Rotating Coordinate Systems

Time derivatives in a Rotating Frame

Let's recall, for a vector $\underline{A}$ of fixed length rotating about origin with constant angular velocity $\underline{\omega}$, rate of change of $\underline{A}$ is:

$$\boxed{\frac{d\underline{A}}{dt} = \underline{\omega} \times \underline{A}}$$

let $\hat{\underline{i}}, \hat{\underline{j}}, \hat{\underline{k}}$ be unit vectors of inertial reference frame O.

let $\hat{\underline{i}}', \hat{\underline{j}}', \hat{\underline{k}}'$ be unit vectors of rotating reference frame O'

As before: $\frac{d\hat{\underline{i}}'}{dt} = \underline{\omega} \times \hat{\underline{i}}$ $\quad \frac{d\hat{\underline{j}}'}{dt} = \underline{\omega} \times \hat{\underline{j}}'$ $\quad \frac{d\hat{\underline{k}}'}{dt} = \underline{\omega} \times \hat{\underline{k}}'$

An arbitrary vector $\underline{a}$ in frames O and O' is:

$$\underline{a} = a_i \hat{\underline{i}} + a_j \hat{\underline{j}} + a_k \hat{\underline{k}} = a_{i'} \hat{\underline{i}}' + a_{j'} \hat{\underline{j}}' + a_{k'} \hat{\underline{k}}'$$

$$\frac{d\underline{a}}{dt} = \frac{da_i}{dt}\hat{\underline{i}} + \frac{da_j}{dt}\hat{\underline{j}} + \frac{da_k}{dt}\hat{\underline{k}}$$

$$= \frac{da_i'}{dt}\hat{\underline{i}}' + \frac{da_j'}{dt}\hat{\underline{j}}' + \frac{da_k'}{dt}\hat{\underline{k}}' + \underbrace{a_{i'}\frac{d\hat{\underline{i}}'}{dt} + a_{j'}\frac{d\hat{\underline{j}}'}{dt} + a_{k'}\frac{d\hat{\underline{k}}'}{dt}}_{= \underline{\omega} \times \hat{\underline{i}}' + \underline{\omega} \times \hat{\underline{j}}' + \underline{\omega} \times \hat{\underline{k}}'}$$

as shown earlier

$$\therefore \frac{d\underline{a}}{dt} = \frac{da_i'}{dt}\hat{\underline{i}}' + \frac{da_j'}{dt}\hat{\underline{j}}' + \frac{da_k'}{dt}\hat{\underline{k}}' + \underline{\omega} \times \hat{\underline{i}}' + \underline{\omega} \times \hat{\underline{j}}' + \underline{\omega} \times \hat{\underline{k}}'$$

let us introduce the notation:

$$\dot{\underline{a}} = \frac{da_i'}{dt}\hat{\underline{i}}' + \frac{da_j'}{dt}\hat{\underline{j}}' + \frac{da_k'}{dt}\hat{\underline{k}}'$$

i.e the differentiation of only the components and not unit vectors, even if unit vectors are time dependent.

$$\therefore \boxed{\frac{d\underline{a}}{dt} = \dot{\underline{a}} + \underline{\omega} \times \underline{a}}$$

One term for rate of change w.r.t rotating axis and one term for rotating axis themselves.

Equation of Motion in a Rotating Frame

We can find the equation of motion for a particle in a rest frame rotating at constant angular velocity $\underline{\omega}$.

Let a be a position vector $\underline{r}$

$$\frac{d\underline{r}}{dt} = \dot{\underline{r}} + \underline{\omega} \times \underline{r}$$

$$\frac{d^2\underline{r}}{dt^2} = \frac{d}{dt}\left\{ \dot{\underline{r}} + \underline{\omega} \times \underline{r} \right\}$$

$$= \underbrace{\ddot{\underline{r}} + \underline{\omega} \times \dot{\underline{r}}}_{\frac{d}{dt}\dot{\underline{r}}} + \underbrace{\underline{\omega} \times \frac{d\underline{r}}{dt}}_{\frac{d}{dt}(\underline{\omega} \times \underline{r})} = \ddot{\underline{r}} + \underline{\omega} \times \dot{\underline{r}} + \underline{\omega} \times (\dot{\underline{r}} + \underline{\omega} \times \underline{r})$$

$$\frac{d^2\underline{r}}{dt^2} = \ddot{\underline{r}} + 2\underline{\omega} \times \dot{\underline{r}} + \underline{\omega} \times (\underline{\omega} \times \underline{r})$$

