The Kalman filter: the Bayesian Brain lecture 3

COMSM0075 Information Processing and Brain

comsm0075.github.io

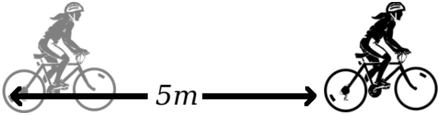
October 2020

Kalman filter

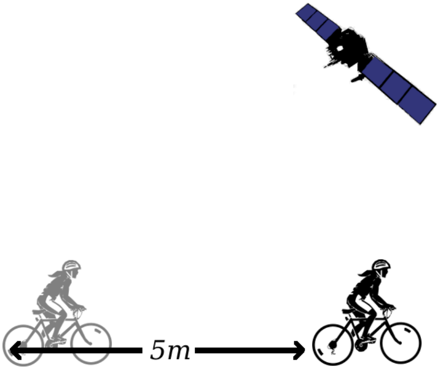
The Kalman filter uses Bayesian fusion to estimate location based on noisy measurement and on dead reckoning.

The problem

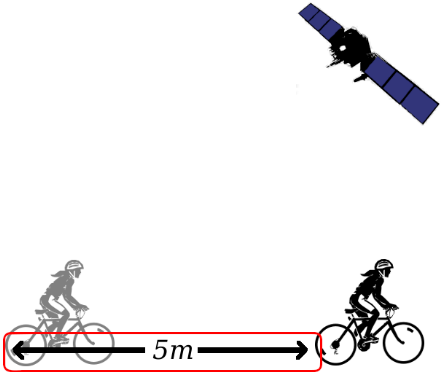
Dead reckoning



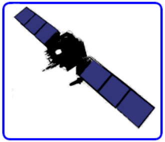
Measurement

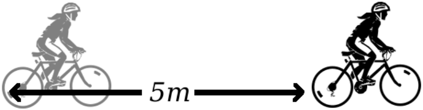


Two sources of position



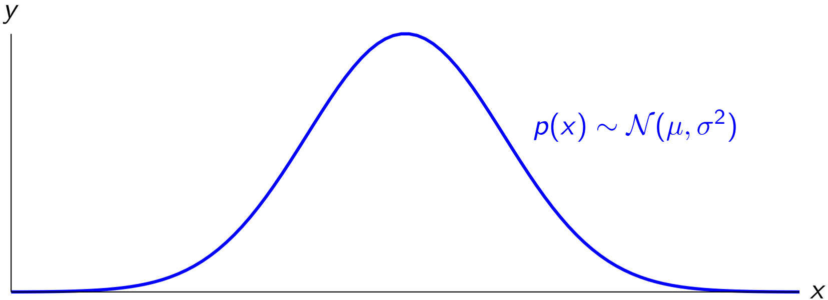
Two sources of position





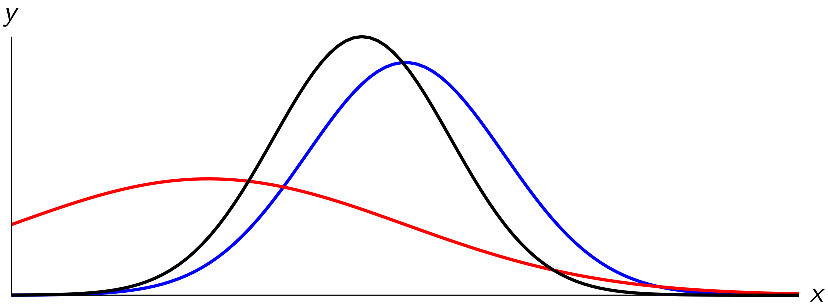
Everything is Gaußian

1) specified by the mean and variance: $\mu = 4$ and $\sigma^2 = 1$



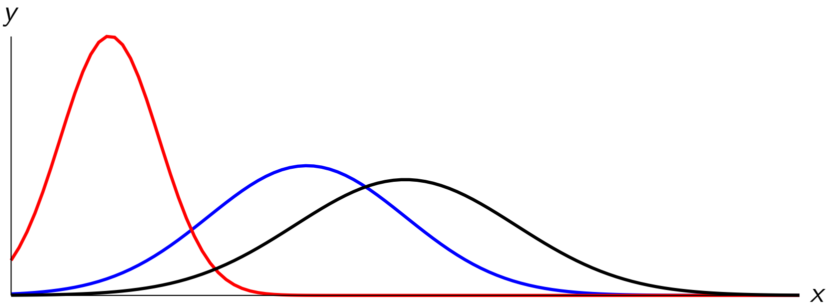
Everything is Gaußian

2) start with Gaußians and stay with Gaußians - Bayesian fusion



Everything is Gaußian

2) start with Gaußians and stay with Gaußian - $Z = X + Y$



Some equations

$$\mathsf{x} = \begin{pmatrix} s \\ v \end{pmatrix}$$

represents the position and speed, sometimes we have random variables:

$$\mathsf{X} = \begin{pmatrix} S \\ V \end{pmatrix}$$

Covariance

$$P_{ij} = \langle (X_i - \bar{x}_i)(X_j - \bar{x}_j) \rangle$$

The idea is to update both the mean and the covariance; which is equivalent for Gaußian distributions to updating the distribution.

Kalman filter

At a time t we have a belief about the position and speed: we believe they are $\bar{\mathrm{x}}$, but our estimate is uncertain, $\bar{\mathrm{x}}$ is the mean of a two-dimensional Gaußian distribution with covariance P representing our belief.

We want to update this to new values for time $t + \delta t$ using two noisy pieces of information: dead reckoning and direct measurement. The new belief will have mean $\bar{\mathrm{x}}_n$ and covariance P_n.

