Electromagnetism in Matter

So now that we have considered EM waves in vacuums, let's think about how they behave in matter. We already saw one example of this in the last section, where EM waves are scattered different amounts depending on their frequency.

Linear, Isotropic, Homogeneous Matter

EM waves in matter become an immensely complicated topic due to non-linear effects, i.e $\underline{E}$ fields strong enough that electrons inside matter become detached from their sites.

We will consider only linear effects to keep things simple. This means $\underline{E}$ fields that are too weak to detach electrons. We shall also assume the matter is isotropic (behaves same in all directions) and homogeneous (behaves same at all points).

We will only consider insulating material, called dielectrics, that do not conduct current and contain no free charge.

Imagine space filled with infinite amount of this material, so without any external charge or current, Maxwell's equations are:

$$\underline{\nabla} \cdot \underline{E} = 0 \qquad \underline{\nabla} \cdot \underline{B} = 0 \qquad \underline{\nabla} \times \underline{E} = -\frac{\partial B}{\partial t} \qquad \underline{\nabla} \times \underline{B} = -\mu \varepsilon \frac{\partial E}{\partial t}$$

where $\mu_0 \to \mu$ and $\varepsilon_0 \to \varepsilon$ since we are no longer in vacuum

we define relative permitivitty or dielectric constant:

$$\boxed{\varepsilon_r = \varepsilon / \varepsilon_0} \qquad \text{where } \varepsilon_r > 1$$

In an infinite, linear, isotropic, homogeneous medium in the absence of free charge and free current, it is trivial to modify Maxwell's equations. So Maxwell's equations again predict EM waves in matter, albeit with a different phase velocity

$$\boxed{v = \frac{1}{\sqrt{\varepsilon\mu}}}$$

We can thus define the refractive index of a medium to be the ratio of EM velocity in the medium and in a vacuum:

$$\boxed{n = \frac{c}{v}}$$

$$\therefore\ n = \frac{c}{1/\sqrt{\varepsilon\mu}} = c\sqrt{\varepsilon\mu} \quad \text{but } c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$$

so $n = \sqrt{\frac{\varepsilon}{\varepsilon_0}}\sqrt{\frac{\mu}{\mu_0}}$ assuming $\mu \approx \mu_0$: $n = \sqrt{\frac{\varepsilon}{\varepsilon_0}} = \underline{\underline{\sqrt{\varepsilon_r}}}$

Boundary Conditions

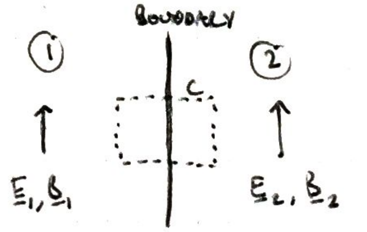
So what happens when EM waves in one medium encounter another medium?

Let's start be deriving boundary conditions at a plane interface between 2 mediums.

Any $\underline{E}$ or $\underline{B}$ field can be split into components parallel or perpendicular to the boundary, and these components must be considered separately.

For the case where the EM waves are at 90° incidence to to boundary, we only need to consider parallel components, simplifying things for us. So we will consider this case.

Consider the $\underline{E}$ and $\underline{B}$ fields $\underline{E}^{(1)}$ and $\underline{B}^{(1)}$ in a medium (1)
and the $\underline{E}$ and $\underline{B}$ fields $\underline{E}^{(2)}$ and $\underline{B}^{(2)}$ in a medium (2).



Let's try applying Faraday's law to the closed loop c:

$$\oint \underline{E}(t,x,y,z) \cdot d\underline{L} = -\frac{\partial \phi_B}{\partial t}$$

If the loop is thin, the magnetic flux through it is 0 so $-\frac{\partial \phi_B}{\partial t} = 0$

Evaluating for the closed loop, we find $\boxed{E_1 = E_2}$

where E_1 and E_2 are parallel components of $\underline{E}^{(1)}$ and $\underline{E}^{(2)}$ at either side of boundary.

Now let's try applying ampere-maxwell law to the closed loop:

$$\oint \underline{B}(t,x,y,z) \cdot d\underline{L} = \mu_0 I + \mu_0 \varepsilon_0 \frac{\partial \phi_E}{\partial t}$$

These will be μ_1 and μ_2 when evaluating this

If the loop is thin, we can once again say $\phi_E = 0$ so $\frac{\partial \phi_E}{\partial t} = 0$

Let's also assume $\mu_1 \approx \mu_2 \approx \mu_0$ and let's set $I = 0$ since no current flows.

so $\oint_c \underline{B} \cdot d\underline{L} \approx 0$ $\therefore$ $\boxed{\underline{B}_1 \approx \underline{B}_2}$

So these are our boundary conditions:

$\underline{\underline{E_1 = E_2}}$ $\underline{\underline{\underline{B}_1 \approx \underline{B}_2}}$

Remember, these have been derived in the absence of free current and free charge, and for a perfect dielectric.

