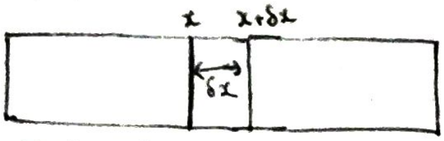
Longitudinal waves

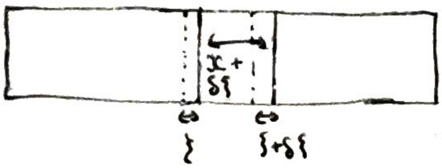
Until now we have only looked at transverse waves in the examples of the guiter string and light waves etc. But what about longitudinal waves? In this section we will cover some examples of such waves:

Sound Waves in an Elastic Medium

Rest:



Displaced:



Consider an elastic medium in which we look at two boundaries. At rest, these have coordinates

$P = x \quad Q = x + \delta x$

After displacement, these have coordinates

$P' = x + \xi \quad Q' = x + \delta x + \xi + \delta \xi$

Since the medium is elastic, any extension of the medium is accompanied by a tension W from Hooke's Law:

$W(x) = EA \dfrac{\delta \xi}{\delta x}$ where E is young's modulus, A is cross-sectional area

If the medium has density ρ, the element we consider has mass $\delta m = \rho A \delta x$

The net force is: $W(x+\delta x) - W(x) = EA \left(\dfrac{\delta \xi}{\delta x}\right)_{(x+\delta x)} - EA \left(\dfrac{\delta \xi}{\delta x}\right)_{(x)}$

We can equate this to Newton's 2nd Law $F = ma$:

$$\rho A \delta x \frac{\partial^2 \xi}{\partial t^2} = EA \left[\left(\frac{\delta \xi}{\delta x}\right)_{(x+\delta x)} - \left(\frac{\delta \xi}{\delta x}\right)_{(x)} \right]$$

$$\rho A \frac{\partial^2 \xi}{\partial t^2} = EA \frac{\left[\left(\frac{\delta \xi}{\delta x}\right)_{(x+\delta x)} - \left(\frac{\delta \xi}{\delta x}\right)_{(x)} \right]}{\delta x}$$ taking limit $\delta x \to 0$

$$\rho A \frac{\partial^2 \xi}{\partial t^2} = EA \frac{\partial^2 \xi}{\partial x^2} \Rightarrow \underline{\underline{\frac{\partial^2 \xi}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 \xi}{\partial x^2}}}$$ we have thus derived the wave equation

v_p here is $v_p = \pm \sqrt{\dfrac{E}{\rho}}$

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 \xi}{\partial x^2} \quad \text{so} \quad v_p = \sqrt{\frac{E}{\rho}}$$

We can actually determine the speed of sound in air from this. Let's start with ideal gas law:

$PV = n_m RT$ so for a small volume element: $P_0 \delta V = \delta n_m R T_0$ (*)

If we expand the volume element: $\delta V' = \delta V + \delta V \frac{\delta \xi}{\delta x}$

$$\delta V' = \delta V \left(1 + \frac{\delta \xi}{\delta x}\right) \quad (**)$$

The new pressure is $P_0 + \delta P$ measured at a new temperature T'

so $(P_0 + \delta P)\delta V' = \delta n_m R T'$ if we assume the process is isothermal we can say $T_0 = T'$ so:

$$(P_0 + \delta P)\delta V' = \delta n_m R T_0$$

$$\delta P = \frac{\delta n_m R T_0}{\delta V'} - P_0 \quad (***)$$

From (*): $\delta n_m = \frac{P_0 \delta V}{R T_0}$ sub into (***):

$$\delta P = \frac{P_0 \delta V}{R T_0} \times \frac{R T_0}{\delta V'} - P_0 = P_0 \frac{\delta V}{\delta V'} - P_0$$

