sin(\pi m) + \sin\left(\frac{\pi m}{2}\right)\right\}$$

$$= \frac{2a_0}{\pi m}\left\{2\sin\left(\frac{\pi m}{2}\right)\right\} \qquad \therefore\ a_m = \frac{4a_0}{\pi m}\sin\left(\frac{\pi m}{2}\right)$$

so $$\underline{\underline{\psi(t) = \sum_m \frac{4a_0}{\pi m}\sin\left(\frac{\pi m}{2}\right)\cos\left(\frac{2\pi m}{T}t\right)}}$$

Triangle Wave Example

Let's consider a triangle wave, antisymmetric about $t=0$, oscillating between $\pm b_0$ with period T:



For this, we can intuitively see that the cosine wave contributions will be 0 since it is antisymmetric. so $a_n = 0$ giving us:

$$\psi(t) = \sum_n b_n \sin(\omega_n t)$$

$$\omega = 2\pi f = \frac{2\pi}{T} \quad \text{so} \quad \omega_n = \frac{2\pi n}{T} \quad \text{so:}$$

$$\psi(t) = \sum_n b_n \sin\left(\frac{2\pi n}{T} t\right)$$

To find b_n we can do a similar thing as for square waves:

$$\int_{-T/2}^{T/2} \psi(t) \sin\left(\frac{2\pi m}{T} t\right) dt = \int_{-T/2}^{T/2} \sum_n b_n \sin\left(\frac{2\pi n}{T} t\right) \sin\left(\frac{2\pi m}{T} t\right) dt$$

but $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

$$\therefore \int_{-T/2}^{T/2} \psi(t) \sin\left(\frac{2\pi m}{T} t\right) dt = \int_{-T/2}^{T/2} \sum \frac{b_n}{2} \left\{\cos\left(\frac{2\pi(m-n)}{T} t\right) - \cos\left(\frac{2\pi(m+n)}{T} t\right)\right\} dt$$

if $m \neq n$ then the integral is 0. if $m = n$, $\cos(0) = 1$ so:

$$\text{if } m=n: \int_{-T/2}^{T/2} \psi(t) \sin\left(\frac{2\pi m}{T} t\right) dt = \int_{-T/2}^{T/2} \frac{b_n}{2} \left\{1 - \cos\left(\frac{4\pi m}{T} t\right)\right\} dt$$

$$\int_{-T/2}^{T/2} \psi(t) \sin\left(\frac{2\pi m}{T} t\right) dt = \frac{b_n}{2} \left[t\right]_{-T/2}^{T/2} = \frac{T b_n}{2}$$

$$\therefore \quad b_m = \frac{2}{T} \int_{-T/2}^{T/2} \psi(t) \sin\left(\frac{2\pi m}{T} t\right) dt$$

if we now put in the triangular wavefunction for $\psi(t)$,

we find $b_m = \frac{8 b_0}{(\pi m)^2} \sin\left(\frac{m\pi}{2}\right)$

$$\text{so} \quad \psi(t) = \sum_m \frac{8 b_0}{(\pi m)^2} \sin\left(\frac{m\pi}{2}\right) \sin\left(\frac{2\pi m}{T} t\right)$$

Alternate forms of Fourier Transform

Just as we've been working with summations, we can also easily represent waves as integrals in the Fourier Transform:

For example: for a function symmetric about $t=0$ (so $b_n = 0$):

$$\psi(t) = \int_{-\infty}^{\infty} a(\omega)\cos(\omega t)\,d\omega \qquad a(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(t)\cos(\omega t)\,dt$$

The $\frac{1}{2\pi}$ factor is arbitrary and we can define it based on whatever is most convenient for the specific problem:

$$\psi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} a(\omega)\cos(\omega t)\,d\omega \qquad a(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \psi(t)\cos(\omega t)\,dt$$

This is the Fourier Transform.

