Some Formal Mathematics

All the examples we've studied so far were solved by computing wave functions and operator expectation values, but we can now put these into a somewhat firmer mathematical footing.

Let's recap on some vector transformations. If $\vec{a} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \vec{b}$$

so while $\vec{a}$ points in $+\vec{z}$ direction, $\vec{b}$ points in $-\vec{y}$ direction.
So we can use matrices to apply transformations (such as rotation) to vectors. We can call the matrix (used here) an "operator of rotation"

Both vectors $\vec{a}$ and $\vec{b}$ here are normalised, i.e
$\vec{a} \cdot \vec{a} = \vec{a}^T \vec{a} = \vec{b} \cdot \vec{b} = \vec{b}^T \vec{b} = 1$

We can define some basis vectors:
$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

These are all mutually orthonormal and complete so that all possible vectors can be made as a linear combination of these:

$$\vec{c} = \sum_{i=1}^{3} \alpha_i \vec{e}_i \qquad \text{orthonormal} : \vec{e}_i \cdot \vec{e}_j = \delta_{ij}$$

A vector space for any finite dimension is a collection of vectors and scalars with addition and scalar multiplication rules (the usual vector rules) under which it is closed.

This is the formal mathematical definition.

Hilbert Space is the vector space that wavefunctions live in. It is an infinite dimensional vector space of square-integrable functions. i.e we can add a number of square-integrable wavefunction solutions to construct a new solution.

New notation

It's getting quite tiring to write out $\int_{-\infty}^{\infty} \psi^* \psi \, dx$ constantly so let's introduce some new notation:

$|f\rangle$ will be "ket" and $\langle f|$ will be "bra" such that:

$$\langle f | g \rangle = \int_{-a}^{b} f^*(x)\, g(x)\, dx$$

where a and b depend on the range of f and g or problem-specific.

if both f and g are square integrable, i.e $\int_{-\infty}^{\infty} |f|^2 dx$ is finite, then $\langle f|g\rangle$ is finite and exists. We can prove this by considering $\langle f-\lambda g | f-\lambda g\rangle$ with complex λ:

$$\langle f-\lambda g | f-\lambda g\rangle = \int_{-\infty}^{\infty} (f-\lambda g)^*(f-\lambda g)\, dx$$

$$= \int_{-\infty}^{\infty} f^* f\, dx - \lambda \int_{-\infty}^{\infty} f^* g \, dx - \lambda^* \int_{-\infty}^{\infty} g^* f\, dx + \lambda^* \lambda \int_{-\infty}^{\infty} g^* g\, dx$$

$$= \langle f|f\rangle - \lambda \langle f|g\rangle - \lambda^* \langle g|f\rangle + |\lambda|^2 \langle g|g\rangle$$

Let's say $\lambda = \dfrac{\langle g|f\rangle}{\langle g|g\rangle}$ so:

note $(\langle g|f\rangle)^* = \langle f|g\rangle$

$$\langle f-\lambda g | f-\lambda g\rangle = \langle f|f\rangle - \frac{\langle g|f\rangle}{\langle g|g\rangle}\langle f|g\rangle - \frac{\langle f|g\rangle}{\langle g|g\rangle}\langle g|f\rangle + \frac{\langle f|g\rangle}{\langle g|g\rangle}\frac{\langle g|f\rangle}{\langle g|g\rangle}\langle g|g\rangle$$

$$= \langle f|f\rangle - \frac{|\langle f|g\rangle|^2}{\langle g|g\rangle} - \frac{|\langle f|g\rangle|^2}{\langle g|g\rangle} + \frac{|\langle f|g\rangle|^2}{\langle g|g\rangle}$$

$$\langle f-\lambda g | f-\lambda g\rangle = \langle f|f\rangle - \frac{|\langle f|g\rangle|^2}{\langle g|g\rangle}$$

The RHS is greater than 0 since all individual terms are either positive or 0

So $\langle f|f\rangle - \dfrac{|\langle f|g\rangle|^2}{\langle g|g\rangle} \geq 0$ so:

$$\boxed{\langle f|f\rangle\langle g|g\rangle \geq |\langle f|g\rangle|^2}$$

This is the cauchy schwartz inequality.

