Dispersion

From previous chapters, we found that the speed of light in a medium is given by $v = \frac{c}{n}$. We have in the past taken n to be a constant for the material, independent of the wavelength of the light, but this is in fact not true! The refractive index n varies with wavelength for all materials, this is called dispersion. Only vacuum is nondispersive.

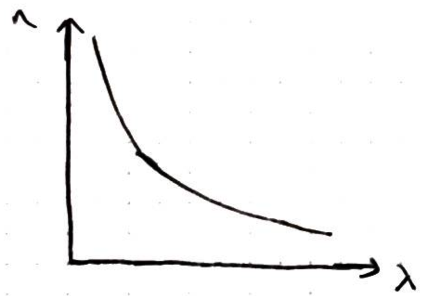
For light of frequency ω, the wavenumber in a vacuum is given by $c = \frac{\omega}{k_0}$. Light of the same frequency ω in a medium with refractive index n satisfies:

$$v = \frac{c}{n} \Rightarrow f\lambda = \frac{c}{n} \Rightarrow 2\pi f \frac{\lambda}{2\pi} = \frac{c}{n} \Rightarrow \frac{\omega}{k} = \frac{c}{n}$$

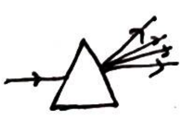
Combining these two: $\frac{\omega}{k} = \frac{\omega}{k_0 n}$

$\therefore$ $\boxed{k = n k_0}$ which gives us $\boxed{n = \frac{\lambda_0}{\lambda}}$

This shows that the refractive index varies with wavelength. This gives a relation like:

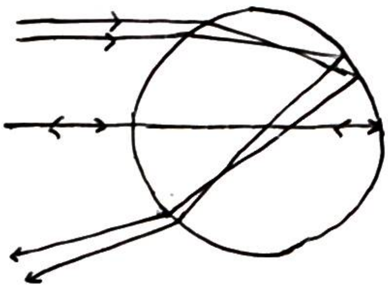


If a ray of white light is incident on a prism, the refractive index of the glass is different for each colour component. From Snell's Law, we know that the angle of refraction depends on the refractive index. Therefore, shorter wavelengths are deviated more. We can define a quantity "amount of dispersion" as the angular spread of the rays.



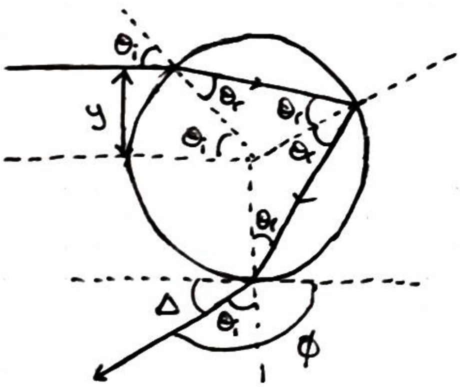
Rainbows

Rainbows are observed when sunlight from behind the observer is refracted and reflected by raindrops in front of the observer.



Multiple parallel rays enter a raindrop at various offsets from the centre axis. These rays undergo refraction, reflection and then further refraction before exiting the drop. Rays travelling straight through the middle are reflected directly out. The other rays have an angle Δ with this central axis. Δ is within 0 and some Δ_{max} dependent on the wavelength of the ray. The <u>longer</u> the wavelength, the larger the Δ_{max}.

So how do we calculate Δ?



Using geometry, we can arrive at the diagram on the left. We will define a "turning angle" ϕ as the total turning of the ray. If y is the offset, then we can make the approximation $\sin\theta_i = \frac{y}{r} = x$ where r is radius

The angles turned by the ray is given by:

At first refraction: $\theta_i - \theta_r$

At reflection: $\pi - 2\theta_r$

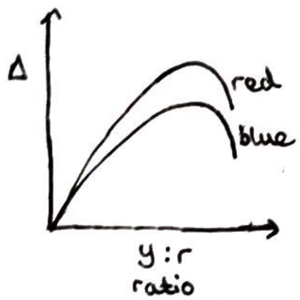
At second refraction: $\theta_i - \theta_r$

This gives a total of $\underline{\underline{\phi = \pi - 4\theta_r + 2\theta_i}}$

So the angle $\Delta = \pi - \phi = \underline{\underline{4\theta_r - 2\theta_i}}$

since $\sin\theta_i = n\sin\theta_r \Rightarrow x = n\sin\theta_r$: $\boxed{\Delta = 4\arcsin\left(\frac{x}{n}\right) - 2\arcsin(x)}$

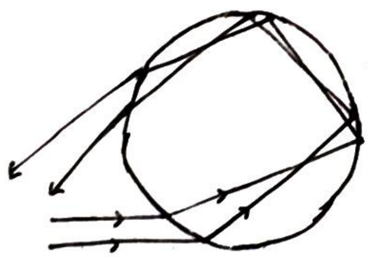
The angle Δ varies with the offset y. The variation can be plotted to look like:



The rays seem to "bunch up" around Δ_{max}. This can be seen from the diagram of rays going in and out of the drop. Notice also that Δ_{max} is higher for longer wavelengths.

