Gravitation and Kepler's Laws

Newton's Law of Universal Gravitation

You'll recall Newton's law of gravitational attraction: $F = \frac{GM_1M_2}{r^2}$
in vector form, the force of particle 2 on particle 1 is:

$$\boxed{\underline{F}_{12} = -\underline{F}_{21} = \frac{GM_1M_2}{r_{12}^2}\hat{\underline{r}}_{12}}$$

where $\underline{r}_{12} = \underline{r}_2 - \underline{r}_1$
is the vector from particle 1 to particle 2

Gravity is a central force and therefore conservative so we can define the potential energy difference between two points as:

$$V(\underline{r}_2) - V(\underline{r}_1) = -\int_{\underline{r}_1}^{\underline{r}_2} \underline{F} \cdot d\underline{r}$$

If we have some two particles, how do we determine the gravitational potential energy of one of these? We can only observe differences in gpe, so we need some reference point that we set to 0. Since there is "0" force when the two particles are infinite distance apart, we set this point as 0 potential energy. So the potential energy U at a point $\underline{r}$ for a particle 2 due to particle 1 is:

$$\boxed{V(r) = -\int_{\infty}^{r} \underline{F} \cdot d\underline{r}' = -\int_{\infty}^{r} -\frac{GM_1M_2}{r'^2}\,dr' = -\frac{GM_1M_2}{r}}$$

We define the gravitational potential as the energy a unit mass would have at a point $\underline{r}$ due to mass M:

$$\boxed{\phi(\underline{r}) = -\frac{GM}{r}}$$

Gravitational field g of particle is:

$$\boxed{g(\underline{r}) = -\frac{GM}{r^2}\hat{\underline{r}}}$$

with the relation:

$$\boxed{g = -\underline{\nabla}\phi}$$

Gravitational Potential Near Earth's Surface

Near the earth's surface, the force on a particle of mass m is given by Newton's 2nd Law:

$\underline{F} = -mg\hat{\underline{k}}$ where $\hat{\underline{k}}$ is unit vector "upwards"

Applying definition of gpe:

$$V(h) - V(0) = -\int_0^h \underline{F} \cdot d\underline{r}$$

$$= -\int_0^h -mg\hat{\underline{k}} \cdot d\underline{r} = -(-mgh)$$

$$= \underline{\underline{mgh}}$$

So we can take gpe near Earth's surface as $V = mgh$

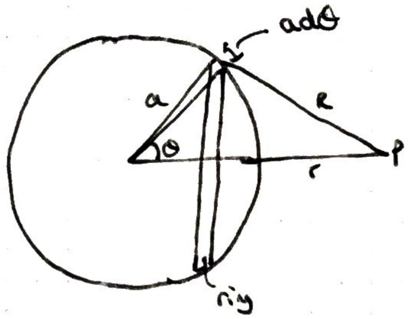
Gravitational Attraction of a Spherical Shell

Outside the shell, the force is the same as if the shell was a point mass located at its C of M, with mass equal to the shell's total mass.

Inside the shell, the force is 0 for a uniform shell.

We can prove this in 2 ways, direct calculation and making use of analogies to EM to use gauss' law.

We will start with direct calculation:



Consider a thin spherical shell of:

radius a

mass per unit area ρ

total mass $M = 4\pi\rho a^2$

The mass of a ring on the surface of the shell, width $a\,d\theta$ is:

$$dM = \underbrace{\rho}_{\text{density}} \times \underbrace{a\,d\theta}_{\text{width}} \times \underbrace{2\pi \times \underbrace{a\sin\theta}_{\text{radius of ring}}}_{\text{circumference of ring}}$$

$$dM = \underbrace{2\pi\rho a^2}_{= M/2}\sin\theta\,d\theta \qquad dM = \frac{M}{2}\sin\theta\,d\theta$$