Newton's law of motion is $\underline{F}_{tot} = m\frac{d^2\underline{r}}{dt^2}$ where $\underline{F}_{tot}$ is the total force acting.

$$\therefore \quad \underline{F}_{tot} = \underbrace{m\ddot{\underline{r}}}_{\text{This is force felt by particle}} + 2\underline{\omega} \times \dot{\underline{r}} + \underline{\omega} \times (\underline{\omega} \times \underline{r})$$

$$\therefore \quad \boxed{m\ddot{\underline{r}} = \underline{F}_{tot} - \overbrace{2m\underline{\omega} \times \dot{\underline{r}}}^{\text{coriolis}} - \overbrace{m\underline{\omega} \times (\underline{\omega} \times \underline{r})}^{\text{centrifugal}}}$$

so force felt by particle = total force acting + some other quantity

This is called apparent or inertial or fictitious force, arising because we are measuring position w.r.t. axes that are themselves rotating (i.e accelerating).

Motion Near Earth's Surface

Let's consider the Earth. Take the weight of an object to be a vector down to centre of the Earth. Take an inertial reference frame O with origin at centre of Earth, and a reference frame O' with origin at Earth centre but rotating with Earth.

Total force on particle is weight + external forces: $\underline{F}_{tot} = \underline{F} + m\underline{g}$

Let $\underline{R}$ be vector from centre to a point on the surface, and $\underline{x}$ be displacement to a particle.

Position vector in O' is:
$$\underline{r} = \underline{R} + \underline{x}$$

Since $\underline{R}$ is fixed in O', $\dot{\underline{R}}' = 0$ $\ddot{\underline{R}}' = 0$ so eqn of motion is:

$$m\ddot{\underline{x}} = \overbrace{\underline{F} + m\underline{g}}^{\underline{F}_{tot}} - 2m\underline{\omega} \times \dot{\underline{x}} - m\underline{\omega} \times (\underline{\omega} \times [\underline{R} + \underline{x}])$$

Let all terms order $x/R \to 0$ since they are very small:

so $\underline{\omega} \times [\underline{R} + \underline{x}] \to \underline{\omega} \times \underline{R}$

and $\underline{g} = \dfrac{-GM}{|\underline{R}+\underline{x}|^3}(\underline{R}+\underline{x}) \longrightarrow -\dfrac{GM}{R^3}\underline{R} = -g\dfrac{\underline{R}}{R}$ since $g = -\dfrac{GM}{R^2}$

so $m\ddot{\underline{x}} = \underline{F} - mg\dfrac{\underline{R}}{R} - 2m\underline{\omega} \times \dot{\underline{x}} - m\underline{\omega} \times (\underline{\omega} \times \underline{R})$

$$= \underline{F} - m\left(g\frac{\underline{R}}{R} - \underline{\omega} \times (\underline{\omega} \times \underline{R})\right) - 2m\underline{\omega} \times \dot{\underline{x}}$$

$\therefore$ $\boxed{m\ddot{\underline{x}} = \underline{F} - m\underline{g}^* - 2m\underline{\omega} \times \dot{\underline{x}}}$

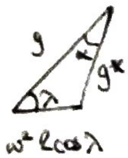
where we define $\underline{g}^*$ as apparent gravity and $-2m\underline{\omega} \times \dot{\underline{x}}$ as Coriolis force

$\boxed{\underline{g}^* = g\dfrac{\underline{R}}{R} - \underline{\omega} \times (\underline{\omega} \times \underline{R})}$ $\boxed{\underline{F}_{coriolis} = -2m\underline{\omega} \times \dot{\underline{x}}}$

Apparent Gravity

$$g^* = -g\frac{\underline{R}}{R} - \underbrace{\underline{\omega} \times (\underline{\omega} \times \underline{R})}_{\text{This is the centripugal term!}} = -g\overset{\overbrace{\hat{}}^{\text{points radially in}}}{\underline{R}} - \underline{\omega} \times (\underline{\omega} \times \underline{R})$$

$$|-\underline{\omega} \times (\underline{\omega} \times \underline{R})| = \omega^2 R \cos\lambda$$ where λ is the latitute



Applying cosine rule:

$$g^{*2} = g^2 + (\omega^2 R \cos\lambda)^2 - 2g\omega^2 R \cos^2\lambda$$

so $g^* = g +$ some quantity

Applying sine rule: $$\frac{\sin\alpha}{\omega^2 R\cos\lambda} = \frac{\sin\lambda}{g^*}$$

so $$\boxed{\alpha = \frac{\omega^2 R}{g} \sin\lambda \cos\lambda}$$ approximating $\sin\alpha \approx \alpha$

This is the deflection angle you would observe if you hung a mass from a spring. The earth turning causes mass to deflect very slightly.

