Relativistic Quantum Mechanics - Klein Gordon Equation

We are interested in interactions of highly energetic particles (electrons) in particle accelerators.
In this section, we will absorb c and $\hbar$ into the units, setting $c=1$ and $\hbar=1$ for convenience.

For a free relativistic particle:

$$E^2 = p^2 + m^2 \qquad (\text{usually } E^2 = p^2c^2 + m^2c^4)$$

in position space the energy-momentum operator is:

$$\hat{p}^\mu \rightarrow i\partial^\mu \qquad \text{so} \qquad (\hat{E}, \hat{p}) = (i\frac{\partial}{\partial t}, -i\underline{\nabla})$$

$$(\text{remember: } \hat{E}\psi = i\hbar\frac{\partial}{\partial t}\psi$$

$$\hat{p}\psi = -i\hbar\frac{\partial}{\partial x}\psi)$$

let's sub $(\hat{E}, \hat{p})$ into $E^2 = p^2 + m^2$

$$i^2\frac{\partial^2}{\partial t^2} = (-i)^2\nabla^2 + m^2$$

$$\Rightarrow -\frac{\partial^2}{\partial t^2} = -\nabla^2 + m^2 \quad \Rightarrow \quad \frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 = 0$$

Note, this isn't very mathematically rigorous. Since we are working with operators, we have to apply them on a function, say $\phi(x)$

$$\Rightarrow \boxed{(\Box + m^2)\phi(x) = 0}$$

This is the Klein Gordon equation

where $\Box = \partial_\mu\partial^\mu = \frac{\partial^2}{\partial t^2} - \nabla^2$

x here is the 4-vector $(t, \underline{x})$

The Klein Gordon equation has plane wave solutions

$$\phi(x) = Ne^{-i(Et - p\cdot\underline{x})}$$

where N is a normalisation constant

Sub this into the equation to find

$$E = \pm\sqrt{|p|^2 + m^2}$$

as expected from the first equation

Problems in the Klein Gordon Equation

The first problem we notice is that the energy is $E = \pm\sqrt{|p|^2 + m^2}$. Notice it can be both +ve and -ve. The negative energy solutions are a problem if we try to interpret $\phi(x)$ as a wave function. With -ve energy, we can extract arbitrarily large amounts of energy from a system by driving it into ever more negative energy states.
Any external perturbation capable of pushing a particle across the energy gap of $2m$ between positive and negative energy will uncover this problem.

We also cannot disregard the -ve states as unphysical since they appear as part of the complete set of states for the Klein Gordon equation.

Another problem arises if we try to find a probability density. We expect density to transform under boosts (since lengths contract) but the ϕ we have is a Lorentz invariant. So $|\phi|^2$ will not transform under boosts, meaning it does not transform like a density. We thus don't have a Lorentz covariant continuity equation like

$$\partial_t \rho + \underline{\nabla} \cdot \underline{J} = 0 \qquad \partial_\mu J^\mu = 0$$

So what is our probability density in this case? We need to find something that satifies the continuity equation.

Let's start with the KG equation. If we multiply the KG equation by ϕ^* and then subtract the complex conjugate of the KG equation multiplied by ϕ:

$$KG\,\phi^* - (KG)^*\phi = \left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi\phi^* + m^2\phi\phi^*$$
$$- \left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi^*\phi - m^2\phi^*\phi$$

$$= \left(\phi^*\frac{\partial^2\phi}{\partial t^2} - \phi\frac{\partial^2\phi^*}{\partial t^2}\right) - \left(\phi^*\nabla^2\phi - \phi\nabla^2\phi^*\right)$$

We thus take $\rho \equiv i\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right)$

and $\underline{J} = -i\left(\phi^*\underline{\nabla}\phi - \phi\underline{\nabla}\phi^*\right)$

interpreting ρ as probability density and $\underline{J}$ as probability current.

However, for a plane wave, we know $\rho = 2|N|^2 E$, from the fact that $\phi(x) = Ne^{-i(Et - \underline{p}\cdot\underline{x})}$. But we previously found $E = \pm\sqrt{|\vec{p}|^2 + m^2}$ which means ρ can be negative.

It isn't possible to have a negative probability!

Feynman - Stückelberg Interpretation

Feynman and Stückelberg proposed a solution linked to Pauli's idea that you cannot directly measure the number of particles; you can only detect them through an interaction. Similarly, you can't observe directly the probability density. You can only directly observe charge density/current and that cannot be negative.

The KG equation has time reversal symmetry so solutions with e^{-iEt} propagate forward in time and e^{iEt} that propagate backward in time.

Normally, we disregard states that propagate backward in time so there are no causality paradoxes. But with the KG equation, we can also have negative E.

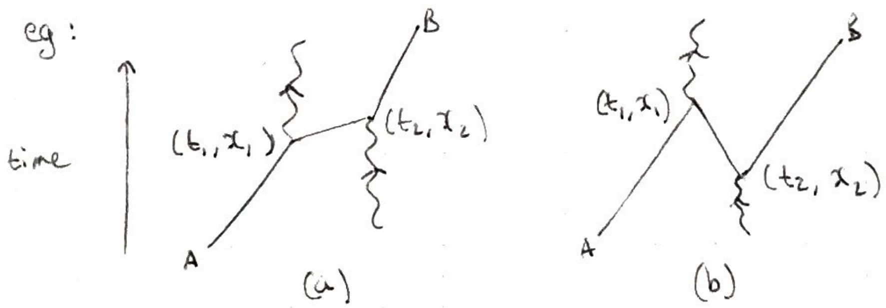
So do $e^{-i(-E)t}$ states propagate forward in time with negative energy or backward in time with positive energy?

Feynman and Stückelberg proposed we keep all states with e^{-iEt}, i.e positive energy moving forward in time. And we also keep $e^{iEt} = e^{-i(-E)t}$, i.e we interpret positive energy moving backward in time as negative energy moving forward in time. This is a prediction of anti-particles!

So this theory is consistent with causality and avoids the aforementioned problems.

It is easier to interpret the emission/absorption of a negative energy particle as the absorption/emission of a positive energy particle.

eg:



Feynman diagrams of pion-photon scatterings.

In (a) the pion has positive energy and travels forward in time, emitting a photon at x_1. If the pion still has positive energy, it will absorb the initial state photon at x_2. The final state is a positive energy pion and a photon.

In (b), the pion has positive energy and travels forward in time, emitting a photon at x_1. However, the energy of the photon is bigger than the energy of the initial pion. Thus, the energy becomes negative and it is forced to travel backwards in time. At x_2, at a previous time, it absorbs the initial state photon, thereby rendering its energy positive. It then moves forward in time. The final state is a positive energy pion and a photon.

(a) and (b) have identical initial and final state.
We can describe (b) in better language using antiparticles:

initial state π^+ and a photon
At t_2 and x_2 photon creates $\pi^+\pi^-$ pair. Both propagate forward in time.
At t_1 and x_1 the π^- is annihilated by the π^+ and produces a photon

The π^+ and photon end up at B. To someone observing in real time, the negative energy moving backward in time is seen as positive energy moving forward in time.

With this interpretation, we have discovered anti-matter!