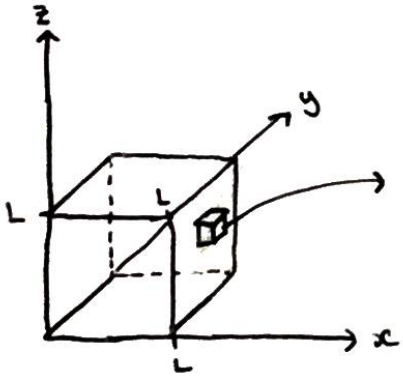
Volume Integrals

Just like with Area, the volume can be calculated by summing the volumes of infinitesimally small cubes. Each cube has a tiny length in the x, y and z directions. For example:



construct a small cube inside the larger cube. This cube has sides, δx δy and δz making the volume of the cube:

$$\delta V = \delta x \, \delta y \, \delta z$$

For the volume of the large cube, we need to sum up all the volumes of the smaller cubes from:

x in the range 0 to L.
y in the range 0 to L.
z in the range 0 to L.

So total volume $= \Sigma \delta V$ which can be written in integral form as:

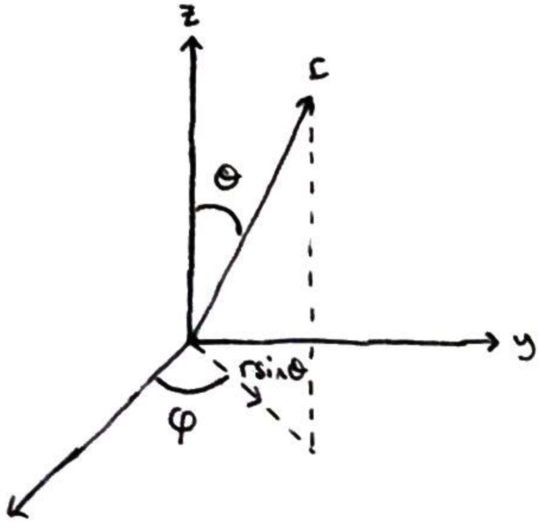
$$V = \int_0^L \int_0^L \int_0^L \delta x \, \delta y \, \delta z$$

This is a triple integral. It doesn't matter which order you evaluate the integrals in:

$$V = [x]_0^L \, [y]_0^L \, [z]_0^L = \underline{\underline{L^3}}$$ which is the volume of a cube.

3-D Polar Coordinates

In problems with spherical symmetry, it is easier to use spherical polar coordinates:



In this system, the limits are:

$0 \leq r < \infty$

$0 \leq \theta \leq \pi$

$0 \leq \varphi \leq 2\pi$

Note that θ is only between 0 and π because angles higher than this can be obtained by altering φ instead.

Now imagine that r reaches the surface of a sphere centred on the origin. To find the volume of the sphere:

Consider a small cube inside the sphere:

The volume of this cube is $\delta V = r^2 \sin\theta \, \delta r \, \delta\theta \, \delta\varphi$

So the volume of the sphere is:

$$V = \int_0^R \int_0^\pi \int_0^{2\pi} r^2 \sin\theta \, d\varphi \, d\theta \, dr$$

$$= \left[\frac{r^3}{3}\right]_0^R \left[\varphi\right]_0^{2\pi} \left[-\cos\theta\right]_0^\pi$$

$$= \underline{\underline{\frac{4}{3}\pi R^3}}$$