The annulus (means ring) causes a potential at point P of:

$$d\Phi = -\frac{G\,dM}{R} = -\frac{GM}{2R}\sin\theta\,d\theta$$

It is convenient to change integration variable from θ to R

using cosine rule: $R^2 = r^2 + a^2 - 2ar\cos\theta$

$$\text{so } \frac{\sin\theta\,d\theta}{R} = \frac{dR}{ar}$$

$$\text{so } \quad d\Phi = -\frac{GM}{2}\frac{\sin\theta\,d\theta}{R} \Rightarrow -\frac{GM}{2}\frac{dR}{ar}$$

integrating from $R = |r-a|$ to $R = r+a$:

$$\phi(r) = -\frac{GM}{2ar}\int_{|r-a|}^{r+a} dR = \begin{cases} -GM/r & \text{for } r \geq a \\ -GM/a & \text{for } r \leq a \end{cases}$$

we differentiate to find field $\underline{g} = \underline{\nabla}\phi$

$$\underline{g}(\underline{r}) = \begin{cases} -\frac{GM}{r}\,\hat{\underline{r}} & \text{for } r \geq a \\ 0 & \text{for } r \leq a \end{cases}$$

This is as we expected. Force is 0 inside the uniform shell and outside, it is as if the shell was a point mass.

We will now obtain this result in perhaps an easier way.

Relabelling Gauss' Law for gravity:

$$\boxed{\int_S \underline{g} \cdot d\underline{S} = -4\pi G \int_V \rho_m \, dV}$$

dS is the surface element. The total area of a sphere is $4\pi r^2$. $\int_V \rho_m \, dV$ is the density integral over the whole volume so ~~it~~ is just M.

$\therefore \quad g \cdot 4\pi r^2 = -4\pi GM$

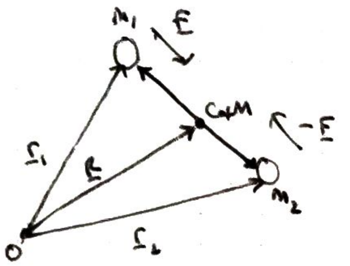
$$\Rightarrow \underline{g} = -\frac{GM}{r^2}\hat{\underline{r}} \quad \text{for } r > a$$

we added the $\hat{r}$ since is is obvious the field is radial.

Alternatively, for $r < a$, $\rho_m = 0$ since there is no density inside shell so $\underline{g} = 0$

Orbits : Preliminary

Two-body Problem : Reduced Mass



consider the setup on the left.

the centre of mass is at position $\underline{R}$

let $\underline{r}$ be the vector from m_2 to m_1 such that:

$$\underline{r} = \underline{r}_1 - \underline{r}_2$$

Let's express the position $\underline{r}_i$ of a particle as the CM location plus some displacement:

$$\underline{r}_1 = \underline{R} + \underline{\rho}_1 \qquad \underline{r}_2 = \underline{R} + \underline{\rho}_2$$

we know:

$$\underline{F} = m_1 \ddot{\underline{r}}_1 \qquad -\underline{F} = m_2 \ddot{\underline{r}}_2$$

$$\Rightarrow \quad m_1 \ddot{\underline{r}}_1 + m \ddot{\underline{r}}_2 = 0$$

$$\therefore \quad M\ddot{\underline{R}} = 0 \quad \text{letting } M = m_1 + m_2$$

This implies CM moves at constant velocity.

$$\ddot{\underline{r}} = \ddot{\underline{r}}_1 - \ddot{\underline{r}}_2 = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\underline{F} = \frac{m_1 + m_2}{m_1 m_2}\underline{F}$$

$$\Rightarrow \quad \underline{F} = \frac{m_1 m_2}{m_1 + m_2}\ddot{\underline{r}} \quad \Rightarrow \quad \boxed{\underline{F} = \mu \ddot{\underline{r}}}$$

where $\boxed{\mu = \dfrac{m_1 m_2}{m_1 + m_2}}$ and is called reduced mass, a funky kind of "average"

The total energy of the system is: $E = \frac{1}{2}M\dot{\underline{R}}^2 + \frac{1}{2}\mu\dot{\underline{r}}^2 + V(r)$ potential energy

Since CM moves at constant velocity, we can set our origin there, giving us

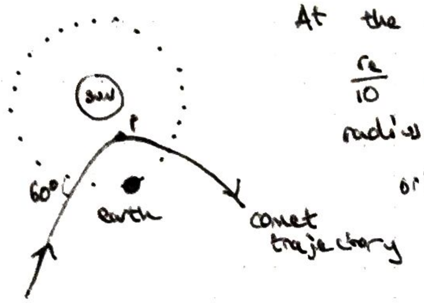
$$\boxed{\begin{aligned} E &= \tfrac{1}{2}\mu\dot{\underline{r}}^2 + V(r) \\ \underline{L} &= \mu \underline{r} \times \dot{\underline{r}} \end{aligned}}$$

in the C of M frame.

