Potential formulation of Electrodynamics

Scalar and Vector Potentials

Let's remind ourselves of Maxwell's Equations:

$\underline{\nabla} \cdot \underline{E} = \frac{1}{\varepsilon_0} \rho$

$\underline{\nabla} \cdot \underline{B} = 0$

$\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$

$\underline{\nabla} \times \underline{B} = \mu_0 \underline{J} + \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}$

In this chapter we will look at solving these for non-zero charge and current density where solutions involve time and space dependent $\underline{E}$ and $\underline{B}$ fields.

Let's start by representing the fields in terms of the scalar and vector potentials.

We know $\underline{B} = \underline{\nabla} \times \underline{A}$ which relied on the fact that $\underline{\nabla} \cdot \underline{B} = 0$ so this is always true.

We thought $\underline{E} = \underline{\nabla} V$ but this relied on the fact that $\underline{\nabla} \times \underline{E} = 0$ which Faraday showed us wasn't true! So what do we do?

Let's start with $\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$ let's sub in $\underline{B} = \underline{\nabla} \times \underline{A}$

$$\underline{\nabla} \times \underline{E} = -\frac{\partial}{\partial t}\{\underline{\nabla} \times \underline{A}\} \Rightarrow \underline{\nabla} \times \underline{E} + \frac{\partial}{\partial t}\{\underline{\nabla} \times \underline{A}\} = 0$$

$$\underline{\nabla} \times \left\{\underline{E} + \frac{\partial \underline{A}}{\partial t}\right\} = 0$$

so we can define a new electric potential since if $\underline{\nabla} \times \underline{X} = 0$ $\underline{X} = \underline{\nabla} V$

$$\therefore \quad \underline{E} + \frac{\partial \underline{A}}{\partial t} = -\underline{\nabla} V$$

$$\therefore \quad \boxed{\underline{E} = -\underline{\nabla} V - \frac{\partial \underline{A}}{\partial t}}$$

In the static case $\frac{\partial \underline{A}}{\partial t} = 0$ so reduces to our old equation.

So if we can calculate $\underline{A}$ and V, it is straightforward to calculate $\underline{B}$ and $\underline{E}$

So we have $\underline{B} = \underline{\nabla} \times \underline{A}$ and $\underline{E} = -\underline{\nabla} V - \frac{\partial \underline{A}}{\partial t}$

These satisfy two of the Maxwell Equations (No monopoles and Faraday's law) but let's try them with the other 2:

Gauss' Law: $\underline{\nabla} \cdot \underline{E} = \frac{1}{\varepsilon_0} \rho$

$$\underline{\nabla} \cdot \left(-\underline{\nabla} V - \frac{\partial \underline{A}}{\partial t}\right) = \frac{1}{\varepsilon_0} \rho$$

$$* \quad -\underline{\nabla}^2 V - \frac{\partial}{\partial t}(\underline{\nabla} \cdot \underline{A}) = \frac{1}{\varepsilon_0} \rho$$

which is true in the static case

Ampère Maxwell Law: $\underline{\nabla} \cdot \underline{B} = \mu_0 \underline{J} + \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}$

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{A}) = \mu_0 \underline{J} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\left\{-\underline{\nabla} V - \frac{\partial \underline{A}}{\partial t}\right\}$$

$$= \mu_0 \underline{J} - \mu_0 \varepsilon_0 \underline{\nabla} \frac{\partial V}{\partial t} - \mu_0 \varepsilon_0 \frac{\partial^2 \underline{A}}{\partial t^2}$$

after some rearranging:

$$* \quad \Rightarrow \mu_0 \varepsilon_0 \frac{\partial^2 \underline{A}}{\partial t^2} - \nabla^2 \underline{A} + \underline{\nabla}\left(\underline{\nabla} \cdot \underline{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t}\right) = \mu_0 \underline{J}$$

So now we have found the two * equations, how do they help us? Well now instead of finding 6 fields (3 components each for $\underline{E}$ and $\underline{B}$) we only need to find 4.

We can also use gauge invariance to neaten these equations!

Gauge Invariance

What is surprising is that despite $\underline{E}$ and $\underline{B}$ being well defined and taking unique values, there are multiple values of V and $\underline{A}$ that give us these. This is due to Gauge invariance.