A perfect conductor has boundary condition $E = 0$ inside the conductor since a perfect conductor has free charge so any $\underline{E}$ field inside the conductor causes free electrons to flow so as to cancel the induced potential

Reflection and Transmission of Waves on a String

Imagine 2 strings of mass per unit length μ_1 and μ_2 with a knot joining them. The phase velocity of a wave in each string is given by:

$$v_1 = \sqrt{\frac{T}{\mu_1}} \qquad v_2 = \sqrt{\frac{T}{\mu_2}}$$ assuming Tension T is same in both strings

Now let's recall the complex harmonic wave solution:

$$\underline{\tilde{f}}(z,t) = Ae^{i(kz - \omega t + \delta)}\,\hat{n}$$

Let's say the wave is polarised in $\hat{x}$ direction so we can drop the $\hat{n}$

The incident wave is:

$$\tilde{f}_I(z,t) = A_I\, e^{i(k_1 z - \omega t + \delta_I)}$$

Some of this gets reflected at the knot and some gets transmitted

The reflected wave is:

$$\tilde{f}_R(z,t) = A_R\, e^{i(-k_1 z - \omega t + \delta_R)}$$

The transmitted wave is:

$$\tilde{f}_T(z,t) = A_T e^{i(k_2 z - \omega t + \delta_T)}$$

This should be quite familiar from Waves last semester.

We have assumed all 3 waves have same angular frequency ω

$\nu = \frac{\omega}{2\pi}$ so the wavelength of the waves in string ① and string ②, and the wavenumbers in string ① and string ② are.

$$\lambda_1 = \frac{v_1}{\nu} \qquad \lambda_2 = \frac{v_2}{\nu} \qquad k_1 = \frac{2\pi}{\lambda_1} \qquad k_2 = \frac{2\pi}{\lambda_2}$$

so $$k_1 = \frac{\omega}{v_1} \qquad k_2 = \frac{\omega}{v_2}$$

which is why we have different wavenumbers in the eqns. Even though ν is the same, k is not.

So the total wave in string ① is:

$$\tilde{f}_1(z,t) = \tilde{f}_I(z,t) + \tilde{f}_R(z,t)$$

The total wave in string ② is:

$$\tilde{f}_2(z,t) = \tilde{f}_T(z,t)$$

We can apply the boundary conditions at the knot where $z=0$:

$$\tilde{f}_1(0,t) = \tilde{f}_2(0,t)$$

$$\left[\frac{\partial \tilde{f}_1(z,t)}{\partial t}\right]_{z=0} = \left[\frac{\partial \tilde{f}_2(z,t)}{\partial t}\right]_{z=0}$$

These should be the same as we derived in the Waves course last semester.

From the first BC:

$$A_I e^{i(-\omega t+\delta_I)} + A_R e^{i(-\omega t+\delta_R)} = A_T e^{i(-\omega t+\delta_T)}$$

$$\Rightarrow \quad \underline{A_I e^{i\delta_I} + A_R e^{i\delta_R} = A_T e^{i\delta_T}}$$

Similarly from the second BC:

$$\underline{k_1 A_I e^{i\delta_I} - k_1 A_R e^{i\delta_R} = k_2 A_T e^{i\delta_T}}$$

We can combine these to get:

$$A_T e^{i\delta_T} = \frac{2v_2}{v_2+v_1} A_I e^{i\delta_I} \qquad A_R e^{i\delta_R} = \frac{v_2-v_1}{v_2+v_1} A_I e^{i\delta_I}$$

Taking the modulus of each we get

$$\underline{\underline{A_T = \frac{2v_2}{v_2+v_1} A_I}} \qquad \underline{\underline{A_R = \left|\frac{v_2-v_1}{v_2+v_1}\right| A_I}}$$

This tells us that $\delta_T = \delta_I$ but it isn't so simple for δ_R:

if $v_2 > v_1$ (string ② is lighter):

$$\delta_R = \delta_I$$

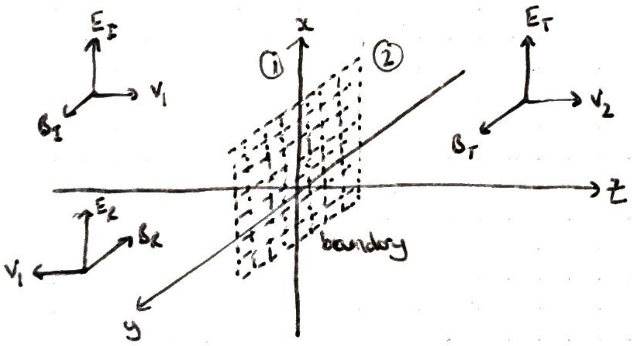
If $v_2 < v_1$ (string ① is lighter):

$$\delta_R = \delta_I + \pi$$

This will be quite handy when we are considering light being reflected at the boundary between two mediums.

