# Electric Potential

Consider the energy needed to bring a charge $+q$ towards another charge $+Q$.


$$\boxed{\text{Work done by the field of } +Q \text{ on } +q = \text{Force experienced by } q \cdot \text{distance moved (in direction of force)}}$$

Since the force experienced by $q$ might not be constant over the whole distance moved, we can break the distance down into lengths of constant force and compute an integral:

$$\boxed{\text{work done, } W = \int_i^f \underline{F} \cdot d\underline{r}}$$

Here, $\underline{r}$ must be in the same direction as $\underline{F}$

In the case of the two particles, the force $\underline{F}$ felt by $q$ and the direction $-\underline{r}$ moved by $q$ are parallel so the dot product becomes: $-\underline{F} \cdot \delta\underline{r} = -|\underline{F}||\delta\underline{r}|\cos 0$
Note that $\cos 0$ is 1 and that the dot product has a negative sign since the direction moved by the particle is in the opposite direction to the felt force. So, to move $+q$ from infinity to $+Q$:

$$\text{Work done} = \int_{\infty}^{R} \underline{F} \cdot d\underline{r} = -\int_{\infty}^{R} |\underline{F}||d\underline{r}| = -\int_{\infty}^{R} \frac{Qq}{4\pi\varepsilon_0 |\underline{r}|^2} \, d\underline{r}$$

$$= -\frac{Qq}{4\pi\varepsilon_0 R}$$

The work done can be defined as the change in potential energy:

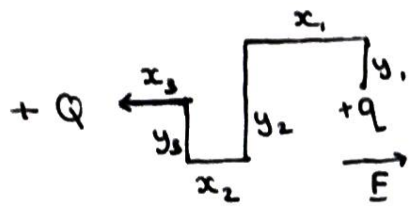
Change in potential, $\Delta U = U_f - U_i$

$= -(U_i - U_f)$ This is the energy to bring a particle from its initial position to its final position

$= -W$

$$\boxed{\therefore \quad \Delta U = -\int_i^f \underline{F} \cdot d\underline{r}}$$

The energy needed to bring the particle from its initial position to its final position is independent of the path taken:




Here, the work done on $+q$ by the field of $Q$ is given by $|\underline{F}||d\underline{r}|\cos\theta$

In the $y_1$, $y_2$ and $y_3$ directions, the $\theta$ value is $90°$ so the work done is $0$. So only the work done over $x_1, x_2$ and $x_3$ matters!

This is what makes the path travelled irrelevant and our lives a whole lot easier!

We can now define Potential as a property of any electric field in the following way:

| |
|---|
| The energy an external agent must provide to move a unit charge from infinity to a point $\underline{r}$. |

Potential, $\phi = -\dfrac{W}{q} = -\dfrac{1}{q}\displaystyle\int_\infty^R \underline{F} \cdot d\underline{r}$

So: $$\boxed{\phi = -\int_\infty^R \underline{E} \cdot d\underline{r}}$$

This is a single value at each point in the field

For a point charge:

$$\phi = -\int_{\infty}^{R} \underline{E} \cdot d\underline{r} = -\int_{\infty}^{R} -|\underline{E}||d\underline{r}|$$

$$= \underline{\underline{\frac{Q}{4\pi\varepsilon_0|\underline{R}|}}}$$

For multiple charges, the total potential is the sum of the potentials of each charge. The order you "bring the charges in from infinity" doesn't matter:

$$\phi_{total} = \sum_{charges} \frac{Q_i}{4\pi\varepsilon_0|\underline{R}_i|}$$

## Potential Difference

The difference in potential between two points is called the voltage, measured in volts:

$$\text{Voltage}, \Delta\phi = \phi_2 - \phi_1$$

$$= -\int_{\infty}^{R_2} \underline{E} \cdot d\underline{r} - -\int_{\infty}^{R_1} \underline{E} \cdot d\underline{r}$$

$$= -\int_{\infty}^{R_2} \underline{E} \cdot d\underline{r} - \int_{R_1}^{\infty} \underline{E} \cdot d\underline{r}$$

$$\boxed{V = -\int_{R_1}^{R_2} \underline{E} \cdot d\underline{r}}$$

If a charge moves through a potential difference, its change in energy is given by $qV$.

