Wave Phenomena

Now we move on from geometric optics (where we treated the light as a ray) to physical optics, where we need to take into account the wave nature of light. This means looking at all sorts of cool things like when two or more waves overlap. This causes a pattern of interference that depends on the relative phases and amplitudes of the overlapping waves.

Interference and Coherent Sources

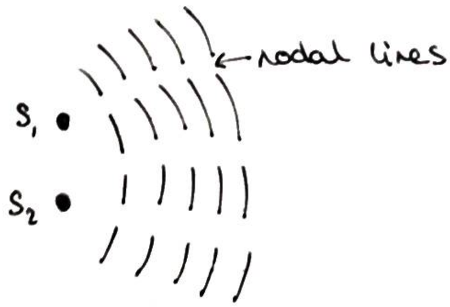
We will really only be looking at interference effects for single-frequency sinusoidal waves. This means monochromatic light. From Chapter 1: Wave Basics, we know that for interference, we just use the principle of superposition and add the wave equations for the light.

To see the interference effects clearly, we need to also consider another property: coherence. Coherent light sources have relative phases that are constant in time. i.e the wave function should look like: $\cos(kx - \omega t + \phi)$ where ϕ, the phase, is a constant and not a function of time.

If the waves are transverse (which light is), to see interference patterns we also need the polarisation to be the same.

Two Source Interference

Consider two sources S_1 and S_2. These produce an interference pattern that looks like:



There are clear "nodal lines" where there is complete destructive interference and there are clear points of maxima. So how do we find where these points are? Clearly it depends on the difference in path length from the source to the point.

Let P_1 and P_2 be the path lengths from S_1 and S_2 to some point r. Recall that $P = \int n \, dr$

If we want points with constructive interference:

$$\boxed{\Delta P = m\lambda \quad \text{where } m = 1, 2 \ldots}$$

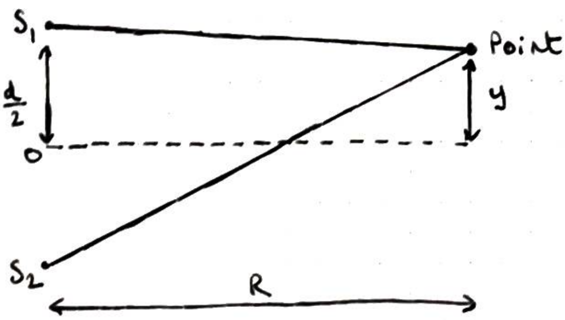
we want the path length to be an integer multiple of λ

If we want points with destructive interference:

$$\boxed{\Delta P = (m + \tfrac{1}{2})\lambda \quad \text{where } m = 1, 2 \ldots}$$

We want the path length to be $\frac{1}{2}$ integer multiple of λ

Consider placing the sources at $(0, \pm \frac{d}{2})$ and look at a point (R, y): Take $n = 1$



so the coordinates are:

$S_1 \; (0, \frac{d}{2})$

$S_2 \; (0, -\frac{d}{2})$

$I \; (R, y)$

So how do we find ΔP?

we can use vector notation:

$S_1 \rightarrow$ Point is $\begin{pmatrix} R \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{d}{2} \end{pmatrix}$

$S_2 \rightarrow$ Point is $\begin{pmatrix} R \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ -\frac{d}{2} \end{pmatrix}$

$$\Delta P = \left| \begin{pmatrix} R \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ -d/2 \end{pmatrix} \right| - \left| \begin{pmatrix} R \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ d/2 \end{pmatrix} \right|$$

$$= \left[R^2 + (y + d/2)^2 \right]^{1/2} - \left[R^2 + (y - d/2)^2 \right]^{1/2}$$

$$= R \left[1 + \left(\frac{y + d/2}{R} \right)^2 \right]^{1/2} - R \left[1 + \left(\frac{y - d/2}{R} \right)^2 \right]^{1/2}$$

Doing the Binomial expansion on this and taking the first few terms:

$$\Delta P \approx \frac{1}{2R} \left[\left(y^2 + yd + \frac{d^2}{4} \right) - \left(y^2 - yd + \frac{d^2}{4} \right) \right]$$

$$\boxed{\Delta P = \frac{yd}{R}}$$

This is the equation for Path Difference for $n = 1$

So the brightest points are found where:

$$\boxed{y = \frac{m \lambda R}{d} \quad \text{where } m = 1, 2, 3 \ldots}$$

The nodal points are found where:

$$\boxed{y = \left(m + \frac{1}{2} \right) \frac{\lambda R}{d} \quad \text{where } m = 1, 2 \ldots}$$

This is Young's Double Slit experiment! Young used a single slit to make light from a source coherent before putting it through a double slit. This is a geometric effect.

