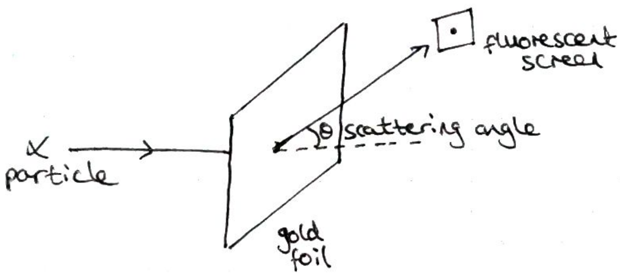
PHYS3002 Nuclei and Particles

Rutherford Scattering

In 1911, Rutherford discovered the nucleus by looking at the scattering of α particles against a thin foil of gold.



In this experiment, we make the assumptions:

- atom contains a nucleus of charge Ze, where Z is atomic number of element
- Nucleus can be treated as point particle
- Nucleus has mass >> than mass of α particle, so nucleus recoil is neglected
- Collision is elastic
- Only classical mechanics and EM apply.

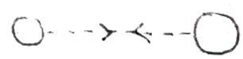
Using these assumptions, it is possible to explain the scattering phenomena observed in this experiment.
Now let's dive into the maths:

We define the distance of closest approach D as the closest distance the α particle can approach directly the nucleus before being repelled by coulomb force. So we can equate the initial kinetic energy of the α particle to the coulomb force felt at a distance D from the nucleus:

$$T = \frac{ze \times Ze}{4\pi \varepsilon_0 D}$$

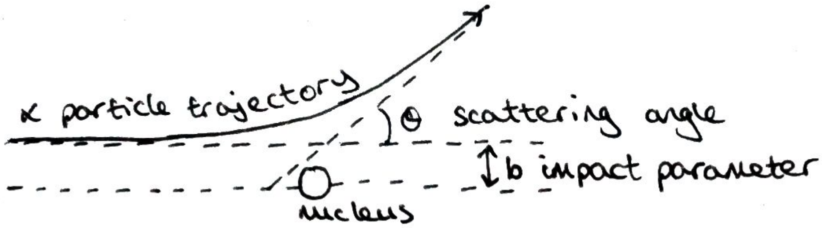
where ze is charge of α particle and Ze is charge of nucleus

$$\therefore \quad D = \frac{zZe^2}{4\pi\varepsilon_0 T}$$



At this point, the α particle would reverse direction.
The scattering angle here is π

But what if the α particle does not approach the nucleus directly? We define an impact parameter b:



The larger the impact parameter b, the smaller the scattering angle. We can relate these with:

$$\tan\left(\frac{\theta}{2}\right) = \frac{D}{2b}$$

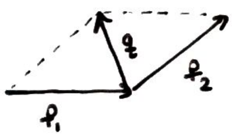
Which has been derived with Newton's 2nd Law, Coulomb's Law and conservation of angular momentum.

Note, when $b=0$, $\theta = \pi$. We derive this below:

If the initial momentum of the α particle is $\underline{p}_1$ and the final momentum (after scattering) is p_2, we can say:

$p = |p_1| = |p_2|$ since we assume elastic scattering and no recoil for nucleus.

so:



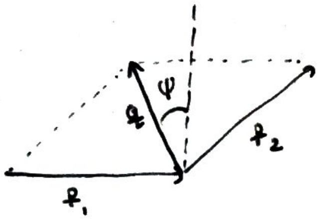
where q is the change in momentum.

Using sin rule: $$\frac{q}{p} = \frac{\sin\theta}{\sin\left[\frac{1}{2}(\pi-\theta)\right]} = 2\sin\left(\frac{\theta}{2}\right)$$

The rate of change of momentum in the direction $\hat{q}$ is the component of force on α particle due to charge of nucleus:

$$F = \frac{zZe^2}{4\pi\varepsilon_0 r^2} \quad \text{but} \quad T = \frac{zZe^2}{4\pi\varepsilon_0 D} \quad \text{so:}$$

$$F = \frac{TD}{r^2}$$



The component of this force in the $\hat{q}$ direction is:
$$f_q(t) = \frac{TD}{r^2}\cos[\psi(t)]$$

so using $\frac{dq}{dt} = F_q(t)$, we find:

$$q = \int \frac{zZe^2}{4\pi\varepsilon_0 r^2}\cos\psi \, dt$$

but $dt = \frac{d\psi}{\frac{d\psi}{dt}} = \frac{d\psi}{\dot{\psi}}$ where $\dot{\psi}$ is obtained from conservation of angular momentum: $L = M_\alpha r^2 \dot{\psi}$

The initial angular momentum is $L = bp$ so $bp = M_\alpha r^2 \dot{\psi}$
$$\Rightarrow \dot{\psi} = \frac{bp}{M_\alpha r^2}$$

$$\therefore \quad q = \int \frac{TD M_\alpha r^2}{r^2 bp}\cos\psi \, d\psi = \int \frac{Dp}{2b}\cos\psi \, d\psi$$

where we have used $T = \frac{p^2}{2M_\alpha}$

Use the limits:
$$\psi = \pm\frac{1}{2}(\pi - \theta)$$

computing the integral:
$$q = \frac{Dp}{2b} 2\sin\left(\tfrac{1}{2}(\pi-\theta)\right)$$

$$\Rightarrow \frac{q}{p} = \frac{D}{2b} 2\sin\left(\tfrac{1}{2}(\pi-\theta)\right)$$ equate to $\frac{q}{p}$ we had earlier

$$2\sin\left(\frac{\theta}{2}\right) = \frac{D}{2b} 2\sin\left(\tfrac{1}{2}(\pi-\theta)\right)$$

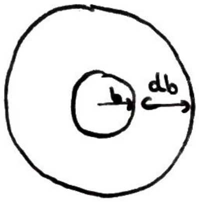
$$\Rightarrow \tan\left(\frac{\theta}{2}\right) = \frac{D}{2b}$$

We have thus derived the relationship between scattering angle θ and impact parameter.

