Wave Basics

Waves are a disturbance of a medium that transfer energy. Waves can have a wide variety of forms but sinusoidal waves are the most general and the type that we will focus on in this course.

These waves can be characterised by the general wave function for a sinusoidal wave:

$$\boxed{\begin{aligned}\psi(x,t) &= A\cos(kx - \omega t + \phi)\\ &= A\cos(k(x - vt) + \phi)\end{aligned}}$$

where A is amplitude, x is displacement, t is time, ϕ is phase shift, k is the wavenumber given by $k = \frac{2\pi}{\lambda}$, and v is the phase velocity given by $v = \frac{\omega}{k}$

Note here that a function of $(x - vt)$ denotes a wave going in the positive x direction and a function of $(x + vt)$ denotes a wave going in the negative x direction.

The reason we have "kx" and "ωt" and such is that the argument of the cosine has to be dimensionless so we have these multipliers to correct the dimensions. So how do we know what the value of k actually is?
We can consider the wave function:

Since a sinusoidal wave is by definition periodic, repeating itself every wavelength λ, the wave function of x is identical to that of $(x+\lambda)$. Also, since the wave is a cosine function, adding a phase shift of 2π should also give an identical function.

From $A\cos[k(x-vt)] = A\cos[kx - kvt]$

$$A\cos[k(x-vt)+2\pi] = A\cos[k(x+\lambda) - kvt]$$

$$A\cos[kx - kvt + 2\pi] = A\cos[kx - kvt + k\lambda]$$

Comparing the arguments of the cosines, it is clear that $k\lambda = 2\pi$ $\quad\therefore\quad$ $\underline{\underline{k = \frac{2\pi}{\lambda}}}$

Similar arguments with the periodicity of the wave in time can be made: $\psi(x,t) = \psi(x, t+T)$ to give $T = \frac{\lambda}{v}$

Since $T = \frac{1}{f}$, this gives $\underline{\underline{v = f\lambda}}$

Wave Equation in 1D

The wave equation for simple systems is:

$$\boxed{\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}}$$

This means any arbitrary pulse of some shape $f(x)$ will satisfy this equation.

The wavefunction $\psi(x,t) = f(x \pm vt)$ then describes this pulse moving with velocity $\pm v$. For sinusoidal waves, this also gives us the $\underline{\underline{\omega = kv}}$ relation for "non-dispersive" systems. For more complicated "dispersive" systems, we instead get some function $\omega = \omega(k)$.

Superposition

The wave equation is linear in ψ, meaning that if $\psi_1(x,t)$ and $\psi_2(x,t)$ are any two solutions of the wave equation, then:
$\psi_1(x,t) + \psi_2(x,t)$ is also a solution of the equation.

In other words, the addition of two waves gives another wave. This is the principle of superposition: If two wave pulses overlap, you simply add the wave functions to find the resultant pulse. Two pulse moving in opposite directions can therefore pass through each other and emerge with their shapes undisturbed.

Beats

A good example of superposition is beats. This is two sinusoidal waves travelling in the same direction with the same amplitude, and almost equal wave numbers and frequencies.
Using the "sum to product" identity:
$\cos A + \cos B = 2\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$, we get

$$\psi(x,t) = 2A\cos\left(\frac{k_1-k_2}{2}x - \frac{\omega_1-\omega_2}{2}t\right)\cos\left(\frac{k_1+k_2}{2}x - \frac{\omega_1+\omega_2}{2}t\right)$$

The first cosine term can be considered as a slowly varying amplitude with dependence on x and t.
Squaring this amplitude and using: $\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$
we get a "beat frequency". This is the sound intensity of the sound wave:

$$\boxed{\text{beat frequency} = |\omega_1 - \omega_2|}$$

Boundary Conditions

In real systems, waves will often strike a boundary of some sort, for example a change in mediums or a rigid surface. At an interface between two mediums, there will typically be some reflection and transmission of the wave. In more than one dimension, the direction may also change (refraction). If a wave moves to a different medium, its phase speed may also change.

Rigid Boundaries - This means the wave cannot propagate at all beyond this point. So the amplitude must be 0 at and beyond the boundary. The wave hits the boundary and is reflected with a change of sign. Usual superposition principle applies. Therefore, at the boundary and beyond, $\psi(x,t) = 0$.

