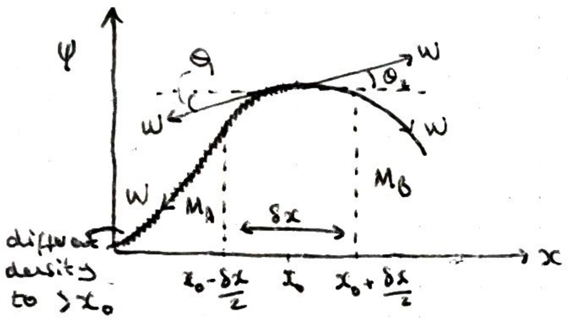
Continuity Conditions

Until now, we have only concerned ourselves with wave propagation in uniform continuous media but what do we do if this is not the case? Well then we will have to make use of continuity conditions that relate to freely propagating waves when they coincide at an interface.

The frayed Guitar String

This is a string of non-uniform density: The outer parts of the string are frayed and thus have a different density to middle parts.



M is mass per unit length

The wave equation for uniform string is:

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{W}{M} \frac{\partial^2 \psi}{\partial x^2}$$

For $x < x_0$: we have a solution ψ_A

$x > x_0$: we have a solution ψ_B

For the labelled δx element, the mass is $\delta m = \frac{M_A + M_B}{2} \delta x$

The force on this δx element is given by:

$$\underbrace{W \sin\theta_2}_{\approx \theta \approx \tan\theta \approx \frac{\delta \psi}{\delta x}} - W \sin\theta_1 = W \left[\left(\frac{\partial \psi_B}{\partial x} \right)_{x_0 + \frac{\delta x}{2}} - \left(\frac{\partial \psi_A}{\partial x} \right)_{x_0 - \frac{\delta x}{2}} \right]$$

The acceleration is given by Newton's 2nd Law: $F = ma \Rightarrow a = \frac{F}{m}$ and here the acceleration is $\frac{\partial^2 \psi}{\partial t^2}$ so:

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{2}{M_A + M_B} \cdot \frac{1}{\delta x} \cdot W \left[\left(\frac{\partial \psi_B}{\partial x} \right)_{x_0 + \frac{\delta x}{2}} - \left(\frac{\partial \psi_A}{\partial x} \right)_{x_0 - \frac{\delta x}{2}} \right]$$

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{2W}{M_A + M_B} \frac{\left(\frac{\partial \psi_B}{\partial x} \right)_{x_0 + \frac{\delta x}{2}} - \left(\frac{\partial \psi_A}{\partial x} \right)_{x_0 - \frac{\delta x}{2}}}{\delta x}$$

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{2W}{M_A + M_B} \frac{\left(\frac{\partial \psi_B}{\partial x}\right)_{x_0 + \frac{\delta x}{2}} - \left(\frac{\partial \psi_A}{\partial x}\right)_{x_0 - \frac{\delta x}{2}}}{\delta x}$$

By taking the limit $\delta x \to 0$, we can learn some information about the continuity conditions.

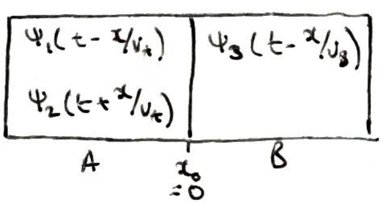
Firstly, at $x = x_0$: $\psi_A(x_0, t) = \psi_B(x_0, t)$ since we know the string is continuous without any breaks (otherwise, we'd have)

Secondly, at $x = x_0$: $\frac{\partial \psi_A}{\partial x}(x_0, t) = \frac{\partial \psi_B}{\partial x}(x_0, t)$ since we also require the gradient to be continuous and $\frac{\partial^2 \psi}{\partial x^2}$ to be finite.

These are thus the continuity conditions of the system.

Determination of reflected and transmitted amplitudes

Just as we worked out the continuity conditions from the wave equation, we can work out information about how much of a wave is reflected and transmitted at an interface from the continuity conditions. Consider the same frayed guitar string, which we split into 2 regions A and B, the interface being the point at which the density changes:



We expect an incident wave ψ_1 moving at velocity v_A to meet the boundary at which point some of the wave will be reflected and some of the wave transmitted with velocity v_B.

