Physics with Four Momentum

In this chapter we will absorb the speed of light c into the units of other quantities, effectively allowing us to set $c=1$. You will see what this means later:

Doppler Effect

What frequency of a photon would be measured by an observer moving relative to the photon.

The photons have four-momentum $p^\mu = (E, \underline{p}) = (hf, -hf, 0, 0)$ if the observer is moving in $+x$ direction relative to the photon. Here, we have used $p = \frac{h}{\lambda} = \frac{hf}{c}$ but we need $-c$ since photon is moving in $-x$ direction so setting $c=-1$, $p = -hf$.

So, if the observer is moving in $+x$ direction with velocity v:

$$p'^\mu = \begin{bmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} hf \\ -hf \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma(1+v)hf \\ -\gamma(1+v)hf \\ 0 \\ 0 \end{bmatrix}$$

Let's just focus on $p'^0 = E' = hf' = \frac{1}{\sqrt{1-v^2}}(1+v)hf$

$$= \sqrt{\frac{(1+v)^2}{(1+v)(1-v)}}\, hf = \sqrt{\frac{1+v}{1-v}}\, hf$$

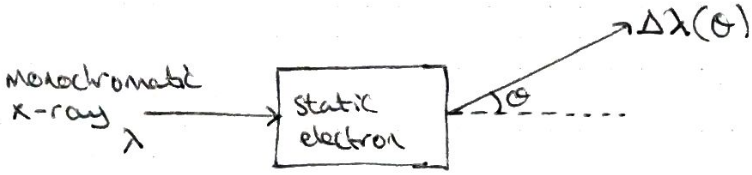
$$\Rightarrow f' = \sqrt{\frac{1+v}{1-v}}\, f$$

adding back the factors of c:

$$\boxed{f' = \sqrt{\frac{1+v/c}{1-v/c}}\, f}$$ which is the familiar equation

The Compton Effect

The compton effect relates the angle of scattering of a photon off a static electron to its final wavelength:



Let's set up the 4-momentums of the particles:

initial photon : $p_{\gamma_i}^{\mu} = \left(\frac{h}{\lambda}, \frac{h}{\lambda}\hat{\underline{x}}\right)$

initial electron : $p_{e_i}^{\mu} = (m_e, \underline{0})$

final photon : $p_{\gamma_f}^{\mu} = \left(\frac{h}{\lambda'}, \frac{h}{\lambda'}\hat{\underline{f}}\right)$ where $\hat{\underline{f}}$ is unit vector in direction of motion (angle θ to $\hat{\underline{x}}$ axis)

final electron : $p_{e_f}^{\mu}$

Conservation of four-momentum tells us:

$$p_{\gamma_i}^{\mu} + p_{e_i}^{\mu} = p_{\gamma_f}^{\mu} + p_{e_f}^{\mu}$$

$$\Rightarrow \quad p_{e_f}^{\mu} = p_{\gamma_i}^{\mu} + p_{e_i}^{\mu} - p_{\gamma_f}^{\mu}$$

but we know $p_{e_f}^{\mu} p_{e_f\mu} = m_e^2$

$$= (p_{\gamma_i}^{\mu} + p_{e_i}^{\mu} - p_{\gamma_f}^{\mu})(p_{\gamma_i\mu} + p_{e_i\mu} - p_{\gamma_f\mu})$$

$$= p_{\gamma_i}^{\mu} p_{\gamma_i\mu} + p_{e_i}^{\mu} p_{e_i\mu} + p_{\gamma_f}^{\mu} p_{\gamma_f\mu} + 2(p_{\gamma_i}^{\mu} p_{e_i\mu} - p_{\gamma_i}^{\mu} p_{\gamma_f\mu} - p_{\gamma_f}^{\mu} p_{e_i\mu})$$

$$= 0 + m_e^2 + 0 + 2\left(\frac{h}{\lambda_i} m_e - 0\right) - 2\frac{h}{\lambda_i}\frac{h}{\lambda_f}(1-\cos\theta) - 2\left(\frac{h}{\lambda_f} m_e - 0\right)$$

in the last stage we used $\underline{p_1^{\mu} p_{2\mu} = p_1^0 p_2^0 - \underline{p_1}\cdot\underline{p_2}}$

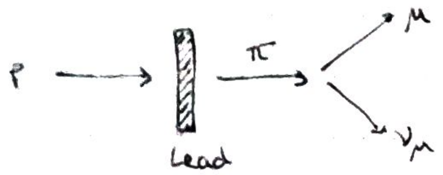
$$\Rightarrow \frac{h}{\lambda_i} m_e - \frac{h}{\lambda_f} m_e = \frac{h}{\lambda_i}\frac{h}{\lambda_f}(1-\cos\theta) \Rightarrow \lambda_f - \lambda_i = \frac{h}{m_e}(1-\cos\theta)$$

Putting the c factor back in:

$$\boxed{\lambda_f - \lambda_i = \frac{h}{m_e c}(1-\cos\theta)}$$

Fixed Target Experiments

We can create fundamental particles by colliding a high energy proton or electron into a fixed target like lead:



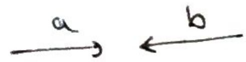
A proton colliding with lead produces a pion that decays into a muon and muon neutrino.

