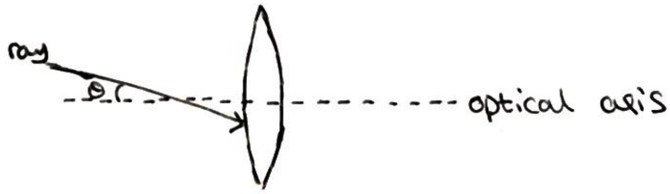
Geometric Optics

If we consider spherical surfaces of optical elements (mirrors and lenses), then the optical axis is a line of rotational symmetry in an optical system:



In this course, we will only be considering paraxial rays:
- These make small angles with the optical axis which allows us to use small angle approximations
- They also remain close to the optical axis so the angle they make with the normal of the lens is small, also allowing for small angle approximations.

The paraxial approximations we will be making are:

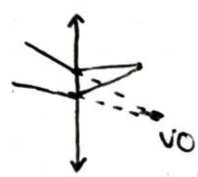
$\underline{\underline{\sin\theta \approx \theta}}$ $\quad$ $\underline{\underline{\cos\theta \approx 1}}$ $\quad$ $\underline{\underline{\tan\theta = \frac{\sin\theta}{\cos\theta} \approx \theta}}$

We will also define certain features in geometric optics:

Real Object - The object from which rays are emitted

Real Image - The point rays converge on

Virtual object - The rays appear to converge at a point

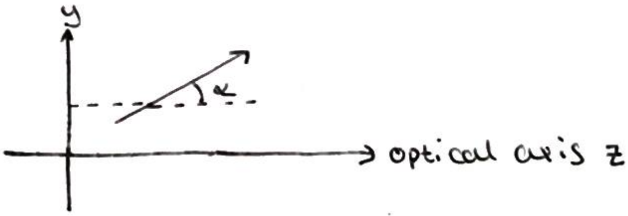


Virtual Image - The rays appear to come from a point.

Matrix Methods for Ray Propagation

We label a light ray by y (its distance from the optical axis) and α (the angle it makes with the optical axis in paraxial approximation).

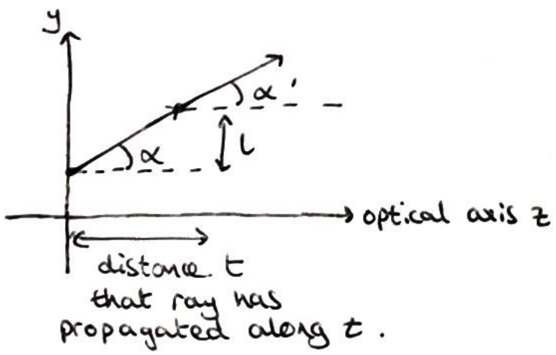
The sign convention is ↑ for positive y and α and ↓ for negative y and α. α is always acute



The ray is described by (y, α)

After a ray goes through some optical element, it emerges transformed to (y', α'). Since the transformation is linear in paraxial approximation, matrices can be used. Each type of optical element will have its own matrix. The ray will propagate as normal between optical elements.

Translation between two points



In this case, the y coordinate has changed so:

$$y' = y + l = y + t(\tan\alpha) = \underline{t + t\alpha}$$

The α has stayed the same so:

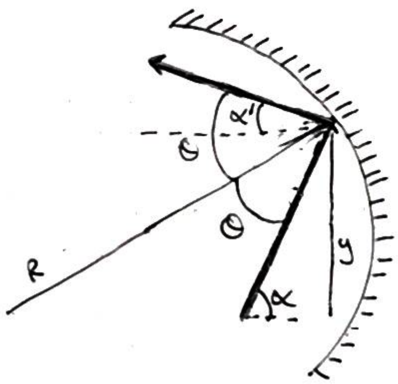
$$\alpha' = \alpha$$

So we can see that the matrix of translation is: $\underline{\underline{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}}}$

such that $\underline{\begin{pmatrix} y' \\ \alpha' \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ \alpha \end{pmatrix}}$

Reflection in a Spherical Mirror

For a concave mirror, a ray enters with (y, α) and emerges as (y', α') reflected by the mirror with radius R



First, we notice that $y' = y$ since the reflection does not change the ray's distance from the axis.

