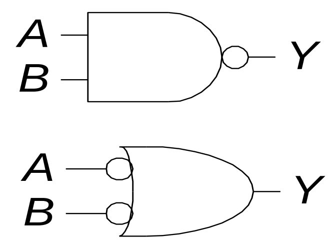
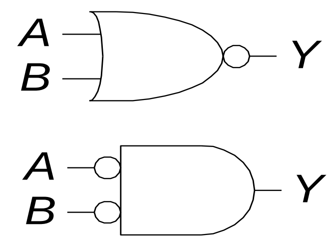
ETE 212/EEE 211
Introduction to Digital Electronics

Boolean Expressions and Canonical Forms

DeMorgan’s Theorem

- $Y = \overline{AB} = \overline{A} + \overline{B}$





- $Y = \overline{A + B} = \overline{A} \bullet \overline{B}$

Complementing Functions

- Use DeMorgan's Theorem:
  1. Interchange AND and OR operators
  2. Complement each constant and literal
- Example: Complement F = $\bar{x}y\bar{z} + x\bar{y}\bar{z}$

$\bar{F} = (x + \bar{y} + z)(\bar{x} + y + z)$

- Example: Complement $G = (\bar{a} + bc)\bar{d} + e$

$\bar{G} = (a(\bar{b} + \bar{c}) + d)\,\bar{e}$

Expression Simplification

- An application of Boolean algebra
- Simplify to contain the smallest number of <u>literals</u> (variables that may or may not be complemented)

$$\mathbf{AB + \overline{A}CD + \overline{A}BD + \overline{A}C\overline{D} + ABCD}$$

$= AB + ABCD + \overline{A}\,C\,D + \overline{A}\,C\,\overline{D} + \overline{A}\,B\,D$

$= AB + AB(CD) + A\,C\,(D + D) + A\,B\,D$

$= AB + \overline{A}\,C + \overline{A}\,B\,D = B(A + \overline{A}D) + \overline{A}C$

$= B\,(A + D) + \overline{A}\,C$ (has only 5 literals)

Canonical Forms

- Minterms and Maxterms
- Sum-of-Minterm (SOM) Canonical Form
- Product-of-Maxterm (POM) Canonical Form
- Representation of Complements of Functions
- Conversions between Representations

Minterms

- <u>Minterms</u> are AND terms with every variable present in either true or complemented form.
- Given that each binary variable may appear normal (e.g., x) or complemented (e.g., $\overline{\mathbf{x}}$), there are 2^n minterms for *n* variables.
- <u>Example:</u> Two variables (X and Y) produce 2 x 2 = 4 combinations:

  $\mathbf{XY}$ (both normal)

  $\mathbf{X\overline{Y}}$ (X normal, Y complemented)

  $\mathbf{\overline{X}Y}$ (X complemented, Y normal)

  $\mathbf{\overline{X}\,\overline{Y}}$ (both complemented)
- Thus there are <u>four minterms</u> of two variables.

Maxterms

- Maxterms are OR terms with every variable in true or complemented form.
- Given that each binary variable may appear normal (e.g., x) or complemented (e.g., $\overline{x}$), there are 2^n maxterms for n variables.
- Example: Two variables (X and Y) produce 2 x 2 = 4 combinations:

$\mathbf{X + Y}$ (both normal)

$\mathbf{X + \overline{Y}}$ (x normal, y complemented)

$\mathbf{\overline{X} + Y}$ (x complemented, y normal)

$\mathbf{\overline{X} + \overline{Y}}$ (both complemented)

Minterms & Maxterms for 2 variables

- Two variable minterms and maxterms.

x	y	Index	Minterm	Maxterm
0	0	0	$m_0 = \overline{x}\,\overline{y}$	$M_0 = x + y$
0	1	1	$m_1 = \overline{x}\,y$	$M_1 = x + \overline{y}$
1	0	2	$m_2 = x\,\overline{y}$	$M_2 = \overline{x} + y$
1	1	3	$m_3 = x\,y$	$M_3 = \overline{x} + \overline{y}$

- The minterm m_i should evaluate to 1 for each combination of x and y.
- The maxterm is the complement of the minterm

