Angular Momenta in Quantum Mechanics

We will begin this section by characterising the properties of the eigenstates of angular momentum. All of the following relations come from the commutation rules for angular momentum.

Ladder Operators: Eigenstates of Angular Momentum

Let's start with the commutator relations of $\hat{J}$ the angular momentum operator:

$$[\hat{J}_x , \hat{J}_y] = i\hbar \hat{J}_z$$

$$[\hat{J}_y , \hat{J}_z] = i\hbar \hat{J}_x$$

$$[\hat{J}_x , \hat{J}_y] = i\hbar \hat{J}_z$$

$$[\hat{J}^2 , \hat{J}_z] = 0$$

so $\hat{J}^2$ and $\hat{J}_z$ pair form a complete set of commuting observables.

We can thus find simultaneous eigenstates for these observables

The eigenvalue of $\hat{J}_z$ is $\hbar m$ where m is a number.

$$\hat{J}_z |m\rangle = \hbar m |m\rangle$$

The eigenstates of $\hat{J}_z$ form an orthonormal set since $\hat{J}_z$ is hermitian

Let's introduce the ladder opperators:

$$\hat{J}_\pm = \hat{J}_x \pm i \hat{J}_y$$

$$\Rightarrow \quad [\hat{J}_z , \hat{J}_\pm] = \pm \hbar \hat{J}_\pm$$

so

$$\hat{J}_z (\hat{J}_\pm |m\rangle) = (\hat{J}_\pm \hat{J}_z + [\hat{J}_z , \hat{J}_\pm]) |m\rangle$$

$$= \hbar (m \pm 1) (\hat{J}_\pm |m\rangle)$$

This looks like an eigenvalue equation, where the eigenstate is $(\hat{J}_\pm |m\rangle)$ with eigenvalue $\hbar(m \pm 1) = \hbar m \pm \hbar$

Since $\hat{J}_z |m\rangle = \hbar m |m\rangle$, the effect of the $\hat{J}_\pm$ operator has been to raise/lower the eigenvalue by $\hbar$.
So we call the $\hat{J}_\pm$ operators ladder operators.

If the system has finite angular momentum, there must be some maximum eigenvalue, which we call j.

So $\hat{J}_+ |j\rangle = 0 \Rightarrow \hat{J}_- \hat{J}_+ |j\rangle = 0$

but
$$\hat{J}_- \hat{J}_+ = (\hat{J}_x - i\hat{J}_y)(\hat{J}_x + i\hat{J}_y)$$
$$= \hat{J}_x^2 + \hat{J}_y^2 + i[\hat{J}_x, \hat{J}_y]$$
$$= \hat{J}_x^2 + \hat{J}_y^2 - \hbar \hat{J}_z$$

so: $(\hat{J}_x^2 + \hat{J}_y^2 - \hbar\hat{J}_z)|j\rangle = 0$ but $\hat{J}_x^2 + \hat{J}_y^2 = \hat{J}^2 - \hat{J}_z^2$

$$\therefore \hat{J}^2 |j\rangle = (\hat{J}_z^2 + \hbar \hat{J}_z)|j\rangle$$
$$= \hat{J}_z \hat{J}_z |j\rangle + \hbar \hat{J}_z |j\rangle$$
$$= \hbar^2 j^2 |j\rangle + \hbar^2 j |j\rangle$$
$$= j(j+1)\hbar^2 |j\rangle$$

So we have found:
$$\hat{J}_z |j, m\rangle = \hbar m |j, m\rangle$$
$$\hat{J}^2 |j, m\rangle = \hbar^2 j(j+1) |j, m\rangle$$

we are able to do this with the "$|j, m\rangle$" because we see that all the lowered states $|m\rangle$ formed by operating $\hat{J}_-$ successively on $|j\rangle$ have the same $\hat{J}^2$ eigenvalue

We are labelling the state by j now because this gives the eigenvalue for $\hat{J}^2$

we can similarly work out $\hat{J}_+ \hat{J}_-$:

$$\hat{J}_+ \hat{J}_- = (\hat{J}_x + i\hat{J}_y)(\hat{J}_x - i\hat{J}_y)$$
$$= \hat{J}_x^{\,2} + \hat{J}_y^{\,2} - i[\hat{J}_x, \hat{J}_y]$$
$$= \hat{J}^2 - \hat{J}_z^{\,2} + \hbar \hat{J}_z$$

This will help us work out $\hat{J}_- |j,m\rangle \propto |j, m-1\rangle$

Letting $|\psi\rangle = \hat{J}_- |j,m\rangle$:

$$\langle\psi|\psi\rangle = \langle j,m | \hat{J}_+ \hat{J}_- | j,m\rangle = \hbar^2 (j+m)(j-m+1)$$

which we square root for eigenvalue of $J_- |j,m\rangle$

$$\therefore \quad \hat{J}_- |j,m\rangle = \hbar\sqrt{(j+m)(j-m+1)}\; |j, m-1\rangle$$