Gauge Invariance is the concept that we can make gauge transformations (i.e offsets) to certain quantities and the results obtained from those quantities remain the same. For example, we can measure gravitational potential of a ball falling from 10 metres to derive Newton's 2nd Law. We can also apply a gauge transformation to gravitational potential, suppose by dropping the ball from 100m (so the starting grav. potential is offset) and we still obtain the same result for Newton's 2nd Law.

We can do this for $\underline{A}$ and V:

$$\boxed{\underline{A} \rightarrow \underline{A} + \underline{\nabla}\phi \qquad V \rightarrow V - \frac{\partial \phi}{\partial t}}$$

Electric and Magnetic fields calculated using the * equations are left invariant after these transformations.

We can choose a function $\phi(\underline{r}, t)$ such that:

$$\boxed{\underline{\nabla} \cdot \underline{A} = -\mu_0 \varepsilon_0 \frac{\partial V}{\partial t}}$$

this eliminates the $\frac{\partial}{\partial t}(\underline{\nabla} \cdot \underline{A})$ term in the 2nd * equation

The choice we make is called the Lorentz gauge. Once we have done it the star equations become:

$$\boxed{\mu_0 \varepsilon_0 \frac{\partial^2 \underline{A}}{\partial t^2} - \underline{\nabla}^2 V = \frac{1}{\varepsilon_0} \rho} \qquad \boxed{\mu_0 \varepsilon_0 \frac{\partial^2 \underline{A}}{\partial t^2} - \nabla^2 \underline{A} = \mu_0 \underline{J}}$$

Really beautiful equations!

We can further neaten this by defining the d'Alembertian operator or box operator:

$$\boxed{\Box^2 = \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} - \nabla^2}$$

so the equations now become

$$\boxed{\Box^2 V = \frac{1}{\varepsilon_0} \rho} \qquad \boxed{\Box^2 \underline{A} = \mu_0 \underline{J}}$$

which are even nicer!

These, together with

$$\boxed{\begin{aligned} \underline{B} &= \underline{\nabla} \times \underline{A} \\ \underline{E} &= -\underline{\nabla} V - \frac{\partial \underline{A}}{\partial t} \end{aligned}}$$

contain all the information of Maxwells Equations.

Formal Solutions

In the static case, $\Box^2 V = \frac{1}{\varepsilon_0}\rho$ and $\Box^2 \underline{A} = \mu_0 \underline{J}$ reduce to:

$$-\nabla^2 V = \frac{1}{\varepsilon_0}\rho \quad \text{and} \quad -\nabla^2 \underline{A} = \mu_0 \underline{J}$$

which are forms of the Poisson equation with known solutions:

$$V(\underline{r}) = \frac{1}{4\pi\varepsilon_0}\iiint_{\text{volume}} \frac{\rho(\underline{R}')}{r'}\,dV'$$

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\iiint_{\text{volume}} \frac{\underline{J}(\underline{R})\,dV'}{r'}$$

But there is no time dependence here. We want to know what happens with time-varying sources. So what do we do?

An additional complication with introducing a time dependence is that electromagnetic information does not travel instantaneously but rather at the speed of light. So if the field changes, the potential does not change instantly; rather, it takes some time to receive the "message"

Taking these time delays into account, we have "retarded potentials" involving the retarded time:

$$\boxed{t_r = t - \frac{r'}{c}}$$

So the time dependent retarded potential solutions are:

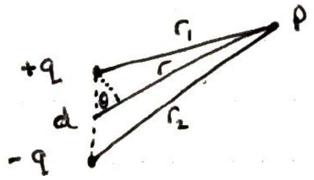
$$\boxed{V(\underline{r}, t) = \frac{1}{4\pi\varepsilon_0}\iiint_{\text{volume}} \frac{\rho(\underline{R}', t_r)}{r'}\,dV'}$$

$$\boxed{\underline{A}(\underline{r}, t) = \frac{\mu_0}{4\pi}\iiint_{\text{volume}} \frac{\underline{J}(\underline{R}', t_r)}{r'}\,dV'}$$

We will now see an example of these for the case of the oscillating electric dipole

Electric Dipole Radiation

consider a dipole:



