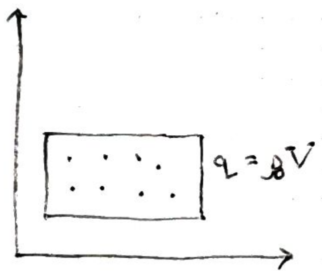
Relativistic Formulation of Electromagnetism

In order to create a relativistic formulation of Electromagnetism, we have to first make some four-vectors, as we did for the laws of dynamics.

Let's start with four-vector current:

Four Vector Current



Consider a uniform distribution of charge in a volume V at rest in some frame.

If charge density is ρ_0 then $q = \rho_0 V$

If we make a boost with speed v relative to the charge, the volume will be smaller due to Lorentz contraction:

$$V' = \frac{V}{\gamma}$$

and $\rho' = \gamma \rho_0$

There will also be a current density since the charges are moving in the new (boosted) inertial reference frame.

We define

$$\boxed{J^\mu = (\rho c, \underline{J})}$$ four vector current density

$$\boxed{J^\mu = \rho_0 u^\mu = \rho_0 \frac{dx^\mu}{d\tau}}$$

The Lorentz invariant length is thus $J^\mu J_\mu = \underline{\underline{\rho_0^2 c^2}}$.

Conservation

$\underline{\nabla} \cdot \underline{J} + \frac{\partial \rho}{\partial t} = 0$ is the conservation of charge equation we found before.

We can write this neater as

$$\boxed{\partial^{\mu} J_{\mu} = 0}$$

where $\partial^{\mu} = \left(\frac{\partial}{\partial x^0}, -\underline{\nabla} \right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\underline{\nabla} \right)$

and $J_{\mu} = (\rho c, -\underline{J})$

The Four Vector ∂^{μ}

$$\boxed{\partial^{\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\underline{\nabla} \right)}$$

Note that it is $-\underline{\nabla}$ and not $\underline{\nabla}$ even though it is an upstairs index on μ.

This transforms as normal:

$\partial'^{\mu} = \Lambda^{\mu}{}_{\nu} \partial^{\nu}$ which would imply:

$$\frac{1}{c}\frac{\partial}{\partial t'} = \gamma \frac{1}{c}\frac{\partial}{\partial t} + \frac{v}{c}\gamma \frac{\partial}{\partial x}$$

$$-\frac{\partial}{\partial x'} = -\gamma \frac{\partial}{\partial x} - \frac{v}{c}\gamma \frac{1}{c}\frac{\partial}{\partial t}$$

We can also define a four-vector version of ∇^2:

$$\boxed{\Box = \partial^{\mu}\partial_{\mu} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2}$$

The Four Vector Potential

Let's now define our magnetic potential in four-vector form.

$$A^\mu = \left(\frac{\phi}{c}, \underline{A}\right)$$

which then let us write the Maxwell's equations as

$$\boxed{\Box A^\mu = \frac{J^\mu}{\varepsilon_0 c^2}} \qquad \text{where} \qquad \Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

The $\mu = 0$ gives us the ϕ equation $\left(-\nabla^2\phi + \mu_0\varepsilon_0 \frac{\partial^2\phi}{\partial t^2} = \frac{\rho}{\varepsilon_0}\right)$ and $\mu = 1, 2, 3$ give the components of $-\nabla^2 \underline{A} + \mu_0\varepsilon_0 \frac{\partial^2 \underline{A}}{\partial t^2} = \mu_0 \underline{J}$

In Lorentz gauge, this becomes

$$\boxed{\partial_\mu A^\mu = 0}$$

Being able to write these equations in four vector notation is evidence that electromagnetism is relativistically invariant.

A Moving Point Charge

An electric charge at rest has four vector potential

$$A^\mu = \left(\frac{\phi}{c}, \underline{A}\right) = \left(\frac{q}{4\pi\varepsilon_0 r c}, \underline{0}\right)$$

We make the charge move by boosting to an inertial frame with speed v in the positive x direction.

$$A'^\mu = \Lambda^\mu_{\ \nu} A^\nu$$

$$\Rightarrow \quad A'^0 = \gamma\left(A^0 - \frac{v}{c} A^x\right)$$

$$\text{so} \quad \phi' = \frac{\gamma q}{4\pi\varepsilon_0 r'}$$

you will see this if you substitute in the relevant terms.

so $\phi' = \dfrac{\gamma q}{4\pi\varepsilon_0 r'}$

but $r^2 = x^2 + y^2 + z^2$ and so $r'^2 = \gamma^2(x+vt)^2 + y^2 + z^2$

$$\Rightarrow \quad \phi' = \frac{\gamma q}{4\pi\varepsilon_0\left(\gamma^2(x'+vt')^2 + y'^2 + z'^2\right)^{1/2}}$$

Now let's do the spacial components (i.e $\mu = 1,2,3$):

since only x component is non-zero $A'^y = A'^z = 0$

$$A'^x = -\gamma \frac{v}{c} A^0 = -\frac{\gamma v}{c^2} \frac{q}{4\pi\varepsilon_0\left(\gamma^2(x'+vt')^2 + y'^2 + z'^2\right)^{1/2}}$$

The Electric field is $\underline{E} = -\underline{\nabla}'\phi' - \dfrac{\partial \underline{A}'}{\partial t'}$

