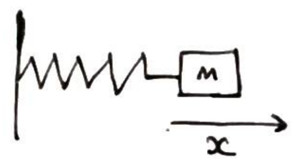
Oscillations

Simple harmonic motion is defined as a periodic motion in which the acceleration is directly proportional to the displacement and acts in the opposite direction to the displacement.

The simplest example of this is a mass on a spring:



The spring is extended in a direction x horizontally. The restoring force is therefore given by: $F = -kx$

So acceleration is given by:

$$a = -\frac{k}{M}x = \frac{d^2x}{dt^2}$$

solving this second order differential equation for x gives:

$$x(t) = A\sin(\omega t + \delta)$$

where A is the amplitude

$\omega = \sqrt{k/M}$

δ is a shift in phase

The phase $\phi = \omega t + \delta$

$$v = \frac{dx(t)}{dt} = A\omega\cos(\omega t + \delta)$$

We get this by differentiating the displacement. Max velocity is given by $A\omega$

$$a = \frac{d^2x(t)}{dt^2} = -A\omega^2\sin(\omega t + \delta)$$

We get this by differentiating the velocity. Max acceleration is given by $A\omega^2$

Note that both v and a (velocity and acceleration) are still functions of time.

Angular Frequency

So where does this ω come from? ω is defined as the angular frequency. We know that the maximum acceleration is given by $a_{max} = -\omega^2 A$ which for a spring will be at the largest displacement x. The largest displacement is when $x = A$ so:

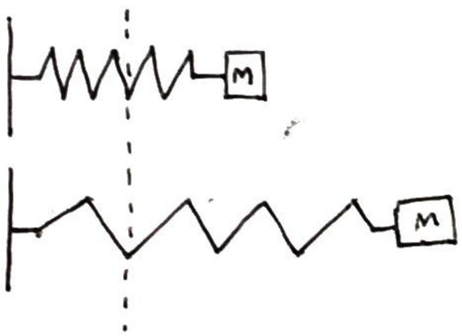
$$a_{max} = -\omega^2 A = -\frac{k}{m} A \quad \therefore \boxed{\frac{k}{m} = \omega^2}$$

The angular frequency is $\frac{2\pi}{T} = 2\pi f$

So we can arrive at the characteristic equation for simple harmonic motion:

$$\boxed{\frac{d^2x}{dt^2} = -\omega^2 x}$$

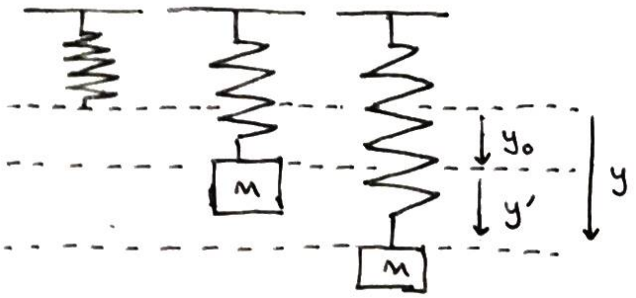
Odd Question



Two identical springs with identical masses are strected different distances and released. One is stretched 5cm and the other 15cm. Which reaches the equilibrium point first?

They reach the equilibrium point at the same time since angular frequency $\omega = \sqrt{\frac{k}{m}}$ does not depend on displacement x.

Mass on a vertical spring



Here, a mass is put on a vertical spring. The spring is stretched a distance y_0 by the weight of the mass. It is further stretched y' (total displacement y) by some initial force.

