Coupled Oscillators

These notes don't cover the basics of simple harmonic motion. If you need a recap, look at Freegarde's Section 5 notes.

Time Translational Invariance

Time translation invariance causes us to use complex exponential time dependence in our trial solutions. Spatial translation invariance gives complex exponential time dependence in our trial solutions as well.

Consider a simple damped harmonic oscillator with eqn of motion:

$$m\ddot{x} = \underbrace{-2m\gamma\dot{x}}_{\text{damping}} \underbrace{-m\omega_0^2 x}_{\text{restoring}}$$

we rearrange this to:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

To solve thise, use ansatz $x = Ae^{\Omega t}$

where A and Ω are in general complex. We can take real parts later when looking for physical solution.

The reason we can take this ansatz is time translational invariance!

if $x(t)$ is a solution then $x(t+c)$ is a solution for any constant c

The simplest possibility is $x(t+c) = f(c)\,x(t)$ for some proportionality constant $f(c)$

We can solve this. Turn the page.

$x(t+c) = f(c)\,x(t)$ differentiate w.r.t. c

$\dot{x}(t+c) = \dot{f}(c)\,x(t)$ set $c=0$

$\dot{x}(t) = \Omega\, x(t)$ where $\Omega = \dot{f}(0)$

This has known solution $x(t) = Ae^{\Omega t}$
which is where our ansatz earlier comes from.

Ω must have a non-zero imaginary part if we want to get oscillatory solutions:
From now on, we will say $\Omega = i\omega$ so $x(t) = Ae^{i\omega t}$

we can't just choose any value for ω, it is determined by demanding $Ae^{i\omega t}$ solves eqn. of motion:

$$(-\omega^2 + 2i\gamma\omega + \omega_0^2)Ae^{i\omega t} = 0$$

For non-trivial solutions, $A \neq 0$ so we require

$$-\omega^2 + 2i\gamma\omega + \omega_0^2 = 0$$

solve this for values of ω

Normal Modes

We are trying to generalise from a single oscillator to a set of coupled oscillators.
For n oscillators with individual positions $x_i(t)$, we denote "position" of whole system with vector $\underline{x}(t)$:

$$\underline{x}(t) = (x_1(t), x_2(t), x_3(t) \dots x_n(t))$$

The differential eqns satisfied by x_i will involve time dependence only as time derivatives so we will be able to use the time translational invariance described previously.

So all the oscillators have same complex exp. time dependence $e^{i\omega t}$

so :

$$\underline{x}(t) = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix} e^{i\omega t} \qquad A_i \text{ are constants}$$

which desribes situation where all oscilators have same frequency but in general different amplitudes and phases.

The osillator displacements are in fixed ratios determined by A_i.

This kind of motion is called a normal mode.

The overall normalisation is arbitrary, i.e multiplying all A_i by the same constant still gives the same normal mode.

Our job is to determine which ω are allowed and then determine a set of A_i for allowed ω.

For n oscillators obeying 2nd order coupled equations, there are $2n$ independent solutions.

Coupled Oscillators

Consider a set of coupled oscilators described by coordinates $q_1 \dots q_n$. The potential $V(q)$ will be complicated. Consider small oscillations about a position of stable equilibrium, which, we can take to occur when $q_i = 0$

Expanding the potential in a taylor series about this point:

$$V(q) = V(0) + \sum_i \frac{\partial V}{\partial q_i}\bigg|_0 q_i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 V}{\partial q_i \partial q_j}\bigg|_0 q_i q_j + \dots$$

We add a constant to V so we can choose $V(0) = 0$.

Since we are at a position of equilibrium, all first derivative terms vanish.

We can also drop all higher derivatives, since they will be very small.

We define $K_{ij} = \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_0$ so:

$$V(q) = \sum_{ij} K_{ij}\, q_i\, q_j$$

$$F_i = -\frac{\partial V}{\partial q_i} = -\sum_j K_{ij}\, q_j$$

Thus the equations of motion are:

$$\underline{M_i \ddot{q}_i = -\sum_j^N K_{ij}\, q_j}$$

For $i = 1 \dots n$

Here M_i are the masses of the oscillators and K is a matrix of spring constants.

$$\underline{M} = \begin{bmatrix} M_1 & 0 & \dots & 0 \\ 0 & M_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & M_n \end{bmatrix} \qquad \underline{K} = \begin{bmatrix} K_{11} & K_{12} & \dots & K_{1n} \\ K_{21} & K_{22} & \dots & K_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ K_{n1} & K_{n2} & \dots & K_{nn} \end{bmatrix}$$

Likewise $\underline{q}$ and $\underline{\ddot{q}}$ are column vectors

$$\underline{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} \qquad \underline{\ddot{q}} = \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \ddot{q}_3 \end{bmatrix}$$

With this notation: $\underline{M}\,\underline{\ddot{q}} = -\underline{K}\,\underline{q}$ or:

$$\underline{\ddot{q}} = -\underline{M}^{-1}\underline{K}\,\underline{q}$$

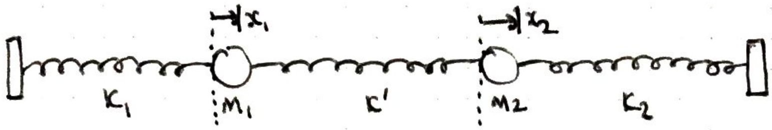
Now we look for a normal solution $\underline{q} = \underline{A} e^{i\omega t}$ where $\underline{A}$ is a column vector

we have: $\underline{\ddot{q}} = -\omega^2 \underline{q}$, so cancelling $e^{i\omega t}$ factors gives:

$$\boxed{\underline{M}^{-1}\underline{K}\,\underline{A} = \omega^2 \underline{A}}$$

This is the eigenvalue equation. ω^2 values are eigenvalues of $\underline{M}^{-1}\underline{K}$ and $\underline{A}$ is eigenvectors.

