Lorentz Transformations

We have seen the Lorentz transformations in Special Relativity in previous modules, so for simplicity I will just state them.

For an inertial frame S' moving with speed v in the x direction:

$$\boxed{\begin{aligned} t' &= \gamma\left(t - \frac{v}{c^2}x\right) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}}$$

where $\boxed{\gamma = \sqrt{\frac{1}{1 - v^2/c^2}}}$

This leads us to two observed phenomena: Time dilation and Length contraction

Time Dilation

Let's imagine an observer at the origin in the frame S. If she flashes a light at $t=0$ and then again at $t=1$, the events have coordinates $(x=0, t=0)$, $(x=0, t=1)$

An observer in the frame S' moving at velocity v in the x direction sees $(x'=0, t'=0)$, $(x' = -\gamma v t, t' = \gamma)$

so since $\gamma \geq 1$, $t' \geq t$ so the observer in the moving frame observes the length of time between the two events longer than 1 second.

"A moving clock runs slow"

Lorentz Contraction

Consider a ruler of length L at rest in the frame S.
An observer in S could make an instantaneous measurement of the positions of both ends of the ruler to deduce the size. i.e $(t=0,\ x=0),\ (t=0,\ x=L)$

An observer in a frame S' would not see this measurement as instantaneous however. To an observer in S', the ruler is moving so the measurements are made at different times, shown by the transformation: $(t'=0,\ x'=0)$ $(t'=-\gamma \frac{v}{c^2}L,\ x'=\gamma L)$

But the observer in this frame S' cannot easily deduce the length from this since the measurements were made at different times. So, the second measurment must be made at $t'=0$

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) = 0 \quad \Rightarrow \quad \gamma t - \gamma\frac{v}{c^2}x = 0,$$

$$\Rightarrow \quad t = \frac{v}{c^2}L$$

So if the observer in S' makes the measurement when $t=\frac{v}{c^2}L$ in S, then she would be making the measurment when $t'=0$.

So what is x' when $t=\frac{v}{c^2}L$?:

$$x' = \gamma(x - vt) = \gamma\left(L - \frac{v^2}{c^2}L\right) = \frac{L}{\gamma}$$

So $(t'=0,\ x'=0)$ and $(t'=0,\ x'=\frac{L}{\gamma})$

So the length measured is $\frac{L}{\gamma}$ in S' which is $\leq L$!

"Moving objects contract in the direction of motion"