The Laws of Dynamics

Since we are now working with special relativity, simple newtonian laws of dynamics like $F = \frac{dp}{dt} = ma$ don't work since different observers will not agree on time.

So we need to construct laws that:

- contain only Lorentz invariant quantities
- or take form $X^{\mu} = Y^{\mu}$ where indices are balanced.

Four Velocity

We can no longer use velocity as $v = \frac{dx^{\mu}}{dt}$ since both x^{μ} and t transform. We need to use a measure of time that is Lorentz invariant, ie proper time. Time measured on a clock that is in the rest frame of the moving object. We call this time τ

4-velocity is defined as $\boxed{u^{\mu} = \frac{dx^{\mu}}{dt}}$

It transforms like: $\underline{\underline{u'^{\mu} = \Lambda^{\mu}{}_{\nu} u^{\nu}}}$ as required.

Note: $u^{\mu} = \frac{dx^{\mu}}{d\tau} = \frac{dx^{\mu}}{dt}\frac{dt}{d\tau}$ but $\frac{dt}{d\tau} = \gamma$ (since $t = \gamma\tau$)

so the components of 4-velocity are given by $u^{\mu} = \gamma\frac{dx^{\mu}}{dt}$

$$\boxed{u^{\mu} = \gamma(c, v_x, v_y, v_z)}$$

The Lorentz invariant length of 4-velocity is:

$$\boxed{u^{\mu}u_{\mu} = \gamma^2(c^2 - |\underline{v}|^2) = c^2}$$

Four Acceleration

We define 4-acceleration simply as:

$$\boxed{a^\mu = \frac{du^\mu}{d\tau}}$$

Four Momentum

Four momentum is given by:

$$\boxed{p^\mu = mu^\mu = m\frac{dx^\mu}{d\tau}}$$

which transform as normal like $p'^\mu = \Lambda^\mu{}_\nu p^\nu$

We can notice something interesting from this. Since $u^0 = \gamma c$,

$$p^0 = mc\gamma = mc\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \approx mc\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \dots\right)$$
$$= mc + \frac{1}{2}m\frac{v^2}{c}$$

so $p^0 c = mc^2 + \underbrace{\tfrac{1}{2}mv^2}_{\text{This is kinetic energy}}$

so we can intepret $p^0 c$ as the relativistic version of energy

$\Rightarrow$ $\underline{E_{rest} = mc^2}$

so we can write the components of 4-momentum:

$$p^\mu = \left(\frac{E}{c}, \underline{p}\right) = mu^\mu = m\gamma(c, \underline{v})$$

The relativistic expression of energy is thus $\underline{E = \gamma mc^2}$

and kinetic energy is $\underline{T = (\gamma - 1)mc^2}$

The Lorentz invariant length is:

$$\boxed{p^\mu p_\mu = \frac{E^2}{c^2} - |\underline{p}|^2 = m^2c^2}$$

Laws of Dynamics

So what is the relativistic form of Newton's 2nd Law using the new notation?

$$\boxed{f^\mu = \frac{dp^\mu}{d\tau}}$$

This is manifestly Lorentz invariant and tells us when $f^\mu = 0$ that p^μ = constant.