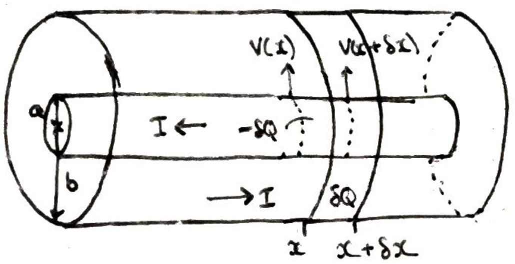
Further Wave Equations

Waves along a Coaxial Cable



Consider a coaxial cable, made up of a conducting copper cylinder surrounded by an insulating material and a copper sleeve. The copper sleeve and cylinder are separated by the insulating material.

This will act like a capacitor, since an electric field will exist between the two copper parts if there is a charge in one. They will also exhibit inductance so the current through the conductors takes a finite amount of time to "react" to any changes in voltage. These phenomena can be explained using Gauss' Law and Faraday's law and thus we can construct a wave equation.

Constructing a Wave Equation: Coaxial Cable

The inner conductor has radius a and the outer conductor has radius b. The voltage between the two conductors at any point x is $V(x)$. Equal but opposite charges are assumed in each conductor $\pm \delta Q$ so the current is equal but flows in opposite directions.

The net current leaving a region is equivalent to the change in charge over time:

$$\frac{\partial}{\partial t}\delta Q(x) = I(x) - I(x+\delta x)$$

$$\delta Q(x) = C\delta x\, V(x)$$ This is the definition of capacitance. C here is capacitance per unit length

$$\Rightarrow \frac{\partial}{\partial t} C\delta x\, V(x) = I(x) - I(x+\delta x)$$

take lim $\delta x \to 0$

$$C\frac{\partial V}{\partial t} = -\frac{I(x+\delta x) - I(x)}{\delta x} \Longrightarrow C\frac{\partial V}{\partial t} = -\frac{\partial I}{\partial x}$$

$C\frac{\partial V}{\partial t} = -\frac{\partial I}{\partial x}$ ① This is still not a wave equation since the left hand side and right hand side are differentiation different variables (I and V)

So how do we change variables? Let's consider Faraday's Law:

$$\oint \underline{E}\cdot d\underline{L} = -\frac{\partial \Phi}{\partial t} \quad \Rightarrow -\frac{\partial}{\partial t}\delta\Phi = V(x+\delta x) - V(x)$$

$$\Rightarrow \frac{\partial}{\partial t}\delta\Phi = V(x) - V(x+\delta x) \quad (*)$$

Now let's consider the definition of self-inductance:

$$\delta\Phi(x) = L\delta x I(x) \quad \Rightarrow \quad L\delta x\frac{\partial I}{\partial t} = \frac{\partial}{\partial t}\delta\Phi$$ subbing in (*):

L is inductance per unit length

$$\Rightarrow L\delta x\frac{\partial I}{\partial t} = V(x) - V(x+\delta x)$$

$$\therefore L\frac{\partial I}{\partial t} = \frac{V(x) - V(x+\delta x)}{\delta x} \quad \Rightarrow \quad L\frac{\partial I}{\partial t} = -\frac{\partial V}{\partial x} \quad ②$$

① $C\frac{\partial V}{\partial t} = -\frac{\partial I}{\partial x}$ ② $L\frac{\partial I}{\partial t} = -\frac{\partial V}{\partial x}$

$\hookrightarrow C\frac{\partial^2 V}{\partial t^2} = -\frac{\partial^2 I}{\partial x\partial t}$ $\hookrightarrow L\frac{\partial^2 I}{\partial x\partial t} = -\frac{\partial^2 V}{\partial x^2}$

$\hookrightarrow \frac{\partial^2 I}{\partial x\partial t} = -C\frac{\partial^2 V}{\partial t^2}$ $\hookrightarrow \frac{\partial^2 I}{\partial x\partial t} = -\frac{1}{L}\frac{\partial^2 V}{\partial x^2}$

Putting these together: $-\frac{1}{L}\frac{\partial^2 V}{\partial x^2} = -C\frac{\partial^2 V}{\partial t^2}$

$$\therefore \underline{\underline{\frac{\partial^2 V}{\partial t^2} = \frac{1}{LC}\frac{\partial^2 V}{\partial x^2}}}$$

We have thus constructed a wave equation.