EM waves at Normal Incidence

Now let's try doing the same for EM waves.
Consider an EM wave traveling through a transparent dielectric and incident on another transparent dielectric:



Some of the wave is reflected at the boundary and some is transmitted

The incident waves:

$$\tilde{E}_I(x,y,z,t) = E_I\,\hat{x}\,e^{i(k_1 z - \omega t + \delta_I)}$$

$$\tilde{B}_I(x,y,z,t) = \frac{1}{v_1}E_I\,\hat{y}\,e^{i(k_1 z - \omega t + \delta_I)}$$

$\tilde{B} = \frac{1}{c}(\hat{z} \times E)$ so:

The reflected waves:

$$\tilde{E}_R(x,y,z,t) = E_R\,\hat{x}\,e^{i(-k_1 z - \omega t + \delta_R)}$$

$$\tilde{B}_R(x,y,z,t) = -\frac{1}{v_1}E_R\,\hat{y}\,e^{i(-k_1 z - \omega t + \delta_R)}$$

The transmitted waves:

$$\tilde{\underline{E}}_T(x,y,z,t) = E_T \hat{\underline{x}} e^{i(k_2 z - \omega t + \delta_T)}$$

$$\tilde{\underline{B}}_T(x,y,z,t) = \frac{1}{v_2} E_T \hat{\underline{y}} e^{i(k_2 z - \omega t + \delta_T)}$$

The respective velocities are:

$$v_1 = \frac{1}{\sqrt{\mu_1 \varepsilon_1}} = \frac{c}{n_1} \qquad v_2 = \frac{1}{\sqrt{\mu_2 \varepsilon_2}} = \frac{c}{n_2}$$

$$k_1 = \frac{\omega}{v_1} \qquad k_2 = \frac{\omega}{v_2}$$

$$\tilde{\underline{E}}_1 = \tilde{\underline{E}}_I + \tilde{\underline{E}}_R \qquad \tilde{\underline{B}}_1 = \tilde{\underline{B}}_I + \tilde{\underline{B}}_R$$

$$\tilde{\underline{E}}_2 = \tilde{\underline{E}}_T \qquad \tilde{\underline{B}}_2 = \tilde{\underline{B}}_T$$

We can now construct boundary conditions:

$$\tilde{\underline{E}}_1(x,y,z=0,t) = \tilde{\underline{E}}_2(x,y,z=0,t)$$

$$\tilde{\underline{B}}_1(x,y,z=0,t) = \tilde{\underline{B}}_2(x,y,z=0,t)$$

Boundary conditions

By substituting the wavesolutions, we obtain results analogous to the waves on a string example:

Note, the boundary condition for $\underline{B}$ is analogous to the derivative BC for waves on a string

$$E_I e^{i\delta_I} + E_R e^{i\delta_R} = E_T e^{i\delta_T}$$

$$\frac{1}{v_1} E_I e^{i\delta_I} - \frac{1}{v_1} E_R e^{i\delta_R} = \frac{1}{v_2} E_T e^{i\delta_T}$$

Rearranging these:

$$\underline{E_T e^{i\delta_T} = \frac{2v_2}{v_2+v_1} E_I e^{i\delta_I}}$$

$$\underline{E_R e^{i\delta_R} = \frac{v_2 - v_1}{v_2+v_1} E_I e^{i\delta_I}}$$

We can take the modulus of these to give:

$$\boxed{E_T = \frac{2n_1}{n_1 + n_2} E_I} \qquad \boxed{E_R = \left| \frac{n_1 - n_2}{n_1 + n_2} \right| E_I}$$

So we see by taking the arguments both sides

$$\boxed{\delta_T = \delta_I}$$

but it again isn't so simple for the reflected wave.

if $n_2 < n_1$:

$$\boxed{\delta_R = \delta_I}$$

4 $n_2 > n_1$

$$\boxed{\delta_R = \delta_I + \pi}$$

So when reflected at boundary between lower n and higher n, there is a phase shift $+\pi$

We already knew this from Waves, Light and Quanta!

