Waves in 2 and 3 dimensions

Now we will briefly look at how to extend what we've covered so far in 2 or 3 dimensions.

In 1D, our wave equation for a long string was:

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{W}{M}\frac{\partial^2 \psi}{\partial x^2}$$

extending this to 3D:

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{W}{M}\left\{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}\right\} \Rightarrow \underline{\underline{\frac{\partial^2 \psi}{\partial t^2} = \frac{W}{M}\nabla^2 \psi}}$$

A general form in 3D would be: $\underline{\underline{\frac{\partial^2 \psi}{\partial t^2} = v_p^2 \nabla^2 \psi}}$

When considering wave propagation in more than 1D, it might be easiest to separate the variables. eg.

$$\psi(x, y, z, t) = X(x)Y(y)Z(z)T(t)$$

so we can write the 3D wave equation as:

$$XYZ\frac{d^2T}{dt^2} = v_p^2\left\{YZT\frac{d^2X}{dx^2} + XZT\frac{d^2Y}{dy^2} + XYT\frac{d^2Z}{dz^2}\right\}$$

if we say $\frac{d^2X}{dx^2} = Xa$ $\frac{d^2Y}{dy^2} = Yb$ $\frac{d^2Z}{dz^2} = Zc$, we can sub these in and divide both sides by $XYZT$ to get:

$$\frac{1}{T}\frac{d^2T}{dt^2} = v_p^2\{a + b + c\}$$

So what are a, b and c? For standing waves, when we separated the variables, we said straight away that it was $-k^2$ but we will actually show that here:

trial solution $\Rightarrow X = X_0 e^{ik_x x}$ so $\frac{d^2X}{dx^2} = -X_0 k_x^2 e^{ik_x x} = aX$

$\therefore$ $\underline{a = -k_x^2}$ similarly, $\underline{b = -k_y^2}$ and $\underline{c = -k_z^2}$

If we let $T = T_0 e^{-i\omega t}$, we can write the full 3D trial solution as:

$$\psi = \psi_0 e^{ik_x x} e^{ik_y y} e^{ik_z z} e^{-i\omega t}$$

$$= \psi_0 e^{i(k_x x + k_y y + k_z z - \omega t)}$$

$$= \underline{\psi_0 e^{i(\underline{k} \cdot \underline{r} - \omega t)}} \qquad |k| = \frac{2\pi}{\lambda} \text{ as normal}$$

where $\underline{k} = k_x \hat{\underline{x}} + k_y \hat{\underline{y}} + k_z \hat{\underline{z}}$ and $\hat{\underline{k}}$ is in the direction of wave propagation. So we have thus constructed a general 3D trial solution in complex exponential form.

To be honest, the best way to understand waves in 2D and 3D is to have a go at some questions. We will come across some examples later in the course.