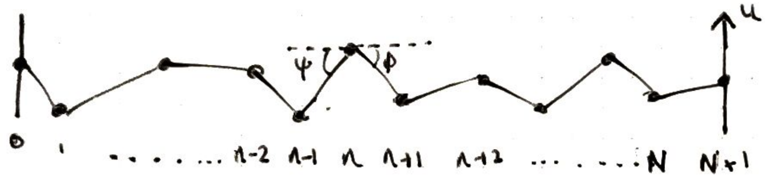
Normal Modes of a Beaded String

Equation of Motion

Consider a string stretched to tension T, carrying N beads, each of mass M:



The beads are equally spaced a distance a apart

Consider the n^{th} bead. The forces acting on it are described by:

$$M\ddot{u}_n = -T(\sin\psi + \sin\phi)$$

if displacements are small $\sin\psi \approx \tan\psi = \frac{opp}{adj} = \frac{u_n - u_{n-1}}{a}$

similarly $\sin\phi \approx \tan\phi = -\frac{u_{n+1} - u_n}{a}$

$$\therefore \quad M\ddot{u}_n = -T\left(\frac{u_n - u_{n-1}}{a} - \frac{u_{n+1} - u_n}{a}\right)$$

$$\ddot{u}_n = \frac{T}{M}\left(\frac{u_{n+1} - 2u_n + u_{n-1}}{a}\right)$$

$$\boxed{\ddot{u}_n = \frac{T}{Ma}\left(u_{n+1} - 2u_n + u_{n-1}\right)}$$

if this were simply the long string example, we would take limit $a \to 0$ and construct wave eqn. But not here.

We can now incorporate boundary conditions:

$u_0 = 0 \quad u_{N+1} = 0$

similarly, for masses on springs:

$$\boxed{\ddot{u}_n = \frac{k}{M}\left(u_{n+1} - 2u_n + u_{n-1}\right)}$$

Normal Modes

If we want normal modes, these are motions where all beads oscillate with same ω. So we have solution:

$$u_n = A_n e^{i\omega t}$$ as we had in previous chapter

For some set of coefficients A_n.

$\therefore \quad \dot{u}_n = i\omega A_n e^{i\omega t} \qquad \ddot{u}_n = -\omega^2 A_n e^{i\omega t}$ | sub this into eqn of motion

$$-\omega^2 A_n e^{i\omega t} = \frac{T}{ma}\left(A_{n+1} - 2A_n + A_{n-1}\right)e^{i\omega t}$$

$$\Rightarrow \quad \omega^2 A_n = \frac{T}{ma}\left(-A_{n+1} + 2A_n - A_{n-1}\right)$$

This is the recurrence relation for A_n, it is a discrete form of the diff eqn.

The boundary conditions are now $A_0 = A_{N+1} = 0$

We could solve for A_n like we did in previous chapter. Instead let's use some physical insight so we don't have to do a lot of work:

- Suppose we have an infinite line of beads. There is now spatial translation invariance. This will make it easy to find normal modes on an infinite system.
- Each bead is affected only by its nearest neighbours. So if we can find a combination of normal modes that satisfies $A_0 = A_{N+1} = 0$, we will have satisfied the whole system.

Infinite System: Translation Invariance

It is quite obvious that an infinite system of this kind has translation invariance.

Suppose one mode of the string has been found and has a set of displacement amplitudes A_n.

Now shift to the left one step. Translational invariance means it should look the same here, so if A_n gave us a mode with frequency ω, the shifted A_n' should give another mode with the same ω.

$$A_n' = A_{n+1} \quad \text{also gives a mode.}$$

Let's look for translationally invariant modes so

$$A_n' = A_{n+1} = h A_n$$

for some constant h, i.e new amplitudes are proportional to the old ones. Applying this repeatedly gives us:

$$A_n = h^n A_0$$ where A_0 is arbitrary and sets the overall scale.

Given this set of A_n, we can find the corresponding ω by subbing into eqn. of motion: $\omega^2 A_n = \frac{T}{Ma}(-A_{n+1} + 2A_n - A_{n-1})$

$$\omega^2 h^n A_0 = \frac{T}{Ma}\left(-h^{n+1}A_0 + 2h^n A_0 - h^{n-1}A_0\right) \qquad [\div h^n A_0]$$

$$\omega^2 = \frac{T}{Ma}\left(-h + 2 - \frac{1}{h}\right)$$ we find it convenient to set $h = e^{i\theta}$

$e^{i\theta} + e^{-i\theta} = 2\cos\theta$ so:

$$\omega^2 = \frac{T}{Ma}(2 - 2\cos\theta) \Rightarrow \omega^2 = \frac{2T}{Ma}(1-\cos\theta)$$
$$= \frac{2T}{Ma}\left(2\sin^2\frac{\theta}{2}\right)$$

so $\boxed{\omega^2 = \frac{4T}{ma}\sin^2\frac{\theta}{2}}$

if we fix ω, the amplitudes A_n must be a linear combination of h and $1/h$ i.e:

$$A_n = \alpha h^n + \beta h^{-n} \quad \text{where } \alpha \text{ and } \beta \text{ are constants}$$

The displacement of the n^{th} bead is given by:

$$u_n = (\alpha e^{in\theta} + \beta e^{-in\theta})e^{i\omega t}$$

Finite System: Boundary Conditions

The value of θ is fixed by the boundary condition, which in turn fixes ω. For N beads with the ends fixed:

$$u_0 = 0 \quad u_{N+1} = 0$$

if $u_0 = 0$, then we require $\alpha = -\beta$ which makes $u_n \propto \sin(n\theta)$

The boundary condition at $N+1$ gives us:

$$\sin[(N+1)\theta] = 0$$

which gives us $\theta = \frac{M\pi}{N+1}$ where $M = 1, 2, 3 \dots$ an integer labelling the modes.

$\boxed{\theta = \frac{M\pi}{N+1}}$ So now we can work out all the normal modes!

And since $u_n = \sin(n\theta)e^{i\omega t}$, we can also work out the displacement of <u>any</u> bead!

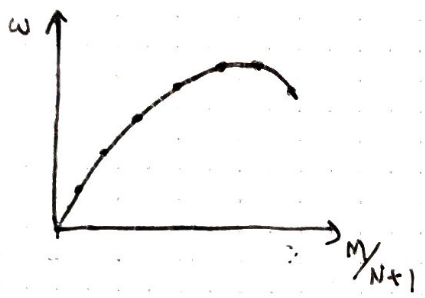
We have thus found a complete solution which we will state in full on the next page.

The Set of Modes

The set of modes have frequencies given by:

$$\boxed{\omega_M = 2\sqrt{\frac{T}{Ma}}\,\sin\left(\frac{M\pi}{2(N+1)}\right)}$$

For a string with 6 beads, we expect 6 normal modes
This is because of the sinusoid in the ω_M expression



As you can see, the modes are repeated at higher M.
So the sinusoid makes it so there is a maximum frequency, beyond which all modes are just repeats.