PHYS 2001: Electromagnetism

Vector Analysis : Basics

By now we should all know some vector basics, i.e. how to add vectors and how to do cross and dot products:

$$\boxed{\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}|\cos\theta}$$

$$\boxed{\underline{a} \times \underline{b} = |\underline{a}||\underline{b}|\sin\theta \, \hat{\underline{n}}}$$

where $\hat{\underline{n}}$ is unit vector perp. to both $\underline{a}$ and $\underline{b}$

We can thus move onto more complicated concepts:

Vector Fields

A vector field is a vector associated with every point is space, denoted $\underline{v}(x,y,z)$. We can break the vector field $\underline{v}$ into components in x y and z:

$$\underline{v}(x,y,z) = v_x(x,y,z)\,\hat{\underline{x}} + v_y(x,y,z)\,\hat{\underline{y}} + v_z(x,y,z)\,\hat{\underline{z}}$$

The Meaning of Grad

The grad can be considered to be a "vector differentiation"; it uses the nabla operator $\underline{\nabla}$. It returns the rate of change of a scalar field in each direction:

$$\boxed{\underline{\nabla} = \hat{\underline{x}}\frac{\partial}{\partial x} + \hat{\underline{y}}\frac{\partial}{\partial y} + \hat{\underline{z}}\frac{\partial}{\partial z}}$$

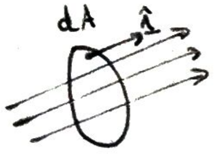
so the grad of scalar field $T(x,y,z)$ is:

$$\underline{\nabla} T = \hat{\underline{x}}\frac{\partial T(x,y,z)}{\partial x} + \hat{\underline{y}}\frac{\partial T(x,y,z)}{\partial y} + \hat{\underline{z}}\frac{\partial T(x,y,z)}{\partial z}$$

Note that this returns a vector field

Flux

An important concept with vector fields is Flux. Flux can be thought of as the number of field lines passing through a unit area. We can write this mathematically:



So some field lines are passing through the unit area. But we want the magnitude along the direction perpendicular to the surface (parallel to $\hat{n}$)

So infinitesimal flux $d\Phi = \underline{v}(x,y,z) \cdot \hat{n}\, dA$

we can define an area vector $d\underline{A} = \hat{n}\, dA$ for convenience so:

$$\boxed{d\Phi = \underline{v} \cdot d\underline{A}} \equiv \boxed{d\Phi = |\underline{v}||d\underline{A}|\cos\theta}$$

since we care about projection of $\underline{v}$ along $\hat{n}$

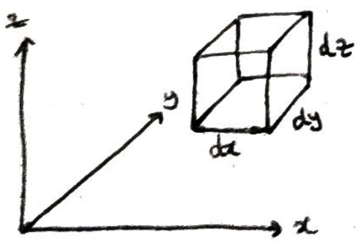
Gauss' Divergence Theorem

The divergence of a vector field is defined as its dot product with nabla. So for a field $\underline{v}(x,y,z) = v_x(x,y,z)\hat{x} + v_y(x,y,z)\hat{y} + v_z(x,y,z)\hat{z}$ the divergence is:

$$\boxed{\underline{\nabla} \cdot \underline{v} = \frac{\partial v_x(x,y,z)}{\partial x} + \frac{\partial v_y(x,y,z)}{\partial y} + \frac{\partial v_z(x,y,z)}{\partial z}}$$

Note that this returns a scalar field

So what does this mean physically? Consider a box element with wire edges placed in the middle of flowing water. How much water flows through each face? This is just the flux through that face. What is the total flux through the box?. This is the sum of the fluxes through each face.



Total flux $d\Phi = \sum_{\text{walls}} \underline{v}(x,y,z) \cdot d\underline{A}$

where $\underline{v}$ is volume of water

This gives us:

$$d\Phi = [V_x(x+dx, y, z) - V_x(x,y,z)]\,dy\,dz$$
$$+ [V_y(x, y+dy, z) - V_y(x,y,z)]\,dx\,dz$$
$$+ [V_z(x, y, z+dz) - V_z(x,y,z)]\,dx\,dy$$

which we can write as:

$$d\Phi = \underbrace{\left\{\frac{\partial V_x(x,y,z)}{\partial x} + \frac{\partial V_y(x,y,z)}{\partial y} + \frac{\partial V_z(x,y,z)}{\partial z}\right\}}_{\text{Note this is the divergence of } \underline{v}} \underbrace{dx\,dy\,dz}_{\text{Note this is volume of cube}}$$

$$\therefore\ d\Phi = (\underline{\nabla} \cdot \underline{v}(x,y,z))\,dV$$

We can actually intuitively tell this is 0 since all the water flowing into the box also flows out. So net flow $= 0$.

This would be non-zero if there is, say, a pipe inside the box allowing water to flow out that didn't flow in.

$$d\Phi = (\underline{\nabla} \cdot \underline{v}(x,y,z))\,dV = \sum_{\text{walls}} \underline{v}(x,y,z) \cdot d\underline{A}$$

We turn this into an integral to give us:

$$\boxed{\iiint_{\text{volume}} \underline{\nabla} \cdot \underline{v}(x,y,z)\,dV = \iint_{\substack{\text{closed}\\ \text{surface}}} \underline{v}(x,y,z) \cdot d\underline{A}}$$

This is the divergence theorem

"The net flux out of a closed surface is equal to the volume integral of the divergence over the region inside the closed surface."