Let's consider this in two frames: the lab frame and the centre of mass frame.

Lab frame

$p_a^\mu = (E_a, p_a)$ $\quad$ $p_b^\mu = (M_b, 0)$

Centre of Mass Frame



So what boost is needed to move from lab frame to CoM frame? Let's say we need to boost 4-momenta by v:

$$p_a^{\mu\prime} = \begin{pmatrix} \gamma & -\gamma v \\ -\gamma v & \gamma \end{pmatrix}\begin{pmatrix} E_a \\ p_a \end{pmatrix} = \begin{pmatrix} \gamma(E_a - v p_a) \\ \gamma(p_a - v E_a) \end{pmatrix}$$

$$p_b^{\mu\prime} = \begin{pmatrix} \gamma & -\gamma v \\ -\gamma v & \gamma \end{pmatrix}\begin{pmatrix} M_b \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma M_b \\ -\gamma v M_b \end{pmatrix}$$

In the CoM frame, momenta must be equal and opposite:

$$p_a^{\prime x} = -p_b^{\prime x} \Rightarrow \gamma(p_a - v E_a) = \gamma v M_b$$

$$\Rightarrow p_a = v(M_b + E_a) \quad \text{so} \quad v = \frac{p_a}{M_b + E_a}$$

Adding in factors of c:

$$\boxed{\frac{v}{c} = \frac{p_a}{M_b c + E_a/c}}$$

This is the boost necessary to move from lab frame to CoM frame.

If after the boost, the particles are ultrarelativistic such that $E_a \approx |P_a| = |P_b| \approx E_b$ then total available energy is:

$$E_{COM} = 2\gamma M_b c = \sqrt{\frac{4M_b^2 c^2}{1-\left(\frac{p_a}{M_b c + E_a/c}\right)}}$$

$$= \sqrt{\frac{4M_b^2 c^2 (M_b c + E_a/c)^2}{(M_b c + E_a/c)^2 - P_a^2}}$$

if we take the limit $E_a \gg M_b c^2, M_a c^2$
i.e the majority of energy is <u>not</u> rest mass energy, then

$$E_{COM} = \sqrt{\frac{4M_b^2 E_a^2}{2M_b E_a}}$$

$$\boxed{E_{COM} = \sqrt{2M_b E_a}}$$

This is also obtainable by calculating invariant rest mass of whole system in original coordinates.

$$P^{\mu}_{TOT} P_{TOT\,\mu} = M_{TOT}^2 c^2$$
$$= (P_a^{\mu} + P_b^{\mu})(P_{a\mu} + P_{b\mu})$$
$$= P_a^{\mu} P_{a\mu} + P_b^{\mu} P_{b\mu} + 2P_a^{\mu} P_{b\mu}$$

which when $E_a \gg$ rest masses gives:

$$M_{TOT}^2 c^2 = 2\frac{E_a}{c}\frac{M_b}{c}c^2 \Rightarrow M_{TOT}^2 c^2 = 2E_a M_b$$

$$\Rightarrow E_{COM} = \sqrt{2M_b E_a}$$

as we found before

GZK Bound

Active galaxies can accelerate protons to high energies but we do expect to see a maximum energy because higher energy protons can interact with photons from the cosmic microwave background radiation ($T \sim 3K$ so $E_\gamma = k_B T \sim 8 \times 10^{-4}$ eV)

$$p\gamma \rightarrow \Delta \rightarrow \pi^+ n$$ This is the proton photon interaction

this Δ is a short lived particle with $M_\Delta \sim 1.2 \text{GeV}/c^2$

Let's assign initial 4-momenta to the proton and photon

$$P_p^\mu = (E_p, k, 0, 0) \qquad P_\gamma^\mu = (h\nu, -h\nu, 0, 0)$$

due to conservation of 4-Momenta:

$$P_\Delta^\mu = P_p^\mu + P_\gamma^\mu$$ squaring:

$$P_\Delta^\mu P_{\Delta\mu} = P_p^\mu P_{p\mu} + P_\gamma^\mu P_{\gamma\mu} + P_p^\mu P_{\gamma\mu} + P_\gamma^\mu P_{p\mu}$$

$$M_\Delta^2 = M_p^2 + M_\gamma^2 + 2E_p h\nu - 2k(-h\nu)$$

for a relativistic proton $E_p \simeq k$ so:

$$E_p = \frac{M_\Delta^2 - M_p^2}{4h\nu} \gtrsim \underline{\underline{2 \times 10^{20} \text{ GeV}}}$$

So protons with this energy or higher will interact with CMB photons. So we shouldn't see protons with energy higher than this from active galaxies.

Surprisingly, we see higher energy protons in observations

Why we do is an open question currently (2019)