Since the exit angle Δ is larger for longer wavelengths, red is at the top of the rainbow!

Just as with light that enters from the top of the drop, rainbows can also be formed when light enters from the bottom of the drop. In this case, the light is reflected a second time and so the rainbow formed is called a <u>secondary rainbow</u>.



The light rays are reflected twice inside the raindrop. The exit angle of this is given by $\phi - \pi$ since the ray is entering from the bottom. So:

$$\boxed{\Delta_{secondary} = \pi + 2\arcsin(x) - 6\arcsin\left(\frac{x}{\lambda}\right)}$$

This makes the the exit angles of the rays larger than the exit angles for primary rainbows. Secondary rainbows are therefore usually larger than primary rainbows. However, since the rays are reflected twice, more light is lost. Secondary rainbows are therefore usually fainter.

Polarisation

In classical physics, we consider light to be an electromagnetic wave, with a $\underline{E}$ field component and a $\underline{B}$ field component at right angles to each other. Both fields are at right angles to the direction of propagation, $\underline{k}$: $\underline{E} \times \underline{B} \propto \underline{k}$ The cross product of $\underline{E}$ and $\underline{B}$ points in the direction of propagation.

Since the electromagnetic wave has both an $\underline{E}$ component and a $\underline{B}$ component, it is described by two wavefunctions:

$$\boxed{\begin{aligned} \underline{E}(z,t) &= \hat{\underline{x}}\, E_{max} \cos(kz - \omega t) \\ \underline{B}(z,t) &= \hat{y}\, B_{max} \cos(kz - \omega t) \end{aligned}}$$

Both are standard sinusoidal wavefunctions perpendicular to each other.

Common electromagnetic radiation detectors usually only respond to the $\underline{E}$ component so, out of convention, we will only show this component in diagrams. This makes drawing the waves a lot easier. So when a EM wave is polarised, we say the polarisation is in the direction of the electric field vector $\underline{E}$.

After polarisation, if we look back along the direction of propagation of a wave, we see the electric field oscillating along a line.

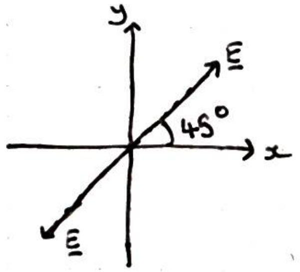
But what happens if we combine two linear polarisations?

Combining two linear polarisations would look, in general, like:

$$\underline{E}(z,t) = E_1 \hat{\underline{x}} \cos(kz-\omega t) + E_2 \hat{\underline{y}} \cos(kz-\omega t + \phi)$$

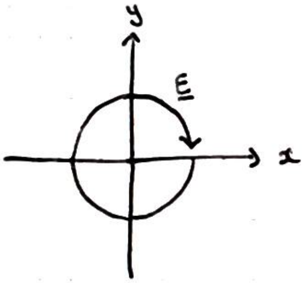
Now, consider the special case when $E_1 = E_2 = E_0$

when $\phi = 0$: $\underline{E}(z,t) = E_0 \hat{\underline{x}} \cos(kz-\omega t) + E_0 \hat{\underline{y}} \cos(kz-\omega t)$



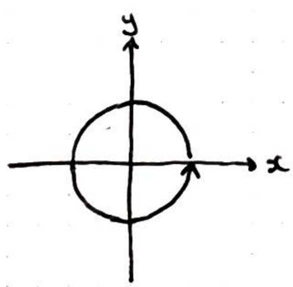
This results in linear polarisation at 45° to the x axis. This makes intuitive sense since the two polarisations are in phase and equal in magnitude. So the total polarisation is the midpoint of the two individual ones.

when $\phi = -\frac{\pi}{2}$: $\underline{E}(z,t) = E_0 \hat{\underline{x}} \cos(kz-\omega t) + E_0 \hat{\underline{y}} \sin(kz-\omega t)$



This results in right circular polarisation. consider looking back down the direction of propagation. choosing a time t such that $kz-\omega t = 0$, the $\underline{E}$ field is along the positive x axis. increasing t makes $kz - \omega t < 0$ so the cosine term decreases and the sine term becomes < 0. This makes the $\underline{E}$ field point lower. Continuing this, the field traces out a clockwise circle.

when $\phi = +\frac{\pi}{2}$: $\underline{E}(z,t) = E_0 \hat{\underline{x}} \cos(kz-\omega t) - E_0 \hat{\underline{y}} \sin(kz-\omega t)$



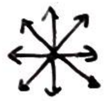
This results in left circular polarisation. This is for similar reasons as right circularly polarised waves but in the opposite direction.

It might be hard to visualise circular polarisation

Polarisation Filters

For mechanical waves like sound or water waves, we can physically obstruct certain polarisation directions and allow only the required directions to pass. This can be done by using a polarising filter like a plane boundary with a slot. We can do the same with light.