Change in pressure = $\frac{\text{Force}}{\text{Area}}$ so $\delta P = \frac{-W(x)}{A}$ (minus since tension acts inwards)

$$\therefore W(x) = -A\delta P$$

$$= -A\left(P_0 \frac{\delta V}{\delta V'} - P_0\right)$$

from (**): $\frac{\delta V}{\delta V'} = \left(1 + \frac{\delta \xi}{\delta x}\right)^{-1}$ sub into

$$\therefore W(x) = -A P_0\left(\left(1 + \frac{\delta \xi}{\delta x}\right)^{-1} - 1\right)$$

For shallow waves, i.e. $\frac{\delta \xi}{\delta x} \ll 1$, this becomes:

$$W(x) \approx A P_0 \frac{\delta \xi}{\delta x}$$

So we have $W(x) = AP_0 \frac{\delta\xi}{\delta x}$

if we compare this to Hooke's Law $W(x) = AE\frac{\delta\xi}{\delta x}$, we can see that $E = P_0$, so the young's modulus of air is its atmospheric pressure. So we now know $E = P_0 = 10^5$ Pa

if we measure $\rho = 1.29\,\text{kgm}^{-3}$, we find $\underline{V_p = 280\,\text{ms}^{-1}}$:(

Unfortunately this is not very close so let's try a different way

Let's consider an adiabatic process $PV^\gamma = \text{constant} \Rightarrow P_0 \delta V^\gamma = \text{constant}$

After expansion: $(P_0 + \delta P)\delta V'^\gamma = P_0 \delta V^\gamma$ so $\delta P = P_0 \frac{\delta V^\gamma}{\delta V'^\gamma} - P_0$

From $\delta P = \frac{-W}{A}$: $W(x) = -A\delta P$

$$= -AP_0\left[\left(\frac{\delta V}{\delta V'}\right)^\gamma - 1\right] \quad \text{but} \quad \frac{\delta V}{\delta V'} = \left(1 + \frac{\delta\xi}{\delta x}\right)^{-1}$$

so $W(x) = -AP_0\left[\left(1 + \frac{\delta\xi}{\delta x}\right)^{-\gamma} - 1\right]$

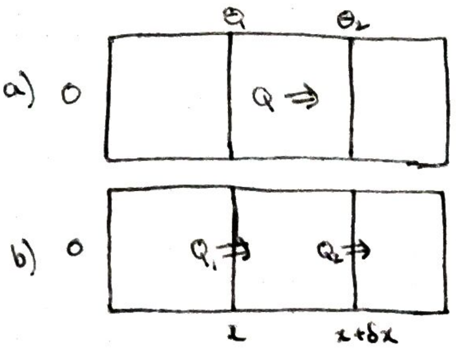
if $\frac{\delta\xi}{\delta x} \ll 1$, we have $W(x) = AP_0\gamma\frac{\delta\xi}{\delta x}$

which tells us $E = P_0\gamma$. For air $\gamma = 1.4$ which gives us a phase velocity $V_p = \sqrt{\frac{\gamma P_0}{\rho}} = 329\,\text{ms}^{-1}$:)

This is much closer to the the experimental value.

Thermal Waves (Diffusion)

Thermal waves are an example of the propagation of a disturbance that is neither longitudinal or transverse.



consider a bar with uniform cross-sectional area A, thermal conductivity K, density ρ and specific heat capacity c. we will look at a slice length δx.

If the temperature at x is θ_1 and the temperature at $x+\delta x$ is θ_2, we can use Fouriers law of thermal conductivity to say:

The heat flowing per second Q is : $Q = \frac{KA}{\delta x}(\theta_1 - \theta_2)$

Taking the limit $\delta x \to 0$: $Q(x) = -K.A \frac{\partial \theta}{\partial x}$

