Mutual information: lecture 4

COMSM0075 Information Processing and Brain

`comsm0075.github.io`

September 2020

The chain rule for entropy

$$H(X, Y) = H(X) + H(Y|X)$$

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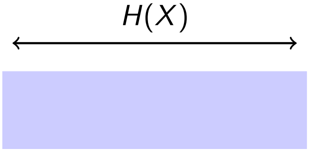

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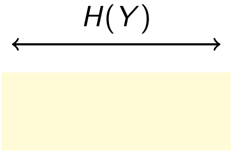
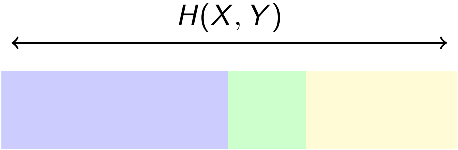


$$H(X,Y) = H(X) + H(Y|X)$$



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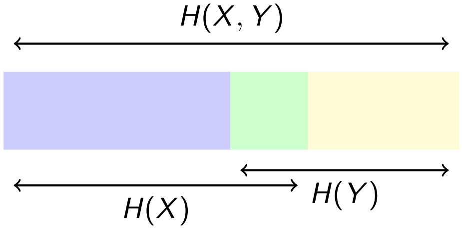

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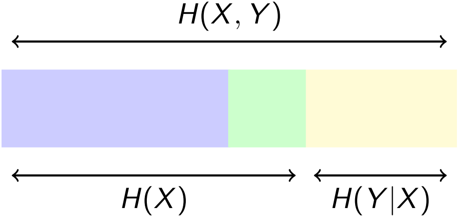
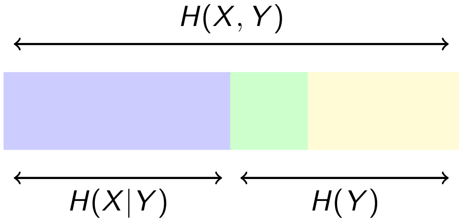
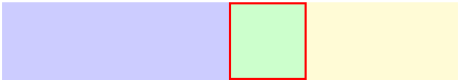


$$H(X, Y) = H(Y) + H(X|Y)$$



Mutual information



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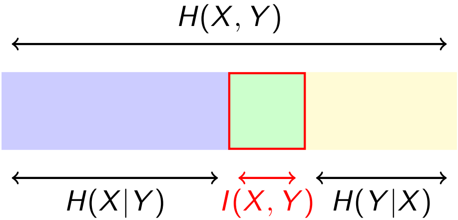

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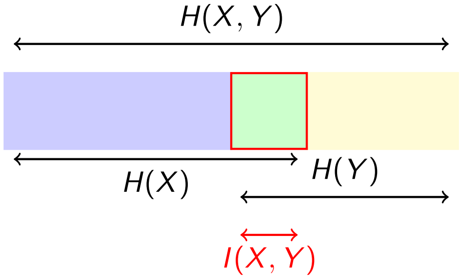
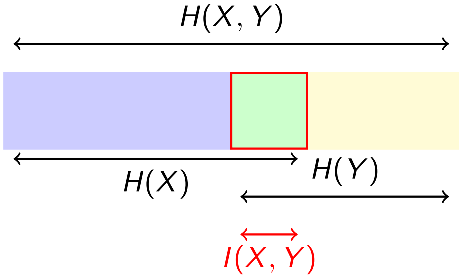
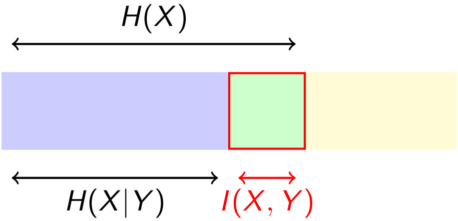


Mutual information



$$I(X,Y) = H(X) + H(Y) - H(X,Y)$$

Mutual information

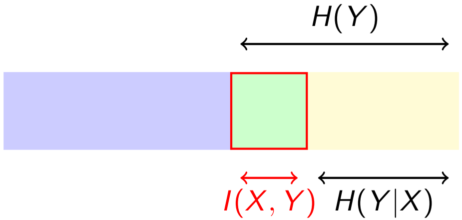


$$I(X,Y) = H(X) + H(Y) - H(X,Y)$$

then substitute $H(X,Y) = H(Y) + H(X|Y)$ to get

$$I(X,Y) = H(X) - H(X|Y)$$

Mutual information



$$I(X,Y) = H(X) + H(Y) - H(X,Y)$$

then substitute $H(X,Y) = H(Y) + H(X|Y)$ to get

$$I(X,Y) = H(Y) - H(Y|X)$$

All the mutual informations

- $I(X, Y) = H(X) + H(Y) - H(X, Y)$
- $I(X, Y) = H(X) - H(X|Y)$
- $I(X, Y) = H(Y) - H(Y|X)$
- $I(X, Y) = H(X, Y) - H(X|Y) - H(Y|X)$

Mutual information - direct formula

By substituting the formulas for $H(X)$, $H(Y)$ and $H(X, Y)$ we get

$$I(X, Y) = \sum_{i,j} p_{X,Y}(x_i, y_j) \log_2 \frac{p_{X,Y}(x_i, y_j)}{p_X(x_j) p_Y(y_j)}$$

Example

	x_0	x_1
y_0	$1/4$	$1/4$
y_1	$1/2$	0

has $H(X, Y) = 3/2$, $H(X) \approx 0.81$ and $H(Y) = 1$ so

$$I(X, Y) \approx 0.31$$

Mutual information - independent variables

$$I(X,Y)=\sum_{i,j}p_{X,Y}(x_i,y_j)\log_2\frac{p_{X,Y}(x_i,y_j)}{p_X(x_j)p_Y(y_j)}$$

but if X and Y are independent $p_{X,Y}(x_i,y_j)=p_X(x_j)p_Y(y_j)$ and hence

$$I(X,Y)=0$$

Mutual information - independent variables

In fact

$$I(X, Y) \geq 0$$

with equality if and if X and Y are independent.

Mutual information - independent variables

In fact

$$I(X, Y) \geq 0$$

with equality if and if X and Y are independent; note that this is equivalent to

$$H(X) \geq H(X|Y)$$

with equality if and if X and Y are independent; as claimed earlier.

Correlation

$$C(X,Y) = \frac{\langle (X-\mu_X)(Y-\mu_Y)\rangle}{\sigma_X \sigma_Y}$$

where μ_X is the average of X and σ_X is its standard deviation with similar notation for Y.

Correlation

Consider

	-1	0	1
1	$1/4$	0	$1/4$
0	0	$1/2$	0

then

$$C(X, Y) = 0$$

whereas $I(X, Y) = 1$.