Coriolis Force

$F_{coriolis} = -2m\underline{\omega} \times \dot{\underline{x}}$ is a fictitious force

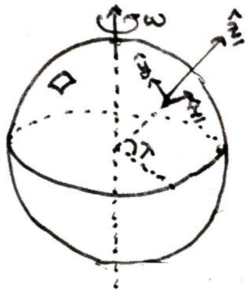
We can see it acts perpendicular to direction of motion and is dependent on speed.

Coriolis force is observed by an observor in a rotating reference frame, it is not a "real" force but a fictitious one.

If you consider a particle moving diametrically across a rotating disk, an observor in an inertial reference frame will see straight motion.

But an observer in a rotating reference frame (say that of the disk) will see curved motion and may incorrectly conclude there is an external force acting on the particle. This is the coriolis force.

To study it we choose axes:



$\hat{z}$ points radially outwards.
$\hat{x}$ points East
$\hat{y}$ points North

Equation of motion become:

$$m\ddot{x} = F_x - 2m\omega(\dot{z}\cos\lambda - \dot{y}\sin\lambda)$$
$$m\ddot{y} = F_y - 2m\omega\dot{x}\sin\lambda$$
$$m\ddot{z} = F_z - mg^* + 2m\omega\dot{x}\cos\lambda$$

Free Fall - Effects of the Coriolis Term

For a particle in free fall, the force $\underline{F}$ disappears from eqn of motion so $\ddot{\underline{x}} = \underline{g}^* - 2\underline{\omega} \times \dot{\underline{x}}$

In this section, we approximate $\underline{g}^* \approx \underline{g}$

we can integrate eqn. of motion w.r.t. time with initial conditions $\underline{x} = \underline{a}$ and $\dot{\underline{x}} = \underline{v}$ at $t=0$, corresponding to a particle projected from point $\underline{a}$ with velocity $\underline{v}$

$$\therefore \quad \dot{\underline{x}} = \underline{v} + \underline{g}t - 2\underline{\omega} \times (\underline{x} - \underline{a})$$

we can use the suvat $s = ut + \frac{1}{2}gt^2$
so $\underline{x} = \underline{a} + \underline{v}t + \frac{1}{2}\underline{g}t^2$
sub into cross product

$$\dot{\underline{x}} = \underline{v} + \underline{g}t - 2\underline{\omega} \times \left(\underline{v}t + \frac{1}{2}\underline{g}t^2\right)$$

integrate again with same initial conditions for the solution:

$$\boxed{x = \underline{a} + \underline{v}t + \frac{1}{2}\underline{g}t^2 - \underline{\omega} \times \left(\underline{v}t^2 + \frac{1}{3}\underline{g}t^3\right)}$$

Example: particle dropped from a tower

Consider a particle dropped from rest from a tower of height h, with initial position $\underline{a}$

initial condition: $\underline{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\underline{a} \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}$

Using $\underline{\omega} \times \underline{g} = -\omega g \cos\lambda \, \hat{\underline{x}}$, so the xyz components of $\underline{x}$ are:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} t + \frac{1}{2} g t^2 \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + \frac{1}{3} \omega g t^3 \cos\lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

This is just subbing values into eq on previous page

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} - \frac{1}{2} g t^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{3} \omega g t^3 \cos\lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

This hits the ground when $z = 0$ which is when:

$$h - \frac{1}{2} g t^2 = 0$$

$$\Rightarrow t = \sqrt{\frac{2h}{g}}$$

The x component at this time is:

$$\frac{1}{3} \omega \cos\lambda \left(\frac{8h^3}{g} \right)^{1/2}$$

There is no y component.

So the particle hits the ground a little more in $\hat{\underline{x}}$ direction (East) then base of tower.

Example: Shell fired from a canon

A shell is fired North with speed v from a canon with elevation angle $\frac{\pi}{4}$. origin is set at canon.

