PHYS2006 Classical Mechanics

Motion of Systems of Particles

For a system of N particles, with masses m_i each, we can calculate total vector force $\underline{F}$ and vector momentum $\underline{p}$ by simply summing those for each particle:

$$\boxed{\underline{F} = \sum_i^N \underline{f}_i} \qquad \boxed{\underline{p} = \sum_i^N \underline{p}_i}$$

Newton's 2nd Law applies to each particle as $\boxed{\underline{f}_i = \dot{\underline{p}}_i}$ and so applies to the total system as:

$$\boxed{\underline{F} = \dot{\underline{p}}}$$

So, what is the force $\underline{f}_i$ on each particle? We can divide the force on each particle into two parts, the external force applied on it $\underline{f}_i^{ext}$ and the force on it due to the other particles; $\underline{f}_{ij}$ is the force on particle i due to particle j, so $\underline{f}_{ij} = -\underline{f}_{ji}$

$$\therefore \boxed{\underline{f}_i = \underline{f}_i^{ext} + \sum_{i \neq j}^N \underline{f}_{ij}}$$

However, if we use this formula in the formula for total force $\underline{F}$, we find:

$$\underline{F} = \sum_i^N \left\{ \underline{f}_i^{ext} + \sum_{i \neq j}^N \underline{f}_{ij} \right\} = \sum_i^N \underline{f}_i^{ext} + \underbrace{\sum_i^N \sum_{i \neq j}^N \underline{f}_{ij}}$$

This vanishes since every $\underline{f}_{ij}$ and $\underline{f}_{ji}$ cancel out

so we are left with $\boxed{\underline{F} = \sum_i^N \underline{f}_i^{ext} = \underline{F}^{ext}}$ $\therefore \boxed{\underline{F}^{ext} = \dot{\underline{p}}}$

So the total external force is equal to the rate of change of total linear momentum of the system. The internal forces all cancel due Newton's 3rd Law. For conservation of momentum, $\dot{\underline{p}} = 0$ so we require $\underline{F}^{ext} = 0$ for CLM.

Centre of Mass

If we deal with a single point mass, we don't really need to worry about centre of mass, we take C of M to be the position of the point mass. But what about in a system of particles of mass m_i each?

if each particle has a position defined by vector $\underline{r}_i$, we can write C of M $\underline{R}$ as:

$$\boxed{\underline{R} = \frac{\sum m_i \underline{r}_i}{\sum m_i} = \frac{1}{M}\sum_{i=1}^{N} m_i \underline{r}_i}$$

where $M = \sum m_i$, total mass

To find the velocity of the centre of mass, we can differentiate $\underline{R}$ w.r.t. time to get $\dot{\underline{R}} = \frac{1}{M}\sum_{i=1}^{N} m_i \dot{\underline{r}}_i$

we can thus relate this to the linear momentum of the system:

$$\boxed{\underline{P} = M\dot{\underline{R}} = \sum_i^N m_i \dot{\underline{r}}_i}$$

i.e the sum of the momentums of each particle, as found earlier.

Since $\underline{F}^{ext} = \dot{\underline{P}}$, we can say $\boxed{\underline{F}^{ext} = M\ddot{\underline{R}} = \sum_i^N m_i \ddot{\underline{r}}_i}$

So we see that if $\underline{F}^{ext} = 0$, the C of M moves with constant velocity, i.e

> The linear momentum of a system subject to no external force is conserved.

we also see that if $\underline{F}^{ext} \neq 0$, the C of M moves as if the total system were a point mass centred at the C of M.

We can also define coordinates relative to the C of M. For example, if ρ_i is the location of the i^{th} particle relative to the C of M:

$$\boxed{\underline{r}_i = \underline{R} + \rho_i}$$

NB:// $\therefore \underline{R} = \frac{1}{M}\sum_i^N m_i(\underline{R} + \rho_i)$

$$\underline{R} = \frac{1}{M}\sum m_i \underline{R} + \frac{1}{M}\sum m_i \rho_i$$

$$\underline{R} = \underline{R} + \frac{1}{M}\sum m_i \rho_i$$

$$\therefore \sum m_i \rho_i = 0$$

C of M condition

Kinetic Energy of a System of Particles

The kinetic energy T of a particle is $T = \frac{1}{2} m_i \dot{\underline{r}}_i^2$ so for the total system it is:

