Bayesian fusion: the Bayesian Brain lecture 2

COMSM0075 Information Processing and Brain

`comsm0075.github.io`

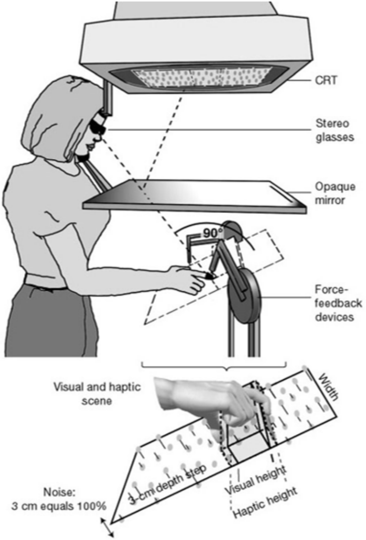
October 2020

Bayesian fusion

The main topic here is Bayesian fusion *but we will introduce it by talking about a nice experiment by Ernst and Banks which demonstrates Bayesian inference in human perception.*

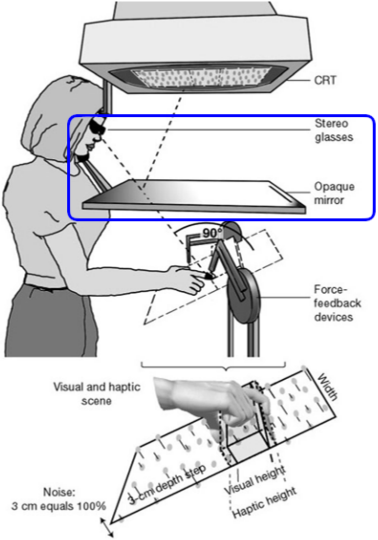
MO Ernst and MS Banks (2002) Humans integrate visual and haptic information in a statistically optimal fashion, 415:429 Nature

Ernst and Banks



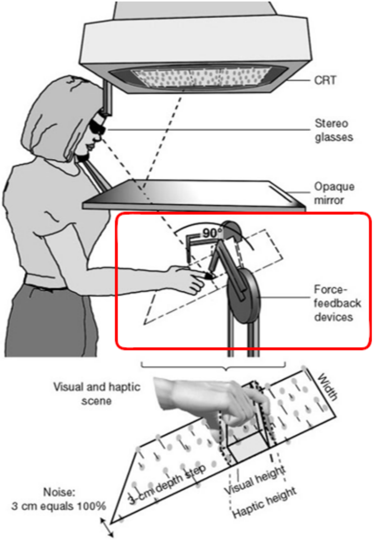
Pictures from Ernst and Banks

Ernst and Banks - vision



Pictures from Ernst and Banks

Ernst and Banks - touch



Pictures from Ernst and Banks

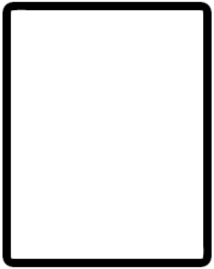
Ernst and Banks

- x is the **true** height of the box.
- v is the **visual** estimate of the height of the box.
- x is the **haptic** estimate of the height of the box.

haptic: of or relating to the sense of touch

Estimates are noisy

$$p(v|x)$$



Estimates are noisy

$$p(v|x)$$

Markov chain

$$V \to X \to H$$

or

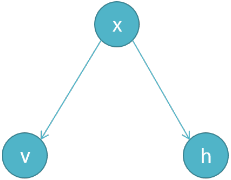
$$p(v, h|x) = p(v|x)p(h|x)$$

Markov chain - also called a directed acyclic graph

$$V \rightarrow X \rightarrow H$$

$$V \leftarrow X \leftarrow H$$

or



Markov chain

$$p(v, h|x) = p(v|x)p(h|x)$$

so

$$p(v, h, x) = p(v, h|x)p(x) = p(v|x)p(h|x)p(x)$$

Posterior judgment

$$p(x|v,h) = \frac{p(v,h|x)p(x)}{p(v,h)} = \frac{p(v|x)p(h|x)p(x)}{p(v,h)}$$

MAP estimate

Ultimately the participant is asked to estimate the height; we assume that they use $p(x|v, h)$ and respond with the value of x which has the maximum value of $p(x|v, h)$:

$$\hat{x} = \arg\max_{x} p(x|v, h)$$

This is the **maximum a posteriori** estimate or **MAP** estimate.

