Work and Energy

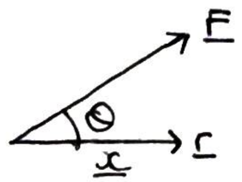
If a force $\underline{F}$ displaces a particle through a small (infinitesimal) displacement $\delta \underline{r}$, then we define the work done by the force as:

$$\boxed{\delta W = \underline{F} \cdot \delta \underline{r}}$$

Note that this is the scalar product of force and displacement and thus Work Done is a scalar.

From the rules of the scalar product, we can also write this as:

$$\boxed{\delta W = |\underline{F}| \, |\delta \underline{r}| \cos\theta}$$



This makes sense intuitively since $\underline{x}$ is $\underline{F}\cos\theta$ so this is just resolving $\underline{F}$ in the direction of the displacement and working out the scalar product (like in the first formula). Remember that the scalar product of two vectors with the same direction is the multiple of their magnitudes.

From the formulas, we can sum the work done to give total work done as:

$$\text{Work Done } W = \Sigma \delta W = \Sigma \underline{F} \cdot \delta \underline{r}$$

$$\boxed{W = \int_{r_i}^{r_f} \underline{F} \cdot d\underline{r}}$$

This is the same as: $W = \int F \cdot 1 \, dr$ which would give us $[F \cdot \underline{r}]_{r_i}^{r_f}$

Constant Force in One Dimension

If the force is constant over time and uniform over position, the integral just gives the scalar product of force and displacement.

Varying Force in One Dimension

If the force changes with position, we need to say that $\underline{F}$ is dependent on x so we need to express $\underline{F}$ as a function of x:

$F(x) = \underline{F} \cdot \underline{r}$ where $\underline{F}$ is the force at that position. Note that $F(x)$ is a scalar quantity.

We therefore write work done as $W_{i \to f} = \int F(x)\,dx$. So the work done for a force varying with position is the area under a graph of $F(x)$ against x, force against distance.

Instantaneous Power

Instantaneous means at one instance instead of over a period of time. So instantaneous power is the rate of doing work at one instance in time. Therefore:

$\boxed{\text{Power} = \frac{dW}{dt}}$ but we know that $dW = \underline{F} \cdot d\underline{r}$

Therefore, $\text{Power} = \frac{\underline{F} \cdot d\underline{r}}{dt} = \underline{F} \cdot \frac{d\underline{r}}{dt}$ but $\frac{d\underline{r}}{dt}$ is defined as velocity, $\underline{v}$. So to conclude: $\boxed{P = \underline{F} \cdot \underline{v}}$

Kinetic Energy

$$\text{Work Done} = \int_{r_i}^{r_f} \underline{F} \cdot d\underline{r}$$

In order to introduce "dt" we can write:

$$\text{Work Done} = \int_{t_i}^{t_f} \underline{F} \cdot \frac{d\underline{r}}{dt} \times dt = \int_{t_i}^{t_f} \underline{F}\,\dot{\underline{r}}\,dt$$

Note that $\dot{\underline{r}}$ is an abbreviation for $\frac{d\underline{r}}{dt}$ and is thus equivalent to velocity $\underline{v}$.

From Newton's Second Law:

$\underline{F} = m\underline{a}$ where $\underline{a} = \frac{d\underline{v}}{dt} = \frac{d}{dt}\left(\frac{d\underline{r}}{dt}\right)$ so it can be rewritten as $\ddot{r}$. So we can rewrite Newton's Second Law as: $\underline{F} = m\ddot{\underline{r}}$. Therefore, work done is:

$$\text{Work Done} = \int_{t_i}^{t_f} m\ddot{r} \cdot \dot{r}\,dt \text{ which is: } \boxed{\text{Work Done} = m\int_{t_i}^{t_f} \ddot{r} \cdot \dot{r}\,dt}$$

A Note on the Integrand

We can often simplify an integral by considering what the integrand is a derivative of. For example, in this case we see the integrand in the form $x . \dot{x}$ which we intuitively know is the derivative using product rule.

$$\frac{d}{dt}(\dot{\underline{r}} \cdot \dot{\underline{r}}) = \ddot{\underline{r}} \cdot \dot{\underline{r}} + \dot{\underline{r}} \cdot \ddot{\underline{r}} = 2\ddot{\underline{r}} \cdot \dot{\underline{r}}$$ so the integrand is:

$$\ddot{\underline{r}} \cdot \dot{\underline{r}} = \frac{1}{2}\frac{d}{dt}(\dot{\underline{r}} \cdot \dot{\underline{r}})$$

Returning to the Problem

$$\text{Work Done} = m\int_{t_i}^{t_f} \ddot{\underline{r}} \cdot \dot{\underline{r}}\, dt$$

$$= \frac{1}{2} m \int_{t_i}^{t_f} \frac{d}{dt}\left(\dot{\underline{r}} \cdot \dot{\underline{r}}\right) dt$$

when we integrate $\frac{d}{dx}(x)\,dx$, we get back to the original x:

$\text{Work Done} = \frac{1}{2} m \left[\dot{\underline{r}} \cdot \dot{\underline{r}}\right]_{t_i}^{t_f}$ but we know that $\dot{\underline{r}} = \underline{v}$ so:

$\text{Work Done} = \frac{1}{2} m \left[\underline{v} \cdot \underline{v}\right]_{t_i}^{t_f}$. The scalar product of two of the same vector can be written as the square of the modulus of the vector. So:

$$\text{Work Done} = \frac{1}{2} m \left[|\underline{v}|^2\right]_{t_i}^{t_f}$$

We can see something similar to $T = \frac{1}{2} m |\underline{v}|^2$ in this equation, where T is Kinetic Energy. Therefore:

$$\boxed{\text{Work Done} = T(t_f) - T(t_i)}$$

This is known as the work-energy theorem: Work Done is the change in kinetic energy.