Sinusoidal Waveforms

Sometimes it is very convenient to consider sinusoidal solutions to wave equations. There are many real-world examples in which waves are (or can be approximated to be) sinusoidal. Sinusoids also let us think about standing wave solutions.

Sinusoidal travelling waves

Consider a general sinusoidal travelling wave : $\psi(x,t) = A\sin(ku)$
where $u = x - vt$

$$\therefore \quad A\sin(k(x-vt)+\phi) = A\sin(kx - kvt + \phi)$$

$$A\sin(k(x-vt)+\phi+2\pi) = A\sin(k(x+\lambda) - kvt + \phi)$$ since it is periodic over 2π as well as λ

$$A\sin(kx - kvt + \phi + 2\pi) = A\sin(kx - kvt + \phi + k\lambda)$$

$$\therefore \quad k\lambda = 2\pi \Rightarrow \underline{\underline{k = \frac{2\pi}{\lambda}}}$$

Similar arguments with periodicity in time, i.e $t = t + T$ gives us $T = \frac{\lambda}{v}$ but $T = \frac{1}{f}$ so $\underline{\underline{v = f\lambda}}$

Combining these and using $2\pi f = \omega$ gives us $\boxed{\omega = vk}$

The v here is known as phase velocity.

If we use sinusoidal waves as our general form, we obtain a "dispersion relation". i.e a relation between the quantities in the wave equation and ω and k. You <u>must</u> state this as "provided [dispersion relation] is true, trial form is a solution". Otherwise you won't get any marks!

Energy of a Wave Motion

We know that waves convey energy, but how do we calculate this energy? Let's consider the waves on a long string example.

The kinetic energy K of a small element on the string with mass δm is: $K = \frac{1}{2} \delta m \, V_t^2$ where V_t is transverse velocity

$\delta m = M \delta x$ where M is mass per unit length

$$\therefore K = \frac{1}{2} M \, \delta x \, V_t^2 \Rightarrow \therefore K = \underline{\underline{\frac{M \delta x}{2} \left(\frac{\partial \psi}{\partial t} \right)^2}}$$

Now we can consider the potential energy. To calculate this, we must first work out what the work done is in stretching the string a length δx against its tension W.

$$\text{length after stretch} = \sqrt{\delta x^2 + \delta \psi^2} = \sqrt{\delta x^2 + \left(\delta x \frac{\partial \psi}{\partial x} \right)^2}$$

$$\begin{aligned} \text{extension} &= \text{length after stretched} - \text{initial length} \\ &= \sqrt{\delta x^2 + \left(\delta x \frac{\partial \psi}{\partial x} \right)^2} - \delta x \\ &= \delta x \left[\sqrt{1 + \left(\frac{\partial \psi}{\partial x} \right)^2} - 1 \right] \quad \text{By binomial expansion:} \\ &\approx \frac{1}{2} \delta x \left(\frac{\partial \psi}{\partial x} \right)^2 \end{aligned}$$

$$\text{Potential} = \int \underline{F} \cdot d\underline{r} \Rightarrow \text{Force} \times \text{distance}$$

$$U = \underline{\underline{\frac{W \delta x}{2} \left(\frac{\partial \psi}{\partial x} \right)^2}}$$

By taking $\psi(x - v_p t) = \psi(u)$, we can get:

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial u} \frac{\partial u}{\partial x} \Rightarrow \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial u}$$

$$\frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial u} \frac{\partial u}{\partial t} \Rightarrow \frac{\partial \psi}{\partial t} = -v_p \frac{\partial \psi}{\partial u}$$

We can substitute these into K and U to find:

$$K = \frac{M\delta x}{2} v_p^2 \left(\frac{\partial \psi}{\partial u}\right)^2 \qquad U = \frac{W\delta x}{2}\left(\frac{\partial \psi}{\partial u}\right)^2$$

since we previously found $v_p^2 = \frac{W}{M}$, we get this only be equating $K = U$. so the total energy is an equal contribution of both: $\underline{\underline{\mathcal{E} = K + U}}$

$$\underline{\underline{\mathcal{E} = W\left(\frac{\partial \psi}{\partial u}\right)^2 \delta x}} = W\left(\frac{\partial \psi}{\partial x}\right)^2 \delta x$$

This lets us work out energy density $\frac{\mathcal{E}}{\delta x}$:

$$\epsilon = \underline{\underline{W\left(\frac{\partial \psi}{\partial u}\right)^2}} = W\left(\frac{\partial \psi}{\partial x}\right)^2$$

Power transmitted in Wave Motion

Now that we have energy, we can easily extend this to Power.

The power at a particular point is the energy passing that point per unit time. $P = \frac{E}{t} = \frac{\epsilon \times \text{distance}}{\text{time}}$ = energydensity x phase velocity

So $P = \epsilon v_p = W v_p \left(\frac{\partial \psi}{\partial x}\right)^2$

Standing Waves

Whilst travelling waves have the form $f(x - vt)$
standing waves have the form $X(x)\,T(t)$, i.e, the variables are separated.

So if we have $\psi(x,t) = X(x)\,T(t)$, we get:

$$\frac{\partial \psi}{\partial x} = \frac{\partial X}{\partial x} T(t) \qquad \frac{\partial \psi}{\partial t} = \frac{\partial T}{\partial t} X(x)$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 X}{\partial x^2} T(t) \qquad \frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 T}{\partial t^2} X(x)$$

So if take the example of the long string, and use it for a guitar string in which standing waves are generated:

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{W}{M}\frac{\partial^2 \psi}{\partial x^2} \Rightarrow M\frac{\partial^2 \psi}{\partial t^2} = W\frac{\partial^2 \psi}{\partial x^2}$$

subbing in the derivatives:

$$MX(x)\frac{\partial^2 T}{\partial t^2} = WT(t)\frac{\partial^2 X}{\partial x^2} \Rightarrow \frac{M}{W}\cdot\frac{1}{T}\frac{\partial^2 T}{\partial t^2} = \frac{1}{X}\frac{\partial^2 X}{\partial x^2}$$

Since the left hand side does not vary with X and the right hand side does not vary with T, for them to be equal to each other they must be equal to some constant, say K.

$$\therefore \frac{M}{W}\frac{1}{T}\frac{\partial^2 T}{\partial t^2} = \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -K^2$$

let $= -K^2$ for later convenience

This gives 2 diff. eqns.

① $\frac{M}{W}\frac{1}{T}\frac{\partial^2 T}{\partial t^2} = -K^2 \Rightarrow T(t) = T_0 \sin(\omega t + \Theta)$

② $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -K^2 \Rightarrow X(x) = X_0 \sin(Kx + \phi)$

These are the solutions to the differential eqns.

where $\omega = \sqrt{\frac{W}{M}}K$, giving $\frac{\omega}{K} = v_p = \sqrt{\frac{W}{M}}$ as expected

so the general solution is:

$$\psi(x,t) = \psi_0 \sin(Kx + \phi)\sin(\omega t + \Theta)$$

which is what we expected having completed the Waves, Light & Quanta course last year.

A note about standing waves:

Since we know the identity $\sin(Kx)\sin(\omega t) = \frac{1}{2}[\cos(Kx - \omega t) - \cos(Kx + \omega t)]$ we can say the sum of 2 travelling waves makes a standing wave.

Since we know the identity $\cos(Kx)\cos(\omega t) + \sin(Kx)\sin(\omega t) = \cos(Kx - \omega t)$ two superposed standing waves make a travelling wave.