We can use the general template to say $V_p = \sqrt{\frac{1}{LC}}$

Since this is an electromagnetic wave, we know that $V_p = c$. We can actually show this!

speed of light

Finding phase velocity : Coaxial cable

Starting from Gauss' Law: $\oint \underline{E} \cdot d\underline{A} = \frac{Q_{enc}}{\varepsilon_0}$

$$\therefore \quad E(r)\, 2\pi r \,\delta x = -\frac{\delta Q}{\varepsilon_0} \Rightarrow E(r) = -\frac{\delta Q}{2\pi \,\delta x\, \varepsilon_0} \frac{1}{r}$$

We know $V(x) = -\int_a^b E(r)\, dr$

$$\text{so} \quad V(x) = \frac{\delta Q}{2\pi \,\delta x\, \varepsilon_0} \int_a^b \frac{1}{r}\, dr$$

$$V(x) = \frac{\delta Q}{2\pi \,\delta x\, \varepsilon_0} \ln(b/a)$$

$$\therefore \quad \delta Q = \underbrace{\frac{2\pi \varepsilon_0}{\ln(b/a)}}_{C \text{ since we know}} \delta x\, V(x) \qquad \delta Q = C\,\delta x\, V(x)$$

$$C = \frac{2\pi \varepsilon_0}{\ln(b/a)}$$

Now using Ampere's Law: $\oint \underline{B} \cdot d\underline{l} = \mu_0 I$

$$\therefore \quad 2\pi r\, B(r) = \mu_0 I \Rightarrow B(r) = \frac{\mu_0 I}{2\pi r}$$

We know $\delta \Phi(x) = \delta x \int_a^b B(r)\, dr$

$$\text{so} \quad \delta\Phi(x) = \delta x \frac{\mu_0 I}{2\pi} \int_a^b \frac{1}{r}\, dr$$

$$\delta\Phi(x) = \underbrace{\frac{\mu_0 \ln(b/a)}{2\pi}}_{= L \text{ since we know}} \delta x\, I \qquad \delta\Phi(x) = L\,\delta x\, I(x)$$

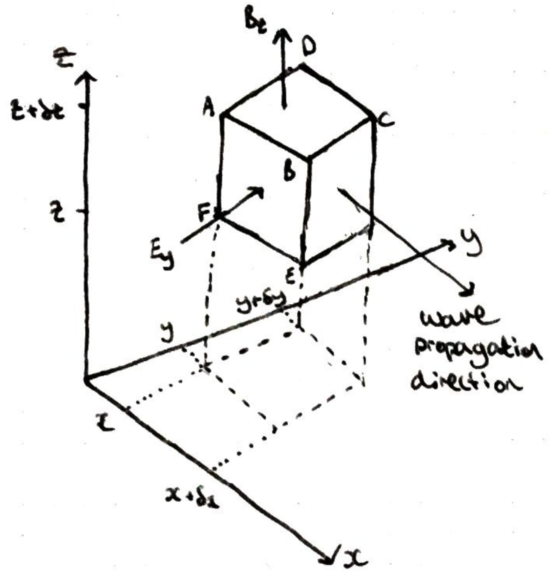
$$L = \frac{\mu_0}{2\pi} \ln(b/a)$$

$$\text{so} \quad \frac{1}{LC} = \frac{2\pi}{\mu_0 \ln(b/a)} \cdot \frac{\ln(b/a)}{2\pi \varepsilon_0} = \frac{1}{\mu_0 \varepsilon_0} = c^2 \text{ by definition}$$

$\uparrow$ speed of light

$$\therefore \quad \sqrt{\frac{1}{LC}} = V_p = \underline{\underline{\text{speed of light}}}$$

Constructing a Wave Equation: Electromagnetic Waves



Consider an element of free space in a region of a propagating planar electromagnetic wave. The element has sides of length δz, δy and δx

We can follow similar steps to those for the coaxial cable but in this case, no conductors are present.

