Wave Equations and their Solutions

For any system, it is possible to write a wave equation that describes the physics in the form of a partial differential equation, relating time and spacial derivatives of the wavefunction.

To understand the physics behind any system, we follow the same procedure:

$\boxed{\text{Use specific physics to write a partial diff. equation, i.e wave eqn. of the system}}$ ①

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$\boxed{\text{Substitute General. Trial soln. into the wave equation}}$ ②

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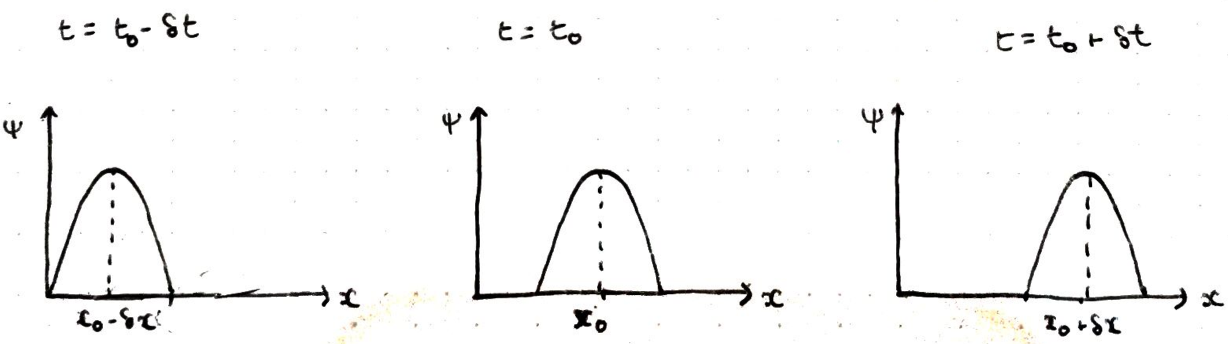
$\boxed{\text{Find the parameter values for which trial form is a soln.}}$ ③

So how do we do ①?

Constructing a Wave Equation: Arbitrary Travelling Wave

Consider some arbitrary wave $\psi(x,t)$:



We can see how the wave travels in space over time.

If we say the wave travels at a constant phase velocity v, it is trivial to notice $\delta x = v\,\delta t$

The value of ψ is equal at $(x_0 - \delta x, t_0 - \delta t)$ and (x_0, t_0) so: This is actually true for x_0 is any x and t_0 is any t so:

$$\psi(x - \delta x, t - \delta t) = \psi(x, t)$$

[subtract $\psi(x, t-\delta t)$ and divide by δt]

$$\frac{\psi(x - \delta x, t - \delta t) - \psi(x, t - \delta t)}{\delta t} = \frac{\psi(x, t) - \psi(x, t - \delta t)}{\delta t}$$

$$\frac{\psi(x, t) - \psi(x, t - \delta t)}{\delta t} = v\,\frac{\psi(x - \delta x, t - \delta t) - \psi(x, t - \delta t)}{\delta x}$$

since $\frac{1}{\delta t} = \frac{v}{\delta x}$

$$\frac{\psi(x, t) - \psi(x, t - \delta t)}{\delta t} = -v\,\frac{\psi(x, t - \delta t) - \psi(x - \delta x, t - \delta t)}{\delta x}$$

If we take the limits $\delta t \to 0$ and $\delta x \to 0$, then the left side term becomes $\frac{\partial \psi}{\partial t}$ and the right term $\frac{\partial \psi}{\partial x}$:

$$\frac{\partial \psi}{\partial t} = -v\,\frac{\partial \psi}{\partial x}$$

Thus, we have constructed a wave equation

We can actually generalise this to be:

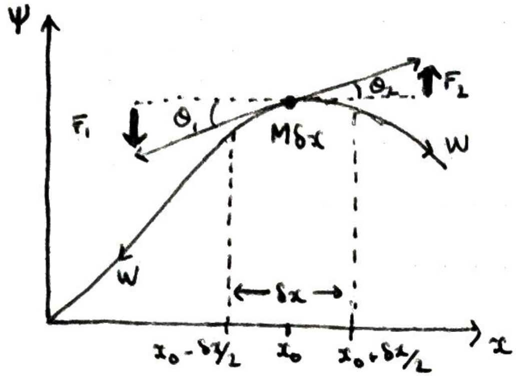
$$\boxed{\frac{\partial^n \psi}{\partial t^n} = (-v)^n \frac{\partial^n \psi}{\partial x^n}}$$

for any even n.