The Continuity Equation

Previously we considered an incompressible fluid, water, which is why we could make the statement that net flux is 0, since all water flowing in flows out. But if we take a compressible fluid like air, then this statement is not necessarily true! Because air can be compressed, the density of the air may be different at different places in the box, so the volume flowing in is not necessarily the volume flowing out.

However, the mass is conserved, so mass flowing in is mass flowing out. And thus we can define a continuity condition. If ρ is the density of the fluid, the mass leaving the box in a time dt is:

$$\underline{\underline{\underline{\nabla} \cdot (\rho \underline{v})\, dV\, dt}}$$

which has to be equal to the decrease in mass of the fluid inside the box:

$$= \underline{\underline{-d\rho\, dV}}$$

$$\therefore \quad \underline{\nabla} \cdot (\rho \underline{v})\, dV\, dt + d\rho\, dV = 0$$

$$\boxed{\underline{\nabla} \cdot (\rho \underline{v}) + \frac{d\rho}{dt} = 0}$$

This is our mass continuity condition

Stokes' Curl Theorem

If we consider a bath full of water and we use our hands to rotate the water, we create small whirlpools in the water. A curl at the point (x, y, z) corresponds to a vortex of water which causes the water to rotate around that point.

The curl is defined mathematically as the cross product of $\underline{\nabla}$ and a vector field:

$$\boxed{\text{curl } \underline{v}(x,y,z) = \underline{\nabla} \times \underline{v}(x,y,z)}$$

$$\underline{\nabla} \times \underline{v}(x,y,z) = \begin{vmatrix} \hat{\underline{x}} & \hat{\underline{y}} & \hat{\underline{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$

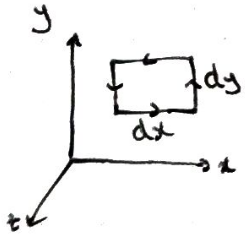
So what does this mean physically? Suppose we want to find the amount of rotation of water around a small rectangular loop's perimeter. If we consider each leg of the loop separately:

Let $\hat{\underline{n}}$ be direction of leg and dL be length of leg:

$\underline{v}(x,y,z) \cdot \hat{\underline{n}}\, dL$ is the velocity of water in the direction of that leg multiplied by the length of the leg. Letting $d\underline{L} = \hat{\underline{n}}\, dL$ we can also write this as $\underline{v}(x,y,z) \cdot d\underline{L}$

So over the whole loop we have $\sum_{\text{legs}} \underline{v}(x,y,z) \cdot d\underline{L}$

Let's consider a similar situation on a coordinate plane



$$\sum_{\text{legs}} \underline{v}(x,y,z) \cdot d\underline{L} = [v_x(x,y,z) - v_x(x,y+dy,z)]\, dx + [v_y(x+dx,y,z) - v_y(x,y,z)]\, dy$$

This can also be written:

$$\sum_{legs} \underline{v}(x,y,z)\cdot d\underline{L} = -\partial V_x \, dx + \partial V_y \, dy$$

$$= -\frac{\partial V_x}{\partial y} \, dy dx + \frac{\partial V_y}{\partial x} \, dx dy$$

$$\sum_{legs} \underline{v}(x,y,z)\cdot d\underline{L} = \underbrace{\left[\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right]}_{\substack{\text{This is the } \hat{\underline{z}} \\ \text{component of the} \\ \text{curl of } \underline{v}}} \underbrace{dx dy}_{\substack{\text{This is} \\ \text{area of loop}}}$$

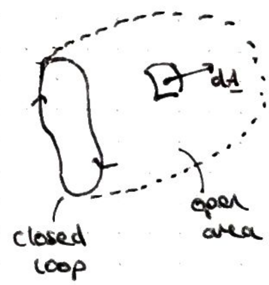
The normal to the plane of the loop is in the $\hat{\underline{z}}$ direction, so this whole thing can be written as:

$$\underbrace{\sum_{legs} \underline{v}(x,y,z)\cdot d\underline{L}}_{\substack{\text{This is the amount} \\ \text{of anticlockwise rotation}}} = \underbrace{\left[\underline{\nabla} \times \underline{v}(x,y,z)\right] \cdot d\underline{A}}_{\substack{\text{curl of velocity vector along} \\ \text{normal to area multiplied} \\ \text{by area}}}$$

Note, we took normal to area to be in $\hat{\underline{z}}$ direction and not $-\hat{\underline{z}}$ direction since we used right hand screw rule.

We can write this as an integral:

$$\boxed{\iint_{\text{open area}} (\underline{\nabla} \times \underline{v}(x,y,z)) \cdot d\underline{A} = \oint_{\text{closed loop}} \underline{v}(x,y,z)\cdot d\underline{L}}$$



"The surface integral of the curl of a vector field is equal to the line integral of the field over a closed loop."