in real orbiting systems (like earth and sun) one is much higher mass than the other so $\dfrac{M_{sun}M_{earth}}{M_{sun} + M_{earth}} \approx M_{sun}$

Two-body Problem : Examples

The comet :



consider a comet on the trajectory shown.
At the closest point to the sun, the comet is $\frac{r_e}{10}$ away from sun where r_e is earth's orbit radius. The comet makes 60° on crossing earth's orbit with $v = 50\,kms^{-1}$
What is v at point P?

The key is to realise angular momentum of comet about sun is fixed.

$$\underline{L} = \underline{r} \times \underline{p} = \underline{r} \times m\underline{v}$$

At the closes point, the velocity is tangential only:

$$|\underline{r} \times \underline{v}| = r_{min} v_{max}$$

at crossing point: $|\underline{r} \times \underline{v}| = r_e v \sin 30°$

$$\therefore \quad r_{min} v_{max} = \frac{r_e v}{2} \quad \Rightarrow \quad v_{max} = 5v = \underline{\underline{250\,kms^{-1}}}$$

Cygnus X1 :

Cy X1 is a binary system of a supergiant star of $25\,M_\odot$ and a black hole $10\,M_\odot$, each in circular orbit about CM with $T = 5.6$ days. What is distance between 2 bodies?

Here, we use $\underline{F} = \mu \ddot{\underline{r}}$ where $\ddot{\underline{r}} = \underline{r}\,\omega^2$

$$\text{so} \quad \frac{GMm}{r^2} = \frac{Mm}{M+m} r\omega^2$$

rearrange for r and sub in values:

giving $r = 3 \times 10^{10}\,m$

Kepler's Laws

Kepler's Laws are:

1) The orbits of the planets are ellipses with the Sun at one focus
2) The radius vector from the Sun to a planet sweeps out equal areas in equal times
3) $T^2 \propto a^3$

So how do we derive these? Let's start with

2nd Law:

$$L = Mr^2\omega = Mr^2\dot{\theta} = \text{constant} \quad \text{(due to conservation laws)}$$

The area swept out by the radius vector from Sun to planet is $\frac{1}{2}r^2\theta$ so:

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{L}{2M} = \underline{\underline{\text{constant}}}$$

This is the proof for the 2nd Law

Orbit Equation:

We need this to derive the 1st and Third Laws.

Start from the radial equation of motion (with $K = GMm$):

$$\ddot{r} - r\dot{\theta}^2 = -\frac{K}{mr^2} \quad \Big| \text{ use } \dot{\theta} = \frac{L}{mr^2} \text{ to eliminate } \dot{\theta}$$

$$\ddot{r} - \frac{L^2}{m r^3} = -\frac{K}{mr^2}$$

Now use $\frac{d}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta} = \frac{L}{mr^2}\frac{d}{d\theta}$ to obtain diff. eqn for r in terms of θ

use $u = \frac{1}{r}$ to get:

$$\boxed{\frac{d^2u}{d\theta^2} + u = \frac{mK}{L^2}}$$

1st Law:

Starting with the orbit equation $\frac{d^2u}{d\theta^2} + u = \frac{mk}{L^2}$

A solution to this is $u = \frac{1}{r} = \frac{mk}{L^2}(1 + e\cos\theta)$

which for $0 < e < 1$ gives an <u>ellipse</u> with semi latus rectum $l = \frac{L^2}{mk}$

This is the proof for the 1st Law

3rd Law:

Starting from the 2nd Law $\frac{dA}{dt} = \frac{L}{2m}$ = constant

we integrate over the whole orbital period T:

$$\int dA = \int_0^T \frac{L}{2m}\,dt \qquad A = \pi ab$$

$$A = \frac{L}{2m}T \Rightarrow T = \frac{2mA}{L} \qquad T = \frac{2m\pi ab}{L}$$

subbing in b in terms of A:

$$\boxed{T^2 = \frac{4\pi^2}{GM}a^3}$$

This is the proof for 3rd Law

Scaling Argument for Kepler's 3rd Law

Suppose you have r, θ as functions of t and solutions to orbit equation

$$\ddot{r} - r\dot{\theta}^2 = -\frac{k}{mr^2}$$

Now scale: $r' = \alpha r$ $\quad t' = \beta t$ and sub these into orbit eqn:

