Electromagnetic Potentials

We use potentials as a mathematical trick to make Maxwell's equations easier to solve.

Electrostatic potential

We define $\boxed{\underline{E} = - \underline{\nabla} \phi_E}$

which gives us the results: $\underline{\nabla} \times \underline{\nabla} \phi = 0$ (from $\underline{\nabla} \times \underline{E} = 0$) when there is no magnetic field

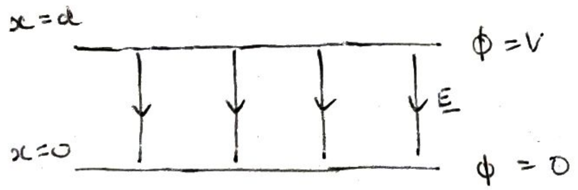
$$-\nabla^2 \phi = \frac{\rho}{\varepsilon_0} \quad \left(\text{from } \underline{\nabla} \cdot \underline{E} = \frac{\rho}{\varepsilon_0}\right)$$

By integrating, we find:

$$\phi = - \int_{\infty}^{x} \underline{E} \cdot d\underline{l}$$

We can now do some examples to see where this is useful.

Example 1: Infinite Parallel capacitor



between the plates, there is no charge so $-\nabla^2 \phi = \frac{\rho}{\varepsilon_0}$ reduces to: $\nabla^2 \phi = 0$

so $\nabla^2 \phi = 0 \Rightarrow \frac{d^2}{dx^2} \phi = 0$ since we are working only with x

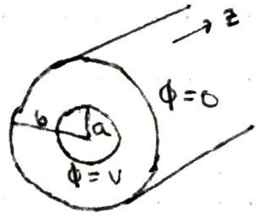
Integrating twice $\phi = Ax + C$ with A, C constants.
setting $\phi = 0$ at $x = 0$ and $\phi = V$ at $x = d$, we obtain:

$$\underline{\phi = \frac{V}{d} x}$$

And finally we obtain the electric field $\underline{E} = - \nabla \phi_E$

$$\Rightarrow \underline{E} = \left(-\frac{V}{d}, 0, 0\right)$$

Example 2 : Coaxial Cable



inside the cable, there are no charges

so $-\nabla^2 \phi = \frac{\rho}{\varepsilon_0}$ reduces to $\nabla^2 \phi = 0$

$$\nabla^2 = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)$$

so $$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = 0$$

$$\Rightarrow \quad \phi = A\ln r + C$$

fixing $\phi = 0$ at $r = b$ and $\phi = V$ at $r = a$

$$\phi(r) = -\frac{V}{\ln(b/a)}\ln(r - \ln b)$$

we can now use $\underline{E} = -\nabla \phi_E$ to work out the field.

Magnetic Vector Potential

Since $\underline{\nabla} \times \underline{B} = \mu_0 \underline{J}$

we cannot use a scalar potential such as $B = \underline{\nabla}\phi$ since $\underline{\nabla} \times \underline{\nabla}\phi = 0$ instead of $\mu_0 \underline{J}$. So we need the magnetic potential to be a vector.

$\underline{\nabla} \cdot \underline{B} = 0$ so if we let $\underline{B} = \underline{\nabla} \times \underline{A}$:

$\underline{\nabla} \cdot (\underline{\nabla} \times \underline{A}) = 0$ using the $\underline{\nabla} \cdot (\underline{\nabla} \times \underline{F})$ identity, as required.

So the magnetic vector potential is:

$$\boxed{\underline{B} = \underline{\nabla} \times \underline{A}}$$

A new Electric Potential

You may notice we used $\underline{\nabla} \times \underline{E} = 0$ in our definition of electric potential. But $\underline{\nabla} \times \underline{E} = -\frac{\partial B}{\partial t}$, so our current electric potential only works in the absence of magnetic fields. Perhaps we can find a more general expression for electric potential.

we find $\boxed{\underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t}}$

$$\underline{B} = \underline{\nabla} \times \underline{A}$$

if we try $\underline{\nabla} \times \underline{E}$, we find:

$$\underline{\nabla} \times \underline{E} = -\underline{\nabla} \times \underline{\nabla}\phi - \frac{\partial}{\partial t}(\underline{\nabla} \times \underline{A})$$

$$= -\frac{\partial B}{\partial t} \quad \text{as required}$$

and $\underline{\nabla} \cdot \underline{E} = -\nabla^2\phi - \frac{d(\underline{\nabla} \cdot \underline{A})}{dt} = \frac{\rho}{\varepsilon_0}$.