We can also notice that:

$\theta - \alpha' = \alpha - \theta = \frac{y}{R}$ using small angle approximation

Using simultaneous equations, we get:

$\alpha' = -\frac{2}{R} y + \alpha$

This gives $y' = y$ $\alpha' = -\frac{2}{R} y + \alpha$ so the matrix of transformation is:

$$\begin{pmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{pmatrix}$$ for a concave spherical mirror

Similarly, for a convex mirror, all the same arguments apply but the simultaneous equation is now:

$\theta - \alpha = \alpha' - \theta = \frac{y}{R}$ giving:

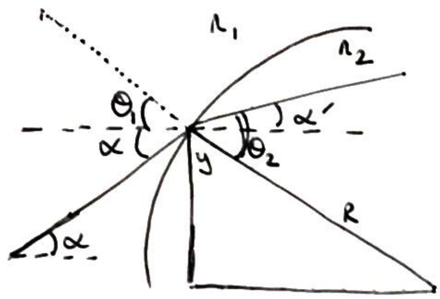
$y' = y$ and $\alpha' = \frac{2}{R} y + \alpha$. So the matrix of transformation is:

$$\begin{pmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{pmatrix}$$ for a convex spherical mirror

The symmetry here is obvious and so a sign convention can be adopted. R is positive if the centre of curvature is on the same side as the reflected ray and negative if it is on the opposite side. This means we can use the same equation

$\begin{pmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{pmatrix}$ for both spherical mirrors but use negative or positive R depending on whether the mirror is convex or concave. This is for reflection only.

Refraction at Spherical Surfaces



For a convex surface:

From this, we again notice that $y = y'$ since the refraction does not change the ray's distance from the axis.

We can also notice that:
$\theta_1 - \alpha = \theta_2 - \alpha' = \frac{y}{R}$ using small angle approximation

Using Snell's law:
$n_1 \sin\theta_1 = n_2 \sin\theta_2 \rightsquigarrow n_1\theta_1 = n_2\theta_2$

$n_1\left(\frac{y}{R} + \alpha\right) = n_2\left(\frac{y}{R} + \alpha'\right)$ which rearranges for α' to give:

$$\alpha' = \frac{y}{R}\left(\frac{n_1}{n_2} - 1\right) + \frac{n_1}{n_2}\alpha$$

This gives the matrix of transformation to be: $\underline{\underline{\begin{pmatrix} 1 & 0 \\ \frac{1}{R}\left(\frac{n_1}{n_2} - 1\right) & \frac{n_1}{n_2} \end{pmatrix}}}$

For a concave surface, we get the same matrix but with a $-R$ instead. This gives another sign convention for refraction. For refraction, R is positive if the centre of curvature is on the same side as the refracted ray and negative for the opposite side. This is similar to the convention for reflection so in general:

R is positive if the centre of curvature is on the same side as the transformed ray and negative if it is on the opposite side.

Refraction with Plane

This can be considered to be the same as refraction on a curved surface with a radius of infinity. So the matrix of transformation is given by:

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{pmatrix}$$

Thin Lens

A thin lens is a lens with two spherical surfaces with R_1 and R_2 radii and a thickness t:

An incoming ray would be transformed 3 times, first by the first surface, then translated through the lens and then finally by the second surface.

This would give a matrix $\begin{pmatrix} 1 & 0 \\ \frac{1}{R_2}\left(\frac{n_b}{n_a}-1\right) & \frac{n_b}{n_a} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{R_1}\left(\frac{n_a}{n_b}-1\right) & \frac{n_a}{n_b} \end{pmatrix}$

We will simplify this, however, using the thin lens approximation, where we assume that the two spherical surfaces are close enough that we can neglect their separation, t.

The matrix therefore becomes:

$$\begin{pmatrix} 1 & 0 \\ \frac{1}{R_2}\left(\frac{n_b}{n_a}-1\right) & \frac{n_b}{n_a} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{R_1}\left(\frac{n_a}{n_b}-1\right) & \frac{n_a}{n_b} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \left(\frac{n_b}{n_a}-1\right)\left(\frac{1}{R_2}-\frac{1}{R_1}\right) & 1 \end{pmatrix}$$

We can further simplify this by assuming the lens sits in air or in a vacuum so $n_a \approx 1$ and we can write $n_b = n$.

The thin lens matrix therefore becomes:

$$\underline{\underline{\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}}}$$

where f is a quantity defined by:

$$\boxed{\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$$

This is known as the lensmaker's equation. f is often called the focal length of the lens.

