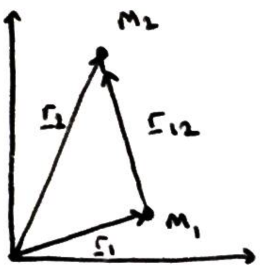
Newton's Law of Gravitation

For two particles with masses m_1 and m_2 separated by distance $\underline{r}$, there is a mutual force of attraction between the two particles with magnitude:

$$\boxed{|\underline{F}| = \frac{G m_1 m_2}{|\underline{r}|^2}}$$

where G is the gravitational constant
$G = 6.67 \times 10^{-11} \, m^3 \, kg^{-1} \, s^{-2}$

example



The force of m_1 on m_2, $\underline{F}_{12}$

$$\underline{F}_{12} = \frac{G m_1 m_2}{|\underline{r}_{12}|^2} \hat{\underline{r}}_{12}$$

where $\hat{\underline{r}}_{12} = \dfrac{\underline{r}_{12}}{|\underline{r}_{12}|}$

Mass

Mass, in physics, commonly refers to two properties of matter: inertial mass and gravitational mass. These have been shown experimentally to be equal to each other.

Inertial mass determines how a body accelerates as a result of a force being directly applied to it.

Gravitational mass determines how much gravitational force a body generates and how it is affected by the gravity of other masses.

Superposition of Gravitational Forces

Gravitational forces from multiple bodies add vectorially. If a particle labelled 1 of mass m_1 at $\underline{r}_1$ is attracted by particles 2, 3 ... n with masses $m_2, m_3 \ldots m_n$ and positions $\underline{r}_2, \underline{r}_3 \ldots \underline{r}_n$, then the total force on 1 is:

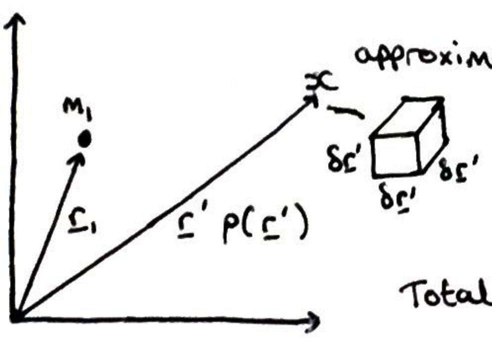
$$\underline{F} = \underline{F}_{12} + \underline{F}_{13} + \ldots + \underline{F}_{1n}$$

We can write this as:

$$\underline{F} = \sum_{2}^{n} \underline{F}_{1i}$$

$$\boxed{\underline{F} = \sum_{2}^{n} \frac{G m_1 m_i}{|\underline{r}_{1i}|^2} \hat{\underline{r}}_{1i}}$$

For a continuous distribution of other masses, this becomes an integral. If the masses have mass distribution $\rho(\underline{r})$ and lie along a vector $\underline{r}'$, then:



approximate small cubes on distribution:

with volume $(\delta \underline{r}')^3$

so mass of cube is: $\delta m = \rho(\underline{r}')(\delta r')^3$

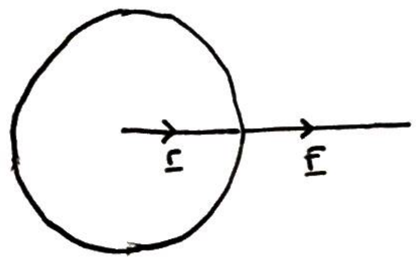
Total force can be written as the sum of the forces due to tiny masses δm

$$\underline{F} = \sum \delta F = \sum_{\text{masses}} \frac{G m_1 \delta m}{|\underline{r}' - \underline{r}_1|^2} \left(\frac{\underline{r}' - \underline{r}_1}{|\underline{r}' - \underline{r}_1|} \right)$$

$$\boxed{\underline{F} = \int_{0}^{x} \frac{G m_1 \rho(\underline{r}')\, d^3 \underline{r}'}{|\underline{r}' - \underline{r}_1|^3} (\underline{r}' - \underline{r}_1)}$$

Central Forces

If the force lies along the radius vector (like with gravity) then angular momentum is conserved.
This is because:



here: $\underline{r} \times \underline{F} = |\underline{r}||\underline{F}|\sin\theta$ but $\theta = 0$

$\therefore \underline{r} \times \underline{F} = 0$

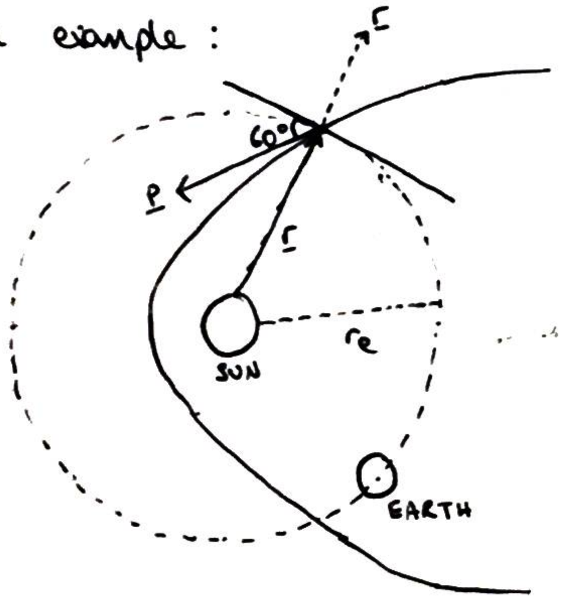
$\tau = \underline{r} \times \underline{F}$ so $\tau = 0$ but τ can be defined as rate of change of angular momentum.