Let's finally try $\underline{\nabla} \times \underline{B} = \underline{\nabla} \times \underline{\nabla} \times \underline{A}$

$$= \underline{\nabla}(\underline{\nabla} \cdot \underline{A}) - \nabla^2 \underline{A} \quad \text{(from an identity)}$$

$$= \mu_0 \underline{J} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\left(-\frac{\partial \underline{A}}{\partial t} - \underline{\nabla}\phi\right) \quad \text{[from Maxwell eqn]}$$

rearranging: $-\nabla^2 \underline{A} + \mu_0 \varepsilon_0 \frac{\partial^2 \underline{A}}{\partial t^2} = \mu_0 \underline{J} - \underline{\nabla}\left(\underline{\nabla} \cdot \underline{A} + \mu_0 \varepsilon_0 \frac{\partial \phi}{\partial t}\right)$

This last eqn is hard to solve so we will make use of gauge invariance, explained overleaf.

Gauge Transformations

If we make the transformations

$$\underline{A} \rightarrow \underline{A} + \underline{\nabla}\psi$$

$$\text{or} \quad \phi \rightarrow \phi - \frac{\partial \psi}{\partial t}$$

$\underline{E}$ and $\underline{B}$ are invariant. we can make use of this fact to solve the equation on the previous page.

if we do $\underline{\nabla} \cdot \underline{A} \rightarrow \underline{\nabla} \cdot (\underline{A} + \underline{\nabla}\psi) = \underline{\nabla} \cdot \underline{A} + \nabla^2 \psi$

we can choose ψ to aid us in solving our equations.

If we choose it such that

$$\underline{\nabla} \cdot \underline{A} = -\mu_0 \varepsilon_0 \frac{\partial \phi}{\partial t}$$

then the equation $-\nabla^2 \phi - \frac{d(\underline{\nabla} \cdot \underline{A})}{dt} = \frac{\rho}{\varepsilon_0}$ simplifies to:

$$\boxed{-\nabla^2 \phi + \mu_0 \varepsilon_0 \frac{\partial^2 \phi}{\partial t^2} = \frac{\rho}{\varepsilon_0}}$$

and the equation $-\nabla^2 \underline{A} + \mu_0 \varepsilon_0 \frac{\partial^2 \underline{A}}{\partial t^2} = \mu_0 \underline{J} - \underline{\nabla}(\underline{\nabla} \cdot \underline{A} + \mu_0 \varepsilon_0 \frac{\partial \phi}{\partial t})$

simplifies to:

$$\boxed{-\nabla^2 \underline{A} + \mu_0 \varepsilon_0 \frac{\partial^2 \underline{A}}{\partial t^2} = \mu_0 \underline{J}}$$

Notice that in free space where $\underline{J} = 0$ and $\rho = 0$, these equations are:

$$\nabla^2 \phi = \mu_0 \varepsilon_0 \frac{\partial^2 \phi}{\partial t^2} \quad \text{and} \quad \nabla^2 \underline{A} = \mu_0 \varepsilon_0 \frac{\partial^2 \underline{A}}{\partial t^2}$$

Wave equations! These have solutions $\underline{A}(\underline{r}, t) = \underline{A}_0 e^{i(\omega t - \underline{k} \cdot \underline{r})}$

where $\frac{\omega^2}{k^2} = c^2 = \frac{1}{\mu_0 \varepsilon_0}$

$$\Rightarrow c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

We have found the wave equations for light.