Initial conditions: $\underline{v} = \frac{v}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}$ $\quad \underline{a}\begin{pmatrix}0\\0\\0\end{pmatrix}$

$\underline{\omega} \times \underline{v} = \frac{\omega v}{\sqrt{2}}(\cos\lambda - \sin\lambda)\,\hat{\underline{x}}$ so xyz components of $\underline{x}$ are:

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} + \frac{v}{\sqrt{2}}t\begin{pmatrix}0\\1\\1\end{pmatrix} + \frac{1}{2}gt^2\begin{pmatrix}0\\0\\-1\end{pmatrix} + \frac{1}{3}wgt^3\cos\lambda\begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{wvt^2}{\sqrt{2}}(\cos\lambda - \sin\lambda)\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \frac{v}{\sqrt{2}}t\begin{pmatrix}0\\1\\1\end{pmatrix} - \frac{1}{2}gt^2\begin{pmatrix}0\\0\\1\end{pmatrix} + \frac{1}{3}wgt^3\cos\lambda\begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{wvt^2}{\sqrt{2}}(\cos\lambda - \sin\lambda)\begin{pmatrix}1\\0\\0\end{pmatrix}$$

The shell hits ground when $z=0$ which is when

$$\frac{v}{\sqrt{2}}t - \frac{1}{2}gt^2 = 0 \qquad z=0 \text{ when } t=0 \text{ initially}$$

$$t\left(\frac{v}{\sqrt{2}} - \frac{1}{2}gt\right) = 0 \quad \text{so} \quad t=0$$

or $\frac{v}{\sqrt{2}} - \frac{1}{2}gt = 0 \Rightarrow \frac{1}{2}gt = \frac{v}{\sqrt{2}}$ $\quad t = \frac{2}{\sqrt{2}}\frac{v}{g} \Rightarrow \underline{t = \sqrt{2}\frac{v}{g}}$

putting this into the eqn.

x component is:

$$\frac{\sqrt{2}\,\omega v^3}{3g^2}(3\sin\lambda - \cos\lambda)$$

If $3\sin\lambda > \cos\lambda$, the deflection is due east. Else it is due west. λ is latitude so deflection depends on initial position of canon on Earth.

Focault's Pendulum

If you set up a pendulum at the North Pole, an observer on Earth would notice the plane of oscillation rotate backwards at angular velocity $-\omega$.

The phenomenon is less pronounced at lower latitudes, and non-existent at the equator.

The plane of oscillation rotates at angular velocity $-\omega \sin\lambda$

We will now derive this.

$\underline{g}^* = -g\hat{\underline{z}}$. We will assume $g^* \approx g$ here.

The pendulum has length L.

$\underline{x}$ is displacement of bob from bottom of swing.

ignoring $\hat{\underline{z}}$ terms, equations of motion are:

$$\ddot{x} = F_x - 2\omega(\dot{z}\cos\lambda - \dot{y}\sin\lambda)$$
$$\ddot{y} = F_y - 2\omega\dot{x}\sin\lambda$$

where $\underline{F}$ is tension in wire.

$F_x = -\frac{Mgx}{L}$ $\quad F_y = -\frac{Mgy}{L}$

ignore all z terms so:

$$\ddot{x} = -\omega_0^2 x + 2\omega\sin\lambda\,\dot{y}$$
$$\ddot{y} = -\omega_0^2 y - 2\omega\sin\lambda\,\dot{x}$$

where $\omega_0 = \frac{g}{L}$

To solve this, define complex quantity $\alpha = x + iy$, so that we combine both equations into one:

$$\ddot{\alpha} + 2i\omega\sin\lambda\,\dot{\alpha} + \omega_0^2\alpha = 0$$

Look for solution in form $\alpha = Ae^{ipt}$. This gives solution provided:

$$p = -\omega\sin\lambda \pm \sqrt{\omega_0^2 + \omega^2\sin\lambda}$$
$$\approx -\omega\sin\lambda \pm \omega_0 \text{, making use of } \omega_0 \gg \omega\sin\lambda$$

So general solution is

$$\alpha = \left(Ae^{i\omega_0 t} + Be^{-i\omega_0 t}\right)e^{-i(\omega\sin\lambda)t}$$

with appropriate initial conditions, the solution is:

$$\boxed{\alpha = ae^{-i(\omega\sin\lambda)t}\cos(\omega_0 t)}$$

describes rotation of plane of oscillation

usual periodic swing