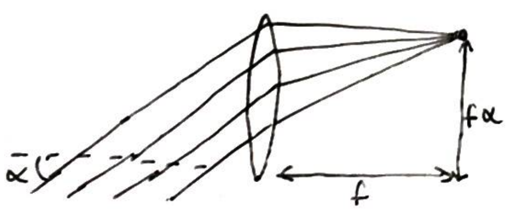
Focal Lengths and Focal Planes

Consider an incoming ray transformed by a thin lens which then propagates a distance f beyond the lens. The matrix for this is given by:

$$\begin{pmatrix} y' \\ \alpha' \end{pmatrix} = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} y \\ \alpha \end{pmatrix}$$

$$= \begin{pmatrix} 0 & f \\ -\frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} y \\ \alpha \end{pmatrix} \Rightarrow y' = f\alpha \quad \alpha' = -\frac{y}{f} + \alpha$$

Note that y' does not depend on y. This means any ray incident at α is refracted to the same point:



All the parallel rays incident at α are refracted to a point $f\alpha$

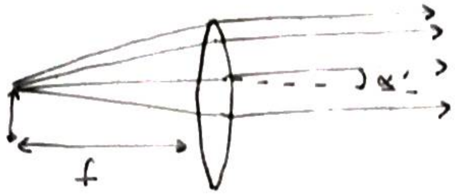
All rays from an object an infinite distance away are focused on a point, the image distance away. The image distance is f and the point on the optical axis a distance f away is called the focal point.

Now consider incoming rays starting out a distance f from the lens:

$$\begin{pmatrix} y' \\ \alpha' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ \alpha \end{pmatrix}$$

$$= \begin{pmatrix} 1 & f \\ -\frac{1}{f} & 0 \end{pmatrix} \begin{pmatrix} y \\ \alpha \end{pmatrix} \Rightarrow y' = y + f\alpha \quad \alpha' = -\frac{y}{f}$$

Here, α' does not depend on α so all the incoming rays are refracted parallel to each other.



where $\alpha' = -\frac{y}{f}$

From the matrix for a spherical mirror:

$\begin{pmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{pmatrix}$, we can get the result for a thin lens if we set:

$\frac{2}{R} = \frac{1}{f} \Rightarrow f = \frac{R}{2}$. So we can use focal lengths for spherical mirrors as well.

Image Formation

To form an image of an object at P, all the rays from a single point P must end up at a single point P'. If we consider some optical element with a matrix of transformation $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we can work out the image forming condition.

The rays must propagate to the element, be transformed by the element, then propagate to a point.

$$\therefore \begin{pmatrix} y' \\ \alpha' \end{pmatrix} = \begin{pmatrix} 1 & s' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ \alpha \end{pmatrix}$$

This gives $y' = (A + Cs')y + (As + B + Css' + Ds')\alpha$

$$\alpha' = Cy + (Cs + D)\alpha$$

For all the outgoing rays to reach a point, the α should not matter since α varies so we would not have an image if it mattered.

So the image forming condition is:

$$\boxed{As + B + Css' + Ds' = 0}$$

Refraction Equation

First consider a spherical refracting surface for which

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{R}\left(\frac{n_1}{n_2}-1\right) & \frac{n_1}{n_2} \end{pmatrix}$$

So the image forming condition is:

$$s + \frac{1}{R}\left(\frac{n_1}{n_2}-1\right)ss' + \frac{n_1}{n_2}s' = 0$$

Divide through by ss' and multiply by n_2

$$\frac{n_2}{s'} + \frac{n_1-n_2}{R} + \frac{n_1}{s} = 0$$

$$\therefore \boxed{\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2-n_1}{R}}$$

This is the refraction equation for a spherical refracting surface.

