Electromagnetic Waves in the Vacuum

As we found in the last section, $\underline{E}$ and $\underline{B}$ fields are real and have physical presence. Maxwell's equations predict EM waves and this next chapter examines these waves in a vacuum.

Waves on a String

After the waves module, you should be very familiar with this by now so I won't cover it in detail.

The wave equation is:

$$\frac{\partial^2 f(z,t)}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f(z,t)}{\partial t^2} \qquad v = \sqrt{\frac{T}{\mu}}$$

where T is tension and μ is mass per unit length.

The particularly nice solutions to the wave equation are called harmonics and take the form:

$f(z,t) = A\cos(kz - \omega t + \delta)$ for right moving wave

$f(z,t) = A\cos(-kz - \omega t + \delta)$ for left moving wave

The wavenumber k is $k = \frac{2\pi}{\lambda}$

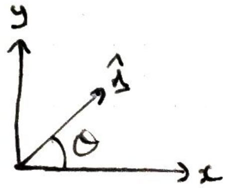
Angular frequency ω is $\omega = 2\pi\nu$

If we want to add waves together, it is perhaps easiest in exponential form: $e^{i\theta} = \cos\theta + i\sin\theta$

so $f(z,t) = \mathrm{Re}\left\{Ae^{i(\pm kz - \omega t + \delta)}\right\}$

We can do operations with the complex wavefunction and convert back to real for physical solutions: $\tilde{f}(z,t) = Ae^{i(kz-\omega t+\delta)}$

Waves on a string are transverse waves and can be polarised in some direction:



The direction $\hat{n}$ is a polarisation direction a wave in the plane with direction of propagation $\hat{z}$ can be polarised on

$$\hat{n} = \hat{x}\cos\theta + \hat{y}\sin\theta$$

The real wavefunction is therefore described as:

$$\underline{f}(z,t) = f(z,t)\,\hat{n} \quad \text{where } \underline{f}(z,t) \text{ satisfies wave eqn.}$$

$$\underline{f}(z,t) = \underline{A}\cos(kz - \omega t + \delta) \quad \text{where } \underline{A} = A\hat{n}$$

Maxwell's Prediction & Electromagnetic Waves

Let's try and do something similar for EM waves. Maxwell's Equations without any charges or currents are:

$$\underline{\nabla}\cdot\underline{E} = 0$$

$$\underline{\nabla}\cdot\underline{B} = 0$$

$$\underline{\nabla}\times\underline{E} = -\frac{\partial B}{\partial t}$$

$$\underline{\nabla}\times\underline{B} = \mu_0\varepsilon_0\frac{\partial \underline{E}}{\partial t}$$

Using an identity

$$\underline{\nabla}\times(\underline{\nabla}\times\underline{A}) = \underline{\nabla}(\underline{\nabla}\cdot\underline{A}) - \nabla^2\underline{A}:$$

$$\underline{\nabla}\times\underbrace{(\underline{\nabla}\times\underline{E})}_{= -\frac{\partial B}{\partial t}} = \underline{\nabla}(\underbrace{\underline{\nabla}\cdot\underline{E}}_{=0}) - \nabla^2\underline{E}$$

$$\therefore \quad \underline{\nabla}\times\left(-\frac{\partial \underline{B}}{\partial t}\right) = -\nabla^2\underline{E} \quad \Rightarrow \quad \frac{\partial}{\partial t}(\underline{\nabla}\times\underline{B}) = \nabla^2\underline{E}$$

$$\therefore \quad \mu_0\varepsilon_0\frac{\partial^2 \underline{E}}{\partial t^2} = \nabla^2\underline{E} \quad \Rightarrow \quad \boxed{\nabla^2\underline{E} = \frac{1}{c^2}\frac{\partial^2\underline{E}}{\partial t^2}}$$

We have constructed a wave equation!

Similarly, starting with $\underline{\nabla} \times (\underline{\nabla} \times \underline{B})$, we find:

$$\boxed{\nabla^2 \underline{B} = \frac{1}{c^2} \frac{\partial^2 \underline{B}}{\partial t^2}}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ the laplacian operator

Since we can construct these wave equations in a vacuum, Maxwell's equations imply that empty space supports the propagation of electromagnetic waves, travelling at speed c. Thus, light <u>is</u> an electromagnetic wave.

EM waves are 3 dimensional planar waves but this is hard to visualise, especially if we consider orthogonality and polarisation.

The 3D harmonic planar wave solutions to the $\underline{E}$ wave equation travelling in the $+\hat{\underline{z}}$ direction can be written:

$$\underline{E}(x,y,z,t) = \underline{E_0} \cos(kz - \omega t + \delta)$$

where amplitude E_0 takes constant value over the xy plane. Since EM waves are transverse, E_0 must lie in the xy plane.

We can prove EM waves are transverse:

complex harmonic wave solutions are:

$$\left.\begin{aligned} \underline{\tilde{E}}(x,y,z,t) &= \underline{E}_0 e^{i(kz - \omega t + \delta_E)} \\ \underline{\tilde{B}}(x,y,z,t) &= \underline{B}_0 e^{i(kz - \omega t + \delta_B)} \end{aligned}\right\} \text{ solutions with } c = \frac{\omega}{k}$$

There are 4 Maxwell's Equations but the wave equations were derived with just 2. So we might be able to use the other 2 somehow.