The total force down in the maximum stretched position is given by: $F = ma = -ky + mg$

$$\therefore m\frac{d^2y}{dt^2} = -ky + mg$$

The extra $+mg$ term makes this look like it isn't just SHM but that is because we are looking at displacement y from the unstretched position instead of y' from the equilibrim position (y_0 away from the unstretched position)

The equilibrium happens when there is no force so:

$-ky_0 + mg = 0 \quad \therefore \quad y_0 = \frac{mg}{k}$

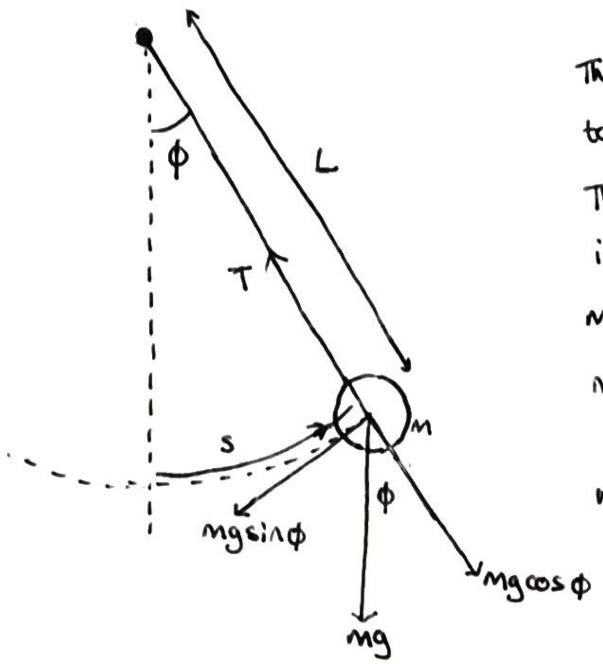
Also: $y = y_0 + y'$ Substituting these two into the original:

$$m\frac{d^2}{dt^2}\Big(y' + \underbrace{\frac{mg}{k}}_{\text{constant}}\Big) = -k\Big(y' + \frac{mg}{k}\Big) + mg$$

$$\Rightarrow m\frac{d^2y'}{dt^2} = -ky'$$ which is the equation for simple harmonic motion!

$$\frac{d^2y}{dt^2} = -\frac{k}{m}y'$$

Simple Pendulum



The tension here must be equivalent to the weight: $T = mg\cos\phi$

The other component of the weight is parallel to the direction of motion and so, by Newton's 2nd Law:

$$m\frac{d^2s}{dt^2} = -mg\sin\phi$$

Where $s = L\phi$ (the arc length)

Therefore: $$mL\frac{d^2\phi}{dt^2} = -mg\sin\phi$$

$$\frac{d^2\phi}{dt^2} = -\frac{g}{L}\sin\phi$$

but for small angles, $\sin\phi \approx \phi$

$$\therefore \boxed{\frac{d^2\phi}{dt^2} = -\frac{g}{L}\phi}$$

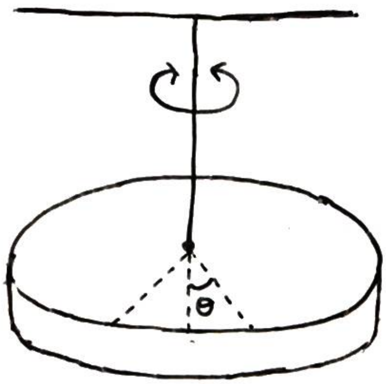
This is simple harmonic motion where $\omega^2 = \frac{g}{L}$

The period of this pendulum is given by $T = \frac{2\pi}{\omega}$

$$\boxed{T = 2\pi\sqrt{\frac{L}{g}}}$$

Note that this is independent of the mass!

Torsional Pendulum



Here, $\underline{\tau = -k\theta}$ where k is the rotational spring constant.

since $\tau = \frac{dL}{dt}$ where $L = I\omega$

$$\therefore \quad \tau = I\frac{d\omega}{dt} = \underline{I\frac{d^2\theta}{dt^2}}$$

Combining these two equations, we get:

$$-k\theta = I\frac{d^2\theta}{dt^2} \qquad \therefore \quad \boxed{\frac{d^2\theta}{dt^2} = -\frac{k}{I}\theta}$$

This is simple harmonic motion where $\omega^2 = \frac{k}{I}$

Energy in Simple Harmonic Motion

In Simple Harmonic Motion the kinetic energy and potential energy can change but the total mechanical energy remains constant and is thus independent of time.

