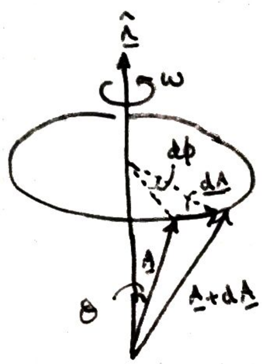
Rotational Motion of Rigid Bodies

Rotations and Angular Velocity

An interesting thing about rotations is that they do not commute. A rotation $R(\hat{n}, \theta)$ is specified by an axis of rotation and an angle of rotation. We cannot combine this with another rotation and expect it to work: $R_x R_y \neq R_y R_x$
So with successive rotations, the order matters.

Infinitesimal rotations do commute however:



Consider a vector $\underline{A}$ rotated about $\hat{n}$ by an angle $d\phi$
The change $d\underline{A}$ is shown. $d\underline{A}$ is perpendicular to both $\underline{A}$ and $\hat{n}$

$|d\underline{A}| = A\sin\theta \, d\phi$ so therefore:

$$d\underline{A} = \hat{n} \times \underline{A} \, d\phi$$

If we apply another infinitesimal rotation $d\underline{A}'$, the final vector is $\underline{A} + d\underline{A} + d\underline{A}'$
The change is $d\underline{A} + d\underline{A}' = d\underline{A}' + d\underline{A}$ since vector addition commutes.

Therefore, infinitesimal rotations commute.
If we fix length of $\underline{A}$ and think of it as a position vector rotating around axis with angular velocity $d\phi/dt$, we see that it describes a particle rotating in a circle about the axis.

velocity of particle is: $\boxed{\underline{v} = \frac{d\underline{A}}{dt} = \hat{n} \times \underline{A}\,\dot{\phi}}$

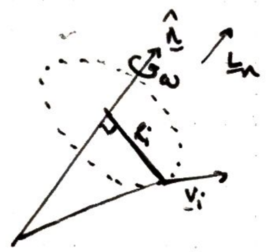
We define angular velocity $\underline{\omega} = \dot{\phi}\,\hat{n}$

so $$\boxed{\frac{d\underline{A}}{dt} = \underline{\omega} \times \underline{A}}$$

Moment of Inertia

This chapter looks at rigid bodies, so relative position of all particles in system are fixed. So describing one particle's motion is sufficient to describe the whole body's.

consider:



consider n particles in a system like above, the total kinetic energy is:

$$T = \sum_i^n \tfrac{1}{2} m_i v_i^2 = \sum_i^n \tfrac{1}{2} m_i R_i^2 \omega^2 = \tfrac{1}{2} I \omega^2$$

we made use of:
$\underline{v} = \underline{\omega} \times \underline{R} \Rightarrow v^2 = \omega^2 R^2$
from $\frac{d\underline{A}}{dt} = \underline{\omega} \times \underline{A}$

$\therefore$ Moment of Inertia $\boxed{I = \sum_i m_i R_i^2}$

For a continuous mass distribution, we extend to integrals:

$I = \int_{body} R^2 dm$ where $dm = \rho d^3\underline{r}$ so:

$$\boxed{I = \int_{body} R^2 \rho \, d^3 \underline{r}}$$

we can also use the radius of gyration k:

$$\boxed{I = Mk^2}$$

L_n in the diagram shows the angular momentum defined as

$$\boxed{\underline{L} = \underline{R} \times \underline{P}} = \underline{r} \times M\underline{v} = m(\underline{r} \times \underline{v})$$
$$= m(\underline{r} \times (\underline{\omega} \times \underline{r}))$$
$$= m[(\underline{r} \cdot \underline{r})\underline{\omega} - \underbrace{(\underline{r} \cdot \underline{\omega})}_{=0 \text{ since } \perp}\underline{r}]$$
$$= mr^2 \underline{\omega} = I\underline{\omega}$$

$$\boxed{L_n = I\omega}$$

where n denotes the axis of rotation.