This gives us:

$$E'^x = \frac{q\gamma}{4\pi\varepsilon_0} \frac{(x'+vt')}{\left(\gamma^2(x'+vt')^2 + y'^2 + z'^2\right)^{3/2}}$$

$$E'^y = \frac{q\gamma}{4\pi\varepsilon_0} \frac{y'}{\left(\gamma^2(x'+vt')^2 + y'^2 + z'^2\right)^{3/2}}$$

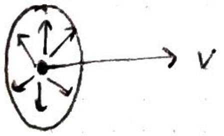
$$E'^z = \frac{q\gamma}{4\pi\varepsilon_0} \frac{z'}{\left(\gamma^2(x'+vt')^2 + y'^2 + z'^2\right)^{3/2}}$$

if we look at the ultrarelativistic case were $v \simeq c$:

$$E'^x \approx \frac{q}{4\pi\varepsilon_0(x'+vt')^2} \frac{1}{\gamma^2}$$

since γ is large, this is smaller than that of a stationary point charge. (in fact it is ~ 0)

However, there will not be a $\frac{1}{\gamma^2}$ term in E'^y or E'^z so we effectively have a disk (torus) of a field:



Electromagnetic Field Strength Tensor

Remember that $\underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t}$ so each component is given by:

$$\frac{E^i}{c} = \partial^i A^0 - \partial^0 A^i$$

similarly $\underline{B} = \underline{\nabla} \times \underline{A}$

so $B^i = \partial^j A^k - \partial^k A^j$

So we conclude that $\underline{E}$ and $\underline{B}$ fields are both described by the electromagnetic field strength tensor.

$$\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu}$$

$$\boxed{F^{\mu\nu} = \begin{bmatrix} 0 & -\frac{E^1}{c} & -\frac{E^2}{c} & -\frac{E^3}{c} \\ \frac{E^1}{c} & 0 & -B^3 & B^2 \\ \frac{E^2}{c} & B^3 & 0 & -B^1 \\ \frac{E^3}{c} & -B^2 & B^1 & 0 \end{bmatrix}}$$

where μ counts the row and ν counts the column

Maxwell's equations in terms of $F^{\mu\nu}$ are given by:

$$\boxed{\partial_\mu F^{\mu\nu} = \mu_0 J^\nu}$$

and

$$\boxed{\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0}$$

Lorentz Transformations of Electric and Magnetic Fields

The EM field strength tensor transforms like:

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} F^{\alpha\beta}$$

For example, a boost with speed v in $+z$ direction:

$\Lambda^{\mu}{}_{\nu}$ is given by $\Lambda^{\mu}{}_{\nu} = \begin{bmatrix} \gamma & 0 & 0 & -\frac{v}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{v}{c}\gamma & 0 & 0 & \gamma \end{bmatrix}$

So if we want to work out how E^1 transforms, we need to see how the $\frac{E^1}{c}$ element transforms in $F^{\mu\nu}$

The $\frac{E^1}{c}$ element is F^{10}:

$$\frac{E'^1}{c} = F'^{10} = \Lambda^1{}_\alpha \Lambda^0{}_\beta F^{\alpha\beta}$$

$$= \Lambda^0{}_\beta \left(\Lambda^1{}_0 F^{0\beta} + \Lambda^1{}_1 F^{1\beta} + \Lambda^1{}_2 F^{2\beta} + \Lambda^1{}_3 F^{3\beta}\right)$$

$$= \Lambda^0{}_\beta F^{1\beta} \quad \text{since the others are 0}$$

$$= \Lambda^0{}_0 F^{10} + \Lambda^0{}_1 F^{11} + \Lambda^0{}_2 F^{12} + \Lambda^0{}_3 F^{13}$$

$$\underline{\frac{E'^1}{c} = \gamma\left(\frac{E^1}{c} - \frac{v}{c} B^2\right)}$$

So we know how the $\frac{E^1}{c}$ transforms.

We can repeat this for all the non-zero elements in $F^{\mu\nu}$ to find:

$E'^1/c = \gamma(E^1/c - v/c\, B^2)$	$B'^1 = \gamma(B^1 + v/c\, E^2/c)$
$E'^2/c = \gamma(E^2/c - v/c\, B^1)$	$B'^2 = \gamma(B^2 - v/c\, E^1/c)$
$E'^3/c = E^3/c$	$B'^3 = B^3$

Relativistic Force Law

Classically, electromagnetic force is given by

$$\underline{F} = q(\underline{E} + \underline{v} \times \underline{B})$$

in the x component:

$$F^1 = q(E^1 + v^2 B^3 - v^3 B^2)$$
$$= q(cF^{10} - v^2 F^{12} - v^3 F^{13})$$
$$= q(cF^{10} - v^1 F^{11} - v^2 F^{12} - v^3 F^{13})$$

we added this for symmetry
we can do this since $-v^1 F^{11} = 0$

since (c, v^1, v^2, v^3) are the non-relativistic limit of u^μ, we are led to:

$$\boxed{f^\mu = q u_\nu F^{\mu\nu}}$$

This is the relativistic electromagnetic for equation.

What is the non-relativistic limit of time-like component of force?

$$f^0 = q(u_0 F^{00} - u_1 F^{10} - u_2 F^{20} - u_2 F^{30})$$
$$= q\gamma \frac{\underline{v} \cdot \underline{E}}{c}$$
$$= -\frac{q\gamma}{c} \underline{v} \cdot (\underline{E} + \underline{v} \times \underline{B})$$

where we use $\underline{v} \cdot \underline{v} \times \underline{B} = 0$

Taking $v \ll c$, we get $f^0 = q\underline{v} \cdot \underline{E}$ which is what we expect in the non-relativistic limit.