Nuclear Size and Shape

In the previous section, we assumed the nucleus was a point-like charge. However, this is not true, and we should be modelling it with a charge distribution $\rho(\underline{r})$.

In QM, we have: $\rho(\underline{r}) = Ze|\Psi(\underline{r})|^2$

where Z is an atomic number, and $\Psi(\underline{r})$ is the wave fn. of one of the protons. Note: the term "Nuclear Radius" is not very precise as the wavefn. would extend over all space.

How do we probe this charge distribution? We can use high energy electrons [since it is difficult to produce such high-energy α particles].

How does the $\frac{d\sigma}{d\Omega}\Big|_{Rutherford}$ we calculated in the previous section change if we use high energy electrons instead? Well, we firstly replace Z with 1. [since an electron has charge $-1e$]
The electrons also move relativistically with v close to c.
The combined result was first calculated by Mott:

$$\frac{d\sigma}{d\Omega}\Big|_{Mott} = \frac{d\sigma}{d\Omega}\Big|_{Rutherford}\left(1-\frac{v^2}{c^2}\sin^2\left(\frac{\theta}{2}\right)\right)$$

But we're not done yet!
We need to add another correction factor called the "electric form factor":

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}\Big|_{Mott}|F(q^2)|^2$$

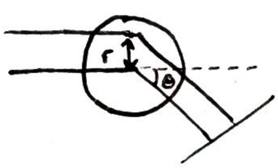
where q is the momentum transferred to the electron in the scattering and $|\underline{q}| = 2p\sin\left(\frac{\theta}{2}\right)$

So what is this electric form factor?

So let's try and understand this electric form factor.
We will first recall that an electron has a de Broglie wavelength $\lambda = \frac{h}{p}$ and when this wavelength is of the order of the "nuclear radius" we get a diffraction pattern.
Proper quantum mechanical treatment shows us that the electric form factor is actually the fourier transform of the charge distribution. For a spherically symmetric charge distribution, this leads to

$$F(q^2) = \frac{4\pi\hbar}{Zeq} \int r\rho(r) \sin\left(\frac{qr}{\hbar}\right) dr$$

Why is this?
Consider:



consider, the part of the wavefront that passes through the nucleus at a distance r from the centre and is scattered at an angle θ, will travel further than the part of the wavefront that passes through the centre by an amount proportional to r.

For this reason, it has a phase change relative to wave passing through centre. The phase change is dependent on θ and equal to $\frac{qr}{\hbar}$ so different parts of the wavefront have different phases. Summing these up gives us the diffraction pattern. The contribution to the amplitude from a part that passes through a distance r from centre is proportional to $\rho(r)$, at r. The total scattering amplitude is therefore the sum of the amplitudes from all these different parts, which is what the integral is doing.

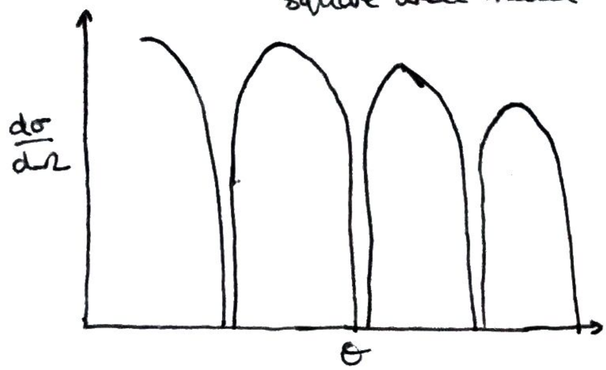
So now let's compute $F(q^2)$ using $\rho(r) = \begin{cases} \frac{3Ze}{4\pi R^3} & \text{for } r < R \\ 0 & \text{for } r > R \end{cases}$

which gives us:

$$F(q^2) = 3\left(\frac{\hbar}{qR}\right)^3 \left(\sin(qR/\hbar) - \frac{qR}{\hbar}\cos(qR/\hbar)\right)$$

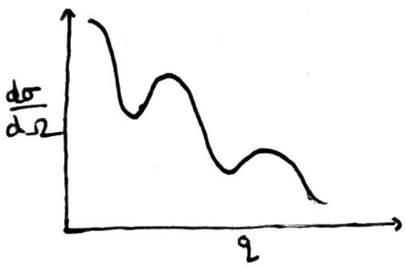
If we sub this into $\frac{d\sigma}{d\Omega}$, and do a plot, we see:

square well model



But this is not what we actually observe in experiments.