Just as the system has a top state $|j\rangle$, there must be a bottom state. This must be when $m = -j$

so therefore $-j \le m \le j$ in integer steps of m.

so therefore j must be integer or half integer, with total number of states being $2j+1$.

similarly: $$\hat{J}_+ |j,m\rangle = \hbar\sqrt{(j-m)(j+m+1)}\; |j, m+1\rangle$$

Sum of Angular Momenta

Consider two angular momenta $\underline{J}_1$ and $\underline{J}_2$ and their sum $\underline{J} = \underline{J}_1 + \underline{J}_2$. Let's start by defining the basis of the angular momenta. Note, for convenience will omit the j_1, j_2 labels so $|j_1, m_1\rangle$ is written simply as $|m_1\rangle$.

$$\hat{J}_{1,z} |m_1\rangle |m_2\rangle = \hbar m_1 |m_1\rangle |m_2\rangle$$
$$\hat{J}_{2,z} |m_1\rangle |m_2\rangle = \hbar m_2 |m_1\rangle |m_2\rangle$$
$$\hat{J}_1^2 |m_1\rangle |m_2\rangle = \hbar^2 j_1(j_1+1) |m_1\rangle |m_2\rangle$$
$$\hat{J}_2^2 |m_1\rangle |m_2\rangle = \hbar^2 j_2(j_2+1) |m_1\rangle |m_2\rangle$$

We want to reorganise these states into eigenstates $|J, M\rangle$ of the total angular momentum operators $\hat{J}^2$ and $\hat{J}_z^{\,2}$. Since

$$\hat{J} = \hat{J}_1 + \hat{J}_2$$

$$\hat{J}_z = \hat{J}_{1,z} + \hat{J}_{2,z}$$

so $|J, M\rangle$ will simultaneous eigenstates of $\hat{J}_1^{\,2}$, $\hat{J}_2^{\,2}$, $\hat{J}^2$ and $\hat{J}_z$

$$\hat{J}_z |M_1\rangle |M_2\rangle = (\hat{J}_{1,z} + \hat{J}_{2,z}) |M_1\rangle |M_2\rangle$$

$$= \hbar M_1 |M_1\rangle |M_2\rangle + \hbar M_2 |M_1\rangle |M_2\rangle$$

$$= \hbar (M_1 + M_2) |M_1\rangle |M_2\rangle$$

We take M to be $M_1 + M_2$. But since M_1 and M_2 can take many values, there are multiple ways to obtain a particular value of M. eg. for $M = \frac{1}{2}$, we can have $M_1 = 0$, $M_2 = \frac{1}{2}$ or $M_1 = 1$, $M_2 = -\frac{1}{2}$, both work for $j_1 = 0$, $j_2 = \frac{1}{2}$.

This is true for all states except the maximum and minimum M, for which there is only one way to obtain it. The top state must have $J = M$, the first state in the new basis is written:

$$|J, M\rangle = |j_1 + j_2, j_1 + j_2\rangle = |j_1\rangle |j_2\rangle$$

We can build all the other states $|j_1 + j_2, M\rangle$ for $-j_1 - j_2 \leq M \leq j_1 + j_2$ using the lowering operators multiple times.

There is a really good problem sheet question which helps you to understand this. I recommend you hunt it down and have a go.

So if we want $|j_1+j_2, j_1+j_2-1\rangle$:

using $$\hat{J}_-|j,M\rangle = \hbar\sqrt{(j+M)(j-M+1)}\,|j,M-1\rangle$$

so $$\hat{J}_-|j_1+j_2, j_1+j_2\rangle = \hbar\sqrt{(j_1+j_2+j_1+j_2)(j_1+j_2-j_1-j_2+1)}\;|j_1+j_2; j_1+j_2-1\rangle$$

$$= \hbar\sqrt{2j_1+2j_2}\;|j_1+j_2, j_1+j_2-1\rangle$$

$$\therefore\; |j_1+j_2, j_1+j_2-1\rangle = \frac{1}{\hbar\sqrt{2(j_1+j_2)}}\hat{J}_-|j_1+j_2, j_1+j_2\rangle$$

$$= \frac{1}{\hbar\sqrt{2(j_1+j_2)}}\left(\hat{J}_{1,-}+\hat{J}_{2,-}\right)|j_1\rangle|j_2\rangle$$

$$= \sqrt{\frac{j_1}{j_1+j_2}}\,|j_1-1\rangle|j_2\rangle + \sqrt{\frac{j_2}{j_1+j_2}}\,|j_1\rangle|j_2-1\rangle$$