We say even n since otherwise we would have to also take into account the sign (and thus direction) of v.

Constructing a Wave Equation: Long String

Consider a taut string with mass per unit length M



W is the tension force on the string

F_1 and F_2 are vertical components of the forces acting on a point x_0

$M\delta x$ is the mass of the string in the infinitesimal

If θ is small, we can ignore horizontal components as they will be approximately equal and opposite (so will cancel out).

$F_1 = W\sin\theta_1 \quad F_2 = W\sin\theta_2$ so the net force is given by:

$$W\sin\theta_2 - W\sin\theta_1 = W(\sin\theta_2 - \sin\theta_1)$$

Since $\sin\theta = \frac{opp}{adj} = \frac{\psi}{x}$ here, if θ is small, we can say:

$\sin\theta = \frac{\partial\psi}{\partial x}$ so:

$$W(\sin\theta_2 - \sin\theta_1) = W\left[\left(\frac{\partial\psi}{\partial x}\right)_{x+\frac{\delta x}{2}} - \left(\frac{\partial\psi}{\partial x}\right)_{x-\frac{\delta x}{2}}\right]$$

From Newton's 2nd Law: $F = ma \Rightarrow$ net force $= M\delta x \times \frac{\partial^2\psi}{\partial t^2}$

$$\therefore \quad M\delta x \frac{\partial^2\psi}{\partial t^2} = W\left[\left(\frac{\partial\psi}{\partial x}\right)_{x+\frac{\delta x}{2}} - \left(\frac{\partial\psi}{\partial x}\right)_{x-\frac{\delta x}{2}}\right]$$

$$\frac{\partial^2\psi}{\partial t^2} = \frac{W}{M}\,\frac{\left[\left(\frac{\partial\psi}{\partial x}\right)_{x+\delta x/2} - \left(\frac{\partial\psi}{\partial x}\right)_{x-\frac{\delta x}{2}}\right]}{\delta x}$$

Taking the limit $\delta x \to 0$, we get:

$$\underline{\underline{\frac{\partial^2\psi}{\partial t^2} = \frac{W}{M}\frac{\partial^2\psi}{\partial x^2}}}$$

We have thus derived a wave equation

Substituting General Solutions: Long String

We can immediately compare our long string wave equation to the general wave equation for an arbitrary wave and extract $v = \pm\sqrt{\frac{W}{M}}$ but we can also obtain this in our more rigorous way (from the earlier flow chart).

We define a general travelling wave solution $\psi(x,t) = \psi(x - vt)$

For convenience: $u = x - vt \Rightarrow \psi(x-vt) = \psi(u)$

$$\frac{\partial \psi}{\partial t} = \frac{\partial u}{\partial t}\frac{\partial \psi}{\partial u} = -v\frac{\partial \psi}{\partial u} \Rightarrow \underline{\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial u^2}}$$

$$\frac{\partial \psi}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial \psi}{\partial u} = \frac{\partial \psi}{\partial u} \Rightarrow \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u}\cdot\frac{\partial \psi}{\partial u}$$

$$\underline{\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \psi}{\partial u^2}}$$

Substituting into the wave eqn. $\frac{\partial^2 \psi}{\partial t^2} = \frac{W}{M}\frac{\partial^2 \psi}{\partial x^2}$:

$$v^2 \frac{\partial^2 \psi}{\partial u^2} = \frac{W}{M}\frac{\partial^2 \psi}{\partial u^2} \Rightarrow v^2 = \frac{W}{M} \quad \text{so} \quad \underline{\underline{v = \pm\sqrt{\frac{W}{M}}}}$$

This is just as we expected.