In experiments, we see:



This is an example of electrons of energy 1.04 GeV against Ca nucleus.

So what did we do wrong in our model? Well, we said that as $r > R$ the distribution goes to 0, but in reality it never goes to 0. It decreases rapidly and gets closer and closer to 0.

We can obtain an estimate for nuclear radius R by saying we expect the first minima when $\frac{qR}{\hbar} \approx \pi$

A more realistic charge distribution is the Saxon Woods distribution which is:

$$\rho(r) \propto \frac{1}{1 + \exp((r-R)/\delta)}$$

δ here is "surface depth"

saxon woods distribution

This gives us:

This has dips and no zeros and is closer to the experimental data.

In fact, this model fits data from most nuclei well, and gives:

$$R = (1.18\,A^{1/3} - 0.48)\text{ fm}$$

$$\delta = 0.4 - 0.5 \text{ fm} \quad \text{for } A > 40$$

δ (surface depth) is a measure of the range in r over which the ρ changes from the order of its value at the centre to much smaller than this value.

Electric Quadrupole Moments

We assumed the charge distribution is spherically symmetric so:

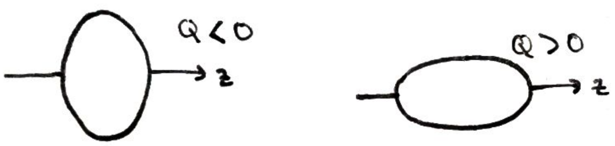
$$\langle x^2 \rangle = \langle y^2 \rangle = \langle z^2 \rangle = \frac{1}{3}\langle r^2 \rangle$$

where $\langle x^2 \rangle = \frac{1}{Ze} \int x^2 \rho(\underline{r})\, d^3\underline{r}$

However, many nuclei are not spherically symmetric and they have an electric quadrupole moment defined w.r.t an axis z as

$$Q = \int (3z^2 - r^2)\, \rho(\underline{r})\, d^3\underline{r}$$

Q/e has dimensions area so is quoted in barns.
The shape of the nucleus depends on Q:



However, even for these nuclei, the electric dipole moment:

$$\underline{d} = \int \underline{r}\, \rho(\underline{r})\, d^3\underline{r}$$

is almost 0. This is because, to a very good approximation, the proton in a nucleus is a parity eigenstate so:

$$\Psi(\underline{r}) = \pm \Psi(-\underline{r})$$

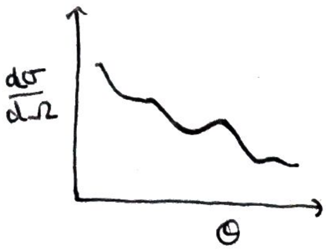
which implies $\rho(\underline{r}) = \rho(-\underline{r})$ so electric dipole vanishes in symmetric integration.

Strong Force Distribution

Strong force holds neutrons and protons together and is strong enough to overcome electrostatic repulsion between protons, though it extends only over a short range.

We can perform scattering experiments with high energy neutrons to probe the strong force distribution.

In this case, the $F(q^2)$ is the form-factor associated with strong force.



This is the experimental data from scattering of neutrons with 14MeV against Ni target.

The Saxon-Woods model is also useful here and gives a nuclear radius for large A of:

$$R = 1.2\, A^{1/3}\ \text{fm}$$

and surface depth:

$$\delta = 0.75\ \text{fm}$$

So the strong force extends over approximately same region as nuclear charge and "volume" of nucleus is proportional to number of nucleons.