Nuclear Shell Model

Magic Numbers

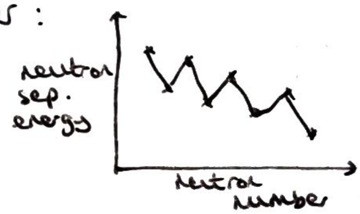
The Binding Energy formula we came up with in the previous chapter still isn't perfect. It underestimates the binding energy of nuclei with number of neutrons or protons equal to one of the following numbers:

2, 8, 20, 28, 50, 82, 126

These numbers are called "magic numbers".
A nuclei is doubly magic if both N and Z equal one of the magic numbers.

Some properties of magic nuclei are:

- The neutron / proton separation energy [the energy needed to remove the last neutron/proton] peaks if N/Z is equal to a magic number:



- There are more stable isotopes if Z is a magic number
- There are more stable isotones if N is a magic number.
- If N is a magic number, cross-section for neutron absorption is much lower than other nucleides.
- Energies of excited states much higher than ground state if N or Z or both are magic numbers
- Elements with magic Z have higher natural abundance than those of nearby elements.

Shell Model

So what are these magic numbers? To explain this we have to consider the shell model. In this model, each nucleon is moving in some potential and the energy levels are are classified in terms of quantum numbers n, l, j.

If we neglect spin, the wavefunction for a spherically symmetric potential is:

$$\Psi_{nlm} = R_{nl}(r)\, Y_l^m(\theta, \varphi)$$

where n is the principle quantum no.
l is the orbital angular momentum quantum no.
m is the magnetic quantum no.

The energy eigenvalues will be degenerate in m.
These energy levels will be grouped together in shells with large energy gaps between each shell.

In the ground state, the nucleons fill up the available energy levels from the bottom upwards with two protons/neutrons in each proton/neutron level.

So what are the properties of the potential the nucleons are moving in? Saxon-Woods model is a reasonable guess!

$$V(r) = -\frac{V_0}{1 + \exp((r-R)/\delta)}$$

For this potential, the lowest energy level turns out to be 1s $(n=1,\ l=0)$ which can contain up to 2 neutrons or protons. Next is 1p which can contain up to 6 protons or neutrons.

This explains the first 2 magic numbers 2 and $2+6 = 8$.

The next level is 1d which is close enough to 2s that they form the same shell, giving us the next magic number:

$$8 + (2 + 10) = 20$$

The next two levels are 1f and 2p which form the same shell giving us a further $14 + 6 = 20$ neutrons or protons. But the next magic number is not 40 as we might expect from this logic but it is 50.

This is because of spin-orbit coupling, which you should remember from atomic physics. Here, the nuclear potential itself is proportional to $\underline{L} \cdot \underline{S}$: $V(r) \rightarrow V(r) + W(r)\, \underline{L} \cdot \underline{S}$

We define a quantum number j which takes values $j = L \pm \frac{1}{2}$ depending on the spin. The spin-orbit coupling term leads to an energy shift proportional to

$$j(j+1) - L(L+1) - S(S+1) \quad \text{for } S = \tfrac{1}{2}$$

Note:// unlike with Atomic Physics, here states with higher j have lower energy.

This energy shift effect leads to energy levels crossing over into different shells. Eg. above the 2p state we have 1g $(L = 4)$ which splits into $1g_{9/2}$ and $1g_{7/2}$ since $j = L \pm \frac{1}{2}$

The energy level of $1g_{9/2}$ is low enough so it joins the shell below which now consists of: $1f_{7/2}, 2p_{3/2}, 1f_{5/2}, 2p_{1/2}, 1g_{9/2}$

This can take $(2j+1)$ protons/neutrons for each state:

$$8 + 4 + 6 + 2 + 10 = 30$$

So the next magic number is $20 + 30 = \underline{50}$ as we know it is from experiments.

Spin and Parity of Nuclear Ground States

Nuclear states have an intrinsic spin and well defined parity $\eta = \pm 1$. There are some rules to determine the spin and parity.

These rules use the fact that protons and neutrons tend to pair up and each pair has spin 0 and parity +1 :

- Even Z Even A means 0 spin , $\eta = +1$
- Odd A means one unpaired nucleon. Spin is equal to j value of unpaired nucleon , parity is $(-1)^L$ where L is orbital angular momentum of unpaired nucleon .
- Odd Z , Even A means one unpaired proton with j_1 and one unpaired neutron with j_2. Spin is a range of values between $|j_1 - j_2|$ and $|j_1 + j_2|$ in integer steps. Parity is $(-1)^{L_1 + L_2}$

Magnetic Dipole Moments

Nuclei with odd no. of neutrons or protons possess an intrinsic spin and also generally have a magnetic dipole moment.

The unit of magnetic dipole moment is the "nuclear magneton"

$$\mu_N = \frac{e\hbar}{2M_p}$$

defined such that a proton with orbital ang. mom. 1 has magnetic dipole moment $1\,\mu_N$.

