Magnetostatics

Let's start by considering the magnitude of the force of attraction between two parallel current elements idl and IdL:

$idl \uparrow \quad \uparrow IdL$

$\longleftrightarrow$
r

$$\boxed{d^2F = \frac{\mu_0}{4\pi}\frac{idl\,IdL}{r^2}}$$

where μ_0 is the permeability of free space

$\mu_0 = 4\pi \times 10^{-7}\ TmA^{-1}$

Note the similarities between this and coulomb's law.

Let's now put this into vector form. We can do this by making the length elements into vectors going in the $\hat{\imath}$ direction:

$d\underline{l} = dl\hat{\imath} \quad d\underline{L} = dL\hat{\imath}$

we also need to introduce a vector $\hat{\underline{r}}$ for the direction the force is acting in.

$Id\underline{L}$ is placed at origin and $id\underline{l}$ is at point $\underline{r}$

$$\boxed{d^2\underline{F} = \frac{\mu_0}{4\pi}\frac{id\underline{l} \times (Id\underline{L} \times \hat{\underline{r}})}{r^2}}$$

The double cross product is quite tedious and makes calculation tricky.

To make this easier, it might be appropriate to consider the magnetic field at point $\hat{\underline{r}}$ generated by the current element $Id\underline{L}$ at the origin:

$$\boxed{d\underline{B} = \frac{\mu_0}{4\pi}\frac{Id\underline{L} \times \hat{\underline{r}}}{r^2}}$$

We can then compute the force with:

$$\boxed{d^2\underline{F} = id\underline{l} \times d\underline{B}} \quad *$$

A current element $id\underline{l}$ is equivalent to a point charge q moving with velocity $\underline{v}$: $\boxed{id\underline{l} = q\underline{v}}$ we can sub this into $*$ to obtain the famous:

$$\boxed{\underline{F} = q\underline{v} \times \underline{B}}$$

Biot-Savart Law

Our definition of magnetic field on the previous page is the first step to the Biot-Savart law which allows us to calculate the magnetic field $d\underline{B}$ at a point $\underline{r}$ due to a current element $Id\underline{L}$ at the point $\underline{R}'$.

The distance from $\underline{R}'$ to $\underline{r}$ is $\underline{r}' = \underline{r} - \underline{R}'$

$$\therefore \quad \boxed{d\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi} \frac{Id\underline{L} \times \hat{\underline{r}}'}{r'^2}}$$

We can integrate this to obtain the total magnetic field over a whole length of wire!

$$\boxed{\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi} \int \frac{Id\underline{L} \times \hat{\underline{r}}'}{r'^2}}$$

This is the Biot-Savart Law.

Volume Currents

The Biot-Savart Law above assumes an infinitely thin wire, but what if this is not the case? We have to do something similar to the continuous charge distribution, except instead for current.

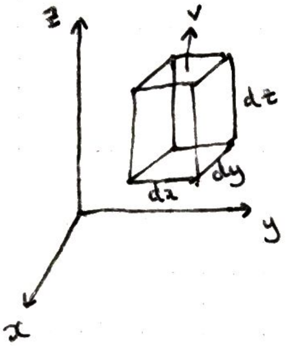
As we defined before $id\underline{L} = q\underline{v}$, so a current element is described by charge and velocity of the charge. So we can define $\underline{J}$ volume current density as:

$$\boxed{\underline{J} = \rho \underline{v}}$$

where ρ is charge density

But what does $\underline{J}$ physically mean?

Consider a wire cuboid element with dimensions as shown below, oriented with a vector $\underline{v}$ describing the motion of charge:



In a time dt, an amount of charge dQ flows through the cuboid. $\therefore$ $dz = v dt$

$dQ = \rho dV$ where $dV = dx dy dz$

$\therefore$ $dQ = \rho dx dy dz$ but $dz = v dt$

$dQ = \rho v dx dy dt$

since $dI = \frac{dQ}{dt}$: $dI = \rho v dx dy$

$$\frac{dI}{dx dy} = \rho v = J$$

So therefore the <u>volume</u> current density is the electric current per unit <u>area</u> in the direction $\underline{v}$ per unit area perpendicular to $\underline{v}$.