True evolution

The position changes according to

$$s \to s_a = s + v\delta t$$

We are assuming the speed is constant; it is easy to include intentional changes of speed by including a **control vector**; we don't do that here.

$$v \to v$$

Dead reckoning

$$\mathsf{X}_d = F\mathsf{X} + \mathsf{W}$$

where F is the motion matrix

$$F = \begin{pmatrix} 1 & \delta t \\ 0 & 1 \end{pmatrix}$$

and W is zero mean Gaußian noise with covariance matrix Q .

Dead reckoning

$$\mathsf{X}_d = F\mathsf{X} + \mathsf{W}$$

Now take the average:

$$\bar{\mathsf{x}}_d = F\bar{\mathsf{x}}$$

Change in covariance

Let's motivate this with a one-dimensional example; say we have a variable

$$Y \sim \mathcal{N}(\bar{y}, p)$$

and say the true update has the form

$$y_a = fy$$

where f is a constant, so

$$Y_d = fY + U$$

and $U \sim \mathcal{N}(0, q)$.

Change in covariance

$$Y_d = fY + U$$

has y_d drawn from the sum of two Gaußians which is also Gaußian with mean and variance that can be calculated directly:

$$\bar{y}_d = \langle Y_d \rangle = \langle fY + U \rangle = f \langle Y \rangle = f \bar{y}$$

and

$$\begin{aligned} \langle (Y_d - \bar{y}_d)^2 \rangle &= \langle (fY + U)^2 \rangle - \bar{Y}_d^2 = f^2 \langle Y^2 \rangle + \langle U^2 \rangle - f^2 \bar{y}^2 \\ &= f^2 (p + \bar{y}^2) + q - f^2 \bar{y} = f^2 p + q \end{aligned}$$

Change in covariance

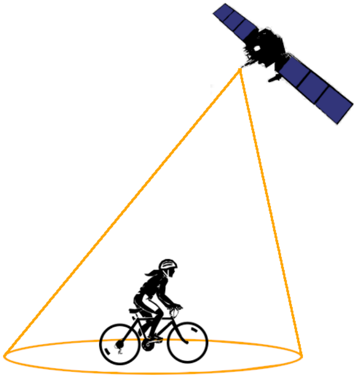
The one-dimensional case:

$$p_d = f^2 p + q$$

In our two-dimensional case simple algebra gives

$$P_d = FPF^T + Q$$

Measurement



Measurement

$$X_s \sim \mathcal{N}(x_a, R)$$

where R is the covariance in the noise in our sensor. More complex models of the sensor noise are often considered, but these are a straightforward extension of what we do here.