The elastic potential energy in a spring is given by:
$U = \frac{1}{2}kx^2$

The kinetic energy in a spring is given by:
$T = \frac{1}{2}mv^2$

subbing $x(t) = A\sin(\overbrace{\omega t + \delta}^{\phi})$

$v(t) = \omega A\cos(\omega t + \delta)$

$= \frac{k}{m}A\cos(\omega t + \delta)$

Total Energy $E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$

$$E = \frac{1}{2}kA^2(\sin^2\phi + \cos^2\phi) \qquad \therefore \quad \boxed{E = \frac{1}{2}kA^2}$$

where k can also be expressed as $\omega^2 m$

Overdamping is even more extreme than critical damping. Here, the velocity approaches 0 as the system approaches the equilibrium position (albeit very slowly!) An example of this could be a pendulum submerged in treacle.

Note that the difference between overdamped and critically damped is that critically damped doesn't really affect the speed of the system.

Linear damping

Linear damping describes a damping force that is proportional to the velocity: $F_{damp} = -bv$

Here, b is a damping constant. This force is non-conservative and always acts against the direction of motion.

Using Newton's Second Law, we can write that the force on a spring that is damped is:

$$F = ma = -kx - bv$$

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$

This is a second order differential equation that can be solved using general methods and has 3 solutions, one for each type of damping.

Underdamped Solution

$$\boxed{x = A_0 e^{\frac{-bt}{2m}} \cos(\omega t + \delta)}$$

b is the damping constant and m is the mass of the system.

This solution only works if the angular frequency is set to a <u>fixed</u> value determined by other properties of the system.

$$\boxed{\omega = \omega_0 \sqrt{1 - \left(b/2m\omega_0\right)^2}}$$

where ω_0 is the value for no damping: $\omega_0 = \sqrt{k/m}$

For weak damping, i.e. damping very much underdamped with $b \ll 2m\omega_0$, we find that $\boxed{\omega \approx \omega_0}$

Critical Damping and Overdamping

Since $\omega = \omega_0 \sqrt{1 - (b/2m\omega_0)^2}$, critical damping is when $\omega = 0$. This happens when $\boxed{b = 2m\omega_0}$.

If b is much higher, the ω value is very small so the slower the system will reach the equilibrium position.

Energy versus Time

If damping is involved, the total energy of the system will also decay over time. For weak damping, the energy at any one time can still be approximated to: $E = \frac{1}{2} m \omega^2 A^2$

But in weak damping, $A \doteq A_0 e^{-bt/2m}$ so:

$$\boxed{E(t) = \frac{1}{2} m \omega^2 A_0^{\,2} e^{-bt/m}} \Rightarrow \boxed{E = E_0 e^{-t/\tau}}$$

where $\tau = m/b$, the time taken for the oscillation to decrease by $1/e$.

Q factor

A damped oscillator is defined by its Quality Factor (Q factor) defined as $Q = \omega_0 \tau$

This is a dimensionless quantity. For small damping Q is large and $\omega_0 \approx \omega$.

$$\therefore \quad Q \approx \omega \tau \quad \Rightarrow \quad \underline{\underline{Q \approx \frac{2\pi}{T} \tau}}$$

The change in energy in one cycle can be calculated by differentiating $E = E_0 e^{-t/\tau}$

$$\frac{dE}{dt} = -\frac{1}{\tau} E_0 e^{-t/\tau}$$ for one cycle, $dt = T$

$$\therefore \quad dE = -\frac{T}{\tau} E_0 e^{-t/\tau} = -\frac{T}{\tau} E$$

So the energy change per cycle is :

$$\boxed{dE = -\frac{T}{\tau} E}$$

Since $Q \simeq 2\pi \frac{\tau}{T}$, we find that :

$$\boxed{\frac{dE}{E} \simeq -\frac{2\pi}{Q}}$$

where $\frac{dE}{E}$ is the fraction of energy lost in each cycle .