Maxwell's Equations

We have thus found all four of Maxwell's Equations:

$$\boxed{\int_{cs} \underline{E} \cdot d\underline{A} = \frac{q_{enc}}{\varepsilon_0}}$$ Gauss' Law

$$\boxed{\int_{cs} \underline{B} \cdot d\underline{A} = 0}$$ No magnetic monopoles

$$\boxed{\int_{\text{closed loop}} \underline{E} \cdot d\underline{L} = -\frac{d\Phi_B}{dt}}$$ Faraday's Law

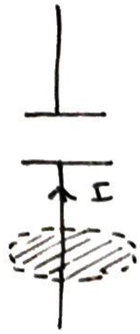
$$\boxed{\int_{\text{closed loop}} \underline{B} \cdot d\underline{L} = \mu_0 I}$$ Ampére's Law

But Maxwell was unhappy with these as there was a lack of symmetry between the equations. There is nothing to be done about Magnetic Gauss' Law since magnetic monopoles do not exist. But, just as Faraday's Law has rate of change of magnetic flux, Maxwell thought Ampére's Law should have a term for rate of change of electric flux:

$$\int \underline{B} \cdot d\underline{L} = \mu_0 I + \text{"}\frac{d\Phi_E}{dt}\text{"} \; ?$$ Is this true?

Maxwell made the argument that Ampére's Law wasn't perfect for time-dependent problems and needed some modification.

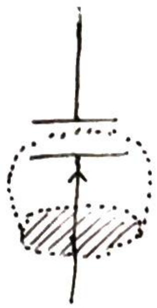
For example, in the problem of charging a capacitor:



Ampère's law says that $\int \underline{B} \cdot d\underline{L} = \mu_0 I_{enc}$ but what exactly is $I_{enclosed}$? We could say that $I_{enc} = \int_{\text{closed surface}} \underline{J} \cdot d\underline{A}$ where $\underline{J}$ is the current density.

This is the current per unit area in the wire and the integral is done over any surface bounded by the loop.

This means we can "blow out" the loop into a 3d surface and the current enclosed will remain the same inside the surface bounded by the loop. But what if we keep going?



Oh no! Now the current enclosed has become 0 since no current is actually flowing in between the plates. But we know that a magnetic field is still being generated through the ampèrian loop.

So something must exist between the plates that equals I. We can use Gauss' law to find out what.

Applying Gauss' Law between the plates: $\int \underline{E} \cdot d\underline{A} = \dfrac{q}{\varepsilon_0}$

Taking the time derivative of both sides:

$$\frac{d}{dt} \int \underline{E} \cdot d\underline{A} = \frac{d}{dt} \frac{q}{\varepsilon_0} \Rightarrow \frac{d}{dt} \int \underline{E} \cdot d\underline{A} = \frac{I}{\varepsilon_0}$$

but $\int \underline{E} \cdot d\underline{A}$ is electric flux

$$\therefore \quad \frac{d\Phi_E}{dt} = \frac{I}{\varepsilon_0}$$

This gives the "displacement current" between the plates.

So the corrected Maxwell equation is:

$$\boxed{\int \underline{B} \cdot d\underline{L} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}}$$

Corrected Ampère's Law

Maxwell's Equations in a vacuum

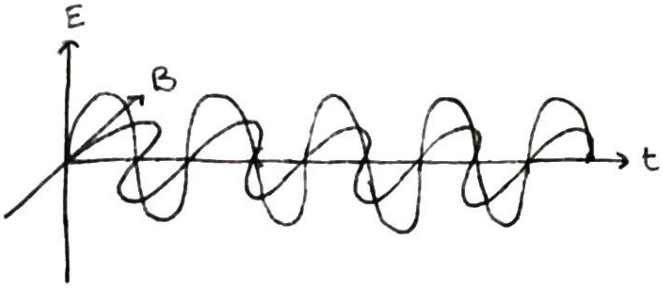
In the absence of charges, Maxwell's equations reduce to

$$\boxed{\begin{array}{ll} \int \underline{E} \cdot d\underline{A} = 0 & \int \underline{B} \cdot d\underline{A} = 0 \\ \int \underline{E} \cdot d\underline{L} = -\dfrac{d\Phi_B}{dt} & \int \underline{B} \cdot d\underline{L} = \mu_0 \varepsilon_0 \dfrac{d\Phi_E}{dt} \end{array}}$$

Surprisingly these can have non zero solutions, making it possible to have electric and magnetic fields without any charges.

