The mutual information: information theory lecture 9

COMSM0075 Information Processing and Brain

`comsm0075.github.io`

October 2020

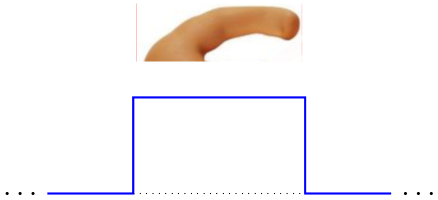
A probability density is a density

The probability density isn't any old function. It is a density.

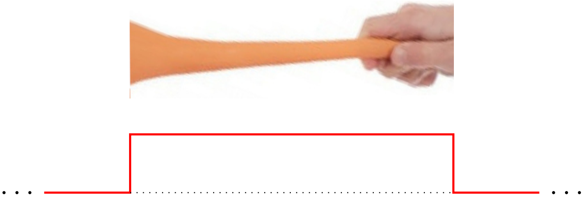
Density

Density

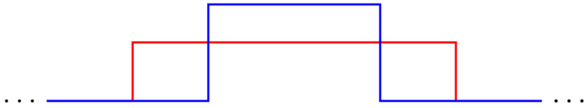
Density



Density



Density



Change of variable

$$P(x \in [x_0, x_1)) = \int_{x_0}^{x_1} p_X(x) dx$$

Now what happens if we do a change of variable to $y(x)$.

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Now what happens if we do a change of variable to $y(x)$. For simplicity, assume $y(x)$ is strictly monotonic so we can invert to get $x(y)$.

Change of variable

Remember how to change variables in an integral:

$$dx = \left| \frac{dx}{dy} \right| dy$$

so if $y_0 = y(x_0)$ and $y_1 = y(x_1)$ we have

$$P(y \in [y_0, y_1)) = \int_{y_0}^{y_1} p_X(x(y)) \left| \frac{dx}{dy} \right| dy$$

Change of variable - Jacobian

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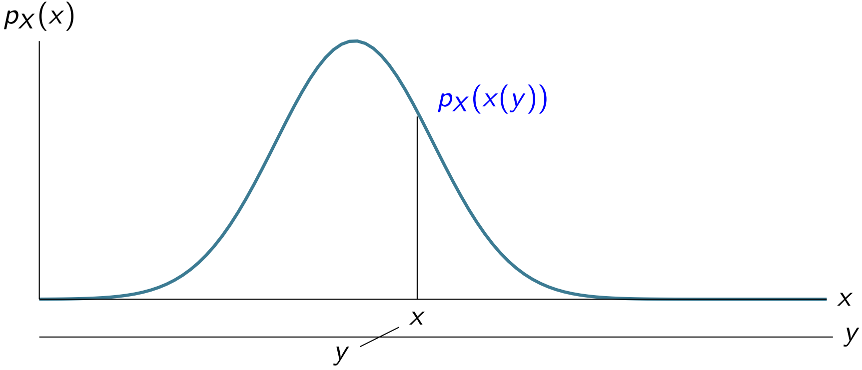

The y probability density

$$P(y \in [y_0, y_1)) = \int_{y_0}^{y_1} p_Y(y) dy$$

The probability shouldn't change

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hence

$$p_Y(y) = \frac{p_X(x(y))}{|dy/dx|}$$

This behaviour is the definition of a density.

The entropy is not invariant under a change of variable!

$$h(Y) = h(X) + \int p_X(x) \log_2 \left| \frac{dy}{dx} \right| dx$$

Scaling

Using $y = ax$ in this formula

$$h(aX) = h(X) + \log |a|$$

The mutual information

$$I(X, Y) = h(X) + h(Y) - h(X, Y)$$

or

$$I(X, Y) = \int p_{X,Y}(x, y) \log_2 \frac{p_{X,Y}(x, y)}{p_X(x) p_Y(y)} dxdy$$

The mutual information has nice properties

$$I(X, Y) = h(X) + h(Y) - h(X, Y)$$

is invariant under a change of variable; roughly speaking the Jacobian factors cancel!

The mutual information is the same for mutual information

$$I(X^{\delta_x}, Y^{\delta_y}) \to I(X, Y)$$

as δ_x and δ_y approach zero.

The mutual information is non-negative

$$I(X, Y) \geq 0$$

Channel capacity theory