Time Independant Schrodinger Equation

In this section, we will be learning how to compute wave functions for various toy examples. We compute wave functions by solving the Schrodinger Equation:

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x,t)}{\partial x^2} + V(x)\Psi(x,t)$$

The particular physical set up (eg. square well or harmonic oscillator) is defined by the form of the potential $V(x)$.

A simplification we will make throughout the course is that the potential will be taken to be time-independent, hence why it is only $V(x)$, not $V(x,t)$. If we make this assumption, we can make the factorisation ansatz:

$$\Psi(x,t) = \psi(x)\varphi(t)$$

i.e, the wavefunction can be separated into the product of two wave functions, one which depends only on x and the other depends only on t.

Substituting into the SE:

$$i\hbar\, \psi(x) \frac{\partial \varphi(t)}{\partial t} = -\frac{\hbar^2}{2m} \varphi(t) \frac{\partial^2 \psi(x)}{\partial x^2} + V(x)\psi(x)\varphi(t)$$

Dividing both sides by $\psi(x)\varphi(t)$:

$$\underbrace{i\hbar \frac{1}{\varphi} \frac{\partial \varphi}{\partial t}}_{\text{depends only on } t} = \underbrace{-\frac{\hbar^2}{2m} \frac{1}{\psi} \frac{\partial^2 \psi}{\partial x^2} + V}_{\text{depends only on } x}$$

Since the LHS and RHS depend on different variables, they must be equal to some constant. We call this constant E since we suspect it could be energy.

(1) $i\hbar \dfrac{\partial \varphi}{\partial t} = E\varphi$

(2) $-\dfrac{\hbar^2}{2m} \dfrac{\partial^2 \psi}{\partial x^2} + V\psi = E\psi$

We can solve ① immediately to give:

$$i\hbar \frac{\partial \varphi}{\partial t} = E\varphi \Rightarrow \frac{d\varphi}{\varphi} = \frac{E}{i\hbar} dt = -\frac{iE}{\hbar} dt$$

$$\int_0^t \frac{d\varphi}{\varphi} = \int_0^t -\frac{iE}{\hbar} dt \Rightarrow \ln(\varphi(t)) - \ln(\varphi(0)) = -\frac{iEt}{\hbar}$$

$$\therefore \varphi(t) = Ce^{-iEt/\hbar}$$

We can therefore say $\boxed{\Psi(x,t) = \psi(x) e^{-iEt/\hbar}}$

Note that the constant was dropped since any constants multiplying a wave equation will be determined by normalisation.

So we have succeeded in separating the wavefunction into spacial-dependent and time-dependent parts. We can solve for spacial-dependent and simply multiply by our found time dependence for a complete solution. So how do we work out the spacial-dependent solution? From ②!

We call ② : $\boxed{-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi = E\psi}$ the time independent Schrodinger Equation

We solve this for a V specific to the physical set up to find the space dependent part of the solution. We will look at how to do this for some generic cases.

Some observations of what we've done:

- Separable solutions, i.e $\Psi(x,t) = \psi(x)e^{-iEt/\hbar}$ correspond to stationary states. The probability density for stationary states is time-independent even though the wave function itself isn't.

  We can see this as $|\Psi|^2 = \Psi^*\Psi = \psi^*\psi \underbrace{e^{-iEt/\hbar} e^{iEt/\hbar}}_{=1} = \psi^*\psi$

  So there is no time-dependence in the norm-square.

  As a result, the expectation values of operators, like $\langle x \rangle$ and $\langle p \rangle$ are also time-independent:

  $$\langle p \rangle = \int \Psi^* \hat{p} \Psi \, dx dt = \int \psi^* \hat{p} \psi \, dx$$

- Separable solutions have definite energy. We know from classical mechanics that the total energy is given by $H = \frac{p^2}{2m} + V$, called the Hamilton. In quantum mechanics, we make this into the Hamiltonian operator:

$$\boxed{\hat{H} = \frac{\hat{p}^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)}$$

We can therefore write the time independent SE as:

$$\underline{\underline{\hat{H}\psi = E\psi}}$$

We can check that this energy is definite by calculating the variance:

$$\langle H \rangle = \int \psi^* \hat{H} \psi \, dx = \int \psi^* E \psi \, dx = E \int \psi^* \psi \, dx = \underline{\underline{E}}$$

$$\langle H^2 \rangle = \int \psi^* \hat{H}^2 \psi \, dx = \int \psi^* E . \hat{H} \psi \, dx = E^2 \int \psi^* \psi \, dx = \underline{\underline{E^2}}$$

$\sigma_H = \langle H^2 \rangle - \langle H \rangle^2 = \underline{\underline{0}}$ so the energy is <u>definite</u> in a stationary state.

- The general solution is a linear combination of separable solutions.

There are infinitely many solutions to the TISE, one for each allowed value of E. We could label them E_1 for ψ_1, E_2 for ψ_2 and so on such that $\Psi(x,t) = \sum_n c_n \psi_n e^{-iE_n t/\hbar}$

where c_n is a constant that defines the proportional contribution for each stationary state.

Recipe for Solving Toy examples

1) Solve the time-independent SE for the potential specific to the physical problem to obtain $\psi_1, \psi_2, \psi_3 \ldots$ for $E_1, E_2, E_3 \ldots$

2) Write down general solution $\Psi(x,0) = \sum_n c_n \psi_n(x)$ and determine c_n

3) Write time dependent solution as $\Psi(x,t) = \sum_n c_n \psi_n(x) e^{-iE_n t/\hbar}$