The potential at the point P is the sum of the potentials at that point due to $+q$ and $-q$:

$$V = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q}{r_1} - \frac{q}{r_2}\right\}$$

Now consider if the charges $\pm q$ vary as: $\underline{q(t) = q_0 \cos\omega t}$

At time t, charge upper is $+q(t)$ and charge lower is $-q(t)$

The current through dotted line is $\underline{I(t) = \frac{dq(t)}{dt} = -\omega q_0 \sin\omega t}$

We have to now consider the retarded potential as discussed on the previous page. Retarded time t_r is $t_r = t - \frac{r'}{c}$ so:

$$V = \frac{q_0}{4\pi\varepsilon_0}\left\{\frac{\cos[\omega(t-\frac{r_1}{c})]}{r_1} - \frac{\cos[\omega(t-\frac{r_2}{c})]}{r_2}\right\}$$

if we make the assumption $d \ll r$ and $d \ll \frac{c}{\omega}$, after <u>lots</u> of algebra we have:

$$V \approx \frac{q_0}{4\pi\varepsilon_0} d\cos\theta \left\{\frac{1}{r^2}\cos\left[\omega(t-\frac{r}{c})\right] - \frac{1}{r}\frac{\omega}{c}\sin\left[\omega(t-\frac{r}{c})\right]\right\}$$

↑this dominates at large r

so $$\underline{\underline{V \approx -\frac{q_0 d}{4\pi\varepsilon_0}\frac{\cos\theta}{r}\frac{\omega}{c}\sin\left[\omega(t-\frac{r}{c})\right]}}$$

We can also calculate the magnetic vector potential $\underline{A}$ in a similar way, continued overleaf.

$$\underline{A} = \frac{\mu_0}{4\pi} \frac{I\,d\underline{l}}{r}$$

so putting in $I(t)$ and making changes for retarded time:

$$\underline{A} = -\hat{\underline{z}} \frac{\mu_0}{4\pi} \frac{\omega q_0 d}{r} \sin\left[\omega\left(t - \frac{r}{c}\right)\right]$$

so, using $\underline{E} = -\underline{\nabla} V - \frac{\partial A}{\partial t}$ and $\underline{B} = \underline{\nabla} \times \underline{A}$, we can work out the fields:

$$\boxed{\underline{E} \approx -\frac{\mu_0 \omega^2 q_0 d}{4\pi} \left(\frac{\sin\theta}{r}\right) \cos\left[\omega\left(t - \frac{r}{c}\right)\right] \hat{\underline{\theta}}}$$

$$\boxed{\underline{B} \approx -\frac{\mu_0 \omega^2 q_0 d}{4\pi c} \left(\frac{\sin\theta}{r}\right) \cos\left[\omega\left(t - \frac{r}{c}\right)\right] \hat{\underline{\phi}}}$$

$\hat{\underline{\theta}}$ is a unit vector in direction of increasing θ and $\hat{\underline{\phi}}$ is unit vector in direction of increasing ϕ

Where θ and ϕ are coordinate angles in spherical coordinates.

$\underline{E}$ and $\underline{B}$ are in phase, mutually perpendicular and transverse to unit vector $\hat{\underline{r}}$. We also find $E_0 / B_0 = c$ where E_0 and B_0 are amplitudes of $\underline{E}$ and $\underline{B}$.

So these satisfy all conditions of EM waves but they are spherical, not planar!

we can calculate Poynting Vector: $\underline{S} = \left(\frac{\mu_0 \omega^4 q_0^2 d^2}{16\pi^2 c}\right) \frac{\sin^2\theta}{r^2} \cos^2\left[\omega\left(t - \frac{r}{c}\right)\right] \hat{\underline{r}}$

so $\underline{S}_{ave} = \left(\frac{\mu_0 \omega^4 q_0^2 d^2}{32\pi^2 c}\right) \frac{\sin^2\theta}{r^2} \hat{\underline{r}}$ This takes the form of a torus

$\underline{P}_{ave} = \left(\frac{\mu_0 \omega^4 q_0^2 d^2}{12\pi c}\right)$ given by integrating $\underline{S}_{ave}$ over large sphere of radius r, where $r \to \infty$

Fun fact, the sky appears blue due to this ω^4 dependence. The larger the frequency the higher the scattering power so blue is scattered more than red.