Pressure

Pressure is defined as a force applied to an area.

$$\boxed{P = \frac{F}{A}}$$

with units Nm^{-2} or Pa for pascals

Pressure is a scalar quantity, it has no direction. Since only the component perpendicular to the area contributes to the pressure, we can omit the other component and use Force as a scalar to keep Pressure as a scalar.

We also measure pressure in other units:

1 atmosphere = 101.325 kPa

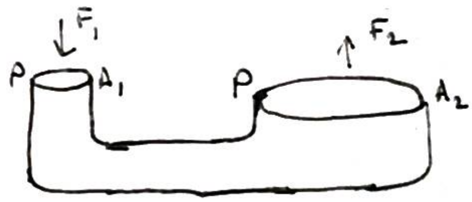
1 torr (mm Hg) = 133.3 Pa

1 bar = 10^5 Pa = 0.987 atm

Hydrostatic Laws

Pascal's Law - Pressure is transmitted undiminished through a fluid. It acts upon every part of the confining vessel at right angles to its interior surface.

This can be seen diagrammatically:



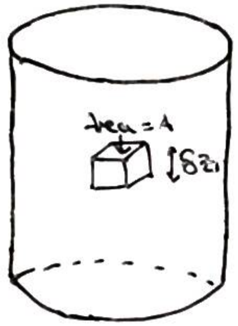
The force F_1 causes the force F_2. Since Pascal's Law tells us Pressure P is same throughout the fluid:

$$P = \frac{F_1}{A_1} = \frac{F_2}{A_2} \quad \therefore \quad \underline{F_2 = \frac{A_2}{A_1} F_1}$$

Since $A_2 > A_1$, the force $F_2 > F_1$ which has some useful applications in hydraulic presses.

Archimedes' Principle - An object immersed in a fluid experiences a force equal in magnitude and opposite in direction to the weight of the displaced fluid.

This can be seen diagramatically:



consider a small volume element with mass δm submerged in a fluid. The force of gravity acting on it is given by:

$$\delta F_g = -g\,\delta m$$

The other force, the buoyancy force, acting on it is caused by the pressure of the fluid and is equivalent to the weight of displaced fluid, according to Archimedes' Principle.

$$\delta F_b = g\,\delta m_{fluid}$$ where $\delta m_{fluid} = \rho\,\delta V$ where ρ is density of fluid

$$\delta F_b = g\rho\,\delta V$$

$$\therefore F_{net} = -g\,\delta m + g\rho\,\delta V$$

Writing this as an integral:

$$F_{net} = -mg + g\int\rho\,dV$$

and $\boxed{F_b = -g\int\rho\,dV}$ This equation is Archimedes' Principle in Mathematical form

example: A balance scale has on one pan a block of pure gold of mass M_{Au} and is perfectly balanced by a block of pure silver of mass M_{Ag} on the other pan. By taking into account air buoyancy, find the ratio of $M_{Ag} : M_{Au}$.

$$F_{Ag} = F_{Au}$$

$$F_{Ag} = -M_{Ag}g + \rho_{air}V_{Ag}g$$

$$F_{Au} = -M_{Au}g + \rho_{air}V_{Au}g$$

$$\therefore -M_{Ag} + \rho_{air}V_{Ag} = -M_{Au} + \rho_{air}V_{Au}$$

$$\rightarrow M_{Ag}\left(-1 + \frac{\rho_{air}}{\rho_{Ag}}\right) = M_{Au}\left(-1 + \frac{\rho_{air}}{\rho_{Au}}\right)$$

$$\therefore M_{Ag}/M_{Au} = \frac{1 - \rho_{air}/\rho_{Au}}{1 - \rho_{air}/\rho_{Ag}} \quad *$$

Hydrostatic Equilibrium

The reason for the buoyant force is that the pressure under the volume element $\delta V = A\delta z$ is higher than the pressure at the top.

$$P(z) \cdot \text{Area} - P(z+\delta z) \cdot \text{Area} = F_b = \rho g \underbrace{\delta V}_{\text{Area} \times \delta z}$$

$$P(z+\delta z) \cdot A - P(z) \cdot A = -\rho g A \delta z$$

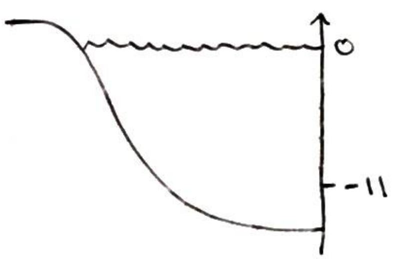
$$P(z+\delta z) - P(z) = -\rho g \delta z$$

$$\boxed{\delta P = -\rho g \delta z}$$

$\therefore$ $\boxed{\dfrac{dP}{dz} = -\rho g}$ This is the hydrostatic equation and describes the change in pressure with respect to z.