The component of torque along this axis is:

$$\boxed{\tau_n = \frac{dL_n}{dt} = I\dot{\omega} = I\ddot{\phi}}$$

Parallel Axis Theorem

let I_{CM} be the moment of inertia about axis through CofM

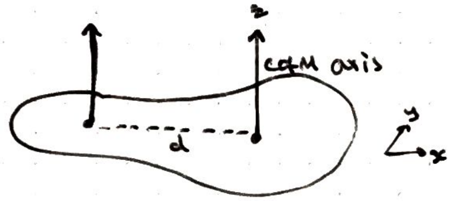
let I be moment of inertia about parallel axis at distance d from axis through CofM.

The Parallel Axis Theorem states:

$$\boxed{I = I_{CM} + Md^2}$$

where M is total mass.

This is very useful! To prove this consider:



since $r_i = \sqrt{x_i^2 + y_i^2}$

so $I = \sum_i m_i(x_i^2 + y_i^2)$

let position of particle w.r.t. CofM axis be (ρ_{ix}, ρ_{iy})

so $d = x_i - \rho_{ix} \Rightarrow x_i = d + \rho_{ix}$

so
$$\begin{aligned} I &= \sum_i m_i\left((d+\rho_{ix})^2 + \rho_{iy}^2\right) \\ &= \sum_i m_i\left(\rho_{ix}^2 + \rho_{iy}^2 + d^2 + 2\rho_{ix} d\right) \\ &= \sum_i m_i(\rho_{ix}^2 + \rho_{iy}^2) + \sum_i m_i d^2 + \underbrace{2d\sum_i m_i \rho_{ix}}_{\text{this vanishes by definition of CofM}} \\ &= \underline{\underline{I_{CM} + Md^2}} \end{aligned}$$

Perpendicular Axis Theorem

For thin flat plates of arbitrary shapes lying in the xy plane, if we take I_x, I_y, I_z to be moments of inertia about the x, y, z axis respectively:

$$\boxed{I_z = I_x + I_y}$$

This is also very useful.
To prove this:

$$I_x = \sum_i^N m_i y_i^2 \quad I_y = \sum_i^N m_i x_i^2$$

[since plate is thin, we don't worry about particles in 3D]

$$I_z = \sum_i^N m_i (x_i^2 + y_i^2)$$

[since now we are rotating in 3D space, we have to consider both x and y position]

$$= \sum_i^N m_i x_i^2 + \sum_i^N m_i y_i^2$$

$$I_z = I_y + I_x$$

Examples

To find the moment of inertia of a uniform thin rod of length $2a$ about an axis perpendicular to the rod through its centre of mass:

Let mass per unit length be ρ

Let $x = 0$ be centre of mass position so $-a \le x \le a$

$dm = \rho dx$

$dI = x^2 dm$

$$I_{cm} = \int_{rod} dI = \int_{-a}^{a} \rho x^2 dx = \rho \left[\frac{x^3}{3}\right]_{-a}^{a}$$

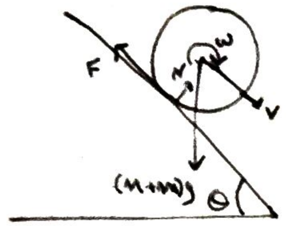
$$= \rho \left\{ \frac{a^3}{3} + \frac{a^3}{3} \right\} = \frac{2}{3} \rho a^3$$

To find moment of inertia about axis through end of rod:

Apply parallel axis theorem where $d = a$

$$I = \frac{2}{3} \rho a^3 + Ma^2 = \frac{2}{3} \rho a^3 + 2\rho a^3 = \frac{8}{3} \rho a^3$$

Consider a spoked wheel, radius a with a thin rim of mass M and n spokes, each of mass m. Think of the spokes as thin rods. The wheel rolls down a hill of inclination θ. What is the linear acceleration of the C of M?



if angular velocity is ω, then speed (from no-slip condition) is $v = a\omega$ where a is radius.

From Chapter 1: $\tau_{cm}^{ext} = I_{cm}\dot{\omega}$

since $L_{cm}^{ext} = I_{cm}\omega$ and $\tau = \frac{dL}{dt}$

I_{cm} is unvarying

I_{cm} of rim is Ma^2. I_{cm} of individual spoke is $\frac{1}{3}ma^2$ (from question on previous page, but with rod length $= a$).