The net heat flowing in is $Q_1 - Q_2$ and this will give a change in temperature we can derive from $Q = \delta m c \frac{\partial \theta}{\partial t}$ (since Q is per second)

$$\therefore \frac{\partial \theta}{\partial t} = \frac{Q}{\delta m c} = \frac{Q(x) - Q(x+\delta x)}{c \rho A \delta x}$$ taking the limit $\delta x \to 0$:

$$\frac{\partial \theta}{\partial t} = -\frac{1}{c\rho A}\frac{\partial Q}{\partial x}$$ but $Q = -KA\frac{\partial \theta}{\partial x} \Rightarrow \frac{\partial Q}{\partial x} = -KA\frac{\partial^2 \theta}{\partial x^2}$

$$\therefore \underline{\underline{\frac{\partial \theta}{\partial t} = \frac{K}{c\rho}\frac{\partial^2 \theta}{\partial x^2}}}$$

we have thus derived the wave equation for thermal waves, this particular eqn is often called the diffusion equation.

This equation is a little different to the ones we've met before since the temporal derivative and spacial derivatives have different orders.

so what would be a solution to this equation?

Let's start with $\theta(x,t) = X(x)T(t)$

This gives $\frac{\partial \theta}{\partial t} = X\frac{\partial T}{\partial t}$ $\quad \frac{\partial^2 \theta}{\partial x^2} = T\frac{\partial^2 X}{\partial x^2}$

$$\frac{\partial \Theta}{\partial t} = \frac{K}{c\rho}\frac{\partial^2 \Theta}{\partial x^2} \quad (*)$$

$$\frac{\partial \Theta}{\partial t} = X\frac{\partial T}{\partial t} \qquad \frac{\partial^2 \Theta}{\partial x^2} = T\frac{\partial^2 X}{\partial x^2}$$ sub these into $(*)$

$$X\frac{\partial T}{\partial t} = \frac{K}{c\rho} T \frac{\partial^2 X}{\partial x^2}$$

$$\Rightarrow \frac{1}{T}\frac{\partial T}{\partial t} = \frac{K}{c\rho}\frac{1}{X}\frac{\partial^2 X}{\partial x^2}$$

The two sides depend on different variable so they must be be equal to some constant. We can call this constant $-i\omega$, for reasons we will see soon.

$$\therefore \frac{1}{T}\frac{\partial T}{\partial t} = -i\omega \quad (1)$$

and $$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -i\omega\frac{c\rho}{K} \quad (2)$$

(1) has the solution $T = T_0 e^{-i\omega t}$ which is why we chose that constat.

if we insert the trial $X = X_0 e^{ikx}$ into (2), we find:

$$-k^2 = \frac{i\omega c\rho}{K} \Rightarrow k = \pm\sqrt{\frac{\omega c\rho}{2K}}(1+i)$$ using the identity $\sqrt{i} = \frac{1+i}{\sqrt{2}}$

let $k_0 = \sqrt{\frac{\omega c\rho}{2K}}$ so $k = \pm k_0(1+i)$

so if $\Theta(x,t) = X(x)T(t)$ then

$$\Theta(x,t) = \underbrace{X_0 T_0}_{\Theta_0} \exp[i(\pm k_0 \pm ik_0)x]\exp[-i\omega t]$$

$$\Theta(x,t) = \Theta_0 \exp[i(\pm k_0 x - \omega t)]\exp[\mp k_0 x]$$

let $\phi = \mathrm{Arg}[\Theta_0]$, a phase shift

$$\Theta(x,t) = |\Theta_0|\exp[i(\pm k_0 x - \omega t + \phi)]\exp[\mp k_0 x]$$

This is complex but our wave is physical so we only need to consider the real part given by $Re(z) = \frac{z+z^*}{2}$

$$\therefore Re(\Theta) = \frac{|\Theta|}{2}\Big\{\exp[i(\pm k_0 x - \omega t + \phi)]\exp[\mp k_0 x] + \exp[-i(\pm k_0 x - \omega t + \phi)]\exp[\mp k_0 x]\Big\}$$

$$= |\Theta|\cos(\pm k_0 x - \omega t + \phi)\exp(\mp k_0 x)$$ which is the solution to the wave eqn.

$$V_p = \frac{\omega}{k_0} = \sqrt{\frac{2K\omega}{c\rho}}$$