There will be one other way of obtaining $M = j_1+j_2-1$. We can find it by imposing orthogonality, since we know it has to be orthogonal to this state. Since this state will be the maximum remaining value of M, we know it will correspond to a state $J = M = j_1+j_2-1$

By inspection, we can see that such a state is obtained by swapping a sign and two coefficients:

$$|j_1+j_2-1, j_1+j_2-1\rangle = -\sqrt{\frac{j_2}{j_1+j_2}}\,|j_1-1\rangle|j_2\rangle + \sqrt{\frac{j_1}{j_1+j_2}}\,|j_1\rangle|j_2-1\rangle$$

If we act with the lowering operator $2(j_1+j_2-1)$ times we generate <u>all</u> the states $|j_1+j_2-1, M\rangle$ for $-j_1-j_2+1 \le M \le j_1+j_2-1$

If either j_1 or j_2 is spin half, we can stop there. But if $j_1 \ge 1$ and $j_2 \ge 1$, then at $M = j_1+j_2-2$, there are three states to take care of:

$|j_1\rangle|j_2-2\rangle \quad |j_1-1\rangle|j_2-1\rangle \quad |j_1-2\rangle|j_2\rangle$

Please turn over

In this case we have $J = j_1 + j_2 - 2$, whose maximum M state is $|j_1+j_2-2, j_1+j_2-2\rangle$. Proceeding in this way we would generate states for all $j_1 - j_2 \leq J \leq j_1 + j_2$ and the corresponding $-M \leq J \leq M$. That's a lot of states!

These coefficients we are working out are known as Clebsch-Gordan coefficients. Their physical meanings are probability amplitudes. e.g, the probability of a system with $J = j_1 + j_2 - 1$ is measured to have $M_1 = j_1 - 1$ and $M_2 = j_2$ is $\left(-\sqrt{\frac{j_2}{j_1+j_2}}\right)^2 = \underline{\frac{j_2}{j_1+j_2}}$

Sum of Angular Momenta 1 and $\frac{1}{2}$

Let's apply all this to the example of $j_1 = 1$ and $j_2 = \frac{1}{2}$

for $j_1 = 1$, $M_1 = -1, 0, 1$ for $j_2 = \frac{1}{2}$, $M_2 = -\frac{1}{2}, \frac{1}{2}$

so our basis vectors written as $|M_1\rangle|M_2\rangle$ are:

$|1\rangle|\frac{1}{2}\rangle$ so $M = M_1 + M_2 = \frac{3}{2}$

$|1\rangle|-\frac{1}{2}\rangle$ $M = M_1 + M_2 = \frac{1}{2}$

$|0\rangle|\frac{1}{2}\rangle$ $M = M_1 + M_2 = \frac{1}{2}$

some combination of these makes $|\frac{3}{2}, \frac{1}{2}\rangle$

$|0\rangle|-\frac{1}{2}\rangle$ $M = M_1 + M_2 = -\frac{1}{2}$

$|-1\rangle|\frac{1}{2}\rangle$ $M = M_1 + M_2 = -\frac{1}{2}$

some combination of these makes $|\frac{3}{2}, -\frac{1}{2}\rangle$

$|-1\rangle|-\frac{1}{2}\rangle$ $M = M_1 + M_2 = -\frac{3}{2}$

The highest one, $M = \frac{3}{2}$, only has one way of being made. So we know it is already an eigenstate:

$$|J = \tfrac{3}{2}, M = \tfrac{3}{2}\rangle = |1\rangle|\tfrac{1}{2}\rangle$$

Now we want to build the other states with $J = \frac{3}{2}$

so $|J=\frac{3}{2}, M=\frac{3}{2}\rangle$, $|J=\frac{3}{2}, M=\frac{1}{2}\rangle$, $|J=\frac{3}{2}, M=-\frac{1}{2}\rangle$, $|J=\frac{3}{2}, M=-\frac{3}{2}\rangle$
we have

so:

$$|\tfrac{3}{2},\tfrac{1}{2}\rangle = \frac{1}{\hbar\sqrt{3}}\hat{J}_-|\tfrac{3}{2},\tfrac{3}{2}\rangle$$

$$= \frac{1}{\hbar\sqrt{3}}(\hat{J}_{1-}+\hat{J}_{2-})|1\rangle|\tfrac{1}{2}\rangle$$

$$= \frac{1}{\hbar\sqrt{3}}|\tfrac{1}{2}\rangle\underbrace{\hat{J}_{1-}|1\rangle}_{j_1=1,\ M_1=1} + \frac{1}{\hbar\sqrt{3}}|1\rangle\underbrace{\hat{J}_{2-}|\tfrac{1}{2}\rangle}_{\text{so } j_2=\frac{1}{2},\ M_2=\frac{1}{2}}$$

$$= \frac{1}{\hbar\sqrt{3}}\times\hbar\sqrt{2\times1}\,|0\rangle|\tfrac{1}{2}\rangle + \frac{1}{\hbar\sqrt{3}}\times\hbar\sqrt{1\times1}\,|1\rangle|-\tfrac{1}{2}\rangle$$

$$\underline{|\tfrac{3}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|0\rangle|\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\,|1\rangle|-\tfrac{1}{2}\rangle}$$