MAP estimate

$$p(x|v,h) = \frac{p(v,h|x)p(x)}{p(v,h)} = \frac{p(v|x)p(h|x)p(x)}{p(v,h)}$$

It is assumed that the prior is uniform, so $p(x)$ is constant over some range of possible x, so if x is in that range

$$p(x|v,h) \propto \frac{p(v|x)p(h|x)}{p(v,h)}$$

Bayesian fusion

Let's assume the noise modelled by V and H are Gaußian so

$$\begin{aligned}(V|X=x) &\sim \mathcal{N}(x,\sigma_v^2)\\ p(v|x) &= \frac{1}{\sqrt{2\pi\sigma_v^2}}e^{-\frac{(v-x)^2}{2\sigma_v^2}}\end{aligned}$$

and

$$\begin{aligned}(H|X=x) &\sim \mathcal{N}(x,\sigma_h^2)\\ p(h|x) &= \frac{1}{\sqrt{2\pi\sigma_h^2}}e^{-\frac{(h-x)^2}{2\sigma_h^2}}\end{aligned}$$

It is also assumed that the participant has an estimate of the size of the noise, that is, the participant knows σ_v^2 and σ_h^2.

Bayesian fusion

So, ignoring any normalization factor

$$p(x|v,h) \propto p(v|x)p(h|x)$$

so we need to multiply the two Gaussians:

$$p(x|v,h) \propto e^{-\frac{(h-x)^2}{2\sigma_h^2}} e^{-\frac{(v-x)^2}{2\sigma_v^2}}$$

Looking at the exponent we get

$$-\frac{(h-x)^2}{2\sigma_h^2} - \frac{(v-x)^2}{2\sigma_v^2} = -\left(\frac{1}{2\sigma_h^2} + \frac{1}{2\sigma_v^2}\right)x^2 + \left(\frac{h}{\sigma_h^2} + \frac{v}{\sigma_v^2}\right)x + A$$

where A is other stuff with no xs.

Bayesian fusion

$$-\frac{(h-x)^2}{2\sigma_h^2} - \frac{(v-x)^2}{2\sigma_v^2} = -\left(\frac{1}{2\sigma_h^2} + \frac{1}{2\sigma_v^2}\right)x^2 + \left(\frac{h}{\sigma_h^2} + \frac{v}{\sigma_v^2}\right)x + A$$

In short

$$p(x|v,h) \sim \mathcal{N}(\bar{x}, \sigma)$$

where

$$\frac{1}{\sigma^2} = \frac{1}{\sigma_v^2} + \frac{1}{\sigma_h^2}$$

and

$$\bar{x} = \frac{\sigma^2}{\sigma_v^2}v + \frac{\sigma^2}{\sigma_h^2}h$$

Bayesian fusion

$$\frac{1}{\sigma^2} = \frac{1}{\sigma_v^2} + \frac{1}{\sigma_h^2}$$

Let

$$\lambda = \frac{\sigma^2}{\sigma_h^2}$$

then

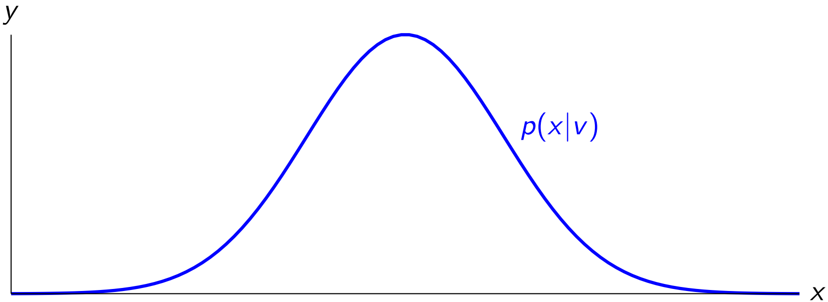
$$1 - \lambda = \frac{\sigma^2}{\sigma_v^2}$$

