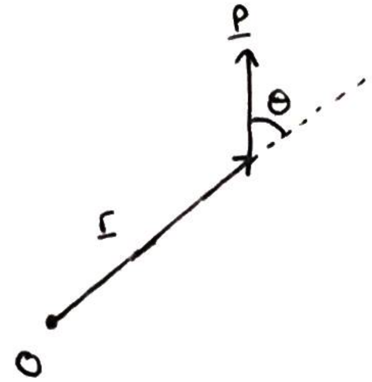
Angular Momentum



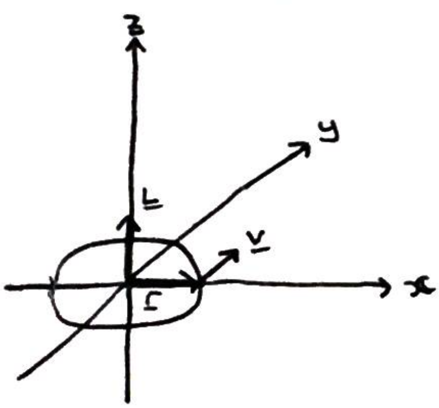
Vector $\underline{r}$, from point of origin to position of particle at a given time.

vector $\underline{p}$ indicates direction of motion of particle. It is the linear momentum vector.

The angular momentum $\underline{L}$ is given by: $\boxed{\underline{L} = \underline{r} \times \underline{p}}$

Note that this is the cross product which means angular momentum is perpendicular to both $\underline{r}$ and $\underline{p}$. If $\underline{r}$ and $\underline{p}$ are parallel, then the angular momentum vanishes since $\underline{L} = |\underline{r}||\underline{p}| \sin\theta \, \hat{\underline{n}}$ and $\sin\theta$ will be 0. This makes sense intuitively since if $\underline{p}$ is parallel to $\underline{r}$, the the momentum vector will be pointing directly to or away from the origin, meaning there is no turning point about the origin.

Circular Motion



For a particle moving at constant speed in a circle of constant radius $\underline{r}$, the linear momentum vector points tangential to the circle, making it perpendicular to the radius. We can find $\underline{L}$ by:

$$\underline{L} = \underline{r} \times \underline{p} = |\underline{r}||\underline{p}| \sin\theta \quad \text{where } \theta = \frac{\pi}{2}$$

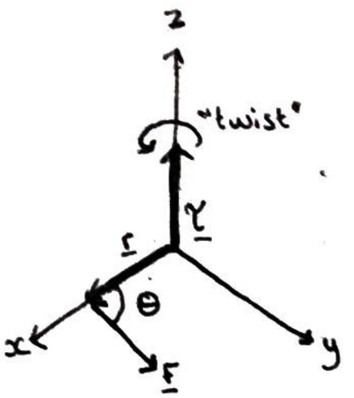
$$\therefore \underline{L} = |\underline{r}||\underline{p}| = m|\underline{v}||\underline{r}|$$

Torque

Torque describes the turning force about the origin. If the force $\underline{F}$ acts through a point $\underline{r}$ then the torque (or moment about the origin) is defined to be:

$$\boxed{\underline{\tau} = \underline{r} \times \underline{F}}$$

Note that this is cross product and thus the vector $\underline{\tau}$ will be perpendicular to both $\underline{r}$ and $\underline{F}$.



Here $\underline{\tau}$ is perpendicular to both $\underline{r}$ and $\underline{F}$. Although it may seem counterintuitive to think of the "turning force" as perpendicular to the applied force, it might make more sense if torque is thought of as a "twist" at the end of a perpendicular rod at the origin. It's like a screwdriver turning whereas $\underline{F}$ is like a spanner turning.

Newton's Second Law for Angular Momentum

For a particle of mass m at position $\underline{r}$ and velocity $\underline{v}$, the linear momentum $\underline{p} = m\underline{v}$ and the angular momentum $\underline{L} = \underline{r} \times \underline{p}$. Differentiating this with respect to time:

$$\frac{d}{dt}(\underline{r} \times \underline{p}) = \underbrace{\dot{\underline{r}} \times \underline{p}} + \underbrace{\underline{r} \times \dot{\underline{p}}}$$

This is $\underline{v} \times \underline{p} = \underline{v} \times m\underline{v} = 0$ This is $\underline{r} \times \frac{d\underline{p}}{dt}$ but $\frac{d\underline{p}}{dt} = \underline{F}$

$$\therefore \frac{d}{dt}(\underline{r} \times \underline{p}) = \underline{r} \times \underline{F} = \underline{\tau}$$

Hence:

$$\boxed{\tau = \frac{d\underline{L}}{dt}}$$

τ can be defined as the rate of change of angular momentum.

When there is no torque, the angular momentum is constant. This is the angular counterpart to Newton's Second Law:

> The total angular momentum of a system of particles subject to no net external torque is conserved.

Rotational Inertia

Angular Momentum is the rotational analogue of linear momentum. So rotational inertia, I, is the analogue of mass and the angular velocity, $\underline{\omega}$, is the analogue of linear velocity. ∴ we can write as magnitudes:

$$\boxed{|\underline{L}| = |I||\underline{\omega}|}$$

where $I = mr^2$ and $\underline{\omega} = 2\pi \underline{v}$

Therefore, if we change $|I|$, we need to change $|\underline{\omega}|$ so Angular Momentum is conserved.