Similarly:

$$|\tfrac{3}{2},-\tfrac{1}{2}\rangle = \frac{1}{2\hbar}J_-|\tfrac{3}{2},\tfrac{1}{2}\rangle$$

$$= \frac{1}{2\hbar}(\hat{J}_{1-}+\hat{J}_{2-})\left(\sqrt{\tfrac{2}{3}}|0\rangle|\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}|1\rangle|-\tfrac{1}{2}\rangle\right)$$

$$= \frac{1}{2\hbar}\Big(\sqrt{\tfrac{2}{3}}\hat{J}_{1-}|0\rangle|\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}|0\rangle\hat{J}_{2-}|\tfrac{1}{2}\rangle$$
$$+\sqrt{\tfrac{1}{3}}\hat{J}_{1-}|1\rangle|-\tfrac{1}{2}\rangle + \underbrace{\sqrt{\tfrac{1}{3}}|1\rangle\hat{J}_{2-}|-\tfrac{1}{2}\rangle}_{\text{this does not exist since the lowest value of } M_2=-\frac{1}{2}}\Big)$$

$$= \frac{1}{2}\left(\frac{2}{\sqrt{3}}|-1\rangle|\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}|0\rangle|-\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}|0\rangle|-\tfrac{1}{2}\rangle\right)$$

$$\underline{|\tfrac{3}{2},-\tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}|-1\rangle|\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}|0\rangle|-\tfrac{1}{2}\rangle}$$

There is only one way to make $M=-\frac{3}{2}$ so:

$$\underline{|\tfrac{3}{2},-\tfrac{3}{2}\rangle = |-1\rangle|-\tfrac{1}{2}\rangle}$$

we thus have completed subspace $J=\frac{3}{2}$. We are left with $J=\frac{1}{2}$ (remember $j_1-j_2 \leq J \leq j_1+j_2$)

We know that

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle = \gamma|0\rangle|\tfrac{1}{2}\rangle + \delta|1\rangle|-\tfrac{1}{2}\rangle$$

for some numbers γ and δ. This must be orthogonal to $|\tfrac{3}{2},\tfrac{1}{2}\rangle$.

So, $\langle\tfrac{3}{2},\tfrac{1}{2}|\tfrac{1}{2},\tfrac{1}{2}\rangle$ tells us $\delta = -\gamma\sqrt{2}$

and we know $\gamma^2+\delta^2=1$ so: $\gamma = -\frac{1}{\sqrt{3}}$

so: $$|\tfrac{1}{2},\tfrac{1}{2}\rangle = -\frac{1}{\sqrt{3}}|0\rangle|\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}|1\rangle|-\tfrac{1}{2}\rangle$$

And finally, we need to determine $|\tfrac{1}{2},-\tfrac{1}{2}\rangle$ using the lowering operator.

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \frac{1}{\hbar}\hat{J}_-|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$= \frac{1}{\hbar}(\hat{J}_{1-}+\hat{J}_{2-})\left(-\frac{1}{\sqrt{3}}|0\rangle|\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}|1\rangle|-\tfrac{1}{2}\rangle\right)$$

$$= \frac{1}{\hbar}\left\{-\frac{1}{\sqrt{3}}\hat{J}_{1-}|0\rangle|\tfrac{1}{2}\rangle - \frac{1}{\sqrt{3}}|0\rangle\hat{J}_{2-}|\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\hat{J}_{1-}|1\rangle|-\tfrac{1}{2}\rangle\right\}$$

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}|-1\rangle|\tfrac{1}{2}\rangle + \frac{1}{\sqrt{3}}|0\rangle|-\tfrac{1}{2}\rangle$$

So we now have all the states in the addition of $j_1=1$ and $j_2=\tfrac{1}{2}$:

$$|\tfrac{3}{2},\tfrac{3}{2}\rangle \quad |\tfrac{3}{2},\tfrac{1}{2}\rangle \quad |\tfrac{3}{2},-\tfrac{1}{2}\rangle \quad |\tfrac{3}{2},-\tfrac{3}{2}\rangle \quad |\tfrac{1}{2},\tfrac{1}{2}\rangle \quad |\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

And there are 6, as expected. These can be tabulated in a Clebsch-Gordan table.

I recommend doing practice questions for these as it is the best way to learn.