so

$$\bar{x} = (1 - \lambda)v + \lambda h$$

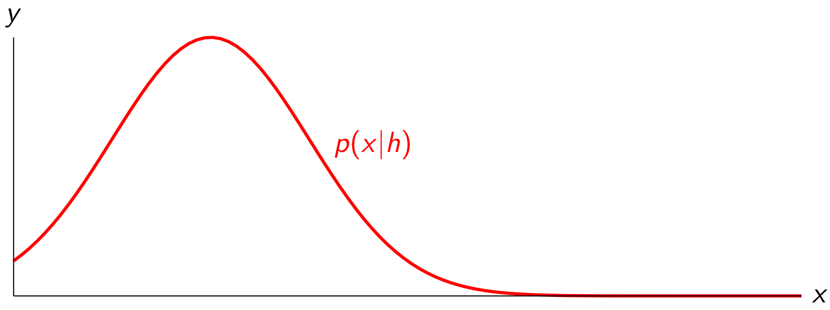
Bayesian fusion

$v = 4$ and $\sigma_v = 1$



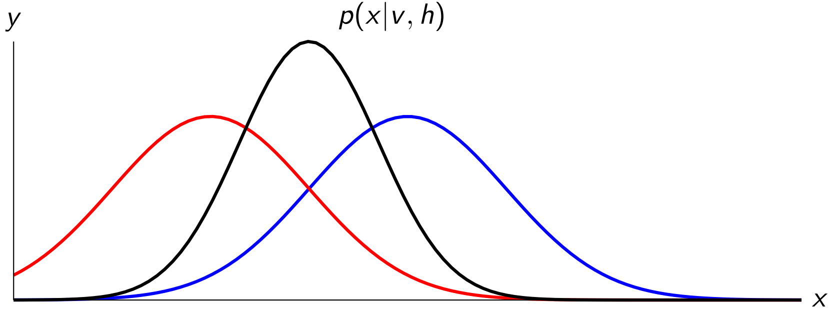
Bayesian fusion

$h = 2$ and $\sigma_h = 1$



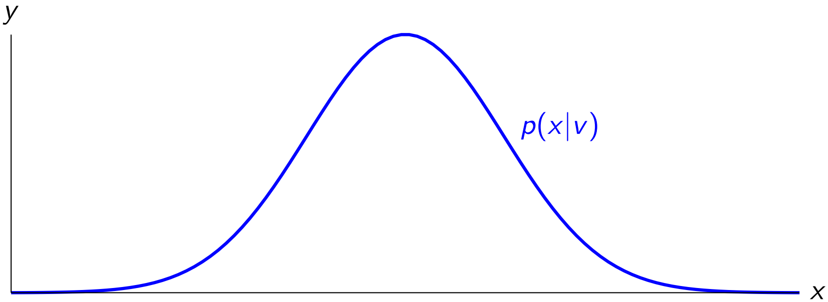
Bayesian fusion

$\bar{x} = 3$ and $\sigma = 0.71$



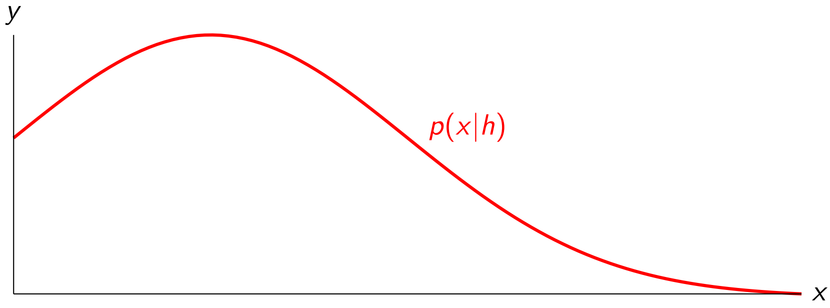
Bayesian fusion

$v = 4$ and $\sigma_v = 1$



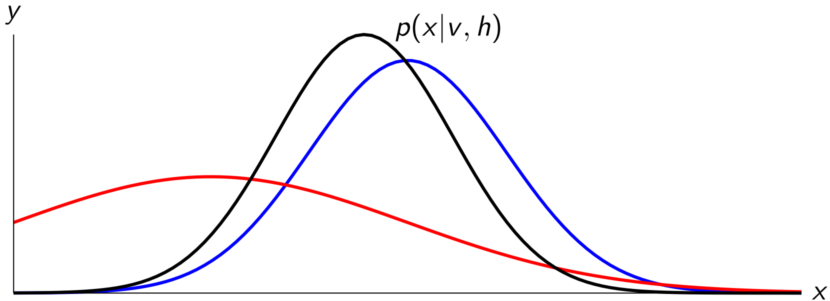
Bayesian fusion

$h = 2$ and $\sigma_h = 2$



Bayesian fusion

$\bar{x} = 3.56$ and $\sigma = 0.9$



Ernst and Banks

Multiple trials in which the participants are asked to pick which of two blocks are larger. They use this to estimate the participant's estimate of the height ξ and fit this to

$$\xi = (1-\mu)v + \mu h$$

and they then compare μ to λ:

$$\bar{x} = (1-\lambda)v + \lambda h$$

Ernst and Banks

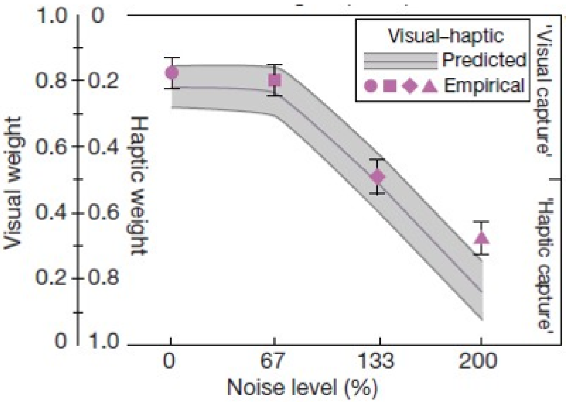


Figure from Ernst and Banks