Note, we define z to be positive $\uparrow$ direction.

example: What is the pressure at the bottom of the Mariana Trench, 11km below sea level?



$$\frac{dP}{dz} = -\rho g$$

$$\int_{P_{atm}}^{P(z)} dP = -\int_0^z \rho g \, dz$$

$$P(z) - P(atm) = -[\rho g z]_0^z$$

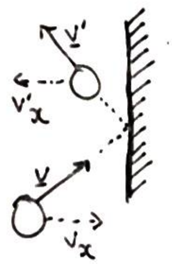
$\therefore P(z) = P(atm) - \rho g z$

so when $z = -11$ km and $P(atm) = 101325$ Pa and $\rho_{water} = 1 \times 10^3 \, kg m^{-3}$,

$P(-11) = 108$ MPa

Pressure in Kinetic Theory of Gases

Consider a particle colliding elastically with a wall:



Notice here that $v_x' = -v_x$. Every other component of the velocity remains same after collision. So the total momentum change is given by: $\Delta p = -2mv_x$

Average force exerted on the wall is given by Newton's 2nd Law:

$$F = \frac{dP}{dt} = \left\langle \frac{\Delta P}{\Delta \tau} \right\rangle$$

Notice that we use expectation value since this is the average force over many collisions, $\Delta\tau$ being time between collisions

Δp obviously depends on the velocity of the molecules but so does Δt as faster molecules means less time between collisions.

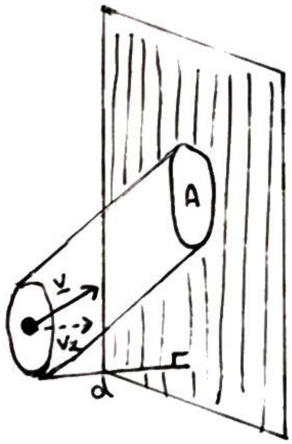
So we will start by restricting the analysis of this system to molecules with velocity $\underline{v}$. i.e only looking at molecules with $\underline{v}$ in the range $[v_x, v_x + dv_x]$, $[v_y, v_y + dv_y]$ and $[v_z, v_z + dv_z]$. This range corresponds to a 3-dimensional volume element d^3v in the velocity-space. The total number of molecules in this velocity-volume is:

$$\text{number} = N\rho(v)\, d^3 v$$

N ← total number of molecules

since $\rho(v)\, d^3v$ is the fraction of molecules whose velocity is in the chosen range.

Next we will consider that the wall upon which the collisions are occuring has area A. We construct a volume by sweeping this area from the wall over a distance d, perpendicular to the wall.



The cylinder is a volume made by sweeping the area of collisions over a distance d perpendicular to the wall. By construction all the molecules selected in a given velocity range that are in this cylinder will collide with the wall. The cylinder has volume A.d

The molecule furthest from the wall (shown on diagram) will collide with the wall at a time $t = \frac{d}{v_x}$

The final step is to work out how many particles with the chosen velocity $\underline{v}$ are in the volume. The total number of particles in the volume is given by $\frac{A.d}{V}$ where V is the total volume of the container. This is because the particles are uniformly distributed throughout the container, an assumption of Kinetic Theory of Gases. The total number of particles with velocity $\underline{v}$ in the cylinder is :

$$\left(\frac{Ad}{V}\right) N \rho(\underline{v})\, d^3v$$

Now we can work out Collision Rate, R

$$R = \frac{\text{No. Molecules}}{\text{unit time}} = \frac{\frac{Ad}{V} N \rho(\underline{v})\, d^3 v}{d/v_x} = \underline{\underline{\frac{N}{V} A \rho(\underline{v})\, v_x\, d^3 v}}$$

Average time between collisions, $\tau = \frac{1}{R}$

$$\therefore \boxed{R = \frac{1}{\tau} = n A \rho(\underline{v})\, v_x\, d^3 v}$$

where n is the number density $n = \frac{N}{V}$

From this, we can compute the force on the area A by adding the contribution at all velocities.

$$F = \int_{\text{velocities}} \frac{\Delta p}{\tau} = \int_{\text{velocities}} \Delta p R = \int_{-\infty}^{\infty} 2mv_x \frac{N}{V} \rho(\underline{v}) v_x A \, d^3v$$

$$= 2Amn \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z \int_{0}^{\infty} v_x^2 \rho(\underline{v}) \, dv_x$$

we only add the velocities of molecules with $v_x > 0$ since these are the only ones that will hit the wall. All values for the other components is fine

$$\therefore \; F = Amn \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z \int_{-\infty}^{\infty} v_x^2 \rho(\underline{v}) \, dv_x$$

since $2\int_{0}^{\infty} v_x^2 \rho(\underline{v}) \, dv_x = \int_{-\infty}^{\infty} v_x^2 \rho(\underline{v}) \, dv_x$ as we assume the velocity distribution is symmetric.

$$= Amn \cdot 1 \cdot 1 \cdot \langle v_x^2 \rangle$$

$$= Amn \langle v_x^2 \rangle$$

These become 1 since the $\rho(\underline{v})$ separates to $\rho(v_x, v_y, v_z)$ and the integral from $-\infty$ to ∞ of any of the components is 1

$$\frac{F}{A} = P = mn \langle v_x^2 \rangle$$ but $\langle v_x^2 \rangle = \frac{1}{3} \langle v^2 \rangle$

$$\boxed{P = \frac{1}{3} mn \langle v^2 \rangle} \Rightarrow P = \frac{1}{3} \frac{N}{V} m \langle v^2 \rangle$$

so $\boxed{PV = \frac{1}{3} mN \langle v^2 \rangle}$ but $U = \frac{1}{2} mN \langle v^2 \rangle$

so $\boxed{PV = \frac{2}{3} U}$

Note that since $n = \frac{N}{V}$, $nm = N \frac{m}{V} = N \rho_{\text{particle}}$

$\therefore nm =$ density of the gas.