Complex Fourier Transforms

We have until now used sine and cosine functions as our base sets but it is also possible to write this as a complex exponential:

$$\boxed{a(\omega)\cos(\omega t) + b(\omega)\sin(\omega t) = g(\omega)e^{-i\omega t}} \quad \text{where} \quad g(\omega) = a(\omega) - ib(\omega)$$

This is because:

$$g(\omega)e^{-i\omega t} = a(\omega)[\cos(\omega t) + i\sin(\omega t)] - ib(\omega)[\cos(\omega t) + i\sin(\omega t)]$$

$$= a(\omega)\cos(\omega t) + ia(\omega)\sin(\omega t) - ib(\omega)\cos(\omega t) + b(\omega)\sin(\omega t)$$

$$= \underbrace{a(\omega)\cos(\omega t) + b(\omega)\sin(\omega t)}_{\text{real}} + \underbrace{i[a(\omega)\sin(\omega t) - b(\omega)\cos(\omega t)]}_{\text{imaginary so neglected for non-complex } \psi}$$

So we write the fourier transform as:

$$\boxed{\psi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)e^{+i\omega t}\,d\omega} \qquad \boxed{g(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \psi(t)e^{-i\omega t}\,dt}$$

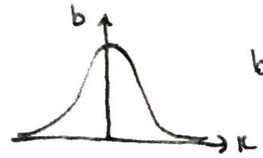
Conjugate Variables

Our fourier transforms until now always have been between the t and ω variables. We can call these variables "conjugate variables". Other conjugate variables also exist like x and k:

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k) e^{+ikx} \, dk$$

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x) e^{-ikx} \, dx$$

Let's try the fourier transform of a gaussian:



width is $\frac{1}{k_0}$

$$b(k) = e^{-(k/k_0)^2}$$

What would this look like in position space?

Because this is symmetric about $k=0$, we can assume there are no sine components.

$$\text{so} \quad \psi(x) = \int_{-\infty}^{\infty} \exp\left(-\left(\frac{k}{k_0}\right)^2\right) \cos(kx) \, dk$$

$$= \int_{-\infty}^{\infty} \exp\left(-\left(\frac{k}{k_0}\right)^2\right) \cdot \frac{1}{2}\left[e^{ikx} + e^{-ikx}\right] dk$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \exp\left[-\left(\frac{k}{k_0}\right)^2 + ikx\right] + \exp\left[-\left(\frac{k}{k_0}\right)^2 - ikx\right] dk$$

$$= \frac{1}{2}\left\{ \underbrace{\int_{-\infty}^{\infty} \exp\left[-\left(\frac{k}{k_0}\right)^2 + ikx\right] dk} + \int_{-\infty}^{\infty} \exp\left[-\left(\frac{k}{k_0}\right)^2 - ikx\right] dk \right\}$$

if we make the substitution $k \Rightarrow -k$, we get $dk \Rightarrow -dk$ but we make it positive again by flipping the integral limits: $\int_{0}^{5} x \, dx = -\int_{5}^{0} dx$ which doesn't change anything since the limits are to infinity so both integrals become identical:

$$\psi(x) = \int_{-\infty}^{\infty} \exp\left[-\left(\frac{k}{k_0}\right)^2\right] \exp\left[-ikx\right] dk$$

$$\Psi(x) = \int_{-\infty}^{\infty} \exp\left[-\left(\frac{k}{k_0}\right)^2\right] \exp\left[-ikx\right] dk$$

Add and subtract $\left(\frac{ik_0 x}{2}\right)^2$ to exponents

$$= \int_{-\infty}^{\infty} \exp\left[-\left\{\left(\frac{k}{k_0}\right)^2 + ikx + \left(\frac{ik_0 x}{2}\right)^2\right\}\right] \exp\left[\left(\frac{ik_0 x}{2}\right)^2\right] dk$$