So how do we work out the intensity at the point?

At the point, we are interested in the sum of the two waves:

$$E\cos(kr_1 - \omega t) + E\cos(kr_2 - \omega t)$$

where r_1 and r_2 are the distances from the source to the point. $k = \frac{2\pi}{\lambda}$. As you can see, there isn't a ϕ term here, so we need to put it in.

$\theta = kr_2 - \omega t$ and $\phi = \frac{ykd}{R}$ so we get:

$$E\cos(\theta + \phi) + E\cos\theta.$$

Now using addition identity $\cos A + \cos B = 2\cos\frac{A-B}{2}\cos\frac{A+B}{2}$

$$= \underbrace{2E\cos\left(\frac{\phi}{2}\right)}_{\text{amplitude}} \underbrace{\cos\left(\theta + \frac{\phi}{2}\right)}_{\text{time dependence}}$$

the average of the time dependence, which for a cosine $= \frac{1}{2}$

Intensity is given by amplitude^2 x ⟨time dependence⟩, and is defined as average power per unit area

$\therefore$ Intensity $I(y) = I_0\cos^2\left(\frac{\phi}{2}\right) = I_0\cos^2\left(\frac{y\pi d}{\lambda R}\right)$

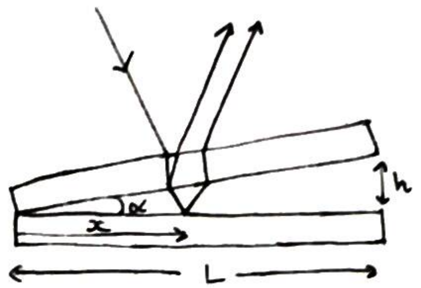
$$\boxed{I = I_0\cos^2\left(\frac{y\pi d}{\lambda R}\right)}$$

Notice that this is a maximum for $\frac{y\pi d}{\lambda R} = m\pi$ where $m = 1, 2 \ldots$

This gives us the familiar $y = \frac{m\lambda R}{d}$ for maxima.

Thin Film Interference

Interference effects can also be seen using a thin wedge-shaped film or a thin wedge of air.



The diagram shows two flat glass plates of length L separated by a thin wedge of air. The angle between the plates is α and is very small.

Some of the incident ray is reflected back by the first glass plate. The rest is reflected back by the second glass plate. This introduces a difference in Path Length between the outgoing rays, ΔP. The path length difference is 2 times the air gap, since this is the extra length one ray travels.
Taking $\sin\alpha \approx \alpha$ since α is small, at some distance x:

$$\boxed{\Delta P = 2x\alpha}$$

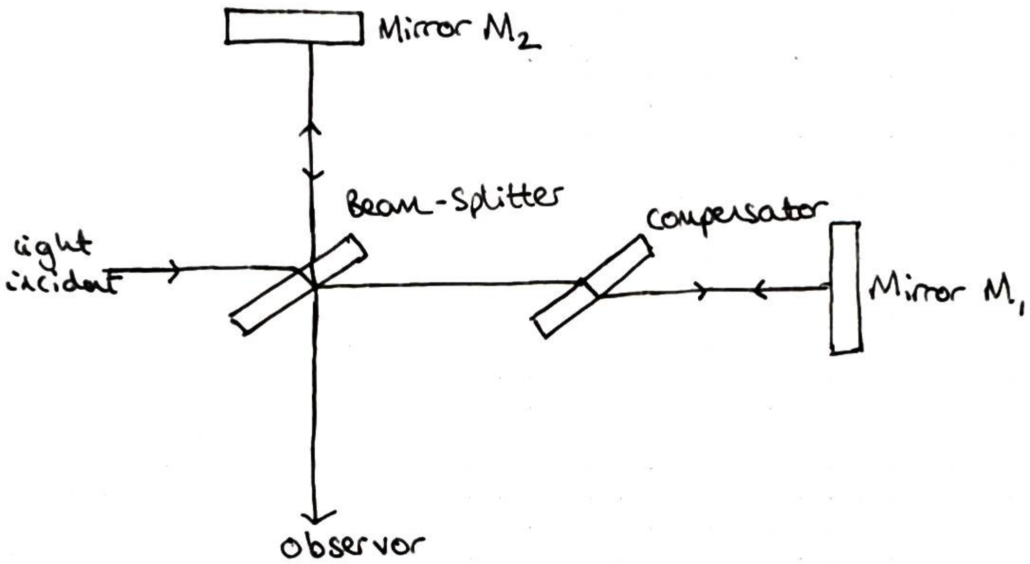
It is <u>not</u> true that the bright fringes are found when $\Delta P = m\lambda$ as with previous interference patterns.