So a small current element can be considered in 3D are a small volume element of "charged fluid".

$$I d\underline{L} = \underline{J} dV$$

We can use this substitution in the Biot-Savart Law.

A volume current element at $\underline{R}'$ can be described as: $I d\underline{L} = \underline{J}(\underline{R}') dV$

So the magnetic field at point $\underline{r}$ given by this volume current element at $\underline{R}'$ is:

$$\boxed{d\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi} \frac{\underline{J}(\underline{R}') dV' \times \hat{\underline{r}}'}{r'^2}}$$ where $\underline{r}' = \underline{r} - \underline{R}'$

For a full volume we integrate:

$$\boxed{\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi} \iiint_{\text{volume}} \frac{\underline{J}(\underline{R}') dV' \times \hat{\underline{r}}'}{r'^2}}$$

Since there are no additional sources of charge inside a conducting wire, we can expect charge to be conserved. So let's pull out the continuity equation from chapter 1:

$$\underline{\nabla} \cdot (\rho \underline{V}) + \frac{\partial \rho}{\partial t} = 0$$

so in our case, this becomes:

$$\boxed{\underline{\nabla} \cdot \underline{J} + \frac{\partial \rho}{\partial t} = 0}$$

In the static case where ρ does not change with time:

$$\boxed{\underline{\nabla} \cdot \underline{J} = 0}$$

electric current density does not diverge. What flows in must flow out! | only for static case

Ampére's Law

We all remember Ampére's Law from last year: $\boxed{\oint \underline{B} \cdot d\underline{L} = \mu_0 I}$

Now let's extend this by applying it inside a "charged fluid" with an associated volume current density $\underline{J}$

To do this consider Stokes' Theorem: $\oint \underline{V} \cdot d\underline{L} = \iint_{\substack{\text{open}\\\text{surface}}} (\underline{\nabla} \times \underline{V}) \cdot d\underline{A}$

$$\therefore \oint \underline{B} \cdot d\underline{L} = \iint_{\substack{\text{open}\\\text{surface}}} (\underline{\nabla} \times \underline{B}(x,y,z)) \cdot d\underline{A}$$

The open surface is an area attached to the loop:

$$\mu_0 I = \iint (\underline{\nabla} \times \underline{B}(x,y,z)) \cdot d\underline{A}$$

This is the current enclosed in an amperian loop which can also be described as $I = \iint_{\substack{\text{open}\\\text{surface}}} \underline{J} \cdot d\underline{A}$

$$\therefore \iint (\underline{\nabla} \times \underline{B}) \cdot d\underline{A} = \mu_0 \iint \underline{J} \cdot d\underline{A}$$

| equating the integrands we get:

$$\boxed{\underline{\nabla} \times \underline{B}(x,y,z) = \mu_0 \underline{J}}$$

This is another cool result: $\boxed{\text{curl } B = \mu_0 \underline{J}}$ The differential form of Amperes Law

There are No Magnetic Monopoles !!!

We saw from the differential form of Gauss' Law that the div of $\underline{E}$ is non-zero. So it is possible for there to be electric monopoles (singularly charged particles like $+q$ or $-q$).

Maybe there is something like that for magnetism? There is not!

$$\boxed{\nabla \cdot \underline{B}(x, y, z) = 0} \qquad \boxed{\iint_{S \text{ closed}} \underline{B}(x, y, z) \cdot d\underline{A} = 0}$$

Magnetic Vector Potential

We know that $\underline{E} = -\text{grad } V$, so the electric scalar potential can be found from $\underline{E}$. Is there something similar for $\underline{B}$?

