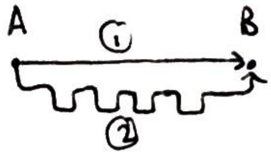
Potential Energy and Energy Conservation

For a conservation force, work done by the force over a displacement between two given points is independent of the path taken.



Work done to move from A to B is the some whether path 1 or path 2 is taken.

A conservation force is a force that "gives back" the work done against it. An example would be when climbing a mountain. The work done climbing a mountain is given back in the form of gravitational potential energy. A force like friction is not a conservation force since the work done against it is not given back and is lost as heat and sound energy.

Potential Energy

> Potential Energy, U, is defined as the difference in work done between any two points.

Since we only see potential energy as a difference, we choose an arbitrary point in space to represent 0 potential energy (or ∞ potential energy for astrophysics) and compare all other points to this. This gives us a formula for Potential Energy:

$$\boxed{U(\underline{r}) = -\int_{r_0}^{r_1} \underline{F}(\underline{r}) \cdot d\underline{r}}$$

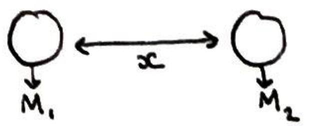
This is measured in Joules, J. The -ve sign is important and can be justified by thinking about doing work against gravity meaning an increase in potential energy.

Deriving Force from Potential Energy

$$U(\underline{r}) = -\int_{r_0}^{r_1} \underline{F}(\underline{r}) \cdot d\underline{r}$$ differentiate both sides with respect to $\underline{r}$.

$$\frac{dU(\underline{r})}{d\underline{r}} = -\underline{F}(\underline{r}) \quad \therefore \quad \boxed{\underline{F}(r) = \frac{dU(\underline{r})}{d\underline{r}}}$$

For an example with gravity:



Working in only 1 dimension, we don't need to worry about vectors. We can therefore write the graviational potential as:

$$U(x) = \frac{GM_1M_2}{x}$$

Note that here, we are taking the reference point at infinity. This is a common practice in astrophysics.

Differentiating both sides with respect to x gives:

$$\frac{dU(x)}{dx} = \frac{GM_1M_2}{x^2}$$ } This is the force, Newton's Law of Gravitation

Some Common Potential Energies

For a spring, we start with Hooke's Law in one dimension:

$F = -kx$ where k is the spring constant.

$$U(x) = -\int_0^{x_1} -kx \, dx = \underline{\underline{\frac{1}{2}kx^2}}$$

For gravitational potential energy near the Earth's surface:

$\underline{F} = -mg\hat{\underline{k}}$ where $\hat{\underline{k}}$ is the unit vector radial to the Earth.

Note that in this case, $F_x = 0$, $F_y = 0$ and $F_z = -mg$. So:

$$U(h) = -\int_0^h F_z \, dz = \underline{\underline{mgh}}$$

Energy Conservation

If the forces acting are conservative, we can derive something called a conservation law. This is a way to find the value of a particular quantity, say energy, that is always constant over time in the system.

For example, for Newton's Second Law:

$$\underline{F} = m\frac{d\underline{v}}{dt} \qquad F = -\frac{dU}{dx} \qquad \text{Therefore:} \quad m\frac{d\underline{v}}{dt} = -\frac{dU}{dx}$$

$$m\frac{d\underline{v}}{dt} + \frac{dU}{dx} = 0 \qquad [\text{multiply by } \underline{v}]$$

$$m\underline{v}\frac{d\underline{v}}{dt} + \underline{v}\frac{dU}{dx} = 0$$

$$\text{But } \frac{dU}{dt} = \frac{dU}{dx} \times \overbrace{\frac{dx}{dt}}^{\underline{v}} \qquad \text{So } \underline{v}\frac{dU}{dx} = \frac{dU}{dt}$$

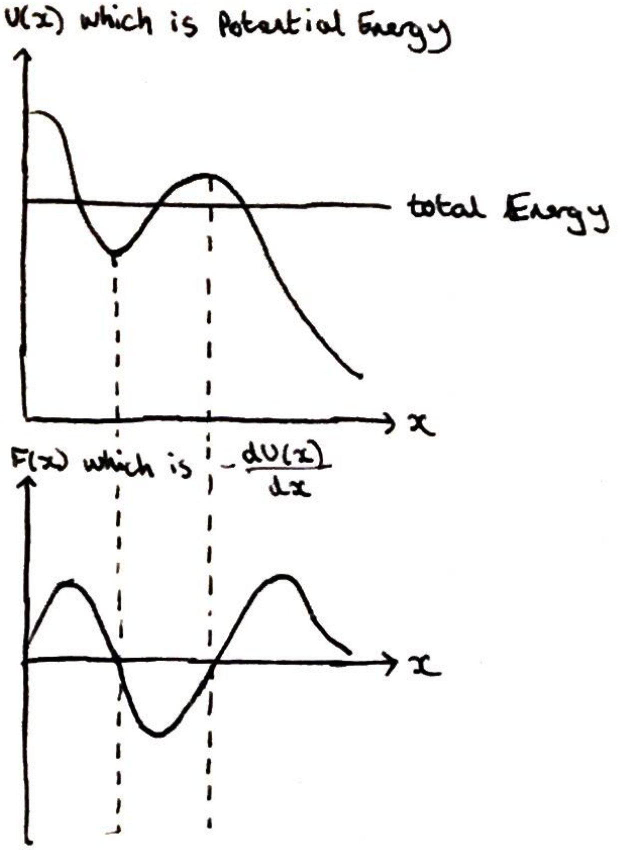
$$\therefore m\underline{v}\frac{d\underline{v}}{dt} + \frac{dU}{dt} = 0$$

$$\text{But } m\underline{v} = \frac{d}{dt}\left(\frac{m\underline{v}^2}{2}\right)$$

$$\therefore \frac{d}{dt}\left(\frac{1}{2}m\underline{v}^2 + U\right) = 0$$

If the derivative with respect to time is 0, then the change in the quantity over time is 0, showing this quantity is a constant with time. In this case, the quantity is $\frac{1}{2}m\underline{v}^2 + U$ which is kinetic energy + potential energy. We can therefore say the total energy in the system is always the same so energy is conserved.

Change in Potential Energy in One Dimension



It might be easy to think about a rollercoaster moving in the x direction. Here, we have graphs for Potential Energy U(x) against displacement x and a graph of Force F(x) against displacement. As can be seen, the turning points of potential energy line up with the 0 points on the graph of force. This is true mathematically since the gradient at these points is 0 so $F(x) = 0$. It also makes sense intuitively since no change in PE implies there is no force causing a change.

At the points where Potential Energy is equal to the total energy, kinetic energy must be equal to 0 since $E = P.E + K.E$. The force at these points cannot be generalised to every system and must be worked out for individual cases.

When PE is a minima - at a turning point, the force is called a restoring force, tending to push you back to the the point where PE is a minima. You can see this on the graph where Force is ← to the right of the turning point and → to the left of the turning point. Here, $\frac{d^2U(x)}{dx^2} > 0$. This is called a stable equilibrium.

When PE is a minima at a t.p., the force pushes you away from the turning point. This is unstable equilibrium and $\frac{d^2U(x)}{dx^2} < 0$.

For $\frac{d^2U(x)}{dx^2} = 0$, we have to look at higher order derivatives.