We have shown that if f and g are square integrable then $\langle f|g\rangle$ is finite. Since all functions in Hilbert space are by definition square integrable, $\langle f|g\rangle$ is finite in all Hilbert space.

Observables

In this new notation we write the expectation value of an operator $\hat{Q}$ as:

$$\underline{\underline{\langle \hat{Q}\rangle = \langle \psi|\hat{Q}\psi\rangle}} = \int_{-\infty}^{\infty} \psi^* \hat{Q}\psi \, dx$$

We expect physical operators to be real so $\langle Q\rangle^* = \langle Q\rangle$

$\langle \hat{Q}\rangle^* = (\langle \psi|\hat{Q}\psi\rangle)^* = \langle \hat{Q}\psi|\psi\rangle$

so for physical operator $\hat{Q}$: $\langle \psi|\hat{Q}\psi\rangle = \langle \hat{Q}\psi|\psi\rangle$ (1)

We call this property Hermitian and we require all physical properities to be found with Hermitian operators. The ladder operator is non-hermitian. Note, even for different wave functions ψ' and ψ, if $\hat{Q}$ is hermitian:

$\langle \psi'|\hat{Q}\psi\rangle = \langle \hat{Q}\psi'|\psi\rangle$ (2)

So to be Hermitian, (1) and (2) must be fullfilled.

Let's see if the momentum operator is hermitian:

$$\langle \psi'|\hat{p}\psi\rangle = \int_{-\infty}^{\infty} \psi'^* \frac{\hbar}{i}\frac{\partial}{\partial x}\psi \, dx \quad \text{integrating by parts}$$

$$= \int_{-\infty}^{\infty} \frac{\hbar}{i}\frac{\partial}{\partial x}(\psi'^*\psi)\,dx + \int_{-\infty}^{\infty} -\frac{\hbar}{i}\frac{\partial}{\partial x}\psi'^*\psi \, dx$$

$$= \frac{\hbar}{i}\left[\psi'^*\psi\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} -\frac{\hbar}{i}\frac{\partial}{\partial x}\psi'^*\psi \, dx$$

so $\langle \psi' | \hat{p}\, \psi \rangle = \underbrace{\left[\frac{\hbar}{i} \psi'^* \psi \right]_{-\infty}^{\infty}} + \int_{-\infty}^{\infty} - \frac{\hbar}{i} \frac{\partial}{\partial x} \psi'^* \psi$

= 0 since otherwise
the functions wouldn't
be square integrable

$$\therefore \langle \psi' | \hat{p}\, \psi \rangle = \int_{-\infty}^{\infty} - \frac{\hbar}{i} \frac{\partial}{\partial x} \psi'^* \psi \, dx = \int_{-\infty}^{\infty} \left(\frac{\hbar}{i} \frac{\partial}{\partial x} \psi' \right)^* \psi \, dx$$

$\langle \psi' | \hat{p}\, \psi \rangle = \underline{\underline{\langle \hat{p}\, \psi' | \psi \rangle}}$. So the momentum operator is hermitian

Determinant states

Can we prepare a system so that every single time we measure it with an operator $\hat{Q}$, we find the same constant q? i.e, where the variance of Q is 0?

$$\sigma^2 = \langle Q \rangle^2 - \langle Q^2 \rangle = \langle (Q - \langle Q \rangle)^2 \rangle$$
$$= \langle \psi | (\hat{Q} - \langle \hat{Q} \rangle)^2 \psi \rangle = \langle \psi | (\hat{Q} - q)^2 \psi \rangle$$

if hermitian then:

$\sigma^2 = \langle (\hat{Q} - q) \psi | (\hat{Q} - q) \psi \rangle$ which we require $= 0$

This only works if $\psi = 0$ which is trivial or if

$\underline{\underline{\hat{Q} \psi = q \psi}}$ This is called the Eigenvalue equation.

q here is an eigenvalue. We call the set of all eigenvalues of $\hat{Q}$, its spectrum. A spectrum is degenerate if an eigenvalue appears multiple times.