Unpolarised light is somewhat of a misnomer since in reality, at any given time, the light source will have a definite direction to the $\underline{E}$ field. What makes it "unpolarised" is that the direction of the $\underline{E}$ field is changing rapidly with all possible directions being attained in a sufficient time interval. A better name, therefore, might be "randomly polarised" light. Either way, it is represented as



A common polariser used for light is a material that selectively absorbs certain polarisation components much more strongly than others. This is called dichroism. So what happens to the intensity of unpolarised light as it passes through a filter?

The intensity of randomly polarised light <u>halves</u> after passing through a polariser, no matter the orientation of the filter.

What about if linearly polarised light passes though a filter? Then we use Malus' Law:

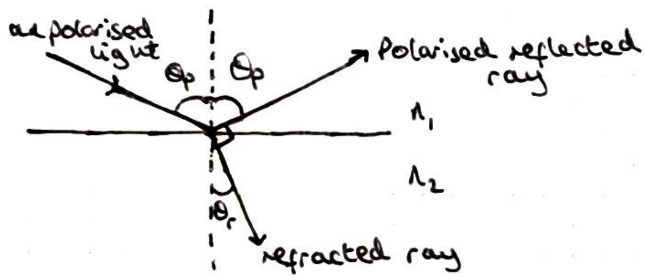
$$\boxed{I_{transmitted} = I_{incident} \cos^2 \phi}$$

where ϕ is the angle between the polarisation axis of the light and the polarisation axis of the filter.

Ideally, $I_{transmitted} = 0$ if $\phi \neq 0$ but this is practically unattainable.

Polarisation by Reflection

Unpolarised light can also be polarised by reflection! This only happens when the reflected ray and the refracted ray are at $90°$ to each other. The reflected ray therefore becomes 100% linearly polarised in the plane perpendicular to the plane of incidence.



The angle of incidence is some angle θ_p that allows for $\theta_p + \theta_r + 90 = 180$

$$\theta_p + \theta_r = 90°$$

It is therefore called the polarising angle.

Since $\theta_p + \theta_r = 90 \therefore \theta_r = 90 - \theta_p$

Using Snell's Law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$

$$n_1 \sin\theta_p = n_2 \sin\theta_r \Rightarrow n_1 \sin\theta_p = n_2 \sin(90 - \theta_p)$$

$\therefore n_1 \sin\theta_p = n_2 \cos\theta_p$ So the polarising angle is given by:

$$\boxed{\tan\theta_p = \frac{n_2}{n_1} \Rightarrow \theta_p = \arctan\left(\frac{n_2}{n_1}\right)}$$

Birefringence

Birefringence is a phenomena in which a material has a different refractive index for different directions of polarisation. A good example of this is in a calcite ($CaCO_3$) crystal. This has:

$$n = \begin{cases} 1.658 & \text{in } \hat{x} \text{ direction} \\ 1.486 & \text{in } \hat{y} \text{ direction.} \end{cases}$$

This means the outgoing wave will have one component "lagging" the other.

A wave propagating in the $\hat{z}$ direction will have its x-component see a larger n. The wavespeed of this component will therefore be slower than for the y-component. The x-component will lag the y-component. The crystal is therefore said to be a <u>phase retarder</u> with its <u>slow axis</u> oriented in the x direction.

Consider an incoming light wave, plane-polarised at $\frac{\pi}{4}$ to the x-axis and with equal amplitudes for x and y components. If the two refractive indices are n_x and n_y, the wavenumbers are: $k_x = \frac{n_x \omega}{c}$, $k_y = \frac{n_y \omega}{c}$

so at distance z into the crystal, the wave will have the form:

$$\underline{E}(z,t) = E_0 \hat{\underline{x}} \cos(k_x z - \omega t) + E_0 \hat{\underline{y}} \cos(k_y z - \omega t)$$

$$= \underline{\underline{E_0 \hat{\underline{x}} \cos\left(\frac{\omega}{c} n_x z - \omega t\right) + E_0 \hat{\underline{y}} \cos\left(\frac{\omega}{c} n_y z - \omega t\right)}}$$

This gives a phase difference between the components of:

$$\boxed{\phi = \frac{\omega}{c}(n_x - n_y) z} \quad \boxed{\phi = \frac{2\pi}{\lambda_0}(n_x - n_y) z}$$

For $n_x > n_y$, $\phi > 0$ so x lags y. If $n_x < n_y$, $\phi < 0$ so y lags x.

For $\phi = \frac{\pi}{2}$, z must equal some specific d.

$$\frac{\pi}{2} = \frac{2\pi}{\lambda_0}(n_x - n_y) d \Rightarrow \boxed{d = \frac{\lambda_0}{4(n_x - n_y)}}$$

so if the thickness is d, the wave will emerge with one component being $\frac{\pi}{2}$ out of phase.

$\frac{\pi}{2}$ out of phase corresponds to circular polarisation as seen before. A crystal of this thickness is called a <u>quarter-wave plate</u> and can turn a linearly polarised wave into a circularly polarised one.