Four Vectors

In the previous section, we worked on a ct-x plane.
So in "3D", we would have a ct-xyz plane
We can express this as a four vector with einstein notation

$$x^\mu = (ct, x, y, z) = (x^0, x^1, x^2, x^3)$$

The index μ takes on values 0, 1, 2, 3 corresponding to the components. We can write the transformation matrix from the previous section as:

$$x^\mu \rightarrow x'^\mu = \begin{bmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

The Lorentz Invariant Length is:

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 = (ct)^2 - |\underline{x}|^2$$

Index Convention

We can write the transformation matrix above as

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu \, x^\nu$$ (note, the ν indices are 'balanced' as one is on top and one is bottom)

$\Lambda^\mu{}_\nu$ has an upstairs index and a downstairs index.
if μ and ν have values $\{0, 1, 2, 3\}$ then we know $\Lambda^\mu{}_\nu$ is a 4x4 matrix.

Λ^{μ} ← counts the rows

Λ_{ν} ← counts the columns

Lorentz Invariant Length

We previously found that the Lorentz invariant length is $(ct)^2 - |\underline{x}|^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$ but we can write this much simpler.

If we define a "metric" $g_{\mu\nu}$ as:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

then $x_\mu = g_{\mu\nu} x^\nu = (ct, -x, -y, -z)$

Note, $x^\mu = (ct, x, y, z)$ and $x_\mu = (ct, -x, -y, -z)$

See how the vector changes depending on whether we use an upstairs or downstairs index.

$$x^\mu x_\mu = (ct)^2 - |\underline{x}|^2$$

so the Lorentz invariant length is $\underline{x^\mu x_\mu}$