$$T = \sum_i^N \frac{1}{2} m_i \dot{\underline{r}}_i^2 \quad \text{but} \quad \underline{r}_i = \underline{R} + \rho_i \quad \text{so:}$$

$$T = \sum_i^N \frac{1}{2} m_i (\dot{\underline{R}} + \dot{\rho}_i)^2 = \sum_i^N \frac{1}{2} m_i (\dot{\underline{R}}^2 + 2\dot{\underline{R}}\cdot\dot{\rho}_i + \dot{\rho}_i^2)$$

$$T = \underbrace{\sum_i^N \frac{1}{2} m_i \dot{\underline{R}}^2}_{=\frac{1}{2}M} + \underbrace{\sum_i^N m_i \dot{\rho}_i \cdot \dot{\underline{R}}}_{=0 \text{ since } \sum m_i \rho_i = 0} + \sum_i^N \frac{1}{2} m_i \dot{\rho}_i^2$$

$$\therefore \quad T = \frac{1}{2} M \dot{\underline{R}}^2 + \sum_i^N \frac{1}{2} m_i \dot{\rho}_i^2 \quad \text{which we write as:}$$

$$\boxed{T = \frac{1}{2} M \dot{\underline{R}}^2 + T_{CofM}}$$

So the first term is the KE from the motion of the centre of mass. But what is the second term T_{CofM}?

KE is different in different reference frames since the velocity of each particle is different in different reference frames. But T_{CofM}, the KE w.r.t. the C of M is the same in all reference frames and is an "internal" kinetic energy of the system. The sum of T_{CofM} and the potential energy due to internal interactions is the total internal energy U used in thermodynamics. Proved overleaf consider the transformation from a frame S to frame S' moving at velocity v w.r.t S :

$\underline{r}_i \rightarrow \underline{r}_i' = \underline{r}_i - \underline{v}t$ so C of M transforms:

$$\underline{R} = \frac{1}{M}\sum m_i \underline{r}_i \rightarrow \underline{R}' = \frac{1}{M}\sum m_i \underline{r}_i' = \frac{1}{M}\sum m_i(\underline{r}_i - \underline{v}t)$$

$$\therefore \underline{R}' = \underline{R} - \underline{v}t$$

so since individual particle are transformed by $-\underline{v}t$ and the C of M is transformed by $-\underline{v}t$, the positions and velocities relative to C of M are unchanged, so T_{CofM} is the same in all reference frames.

System of Two Particles

Let's consider two particles with positions $\underline{r}_1$ and $\underline{r}_2$, and velocities $\underline{u}_1 = \dot{\underline{r}}_1$ and $\underline{u}_2 = \dot{\underline{r}}_2$ respectively.

We know we can write $\underline{r}_i = \underline{R} + \underline{\rho}_i$ so:

$$\underline{u}_1 = \dot{\underline{R}} + \dot{\underline{\rho}}_1 \qquad \underline{u}_2 = \dot{\underline{R}} + \dot{\underline{\rho}}_2 \qquad \Rightarrow \underline{u}_1 - \underline{u}_2 = \dot{\underline{\rho}}_1 - \dot{\underline{\rho}}_2 \quad (1)$$

C of M condition is $\sum m_i \underline{\rho}_i = 0$ so $m_1\underline{\rho}_1 + m_2\underline{\rho}_2 = 0$

$\Rightarrow m_1\dot{\underline{\rho}}_1 + m_2\dot{\underline{\rho}}_2 = 0 \quad (2)$ solving these simultaneous equations:

$$\dot{\underline{\rho}}_1 = \frac{m_2(\underline{u}_1 - \underline{u}_2)}{m_1 + m_2} \qquad \dot{\underline{\rho}}_2 = \frac{-m_1(\underline{u}_1 - \underline{u}_2)}{m_1 + m_2}$$

sub into $T = \frac{1}{2}M\dot{\underline{R}}^2 + \sum \frac{1}{2} m_i \dot{\underline{\rho}}_i^2$

$$T = \frac{1}{2}(m_1 + m_2)\dot{\underline{R}}^2 + \frac{1}{2}\underbrace{\frac{m_1 m_2}{m_1 + m_2}}_{\text{This is "reduced mass"}}\underbrace{(\underline{u}_1 - \underline{u}_2)}_{\text{This is relative velocity}}{}^2$$

Rocket Motion

At time t At time $t+\delta t$

Consider a rocket at a time t and then immediately after at time $t+\delta t$. At time t, the rocket moves with velocity $\underline{v}$ and has a mass m. At a time $t+\delta t$, the rocket has burned fuel and its velocity has increased to $v+\delta v$. Its mass has decreased to $m+\delta m$. Note that δm is a negative quantity. A mass $-\delta m$ is ejected from the rocket. The mass is ejected with velocity $-\underline{u}$ w.r.t. the rocket, so the velocity of the mass in an outside ref. frame is $\underline{v}-\underline{u}$.