Lets start with Faraday's law for a magnetic field in z direction

$$\oint \underline{E}\cdot d\underline{l} = -\frac{\partial \Phi}{\partial t} = -\frac{\partial}{\partial t}\int_{Area} \underline{B}\cdot d\underline{A} = \int_A^B \underline{E}\cdot d\underline{l} + \int_B^C \underline{E}\cdot d\underline{l} + \int_C^D \underline{E}\cdot d\underline{l} + \int_D^A \underline{E}\cdot d\underline{l}$$

but $A \to B$ and $C \to D$ are perpendicular to $\underline{E}$ so $=0$

$$\therefore \quad -\frac{\partial}{\partial t}\int_A \underline{B}\cdot d\underline{A} = \int_B^C \underline{E}\cdot d\underline{l} + \int_D^A \underline{E}\cdot d\underline{l}$$

$$-\frac{\partial B_z}{\partial t}\int d\underline{A} = \int_B^C \underline{E}\cdot d\underline{l} + \int_D^A \underline{E}\cdot d\underline{l} \Rightarrow \int_B^C \underline{E}\cdot d\underline{l} + \int_D^A \underline{E}\cdot d\underline{l} = -\frac{\partial B_z}{\partial t}\delta x \delta y$$

$$\therefore \int_y^{y+\delta y} E_y(x+\delta x)\, dy + \int_{y+\delta y}^{y} E(x)\, dy = -\frac{\partial B_z}{\partial t}\delta x \delta y$$

$$E_y(x+\delta x)\delta y - E(x)\delta y = -\frac{\partial B_z}{\partial t}\delta x \delta y$$

$$\frac{E_y(x+\delta x) - E_y(x)}{\delta x} = -\frac{\partial B_z}{\partial t} \Rightarrow \underline{\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}} \quad (1)$$

Still not a wave eqn. We need a change of variable

We can now apply Ampére's Law: $\oint \underline{B}\cdot d\underline{l} = \int_{Area}\left(\mu_0 \underline{J} + \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}\right)\cdot d\underline{A}$

In this case $\underline{J} = 0$ since no current is flowing.

$$\therefore \underbrace{\int_A^B B_x(z+\delta z)\, dx}_{=0 \text{ since perp.}} + \int_B^E B_z(x+\delta x)\, dz + \underbrace{\int_E^F B_x(z)\, dx}_{=0 \text{ since perp}} + \int_F^A B_z(x)\, dz$$

$$\Rightarrow \int_B^E B_z(x+\delta x)\, dz + \int_F^A B_z(x)\, dz = \int_{\text{Area}} \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t} \cdot d\underline{A}$$

$$-B_z(x+\delta x)\,\delta z + B_z(x)\,\delta z = \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t} \delta x \delta z$$

$$\frac{B_z(x+\delta x) - B_z(x)}{\delta x} = -\mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}$$

$$\therefore -\mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t} = \frac{\partial B_z}{\partial x} \quad (2)$$

(1) $\dfrac{\partial E_y}{\partial x} = -\dfrac{\partial B_z}{\partial t}$ (2) $-\mu_0 \varepsilon_0 \dfrac{\partial E_y}{\partial t} = \dfrac{\partial B_z}{\partial x}$

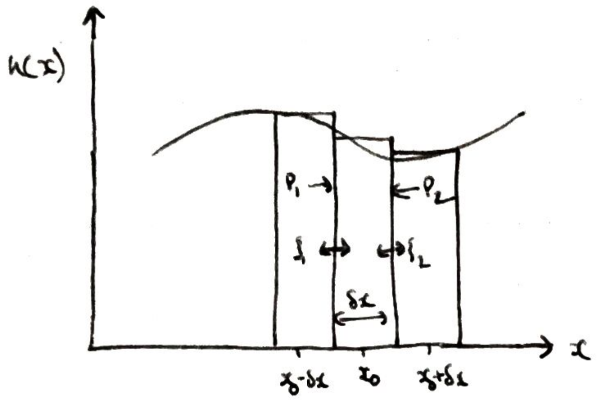
↳ $\dfrac{\partial^2 E_y}{\partial x^2} = -\dfrac{\partial^2 B_z}{\partial x \partial t}$ ↳ $-\mu_0 \varepsilon_0 \dfrac{\partial^2 E_y}{\partial t^2} = \dfrac{\partial^2 B_z}{\partial x \partial t}$

combining these: $\dfrac{\partial^2 E_y}{\partial x^2} = \mu_0 \varepsilon_0 \dfrac{\partial^2 E_y}{\partial t^2}$

$$\Rightarrow \frac{\partial^2 E_y}{\partial t^2} = \frac{1}{\mu_0 \varepsilon_0} \frac{\partial^2 E_y}{\partial x^2}$$

