Tensors

The four momentum we have been working with is part of a family of objects called Tensors, which can have more than one index.

Consider angular momentum (non-relativistic):

$$\underline{L} = \underline{r} \times \underline{p}$$

so
$$L^1 = yP_z - zP_y$$
$$L^2 = zP_x - xP_z$$
$$L^3 = xP_y - yP_x$$

Relativistically, we express this as a tensor:

$$L^{\mu\nu} = x^\mu p^\nu - p^\mu x^\nu$$

where $x^1, x^2, x^3 = x, y, z$ respectively
$p^1, p^2, p^3 = P_x, P_y, P_z$ respectively

so eg. $L^{12} = x^1 p^2 - p^1 x^2 = xP_y - yP_x = L^3$

Tensors have many properties, some of which are:

- Under Lorentz transformations $L'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta L^{\alpha\beta}$
- Lorentz invariant: $L^{\mu\nu} L_{\mu\nu} = (L^{00})^2 - (L^{01})^2 + (L^{11})^2 + \ldots = \text{constant}$

There will be 16 terms in the final matrix here, but the diagonal terms will be 0. So there are really only 6 independent terms.

Note the metric $g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ is a tensor