Electrostatics

Let's start with a recap on Coulomb's Law!

$$\boxed{\underline{F} = \frac{1}{4\pi\varepsilon_0} \frac{Qq}{r^2} \hat{\underline{r}}}$$ where $r = |\underline{r}|$

So if we have more than 2 point charges, we can find the force on any 1 of the charges by adding the contribution of the forces due to the other 2 charges.

$$\underline{F} = \sum_i \underline{F}_i = \sum_i \frac{1}{4\pi\varepsilon_0} \frac{q Q_i}{r_i^2} \hat{\underline{r}}_i$$

If we move the charge q around, it will experience different forces at different position. In this way, we can define the electric field $\underline{E}$, a vector field which defines a vector force $\underline{F}$ per unit charge at each point in space:

$$\boxed{\underline{E}(\underline{r}) = \frac{\underline{F}(\underline{r})}{q} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{\underline{r}}}$$

Coulomb's Law for a Charge Density

So above, we were working with test charges. But this isn't how most systems behave. Usually we have to consider a continuous charge distribution.

In these cases we have to think about charge density at each point $\underline{r}'$ given by $\rho(\underline{r}')$

Let's calculate electric field $\underline{E}(\underline{r})$ for a static continuous 3D charge distribution:

Let's first consider the contribution of charge dQ' from a volume element dV' at the point $\underline{R}'$

So what is the field at $\underline{r}$ due to dQ' at $\underline{R}'$?
Well first we define the vector from $\underline{R}'$ to $\underline{r}$:

$$\underline{r}' = \underline{r} - \underline{R}'$$

Field at $\underline{r}$ due to dQ':

$$d\underline{E}(\underline{r}) = \frac{1}{4\pi\varepsilon_0} \frac{dQ'}{r'^2} \hat{\underline{r}}'$$

but we can write:

$$dQ' = \rho(\underline{R}')\, dV'$$

$$d\underline{E}(\underline{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\rho(\underline{R}')}{r'^2} \hat{\underline{r}}'\, dV'$$

so

$$\boxed{\underline{E}(\underline{r}) = \iiint_{\text{volume}} \frac{1}{4\pi\varepsilon_0} \frac{\rho(\underline{R}')}{r'^2} \hat{\underline{r}}'\, dV'}$$

We can use an expression for $\rho(\underline{R}')$ to evaluate this now!

Gauss' Law

You may remember Gauss' Law from last year:
The net flux through a closed surface is equal to the charge enclosed over ε_0:

$$\boxed{\oint \underline{E} \cdot d\underline{A} = \frac{q_{enclosed}}{\varepsilon_0}}$$

We will make this a little more mathematically rigorous.

Let's start with the definition of element flux:

$$\boxed{d\Phi_E = \underline{E}(x,y,z) \cdot d\underline{A}}$$

and Gauss' Law is

$$\boxed{\Phi_E = \frac{Q}{\varepsilon_0}}$$

so

$$\boxed{\iint_{S_{closed}} \underline{E}(x,y,z) \cdot d\underline{A} = \frac{1}{\varepsilon_0} Q}$$

This is the new notation we will follow in this course.

We can now use Gauss Divergence Theorem which you'll remember is:

$$\iiint_{volume} \underline{\nabla} \cdot \underline{v}(x,y,z)\, dV = \iint_{S_{closed}} \underline{v}(x,y,z) \cdot d\underline{A}$$

So we can rewrite Gauss' Law in this form:

$$\boxed{\iint_{S_{closed}} \underline{E}(x,y,z) \cdot d\underline{A} = \iiint_{volume} \underline{\nabla} \cdot \underline{E}(x,y,z)\, dV}$$

we can see from this that:

$$\iiint_{volume} \underline{\nabla} \cdot \underline{E}(x,y,z)\, dV = \frac{1}{\varepsilon_0} Q = \frac{1}{\varepsilon_0} \iiint_{volume} \rho(x,y,z)\, dV$$

$$\therefore \boxed{\underline{\nabla} \cdot \underline{E} = \frac{1}{\varepsilon_0} \rho(x,y,z)}$$

This is really cool!

$$\boxed{\mathrm{div}\, E = \frac{1}{\varepsilon} \rho}$$

This is the differential form of Gauss' Law

Electric Potential

Lets start by remembering the definition of work done:

$$\boxed{dW = \mathbf{F} \cdot d\mathbf{r}} \equiv \boxed{dW = |\mathbf{E}||\mathbf{r}| \cos\theta}$$

since $\mathbf{F} = q\mathbf{E}$ we can also write work as:

$$\boxed{dW = q\,\mathbf{E}(\mathbf{r}) \cdot d\mathbf{r}}$$

We define the electric potential as the work needed to move a test charge from $\mathbf{r}$ to infinity. We obtain an expression for this simply by dividing work by charge:

$$\frac{dW}{q} = \mathbf{E}(\mathbf{r}) \cdot d\mathbf{r} \Rightarrow \boxed{V(\mathbf{r}) = \int_{\mathbf{r}}^{\infty} \mathbf{E}(\mathbf{r}') \cdot d\mathbf{r}'}$$

We can also consider moving a test charge <u>from</u> infinity to that point:

$$\boxed{V(\mathbf{r}) = -\int_{\infty}^{\mathbf{r}} \mathbf{E}(\mathbf{r}') \cdot d\mathbf{r}'}$$

The potential diff between $\mathbf{r}_1$ and $\mathbf{r}_2$ is:

$$V(\mathbf{r}_1) - V(\mathbf{r}_2) = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{E}(\mathbf{r}') \cdot d\mathbf{r}'$$

The potential diff between $\mathbf{r}+d\mathbf{r}$ and $\mathbf{r}$ is:

$$dV = V(\mathbf{r}+d\mathbf{r}) - V(\mathbf{r}) = -\mathbf{E}(\mathbf{r}) \cdot d\mathbf{r}$$

We can also write dV as $dV = \nabla V(\mathbf{r}) \cdot d\mathbf{r}$

equating these

$$\boxed{\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})}$$ This is also cool! $$\boxed{\mathbf{E} = -\text{grad } V}$$

An interesting thing about the Coulomb force is that it is conservative. This means that the potential is the same regardless of the path taken from $\underline{r} \to \infty$.
This implies:

$$\boxed{\oint_C \underline{E}(x,y,z) \cdot d\underline{L} = 0}$$

over a closed loop the potential difference is 0

We can now use Stokes' theorem: $\oint \underline{V}(x,y,z) \cdot d\underline{L} = \iint_{\text{open area}} (\underline{\nabla} \times \underline{V}) \cdot d\underline{A}$

so $$\iint_{\text{open area}} (\underline{\nabla} \times \underline{E}(x,y,z)) \cdot d\underline{A} = 0$$

$$\Rightarrow \boxed{\underline{\nabla} \times \underline{E}(x,y,z) = 0}$$

This is another important result!

$$\boxed{\text{curl } \underline{E} = 0}$$

What does this mean? Well if curl $\underline{E}$ is 0 then it implies there are no electric field whirlpools, i.e electric field lines cannot circulate!

As it turns out, Faraday proved this wrong by experimentally getting field lines to circulate!
This implies there is more to these equations than we understand. Perhaps another aspect to EM waves, perhaps the M part is a clue...