Linearity and the Superposition Principle

The wave equations we have seen until now have all been linear in the wave displacement ψ. This means if ψ_1 and ψ_2 are both solutions, then the superposition $\psi = \psi_1 + \psi_2$ is also a solution. In general, we can say for linear solutions that a solution $\psi(x,t)$ is given by:

$$\psi(x,t) = \sum_n c_n \psi_n(x,t)$$ where c_n is some scaling coefficient.

We used this principle of superposition to make standing wave solutions as well as to convert between complex exponential and sinusoidal oscillations.

An example of a non-linear wave equation is: $\frac{\partial^2 \psi}{\partial t^2} = \psi \frac{\partial \psi}{\partial x}$

Here, if we double ψ, i.e $\psi \rightarrow 2\psi$ then the right hand side is quadrupled but the LHS is only doubled.

So it is nonlinear. We can also show this by substituting in $\psi = a\psi_1 + b\psi_2$. This should not satisfy the equation.

Beats

Let's consider a wavefunction that is the superposition of 2 basic wavefunctions:

$$\psi(x,t) = \cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)$$

$$= \cos\left[\frac{k_1+k_2}{2}x + \frac{k_1-k_2}{2}x - \frac{\omega_1+\omega_2}{2}t - \frac{\omega_1-\omega_2}{2}t\right]$$

$$+ \cos\left[\frac{k_1+k_2}{2}x - \frac{k_1-k_2}{2}x - \frac{\omega_1+\omega_2}{2}t + \frac{\omega_1+\omega_2}{2}t\right]$$

$$= \cos\left[\left(\frac{k_1+k_2}{2}x - \frac{\omega_1+\omega_2}{2}t\right) + \left(\frac{k_1-k_2}{2}x - \frac{\omega_1-\omega_2}{2}t\right)\right]$$

$$+ \cos\left[\underbrace{\left(\frac{k_1+k_2}{2}x - \frac{\omega_1+\omega_2}{2}t\right)}_{\alpha} - \underbrace{\left(\frac{k_1-k_2}{2}x - \frac{\omega_1-\omega_2}{2}t\right)}_{\beta}\right]$$

$\therefore \ \psi = \cos(\alpha+\beta) + \cos(\alpha-\beta)$

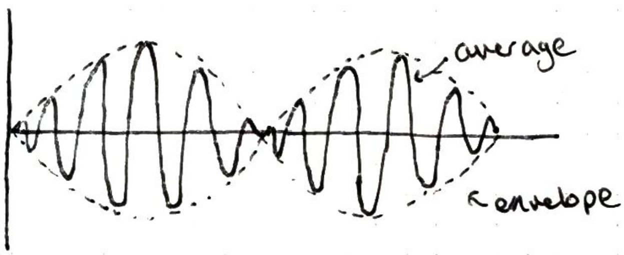
but using the identity $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$:

$$\psi = \cos\alpha\cos\beta - \sin\alpha\sin\beta + \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\psi = \underline{2\cos\alpha\cos\beta}$$

$$\therefore \ \psi(x,t) = 2\underbrace{\cos\left[\left(\frac{k_1+k_2}{2}x - \frac{\omega_1+\omega_2}{2}t\right)\right]}_{\text{average } k,\omega}\underbrace{\cos\left[\left(\frac{k_1-k_2}{2}x - \frac{\omega_1-\omega_2}{2}t\right)\right]}_{\text{envelope}}$$

This gives us the phenomenon of beating:



Group Velocity

From the example of beating above, we can say the phase velocity is the velocity of the "average":

$$V_p = \frac{(\omega_1+\omega_2)/2}{(k_1+k_2)/2} = \frac{\omega}{k} \quad \text{as we expect}$$

the velocity of the envelope, which we call group velocity is:

$$V_g = \frac{(\omega_1-\omega_2)/2}{(k_1-k_2)/2} = \frac{\delta\omega}{\delta k}$$

$$\approx \frac{d\omega}{dk} \text{ as } \delta k \to 0$$

$$\boxed{V_g = \frac{d\omega}{dk}}$$

So we obtain this by differentiating the dispersion relation.