Completing the square

$$= \exp\left[\left(\frac{ik_0 x}{2}\right)^2\right] \int_{-\infty}^{\infty} \exp\left[-\left(\frac{k}{k_0} + \frac{ik_0 x}{2}\right)^2\right] dk$$

sub $\frac{k'}{k_0} = \frac{k}{k_0} + \frac{ik_0 x}{2}$: $dk = dk'$

$$\Psi(x) = \exp\left[\left(\frac{ik_0 x}{2}\right)^2\right] \int_{-\infty}^{\infty} \exp\left[-\left(\frac{k'}{k_0}\right)^2\right] dk'$$

sub $\frac{k'}{k_0} = z$: $dk' = k_0 \, dz$

$$\Psi(x) = \exp\left[-\left(\frac{k_0 x}{2}\right)^2\right] \int_{-\infty}^{\infty} \underbrace{\exp(-z^2)\, k_0 \, dz}_{\text{standard integral } = \sqrt{\pi}}$$

$$\therefore \Psi(x) = \underline{\underline{\sqrt{\pi}\, k_0 \exp\left[-\left(\frac{x k_0}{2}\right)^2\right]}}$$

which is still a gaussian. So the transform of a gaussian is a gaussian! The width this time is $2/k_0$

While we're at it, let's work out the area under the curve:

$$\text{Area} = \int_{-\infty}^{\infty} \sqrt{\pi}\, k_0 \exp\left[-\left(\frac{x k_0}{2}\right)^2\right] dx$$

sub $z = \frac{x k_0}{2} \Rightarrow dx = \frac{2}{k_0} dz$

$$= \sqrt{\pi}\, k_0 \cdot \frac{2}{k_0} \underbrace{\int_{-\infty}^{\infty} \exp\left[-z^2\right] dz}_{\text{standard integral } = \sqrt{\pi}}$$

$$= \sqrt{\pi} \cdot 2\sqrt{\pi} = \underline{\underline{2\pi}}$$

The Dirac-delta function

Now we have found a gaussian of width $\frac{2}{K_0}$ and area 2π, we can think how to make this into a dirac delta function

The $\delta(x)$ function has area 1 and is infinitely thin so we can say:

$$\delta(x) = \lim_{K_0 \to 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left[-\left(\frac{k}{k_0}\right)^2\right] \cos(kx)\, dk$$

$$\boxed{\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos(kx)\, dk}$$

We can shift the position of the function by using $x - x_0$:

$$\delta(x - x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos[k(x - x_0)]\, dk$$ is a δ function centred on x_0

Since we are integrating over symmetric interval, we can add any odd function (say $i\sin(k(x-x_0))$) to the integral and it will still be the same:

$$\delta(x - x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos[k(x - x_0)] + \underbrace{i\sin[k(x - x_0)]}_{\text{odd } = 0}\, dk$$

which we can write as a complex exp.

$$\boxed{\delta(x - x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x - x_0)}\, dk}$$

This is a very important function and we will find it in many places later.

We can also use these to derive the Fourier Transforms. If we think it is possible to compose any waveform with sinusoidal components, since we construct $\delta(x - x_0)$ from sinusoids, we should be able to write any waveform as a series of δ functions

Consider an arbitrary function $\psi(x)$:

$$\psi(x) = \int_{-\infty}^{\infty} \psi(x_0)\, \delta(x - x_0)\, dx_0$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cos[k(x - x_0)]\, \psi(x_0)\, dk\, dx_0$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{\cos(kx)\cos(kx_0) + \sin(kx)\sin(kx_0)\}\, \psi(x_0)\, dk\, dx_0$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos(kx) \underbrace{\int_{-\infty}^{\infty} \psi(x_0) \cos(kx_0)\, dx_0}_{a(k)}\, dk$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \sin(kx) \underbrace{\int_{-\infty}^{\infty} \psi(x_0) \sin(kx_0)\, dx_0}_{b(k)}\, dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} a(k)\cos(kx) + b(k)\sin(kx)\, dk$$

so we have derived our Fourier Transform equations!

