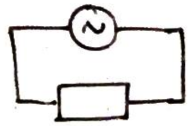
AC Circuits

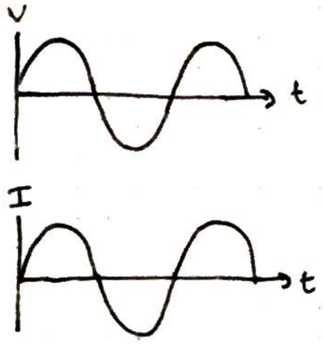
example: Simple resistor circuit



The AC generator provides $V = V_0 \sin\omega t$
so the current is given by:

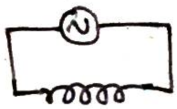
$$I = \frac{V}{R} = \frac{V_0}{R}\sin\omega t$$

Since both I and V are sine waves, they are in phase:



The two waves are in phase with each other

example: simple inductor circuit



The AC generator provides $V = V_0 \sin\omega t$
so the current is given by:

$$L\frac{dI}{dt} = V \Rightarrow \frac{dI}{dt} = \frac{V_0 \sin\omega t}{L}$$

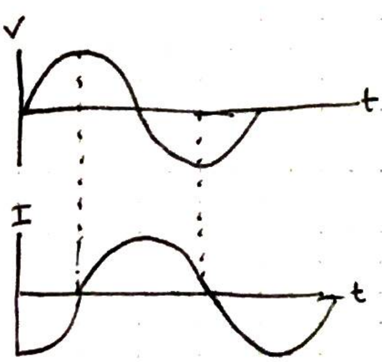
$$\int dI = \int \frac{V_0}{L}\sin\omega t \, dt$$

$$I = \frac{-V_0}{\omega L}\cos\omega t + \text{constant}$$

This is the DC offset which is 0 in this case

$$I = \frac{V_0}{\omega L}\sin\left(\omega t - \frac{\pi}{2}\right)$$

phase shift



The two waves are out of phase with each other by $\frac{\pi}{2}$

The current lags behind the voltage

example: simple capacitor circuit

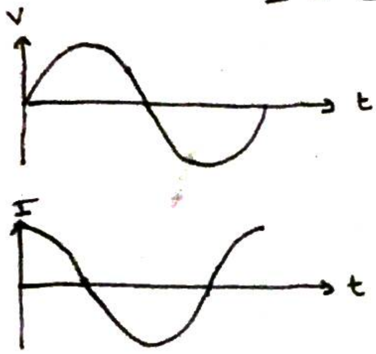
The AC generator provides $V = V_0 \sin\omega t$
so the current is given by:

$Q = CV$

$$I = \frac{dQ}{dt} = C\frac{dV}{dt} \Rightarrow I = C\frac{d}{dt}V_0 \sin\omega t$$

$$I = CV_0\omega\cos\omega t = CV_0\,\omega\sin(\omega t + \tfrac{\pi}{2})$$

↑ phase shift



The two are out of phase by $\frac{\pi}{2}$

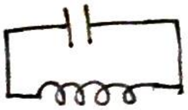
The current leads the voltage

A good way to remember these for inductor and capacitor circuits is by the mnemonic CIVIL

for C: IV (current leads voltage)

for L: VI (voltage leads current)

example: LC circuit



Using Kirchoff's voltage law:

$$L\frac{dI}{dt} + \frac{q}{C} = 0 \Rightarrow L\frac{d^2q}{dt^2} + \frac{q}{C} = 0$$

$$\therefore \frac{d^2q}{dt^2} = -\frac{1}{LC}q$$

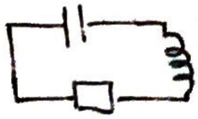
This is an equation for simple harmonic motion: $\frac{d^2q}{dt^2}$ is like acceleration here.

