Electrodynamics

Until now we have only considred static electric and magnetic fields, i.e where $\frac{\partial \rho}{\partial t} = 0$. But we will now move onto the exciting topic of electrodynamics, where we allow the fields to vary with time.

Faraday's Law:

So we have seen that electric current in wires causes circulating $\underline{B}$ fields. We will now explore Faraday's discovery that magnetic fields can produce circulating electric fields.

Faraday discovered this by pushing a magnet through a closed loop of wire and recording a measured current through the wire. The faster the magnet is moved the larger the current and the direction of motion affects the current's bias. This led to Faraday's Law:

$$\boxed{\mathcal{E} = -\frac{\partial \Phi_B}{\partial t}}$$

where $\mathcal{E}$ is the electromotive force induced in the loop
Φ_B is magnetic flux

The electromotive force is defined as:

$$\boxed{\mathcal{E} = \oint_c \underline{E}(t,x,y,z) . d\underline{L}}$$

the integral of the field around a closed loop

The magnetic flux is defined as:

$$\boxed{\Phi_B = \iint_{S_{open}} \underline{B}(t,x,y,z) . d\underline{A}}$$

Which gives us Faraday's Law as:

Faraday's Law:

$$\boxed{\oint_c \underline{E}(t,x,y,z)\cdot d\underline{L} = -\frac{\partial}{\partial t}\iint_{S_{open}} \underline{B}(t,x,y,z)\cdot d\underline{A}}$$

The negative sign in the law is part of Lenz's Law which states that nature abhors a change in flux so the direction of the induced current is in such a direction as to induce flux to oppose the change in flux.

We can write Faraday's law as:

$$\oint \underline{E}(t,x,y,z)\cdot d\underline{L} = \iint -\left(\frac{\partial \underline{B}(t,x,y,z)}{\partial t}\right)\cdot d\underline{A}$$

Using Stokes' theorem: $\oint \underline{v}(x,y,z)\cdot d\underline{L} = \iint_{\substack{open\\surface}} (\underline{\nabla}\times \underline{v}(x,y,z))\cdot d\underline{A}$

LHS is:

$$\oint \underline{E}(t,x,y,z)\cdot d\underline{L} = \iint_{S_{open}} (\underline{\nabla}\times \underline{E}(t,x,y,z))\cdot d\underline{A}$$

$$\Rightarrow \iint_{S_{open}} (\underline{\nabla}\times \underline{E})\cdot d\underline{A} = \iint_{S_{open}} -\left(\frac{\partial \underline{B}}{\partial t}\right)\cdot d\underline{A}$$

$$\Rightarrow \boxed{\underline{\nabla}\times \underline{E}(x,y,z) = -\frac{\partial \underline{B}(x,y,z)}{\partial t}}$$

This is a very interesting connection. This is Faraday's Law in differential form

$$\boxed{\text{curl } \underline{E} = -\frac{\partial \underline{B}}{\partial t}}$$

So a changing magnetic field induces an electric field.

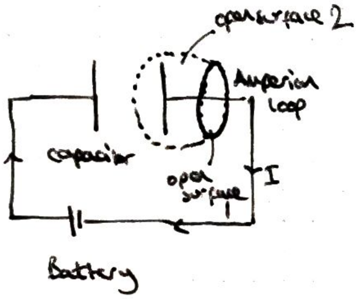
The Ampère Maxwell Law

The complimentary observation to Faraday's law is:

A changing electric field induces a magnetic field.

observed by Maxwell.

Consider the following electrical circuit:



As the capacitor is charging and discharging, there is a non-steady electrical current $I(t)$ flowing through the wire.

Consider an amperian loop around the wire:

$$\oint \underline{B} \cdot d\underline{L} = \mu_0 I$$

we expect $\oint \underline{B} \cdot d\underline{L}$ to be non-zero

Consider two open surfaces, labelled in the diagrame.

From Stokes' theorem: $\oint \underline{B} \cdot d\underline{L} = \iint_{S_{open}} (\underline{\nabla} \times \underline{B}) \cdot d\underline{A} = \mu_0 I$

For open surface 1, this seems fine, and we will find a non-zero $\underline{B}$.
But for open surface 2, the surface goes between the capacitors, so the total current through the open surface is 0! This would give $\underline{B} = 0$ which is not right!

Maxwell realised this and worked out that Ampère's law only works for the static case and is incomplete for the dynamic case.

The rate of increase of electric flux Φ_E between capacitor plate times ε_0 is the current flowing in the wire:

$$I = \varepsilon_0 \frac{\partial \Phi_E}{\partial t}$$

The proof is given overleaf.

The capacitance is given by $\underline{\underline{C = \varepsilon_0 \frac{A}{d}}}$ where A is area of plates, d is distance between them

$\underline{C = \frac{Q}{V}}$ and $\underline{E = \frac{V}{d}}$ These are all the eqns we need

$$I = \frac{\partial Q}{\partial t} = C \frac{\partial V}{\partial t} = Cd \frac{\partial E}{\partial t} = \varepsilon_0 A \frac{\partial E}{\partial t} = \underline{\underline{\varepsilon_0 \frac{\partial \Phi_E}{\partial t}}}$$

$$\therefore \boxed{I = \varepsilon_0 \frac{\partial \Phi_E}{\partial t}}$$

In the static case, this $=0$ which is why we don't worry about it

So the total Ampére-Maxwell Law is:

$$\boxed{\oint \underline{B} \cdot d\underline{L} = \mu_0 I + \mu_0 \varepsilon_0 \frac{\partial \Phi_E}{\partial t}}$$

