Least Action - Newtonian Dynamics

In the optics section, we made use of the principle of least time. The newtonian dynamic equivalent is Hamilton's Principle:

> A particle travels by the path between two points that minimises the Action

The Lagrangian is given here as $\boxed{L = T - V}$

where T is the kinetic energy and V is the potential

so $L = \frac{1}{2}m\dot{x}^2 - V(x)$ for a non-relativistic particle in 1D

The Euler-Lagrange equation is:

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0}$$

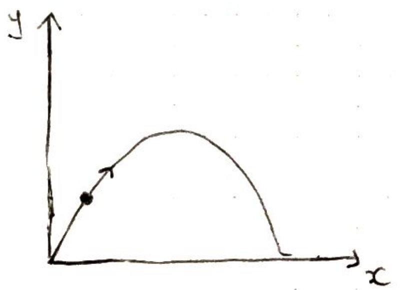
So for a 1D particle:

$$\frac{d}{dt}(m\dot{x}) + \frac{\partial V}{\partial x} = 0$$

Note that this is just Newton's 2nd Law where $F = -\frac{\partial V}{\partial x}$

Note that momentum is given by $p = m\dot{x} = \frac{\partial L}{\partial \dot{x}}$

Let's try an example now with Projectile Motion:



The kinetic energy is given by:

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2$$

The potential energy is given by:

$$V = mgy$$

so the Lagrangian is: $L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mgy$

Since this example is two dimensional, there will be 2 Euler Lagrange equations.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

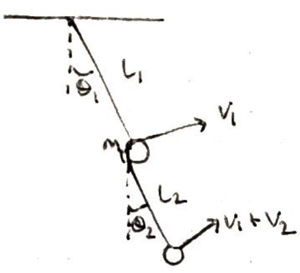
$$\Rightarrow \frac{d}{dt}(m\dot{x}) = 0 \quad \text{so} \quad \underline{\underline{m\ddot{x} = 0}}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

$$\Rightarrow \frac{d}{dt}(m\dot{y}) + mg = 0 \quad \text{so} \quad \underline{\underline{\ddot{y} = -g}}$$

We have thus found the standard Newtonian equations of motion. This shows how easy it is to obtain Newtonian results with this method. What about a system that cannot be solved in the standard way?

eg. Double Pendulum



This would be pretty hard to solve by resolving force components. The EL method is much easier!

$$\underline{v}_{tot} = \underline{v}_1 + \underline{v}_2$$

$$|\underline{v}_{tot}|^2 = \underline{v}_{tot} \cdot \underline{v}_{tot} = (\underline{v}_1 + \underline{v}_2) \cdot (\underline{v}_1 + \underline{v}_2)$$
$$= (L_1\dot{\theta}_1)^2 + (L_2\dot{\theta}_2)^2 + 2L_1\dot{\theta}_1 L_2\dot{\theta}_2 \cos(\theta_2 - \theta_1)$$

where $\theta_2 - \theta_1$ is the angle between $\underline{v}_1$ and $\underline{v}_2$

$$T = \tfrac{1}{2}m_1 v_1^2 + \tfrac{1}{2}m_2 v_{tot}^2$$
$$T = \tfrac{1}{2}m_1 L_1^2\dot{\theta}_1^2 + \tfrac{1}{2}m_2\left[(L_1\dot{\theta}_1)^2 + (L_2\dot{\theta}_2)^2 + 2L_1\dot{\theta}_1 L_2\dot{\theta}_2\cos(\theta_2 - \theta_1)\right]$$

The potential is given by:

$$V = -m_1 g L_1 \cos\theta_1 - m_2 g (L_1\cos\theta_1 + L_2\cos\theta_2)$$

The Lagrangian is $L = T - V$

We have 2 EL equations, one for θ_1 and one for θ_2:

$$\frac{d}{dt}\left\{ m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_2 - \theta_1) \right\}$$

$$- m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2 - \theta_1) + (m_1 + m_2) g l_1 \sin\theta_1 = 0$$

$$\frac{d}{dt}\left\{ m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_2 - \theta_1) \right\}$$

$$+ m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2 - \theta_1) + m_2 g l_2 \sin\theta_2 = 0$$

These look pretty awful but we can simplify them to:

$$(m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 = -(m_1 + m_2) g l_1 \theta_1$$

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 = -m_2 g l_2 \theta_2$$

which each have normal mode solutions of the form:

$$\ddot{\theta}_1 = -\omega^2 \theta_1$$

$$\ddot{\theta}_2 = -\omega^2 \theta_2$$

so the two pendulums oscillate with the same frequency!