Electric fields

One way to think about fields is to think of hills. At every point in space on the hill, there is a value (height) associated with that point. This is an example of a scalar field since every point in space has a one dimensional value associated with it.

Vector fields have 3 numbers (a three dimensional vector) associated with each point in space. An example of this would be wind velocity in the atmosphere.

We can invent our own vector field whose value is a three dimensional vector that tells you the force a 1 coulomb charge would experience if placed in that point in space. This can be written mathematically as:

$$\boxed{\underline{E} = \frac{\underline{F}_Q}{Q} = \frac{Q}{4\pi\varepsilon_0 |\underline{r}|^2}\,\hat{\underline{r}}}$$

This is the electric field: the force on Q, normalised to Q to represent a unit point charge.

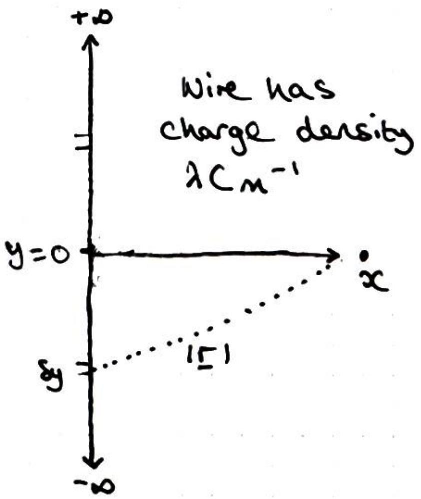
The units are NC^{-1}.

We can draw electric fields with lines whose tangent is in the direction $\underline{E}$ and whose density is proportional to $|\underline{E}|$. Some rules to drawing these lines are:

1. Begin on positive charges and end on negative charges
2. No. lines entering/leaving a charge $\propto$ total charge of object
3. Line density $\propto$ magnitude of field (electric field strength)
4. From a distance, it will always look like a single point charge.

Electric Fields from Multiple Charges

An example with a charged wire of infinite length:



What is $\underline{E}$ at x? In other words, what is the force a unit point charge would experience from the wire's electric field if placed at x?

The formula gives us:

$$E = \frac{Q}{4\pi\varepsilon_0 |\underline{r}|^2}\hat{\underline{r}} = \frac{Q}{4\pi\varepsilon_0 |\underline{r}|^3}\underline{r}$$

We can divide the wire into many small regions δy in length, each with an electric field of $\delta\underline{E}$. Charge Q is given by charge density × length = $\lambda \times \delta y$:

$$\underline{E} = \sum_{-\infty}^{\infty} \frac{\lambda \delta y}{4\pi\varepsilon_0 |\underline{r}|^3}\underline{r}$$

In order to work out $|\underline{r}|$ and $\underline{r}$, we can look at the diagram.

Using Pythagoras: $|\underline{r}| = \sqrt{x^2+y^2}$

Reading from diagram: $\underline{r} = (x, -y, 0)$ NB:// it is $-y$ since you need to go $-y$ of the y value.

We can now write the problem as:

$$\underline{E} = \sum_{-\infty}^{\infty} \frac{\lambda \delta y}{4\pi\varepsilon_0 (x^2+y^2)^{3/2}} \begin{bmatrix} x \\ -y \\ 0 \end{bmatrix}$$

This can then be written as an integral:

$$\underline{E} = \int_{-\infty}^{\infty} \frac{\lambda}{4\pi\varepsilon_0 (x^2+y^2)^{3/2}} \begin{pmatrix} x \\ -y \\ 0 \end{pmatrix} dy$$

A note on standard integrals

Some standard integrals you should know are:

$$\int \frac{x}{(x^2+y^2)^{3/2}}\, dy = \frac{y}{x(x^2+y^2)^{1/2}} + c$$

$$\int \frac{y}{(x^2+y^2)^{3/2}}\, dy = -\frac{1}{(x^2+y^2)^{1/2}} + c$$

These don't have to be memorised as practice with questions will result in familiarity with them. They can also be looked up in integral tables in real-world use.

Back to the problem

To integrate a vector, we divide it into its x, y, z components:

$$E_x = \frac{\lambda}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{x}{(x^2+y^2)^{3/2}}\, dy = \frac{\lambda}{4\pi\varepsilon_0} \left[\frac{y}{x(x^2+y^2)^{1/2}} \right]_{-\infty}^{\infty}$$

when $y \to \infty$, $(x^2+y^2)^{1/2} \to y \quad \therefore \frac{y}{x(x^2+y^2)^{1/2}} \to \frac{1}{x}$

when $y \to -\infty$, $(x^2+y^2)^{1/2} \to -y \quad \therefore \frac{y}{x(x^2+y^2)^{1/2}} \to -\frac{1}{x}$

$$\therefore E_x = \frac{\lambda}{4\pi\varepsilon_0} \left[\frac{y}{x(x^2+y^2)^{1/2}} \right]_{-\infty}^{\infty}$$

$$= \frac{\lambda}{4\pi\varepsilon_0} \left(\frac{1}{x} - \left(-\frac{1}{x} \right) \right) = \underline{\frac{\lambda}{2\pi\varepsilon_0 x}}$$

we now have the x component of $\underline{E}$

For the y component:

$$E_y = \frac{\lambda}{4\pi \varepsilon_0} \int_{-\infty}^{\infty} \frac{-y}{(x^2+y^2)^{3/2}} \, dy$$

$$= \frac{\lambda}{4\pi \varepsilon_0} \left[\frac{1}{(x^2+y^2)^{3/2}} \right]_{-\infty}^{\infty}$$

As $y \to \pm\infty$, $\frac{1}{(x^2+y^2)^{3/2}} \to 0$

$\therefore$ $E_y = 0$. There is no y component to the electric field at x. We can see this intuitively from the diagram since there is an equal amount of charged wire above and below point x so there would be no net force on a unit charge placed at x.