The solution is: $\underline{\underline{q = q_0\cos(\omega t + \delta)}}$

with $\underline{\underline{\omega^2 = \frac{1}{LC}}}$

The charge leaves the capacitor but the inductor reloads it onto the capacitor.

example: LCR discharge circuit



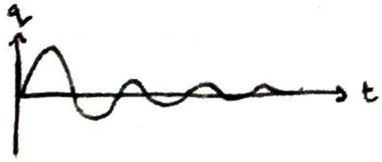
Using Kirchoff's Voltage Law:

$$L\frac{dI}{dt} + \frac{q}{c} + IR = 0$$

$$\Rightarrow L\frac{d^2q}{dt^2} + \frac{q}{c} + R\frac{dq}{dt} = 0$$

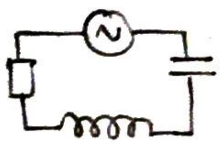
This has solution $q = q_0 e^{-Rt/2L} \cos \omega t$

and is equivalent to damped harmonic motion:



The resistor damps the SHM by removing energy from the circuit as heat.

example: Driven LCR circuit



The AC generator provides a current of $I = I_0 \cos \omega t$ so the voltage drops in each component are:

$$V_R = I_0 R \cos \omega t$$

$$V_L = I_0 L\omega \cos(\omega t + \tfrac{\pi}{2}) = I_0 L\omega \sin \omega t$$

$$V_C = \frac{I_0}{\omega C} \cos(\omega t - \tfrac{\pi}{2}) = -\frac{I_0}{\omega C} \sin \omega t$$

This can be found using CIVIL or by using calculus.

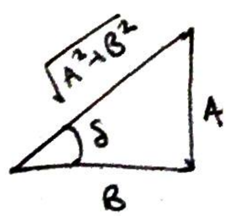
$$V_S = V_R + V_L + V_C \Rightarrow \underline{\underline{V_S = I_0 R \cos \omega t + I_0 \left(L\omega - \frac{1}{\omega C}\right) \sin \omega t}}$$

Write this in the form:

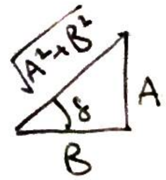
$$\sqrt{A^2+B^2}\left[\frac{A}{\sqrt{A^2+B^2}} \cos \omega t + \frac{B}{\sqrt{A^2+B^2}} \sin \omega t\right]$$

where $A = I_0 R$ and $B = I_0\left(L\omega - \frac{1}{\omega C}\right)$

Now we can draw the triangle:



The reason for drawing the triangle is to obtain expressions for the terms in the voltage equation.



$$\frac{A}{\sqrt{A^2+B^2}} = \frac{\text{opposite}}{\text{hypotenuse}} = \sin\delta$$

$$\frac{B}{\sqrt{A^2+B^2}} = \frac{\text{adjacent}}{\text{hypotenuse}} = \cos\delta$$

$$\therefore \quad V_s = \sqrt{A^2+B^2}\left[\frac{A}{\sqrt{A^2+B^2}}\cos\omega t + \frac{B}{\sqrt{A^2+B^2}}\sin\omega t\right]$$

$$V_s = \sqrt{A^2+B^2}\left[\sin\delta\cos\omega t + \cos\delta\sin\omega t\right]$$

$$\therefore V_s = \sqrt{A^2+B^2}\,\sin(\omega t+\delta)$$

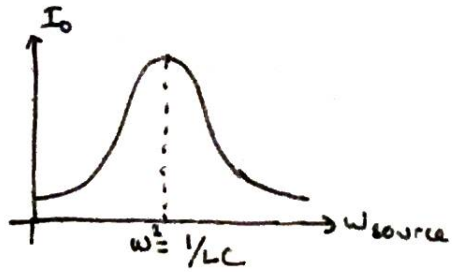
Since this is in the form $V_s = V_0 \sin\omega t$

$$V_0 = \sqrt{A^2+B^2}$$

$$V_0^2 = (I_0 R)^2 + I_0^2\left(\omega L - \frac{1}{\omega C}\right)^2$$

$$\Rightarrow I_0 = \frac{V_0}{\sqrt{R^2+\left(\omega L - \frac{1}{\omega C}\right)^2}}$$

This shows that there is a peak I_0, a "resonance", when the driving frequency matches the SHM frequency of the LC circuit.



The peak is seen here when $\omega^2 = \frac{1}{LC}$ as for the SHM circuit.