Minterms & Maxterms for 3 variables

x	y	z	Index	Minterm	Maxterm
0	0	0	0	$m0 = \bar{x}\,\bar{y}\,\bar{z}$	$M0 = x + y + z$
0	0	1	1	$m1 = \bar{x}\,\bar{y}\,z$	$M1 = x + y + \bar{z}$
0	1	0	2	$m2 = \bar{x}\,y\,\bar{z}$	$M2 = x + \bar{y} + z$
0	1	1	3	$m3 = \bar{x}\,y\,z$	$M3 = x + \bar{y} + \bar{z}$
1	0	0	4	$m4 = x\,\bar{y}\,\bar{z}$	$M4 = \bar{x} + y + z$
1	0	1	5	$m5 = x\,\bar{y}\,z$	$M5 = \bar{x} + y + \bar{z}$
1	1	0	6	$m6 = x\,y\,\bar{z}$	$M6 = \bar{x} + \bar{y} + z$
1	1	1	7	$m7 = x\,y\,z$	$M7 = \bar{x} + \bar{y} + \bar{z}$

Maxterm M_i is the complement of minterm m_i

$M_i = \overline{m_i}$ and $m_i = \overline{M_i}$

Purpose of the Index

- Minterms and Maxterms are designated with an index
- The index number corresponds to a binary pattern
- The <u>index</u> for the minterm or maxterm, expressed as a binary number, is used to determine whether the variable is shown in the true or complemented form
- For Minterms:
  - ‘1’ means the variable is “Not Complemented” and
  - ‘0’ means the variable is “Complemented”.
- For Maxterms:
  - ‘0’ means the variable is “Not Complemented” and
  - ‘1’ means the variable is “Complemented”.

Standard Order

- All variables should be present in a minterm or maxterm and should be listed in the <u>same order</u> (usually alphabetically)
- Example: For variables a, b, c:
  - Maxterms $(a + b + \bar{c})$, $(\bar{a} + b + \bar{c})$ are in standard order
  - However, $(b + \bar{a} + c)$ is NOT in standard order

    $(\bar{a} + c)$ does NOT contain all variables
  - Minterms $(a\ b\ \bar{c})$ and $(\bar{a}\ b\ \bar{c})$ are in standard order
  - However, $(b\ a\ \bar{c})$ is not in standard order

    $(\bar{a}\ c)$ does not contain all variables

Sum-Of-Minterm (SOM)

- Sum-Of-Minterm (SOM) canonical form:
  Sum of minterms of entries that evaluate to '**1**'

x	y	z	F	Minterm
0	0	0	0	
0	0	1	**1**	$m_1 = \bar{x}\,\bar{y}\,z$
0	1	0	0	
0	1	1	0	
1	0	0	0	
1	0	1	0	
1	1	0	**1**	$m_6 = x\,y\,\bar{z}$
1	1	1	**1**	$m_7 = x\,y\,z$

Focus on the '**1**' entries

$$F = m_1 + m_6 + m_7 = \sum(1, 6, 7) = \bar{x}\,\bar{y}\,z + x\,y\,\bar{z} + x\,y\,z$$

Sum-Of-Minterm Examples

- $F(a, b, c, d) = \sum(2, 3, 6, 10, 11)$
- $F(a, b, c, d) = \mathrm{m}_2 + \mathrm{m}_3 + \mathrm{m}_6 + \mathrm{m}_{10} + \mathrm{m}_{11}$

  $\boldsymbol{\bar{a}\,\bar{b}\,c\,\bar{d} + \bar{a}\,\bar{b}\,c\,d + \bar{a}\,b\,c\,\bar{d} + a\,\bar{b}\,c\,\bar{d} + a\,\bar{b}\,c\,d}$

- $G(a, b, c, d) = \sum(0, 1, 12, 15)$
- $G(a, b, c, d) = \mathrm{m}_0 + \mathrm{m}_1 + \mathrm{m}_{12} + \mathrm{m}_{15}$

  $\boldsymbol{\bar{a}\,\bar{b}\,\bar{c}\,\bar{d} + \bar{a}\,\bar{b}\,\bar{c}\,d + a\,b\,\bar{c}\,\bar{d} + a\,b\,c\,d}$

Product-Of-Maxterm (POM)

- Product-Of-Maxterm (POM) canonical form:
  Product of maxterms of entries that evaluate to '**0**'

x	y	z	F	Maxterm
0	0	0	1	
0	0	1	1	
0	1	0	**0**	$M_2 = (x + \overline{y} + z)$
0	1	1	1	
1	0	0	**0**	$M_4 = (\overline{x} + y + z)$
1	0	1	1	
1	1	0	**0**	$M_6 = (\overline{x} + \overline{y} + z)$
1	1	1	1	