Using conservation of linear momentum:

$$m\underline{v} = (m+\delta m)(\underline{v}+\delta\underline{v}) - \delta m(\underline{v}-\underline{u})$$

$$\Rightarrow m\underline{v} = m\underline{v} + m\delta\underline{v} + \delta m\underline{v} + \delta m\delta\underline{v} - \delta m\underline{v} + \delta m\underline{u}$$

$$\Rightarrow \underline{u}\,\delta m + m\delta\underline{v} + \delta m\delta\underline{v} = 0$$

If we take $\delta t \to 0$, then $\delta m\delta\underline{v}$ which is second order infinitesimal drops out. We then divide by m to get:

$$\underline{u}\frac{dm}{m} = -d\underline{v} \qquad \{\text{note } \delta m \to dm \text{ and } \delta v \to dv \text{ due to } \delta t \to 0\}$$

$$\int_{m_i}^{m_f} \underline{u}\frac{dm}{m} = -\int_{\underline{v}_i}^{\underline{v}_f} d\underline{v} \quad \Rightarrow \quad \underline{u}\ln\left(\frac{m_f}{m_i}\right) = -(\underline{v}_f - \underline{v}_i)$$

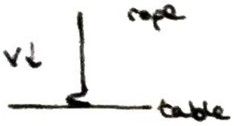
$$\boxed{\underline{v}_f = \underline{v}_i + \underline{u}\ln\left(\frac{m_i}{m_f}\right)}$$

This is the rocket equation. MEMORISE IT!

Rope falling onto a Table

Consider a rope with mass per unit length ρ suspended above a table. When the rope is released, it falls onto the table; what is the force on the table when a length x has fallen onto it?

Before: time t



After: time $t+\delta t$

If we take the total length of the rope to be $a+x$, a is the length of the rope still above the table, x is length on the table.

The weight of the rope falling onto the table is mass × acceleration so $\rho a g$. The normal force exerted by the table is $F(x)$, which we want to determine. Let's consider momentum in vertical direction.

At time t: $\quad p(t) = \rho(a-x)v$

At time $t+\delta t$: $\quad p(t+\delta t) = \rho(a-x-\delta x)(v+\delta v)$

so change of momentum $\delta p = p(t+\delta t) - p(t)$

$$\delta p = \rho(a-x-\delta x)(v+\delta v) - \rho(a-x)v$$
$$= \rho(av + a\delta v - xv - x\delta v - \delta x v - \delta x \delta v) - \rho(av - xv)$$

$$\delta p = \rho(a-x)\delta v - \rho v \delta x$$

$$\Rightarrow \frac{\delta p}{\delta t} = \rho(a-x)\frac{\delta v}{\delta t} - \rho v \frac{\delta x}{\delta t}$$ take limit $\delta t \to 0$

$$\frac{dp}{dt} = \rho(a-x)\dot{v} - \rho v \dot{x}$$

The acceleration is just g

This is just v so we have $-\rho v^2$ but $v^2 = u^2 + 2as$ from suvat so v^2 here is $v^2 = 2gx$

$$\therefore \frac{dp}{dt} = \rho(a-x)g - 2\rho x g = \rho a g - F(x)$$ which is resultant force

so therefore $\underline{\underline{F(x) = 3\rho x g}}$

Angular Motion

we define angular momentum $\underline{L}$ as $\underline{L} = \underline{r} \times \underline{p}$

we define torque $\underline{\tau}$ as $\underline{\tau} = \underline{r} \times \underline{F}$

So we see that $\underline{\tau} = \frac{d\underline{L}}{dt} = \frac{d}{dt}\{\underline{r} \times \underline{p}\} = \underline{r} \times \underline{F}$

