Operators for Wave Motion

Operators are used in this course similarly to in the Quantum Mechanics course. Operators are recipes for determining values of observable quantities. They are denoted with "^". One difference between the QM course and this one is that in QM we always apply operators to normalised wavefunctions. Here, we will normalise the wavefunction simultaneously (you'll see what I mean) when calculating expectation values

Take a generic wavefunction $\psi(x,t) = \psi_0 \exp[i(kx - \omega t)]$

$$\frac{\partial \psi}{\partial t} = -i\omega \psi_0 \exp[i(kx - \omega t)] = -i\omega \psi(x,t)$$

$$\therefore \omega = \frac{\frac{\partial \psi}{\partial t}}{-i\psi(x,t)} = \frac{i\frac{\partial \psi}{\partial t}}{\psi}$$ ← This is the normalisation factor

So we can define angular frequency operator ω as:

$$\hat{\omega}\psi = \omega\psi \quad \text{so} \quad \underline{\underline{\hat{\omega} = i\frac{\partial}{\partial t}}}$$

if we normalise the wavefunction before applying the operator, we don't have to worry about the normalisation factor

$$\frac{\partial \psi}{\partial x} = ik\psi_0 \exp[i(kx - \omega t)]$$

$$\text{so} \quad -i\frac{\partial}{\partial x}\psi = k\psi \quad \text{so} \quad \underline{\underline{\hat{k} = -i\frac{\partial}{\partial x}}}$$

We can work out the average value (expectation value) by integrating over a full range:

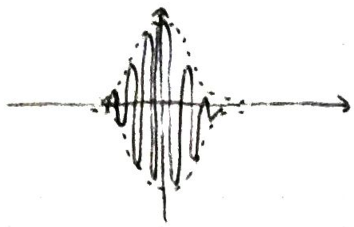
for some observable O, the expectation value is:

$$\langle O \rangle = \frac{\int_{-\infty}^{\infty} \psi^* \hat{O} \psi \, dx}{\int_{-\infty}^{\infty} \psi^* \psi \, dx}$$ ← We are normalising here.

Let's try the example of a Gaussian wave packet:

$$\psi(t) = \psi_0 \exp(-i\omega_0 t)\exp\left(-\left(\frac{t}{t_0}\right)^2\right)$$

which looks like:



What is $\omega(t)$?

$$\hat{\omega}\psi = \omega\psi$$

so: $$\hat{\omega}\psi = i\frac{\partial}{\partial t}\psi$$

$$= \psi_0 i \frac{\partial}{\partial t}\left\{\exp(-i\omega_0 t)\exp\left(-\left(\frac{t}{t_0}\right)^2\right)\right\}$$

$$= \psi_0 i \left\{-i\omega_0 \exp(-i\omega_0 t)\exp\left(-\left(\frac{t}{t_0}\right)^2\right) - \frac{2t}{t_0^2}\exp(-i\omega_0 t)\exp\left(-\left(\frac{t}{t_0}\right)^2\right)\right\}$$

$$= \psi_0\left(\omega_0 - \frac{2it}{t_0^2}\right)\exp(-i\omega_0 t)\exp\left(-\left(\frac{t}{t_0}\right)^2\right)$$

$$\hat{\omega}\psi = \left(\omega_0 - \frac{2it}{t_0^2}\right)\psi$$ since $\hat{\omega}\psi = \omega\psi$, we can say:

$$\omega(t) = \omega_0 - \frac{2it}{t_0^2}$$