Free End - if we thought of a rigid boundary as fixing the end of the string the wave propagates through to a wall, then we can think of a free end as fixing the end of a string to a rod that is free to move perpendicular to the string length. The wave hits the boundary and is reflected without a change in sign. Since the rod is only free to move perpendicularly (so not in x), we can say that at the boundary $\frac{\partial \psi}{\partial x} = 0$

Standing Waves

Consider two sinusoidal waves with identical amplitude, frequency and wave number but travelling in opposite directions :

$$\psi(x,t) = A\cos[\underbrace{kx - \omega t}_{\text{travelling} \rightarrow}] - A\cos[\underbrace{kx + \omega t}_{\text{travelling} \leftarrow}]$$

Using sum to product identities:

$\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$, we get :

$$\psi(x,t) = 2\sin(kx)\sin(\omega t)$$

Although this is a perfectly good wave function, this is no longer a function of $\psi(x \pm vt)$. This means that this is a stationary waveform, i.e the wave has no phase velocity. Also notice that this means at certain x points, the amplitude will be 0. These points are called nodes. Nodes don't oscillate and are stationary. Antinodes oscillate between maximum amplitudes.

Since sine functions are 0 at multiples of π, we know $kx = n\pi$ where n is an integer. This gives us the node points.

$$\boxed{x_{node} = \frac{n\pi}{k}}$$

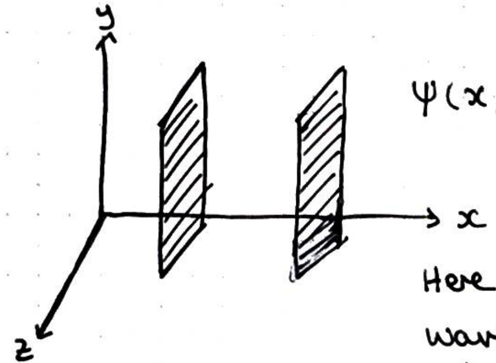
The distance between each node are $\frac{\pi}{k}$ which is $\frac{\lambda}{2}$, half the wavelength.

If boundaries are set to $x=0$ and $x=L$, we know $x=L$ is a node (from rigid boundary conditions). We can therefore find "allowed" wave numbers and frequencies that produce standing waves: $k_n = \frac{n\pi}{L}$, $\lambda_n = \frac{2L}{n}$, $f_n = \frac{nv}{2L}$

Sinusoidal Waves in 3D

The wave function in 3D can be generalised to:
$\psi(\underline{r}, t) = A\cos(\underline{k}\cdot\underline{r} - \omega t)$

This is now describing a plane wave. A simplified version with no y and z dependence is shown:



$\psi(x, y, z, t) = A\cos(k_x x - \omega t)$

wavefronts are surfaces of constant phase.

Here, $\underline{k} = (k_x, 0, 0)$ and the wave is propagating in the x direction only.

Plane waves are one class of solution to the 3-Dimensional wave equation:

$$\boxed{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}}$$

where $|\underline{k}|^2 = k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{v^2}$ so $v = \frac{\omega}{k}$ is still true.

In an electromagnetic wave, there are wave functions for both $\underline{E}$ waves and $\underline{B}$ waves. Maxwell's equations relate the spatial derivatives of $\underline{E}$ to time derivatives of $\underline{B}$ and vice versa. These can then be used to satisfy a wave equation with phase velocity $v = c$, the speed of light.

Spherical Waves

Another class of solution to the 3-D wave equation are spherically symmetric waves. We can thus omit the $\underline{r}$ vector as $\underline{r}$ is the same magnitude in all directions: $\psi(\underline{r}, t) = \psi(r, t)$

These spherically symmetric waves take the form:

$$\boxed{\psi(r,t) = \frac{1}{r} f(kr \pm \omega t)}$$

where f is some arbitrary function. The '$-$' sign represents propagation outward from the origin. A '$+$' sign represents propagation converging on the origin, something that is hard to realise experimentally.

Note the $\frac{1}{r}$ factor corresponds to weakening of the disturbance with distance from the amplitude.