Since $I = \iint_{S_{open}} \underline{J} \cdot d\underline{A}$, we can put this in differential form

$$\Phi_E = \iint_{S_{open}} \underline{E} \cdot d\underline{A} \Rightarrow \frac{\partial \Phi_E}{\partial t} = \iint_{S_{open}} \left(\frac{\partial \underline{E}}{\partial t} \right) \cdot d\underline{A}$$

Using $\oint \underline{B} \cdot d\underline{L} = \iint_{S_{open}} (\underline{\nabla} \times \underline{B}) \cdot d\underline{A}$

$$\therefore \iint_{S_{open}} (\underline{\nabla} \times \underline{B}) \cdot d\underline{A} = \mu_0 \iint_{S_{open}} \underline{J} \cdot d\underline{A} + \mu_0 \varepsilon_0 \iint_{S_{open}} \left(\frac{\partial \underline{E}}{\partial t} \right) \cdot d\underline{A}$$

$$\therefore \boxed{\underline{\nabla} \times \underline{B} = \mu_0 \underline{J} + \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}}$$

$$\boxed{\text{curl}\, \underline{B} = \mu_0 \underline{J} + \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}}$$

Maxwell's Equations

We can now summarise Maxwell's equations in their differential form:

$\underline{\nabla} \cdot \underline{E} = \frac{1}{\varepsilon_0} \rho$	$\equiv \oiint \underline{E} \cdot d\underline{A} = \frac{1}{\varepsilon_0} Q$	Gauss' Law
$\underline{\nabla} \cdot \underline{B} = 0$	$\equiv \oiint \underline{B} \cdot d\underline{A} = 0$	No Magnetic Monopoles
$\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$	$\equiv \oint \underline{E} \cdot d\underline{L} = -\frac{\partial \Phi_B}{\partial t}$	Faraday's Law
$\underline{\nabla} \times \underline{B} = \mu_0 \underline{J} + \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}$	$\equiv \oint \underline{B} \cdot d\underline{L} = \mu_0 I + \mu_0 \varepsilon_0 \frac{\partial \Phi_E}{\partial t}$	Ampere-Maxwell Law

Together with $\underline{F} = q(\underline{E} + (\underline{v} \times \underline{B}))$, they summarise everything we know about Electromagnetism.

The Poynting Vector

You will remember from last year that fields can store energy.

For $\underline{E}$ field:

$$\text{Energy density} = \underline{\underline{\frac{1}{2}\varepsilon_0 E^2}}$$

For $\underline{B}$ field:

$$\text{Energy density} = \underline{\underline{\frac{1}{2}\frac{1}{\mu_0} B^2}}$$

Electromagnetic energy density is the sum of these:

$$\boxed{u_{em} = \frac{1}{2}\left\{ \varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right\}}$$

This is quite remarkable as it implies energy is stored in the field itself, as there are no other quantities in the expression. So the fields aren't just magnetic constructs, but have real physical existence.

An interesting thing we can show is that EM fields have energy that can flow from place to place!
Consider EM fields in a vacuum (i.e no current or charges nearby):

$$\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \qquad \underline{\nabla} \times \underline{B} = \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}$$

A known relation is:

$$\underline{\nabla} \cdot (\underline{E} \times \underline{B}) = \underline{B} \cdot (\underline{\nabla} \times \underline{E}) - \underline{E} \cdot (\underline{\nabla} \times \underline{B})$$

so

$$\underline{\nabla} \cdot (\underline{E} \times \underline{B}) = \underline{B} \cdot \left(-\frac{\partial \underline{B}}{\partial t}\right) - \underline{E} \cdot \left(\mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}\right)$$

but for variable a^2
$da = 2a$
$\Rightarrow a = \frac{1}{2} da$

$$\underline{\nabla} \cdot (\underline{E} \times \underline{B}) = -\frac{1}{2} \frac{\partial B^2}{\partial t} - \mu_0 \varepsilon_0 \frac{1}{2} \frac{\partial E^2}{\partial t}$$

$$\underline{\nabla} \cdot \left(\frac{1}{\mu_0} \underline{E} \times \underline{B}\right) = -\frac{1}{2} \frac{1}{\mu_0} \frac{\partial B^2}{\partial t} - \varepsilon_0 \frac{1}{2} \frac{\partial E^2}{\partial t}$$

$$= -\frac{\partial}{\partial t} \frac{1}{2} \left\{ \frac{1}{\mu_0} B^2 + \varepsilon_0 E^2 \right\}$$

$$\underline{\nabla} \cdot \left(\frac{1}{\mu_0} \underline{E} \times \underline{B}\right) + \frac{\partial}{\partial t} \underbrace{\frac{1}{2} \left\{ \frac{1}{\mu_0} B^2 + \varepsilon_0 E^2 \right\}}_{\text{this is } u_{em}} = 0$$

This has the characteristic form of the continuity equation

$$\underline{\nabla} \cdot \underline{v} + \frac{\partial \rho}{\partial t} = 0$$

where $\rho = u_{em}$ and $\underline{v}$ is $\frac{1}{\mu_0} \underline{E} \times \underline{B}$ which we define as $\underline{S}$

$$\boxed{\underline{\nabla} \cdot \underline{S} + \frac{\partial u_{em}}{\partial t} = 0}$$

we call $\underline{S}$ the <u>poynting vector</u>

We interpret the poynting vector $\underline{S}$ as the energy current and its magnitude gives the energy flow per unit area per unit time.

The poynting vector implies that anywhere in space where there are non-parallel $\underline{E}$ and $\underline{B}$ fields, there will be a flow of electromagnetic energy. This applies even if the fields are static!