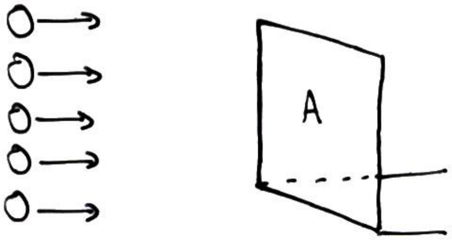
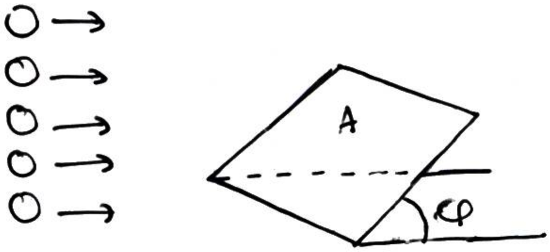
Electric Flux

Consider a continuous barrage of rotten tomatoes!



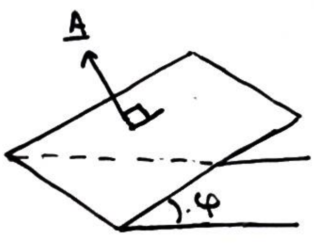
Here, the current density (defined by a vector $\underline{J}$) is defined as the number of tomatoes per metre squared per second.

Note that for an electric field, this is the number of charges per metre squared per second.



Putting the area at an angle means not as many tomatoes will hit the area. So the smaller the φ value the less tomatoes will hit.

From this, we can imagine Area as a vector to allow us to work with the position of the Area. Since the current density is a vector quantity, we know that we need to use a dot product to multiply



we define the Area vector $\underline{A}$ as pointing perpendicular to the plane of the measured area. The angle θ between $\underline{A}$ and the horizontal is

$\theta = 90 + \varphi$

So if the plane of the area is flat on the horizontal $\varphi = 0$ so $\theta = 90$

The area presented to the tomatoes is given by $|\underline{A}|\cos\theta$ so if $\theta = 90$, no area is given to tomatoes

We can therefore compute the number of tomatoes per second as the dot product of the current density and the Area vector. This is the flux, denoted by Φ

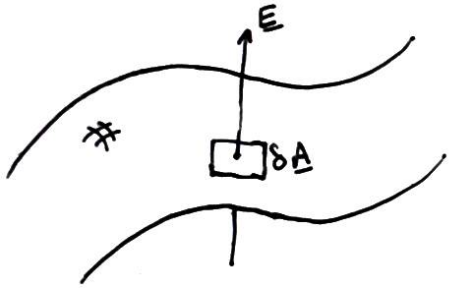
$$\text{Flux} = \underline{J} \cdot \underline{A} = |\underline{J}||\underline{A}| \cos \theta$$

In the case of electric fields, the current density $\underline{J}$ is the number of charges passing through unit area per second. This is the electric field vector. Thus:

$$\boxed{\text{Electric Flux } \Phi = \underline{E} \cdot \underline{A}}$$

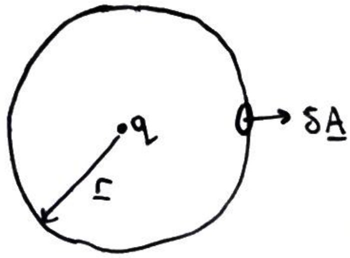
Physically, this is the number of field lines passing through a surface with area $\underline{A}$.

But what do we do if $\underline{A}$ isn't flat (i.e curved) or if $\underline{E}$ varies over $\underline{A}$? We break the surface into little flat pieces with constant $\underline{E}$. We then compute δFlux and add up over the whole area.



$$\boxed{\text{Flux} = \int_{\text{surface}} \underline{E} \cdot dA}$$

Eg. Compute the flux through a spherical surface with a charge q at its centre.



The tiny surface area elements have area $|\delta \underline{A}|$ and all point radially outwards. We can therefore define the tiny area element vector as $|\delta \underline{A}| \hat{\underline{r}}$

Since the charge is at the centre, $|\underline{E}|$ is the same everywhere on the surface and $\underline{E}$ also points radially outward and is therefore parallel to $\delta \underline{A} = |\delta \underline{A}| \hat{\underline{r}}$. We can now compute Flux:

$$\text{Flux} = \int_{\text{Surface}} \underline{E} \cdot d\underline{A}$$

But the $\underline{E}$ is the same everywhere so we can forego the integral sign and just sum over the surface. Also, since $\underline{E}$ and $d\underline{A}$ are parallel, we can write them as a product of magnitudes.

$$\therefore \text{Flux} = \sum_{\text{bits}} |\underline{E}| |\delta \underline{A}|$$

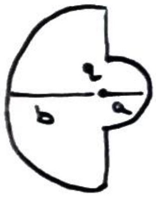
$$= \sum \frac{q}{4\pi \varepsilon_0 |\underline{r}|^2} |\delta \underline{A}| = \frac{q}{4\pi \varepsilon_0 |\underline{r}|^2} \sum |\delta \underline{A}|$$

$$= \frac{q}{4\pi \varepsilon_0 |\underline{r}|^2} \int_0^{4\pi|r|^2} \delta A = \frac{q}{4\pi \varepsilon_0 |\underline{r}|^2} 4\pi |\underline{r}|^2$$

$$= \underline{\underline{\frac{q}{\varepsilon_0}}}$$

Note that this does not depend on $\underline{r}$. It is independent of the radius which implies maybe the flux would be the same through a non spherical surface?

Eg. compute the flux through a surface made of two hemispheres with radii a and b with a charge q in the centre.



The curved sides will have a flux calculated in a similar method as before. The straight sides have $\underline{A}$ perpendicular to $\underline{E}$ so there will be no flux through it.

$$\text{Flux} = \int_{\text{surface}} \underline{E} \cdot d\underline{A} = \frac{q}{4\pi\varepsilon_0 |\underline{a}|^2} \times \frac{1}{2} 4\pi |\underline{a}|^2 + \frac{q}{4\pi\varepsilon_0 |\underline{b}|^2} \times \frac{1}{2} 4\pi |\underline{b}|^2$$

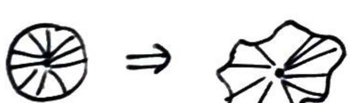
$$= \frac{q}{4\pi\varepsilon_0 |\underline{a}|^2} 2\pi |\underline{a}|^2 + \frac{q}{4\pi\varepsilon_0 |\underline{b}|^2} 2\pi |\underline{b}|^2$$

$$= \frac{q}{2\varepsilon_0} + \frac{q}{2\varepsilon_0} = \frac{q}{\varepsilon_0}$$

This is the same as before, further supporting the idea that the shape of the surface doesn't matter!

Maths Game

Let's play a maths game. Start with a charge q in the centre of a spherical surface. Now deform the surface by moving every point on the surface to a different $\underline{r}$.



(move all points to different r)

Now compute the flux through this weird shape: $\text{Flux} = \int_{\text{surface}} \underline{E} \cdot d\underline{A}$

$$\text{Flux} = \frac{q}{4\pi\varepsilon_0 |\underline{r}|^2} \int_0^{\pi} \int_0^{2\pi} |\underline{r}|^2 \sin\theta \, d\theta \, d\varphi = \frac{q}{\varepsilon_0}$$

So the flux is independent of the shape of the enclosing surface. This is Gauss' Law.