Let's consider an example for an operator $\hat{Q} = i \frac{d}{d\phi}$

$\hat{Q} = i\frac{d}{d\phi}$ where ϕ is an angle on a coordinate system.

Consider a function f that has period 2π and another function g with the same period: $f(\phi) = f(\phi + 2\pi)$ $g(\phi) = g(\phi + 2\pi)$

is $\hat{Q}$ hermitian?

$$\langle f|\hat{Q}g\rangle = \int_0^{2\pi} f^* i\frac{d}{d\phi} g \, d\phi$$ integrate by parts:

$$= \int_0^{2\pi} i\frac{d}{d\phi}(f^* g)\, d\phi - \int_0^{2\pi} i\frac{d}{d\phi} f^* g \, d\phi$$

$$= [if^* g]_0^{2\pi} + \int_0^{2\pi} -i\frac{d}{d\phi} f^* g \, d\phi$$

$$= i\{f^*(2\pi)g(2\pi) - f^*(0)g(0)\} + \int_0^{2\pi} (i\frac{d}{d\phi} f)^* g \, d\phi$$

$$= \int_0^{2\pi} (i\frac{d}{d\phi} f)^* g \, d\phi = \underline{\langle \hat{Q} f|g\rangle}$$ so $\hat{Q}$ is hermitian

The eigenvalue equation is $\hat{Q}f = qf$ so: $i\frac{d}{d\phi} f = qf$

This has asatz $f = Ae^{-iq\phi}$

$f(0) = A = f(2\pi) = Ae^{-2\pi i q}$ which is solved for

$q = 0, \pm 1, \pm 2 \ldots$

This is not degenerate since the spectrum has no repeating values.

Eigenfunctions of a Hermitian operator

The way the eigenfunctions (determinant states) work for Hermitian operators is different for Discrete and Continuous spectra.

Discrete spetra:

The eigenfunctions lie in hilbert space and are physical.

This means:

a) eigenvalues are real:

- Let $\hat{Q}f = qf$ (eigenvalue eqn)
- $\langle f|\hat{Q}f\rangle = \langle \hat{Q}f|f\rangle$ (hermitian)

f is physical so cannot be 0 everywhere so:

① $\langle f|\hat{Q}f\rangle = \int f^* \hat{Q}f \, dx = q\int f^* f \, dx$

② $\langle \hat{Q}f|f\rangle = \int (\hat{Q}f)^* f \, dx = q^* \int f^* f \, dx$

$\therefore q^* = q$ so q is real.

b) eigenfunctions of distinct eigenvalues are orthogonal:

- Let $\hat{Q}f = qf$; $\hat{Q}g = q'g$ with f, g two different eigenfunctions

$\int f^* \hat{Q}g \, dx = q' \int f^* g \, dx$ $\quad \int (\hat{Q}f)^* g \, dx = q^* \int f^* g \, dx$

$\therefore q'\langle f|g\rangle = q^*\langle f|g\rangle = q\langle f|g\rangle$

since q' and q are different, this is only true for $\langle f|g\rangle = 0$ so f and g are orthogonal

c) eigenfunctions of physical operators are complete.

d) if two eigenfunctions share the same eigenvalue, then any linear combination of the eigenfunctions still gives the same eigenvalue.

i.e $\hat{Q}f = qf$ $\quad \hat{Q}g = qg$ i.e f, g have same eigenvalue

if $h = Af + Bg$, then:

$\hat{Q}h = A\hat{Q}f + B\hat{Q}g = q(Af + Bg)$

so $\hat{Q}h = qh$

This is all different to continuous spectra which we will see next.

