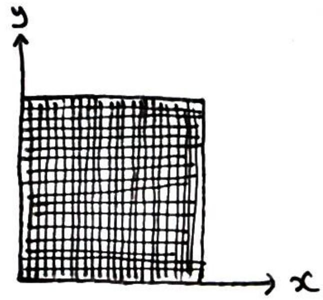
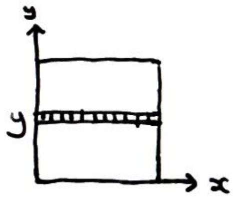
Area Integrals

This is a method of working out the area of a shape using integration. The simplest example of this is to compute the area of a square. To do this, we first split up the larger square into smaller squares:



The width of each square is δx
The height of each square is δy
$\therefore$ The area of the tiny square is $\delta x \delta y$
Total Area $= \Sigma \delta x \delta y$

To do the sum of all the tiny squares, we can first sum one strip and the sum the area of all the strips.



Each square has the same value of y, δy.
So we just need to sum all the δx's.
Area Strip $= \delta y (\Sigma \delta x)$. This can be written as an integral in the form:

Area Strip $= \delta y \int_0^L dx = \delta y L$

Now we can add up all the strips across all y:
Total Area $= \Sigma L \delta y$ which can be written as the integral:

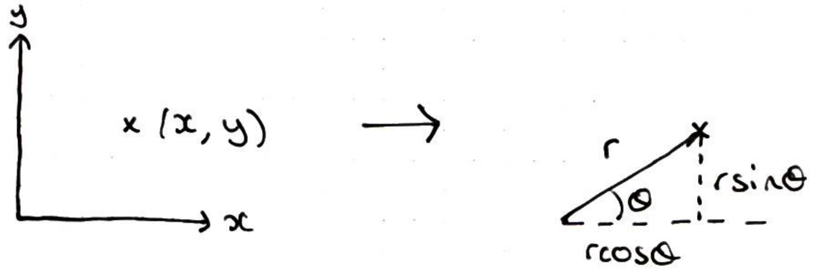
Total Area $= L \int_0^L dy = L^2$ So we have the area of the square is L^2. Shocking!

Notice that we have just done something called a double integral. The first sum $\Sigma \delta x \delta y$ could have been written directly as:

Total Area $= \int_0^L \int_0^L dx\, dy$ The integrals can be done in either order.

2d Polar Coordinates

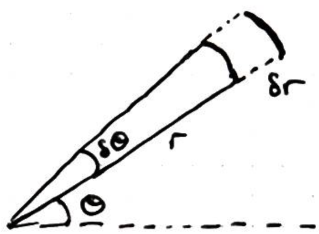
In problems with rotational symmetry, for example when working with a disc, it is sometimes easier to use a polar coordinate system (defined with values of r and θ) instead of a cartesian coordinate system.



Generally, the conversion is: $\boxed{x = r\cos\theta \quad y = r\sin\theta}$

This polar system uses a value for distance from centre, called r, and a value for angle from the horizontal, called θ, in the ranges $0 \leq r \leq \infty$, $0 \leq \theta < 2\pi$.

In the same way that we can divide a shape in a cartesian system into infinitesimal lines or squares, we can do the same in a polar system:



$\delta\theta$ is an infinitesimal increase of rotation from θ and δr is an infinitesimal increase of radius from r.

Note that the arc length of the formed arc is $r\delta\theta$, making each small "square" look like:

, giving each square Area $= r\,\delta\theta\,\delta r$

The line integral can be worked out in the Polar System in much the same way as the cartesian system.

Here is an example with a circle:

Divide circle into small arcs of length $R\delta\theta$.

Summing all the arc lengths:

$$\text{Circumference} = \sum \text{Arc Lengths}$$

$$= \sum_{\theta=0}^{2\pi} R\,\delta\theta \quad \text{which is rewritten as:}$$

$$= \int_0^{2\pi} R\,d\theta$$

$$= R\left[\theta\right]_0^{2\pi}$$

$$= \underline{2\pi R}$$

The same is true for working out the Area Integral:

Divide circle into small 'squares' of area:

Area square $= r\,\delta\theta\,\delta r$

Summing the area of the squares:

$$\text{Area} = \sum \text{Area Squares} \quad \text{which is really 2 sums}$$

$$= \sum_{r=0}^{R}\sum_{\theta=0}^{2\pi} r\,\delta\theta\,\delta r \quad \text{which is rewritten as:}$$

$$= \int_0^R r \int_0^{2\pi} d\theta\, dr$$

$$= \left[\frac{R^2}{2}\right]\left[2\pi\right]$$

$$= \underline{\pi R^2}$$