For a system of N particles, we find total $\underline{L}$ and $\underline{\tau}$ as:

$$\boxed{\underline{L} = \sum_i^N \underline{r}_i \times \underline{p}_i} \quad \boxed{\underline{\tau} = \sum_i^N \underline{r}_i \times \underline{F}_i}$$

Just as before we split $\underline{F}_i$ into the external force $\underline{F}_i^{ext}$ and the force of particle j on particle i $\underline{F}_{ij}$ so:

$$\underline{\tau} = \sum_i^N \underline{r}_i \times \left(\underline{F}_i^{ext} + \sum_{i \neq j}^N \underline{F}_{ij}\right) = \sum_i^N \underline{r}_i \times \underline{F}_i^{ext} + \underbrace{\sum_i^N \sum_{i \neq j}^N \underline{F}_{ij}}$$

The $\underline{F}_{ij}$ and $\underline{F}_{ji}$ cancel out only for central forces

$$\text{so } \underline{\tau} = \sum_i^N \underline{r}_i \times \underline{F}_i^{ext}$$

$$= \frac{d}{dt}\left\{\sum_i^N \underline{r}_i \times \underline{p}_i\right\}$$

$$\text{so } \boxed{\underline{\tau}^{ext} = \dot{\underline{L}}}$$

Angular Momentum About the Centre of Mass

Let's start using our C of M coordinates again: $\underline{r}_i = \underline{R} + \underline{\rho}_i$

$$\underline{L} = \sum_i^N \underline{r}_i \times \underline{p}_i = \sum_i^N \underline{r}_i \times m_i \dot{\underline{r}}_i = \sum_i^N (\underline{R} + \underline{\rho}_i) \times m_i (\dot{\underline{R}} + \dot{\underline{\rho}}_i)$$

$$= \sum \underline{R} \times m_i \dot{\underline{R}} + \underbrace{\sum \underline{R} \times m_i \dot{\underline{\rho}}_i}_{=0 \text{ since } \sum m_i \dot{\underline{\rho}}_i = 0} + \sum \underline{\rho}_i \times m_i \dot{\underline{R}} + \underbrace{\sum \underline{\rho}_i \times m_i \dot{\underline{\rho}}_i}_{=0 \text{ since } \sum m_i \dot{\underline{\rho}}_i = 0}$$

$$\underline{L} = \sum \underline{R} \times m_i \dot{\underline{R}} + \sum \underline{\rho}_i \times m_i \underline{R}$$

$$\underline{L} = \sum \underline{R} \times M_i \dot{\underline{R}} + \sum \underline{\rho}_i \times M_i \dot{\underline{\rho}}_i$$

$$\boxed{\underline{L} = \underline{R} \times M \dot{\underline{R}} + \underline{L}_{C\&M}}$$

so total angular momentum has two terms, the angular momentum of the system about the centre of mass ($\underline{R} \times M \dot{\underline{R}}$) and the "intrinsic" or "spin" angular momentum ($\underline{L}_{C\&M}$), which is the same in all reference frames. This is analogous to our formula for kinetic energy T.

We can take the time derivative to find:

$$\frac{d\underline{L}}{dt} = \underline{R} \times M \ddot{\underline{R}} + \frac{d\underline{L}_{C\&M}}{dt}$$

$$\Rightarrow \frac{d\underline{L}_{C\&M}}{dt} = \frac{d\underline{L}}{dt} - \underline{R} \times M \ddot{\underline{R}}$$

$$= \underline{\tau}^{ext} - \underline{R} \times M \ddot{\underline{R}} = \underline{\tau}^{ext} - \underline{R} \times \underline{F}^{ext}$$

$$= \sum_i^N \underline{r}_i \times \underline{F}_i^{ext} - \sum_i^N \underline{R} \times \underline{F}_i^{ext}$$

$$= \sum_i^N (\underline{r}_i - \underline{R}) \times \underline{F}_i^{ext}$$

$$= \sum_i^N \underline{\rho}_i \times \underline{F}_i^{ext}$$

$$\frac{d\underline{L}_{C\&M}}{dt} = \underline{\tau}_{C\&M}^{ext}$$

so we have found $\boxed{\frac{d\underline{L}}{dt} = \underline{\tau}^{ext}}$ and $\boxed{\frac{d\underline{L}_{C\&M}}{dt} = \underline{\tau}_{C\&M}^{ext}}$

so we can take moments about the origin or the centre of mass. And the angular momentum of a system subject to no external torque is conserved.