For <u>continuous spectra</u>:

We saw in the example of the free particle that the eigenfunction in the case of continuous spectrum is not normaliseable. It is therefore difficult to prove eigenvalues are real and that eigenfunctions are orthogonal. We can, however, analyse eigenfunction properties on case-by-case bases. We will do this for $\hat{x}$ and $\hat{p}$ operators.

Let's make some observations about the dirac delta function:

i) $\delta(cx) = \frac{1}{|c|}\delta(x)$ The usual way of scaling a fn.

ii) $\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dx$ which prove by considering fourier transforms:

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x)e^{-ikx}\,dx = \frac{1}{\sqrt{2\pi}}$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{ikx}\,dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{ikx}\,dk = \underline{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk}$$

Now let's find eigenfunctions and eigenvalues of $\hat{x}$ and $\hat{p}$:

<u>For $\hat{p}$</u>: let $f_p(x)$ be the eigenfunction of $\hat{p}$ so that

$$\underline{\frac{\hbar}{i}\frac{\partial}{\partial x}f_p(x) = p f_p(x)}$$ is the eigenvalue eqn.

$\Rightarrow \frac{\partial}{\partial x}f_p(x) = \frac{ip}{\hbar}f_p(x)$ which has Ansatz $f_p(x) = Ae^{ipx/\hbar}$

The ansatz is not normaliseable so isn't in Hilbert Space. How do we fix this? Let's restrict ourselves to real numbers, with real physical momentum $p^* = p$:

$$\int_{-\infty}^{\infty} f_{p'}(x)^* f_p(x)\,dx = |A|^2\int_{-\infty}^{\infty} e^{i(p-p')x/\hbar}\,dx$$ sub $x' = \frac{x}{\hbar}$ so $dx = \hbar\,dx'$

$$|A|^2\hbar\int_{-\infty}^{\infty} e^{i(p-p')x/\hbar}\,dx$$ which is now in the form of $\delta(p-p')$ so:

$$= |A|^2\hbar\,2\pi\,\delta(p-p')$$ so $A = \frac{1}{\sqrt{2\pi\hbar}}$ for normalisation.

so $$\underline{f_p(x) = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}}$$

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$$

so $\langle f_{p'} | f_p \rangle = \delta(p'-p)$

This is reminiscent of the orthogonality in discrete spectra. The discrete kronecker delta δ_{mn} has been replaced with continuous dirac delta

The momentum eigenfunctions are complete and therefore any square integrable function $f(x)$ is:

$$f(x) = \int_{-\infty}^{\infty} c(p) f_p(x)\, dp = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} c(p) e^{ipx/\hbar}\, dp$$

we determine $c(p)$ using:

$$\langle f_{p'} | f \rangle = \int_{-\infty}^{\infty} c(p) \langle f_{p'} | f_p \rangle\, dp$$

$$= \int_{-\infty}^{\infty} c(p)\, \delta(p-p')\, dp = c(p')$$

while $f_p(x)$ is not normaliseable, restricting ourselves to physical momenta we were able to define an orthogonality and use completeness to determine $c(p)$. Note the eigenfunctions of momentum $f_p(x)$ are sinusoidal with wavelength:

$\lambda = \frac{2\pi\hbar}{p}$ ← de Broglie relation

Now let's consider the position operator $\hat{x} = x$:

<u>For $\hat{x}$:</u> the eigenvalue equation is $\hat{x} g_y(x) = y g_y(x)$

where $g_y(x)$ is the eigenfunction and y is the eigenvalue.