We still have the same equation for $v_p = \sqrt{\frac{W}{M}}$

The total displacement (superposed wave) in region A is

$\psi_A = \psi_1 + \psi_2$ while in region 2 we simply have $\psi_B = \psi_3$

So our continuity condition tells us $\psi_1 + \psi_2 = \psi_3$ (1)

We can also say $\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial x} = \frac{\partial \psi_3}{\partial x}$ at $x = x_0 = 0$ (2)

① $\psi_1(t - x/v_A) + \psi_2(t + x/v_A) = \psi_3(t - x/v_B)$

② $\dfrac{\partial \psi_1}{\partial x} + \dfrac{\partial \psi_2}{\partial x} = \dfrac{\partial \psi_3}{\partial x}$

if we say $\psi(t - x/v) = \psi(u)$ where $u = t - x/v$

$$\frac{\partial \psi}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial \psi}{\partial t} = -\frac{1}{v}\frac{\partial \psi}{\partial t}$$

Therefore, from ②:

$$-\frac{1}{v_A}\frac{\partial \psi_1}{\partial t} + \frac{1}{v_A}\frac{\partial \psi_2}{\partial t} = -\frac{1}{v_B}\frac{\partial \psi_3}{\partial t}$$

③ $\dfrac{1}{v_A}\left[\dfrac{\partial \psi_1}{\partial t} - \dfrac{\partial \psi_2}{\partial t}\right] = \dfrac{1}{v_B}\dfrac{\partial \psi_3}{\partial t}$

Differentiating ① w.r.t t: $\dfrac{\partial \psi_1}{\partial t} + \dfrac{\partial \psi_2}{\partial t} = \dfrac{\partial \psi_3}{\partial t}$ sub into ③

$$\frac{1}{v_A}\left[\frac{\partial \psi_1}{\partial t} - \frac{\partial \psi_2}{\partial t}\right] = \frac{1}{v_B}\left[\frac{\partial \psi_1}{\partial t} + \frac{\partial \psi_2}{\partial t}\right]$$

$$\Rightarrow \frac{\partial \psi_1}{\partial t} \Big/ \frac{\partial \psi_2}{\partial t} = \frac{\frac{1}{v_A} - \frac{1}{v_B}}{\frac{1}{v_A} + \frac{1}{v_B}} = \underline{\underline{\frac{v_B - v_A}{v_B + v_A}}}$$

$$\frac{\partial \psi_1}{\partial t} + \frac{\partial \psi_2}{\partial t} = \frac{\partial \psi_3}{\partial t} \quad \left(\text{divide b.s by } \frac{\partial \psi_1}{\partial t}\right)$$

$$1 + \frac{\partial \psi_2/\partial t}{\partial \psi_1/\partial t} = \frac{\partial \psi_3/\partial t}{\partial \psi_1/\partial t} = 1 + \frac{v_B - v_A}{v_B + v_A} = \underline{\underline{\frac{2v_B}{v_B + v_A}}}$$

if we define the reflection coefficient as the fraction of ψ_1 that is reflected so $\psi_2 = r\,\psi_1$ and the transmission coefficient as the fraction of ψ_1 that is transmitted so $\psi_3 = t\,\psi_1$

$\psi_1/\psi_2 = \dfrac{v_B - v_A}{v_B + v_A}$ so $\underline{\underline{r = \dfrac{v_B - v_A}{v_B + v_A}}}$

$\psi_1/\psi_3 = \dfrac{2v_B}{v_B + v_A}$ so $\underline{\underline{t = \dfrac{2v_B}{v_B + v_A}}}$

so we have derived information on the transmission and reflection of a wave from its continuity conditions

Conservation of Energy at an Interface

We saw in the "Sinusoidal Waveform" section that the power transmitted in a wave motion is given by:

$P = W V_p \left(\frac{\partial \psi}{\partial x}\right)^2 = M V_p \left(\frac{\partial \psi}{\partial t}\right)^2$ where M is the mass per unit length

From our frayed guitar string example, we know there are two tensions W_A in region A and W_B in region B. We can therefore say:

$$\underbrace{W_A V_A \left(\frac{\partial \psi_1}{\partial x}\right)^2}_{\text{Power of incident wave}} = \underbrace{W_A V_A \left(\frac{\partial \psi_2}{\partial x}\right)^2}_{\text{Power of reflected wave}} + \underbrace{W_B V_B \left(\frac{\partial \psi_3}{\partial x}\right)^2}_{\text{Power of transmitted wave}}$$

This is due to the conservation of energy at this interface.