> Reflection at an air-to-glass boundary introduces a phase shift of π. There is no change for glass-to-air.

Including this extra shift, the <u>bright</u> fringes are found:

$\boxed{\Delta P = 2x\alpha = (m+\tfrac{1}{2})\lambda}$ giving fringe separation $\boxed{\Delta x = \dfrac{\lambda}{2\alpha}}$

Michelson - Interferometer



This is a very important device for observing interference patterns and was famously used in the Michelson-Morley experiment that disproved the existence of the Ether.

The beam splitter is a plate of glass with a thin silver coating on the back. It splits the light into 2 rays that bounce off mirrors and return to an observor as one ray, allowing for the observation of an interference pattern. The compensator ensures that the two split beams traverse the same amount of glass before recombining.

M_2 can be moved to observe different patterns of fringes. If m fringes cross a fixed position as M_2 is moved, M_2 will have moved a distance:

$$\boxed{y = m\frac{\lambda}{2}}$$

This allows you to determine λ, by counting m and measuring y.

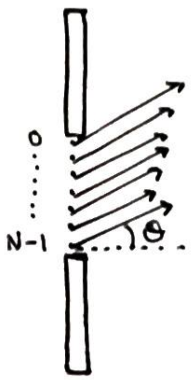
Diffraction

Diffraction is interference from many soures. An example of this is in the double-slit experiment. Although we saw that as interference, but we could easily have seen this as diffraction. This is because a slit will have a finite width and we could view the outgoing light the way Huygen would have: as infinitely many sources spread out over the slit.

There are two important types of diffraction. Consider a source, a single slit, and a screen. If both the source and the screen are close to the slit, we have Fresnel diffraction (near-field diffraction). We have to consider the angles the light is entering and leaving the slit.

However, if the source and screen are far from the slit, we can say that all the rays in and out of the slit are parallel to each other. This is Fraunhofer diffraction (far-field diffraction). This really simplifies analysis so we will only really be looking at this kind of diffraction.

So consider light entering a single slit of width a. As the light leaves the slit, we want to think of the light as infinite sources. To do this, we can divide the slit into N strips, each strip having a thickness of $\frac{a}{N}$, and say a source is in the centre of each strip.



The sources are labelled from 0 to $N-1$ and the light from the sources is emitted at an angle θ. Each source is spaced $\frac{a}{N}$ from its neighbour.

If we say the total amplitude of light emitted from the slit is E_0, then each source has amplitude $\underline{\underline{\frac{E_0}{N}}}$

The path difference between neighbouring sources is:

$$\underline{\underline{\Delta P = \frac{a}{N}\sin\theta}}$$

To find the phase shift γ from this, we multiply by $\frac{2\pi}{\lambda} = k$.

$$\therefore \boxed{\gamma = \frac{1}{N}\frac{2\pi}{\lambda}a\sin\theta} = \frac{\beta}{N} \qquad \text{where } \beta = \frac{2\pi}{\lambda}a\sin\theta$$