# Relation between Potential and Force

If Potential is a single number on each point of an electric field, is it possible to recover information about the field from just the Potential?

To recover 3 vector components of $\underline{E}$ from $\phi$, consider a small local motion, i.e. a small movement within a field:

$$\delta\phi = -\underline{E} \cdot \delta\underline{r} \quad \therefore \quad \underline{E} \cdot \delta\underline{r} = -\delta\phi$$

So for small movements in $x$, $y$ and $z$:

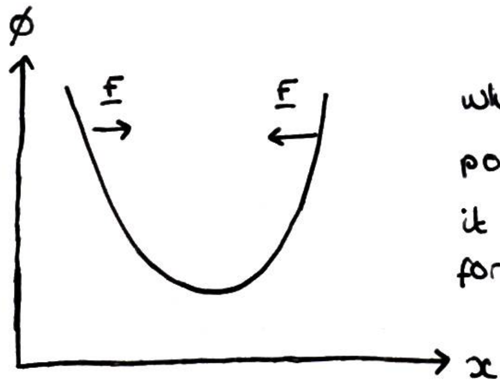
$$\underline{E} \cdot \delta x = -\delta\phi \qquad \underline{E} \cdot \delta y = -\delta\phi \quad \text{and} \quad \underline{E} \cdot \delta z = -\delta\phi$$

Rearranging these: $\underline{E} = -\dfrac{d\phi}{dx}, \; -\dfrac{d\phi}{dy}, \; -\dfrac{d\phi}{dz}$

We can write this as a vector:

$$\boxed{\begin{aligned} \underline{E} &= -\left[\frac{d\phi}{dx}\hat{\underline{i}} + \frac{d\phi}{dy}\hat{\underline{j}} + \frac{d\phi}{dz}\hat{\underline{k}}\right] \\ \underline{E} &= -\underline{\nabla}\phi \quad \text{This is called a "grad"} \end{aligned}}$$

Since the force will act in the same direction as $\underline{E}$, we can also recover information about force:




where $\dfrac{d\phi}{dx} < 0$, the force acts in the positive $x$ direction and where $\dfrac{d\phi}{dx} > 0$ it acts in the negative $x$ direction, for positive charges.

"Postive Charges roll down the hill"

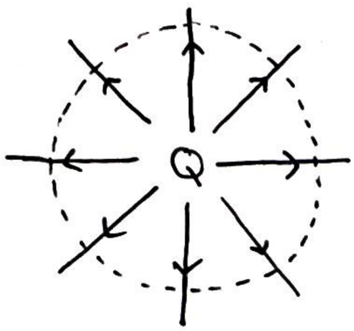
"Negative Charges roll up the hill"

# Equipotentials

Equipotentials are lines in an electric field along which potential does not change. This also means that no additional force acts when moving along these lines

$\underline{E} = -\underline{\nabla}\phi = 0 \qquad \underline{F} = \underline{E}q = -\underline{\nabla}\phi q = 0$

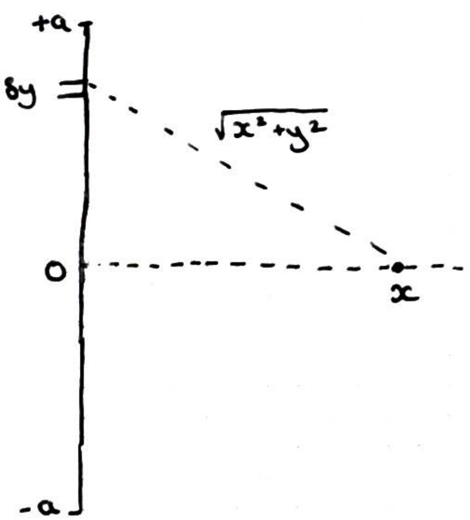
An example of this can be seen on a point charge:




The equipotential lines are always 90° to the $\underline{E}$ lines so there is no change of potential when moving along them which also means no extra force is felt.

Another example would be the surface of a conductor.

