Lagrangian for a Charged Particle

This is non-examinable content in 2019/20

The equation of motion for a charged, moving particle is

$$\frac{d\underline{p}}{dt} = q(\underline{E} + \underline{v} \times \underline{B})$$

The action that reproduces this is $S = \int L \, dt$ where

$$L = \frac{1}{2} m |\dot{\underline{x}}|^2 + q(\dot{\underline{x}} \cdot \underline{A}) - q\phi$$

The Euler-Lagrange equation is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{x}}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\underbrace{\frac{d}{dt}(m\dot{\underline{x}} + q\underline{A})}_{\text{this is } \frac{d\underline{p}}{dt}} - \underline{\nabla}(q\dot{\underline{x}} \cdot \underline{A} - q\phi) = 0$$

$$\Rightarrow \quad \frac{d\underline{p}}{dt} + q\frac{d\underline{A}}{dt} + q\dot{\underline{x}} \cdot \underline{\nabla} \underline{A} - q\nabla(\dot{\underline{x}} \cdot \underline{A}) + q\underline{\nabla}\phi = 0$$

Now we use the identity: $\dot{\underline{x}} \times \underline{\nabla} \times \underline{A} = \underline{\nabla}(\dot{\underline{x}} \cdot \underline{A}) - (\dot{\underline{x}} \cdot \underline{\nabla})\underline{A}$

so we have: $$\frac{d\underline{p}}{dt} = q\left(-\frac{\partial \underline{A}}{\partial t} - \underline{\nabla}\phi\right) + q\dot{\underline{x}} \times \underline{\nabla} \times \underline{A}$$

This is our equation of motion!

Note, the expression for generalised momenta:

$$\underline{p}_{gen} = \frac{\partial L}{\partial \dot{x}} = m\dot{\underline{x}} + e\underline{A}$$

and Hamiltonian:

$$H = \underline{p}_{gen} \cdot \underline{x} - L = \frac{1}{2} m |\dot{\underline{x}}|^2 + e\phi$$

These combine to give four-vector generalised momentum

$$\Rightarrow \quad P^{\mu}_{gen} = m u^{\mu} + e A^{\mu}$$