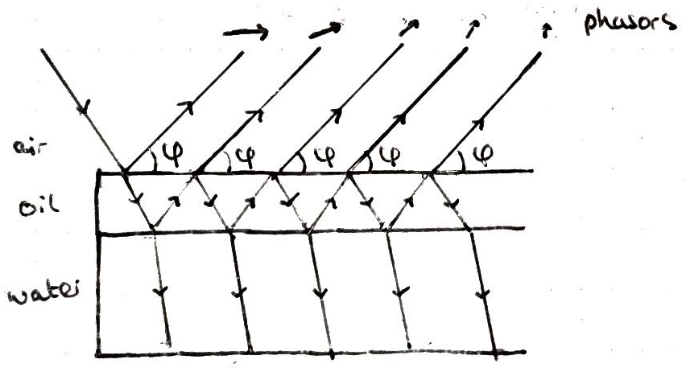
Phasors

This is really best understood with the animations on the slides!

We can use complex numbers as part of the detailed model of how a wave propagates from point to point. The displacement due a sinusoidal wave motion is represented by an arrow with length proportional to the amplitude of the oscillation and orientation indicating the phase. To determine the propagated field at some given point, all the contributing phasors are added together as vectors, i.e they are joined up head to tail

The graphical depiction of phasors is very similar to the depiction of complex numbers on a Argand Diagram, showing we can represent phasors with complex exponentials whose modulus is the amplitude and argument is the phase. The combined contribution of multiple phasors is thus the sum of a series of complex exponentials.

Consider:



This shows how phasors can be used to analyse the light reflected by a thin film of oil on top of some water. As can be seen, each phasor is smaller than the last by some constant factor (which we can call α) due to the lost amplitude as some of the light is not reflected. The phasors by an extra constant angle φ due to the phase shift because of the extra path length. Theoretically, there would be an infinite number of phasors.

If we draw together the infinite number of phasors:

we can completely geometrically find the Phasor resultant and hence the intensity of the observed reflection.

We can also do it mathematically. Mathematically, the contributions are: $a_0,\ \alpha a_0 e^{i\varphi},\ \alpha^2 a_0 e^{2i\varphi},\ \alpha^3 a_0 e^{3i\varphi} \ldots$

if we let $r = \alpha e^{i\varphi}$, then we can write the sum as:

$$E = a_0 + \alpha a_0 e^{i\varphi} + \alpha^2 a_0 e^{2i\varphi} + \alpha^3 a_0 e^{3i\varphi} \ldots$$

$$= a_0 (1 + r + r^2 + r^3 \ldots)$$

To find a convergence, we can do:

$$rE = a_0 (r + r^2 + r^3 \ldots)$$

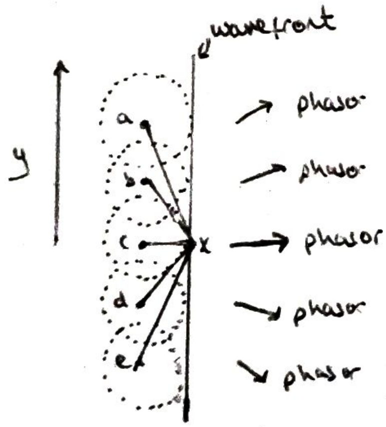
so $E - rE = a_0 \Rightarrow E(1-r) = a_0$ so $E = \dfrac{a_0}{1-r}$

$$E = \frac{a_0}{1-\alpha e^{i\varphi}}$$

Thus, we mathematically found the total light field reflected at angle φ.

Phasors in Huygen's Construction

We can extend our understanding of Huygen's Construction a little by thinking about the total contribution to the disturbance at a given point from a line of radiating secondary sources:



a, b, c, d, e are secondary sources. Dashed lines are spherical wavefronts. The resultant planar wavefront is labelled.

The 5 phasors shown are the contributions of the 5 secondary sources to the centre point on the wavefront, x

The distance of each point (a, b, c, d, e) from x is

$$r = \sqrt{y^2 + x^2}$$

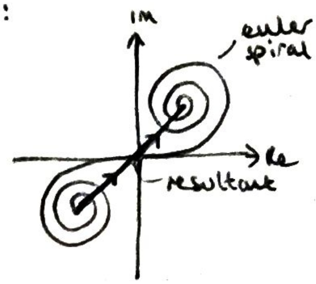
For each wavelength λ, the phasor corresponding to the contribution arriving from that source is written algebraically:

$$A(x,y) = \frac{e^{ikr}}{r}$$ where A is the wave amplitude, $k = \frac{2\pi}{\lambda}$ and

the $\frac{1}{r}$ term is because the further away the secondary source, the weaker its contribution. The total contribution from all secondary sources (not just the 5 labelled) is given by:

$$\int_{-\infty}^{\infty} \frac{e^{ikr}}{r}\, dy$$

The summation of all phasors looks on an argand diagram like:



where $r = \sqrt{x^2+y^2}$

if we make the substitution $x + s^2 = \sqrt{x^2+y^2}$

such that $y^2 = s^4 + 2xs^2$, differentiating b.s. w.r.t. s:

$$2y\frac{dy}{ds} = 4s^3 + 4xs \Rightarrow \frac{dy}{ds} = \frac{4s^3+4xs}{2\sqrt{s^4+2xs^2}}$$

simplifying further and substituting in, our integral becomes:

$$\int_{-\infty}^{\infty} \frac{e^{ikr}}{r}\, dy = \int_{-\infty}^{\infty} \frac{e^{ik(x+s^2)}}{x+s^2} \frac{2(x+s^2)}{\sqrt{2x+s^2}}\, ds = \sqrt{\frac{2}{x}}\, \exp(ikx) \int_{-\infty}^{\infty} \frac{\exp(iks^2)}{\sqrt{1+s^2/2x}}\, ds$$

if x is greater than a few wavelengths so $kx \gg 1$, the denominator can be approximated to $1/\sqrt{2x}$ giving us the integral

$$\int_{-\infty}^{\infty} \exp(iks^2)\, ds$$

This is called the Fresnel integral. It probably won't come up in the exam and we won't need to evaluate it. The important concept to be understood here is that the wave disturbance at a given point is the superposition of wave disturbances travelling all possible routes.

This is why in Huygen's Description, there is no resultant wavefront travelling "backwards" as this possible wavefront is the sum of the wave disturbances travelling all possible routes, which all destructively interfere with each other.