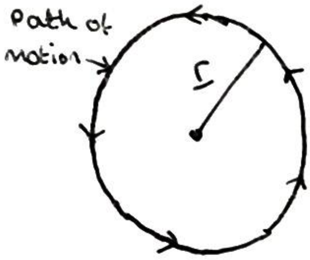
Motion under Forces

If given a specified force $\underline{F}$, it may be possible to find the motion under this force using Newton's 2nd Law. This is done by integrating twice:

$\underline{F} = m.\underline{a}$ integrate both sides twice with respect to time to get the vector of motion $\underline{r}$ from $\underline{a}$.

Circular Motion



if $|\underline{r}|$ remains constant through the object's motion, then the object is undergoing circular motion in two dimensions. To obtain the velocity, we start with:

$\underline{r} \cdot \underline{r} = |\underline{r}|^2$ (differentiating both sides with respect to t.:)

differentiates to 0 since it is a constant

$2\underline{r} \cdot \dot{\underline{r}} = 0$

$\underline{r} \cdot \dot{\underline{r}} = 0$

this is velocity $\underline{v}$

If $\underline{v} \cdot \underline{r} = 0$, then $\underline{v}$ must be orthogonal to $\underline{r}$. Therefore velocity is always perpendicular to the radius. To find the acceleration, we start with:

$\dot{\underline{r}} \cdot \dot{\underline{r}} = |\dot{\underline{r}}|^2$ (differentiating b.s. w.r.t. t:)

$2\dot{r} \cdot \ddot{r} = 0$

$\dot{r} \cdot \ddot{r} = 0$

this is acceleration

Therefore, acceleration is perpendicular to velocity $\underline{v}$ and parallel (anti-parallel) to the radius $\underline{r}$

We can actually derive the centripetal acceleration towards the centre from:

$$\underline{r} \cdot \dot{\underline{r}} = 0$$

velocity is perpendicular to radius. Differentiate both sides with respect to time:

$$\dot{\underline{r}} \cdot \dot{\underline{r}} + \underline{r} \cdot \ddot{\underline{r}} = 0$$

$$\underline{r} \cdot \ddot{\underline{r}} = -\dot{\underline{r}} \cdot \dot{\underline{r}} = -|\underline{v}|^2$$

but $\underline{r}$ and $\ddot{\underline{r}}$ are parallel so:

$$\underline{r} \cdot \ddot{\underline{r}} = |\underline{r}||\ddot{\underline{r}}| \cos 0$$

$$= |\underline{r}||\ddot{\underline{r}}|$$

but $|\ddot{\underline{r}}| = \dfrac{\ddot{\underline{r}}}{\hat{\ddot{\underline{r}}}}$ note that $\hat{\ddot{\underline{r}}}$ is just $\hat{\underline{r}}$ since they are in the same direction

$$\therefore \frac{\ddot{\underline{r}}}{\hat{\underline{r}}} |\underline{r}| = -|\underline{v}|^2$$

$$\therefore \ddot{\underline{r}} = \frac{-|\underline{v}|^2}{|\underline{r}|} \hat{\underline{r}}$$

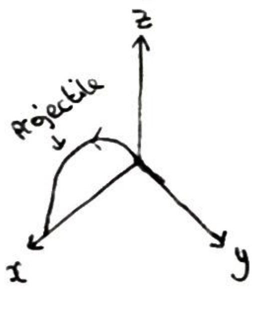
This is the centripetal acceleration towards the centre:

$$\boxed{\underline{a} = \frac{-|\underline{v}|^2}{|\underline{r}|} \hat{\underline{r}}}$$

Note that this has direction $-\hat{\underline{r}}$ which is towards the centre.

Projectile Motion

The force on a particle experiencing projectile motion is a constant given by $m|g|$ where $|g|$ is the strength of the gravitational vector.



