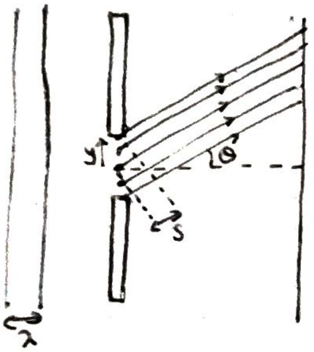
Fraunhofer Diffraction

Fraunhofer diffraction is one of the most important types of diffraction: consider diffraction through a single slit. If the screen we observe the interference pattern on is far from the slit, and the source of light is also far from the slit, we can make the assumption that the rays in and out of the slit are all parallel to each other. This really simplifies our analysis. When such simplification is not possible (i.e when they are close together) it is called Fresnel Diffraction.

Diffraction by a Single slit

Consider light diffracted through a single slit of width a:



using simple geometry, we can say

$S(y) = -y\sin\theta$

we construct the phasor for this system

$\psi(y) = \psi_0 e^{-iky\sin\theta}$ where ψ_0 is observed amplitude

This is for one ray, we need the total contribution from rays all the sources. We therefore need to add all these together in an integral:

$$\psi(\theta) = \int_{-a/2}^{a/2} \psi_0 \exp(-iky\sin\theta)\,dy = \left[\frac{\psi_0 \exp(-iky\sin\theta)}{-ik\sin\theta}\right]_{-a/2}^{a/2}$$

$$= \frac{\psi_0}{k\sin\theta}\left[\frac{1}{-i}\exp\left(\frac{-ika\sin\theta}{2}\right) + \frac{1}{i}\exp\left(\frac{ika\sin\theta}{2}\right)\right]$$

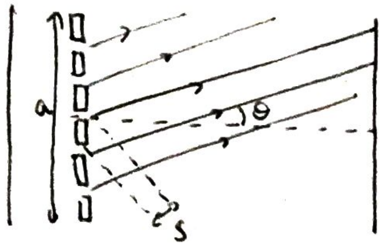
$$= \frac{\psi_0}{k\sin\theta}\, 2 \cdot \frac{1}{2i}\left[\exp\left(\frac{ika\sin\theta}{2}\right) - \exp\left(\frac{-ika\sin\theta}{2}\right)\right]$$

$$= \frac{2\psi_0}{k\sin\theta}\sin\left(\frac{ka\sin\theta}{2}\right)$$

$$\psi(\theta) = \psi_0 a \frac{\sin(\alpha)}{\alpha} \quad \text{where } \alpha = \frac{ka}{2}\sin\theta$$

we have thus found the sum contribution from all the sources at an angle θ

Diffraction Grating



We can think of a diffraction grating as simply an extension of the single slit example. We pretend the whole grating is one large slit, allowing us to start at the same place:

$$\psi(\theta) = \int_{-a/2}^{a/2} \psi_0 \exp(-iky\sin\theta)\,dy$$

However, we haven't accounted for the solid parts of the grating. We thus define a function $t(y)$ that $=0$ or $=1$ depending on the position in y.

$$\therefore \psi(\theta) = \int_{-a/2}^{a/2} \psi_0 t(y) \exp(-iky\sin\theta)\,dy$$

if each small slit in the grating has width w, then

$$\psi(\theta) = \int_{y_1-\frac{w}{2}}^{y_1+\frac{w}{2}} \psi_0 t \exp(-iky\sin\theta)\,dy + \int_{y_2-\frac{w}{2}}^{y_2+\frac{w}{2}} \psi_0 t \exp(-iky\sin\theta)\,dy + \int_{y_3-\frac{w}{2}}^{y_3+\frac{w}{2}} \psi_0 t \exp(-iky\sin\theta)\,dy$$

and so on, where $y_1, y_2 \ldots$ are the midpoints of the slits across the whole grating.

This expression can be simplified by noticing that every integral is identical after computing the integral, except for a phase factor $+y_1$ or $+y_2$ or ... so we can pull that out:

$$\psi(\theta) = \exp(-iky_1\sin\theta)\int_{-\frac{w}{2}}^{\frac{w}{2}} \psi_0 t \exp(-iky\sin\theta)\,dy + \exp(-iky_2\sin\theta)\int_{-\frac{w}{2}}^{\frac{w}{2}} \psi_0 t \exp(-iky\sin\theta)\,dy$$

$$\therefore \psi(\theta) = \underbrace{\left(\sum_{n=-\frac{N}{2}}^{\frac{N}{2}} \exp(-iknd\sin\theta)\right)}_{\text{we can evaluate this part as:}} \psi_0 \int_{-\frac{w}{2}}^{\frac{w}{2}} t\exp(-iky\sin\theta)\,dy$$

where we have used $y_n = nd$ (d is size of solid area in between slits).

$$\sum_{-N/2}^{N/2} \exp(-iknd\sin\theta) = \frac{\sin\left(\frac{N+1}{2}kd\sin\theta\right)}{\sin(\frac{1}{2}kd\sin\theta)}$$

from putting it into the form $\text{sum} = a + ar + ar^2 \ldots$ and using $\text{sum} = \frac{a(1-r^n)}{1-r}$

$$\therefore \psi(\theta) = \frac{\sin\left(\frac{N+1}{2}kd\sin\theta\right)}{\sin(\frac{1}{2}kd\sin\theta)} \psi_0 \int_{-w/2}^{w/2} t\exp(-iky\sin\theta)\,dy$$

$$\psi(\theta) = \frac{\sin\left(\frac{N+1}{2} kd\sin\theta\right)}{\sin\left(\frac{1}{2} kd\sin\theta\right)} \psi_0 \int_{-w/2}^{w/2} t\exp(-iky\sin\theta)\, dy$$

Letting $N' = N+1$, we can write this more helpfully as:

$$\psi(\theta) = \frac{\sin\left(\frac{N'}{2} kd\sin\theta\right)}{\sin\left(\frac{1}{2} kd\sin\theta\right)} \psi_0 \int_{-\frac{w}{2}}^{\frac{w}{2}} t\exp(-iky\sin\theta)\, dy \qquad t=1 \text{ here so:}$$

$$\psi(\theta) = \frac{\sin\left(\frac{N'}{2} kd\sin\theta\right)}{\sin\left(\frac{1}{2} kd\sin\theta\right)} \psi_0 d \frac{\sin\left(\frac{1}{2} kw\sin\theta\right)}{\frac{1}{2} kw\sin\theta}$$

if the space between the slits d is the same as the slit width:

$$\underline{\underline{\psi(\theta) = \frac{\sin\left(\frac{N'}{2} kd\sin\theta\right)}{\sin\left(\frac{1}{2} kd\sin\theta\right)} \psi_0 \frac{\sin\left(\frac{1}{2} kw\sin\theta\right)}{\frac{1}{2} k\sin\theta}}}$$

we find narrow peaks wherever the denominator $\sin\left(\frac{kd\sin\theta}{2}\right)$ approaches 0

$\therefore$ we require $\frac{kd\sin\theta}{2} = m\pi \Rightarrow \underline{\underline{d\sin\theta = m\lambda}}$ for narrow peaks

where m is an integer which serves as diffraction order

So the angle peaks occur at is give by $\underline{\underline{\sin\theta = \frac{m\lambda}{d}}}$