So once again we have obtained $v_p = \sqrt{\dfrac{1}{\mu_0 \varepsilon_0}} = c$
as we expect for the speed of light

Constructing a Wave Equation: Shallow Water Waves



Consider a cross section of the ocean where the wave is the position of the top layer of water, $h(x)$ is the height above ground. Now imagine dividing this into thin vertical slices with width δx. If the disturbance is small enough and the slices are thin enough, we can approximate the disturbance to a constant height within the slice.

As the wave propagates, we can move the boundaries of the slices with it. Since no water travels between the slices, the volume of water in each slice remains constant. i.e. if the boundaries around x_0 move by ξ_1 and ξ_2, then:

$$\text{Volume} = h(x)(\delta x + \xi_2 - \xi_1)\,\delta y = \text{constant}$$

+ve is → and δy is the lateral "width" of the slice.

differentiating w.r.t. time:

$$(\delta x + \xi_2 - \xi_1)\,\delta y \frac{\partial h}{\partial t} + h_0 \delta y \left(\frac{\partial \xi_2}{\partial t} - \frac{\partial \xi_1}{\partial t}\right) = 0$$

↳ 0 because constant diff. to 0

ξ_2 and ξ_1 are small compared to δx so we neglect them.

$$\therefore \frac{\partial h}{\partial t} = -h_0 \frac{\partial \xi_2/\partial t - \partial \xi_1/\partial t}{\delta x}$$

but $\frac{\partial \xi}{\partial t}$ is just the horizontal velocity of the boundary so:

$$\frac{\partial h}{\partial t} = -h_0 \frac{V_{x2} - V_{x1}}{\partial x} \Rightarrow \frac{\partial h}{\partial t} = -h_0 \frac{\partial V_x}{\partial x} \quad (1)$$

This is still not a wave equation

Let's consider the hydrostatic pressure on the boundary due to the difference in height of water.

$$P_1(h(x)) - P_2(h(x')) = (h_1 - h_2)\rho g$$

This is the difference in pressure across the slice.

$P_1(h(x)) - P_2(h(x)) = (h_1 - h_2)\rho g$

if we let $z = h(x)$:

$P_1(z) - P_2(z) = (h_1 - h_2)\rho g$ where ρ is water density and $g = 9.81\,ms^{-2}$

A net horizontal force acts on the slice, resulting in its acceleration

$$\Rightarrow (h_1 - h_2)\rho g\, \partial z\, \delta y = \text{mass} \times \text{acceleration}$$

$$= \rho \times \delta x \delta y \delta z \times \frac{\partial V_x}{\partial t}$$

$$\frac{\partial V_x}{\partial t} = \frac{h_1 - h_2}{\partial x} g$$

$$= -g \frac{h_2 - h_1}{\partial x} \Rightarrow \frac{\partial V_x}{\partial t} = -g\frac{\partial h}{\partial x} \quad (2)$$

(1) $\frac{\partial h}{\partial t} = -h_0 \frac{\partial V_x}{\partial x}$ (2) $\frac{\partial V_x}{\partial t} = -g\frac{\partial h}{\partial x}$

$\hookrightarrow \frac{\partial^2 h}{\partial t^2} = -h_0 \frac{\partial V_x}{\partial x \partial t}$ $\hookrightarrow \frac{\partial^2 V_x}{\partial x \partial t} = -g\frac{\partial^2 h}{\partial x^2}$

$\hookrightarrow -\frac{1}{h_0}\frac{\partial^2 h}{\partial t^2} = \frac{\partial^2 V_x}{\partial x \partial t}$ combining these:

$$\frac{1}{h_0}\frac{\partial^2 h}{\partial t^2} = g\frac{\partial^2 h}{\partial x^2}$$

$$\therefore \frac{\partial^2 h}{\partial t^2} = h_0 g \frac{\partial^2 h}{\partial x^2}$$

we have constructed a wave equation

phase velocity is $\sqrt{h_0 g}$ from the general form