Substituting the wavesolution into the Maxwells Eqn : $\underline{\nabla} \cdot \underline{E} = 0$

$$\underline{\nabla} \cdot \underline{E}_0 e^{i(kz-\omega t+\delta_E)} = 0$$

$$\underline{\nabla} \cdot \{ E_{0x} \hat{\underline{x}} + E_{0y} \hat{\underline{y}} + E_{0z} \hat{\underline{z}} \} e^{i(kz-\omega t+\delta_E)} = 0$$

$$\Rightarrow \frac{\partial}{\partial z} E_{0z} e^{i(kz-\omega t+\delta_E)} = 0$$

since the $\hat{x}$ and $\hat{y}$ terms go to 0 as there is no x or y in exponent

$$ik E_{0z} e^{i(kz-\omega t+\delta_E)} = 0$$

$$\therefore \underline{\underline{E_{0z} = 0}}$$

so for a wave propagating in z direction to have no z component of amplitude, it must be transverse.

we can similarly show:

$$\underline{\underline{B_{0z} = 0}}$$

So EM waves are transverse.

We can also show $\delta_E = \delta_B$, so $\underline{E}$ and $\underline{B}$ waves are in phase.

Furthermore, we can show $\underline{B}$ waves are perpendicular to $\underline{E}$ waves, and are given by:

$$\underline{\underline{\underline{B}_0 = \frac{1}{c} (\hat{\underline{z}} \times \underline{E}_0)}}$$

So the $\underline{B}$ wave is a slave to the $\underline{E}$ wave.

We can generalise our wavesolutions for any arbitrary direction $\hat{\underline{k}}$.

We can define a wave vector $\underline{k}$ from the wavenumber: $\underline{k} = k\hat{\underline{k}}$

so the general solutions are:

$$\underline{\underline{\tilde{\underline{E}}(x,y,z,t) = \underline{E}_0 e^{i(\underline{k}\cdot\underline{r} - \omega t + \delta)}}} \qquad \underline{\underline{\tilde{\underline{B}}(x,y,z,t) = \underline{B}_0 e^{i(\underline{k}\cdot\underline{r} - \omega t + \delta)}}}$$

Energy Flow in EM waves

We know that the energy per unit volume is

$$\boxed{u_{em} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)}$$ where $E = |\underline{E}|$ and $B = |\underline{B}|$

Lets put in our wave solutions and see what happens:

$\underline{E}(x,y,z,t) = \underline{E}_0 \cos(\underline{k}\cdot\underline{r} - \omega t + \delta)$ $\quad \underline{B}(x,y,z,t) = \underline{B}_0 \cos(\underline{k}\cdot\underline{r} - \omega t + \delta)$

$E^2 = E_0^2 \cos^2(\underline{k}\cdot\underline{r} - \omega t + \delta)$

$B^2 = \left[\frac{1}{c}(\hat{\underline{k}} \times \underline{E}_0)\right]^2 \cos^2(\underline{k}\cdot\underline{r} - \omega t + \delta)$ but $|\hat{\underline{k}} \times \underline{E}_0| = 1$ since $\hat{\underline{k}}$ is unit vector

$B^2 = \frac{1}{c^2} E_0^2 \cos^2(\underline{k}\cdot\underline{r} - \omega t + \delta)$

$$\therefore \quad u_{em} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

$$= \frac{1}{2}\varepsilon_0 \left(E^2 + \frac{1}{\mu_0 \varepsilon_0} B^2\right)$$

$$u_{em} = \frac{1}{2}\varepsilon_0 \left(E^2 + c^2 B^2\right)$$ but $c^2 B^2 = E^2$

$$\therefore \quad \boxed{u_{em} = \varepsilon_0 E^2}$$

$$\boxed{u_{em} = \varepsilon_0 E_0^2 \cos^2(\underline{k}\cdot\underline{r} - \omega t + \delta)}$$

Recall the Poynting vector $\underline{S} = \frac{1}{\mu_0} \underline{E} \times \underline{B}$

$$\therefore \quad \underline{S} = \frac{1}{\mu_0}\left\{\underline{E}_0 \cos(\underline{k}\cdot\underline{r} - \omega t + \delta) \times \frac{1}{c}(\hat{\underline{k}} \times \underline{E}_0)\cos(\underline{k}\cdot\underline{r} - \omega t + \delta)\right\}$$

$$= \frac{1}{\mu_0}\frac{1}{c} \underline{E}_0 \times (\hat{\underline{k}} \times \underline{E}_0)\cos^2(\underline{k}\cdot\underline{r} - \omega t + \delta)$$

we use the identity
$\underline{A} \times (\underline{B} \times \underline{C}) = \underline{B}(\underline{A}\cdot\underline{C}) - \underline{C}(\underline{A}\cdot\underline{B})$

$$\underline{S} = \frac{1}{\mu_0}\frac{1}{c}\hat{\underline{k}} E_0^2 \cos^2(\underline{k}\cdot\underline{r} - \omega t + \delta)$$

combining this with the expression for u_{em}:

$$\boxed{\underline{S} = u_{em} c \hat{\underline{k}}}$$

The energy flow per unit time $|\underline{S}|$ is therefore in the direction of wave propagation $\hat{\underline{k}}$.

This seems obvious but the derivation confirms it.

The intensity of the wave is defined as average power per unit area transported by the EM wave. i.e time average magnitude of $\underline{S}$:

$$\boxed{I = \frac{1}{2} c \varepsilon_0 E_0^2}$$