Characteristic Impedence

Starting from our equation for power (but generalising for any i, incident, r, reflected, and t, transmitted, waveforms:

$$W_i v_i \left[\left(\frac{\partial \psi_i}{\partial x}\right)^2 - \left(\frac{\partial \psi_r}{\partial x}\right)^2\right] = W_t v_t \left(\frac{\partial \psi_3}{\partial x}\right)^2$$

$$W_i v_i \left[\left(\frac{\partial \psi_i}{\partial x} + \frac{\partial \psi_r}{\partial x}\right)\left(\frac{\partial \psi_i}{\partial x} - \frac{\partial \psi_r}{\partial x}\right)\right] = W_t v_t \left(\frac{\partial \psi_t}{\partial x}\right)^2$$

but $\frac{\partial \psi_i}{\partial x} = -\frac{1}{v_i}\frac{\partial \psi_i}{\partial t}$ $\quad \frac{\partial \psi_r}{\partial x} = \frac{1}{v_i}\frac{\partial \psi_r}{\partial t}$ $\quad \frac{\partial \psi_t}{\partial x} = -\frac{1}{v_t}\frac{\partial \psi_t}{\partial t}$ $\quad (*)$

so $W_i \left[\left(\frac{\partial \psi_i}{\partial t} + \frac{\partial \psi_r}{\partial t}\right)\left(\frac{\partial \psi_i}{\partial x} + \frac{\partial \psi_r}{\partial x}\right)\right] = W_t \frac{\partial \psi_t}{\partial t}\frac{\partial \psi_t}{\partial x}$

but we have the continuity condition $\frac{\partial \psi_1}{\partial t} + \frac{\partial \psi_2}{\partial t} = \frac{\partial \psi_3}{\partial t}$ subbing in:

$$W_i \left(\frac{\partial \psi_i}{\partial x} + \frac{\partial \psi_r}{\partial x}\right)\left(\frac{\partial \psi_t}{\partial t}\right) = W_t \frac{\partial \psi_t}{\partial t}\frac{\partial \psi_t}{\partial x}$$

$$\therefore W_i \frac{\partial \psi_i}{\partial x} + W_i \frac{\partial \psi_r}{\partial x} = W_t \frac{\partial \psi_t}{\partial x}$$

$$W_i \frac{\partial \psi_i}{\partial x} + W_i \frac{\partial \psi_r}{\partial x} = W_t \frac{\partial \psi_t}{\partial x}$$

Making use of (*) and noticing $v_i = -v_r$ and $W_i = -W_r$

$$\frac{W_i}{v_i} \frac{\partial \psi_i}{\partial t} + \frac{W_r}{v_r} \frac{\partial \psi_r}{\partial t} = \frac{W_t}{v_t} \frac{\partial \psi_t}{\partial t}$$

$\frac{W}{v}$ is a very important quantity we define as Characteristic Impedence, Z. Z can be thougut of as the ratio of force to responce $\frac{\text{force}}{\text{respone}}$.

So our continuity condition become:

$$\psi_i + \psi_r = \psi_t$$

$$Z_i \frac{\partial \psi_i}{\partial t} + Z_r \frac{\partial \psi_r}{\partial t} = Z_t \frac{\partial \psi_t}{\partial t}$$

Since $Z = \frac{W}{v}$ and $v = \sqrt{\frac{W}{M}}$: $Z = \frac{W}{\frac{\sqrt{W}}{\sqrt{M}}} = \sqrt{W}\sqrt{M}$

$\therefore$ $\underline{\underline{Z = \sqrt{WM}}}$ for the string example.