Example: Masses and Springs



Consider the system shown above. The two masses are joined by a spring with spring constant k', and joined to the walls by springs of spring constants k_1 and k_2

The equilibrium position has each spring unstretched.
Use displacements x_1 and x_2 of the masses from their equilibrium position as coordinates.

Force on mass m_1:

$$F_1 = -k_1 x_1 - k'(x_1 - x_2)$$

Force on mass m_2:

$$F_2 = -k_2 x_2 - k'(x_2 - x_1)$$

combining these and putting into matrix form:

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_1 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = - \begin{pmatrix} k_1 + k' & -k' \\ -k' & k_2 + k' \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

which is in the form $\underline{M}\,\ddot{\underline{q}} = -\underline{K}\,\underline{q}$ as we saw on previous page

The eigenvalue equation is

$$\underline{M}^{-1}\underline{K}\,\underline{A} = \omega^2 \underline{A}$$

so our eigenvalue equation is

$$\begin{pmatrix} \frac{k_1 + k'}{m_1} & -\frac{k'}{m_1} \\ \frac{-k'}{m_2} & \frac{k_2 + k'}{m_2} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \omega^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

we can consider the special case: $M_1 = M$ $M_2 = 2M$ $K' = 2K$
$K_1 = K$ $K_2 = 2K$

The eigenvalue equation becomes:

$$\begin{pmatrix} 3K/M & -2K/M \\ -K/M & 2K/M \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \omega^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \frac{M}{K}\omega^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$ letting $\lambda = \frac{M\omega^2}{K}$

$$\begin{pmatrix} 3-\lambda & -2-\lambda \\ -1-\lambda & 2-\lambda \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For there to be a solution:

$$\det \begin{vmatrix} 3-\lambda & -2-\lambda \\ -1-\lambda & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 5\lambda + 4 = 0 \quad \Rightarrow \quad \lambda = 4,\ \lambda = 1$$

These are the eigenvalues

so the eigenfrequencies are:

$\lambda = 4$: $\frac{M\omega^2}{K} = 4$ $\omega = 2\sqrt{\frac{K}{M}}$

$\lambda = 1$: $\frac{M\omega^2}{K} = 1$ $\omega = \sqrt{\frac{K}{M}}$

Eigenvalues $\underline{A}$ are:

$\lambda = 4$: you find $A_2 = -\frac{1}{2}A_1$ so $\underline{A} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\lambda = 1$: you find $A_2 = A_1$ so $\underline{A} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

so we have found our normal modes

In the second one ($\lambda = 1$), the two masses swing in phase and have same amplitude, and the middle spring remains unstretched.

In the first one ($\lambda = 4$), the two masses move out of phase with each other (- amplitude) and one has twice the amplitude of the other.

Weak Coupling and Beats

Now conside the case $M_1 = M_2 = M$ and $K_1 = K_2 = K$

Note $K \neq K'$ necessarily in this case.

From the symmetry of the setup we expect one mode where the masses swing in phase with same amplitude and central spring is unstretched.

We also expect a second mode where the springs are out of phase but oscillations have same amplitude.

This one has higher frequency.

If the connecting spring has spring constant $K' = \varepsilon K$ then:

$$\omega_1 = \sqrt{\frac{K}{M}} \qquad \underline{A}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$\omega_2 = \sqrt{(1+2\varepsilon)\frac{K}{M}} \qquad \underline{A}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

By doing the method on previous page.

When middle spring is very weak $\varepsilon \ll 1$, the two modes have almost same frequency. In this case, we observe beats when a motion contains components from both normal modes.

Suppose we hold M_1 at equilibrium position and move M_2 a distance d and release, the general solution for motion is:

$$\underline{x}(t) = C_1 \underline{A}_1 \cos(\omega_1 t) + C_2 \underline{A}_2 \cos(\omega_2 t) + C_3 \underline{A}_1 \sin(\omega_1 t) + C_4 \underline{A}_2 \sin(\omega_2 t)$$

Since we start from rest, we see $C_3 = C_4 = 0$

Applying initial condition $\underline{x}(0) = \begin{pmatrix}d\\0\end{pmatrix}$:

$$\begin{pmatrix} d \\ 0 \end{pmatrix} = \frac{c_1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{c_2}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

which is solved by $c_1 = c_2 = d/\sqrt{2}$ so the motion is:

$$x_1(t) = \frac{d}{2}\left(\cos(\omega_1 t) + \cos(\omega_2 t)\right)$$

$$x_2(t) = \frac{d}{2}\left(\cos(\omega_1 t) - \cos(\omega_2 t)\right)$$

which we rewrite as:

$$x_1(t) = d\cos\left(\frac{\omega_2 - \omega_1}{2} t\right) \cos\left(\frac{\omega_1 + \omega_2}{2} t\right)$$

$$x_2(t) = d\sin\left(\frac{\omega_2 - \omega_1}{2} t\right) \sin\left(\frac{\omega_1 + \omega_2}{2} t\right)$$

x_1 and x_2 each have a fast oscillation at the average frequency $\frac{\omega_1 + \omega_2}{2}$, modulated by a slow amplitude variance at the difference frequency $\frac{\omega_2 - \omega_1}{2}$

These produce the beats we learned all about last semester.