The intensity of any of the three waves is given by:

$$\boxed{I = \frac{1}{2} v \varepsilon E_0^2}$$

The reflection coefficient is:

$$\boxed{R = \frac{I_R}{I_I} = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2}$$

The transmission coefficient is:

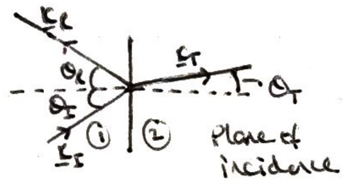
$$\boxed{T = \frac{I_T}{I_I} = \frac{4 n_1 n_2}{(n_1 + n_2)^2}}$$

Note :// $R + T = 1$

Laws of Optics from Maxwell's Equations

We can derive the familiar laws of optics from Maxwell's Equations:

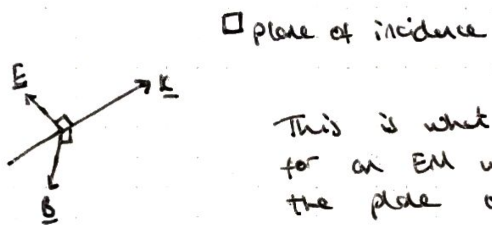
1) The incident, reflected and transmitted wave vectors form a plane called the plane of incidence as shown:



2) The angle of incidence = angle of reflection $\theta_I = \theta_R$

3) Snell's Law : $n_1 \sin\theta_I = n_2 \sin\theta_T$

Now let's consider what happens with the polarisation for the case that the polarisation of incident wave is parallel to the plane of incidence. This is hard to visualise but:



This is what it means for an EM wave to have polaraisation parallel to the plane of incidence

In this case the reflected and transmitted amplitudes are given by Fresnel's Equations:

$$E_T e^{i\delta_T} = \frac{2}{\alpha+\beta} E_I e^{i\delta_I}$$

$$E_R e^{i\delta_R} = \frac{\alpha-\beta}{\alpha+\beta} E_I e^{i\delta_I}$$

where $\alpha = \frac{\cos\theta_T}{\cos\theta_I}$ $\quad \beta = \frac{v_1}{v_2} = \frac{n_1}{n_2}$

At normal incidence, $\alpha = 1$ and these equations reduce to what we had before.

At grazing incidence, $\theta_1 = 90°$ so $\alpha = \infty$ and the wave is totally reflected, since $E_T = 0$

When $\alpha = \beta$, the wave is totally transmitted, as $E_R = 0$. The incident angle θ_I this happens at is called Brewster's angle. This happens only for the case where all waves are polarised in the plane of incidence.

But what if this isn't the case? Consider unpolarised light incident at Brewster's angle. There is a mixture of polarisations, so the part of the light polarised parallel to the plane of incidence will not be reflected, while light polarised perpendicular to this plane will be reflected.

So unpolarised light incident at Brewster's angle gives rise to reflected light polarised perpendicular to the plane of incidence. You will recognise this from Waves, Light and Quanta.

The Waveguide

A waveguide is a hollow metal tube through which EM waves can travel, being reflected off the walls due to the BC that the $\underline{E}$ field is 0 inside a perfect conductor.

This allows to "guide" the wave, hence the name.

For a rectangular tube with dimensions a, b, the conditions for a standing wave are given by components:

$$\lambda_x = \frac{2a}{m} \qquad k_x = \frac{\pi m}{a}$$

$$\lambda_y = \frac{2b}{n} \qquad k_y = \frac{\pi m}{b}$$

where λ_x is wavelength in x direction
λ_y is wavelength in y direction
k_x is wavenumber in x direction
k_y is wavenumber in y direction

m and n are integers.

The remaining component k_z is unconstrained

The wave vector $\underline{k}'$ for the resulting plane wave is:

$$\underline{k}' = \frac{\pi m}{a}\hat{\underline{x}} + \frac{\pi n}{b}\hat{\underline{y}} + k\hat{\underline{z}}$$

Angular frequency of the resulting wave is:

$$\omega = c|\underline{k}'| = c\sqrt{\left(\frac{\pi m}{a}\right)^2 + \left(\frac{\pi n}{b}\right)^2 + k^2}$$

The plane wave propagates in a direction that makes an angle θ with the z axis where:

$$\cos\theta = \frac{k}{|\underline{k}'|}$$

This implies for a given $k_z = k$, only certain angles θ are allowed, corresponding to n and m

Although the plane wave travels at speed c, since it is travelling at an angle to the $\hat{z}$ axis, while the waveguide is oriented with the $\hat{z}$ axis, we define a new velocity for the wave's net velocity down the waveguide.

We call this group velocity:

$$\underline{\underline{v_g = c\cos\theta}}$$

The speed of individual wavefronts is called phase velocity:

$$\underline{\underline{v_p = \frac{c}{\cos\theta}}}$$