Eg. compute the potential due to a wire of length $2a$ and charge density $\lambda$ at a point $x$ from the wire.




As usual, we split the wire into infinitesimally small pieces (that act like point charges) and sum over the whole wire.

Here, $\delta q = \delta y \lambda$

Potential from a point charge is given by:

$$\phi = \frac{Q}{4\pi\varepsilon_0 |\underline{r}|}$$ so in this case:

$$\delta\phi = \frac{\delta q}{4\pi\varepsilon_0 |\underline{r}|} \quad \therefore \quad \phi = \sum \frac{\lambda\,\delta y}{4\pi\varepsilon_0 \sqrt{x^2+y^2}}$$ which we can treat as an integral:

$$\phi = \int_{-a}^{a} \frac{\lambda}{4\pi\varepsilon_0} \frac{dy}{\sqrt{x^2+y^2}} = \frac{\lambda}{4\pi\varepsilon_0}\left[\ln\left(2\left(y+\sqrt{x^2+y^2}\right)\right)\right]_{-a}^{a}$$

$$\underline{\underline{\phi = \frac{\lambda}{4\pi\varepsilon_0} \ln\left[\frac{\sqrt{x^2+y^2}+a}{\sqrt{x^2+y^2}-a}\right]}}$$ This is the potential at a point $x$ from the wire.

Now, let's think about what would happen if the wire was infinitely long. For this, we have to let $a \to \infty$:

$$\phi = \frac{\lambda}{4\pi\varepsilon_0} \ln\left[\frac{2a}{a\left(1+\frac{x^2}{a^2}\right)^{1/2} - a}\right]$$ we can expand this using binomial expansion:

$$\phi = \frac{\lambda}{4\pi\varepsilon_0} \ln\left[\frac{2a}{a - a + 1/2\, x^2/a^2 \cdots}\right]$$

$$= \frac{\lambda}{4\pi\varepsilon_0} \ln \frac{4a^2}{x^2} = \frac{\lambda}{4\pi\varepsilon_0}\left[\ln 4a^2 - \ln x^2\right]$$

but as $a \to \infty$ $\ln 4a^2 \to 0$

$$= -\frac{\lambda}{4\pi\varepsilon_0} \ln x^2 = -\frac{\lambda}{4\pi\varepsilon_0} \cdot 2\ln x$$

$$\underline{\underline{\phi = \frac{-\lambda}{2\pi\varepsilon_0} \ln x}}$$ but remember $\underline{E} = -\underline{\nabla}\phi$

$$= \frac{\lambda}{2\pi\varepsilon_0 |\underline{x}|}\hat{\underline{x}}$$

which is the same as the result we obtained in the Electric Fields chapter!

If we instead wanted to know what would happen if the wire was infinitesimally small, we could take the opposite limit $x >> a$ to obtain:

$$\phi = \frac{\lambda}{4\pi\varepsilon_0} \ln\left[\frac{x+a}{x-a}\right]$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left(\ln(x+a) - \ln(x-a)\right)$$

$$\underline{E} = -\underline{\nabla}\phi = \frac{-\lambda}{4\pi\varepsilon_0}\underbrace{\left[\frac{1}{x+a} - \frac{1}{x-a}\right]}\hat{\underline{x}}$$

$$\hookrightarrow \frac{x-a-(x+a)}{x^2-a^2} = \frac{-2a}{x^2-a^2}$$

$$\underline{E} = \frac{-\lambda}{4\pi\varepsilon_0}\left(-\frac{2a}{x^2-a^2}\right)\hat{\underline{x}}$$

$$= \frac{\lambda 2a}{4\pi\varepsilon_0}\frac{1}{\underbrace{x^2-a^2}}\hat{\underline{x}}$$ since $x^2 >> a^2$, we can write this as just $a^2$:

$$= \frac{\lambda 2a}{4\pi\varepsilon_0 x^2}\hat{\underline{x}}$$ Note here, that $\lambda 2a = Q$

$$\therefore \underline{E} = \frac{Q}{4\pi\varepsilon_0 x^2}\hat{\underline{x}}$$ which is the well known result for a point charge!