We can make the ansatz $g_y(x) = A\delta(x-y)$

Again $g_y(x)$ is not normaliseable but we have Dirac orthogonality

$$\int_{-\infty}^{\infty} g_{y'}^* g_y\, dx = |A|^2 \int_{-\infty}^{\infty} \delta(x-y')\delta(x-y)\, dx = |A|^2 \delta(y-y')$$

For convenience we pick $A = 1$ so:

$\langle g_{y'} | g_y \rangle = \delta(y-y')$ so $g_{y'}$ and g_y are orthogonal

As for the case of the momentum operator, the eigenfunctions of $\hat{x}$ form a complete set and for any square-integrable function we have:

$$f(x) = \int_{-\infty}^{\infty} c(y)\, g_y(x)\, dy = \int_{-\infty}^{\infty} c(y)\, \delta(x-y)\, dy$$

here $\underline{c(y) = f(y)}$

The eigenfunctions of a hermitian operator with a continuous spectrum are not normaliseable. But eigenfunctions for real eigenvalues are Dirac-orthonormal:

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \qquad \langle f_{p'} | f_p \rangle = \delta(p-p')$$

$$g_y(x) = \delta(x-y) \qquad \langle g_{y'} | g_y \rangle = \delta(x-x')$$

Generalised Statistical Interpretation

At the start of the course, we said that when we take a measurement, the wavefunction collapses. But what does this mean exactly? Consider an observable $Q(x,t)$ represented by the hermitian operator $\hat{Q}(\hat{x}, -i\hbar \frac{d}{dx})$ with a discrete spectrum. Using the operator will return an eigenvalue q_n corresponding to an eigenfunction f_n with probability $|c_n|^2$ where $c_n = \langle f_n | \Psi \rangle$

Similarly, for a continuous spectrum, the probability of getting the eigenvalue $q(z)$ corresponding to eigenfunction $f_z(x)$ in range $z, z+dz$ is $|c(z)|^2 dz$ where $c(z) = \langle f_z | \Psi \rangle$

Consider a wavefunction $\Psi(x,t) = \Sigma c_n(t) f_n(x)$

we can make some comments about this:

$$\Psi(x,t) = \sum_n c_n(t) f_n(x)$$

comments:

- total probability $= 1$ so $\sum_n |c_n|^2 = 1$

- expectation value is the sum over all possible outcomes:

  $$\langle Q \rangle = \sum_n q_n |c_n|^2$$

- For position space functions: $g_y(x) = \delta(x-y)$

  $$c(y) = \langle g_y | \Psi \rangle = \int_{-\infty}^{\infty} \delta(x-y)\, \Psi(x,t). = \Psi(y,t)$$

  which tells us the probability of finding a particle in range $y, y+dy$ is $|c(y)|^2\, dy = |\Psi(x,t)|^2\, dy$

  i.e, the normsquared of a wavefunction is its probability distribution.

- For momentum space functions: $f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$

  $$c(p) = \langle f_p | \Psi \rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar}\, \Psi(x,t)\, dx = \Phi(p,t)$$

  we have introduced $\Phi(p,t)$ which is the momentum space wave function, the fourier transform of the position space wave function Ψ. The prob. of measuring a momentum in range $p, p+dp$ is $\int_{-\infty}^{\infty} |\Phi|^2 dp$.

$$\therefore \quad \Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar}\, \Psi(x,t)\, dx \leftrightarrow \Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{+ipx/\hbar}\, \Phi(p,t)\, dp$$

Uncertainty Principle : $\sigma_x \sigma_p \leq \frac{\hbar}{2}$

We haven't actually provided a proper proof for the uncertainty principle. That is what we will do here.