We have thus found the parameter values for which the trial form is a solution.

We should realise that our wave equation has the important property of linearity. i.e if ψ_1 and ψ_2 are solutions, then $(a\psi_1 + b\psi_2)$ is also a solution, where a and b are arbitrary constants.

Finding Particular Solutions : Long String

We can now fit the general solution to the constraints to find a particular solution.
The general solution gave a +ve $|V|$ and a -ve $|V|$ so we can write it as:

$$\psi(x,t) = \psi_+(x - |V|t) + \psi_-(x + |V|t)$$

If we know at time $t=0$ then $V=0$:

$$\psi(x,0) = \psi_+(x) + \psi_-(x) \qquad (*)$$

We can write the derivative as:

$$\frac{\partial\psi(x,0)}{\partial t} = \left(\frac{\partial\psi_+(x-|V|t)}{\partial t}\right)_{(x,0)} + \left(\frac{\partial\psi_-(x+|V|t)}{\partial t}\right)_{(x,0)}$$

for convenience, let $u_+ = x - |V|t$ and $u_- = x + |V|t$

$$\frac{\partial\psi(x,0)}{\partial t} = \frac{\partial u_+}{\partial t}\frac{\partial\psi_+}{\partial u_+} + \frac{\partial u_-}{\partial t}\frac{\partial\psi_-}{\partial u_-}$$

$$\frac{\partial\psi(x,0)}{\partial t} = -|V|\frac{\partial\psi_+}{\partial u_+}(x,0) + |V|\frac{\partial\psi_-}{\partial u_-}(x,0)$$

at $t=0$ $\quad \partial u_+ = \partial x$ and $\partial u_- = \partial x$

$$\therefore \frac{\partial\psi(x,0)}{\partial t} = |V|\left\{\frac{\partial\psi_-(x)}{\partial x} - \frac{\partial\psi_+(x)}{\partial x}\right\}$$

This is the particular solution for $t=0$, $v=0$ conditions

Another condition we can use is that for a plucked guitar string. This is similar as this example but extends it by adding the conditions, at $t=0$ $\frac{\partial\psi}{\partial t} = 0$ and at $x=0$ and $x=l$, $\psi(x,t) = 0$

Finding Particular Solutions : Plucked Long String

As mentioned earlier, the conditions are now:

① At $x=0$ and $x=L$, $\psi(0,t)=0$ and $\psi(L,t)=0$

② At $t=0$, $\psi(x,0) = \psi_0(x,0) = \psi_+(x) + \psi_-(x)$

③ At $t=0$, $\dfrac{\partial \psi(x,0)}{\partial t} = 0$

So continuing from the previous example, we can say $\dfrac{\partial \psi}{\partial t} = 0$, to give us:

$$\frac{\partial \psi_-}{\partial x} = \frac{\partial \psi_+}{\partial x} \Rightarrow \psi_-(x) = \psi_+(x) + c \quad \text{at } t=0$$

But from ①: $\psi_0(x) = 2\psi_+(x) + c$

$$\therefore \psi_+(x) = \tfrac{1}{2}(\psi_0(x) - c) \quad \text{and} \quad \psi_-(x) = \tfrac{1}{2}(\psi_0(x) + c) \quad (*)$$

We previously said $\psi(x,t) = \psi_+(x - |v|t) + \psi_-(x + |v|t)$

Subbing $(*)$ into this:

$$\psi(x,t) = \tfrac{1}{2}\left[\psi_0(x - |v|t) + \psi_0(x + |v|t)\right]$$

You may have noticed that we never used condition ① So the solution we obtained is not complete and only works for $0 < x < L$. Beyond this, we have to use condition ① which would give us:

$$\psi_+(u) = -\psi_-(2L - u) = \psi_+(2L + u)$$

More on this in the boundary conditions chapter