Focus on the '**0**' entries

$$F = M_2 \cdot M_4 \cdot M_6 = \prod (2, 4, 6) = (x+\overline{y}+z)\ (\overline{x}+y+z)\ (\overline{x}+\overline{y}+z)$$

Product-Of-Maxterm Examples

- $F(a, b, c, d) = \prod(1, 3, 6, 11)$
- $F(a, b, c, d) = \mathrm{M}_1 \cdot \mathrm{M}_3 \cdot \mathrm{M}_6 \cdot \mathrm{M}_{11}$

$$(a+b+c+\overline{d})(a+b+\overline{c}+\overline{d})(a+\overline{b}+\overline{c}+d)(\overline{a}+b+\overline{c}+\overline{d})$$

- $G(a, b, c, d) = \prod(0, 4, 12, 15)$
- $G(a, b, c, d) = \mathrm{M}_0 \cdot \mathrm{M}_4 \cdot \mathrm{M}_{12} \cdot \mathrm{M}_{15}$

$$(a+b+c+d)(a+\overline{b}+c+d)(\overline{a}+\overline{b}+c+d)(\overline{a}+\overline{b}+\overline{c}+\overline{d})$$

Observations

- We can implement any function by "ORing" the minterms corresponding to the ‘**1**’ entries in the function table. A minterm evaluates to ‘**1**’ for its corresponding entry.
- We can implement any function by "ANDing" the maxterms corresponding to ‘**0**’ entries in the function table. A maxterm evaluates to ‘**0**’ for its corresponding entry.
- The same Boolean function can be expressed in two canonical ways: Sum-of-Minterms (SOM) and Product-of-Maxterms (POM).
- If a Boolean function has fewer ‘**1**’ entries then the SOM canonical form will contain fewer literals than POM. However, if it has fewer ‘**0**’ entries then the POM form will have fewer literals than SOM.

Converting to Sum-of-Minterms Form

- A function that is not in the Sum-of-Minterms form can be converted to that form by means of a truth table
- Consider $F = \overline{y} + \overline{x}\,\overline{z}$

x	y	z	F	Minterm
0	0	0	**1**	$m_0 = \overline{x}\,\overline{y}\,\overline{z}$
0	0	1	**1**	$m_1 = \overline{x}\,\overline{y}\,z$
0	1	0	**1**	$m_2 = \overline{x}\,y\,\overline{z}$
0	1	1	0	
1	0	0	**1**	$m_4 = x\,\overline{y}\,\overline{z}$
1	0	1	**1**	$m_5 = x\,\overline{y}\,z$
1	1	0	0	
1	1	1	0	

$$F = \sum(0, 1, 2, 4, 5) =$$

$$m_0 + m_1 + m_2 + m_4 + m_5 =$$

$$\overline{x}\,\overline{y}\,\overline{z} + \overline{x}\,\overline{y}\,z + \overline{x}\,y\,\overline{z} +$$

$$x\,\overline{y}\,\overline{z} + x\,\overline{y}\,z$$

Converting to Product-of-Maxterms Form

- A function that is not in the Product-of-Minterms form can be converted to that form by means of a truth table
- Consider again: $F = \overline{y} + \overline{x}\,\overline{z}$

x	y	z	F	Maxterm
0	0	0	1	
0	0	1	1	
0	1	0	1	
0	1	1	**0**	$M_3 = (x+\overline{y}+\overline{z})$
1	0	0	1	
1	0	1	1	
1	1	0	**0**	$M_6 = (\overline{x}+\overline{y}+z)$
1	1	1	**0**	$M_7 = (\overline{x}+\overline{y}+\overline{z})$

$$F = \prod(3, 6, 7) =$$

$$M_3 \cdot M_6 \cdot M_7 =$$

$$(x+\overline{y}+\overline{z})\,(\overline{x}+\overline{y}+z)\,(\overline{x}+\overline{y}+\overline{z})$$

Conversions Between Canonical Forms

x	y	z	F	Minterm	Maxterm
0	0	0	**0**		$M_0 = (x + y + z)$
0	0	1	**1**	$m_1 = \overline{x}\,\overline{y}\,z$	
0	1	0	**1**	$m_2 = \overline{x}\,y\,\overline{z}$	
0	1	1	**1**	$m_3 = \overline{x}\,y\,z$	
1	0	0	**0**		$M_4 = (\overline{x} + y + z)$
1	0	1	**1**	$m_5 = x\,\overline{y}\,z$	
1	1	0	**0**		$M_6 = (\overline{x} + \overline{y} + z)$
1	1	1	**1**	$m_7 = x\,y\,z$	

$$F = m_1 + m_2 + m_3 + m_5 + m_7 = \sum(1, 2, 3, 5, 7) =$$

$$\overline{x}\,\overline{y}\,z + \overline{x}\,y\,\overline{z} + \overline{x}\,y\,z + x\,\overline{y}\,z + x\,y\,z$$

$$F = M_0 \cdot M_4 \cdot M_6 = \prod(0, 4, 6) = x+y+z)(\overline{x}+y+z)(\overline{x}+\overline{y}+z)$$