This makes it clear that electric and magnetic fields really exist and aren't just a representation of the forces on a charge.

So a changing $\underline{E}$ field can generate a $\underline{B}$ field and vice versa:



It turns out these waves move at the speed:

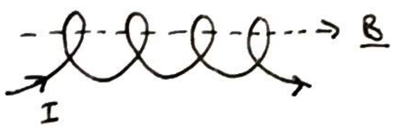
$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \, ms^{-1}$$

i.e, the speed of light!

There will be more on this in the second year.

Inductors

A current carrying coil generates a $\underline{B}$ field which causes a magnetic flux in the coil itself. This is called self-inductance as changes in the flux will induce a current.



$$\boxed{\Phi_B = LI}$$

where L is the coefficient of self inductance, measured in Henrys

Since a change in flux will induce an emf, if I is reduced, an emf is induced that supports I and if I is increased, an emf is induced that opposes I.

for example: The self inductance of a solenoid

n turns per unit length.

In a previous example, we showed

$$|\underline{B}| = \mu_0 n I$$

But since $\Phi_B = \{\int \underline{B}.dA\} \times$ number of turns:

$$\Phi_B = \mu_0 n I \times \underbrace{\pi r^2}_{\text{cross sectional area}} \times \underbrace{nl}_{\text{number of turns}}$$

$$\Phi_B = \mu_0 \pi n^2 r^2 I l$$

$$\therefore \underbrace{L}_{\text{coefficient of inductance}} = \mu_0 \pi n^2 r^2 l$$

Energy stored in a magnetic field

To find out how much energy is needed to create a $\underline{B}$ field, we need to find out how much power was expended as the current in a circuit goes from 0 to I. This is the power that makes $\underline{B}$.

Power $= VI$ but emf $(V) = \dfrac{d\Phi_B}{dt}$ and $\Phi_B = LI$

$\therefore$ Power $= \dfrac{d}{dt}(LI)\, I$

(we don't need to worry about Lenz's law as the direction doesn't matter here).

$= L\dfrac{dI}{dt}\, I$

$$\text{Work Done} = \int_0^t \text{Power}\, dt$$

$$= \int L\frac{dI}{dt} I\, dt = \int_0^I LI\, dI$$

$$= \frac{1}{2}LI^2$$

so for a solenoid, sub in L:

$$\text{Energy stored} = \frac{1}{2}\mu_0 \pi r^2 l n^2 I^2$$

$$= \frac{1}{2}\mu_0 n^2 I^2 \times \underbrace{\pi r^2 l}_{\text{volume}}$$

$$\text{Energy stored per unit volume} = \frac{1}{2}\mu_0 n^2 I^2$$

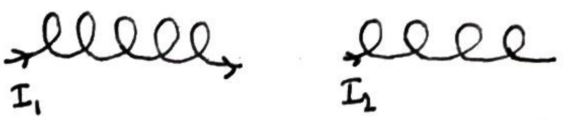
but $\mu_0 I . n$ is $|\underline{B}|$ from a previous example

$$\therefore \boxed{\text{Energy stored per unit volume} = \frac{1}{2}\frac{|\underline{B}|^2}{\mu_0}}$$

This is the general expression for the energy stored in a magnetic field

Mutual Inductance

Two neigbouring solenoids will also cause a magnetic flux in each other.



$$\Phi_2 = M_{12} I_1$$

$$\Phi_1 = M_{21} I_2$$

coefficient of mutual inductance

This is how transformers work since the voltage in the second coil differs from that inducing it in the first by the ratio of the number of turns.