Total I_{cm} of wheel is:

$$I_{cm} = Ma^2 + \frac{n}{3}ma^2$$

$$\therefore \quad \tau_{cm}^{ext} = (Ma^2 + \frac{n}{3}ma^2)\dot{\omega}$$

since $\underline{\tau} = \underline{r} \times \underline{F}$, $\tau = Fa$ so:

$$Fa = (Ma^2 + \frac{n}{3}ma^2)\dot{\omega}$$

from diagram, consider forces parallel to slope:

$$-F + (M+nm)g\sin\theta = (M+nm)a\dot{\omega}$$

we eliminate F to get: $\underline{\underline{a\dot{\omega} = \frac{3(M+nm)g\sin\theta}{6M+4nm}}}$ This is linear acceleration

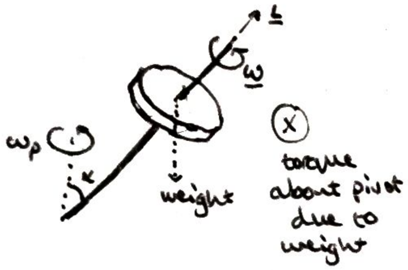
Alternatively, we can use conservation of KE and GPE:

$$\frac{1}{2}(M+nm)v^2 + \frac{1}{2}I_{cm}\omega^2 - (M+nm)gx\sin\theta = \text{constant}$$

let $v = \dot{x} = a\omega$ and differentiate to find $\ddot{x}$ (acceleration)

Precession

Spinning bodies tend to precess under the action of a gravitational torque. We can calculate this precession rate by applying $\tau = dL/dt$ about the pivot.



The torque about pivot due to the weight points into the paper, so it causes the spinning top to precess.

$\underline{\tau} = \underline{r} \times \underline{F}$ where r is vector from pivot to top's CofM and $\underline{F} = mg$, top's weight

$$\therefore \underline{\tau} = \underline{r} \times m\underline{g} \Rightarrow rmg\sin\alpha = \tau$$

Let the angle the top precesses through be ϕ. For an infinitesimal precession $d\phi$, the change in angular momentum dL is:

$$dL = L\,d\phi\sin\alpha$$

$\dot{\phi} = \omega_p$ so $$\frac{dL}{dt} = L\omega_p\sin\alpha$$

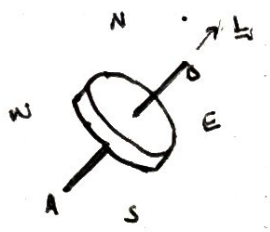
$$\therefore |\underline{\tau}| = L\omega_p\sin\alpha$$

$$rmg\sin\alpha = L\omega_p\sin\alpha$$

$$\boxed{\omega_p = \frac{mgr}{L}}$$

Gyroscopic Navigation

A gyrocompass is a spinning top constrained to have its axis be horizontal with respect to Earth:



When the Earth spins, the axis turns with it, raising up point A and pushing down point B. This causes a torque on the gyroscope perpendicular to $\underline{L}$, pointing between North and West when compass is oriented as shown in diagram.

$\underline{\tau} = \frac{d\underline{L}}{dt}$, so this pushes $\underline{L}$ towards North. If $\underline{L}$ points between N and W, the torque tries to line $\underline{L}$ up with N-S axis.

So the gyroscope oscillates with its spin-direction oscillating about N-S axis. Applying damping cause oscilloscope to settle with its spin always along N-S line. This gives us a useable compass!

Inertia Tensor

We have thus far considered inertia to be a number but it is actually a tensor!

$$\underline{L} = \sum_{i=1}^{N} m_i \left((\underline{r}_i \cdot \underline{r}_i)\underline{\omega} - (\underline{r}_i \cdot \underline{\omega}) \cdot \underline{r}_i \right)$$

we can rewrite this as a matrix eqn.

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} \sum m_i (x_i^2 + y_i^2) & -\sum m_i x_i y_i & -\sum m_i x_i z_i \\ -\sum m_i y_i x_i & \sum m_i (z_i^2 + x_i^2) & -\sum m_i y_i z_i \\ -\sum m_i z_i x_i & -\sum m_i z_i y_i & \sum m_i (x_i^2 + y_i^2) \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$= \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$\underline{L} = \underline{I}\,\underline{\omega}$ where $\underline{I}$ is a tensor

We won't really use it in tensor form in this course, it's just good to know!