

Information Theory lecture 2

COMSM0075 Information Processing and Brain

`comsm0075.github.io`

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Shannon's entropy

For a finite discrete distribution with random variable X , possible outcomes $\{x_1, x_2, \dots, x_n\} \in \mathcal{X}$ and a probability mass function p_X giving probabilities $p_X(x_i)$, the entropy is

$$H(X) = - \sum_{x_i \in \mathcal{X}} p_X(x_i) \log_2 p_X(x_i)$$

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In this definition $p \log_2 p = 0$ when $p = 0$; this makes sense since

$$\lim_{p \rightarrow 0} p \log_2 p = 0$$

Shannon's entropy

For a finite discrete distribution with random variable X , possible outcomes $\{x_1, x_2, \dots, x_n\} \in \mathcal{X}$ and a probability mass function p_X giving probabilities $p_X(x_i)$, the entropy is

$$H(X) = - \sum_{x_i \in \mathcal{X}} p_X(x_i) \log_2 p_X(x_i) = -E_X[\log_2 p_X(X)] = -\langle \log_2 p_X(X) \rangle$$

Shannon's entropy

Shannon's entropy has lots of nice properties, being easy to estimate isn't one.

works on any sample space

The mean of a distribution

$$\langle X \rangle = \sum_{x_i \in \mathcal{X}} p_X(x_i) x_i$$

only works if the x_i live in a vector space.

works on any sample space

Not all sample spaces are vector spaces, trying to work out the average fruit bought in a grocers doesn't make sense because

$$0.25 \times \text{apple} + 0.125 \times \text{banana} + 0.1 \times \text{orange} \dots$$

is nonsense.

works on any sample space

Shannon's entropy is defined on any probability space.

it's always positive

$$H(X) = - \sum_{x_i \in \mathcal{X}} p_X(x_i) \log_2 p_X(x_i)$$

and since $0 \leq p_X(x_i) \leq 1$

$$H(X) \geq 0$$

it's zero if the distribution isn't random

If $p_X(x_i)$ look like $\{0, 0, \dots, 1, \dots, 0\}$ then

$$H(X) = 0$$

uniform distribution

If the distribution is uniform

$$p_X(x_i) = \frac{1}{n}$$

for all x_i where

$$n = \#\mathcal{X}$$

then, since $-\log_2(1/n) = \log_2 n$

$$H(X) = \log_2 n$$

bounds

In fact, not proved here but not difficult to prove,

$$0 \leq H(X) \leq \log_2 n$$

with $H(X)$ only if one probability is one and the rest zero and $H(X) = \log_2 n$ only for the uniform distribution.

bounds

$$0 \leq H(X) \leq \log_2 n$$

That what we want!

$$n = 2$$

Two outcomes, a and b with $p(a) = p$ and $p(b) = 1 - p$ then

$$H = -p \log_2 p - (1 - p) \log_2 (1 - p)$$

$$n = 2$$

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$$H = -p \log_2 p - (1 - p) \log_2 (1 - p)$$

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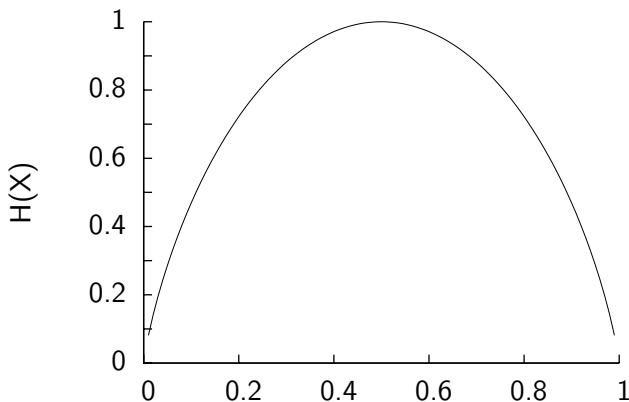
Two outcomes, a and b with $p(a) = p$ and $p(b) = 1 - p$ then

$$H = -p \log_2 p - (1 - p) \log_2 (1 - p)$$

$$n = 2$$

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$$H = -p \log_2 p - (1 - p) \log_2 (1 - p)$$



source coding

The main reason to believe that Shannon's entropy is a good quantity for calculating entropy is its relationship with what is called source coding.

source coding

Consider storing a long sequence of the letters A, B, C and D as binary.

AABACBDA...

a dictionary might look like

A	B	C	D
00	01	10	11

a dictionary might look like

A	B	C	D
00	01	10	11

AABACD...

becomes

000001001011...

a dictionary might look like

A	B	C	D
00	01	10	11

*A**ABACD*...

becomes

000001001011...

a dictionary might look like

A	B	C	D
00	01	10	11

AABACD...

becomes

000001001011...

a dictionary might look like

A	B	C	D
00	01	10	11

*AA**B**A**C**D*...

becomes

*0000**0**1001011*...

a dictionary might look like

A	B	C	D
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AABACD...

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AABACD...

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average bits per letter

$$L = 2$$

say we know the letter frequencies

Now, say we also knew that

A	B	C	D
0.5	0.25	0.125	0.125

So in the message that will be encoded, **A** occurs half the time, **B** a quarter the time and **C** and **D** an eighth of the time.

say we know the letter frequencies

Can we use this information to make L smaller?

say we know the letter frequencies

Can we find an shorter encoding for the most frequent letter: A?

here is a better code

A	B	C	D
0	10	110	111

here is a better code - prefix free code

A	B	C	D
0	10	110	111

here is a better code - prefix free code

A	B	C	D
0	10	110	111

here is a better code

A	B	C	D
0	10	110	111

AABACD...

becomes

00100110111...

here is a better code

A	B	C	D
0	10	110	111

*A**ABACD*...

becomes

00100110111...

here is a better code

A	B	C	D
0	10	110	111

ABACD...

becomes

00100110111...

here is a better code

A	B	C	D
0	10	110	111

AABACD...

becomes

00100110111...

here is a better code

A	B	C	D
0	10	110	111

*AAB**A**CD*...

becomes

*00100**110111*...

here is a better code

A	B	C	D
0	10	110	111

AABACD...

becomes

00100110111...

here is a better code

A	B	C	D
0	10	110	111

AABACD...

becomes

00100110111...

this code is shorter

$$L = 0.5 \times 1 + 0.25 \times 2 + 0.125 \times 3 + 0.125 \times 3 = 1.75$$

this code is shorter

$$L = 0.5 \times 1 + 0.25 \times 2 + 0.125 \times 3 + 0.125 \times 3 = 1.75$$

where 0.5 is the frequency of A and 1 is the length of the code.

this code is shorter

$$L = 0.5 \times 1 + 0.25 \times 2 + 0.125 \times 3 + 0.125 \times 3 = 1.75 < 2$$

Shannon's entropy

$$H(X) = -0.5 \log_2(0.5) - 0.25 \log_2(0.25) - 0.250 \log_2(0.125) = 1.75$$

The source coding theorem

Roughly, for the most efficient code

$$H(X) \leq L < H(X) + 1$$

The source coding theorem

$$H(X) \leq L < H(X) + 1$$

The source coding theorem shows that the entropy $H(X)$ is a lower bound on the average length of a message using the most efficient code.