Flux and Cross-Section

The flux F is defined as: the number of incident particles per unit area per unit second arriving at the target.

Consider the number of particles $dN(b)$ with impact parameter between b and $b+db$. This is the flux in this region multiplied by the area:



$$\therefore \quad dN(b) = F \times 2\pi b\, db$$

Let's differentiate $\tan\left(\frac{\theta}{2}\right) = \frac{D}{2b} \Rightarrow b = \frac{D}{2\tan\left(\frac{\theta}{2}\right)}$

$$\frac{db}{d\theta} = -\frac{D}{4\sin^2\left(\frac{\theta}{2}\right)} \Rightarrow db = -\frac{D}{4\sin^2(\theta/2)}\, d\theta$$

Sub into the original dN eqn:

$$dN(\theta) = F\pi \frac{D^2}{4} \frac{\cos(\theta/2)}{\sin^3(\theta/2)}\, d\theta$$

Note, the minus sign has been dropped because as b increases, θ decreases and $N(\theta)$ must be positive.

We have made substitutions for db and b here.

We define the "differential cross-section" $\frac{d\sigma}{d\theta}$ as the number of scatterings between θ and $\theta + d\theta$ per unit flux, per unit range of angle so:

$$\frac{d\sigma}{d\theta} = \frac{dN(\theta)}{F\, d\theta} = \pi \frac{D^2}{4} \frac{\cos(\theta/2)}{\sin^3(\theta/2)}$$

Usually differential cross section is given w.r.t. the solid angle Ω which is related to scattering angle θ and azimuthal angle φ by:

$$d\Omega = \sin\theta\, d\theta\, d\varphi = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) d\theta\, d\varphi$$

so $\frac{dN}{d\Omega} = F\frac{d\sigma}{d\Omega}$ where we find $\frac{d\sigma}{d\Omega}$ by:

$$\frac{d\sigma}{d\theta} = 2\pi\frac{d^2\sigma}{d\theta d\varphi} \Rightarrow \frac{d^2\sigma}{d\theta d\varphi} = \frac{1}{2\pi}\frac{d\sigma}{d\theta} = \frac{1}{2\pi}\pi\frac{D^2}{4}\frac{\cos(\theta/2)}{\sin^3(\theta/2)}$$

$$\Rightarrow \frac{d^2\sigma}{d\theta d\varphi} = \frac{D^2}{8}\frac{\cos(\theta/2)}{\sin^3(\theta/2)}$$

Now use relation between $d\Omega$ and $d\theta d\varphi$ to find:

$$\frac{d\sigma}{d\Omega} = \frac{D^2}{8}\frac{\cos(\theta/2)}{\sin^3(\theta/2)}\frac{1}{2\sin(\theta/2)\cos(\theta/2)}$$

$$\Rightarrow \underline{\underline{\frac{d\sigma}{d\Omega} = \frac{D^2}{16\sin^4(\theta/2)}}}$$

Note, for convention, we write $\frac{d\sigma}{d\Omega}$ even though really it is $\frac{d^2\sigma}{d\Omega}$

A Note on Units

Differential cross-sections have unit area and are usually quoted in barns: $1 \text{ barn} = 10^{-28} m^2$

Length in nuclear physics is usually quoted in fermi:
$1 \text{ fermi} = 10^{-15} m$

$1 \text{ fermi}^2 = 10 \text{ millibarn}$

$\hbar c = 197.3 \text{ MeV fm}$ so $\frac{e^2}{4\pi\varepsilon_0} = \alpha\hbar c \approx \frac{1}{137} \times 197.3 \text{ MeV fm}$

The distance of closest approach $D = \frac{197.3}{137}\frac{zZ}{T} \text{ fm}$ where T is in MeV

Interpretation of Rutherford Experiment

We can see that the differential cross-section diverges as the scattering angle goes to zero. We also know small scattering angle implies large impact parameter.

However, in gold foil with many nuclei, the distance of the incident α particle from any nucleus can only be at max half the distance between 2 nuclei in the foil. We assume that all nuclei lie in the same plane.

If we assume the mass of a nucleus with atomic number A is Am_p where m_p is number of protons, the total number of nuclei per unit area of foil is:

$$\rho \delta \frac{1}{Am_p}$$

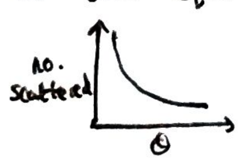
where ρ is density of foil and δ is thickness.

So the fraction of α particle scattered into a small interval of solid angle $d\Omega$ is:

$$\frac{\delta n}{n} = \rho \delta \frac{1}{Am_p} \frac{d\sigma}{d\Omega} d\Omega \qquad (*)$$

where we define the solid angle such that an area element dA at a distance r from the scattering centre subtends a solid angle:

$$d\Omega = \frac{dA}{r^2}$$

So if the detector (with area dA) is placed a distance r from the foil and at an angle θ to the incident direction of α particles, then the fraction of α particles in detector is found by replacing $d\Omega$ by $\frac{dA}{r^2}$ in the eqn $(*)$. This is what we observe in experiments:  which matches the theory!