LHS: $\frac{d^2r'}{dt'^2} - r'\left(\frac{d\theta}{dt'}\right)^2 = \frac{\alpha}{\beta^2}\left(\ddot{r} - r\dot{\theta}^2\right)$ $\quad$ RHS: $-\frac{k}{mr'^2} = \frac{1}{\alpha^2}\left(-\frac{k}{mr^2}\right)$

This is still a solution provided $\beta^2 = \alpha^3$. So for orbits of similar shape, period and radius are related by $T^2 \propto r^3$.

Another proof of Kepler's 3rd Law.

Effective Potential

The total energy is given by the total kinetic energy and the total potential energy.
The total kinetic energy is the sum of the linear motion energy and rotational energy. We also set the reduced mass $\mu = m$ the planet's mass, as an approximation.

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r)$$

We make use of $r^2\dot{\theta} = \frac{L}{m}$ (conservation of angular momentum)

$$\Rightarrow r^2\dot{\theta}^2 = \frac{L^2}{m^2r^2}$$

$$\boxed{E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r)}$$

we recognise this term

What is this?

This is gpe

it seems potential $\boxed{U(r) = \frac{L^2}{2mr^2} + V(r)}$ has an extra term!

This is the centrifugal term! This arises due to conservation of angular momentum.

Let's sub $V(r) = -\frac{k}{r}$:

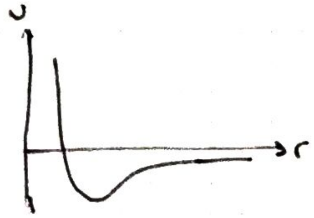
$$U(r) = \frac{L^2}{2mr^2} - \frac{k}{r}$$ and let's use $l = \frac{L^2}{mk}$

$$U(r) = \frac{kl}{2r^2} - \frac{k}{r}$$ This is the effective potential.

$\frac{1}{2}m\dot{r}^2 \geq 0$ in the energy eqn (since negative KE not possible)

so $$E \geq U(r) = \frac{kl}{2r^2} - \frac{k}{r}$$

So if we choose a value for total energy E, we can draw a horizontal line on a U against r curve:



we know motion can only occur where U(r) is below our chosen value of E

The minimum possible total energy for a given L is given by the minimum or the curve. In this instant, r is constant at:

$$r_c = l = \frac{L^2}{mK}$$

so the orbit is a circle and the total energy is:

$$E = -\frac{K}{2l} = -\frac{mK^2}{2L^2}$$

for $\frac{-K}{2l} < E < 0$, we have an eliptical orbit

for $E > 0$, we have a hyperbolic orbit.

For $E = 0$, we have a parabolic orbit.

Orbits in a Yukawa Potential

We saw orbits produced by inverse square law attractive forces are elliptical. But what about a force given by the Yukawa potential:

$$V(r) = -\frac{\alpha e^{-Kr}}{r} \qquad (\alpha > 0, K < 0)$$

Such a potential describes eg. the force of attraction between nucleons in an atomic nucleus. Instead of looking at this through a QM scope, lets treat it classically and see what happens.

The effective potential here is:

$$U(r) = \frac{L^2}{2mr} - \frac{\alpha e^{-Kr}}{r}$$

Note, for $K = 0$ the effective potential reduces to the form of our gravitational one.

If Kr is small compared to 1, then we expect something similar to the gravitational case but will small perturbations. This is called the Rosette Orbit and looks like an elliptical orbit that precesses. In the case of planets, this is usually due to small gravitational influences from other bodies.

At large r, the $\frac{L^2}{2mr}$ dominates so $U(r)$ becomes positive.

Chaos in Planetary Orbits

Why have we only looked at 2-body problems and no 3-body problems?

Well, 3-body problems are really complicated! They are chaotic and thus can only be solved numerically.

There are online simulations of these that are really fun to watch, look them up!