If the relative phase of source 0 is $\phi = kr - \omega t + \phi_0$ at a distance r and time t, the combined disturbance $E(\theta)$ is given by:

$$E(\theta) = \underbrace{\frac{E_0}{N}\cos\phi}_{\text{source } 0} + \underbrace{\frac{E_0}{N}\cos(\phi+\gamma)}_{\text{source } 1} + \underbrace{\frac{E_0}{N}\cos(\phi+2\gamma)}_{\text{source } 2} + \dots$$

$$\therefore E(\theta) = \sum_{n=0}^{N-1}\frac{E_0}{N}\cos(\phi+n\gamma)$$

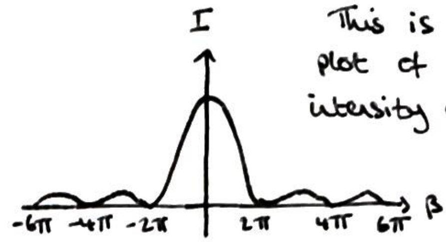
After some complicated maths, we arrive at:

$$\boxed{E(\theta) = E_0\frac{\sin(\beta/2)}{(\beta/2)}\cos(\phi+\beta/2)}$$

The non-cosine terms are the amplitude of the disturbance and the cosine term is the time-dependence

This gives an intensity of :

$$\boxed{I(\theta) = I_0 \left(\frac{\sin(\beta/2)}{(\beta/2} \right)^2}$$



This is the plot of the intensity equation

The intensity will be 0 whenever $\sin\left(\frac{\beta}{2}\right) = 0$ which is whenever $\frac{\beta}{2} = m\pi$

$$\therefore \quad \frac{1}{2}\left(\frac{2\pi}{\lambda} a\sin\theta\right) = m\pi \quad \Rightarrow \quad \boxed{\sin\theta = \frac{m\lambda}{a}} \quad \text{for } m = \pm 1, \pm 2 \ldots$$

This is where there will be 0 intensity.
The maxima will be between these points, i.e when m is a $\frac{1}{2}$ integer.

Rayleigh's Criterion

So the first 0 in the intensity pattern of light diffracted by a single slit of width a occurs when $\sin\theta = \frac{\lambda}{a}$.
With optical instruments like telescopes or cameras, there isn't a slit but an aperture of diameter D. The first 0 for these occur when $\sin\theta = 1.22\frac{\lambda}{D}$.

For two distant point-like sources with small angular separation $\delta\theta$, the sources will be resolvable (distinguishable from each other) through an optical instrument when

$$\boxed{\delta\theta = 1.22\frac{\lambda}{D}}$$

we have used small angle approximation here.

This is the resolving power of the telescope.

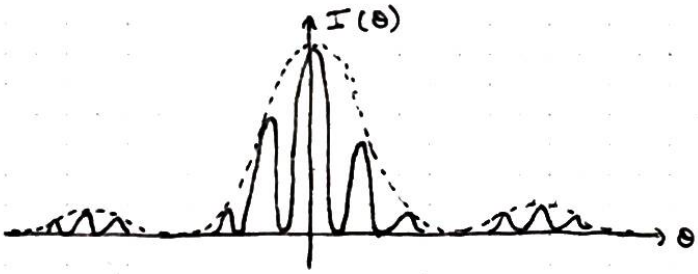
Two Slits of finite Width

When we looked at double-slits previously, we assumed the slits would have infinitesimal width, and therefore neglected it. However, this is not practically achievable. We will now consider two slits with a finite thickness a, spaced a distance d apart such that $a < d$. This yields an intensity distribution:

$$\boxed{I(\theta) = I_0 \cos^2\left(\frac{\phi}{2}\right)\left(\frac{\sin(\beta/2)}{(\beta/2)}\right)^2}$$

where $\phi = \frac{2\pi}{\lambda} d\sin\theta$ and $\beta = \frac{2\pi}{\lambda} a\sin\theta$

This gives an intensity curve that looks like:



As opposed to the uniform sinusoidal wave

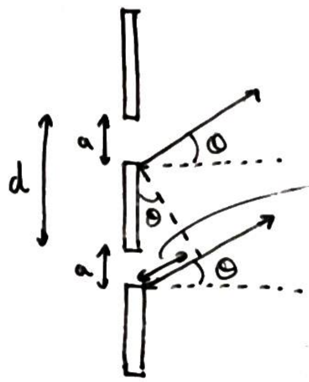
This can be shown by starting with the expression for amplitude from a single slit of thickness d:

$$E(\theta) = E_0 \frac{\sin(\beta/2)}{(\beta/2)} \cos\left(\phi + \frac{\beta}{2}\right)$$

where ϕ here is the phase from the contribution to the amplitude coming from one edge of the slit.

For 2 slits, we add two of these expressions together with phase contributions of ϕ_1 and ϕ_2 from each slit.