Using this result, we can simplify:

$$\begin{pmatrix} 1 & s' \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$ to give:

$$\begin{pmatrix} A+Cs' & As+B+Css'+Ds' \\ C & Cs+D \end{pmatrix} = \begin{pmatrix} -\frac{n_1}{n_2}\frac{s'}{s} & 0 \\ \frac{1}{R}\left(\frac{n_1}{n_2}-1\right) & -\frac{s}{s'} \end{pmatrix}$$

so if $$\begin{pmatrix} y' \\ \alpha' \end{pmatrix} = \begin{pmatrix} -\frac{n_1}{n_2}\frac{s'}{s} & 0 \\ \frac{1}{R}\left(\frac{n_1}{n_2}-1\right) & -\frac{s}{s'} \end{pmatrix}\begin{pmatrix} y \\ \alpha \end{pmatrix}$$

we get $$\underline{\underline{y' = -\frac{n_1 s'}{n_2 s}y}}$$

where s' is image distance and s is object distance

This also gives lateral magnification since y and y' are object height and image height respectively.

$$\underline{\text{Magnification} = \frac{y'}{y} = -\frac{n_1}{n_2}\frac{s'}{s}}$$

Gaussian Lens Equation

We will now attempt to work out a refraction equation for a thin lens.

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

so the image forming condition, given by $As + B + Css' + Ds' = 0$ is:

$s - \frac{1}{f} ss' + s' = 0$ dividing through by ss', we get:

$$\boxed{\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}}$$

This is a very important result known as the Gaussian Lens Equation.

This also works for spherical mirrors if f is set to $\frac{R}{2}$ as previously discussed.

Using this result, we can simplify $\begin{pmatrix} 1 & s' \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ to give:

$$\begin{pmatrix} A + Cs' & As + B + Css' + Ds' \\ C & Cs + D \end{pmatrix} = \begin{pmatrix} 1 - \frac{s'}{f} & 0 \\ -\frac{1}{f} & 1 - \frac{s}{f} \end{pmatrix} = \begin{pmatrix} -\frac{s'}{s} & 0 \\ -\frac{1}{f} & -\frac{s}{s'} \end{pmatrix}$$

so if $\begin{pmatrix} y' \\ \alpha' \end{pmatrix} = \begin{pmatrix} -\frac{s'}{s} & 0 \\ -\frac{1}{f} & -\frac{s'}{s} \end{pmatrix}\begin{pmatrix} y \\ \alpha \end{pmatrix}$

then $\underline{\underline{y' = -\frac{s'}{s} y}}$ where s' is image distance and s is object distance.

This gives the lateral magnification to be:

$$\underline{\text{Magnification} = \frac{y'}{y} = -\frac{s'}{s}}$$

Note that s is defined as distance to optical element and s' is distance past optical element so s is positive if 'before' the element and s' is negative if 'before' the element.

Human Eye

A Normal human eye focuses incoming parallel rays on the retina. The ciliary muscles can work to change the focal length of the eye; this is called accommodation. For relaxed viewing, the ciliary muscles do no work and the object is at infinity, called the far point of the eye:

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \Rightarrow \frac{1}{s'} = \frac{1}{f} - \frac{1}{s}$$

As $s \to \infty$, $\frac{1}{s'} = \frac{1}{f}$ and no work needs to be done by the ciliary muscles

As $\to 0$, $\frac{1}{f} \to \infty$ so the ciliary muscles need to work harder and harder.

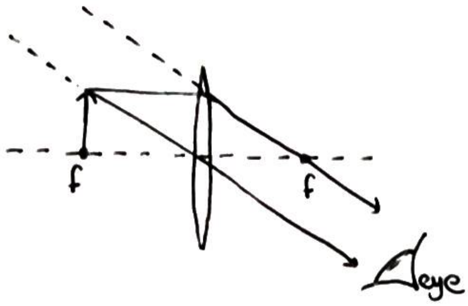
Obviously, we can't see objects that are too close. The near point of the eye is 25cm.

Magnifiers

Previously, we used lateral magnification but now we will use angular magnification M_{ang} defined as the angle subtended at eye by an image divided by the angle subtended at eye by the object.

$$M_{ang} = \frac{\alpha'}{\alpha}$$

For a magnifier, first consider a single thin lens. Imagine an object placed just inside the focal length:



Here, the object is placed just inside the focal length of a thin lens so the virtual image formed is at infinity.

If the object has height h, then the angle subtended at eye by the image is:

$$\boxed{\theta_M = \frac{h}{f}}$$

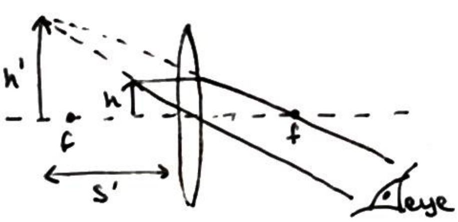
Since this is approximately equal to the angle the rays make on entering the eye.