$E_z = 0$. This also makes sense intuitively as there is no wire (and thus no charge and no force) in the z direction.

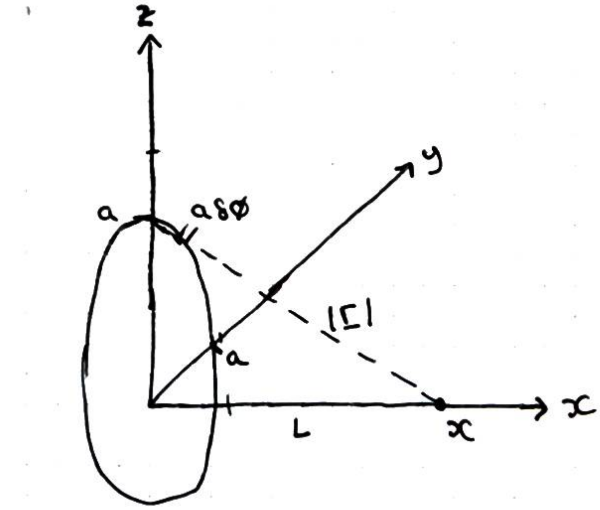
Therefore, $\underline{E}$ at x can be written:

$$\underline{E} \text{ at } x = \frac{\lambda}{2\pi \varepsilon_0 |x|} \hat{\underline{i}}$$

We can generalise this result for any point near the wire since the wire is always infinite, there is always equal amounts of wire above and below the chosen point. The z component also doesn't matter as it is a one dimensional line. The general result is therefore:

$$\underline{E} = \frac{\lambda}{2\pi \varepsilon_0 |\underline{r}|} \hat{\underline{r}}$$

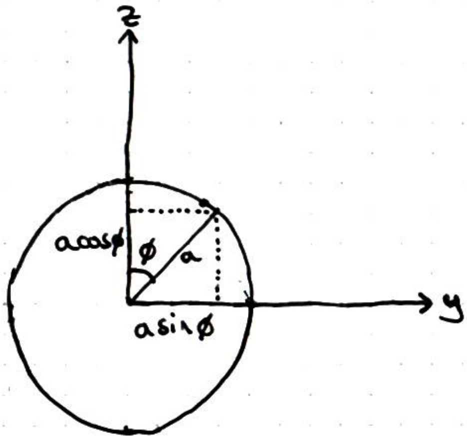
Another example with a charged ring, radius a, total charge Q:



Note, this is a ring NOT a disc.

The ring is on the y-z plane. The radius of the circle is a. ∴ Any arc length is given by $a\phi$ where ϕ is a chosen angle. For an infinitesimally small arc length, we use $a\delta\phi$

To simplify the diagram, we can draw it in only the y-z plane for now:



x is located on a line perpendicular to the ring, going through its centre, L units away from the centre.

Total charge is Q so charge density is Q divided by circumference.

$$\lambda = \frac{Q}{2\pi a}$$

For the distance from x to any point on the ring: using pythagoras $|\underline{r}| = \sqrt{a^2 + L^2}$

The vector $\underline{r}$ is the vector from any point on the ring to x and it is given by the components:

$$\underline{r} = (L, -a\sin\phi, -a\cos\phi)$$

Since Q is the total charge and charge density is given by $\frac{Q}{2\pi a}$, we can work out a small charge on the ring by multiplying the charge density by a small arc length:

$$\delta Q = \lambda \times \text{arc length}$$
$$= \frac{Q}{2\pi a} \times a\,\delta\phi$$
$$= \frac{Q}{2\pi}\,\delta\phi$$

We can now write the formula for electric field:

$$\underline{E} = \sum_{0}^{2\pi} \frac{\delta Q}{4\pi\varepsilon_0 |\underline{r}|^2}\,\hat{\underline{r}} = \sum_{0}^{2\pi} \frac{\delta Q}{4\pi\varepsilon_0 |\underline{r}|^3}\,\underline{r}$$

$$= \sum_{0}^{2\pi} \frac{\frac{Q}{2\pi}\,\delta\phi}{4\pi\varepsilon_0 (a^2+L^2)^{3/2}} \begin{pmatrix} L \\ -a\sin\phi \\ -a\cos\phi \end{pmatrix}$$

we can write this as an integral as:

$$\underline{E} = \int_0^{2\pi} \underbrace{\frac{Q/2\pi}{4\pi\varepsilon_0 (a^2+L^2)^{3/2}}}_{\text{this is all a constant}} \begin{pmatrix} L \\ -a\sin\phi \\ -a\cos\phi \end{pmatrix} d\phi$$

$$\therefore \underline{E} = \frac{Q/2\pi}{4\pi\varepsilon_0 (a^2+L^2)^{3/2}} \begin{bmatrix} L(\phi) \\ a\cos\phi \\ -a\sin\phi \end{bmatrix}_0^{2\pi}$$

$$= \frac{Q/2\pi}{4\pi\varepsilon_0 (a^2+L^2)^{3/2}} \begin{bmatrix} L\,2\pi \\ 0 \\ 0 \end{bmatrix} = \frac{QL}{4\pi\varepsilon_0 (L^2+a^2)^{3/2}}\,\hat{\underline{i}}$$

This is a specific solution for this particular x. A general solution is harder (but not impossibley hard) to get in this case.