Bayesian fusion

We want to put these together.

$$p(\mathrm{x}_a|\mathrm{x}_d, \mathrm{x}_s)$$

From the Bayes rule:

$$p(\mathrm{x}_a|\mathrm{x}_d, \mathrm{x}_s) \propto p(\mathrm{x}_d, \mathrm{x}_s|\mathrm{x}_a) = p(\mathrm{x}_d|\mathrm{x}_a)p(\mathrm{x}_s|\mathrm{x}_a)$$

One-dimensional example

ys where the xs are, little letters for variances, the equivalent of the covariances before.

$$p(y_a|y_d, y_s) \propto p(y_d|y_a)p(y_s|y_a)$$

where $p(y_d|y_a) \sim \mathcal{N}(y_a, p_d)$ and $p(y_s|y_a) \sim \mathcal{N}(y_a, r)$.

This is just Bayesian fusion

$$\frac{1}{p_n} = \frac{1}{p_d} + \frac{1}{r}$$

and this gives the new mean

$$\bar{y}_n = \frac{p_n}{p_d} y_d + \frac{p_n}{r} y_s$$

One-dimensional example

$$\frac{1}{p_n} = \frac{1}{p_d} + \frac{1}{r}$$

with new mean

$$\bar{y}_n = \frac{p_n}{p_d} y_d + \frac{p_n}{r} y_s$$

We can rewrite this:

$$\frac{p_n}{p_d} = 1 - \frac{p_n}{r}$$

so

$$\bar{y}_n = y_d + k(y_s - y_d)$$

where

$$k = \frac{p_n}{r}$$

Thus, the new estimate is the dead reckoning estimate along with a correction coming from the sensor.

Kalman gain

$$\bar{y}_n = y_d + k(y_s - y_d)$$

where

$$k = \frac{p_n}{r} = \frac{p_d}{p_d + r}$$

and

$$\frac{1}{p_n} = \frac{1}{p_d} + \frac{1}{r}$$

and after a bit of algebra

$$p_n = \frac{r p_d}{p_d + r} = \frac{p_d(r + p_d)}{p_d + r} - \frac{p_d^2}{p_d + r} = (1 - k) p_d$$

Back to two-dimensions

$$S = P_d + R$$

and

$$K = P_d S^{-1}$$

This factor, called the **Kalman gain** is clearly the analogue of k above.

$$\bar{\mathrm{x}}_n = \mathrm{x}_d + K(\mathrm{x}_s - \mathrm{x}_d)$$

and

$$P_n = (1 - K)P_d$$

Kalman filter

$$\begin{tikzcd}
\bar{\mathrm{x}}, P \arrow[rr, "\text{dead reckoning}"] & & \mathrm{x}_{\mathrm{d}}, P_d \arrow[r] & \bar{\mathrm{x}}_{\mathrm{n}}, P_n \arrow[lll, dashed, bend right] \\
& \text{measurement} \arrow[r] & \mathrm{x}_{\mathrm{s}}, R \arrow[ur] &
\end{tikzcd}$$

The Kalman filter and the brain

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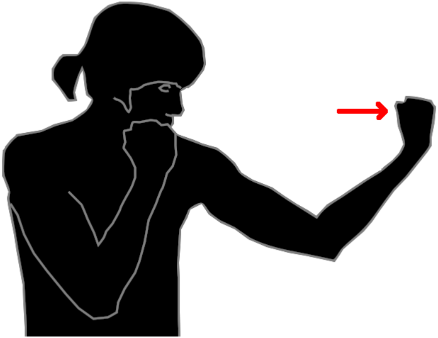
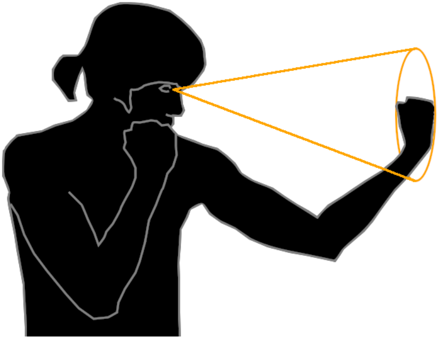


The Kalman filter and the brain



The Kalman filter and the brain