$\therefore \underline{\underline{\frac{dL}{dt} = 0}}$ Angular Momentum is conserved.

So: $\boxed{\text{Angular Momentum of an isolated gravitationally interacting system is conserved.}}$

This works for irregular shaped orbits as well.

An example:



The comet orbiting the sun cuts the path of the earth's orbit at 60°, travelling at 50 kms^{-1}. The closest it gets to the sun is a fraction of 0.312 times the earth's orbital radius. Calculate the speed of the comet at this point.

Ignore the gravitational effect of gravity.

Solution

We will do this by considering angular momentum. Since angular momentum is conserved, the angular momentum will be the same when the comet cuts the earth's orbit and when the comet is at its closest point to the sun.

When the comet cuts the orbit:

$\underline{L} = \underline{r} \times \underline{p}$

$|\underline{L}| = |\underline{r}||\underline{p}| \sin\theta$

$= \frac{1}{2}|\underline{r}||\underline{p}|$

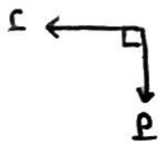
Here: $|\underline{L}| = \frac{1}{2}|r_e| m |V_{initial}|$

When the comet is closest to the sun:

$\underline{L} = \underline{r} \times \underline{p}$

$|\underline{L}| = |\underline{r}||\underline{p}| \sin\theta$

$= |\underline{r}||\underline{p}|$



Here: $|\underline{L}| = 0.312|\underline{r}_e| m |V_{max}|$

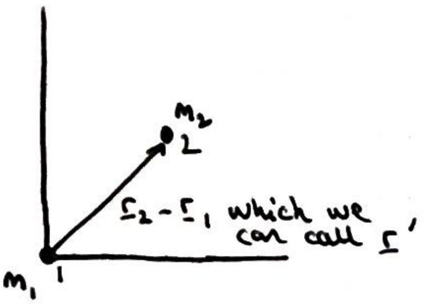
Since Angular Momentum is conserved:

$$\frac{1}{2}|r_e| m |V_{initial}| = 0.312|\underline{r}_e| m |V_{max}|$$

$$\frac{1}{2}|V_{initial}| = 0.312|V_{max}|$$

$$\underline{|V_{max}| = 80.1\ \mathrm{km s^{-1}}}$$

Gravitational Potential Energy



The force of 1 on 2 is:

$$\underline{F}_{12} = \frac{GM_1M_2}{|\underline{r}'|^2}\,\hat{\underline{r}}'$$

$$\therefore \underline{F}_{21} = -\frac{GM_1M_2}{|\underline{r}'|^2}\,\hat{\underline{r}}'$$

To find gravitational potential energy, we consider its definition: work done against the gravitational force:

$$U(\underline{r}) = -\int_{\underline{r}_0}^{\underline{r}} \underline{F}(\underline{r}')\cdot d\underline{r}' = -\int -\frac{GM_1M_2}{|\underline{r}'|^2}\,\hat{\underline{r}}'\cdot d\underline{r}'$$

We can consider the work done against the force from infinity so $\underline{r}_0$ is ∞. $\hat{\underline{r}}'$ and $d\underline{r}'$ are parallel so

$$\hat{\underline{r}}'\cdot d\underline{r}' = d|\hat{\underline{r}}|\,.\,|\underline{r}'|$$ but unit vectors have magnitude 1 so:

$$= d|\underline{r}'|$$

$$\therefore U(\underline{r}) = \int_{\infty}^{\underline{r}} \frac{GM_1M_2}{|\underline{r}'|^2}\,d|\underline{r}'|$$

$$= -\left[\frac{GM_1M_2}{|\underline{r}'|}\right]_{\infty}^{\underline{r}}$$

$$= -\frac{GM_1M_2}{|\underline{r}|} + \underbrace{\frac{GM_1M_2}{\infty}}_{0}$$

$$\boxed{U(\underline{r}) = -\frac{GM_1M_2}{|\underline{r}|}}$$

This is the gravitational potential energy of a particle.

Note that this is -ve since we define ∞ to have 0 gpe and everywhere closer will have lower gpe so it has to be negative.

Escape Speed

The minimum projection speed required to move a particle to infinite separation (never returning to starting body) is called the escape speed. If we ignore other effects and forces like atmosphere, we can calculate the escape speed for earth using energy conservation:

If the particle only just escapes to infinity, the kinetic energy at the point of escape is 0. At infinity, we define gravitational potential energy to be 0. Total Energy is therefore also 0.

$$\frac{1}{2}m|\underline{v}|^2 - \frac{GmM}{|\underline{r}|} = 0$$

$$\underbrace{|\underline{v}|}_{\text{escape speed}} = \sqrt{\frac{2GM}{\underbrace{|\underline{r}|}_{\text{radius of earth}}}}$$

This works out to be $\approx$ 11.2 kms^{-1}

Equating Centripetal and Gravitational Forces

When a particle orbits a body, its centripetal acceleration is induced by the gravitational force. We can therefore equate the forces as:

$$\boxed{\frac{m|\underline{v}|^2}{|\underline{r}|} = \frac{GMm}{|\underline{r}|^2}}$$