Algebraic Conversion to Sum-of-Minterms

- Expand all terms first to explicitly list all minterms
- AND any term missing a variable v with $(v + \overline{v})$
- Example 1: $f = x + \overline{x}\,\overline{y}$ (2 variables)
  $f = x\,(y + \overline{y}) + \overline{x}\,\overline{y}$
  $f = x\,y + x\,\overline{y} + \overline{x}\,\overline{y}$
  $\mathrm{f} = \mathrm{m}_3 + \mathrm{m}_2 + \mathrm{m}_0 = \sum(0, 2, 3)$
- Example 2: $g = a + \overline{b}\,c$ (3 variables)
  $g = a\,(b + \overline{b})(c + \overline{c}) + (a + \overline{a})\,\overline{b}\,c$
  $g = a\,b\,c + a\,b\,\overline{c} + a\,\overline{b}\,c + a\,\overline{b}\,\overline{c} + a\,\overline{b}\,c + \overline{a}\,\overline{b}\,c$
  $g = \overline{a}\,\overline{b}\,c + a\,\overline{b}\,\overline{c} + a\,\overline{b}\,c + a\,b\,\overline{c} + a\,b\,c$
  $g = \mathrm{m}_1 + \mathrm{m}_4 + \mathrm{m}_5 + \mathrm{m}_6 + \mathrm{m}_7 = \sum(1, 4, 5, 6, 7)$

Algebraic Conversion to Product-of-Maxterms

- Expand all terms first to explicitly list all maxterms
- OR any term missing a variable v with $v \cdot \overline{v}$
- Example 1: $f = x + \overline{x}\,\overline{y}$ (2 variables)

  Apply 2nd distributive law:

  $f = (x + \overline{x})\,(x + \overline{y}) = 1 \cdot (x + \overline{y}) = (x + \overline{y}) = \mathrm{M}_1$
- Example 2: $g = a\,\overline{c} + b\,c + \overline{a}\,\overline{b}$ (3 variables)

  $g = (a\,\overline{c} + b\,c + \overline{a})\,(a\,\overline{c} + b\,c + \overline{b})$ (distributive)

  $g = (\overline{c} + b\,c + \overline{a})\,(a\,\overline{c} + c + \overline{b})$ $(x + \overline{x}\,y = x + y)$

  $g = (\overline{c} + b + \overline{a})\,(a + c + \overline{b})$ $(x + \overline{x}\,y = x + y)$

  $g = (\overline{a} + b + \overline{c})\,(a + \overline{b} + c) = \mathrm{M}_5 \,.\, \mathrm{M}_2 = \prod (2, 5)$

Function Complements

- The complement of a function expressed as a sum of minterms is constructed by selecting the minterms missing in the sum-of-minterms canonical form
- Alternatively, the complement of a function expressed by a Sum of Minterms form is simply the Product of Maxterms with the same indices
- Example: Given $F(x, y, z) = \sum (1, 3, 5, 7)$

$$\overline{F}(x, y, z) = \sum (0, 2, 4, 6)$$

$$\overline{F}(x, y, z) = \prod (1, 3, 5, 7)$$

Summary of Minterms and Maxterms

- There are 2^n minterms and maxterms for Boolean functions with n variables.
- Minterms and maxterms are indexed from 0 to $2^n - 1$
- Any Boolean function can be expressed as a logical sum of minterms and as a logical product of maxterms
- The complement of a function contains those minterms not included in the original function
- The complement of a sum-of-minterms is a product-of-maxterms with the same indices

Standard Forms

- Standard Sum-of-Products (SOP) form: equations are written as an OR of AND terms
- Standard Product-of-Sums (POS) form: equations are written as an AND of OR terms
- Examples:
  - SOP: $\mathbf{A\,B\,C + \overline{A}\,\overline{B}\,C + B}$
  - POS: $\mathbf{(A+B)\cdot(A+\overline{B}+\overline{C})\cdot C}$
- These "mixed" forms are neither SOP nor POS
  - $\mathbf{(A\,B + C)\,(A + C)}$
  - $\mathbf{A\,B\,\overline{C} + A\,C\,(A+B)}$

Standard Sum-of-Products (SOP)

- A sum of minterms form for *n* variables can be written down directly from a truth table.
  - Implementation of this form is a two-level network of gates such that:
  - The first level consists of *n*-input AND gates
  - The second level is a single OR gate
- This form often can be simplified so that the corresponding circuit is simpler.