For any observable represented by operator $\hat{A}$:

$$\sigma_A^2 = \langle A^2 \rangle - \langle A \rangle^2 = \underbrace{\langle (A - \langle A \rangle)^2 \rangle}_{\text{we can prove this:}}$$

$$\begin{aligned}
\langle (A - \langle A \rangle)^2 \rangle &= \langle A^2 - 2A\langle A \rangle + \langle A \rangle^2 \rangle \\
&= \langle A^2 \rangle - \langle 2A\langle A \rangle \rangle + \langle \langle A \rangle^2 \rangle \\
&= \langle A^2 \rangle - 2\langle A \rangle^2 + \langle A \rangle^2 \\
&= \underline{\langle A^2 \rangle - \langle A \rangle^2}
\end{aligned}$$

so $\sigma_A^2 = \langle (A - \langle A \rangle)^2 \rangle = \langle (\hat{A} - \langle A \rangle)\Psi \,|\, (\hat{A} - \langle A \rangle)\Psi \rangle$

$= \langle f | f \rangle$ where $f = (\hat{A} - \langle A \rangle)\Psi$

similarly for an operator $\hat{B}$:

$\sigma_B^2 = \langle g | g \rangle$ where $g = (\hat{B} - \langle B \rangle)\Psi$

The cauchy-schwartz inequality is $\langle f|f \rangle \langle g|g \rangle \geq |\langle f|g \rangle|^2$

so $\sigma_A^2 \sigma_B^2 = \underbrace{\langle f|f \rangle}_{\text{real}} \underbrace{\langle g|g \rangle}_{\text{real}} \geq |\underbrace{\langle f|g \rangle}_{\text{complex number which we can call } z}|^2$

if $z = a + ib$, then $z^* = a - ib$ and $|z|^2 = a^2 + b^2$
$= Re(z)^2 + Im(z)^2$

so $|z|^2 \geq Im(z)^2$ so $|z|^2 \geq \frac{z - z^*}{2i}$

$\therefore \quad \sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} [\langle f|g \rangle - \langle g|f \rangle] \right)^2$

At the same time $\langle f|g \rangle = \langle (A - \langle A \rangle)\Psi \,|\, (B - \langle B \rangle)\Psi \rangle$

$$\langle f|g\rangle = \langle (A-\langle A\rangle)\Psi \,|\, (B-\langle B\rangle)\Psi\rangle$$

$$= \langle \Psi | (A-\langle A\rangle)(B-\langle B\rangle)\Psi\rangle$$ since they are hermitian

$$= \langle \Psi | \hat{A}\hat{B}\Psi\rangle - \langle B\rangle\langle\Psi|\hat{A}\Psi\rangle - \langle A\rangle\langle\Psi|\hat{B}\Psi\rangle + \langle A\rangle\langle B\rangle\langle\Psi|\Psi\rangle$$

$$= \langle AB\rangle - \langle B\rangle\langle A\rangle - \langle A\rangle\langle B\rangle + \langle A\rangle\langle B\rangle$$

$$= \langle AB\rangle - \langle B\rangle\langle A\rangle$$

and $\langle g|f\rangle = \langle BA\rangle - \langle A\rangle\langle B\rangle$

$\langle f|g\rangle - \langle g|f\rangle = \langle AB\rangle - \langle BA\rangle = \langle [A,B]\rangle$ where $[A,B]$ is commutator

combining with $\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}\left[\langle f|g\rangle - \langle g|f\rangle\right]\right)^2$:

$$\underline{\underline{\sigma_A\sigma_B \geq \frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle}} \quad *$$

so, if we take $\hat{x}$ and $\hat{p}$:

$$[\hat{x},\hat{p}]\lambda = \hat{x}\hat{p}\lambda - \hat{p}\hat{x}\lambda$$

$$= x\cdot\frac{i}{\hbar}\frac{\partial}{\partial x}\lambda - \frac{i}{\hbar}\frac{\partial}{\partial x}(x\lambda)$$

$$= x\frac{i}{\hbar}\frac{\partial}{\partial x}\lambda - x\frac{i}{\hbar}\frac{\partial}{\partial x}\lambda - \frac{i}{\hbar}\lambda = \underline{-\frac{i}{\hbar}\lambda}$$