Comparing this to the angle subtended at eye by the object when viewed at near point distance is:

$$\boxed{\theta_e = \frac{h}{25cm}}$$

This gives an angular magnification of $\boxed{M_{ang} = \frac{25cm}{f}}$

We can make this magnification even better if we allow the eye to do a little work. Let's make the image not at infinity but at the nearpoint of the eye. Put the eye very close to the lens in this case:



Using the gaussian lens equation

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \Rightarrow s = \frac{s'-f}{s'f}$$

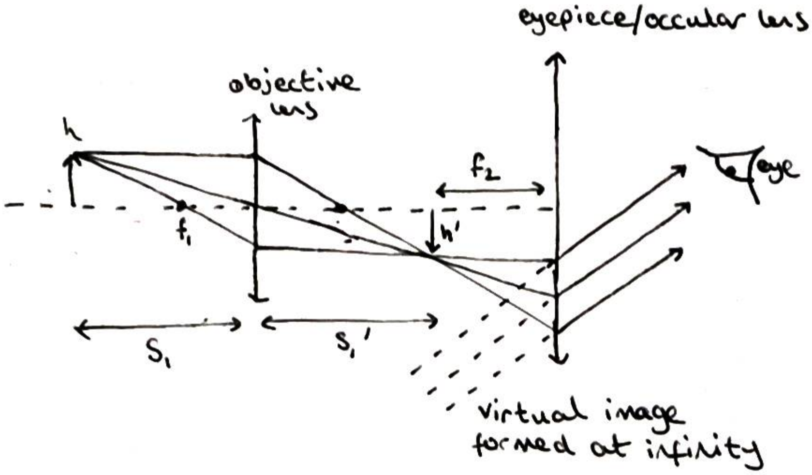
For s' to be 25cm away, $s' = -25cm$

$$\therefore \quad s = \frac{\;\frac{-25cm - f}{25f}\;}{}$$

So $\theta_M = \dfrac{h'}{|s'|} \overset{\text{from congruent triangles}}{=} \dfrac{h}{s} = h\left(\dfrac{1}{f} - \dfrac{1}{s'}\right) = h\left(\dfrac{1}{f} + \dfrac{1}{25cm}\right)$

$\theta_e = \dfrac{h}{25cm}$ as before so $\boxed{M_{ang} = \dfrac{25cm}{f} + 1}$ which is better than before!

Microscopes



This is a diagram of a microscope. The object is placed close to the focal length of the objective lens which forms a real inverted image in the barrel of the microscope. This forms a virtual enlarged image at infinity.

So what is the angular magnification of a microscope?

If the virtual image is very far away, the angle it subtends at the eye is given by $\frac{|h'|}{f_2}$, just like with a magnifier.

The lateral magnification of the objective is $\frac{h'}{h} = -\frac{S_1'}{S_1} \approx \frac{-S_1'}{f_1}$

since the object is close to the focal point

$\therefore \; h' = -\frac{S_1}{f_1} h$

The unaided eye has angular size $\theta_e = \frac{h}{25}$

The virtual image has angular size $\theta_M = \frac{|h'|}{f_2}$

So the angular magnification of the microscope is:

$$M_{ang} = \frac{|h'|/f_2}{h/25\text{cm}} = \frac{25\text{cm}}{f_2}\frac{|h'|}{h} \quad \text{but } h' = -\frac{S_1}{f_1} h$$

$$\therefore \boxed{M_{ang} = \frac{25\text{cm}\; S_1'}{f_1 f_2}}$$

Matrix Description of a Microscope

For the microscope, the object is placed just outside the focal point, such that

$S = f_1 + \varepsilon$ (which we approximated to $S = f_1$)
where ε is very small.

Using Gaussian Lens equation: $\frac{1}{S} + \frac{1}{S_1'} = \frac{1}{f_1} \Rightarrow \frac{1}{S_1'} = \frac{1}{f_1} - \frac{1}{S}$

$$\frac{1}{S_1'} = \frac{1}{f_1} - \frac{1}{f_1+\varepsilon} = \frac{\varepsilon}{f_1(f_1+\varepsilon)} \quad \therefore \; S_1' = \frac{f_1^2+\varepsilon}{\varepsilon} \approx \frac{f_1^2}{\varepsilon}$$

which is indeed larger than f_1.

The real image is just inside the focal length of the eyepiece. The barrel length of the microscope is therefore given by $L = S_1' + f_2$. The virtual image is formed at infinity such that $|S'| \to \infty$.