If the projectile is moving in the x-z plane, then g is given by: $\underline{g} = -|g|\underline{\hat{k}}$. This means the projectile experiences a constant force of $\underline{F} = -m|g|\underline{\hat{k}}$. Using Newton's 2nd Law, we get:

$\underline{F} = m\underline{a} = -m|g|\underline{\hat{k}}$ for force in the vertical direction

$\therefore \underline{a} = -|g|\underline{\hat{k}}$ integrating with respect to time:

$\underline{v} = -|g|t\underline{\hat{k}} + \text{constant}$ [this constant is v_0 since it is the value when $t = 0$]

$\underline{v} = -|g|t\underline{\hat{k}} + v_0$

↳ integrating again:

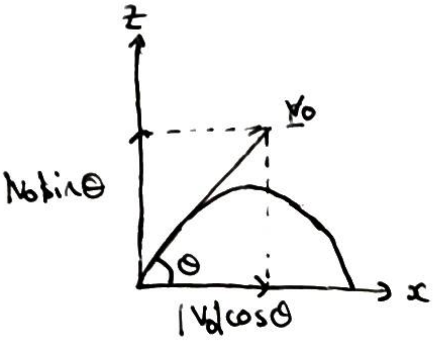
$$\underline{r} = \frac{-|g|t^2\underline{\hat{k}}}{2} + v_0 t + \text{constant}$$

$\underline{r} = -\frac{1}{2}|g|t^2\underline{\hat{k}} + v_0 t + r_0$ [this constant is r_0 since it is the value when $t = 0$]

$$\therefore \boxed{\underline{r}(t) = r_0 + v_0 t - \tfrac{1}{2}|g|t^2\underline{\hat{k}}}$$

This is an equation for the position $\underline{r}$ of the particle as a function of time.

The equation of motion can be obtained by looking at the motion in the x and z directions:



$$\underline{v_0} = \underline{v_0}\sin\theta + \underline{v_0}\cos\theta$$

We will take the initial position $\underline{r_0}$ to be at the origin. We can now decompose the equation for $r(t)$ in the x and z directions.

$$r(t) = \underline{r_0} + \underline{v_0}t - \frac{1}{2}|g|t^2\,\underline{\hat{k}}$$

$$\therefore \underline{r}_x = |\underline{v_0}|\cos\theta\, t \quad \Rightarrow \quad x = |\underline{v_0}|\cos\theta t \qquad (1)$$

$$\underline{r}_z = |\underline{v_0}|\sin\theta\, t - \frac{1}{2}|g|t^2\,\underline{\hat{k}} \Rightarrow z = |\underline{v_0}|\sin\theta\, t - \frac{1}{2}|g|t^2 \quad (2)$$

For simplicity, we can write these as (1) and (2)

$$(2) \div (1): \quad \frac{z}{x} = \frac{|v_0|\sin\theta\, t}{|v_0|\cos\theta\, t} - \frac{\frac{1}{2}|g|t^2}{|v_0|\cos\theta\, t}$$

$$\frac{z}{x} = \tan\theta - \frac{1}{2}|g|t\,\frac{1}{|v_0|\cos\theta}$$

$$z = x\tan\theta - \frac{1}{2}|g|t \cdot \frac{|v_0|\cos\theta\, t}{|v_0|\cos\theta} \times \frac{|v_0|\cos\theta\, t}{|v_0|\cos\theta\, t}$$

$$\boxed{z = x\tan\theta - \frac{1}{2}|g|\frac{x^2}{|v_0|^2\cos^2\theta}}$$

This is the equation of a parabola in the form:

$$z = ax^2 + bx + c$$

Time of flight

We can obtain the time of flight from the equation of the z component.

$$\underline{r}_z = |\underline{V}_0| \sin\theta \, t - \frac{1}{2}|g|t^2 \hat{\underline{k}}$$

The projectile starts flight and finishes flight at $\underline{r}_z = 0$:

$$|\underline{V}_0| \sin\theta \, t - \frac{1}{2}|g|t^2 \hat{\underline{k}} = 0$$

One solution is $t = 0$ (this is the start of flight)

Another solution is the time of flight :

$$\boxed{\text{Time of flight} = \frac{2|\underline{V}_0|\sin\theta}{|g|}}$$

Range of Projectile

This is the position of x at the end of the flight.

Substituting time of flight into the x equation:

$$\underline{r}_x = |\underline{V}_0| \cos\theta \, t \quad \text{where} \quad t = \frac{2|\underline{V}_0|\sin\theta}{|g|}$$

$$\text{range of flight} = \frac{2|\underline{V}_0|^2 \sin\theta \cos\theta}{|g|}$$

$$\therefore \boxed{\text{range of flight} = \frac{|\underline{V}_0|^2 \sin 2\theta}{|g|}}$$

Maximum Height

The maximum height is reached at half the time of flight. Therefore, it is the z component when $t = \frac{|V_0| \sin \theta}{|g|}$

Substituting this into the z equation gives:

$$\boxed{\text{maximum height} = \frac{1}{2} \frac{|V_0|^2 \sin^2 \theta}{|g|}}$$