But what is $\phi_1 + \phi_2$ or $\phi_1 - \phi_2$? $\phi_1 - \phi_2$ will be the phase difference of the contribution from the two slits.



$\Delta P = d\sin\theta$

Phase difference = k × path difference

$$= \frac{2\pi}{\lambda}\,\Delta P$$

$$\phi = \phi_2 - \phi_1 = \frac{2\pi}{\lambda}\, d\sin\theta$$

So adding the expressions for amplitude of each slit gives:

$$E(\theta) = E_0 \frac{\sin(\beta/2)}{(\beta/2)}\left[\cos\left(\phi_1 + \frac{\beta}{2}\right) + \cos\left(\phi_2 + \frac{\beta}{2}\right)\right]$$

Using $\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$:

$$E(\theta) = E_0 \frac{\sin(\beta/2)}{(\beta/2)}\left[2\cos\left(\frac{\phi_1+\phi_2}{2} + \frac{\beta}{2}\right)\cos\left(\frac{\phi}{2}\right)\right]$$

$$= E_0 \frac{\sin(\beta/2)}{(\beta/2)}\left[2\cos\left(\phi_1 + \frac{\beta}{2} + \phi\right)\cos\left(\frac{\phi}{2}\right)\right]$$

$$= \underbrace{2E_0 \frac{\sin(\beta/2)}{(\beta/2)}\cos\left(\frac{\phi}{2}\right)}_{\text{Amplitude}}\ \underbrace{\left[\cos\left(\phi_1 + \frac{\beta}{2} + \phi\right)\right]}_{\text{time dependence}}$$

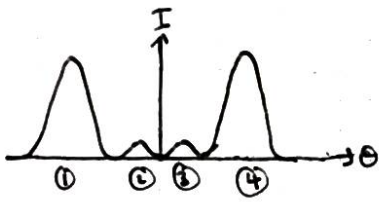
For intensity, we square the amplitude and multiply by the time average of the time dependence.

This gives us the intensity equation we stated previously.

Multiple Narrow Slits

Now let's go back to considering the slits to be infinitely thin. We saw what happens with a single thin slit but what about with many thin slits?

The interference pattern will change with increasing slit numbers and the number of slits can be found by counting the number of peaks (largest peak to largest peak):



so this interference pattern is produced by 4 slits.

If the slit separation is d, then light from adjacent slits will always be in phase when $d\sin\theta = m\lambda$. So there will be lage peaks in the pattern at angles given by:

$$\boxed{\sin\theta = \frac{m\lambda}{d}} \quad \text{for } m = 1, 2, 3 \ldots$$

These are the positions of the largest peaks. These peaks stay in the same place but have more and more smaller peaks in between as the number of slits is increased.

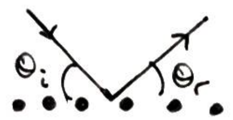
The intensity for N slits is given by:

$$\boxed{I(\theta) = I_0 \left(\frac{\sin(N\phi/2)}{\sin(\phi/2)} \right)^2} \quad \text{where } \phi = \frac{2\pi}{\lambda} d\sin\theta$$

As the number of slits gets larger and larger, we reach the case of a diffraction grating.

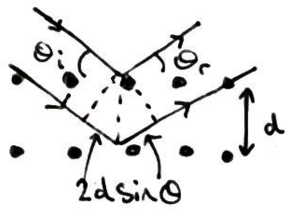
X-ray diffraction

X-rays can be scattered from an array of atoms in a solid, giving a diffraction pattern on a screen. This is called Bragg diffraction.



Consider a single plane of atoms. The rays reflected from the plane of atoms will be in phase if the path lengths differ by an integer number of wavelengths.

This is definitely true when $\theta_i = \theta_r$. So if the angle of incidence is equal to the angle of reflection, for some angle θ, there will be "strong scattering".



Now consider adjacent planes of atoms. We need the path length difference between the rays coming from the two planes to be some integer number of λ.

It is clear that the rays being scattered by the second plane have travelled an extra $2d\sin\theta$ ($d\sin\theta$ incident and $d\sin\theta$ reflected). So, $2d\sin\theta = m\lambda$.

These are the Bragg Conditions:

$$\boxed{\begin{gathered}\theta_i = \theta_r = \theta \\ 2d\sin\theta = m\lambda\end{gathered}}$$

These two conditions lead to a maxima in intensity on the screen.