Bandwidth Theorem

If we take any two conjugate variables, for example x and k or t and ω, then their standard deviations (denoted by Δ) make the relation:

$$\Delta x\, \Delta k \geq \frac{1}{2}$$

$$\Delta t\, \Delta \omega \geq \frac{1}{2}$$

This is related to the uncertainty principle $\Delta x \Delta p \geq \frac{\hbar}{2}$ since energy and momentum are linked to ω and k by:

$$E = \hbar\omega \qquad p = \hbar k$$

so $\Delta x (\Delta p) \frac{1}{\hbar} \geq \frac{1}{2} \Rightarrow \Delta x \Delta p \geq \frac{\hbar}{2}$

$$\Delta t (\Delta E) \frac{1}{\hbar} \geq \frac{1}{2} \Rightarrow \Delta t \Delta E \geq \frac{\hbar}{2}$$

Power and Intensity

In our long string example, we found the power to be:

$$P(x,t) = W v_p \left(\frac{\partial \psi(x,t)}{\partial x} \right)^2$$

For a general waveform $\psi(x,t) = a\cos(kx - \omega t)$:

$$P(x,t) = W v_p \left[ak\sin(kx - \omega t) \right]^2 \quad \text{omitting the minus sign}$$

From $\frac{\omega}{k} = \sqrt{\frac{W}{M}} \Rightarrow W = \frac{\omega^2}{k^2} M$

So $P(x,t) = \omega^2 v_p M a^2 \sin^2(kx - \omega t)$

but $\cos 2A = 1 - 2\sin^2 A$ so $\sin^2 A = \frac{1 - \cos 2A}{2}$:

$$P(x,t) = \omega^2 v_p M a^2 \left\{ \frac{1 - \cos[2(kx - \omega t)]}{2} \right\}$$

we can find the average by simply dropping the oscillatory terms (the average of $\sin(x) = 0$, $\cos(x) = 0$ since symmetric about x axis

so $$\langle P(x) \rangle = \frac{M v_p \omega^2 a^2}{2}$$

We can write $\psi(x)$ as a fourier series with both spacial and time dependence as:

$$\psi(x,t) = \sum_n a_n \cos(k_n x - \omega_n t)$$

where $\omega_n = \frac{2\pi n}{T}$ and $k_n = \frac{\omega_n}{v_p}$

For convenience we'll find power at $x = 0$:

$$P(0,t) = M v_p \left[\sum_n \omega_n a_n \sin(\omega_n t) \right]^2$$

$$= M v_p \sum_m \sum_n \omega_m \omega_n a_m a_n \sin(\omega_m t) \sin(\omega_n t)$$

$$= \frac{M v_p}{2} \sum_m \sum_n \omega_m \omega_n a_m a_n \left\{ \cos[(\omega_m - \omega_n)t] - \cos[(\omega_m + \omega_n)t] \right\}$$

so this is the power at $x = 0$

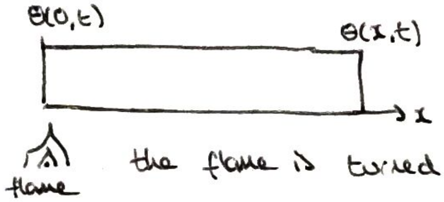
$$\text{Average power} = \frac{M v_p \omega^2 a^2}{2}$$

which we write as:

$$\langle P(0) \rangle = \sum_n \frac{M v_p \omega_n^2 a^2}{2}$$

Fourier Analysis of Dispesive Propagation

Let's consider our thermal waves example only in this case the heat source is turned on and off:



the flame is turned on and off such that at $x=0$, Θ is a square wave

We can write this square wave as $\Theta(0,t) = \sum_n a_n \cos(\omega_n t)$

where $a_n = \frac{4\Theta_0}{\pi n} \sin\left(\frac{n\pi}{2}\right)$ from our square wave example.