We write the image forming condition $AS + B + CSS' + DS' = 0$ as:

$$\frac{AS+B}{S'} + CS + D = 0$$

As $S' \to \infty$ $\quad CS + D = 0$

If
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{L}{f_1} & L \\ \frac{1}{f_1 f_2}(L-f_1-f_2) & 1-\frac{L}{f_2} \end{pmatrix}$$

then
$$CS + D = \frac{S}{f_1 f_2}(L-f_1-f_2) + 1 - \frac{L}{f_2} = \frac{f_1+\varepsilon}{f_1 f_2}(S_1'-f_1) + 1 - \frac{S_1'+f_2}{f_2}$$

since $S = f_1 + \varepsilon$ and $L = S_1' + f_2$

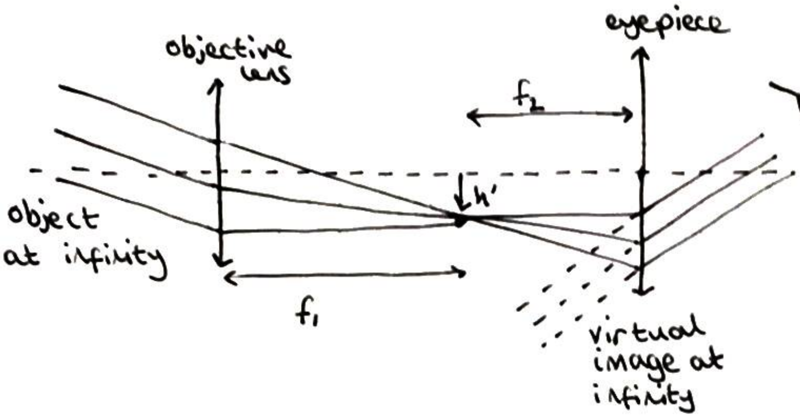
$$= \frac{f_1+\varepsilon}{f_1 f_2}\left(\frac{f_1^2}{\varepsilon}\right) - \frac{f_1(f_1+\varepsilon)}{\varepsilon f_2} \quad \text{since } S_1' = \frac{f_1^2}{\varepsilon}$$

$= \underline{\underline{0}}$ So for a ray $(h\ \alpha)$, $\alpha' = Ch + (CS+D)\alpha$

$$\alpha' = \frac{h}{f_1 f_2}(L-f_1-f_2) = \frac{h}{f_1 f_2}(S_1'-f_1) \approx \frac{hS_1'}{f_1 f_2}$$

So $M_{ang} = \frac{\alpha'}{\alpha} = \frac{\alpha'}{h/25\text{cm}} = \frac{25\text{cm}\, S_1'}{f_1 f_2}$ so the matrix method works!

Telescopes



This is similar to the microscope but the object is very far away. This makes a smaller real image in the barrel of the telescope. This forms a virtual image at infinity.

So what is the angular magnification of the telescope?

The angle subtended by the image is $\frac{h'}{f_2}$ as it is for magnifiers.

The angle subtended by the object is $\frac{-h'}{f_1}$, using the same principle as the magnifier

So the angular magnification of the telescope is given by:

$$M_{ang} = \frac{h'/f_2}{-h'/f_1} \quad \therefore \quad \boxed{M_{ang} = \frac{-f_1}{f_2}}$$

Matrix Description of Telescope

Both the object and virtual image are at infinity $S \to \infty$ and $|S'| \to \infty$

The barrel length of the telescope is $L = f_1 + f_2$

The image forming condition is divided through by SS':

$$\frac{A}{S'} + \frac{B}{SS'} + C + \frac{D}{S} = 0 \qquad \text{As } S \to \infty \text{ and } |S'| \to \infty, \quad C = 0$$

If

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{f_1} & L \\ \frac{1}{f_1 f_2}(L - f_1 - f_2) & 1 - \frac{L}{f_2} \end{pmatrix}$$

then $\frac{1}{f_1 f_2}(L - f_1 - f_2) = 0$ This is true when $L = f_1 + f_2$. For a ray (h, α), $\alpha' = Ch + (Cs + D)\alpha$ so $\alpha' = D\alpha$

So $M_{ang} = \frac{\alpha'}{\alpha} = D = 1 - \frac{L}{f_2} = -\frac{f_1}{f_2}$ So the matrix method works!