Experimentally we find that the magnetic moment of the proton with spin $\frac{1}{2}$ is : $\mu_p = 2.79\,\mu_N = 5.58\,\mu_N S \quad (S = \frac{1}{2})$

For a neutron with spin $\frac{1}{2}$: $\mu_n = -1.91\,\mu_N = -3.82\,\mu_N S \quad (S = \frac{1}{2})$

Applying a magnetic field in the z direction to a nucleus will mean that the unpaired proton with $\underline{j}$, $\underline{L}$ and $\underline{s}$ will give a contribution to z-component of magnetic moment by:

$$\mu^z = (5.58 s^z + L^z)\mu_N$$

As in the case of Zeeman Effect, we can write:

$$\mu^z = \frac{(5.58\langle \underline{s}\cdot\underline{j}\rangle + \langle \underline{L}\cdot\underline{j}\rangle)}{\langle j^2\rangle} j^z \mu_N$$

if we sub in:

$$\langle j^2\rangle = j(j+1)\hbar^2$$

$$\langle \underline{s}\cdot\underline{j}\rangle = \frac{1}{2}(\langle j^2\rangle + \langle \underline{s}^2\rangle - \langle \underline{L}^2\rangle)$$

$$= \frac{\hbar^2}{2}(j(j+1) + s(s+1) - L(L+1))$$

$$\langle \underline{L}\cdot\underline{j}\rangle = \frac{1}{2}(\langle j^2\rangle + \langle \underline{L}^2\rangle - \langle \underline{s}^2\rangle)$$

$$= \frac{\hbar^2}{2}(j(j+1) + L(L+1) - s(s+1))$$

we get:

$$\mu = \frac{5.58(j(j+1)) + s(s+1) - L(L+1) + j(j+1) + L(L+1) - s(s+1)}{2j(j+1)} j\mu_N$$

similarly, for unpaired neutron with $\underline{L}'$, $\underline{j}'$, $\underline{s}$:

$$\mu = -\frac{3.82(j'(j'+1) + s(s+1) - L'(L'+1))}{2j'(j'+1)} j'\mu_N$$

So if we wanted to know μ for ^7_3Li which has an unpaired proton in the $2P_{3/2}$ state ($L=1$, $j=3/2$) then $\mu = 3.79\mu_N$

The experimental value is $3.26\mu_N$ so this estimate is not very good :(and it only gets worse for heavier nuclei.

The reason why this estimate is not very good is not well understood.

Excited States

Like in atomic physics, nuclei can be in excited states, in which one of the neutrons or protons is promoted to a higher energy level.

Sometimes, it is energetically cheaper to promote a nucleon from an inner closed shell (rather than from an outer shell) into a higher energy state.

The nuclear spectrum of states is very complicated and not well understood with the shell model.

Most excited states decay rapidly so their lifetimes cannot be measured. However, some states are "metastable" since they cannot decay without violating selection rules; these states are called isomers and their half-life can be measured.

The Collective Model

Another failing of the Shell Model is that it does not accurately predict electric quadrupole moments. The reason for this is that we assumed a spherically symmetric potential.

The Collective Model generalises the shell model by considering the effect of a non-spherically symmetric potential, which gives us more accurate electric quadrupole moments.

The collective model explains low-lying excited states of heavy nuclei in 2 ways:

- Rotational States

A nucleus whose nucleon density distributions are spherically symmetric, so no quadrupole moment, cannot have rotational excitations

But a non-zero quadrupole moment nucleus can have rotational excitations due to rotation perpendicular to axis of symmetry.

For even-even nucleus (zero spin for ground state) the states have energy $E_{rot} = \frac{I(I+1)\hbar^2}{2\mathcal{I}}$

where $\mathcal{I}$ is the moment of inertia of nucleus about an axis through centre perp. to axis of symmetry.

Rotational energy levels of even-even nucleus can only take even values of I.

If we know, for example, that the rotational energy levels of $^{170}_{72}Hf$ are: 100 keV 321 keV 641 keV

These are in ratio $2\times3 : 4\times5 : 6\times7$

so $I = 2, 4, 6$ respectively

We can then extract $\mathcal{I}$ from the E_{rot} equation.

For odd-odd nuclei, for which ground state I_0 has non-zero spin:

$$E_{rot} = \frac{1}{2\mathcal{I}}\left(I(I+1) - I_0(I_0+1)\right)\hbar^2$$

Where I can take values I_0+1, I_0+2 etc.

- Shape Oscillators

Modes of vibration where the deformation of nucleus osciliates. i.e the electric quadrupole moment oscilates about its mean value.

So if the nucleus' shape is oscillating about the equilibrium, we have simple harmonic motion. The energy levels of such modes are evenly spaced, so if we observe evenly spaced energy levels in the spectrum of a nuclide, we assume it is due to shape oscillations.