From the thermal waves example, we saw that Θ decayed with e^{-kx} so we can write:

$$\Theta(x,t) = \sum_{n=1}^{\infty} a_n \cos[(k_n x - \omega_n t)] \exp[-k_n x]$$

This is in the form of the solution we found for this system.

where $k_n = \sqrt{\frac{\omega_n C \rho}{2K}}$ from the example

What we see here is that writing the wave in fourier form massively simplifies the problem.

Transfer Functions

So we see that at $x = 0$ $\quad \theta(0,t) = \sum_n a_n \cos[\omega_n t]$.

At some arbitrary x: $\quad \theta(x,t) = \sum_n a_n \exp[-K_n x] \cos[K_n x - \omega_n t]$

So we can see that at $x=0$ the amplitude is a_n and at x, the amplitude is $a_n \exp(-K_n x)$ so we can say that the relative amplitude, denoted $\alpha_n(x)$ is:

$$\alpha_n(x) = \exp(-K_n x)$$

The phase offset is $K_n x$ so:

$$\varphi_n(x) = K_n x$$

We define a transfer function as $H_n(x) = \alpha(x) e^{i\varphi(x)}$

so for this system: $\quad H_n(x) = \exp(-K_n x)\exp(iK_n x)$

$$H_n(x) = \exp(-K_n x + iK_n x) = \underline{\exp[-i(1+i)K_n x]}$$

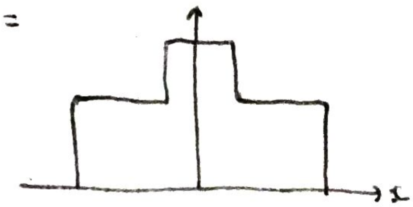
since we know $K_n = \sqrt{\frac{\omega_n C\rho}{2K}}$: $\quad \underline{\underline{H(x) = \exp\left[-i(1+i)\sqrt{\frac{\omega_n C\rho}{2K}}\, x\right]}}$

Thus, we have derived the transfer function for this system.

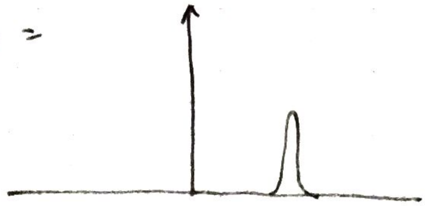
Convolution

Convolution "fits" one curve to another as show below:

$F(x) =$

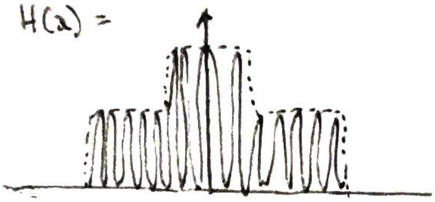


$G(x) =$



So if we denote convolution as $*$ so $H(x) = F(x) * G(x)$

then $H(x) =$



so the $G(x)$ function has been "fit" to the

So what are we actually doing with $*$?

Consider $H_1(x) = F(x_1)\,G(x - x_1)$ Here, we have moved the G function into the F, and adjusted its height to be the height of F at that point. So we have made one small part of H. We need all of H so we can extend this to:

$$\boxed{H(x) = F(x) * G(x) = \int_{-\infty}^{\infty} F(x_1)\,G(x - x_1)\,dx_1}$$

An important link to the Fourier Transform is the convolution theorem: [here Fourier transform is denoted $\mathcal{F}(\,)$]

$$\boxed{\mathcal{F}\{F(x) * G(x)\} = \mathcal{F}\{F(x)\} \cdot \mathcal{F}\{G(x)\}}$$

$$\boxed{\mathcal{F}\{F(x) \cdot G(x)\} = \mathcal{F}\{F(x)\} * \mathcal{F}\{G(x)\}}$$