$\underline{E} = -\text{grad } V \Rightarrow \text{curl } \underline{E} = 0$ since curl grad (anything) $= 0$

Starting from $\text{div } \underline{B} = 0$ we know $\text{div curl } \underline{A} = 0$ where $\underline{A}$ is some vector.

so $\boxed{\underline{B} = \text{curl } \underline{A}}$

where we define $\underline{A}$ as the magnetic vector potential.

In magnetostatics, $\underline{A}$ isn't that useful but it will become very useful when we move onto electrodynamics.

Magnetic Forces from Relativity

It turns out, magnetic fields are a consequence of relativity. There is a great video by Veritasium on YouTube about this called "How Special Relativity makes Magnets Work". Watch it!

Consider: an infinite line of positive charge with charge per unit length λ_0 projected to the right with velocity u relative to lab frame:

$+ + + + + + + \longrightarrow u$

The line will undergo Lorentz contraction due to relativistic effects, so the observor in the lab frame will see charge per unit length:

$$\lambda = \gamma\lambda_0 = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\lambda_0$$

The current is $\frac{\text{charge}}{\text{time}}$ = charge per unit length x velocity

So the observor measures current $I = u\lambda$

Right now, we are thinking of a charged wire, but we really want to consider a neutral wire carrying a charge. So let's consider equal and opposite charge line moving in opposite direction.

$-u \leftarrow - - - - - - -$

$+ + + + + + + \rightarrow u$

r

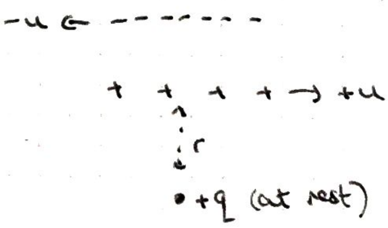
$+q \rightarrow v$

Consider a charge $+q$ a distance r from the wire moving with velocity v.

In the lab frame, the wire is neutral so there is no electrical force on the $+q$ charge.

But we know there will be a magnetic force, so where does this come from?

Consider the frame S' where the $+q$ charge is at rest:



The two lines of charge experience a different length contraction since they have different velocities given by the Lorentz velocity transformation:

$$u_{\pm} = \frac{u \pm v}{1 \mp \frac{uv}{c^2}}$$

So the charge per unit length in this frame S' is:

$$\lambda_{\pm} = \pm \gamma_{\pm} \lambda_0 = \pm \frac{1}{\sqrt{1 - \frac{u_{\pm}^2}{c^2}}} \lambda_0$$

So clearly in this frame the charges do not cancel, so the "wire" isn't neutral in this frame. The net negative charge is: (per unit length)

$$\lambda_{tot} = \lambda_+ + \lambda_- = \frac{-2\lambda u v}{c^2 \sqrt{1 - \frac{v^2}{c^2}}}$$ after some lengthy algebra

The negative charge sets up an electric field at the point r with magnitude:

$$E = \frac{\lambda_{tot}}{2\pi \varepsilon_0 r}$$

So the force on the positive charge $+q$ at the point r is:

$$F' = qE = \frac{\lambda_{tot} q}{2\pi \varepsilon_0 r}$$ attraction <u>to</u> the wire

If there is a force of attraction in this frame, there will also be a force of attraction F in the lab frame S, where the charge has velocity v. We can use a Lorentz velocity transformation to calculate F

$$F = \sqrt{1 - \frac{v^2}{c^2}}\, F'$$

$\therefore$ $F = qv \dfrac{\lambda u}{\varepsilon_0 c^2 \pi r}$ after some algebra. which we rearrange for current as:

$$\boxed{F = qv \frac{I}{\varepsilon_0 c^2 2\pi r}}$$

This is just B so:

$F = qvB$ where $B = \dfrac{\mu_0 I}{2\pi r}$ since $\mu_0 = \dfrac{1}{\varepsilon_0 c^2}$

So we have derived these important equation from relativity!