Standard Sum-of-Products (SOP)

- A Simplification Example:

$$F(A,B,C) = \sum(1,4,5,6,7)$$

- Writing the minterm expression:

$F = \overline{A}\,\overline{B}\,C + A\,\overline{B}\,\overline{C} + A\,\overline{B}\,C + AB\overline{C} + ABC$

- Simplifying:

$F = \overline{A}\,\overline{B}\,C + A\,(\overline{B}\,\overline{C} + \overline{B}\,C + B\,\overline{C} + B\,C)$

$F = \overline{A}\,\overline{B}\,C + A\,(\overline{B}\,(\overline{C} + C) + B\,(\overline{C} + C))$

$F = \overline{A}\,\overline{B}\,C + A\,(\overline{B} + B)$

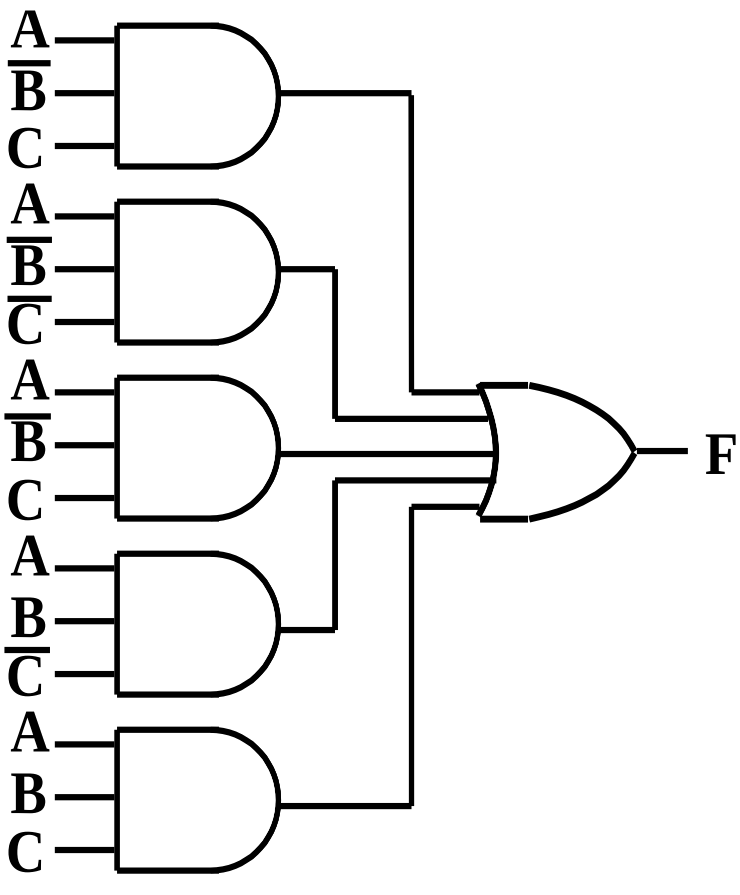
$F = \overline{A}\,\overline{B}\,C + A$

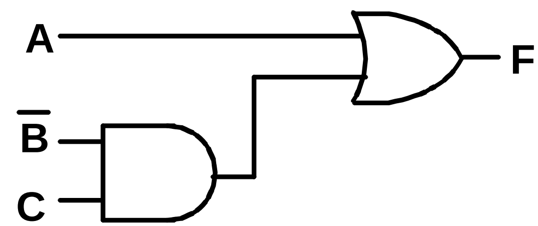
$F = \overline{B}\,C + A$

- Simplified F contains 3 literals compared to 15

AND/OR Two-Level Implementation

- The two implementations for F are shown below





It is quite apparent which is simpler!

SOP and POS Observations

- **The previous examples show that:**
  - Canonical Forms (Sum-of-minterms, Product-of-Maxterms), or other standard forms (SOP, POS) differ in complexity
  - Boolean algebra can be used to manipulate equations into simpler forms
  - Simpler equations lead to simpler implementations
- **Questions:**
  - How can we attain a “simplest” expression?
  - Is there only one minimum cost circuit?
  - The next part will deal with these issues