$\therefore [\hat{x},\hat{p}] = i\hbar$ sub into $*$

$$\sigma_x\sigma_p \geq \frac{1}{2i}i\hbar \Rightarrow \underline{\underline{\sigma_x\sigma_p \geq \frac{\hbar}{2}}}$$

- Two operators are compatible if $[\hat{A},\hat{B}] = 0$

This means if $\hat{A}\Psi = a_i f_i$ then we measure $\hat{B}$ of collapsed wave function: $\hat{B}f_i = b_j f_{ij}$

if we make another measurement after this of A :

$$\hat{A}f_{ij} = \underline{a_i f_i} \quad \text{This is cool!}$$

Energy - Time uncertainty

We can provide bounds for how fast a measured quantity changes appreciably in time.

by product rule:

$$\frac{d}{dt}\langle Q\rangle = \frac{d}{dt}\langle \Psi | \hat{Q}\Psi\rangle = \left\langle \frac{\partial \Psi}{\partial t} \middle| \hat{Q}\Psi\right\rangle + \underbrace{\left\langle \Psi \middle| \frac{\partial \hat{Q}}{\partial t}\Psi\right\rangle}_{\text{assume } 0} + \left\langle \Psi \middle| \hat{Q}\frac{\partial \Psi}{\partial t}\right\rangle$$

as operator $\hat{Q}$ does not change in time

$$\frac{d}{dt}\langle Q\rangle = \left\langle \frac{\partial \Psi}{\partial t} \middle| \hat{Q}\Psi\right\rangle + \left\langle \Psi \middle| \hat{Q}\frac{\partial \Psi}{\partial t}\right\rangle$$

the Hamilton operator is $i\hbar \frac{d}{dt} = \hat{H}$ so:

$$\frac{d}{dt}\langle Q\rangle = \frac{-1}{i\hbar}\langle \hat{H}\Psi | \hat{Q}\Psi\rangle + \frac{1}{i\hbar}\langle \Psi | \hat{Q}\hat{H}\Psi\rangle$$

$$= -\frac{1}{i\hbar}\langle \Psi | \hat{H}\hat{Q}\Psi\rangle + \frac{1}{i\hbar}\langle \Psi | \hat{Q}\hat{H}\Psi\rangle$$

$$= \frac{i}{\hbar}\langle \Psi | [\hat{H},\hat{Q}]\Psi\rangle = \frac{i}{\hbar}\langle [\hat{H},\hat{Q}]\rangle$$

$$\therefore \underline{\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle [\hat{H},\hat{Q}]\rangle} \Rightarrow \langle [\hat{H},\hat{Q}]\rangle = \frac{\hbar}{i}\frac{d\langle Q\rangle}{dt}$$

our general expression for uncertainty principle was $\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i}\langle [\hat{A},\hat{B}]\rangle\right)^2$

so $\sigma_H^2 \sigma_Q^2 \geq \left(\frac{1}{2i}\frac{\hbar}{i}\frac{d\langle Q\rangle}{dt}\right)^2 \Rightarrow \sigma_H^2 \sigma_Q^2 = \left(-\frac{\hbar}{2}\frac{d\langle Q\rangle}{dt}\right)^2$

$$\sigma_H^2 \sigma_Q^2 \geq \left(\frac{\hbar}{2}\right)^2\left(\frac{d\langle Q\rangle}{dt}\right)^2$$

$\sigma_H \sim$ variance of energy of the system

$$\sigma_H \sigma_Q \geq \frac{\hbar}{2}\frac{d\langle Q\rangle}{dt}$$

$$\approx \sigma_E \underbrace{\frac{\sigma_Q\, dt}{d\langle Q\rangle}} \geq \frac{\hbar}{2} \quad \& \quad \sigma_E \sigma_t \geq \frac{\hbar}{2}$$

This is the energy-time uncertainty relation

we interpret this as time taken for system to change by 1σ

An important consequence is that a state that fully exists for a short period of time can't have definite energy. i.e unstable particles don't have definite or well-defined mass.