Least Action - Conservation Laws

If the Lagrangian does not depend on a particular coordinate q_i, then we call q_i an ignorable coordinate and we say its associated generalised momentum is conserved

EL eqn is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \underbrace{\frac{\partial L}{\partial q_i}}_{} = 0$$

$p_i = \frac{\partial L}{\partial \dot{q}_i}$

$\hookrightarrow$ L does not depend on q_i so $\to 0$

so $\frac{dp_i}{dt} = 0 \Rightarrow p_i = \text{constant}$

so momentum in the q_i direction is conserved.

But we can notice something else. If L depends on $\dot{q}_i$ but not q_i, then applying $q_i \to q_i + \text{constant}$ will leave the Lagrangian invariant.

So symmetry (translation invariance) implies conservation of momentum.

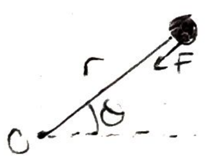
Now let's think about Energy Conservation.

If L does not depend on t, then we say that the Hamiltonian is conserved.

The Hamiltonian is defined as: $\boxed{H = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L}$

This is very important!

Let's do an example with Central forces.



Consider a particle moving subject to a central force in a potential $V(r)$

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}m(r\dot{\theta})^2$$

$$\Rightarrow L = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 - V(r)$$

The EL equation associated with r is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) - \frac{\partial L}{\partial r} = 0$$

$$\Rightarrow \quad m\ddot r - mr\dot\theta^2 - \frac{\partial V}{\partial r} = 0$$

The EL equation associated with θ is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow \quad \frac{d}{dt}(mr^2\dot\theta) = 0$$

Note that this shows us that angular momentum is conserved.

We can also work out the Hamiltonian

$$\frac{\partial L}{\partial \dot r}\dot r = m\dot r^2 \qquad \frac{\partial L}{\partial \dot\theta}\dot\theta = mr^2\dot\theta^2$$

so $$H = \sum_i \frac{\partial L}{\partial \dot q_i}\dot q_i - L$$

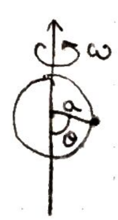
$$= m\dot r^2 + mr^2\dot\theta^2 - \frac{1}{2}m\dot r^2 - \frac{1}{2}mr^2\dot\theta^2 + V(r)$$

$$\Rightarrow \quad H = \frac{1}{2}m\dot r^2 + \frac{1}{2}mr^2\dot\theta^2 + V(r)$$

This is just total energy!

However, it is not always the case that the Hamiltonian is the energy of the system.

For example, consider a bead on a hoop which is vertical near the earth and is rotating at angular speed ω:



The kinetic energy is:

$$T = \frac{1}{2}ma^2\dot\theta^2 + \frac{1}{2}ma^2\sin^2\theta\,\omega^2$$

The potential energy is

$$V = -mga\cos\theta$$

so $$L = \frac{1}{2}m(a^2\dot\theta^2 + a^2\sin^2\theta\,\omega^2) + mga\cos\theta$$

So we can now work out the Hamiltonian:

$$H = \frac{1}{2}ma^2\dot{\theta}^2 - \frac{1}{2}ma^2\sin^2\theta\omega^2 - mga\cos\theta$$

Notice that this is not the same as total energy. Although H is conserved, the total energy is not as an external torque must be applied to keep the hoop spinning.