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Lab Course
Scientific Computing

Worksheet 3

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We again examine a similar population equation as in worksheet 2 but with different parameters

$$\dot{p} = 7\left(1 - \frac{p}{10}\right) \cdot p \tag{1}$$

and with a different initial condition

$$p(0) = 20. \tag{2}$$

The analytical solution is given by

$$p(t) = \frac{200}{20 - 10e^{-7t}}.$$

a) Plot the function $p(t)$ in a graph.

b) Reuse the Euler method and the method of Heun implemented in worksheet 2 to compute approximate solutions for equation (1) with initial conditions (2), end time $t_{end} = 5$, and $\delta t = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$.

Plot your solutions in one graph per method (together with the given solution from **a)**). Plot the function in the range $t \in [0, 5]$ and $p \in [0, 20]$.

c) Implement the following implicit numerical methods with variable stepsize δt and end time t_{end}

- 1) implicit Euler method,
- 2) second order Adams-Moulton method.

for the solution of the initial value problem

$$\dot{y} = f(y), \quad y(0) = y_0$$

as a function of the right hand side f, the first derivative of the right hand side with respect to y, initial value y_0, the stepsize δt and the end time t_{end}. The output of the function is a vector containing all computed approximate values for y. Use an accuracy limit of 10^{-4} for the Newton iteration in each time step. Stick to the signatures of the functions from worksheet 2 as far as possible.

Hint: Examine if the equation to be solved in each time step of these methods is solvable. If necessary, implement a stopping criterion to prevent the Newton solver from trying to solve unsolvable equations. In these cases, stop the time stepping with the method and the time step concerned and do not consider the associated approximations of y in your further examinations.

d) For both methods implemented, compute – as far as possible – approximate solutions for equation (1) with initial conditions (2) and with time steps $\delta t = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$. Plot your solutions in one graph per method (together with the given solution from **a)**).

e) To be able to handle also the cases for which you did not find a solution in **d)**, implement the following linearised versions of the Adams-Moulton method for equation (1)

- 1) linearisation 1:

$$y^{(n+1)} = y^{(n)} + \frac{\delta t}{2}\left(7 \cdot \left(1 - \frac{y^{(n)}}{10}\right) \cdot y^{(n)} + 7 \cdot \left(1 - \frac{y^{(n+1)}}{10}\right) \cdot y^{(n)}\right),$$

- 2) linearisation 2:

$$y^{(n+1)} = y^{(n)} + \frac{\delta t}{2}\left(7 \cdot \left(1 - \frac{y^{(n)}}{10}\right) \cdot y^{(n)} + 7 \cdot \left(1 - \frac{y^{(n)}}{10}\right) \cdot y^{(n+1)}\right),$$

for the solution of the initial value problem

$$\dot{y} = 7 \cdot \left(1 - \frac{y}{10}\right) \cdot y, \quad y(0) = y_0$$

as a function of the initial value y_0, the stepsize δt and the end time t_{end}. The output of the function is a vector with all computed approximate values for y.

f) For both methods implemented, compute approximate solutions for equation (1) with initial conditions (2) and with time steps $\delta t = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$. Plot your solutions in one graph per method (together with the given solution from **a)**).

g) Compare the results of the implicit methods to those computed with the explicit methods:

Compute the approximation error

$$E = \sqrt{\frac{\delta t}{5} \sum_k \left(p_k - p_{k,exact}\right)^2}$$

for each case in **b)** and **d)**, where p_k denotes the approximation of $p(\delta t \cdot k)$, $p_{exact,k}$ the exact values of p at $t = \delta t \cdot k$.

Collect the results in the tables below. Also write all information needed in the tables to the Matlab console in a readable way.

h) For each of the following methods, determine the factor by which the error is reduced if the step size δt is halved.

1) explicit Euler method,
2) method of Heun,
3) implicit Euler method,
4) Adams-Moulton method,
5) Adams-Moulton method – linearisation 1,
6) Adams-Moulton method – linearisation 2

Write down the results in the tabular below.

i) In addition to accuracy, we examine an additional aspect of 'quality' of a method: the *stability*. Descriptively spoken, stability denotes the applicability of a method for varying parameters, whereas at least results similar to the exact/correct solution have to be achieved (In particular, unphysical oscillations should not occur). With this heuristic definition, decide for which of the used values for δt each of the four examined methods is stable (in the case of our problem).

Mark stable cases by a cross in the last tabular. Try to find a simple criterion to determine whether a solution is stable or not and write the result to the Matlab console.

explicit Euler					
δt	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
error					
error red.					

Heun					
δt	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
error					
error red.					

implicit Euler					
δt	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
error					
error red.					

Adams-Moulton					
δt	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
error					
error red.					

Adams-Moulton – linearisation 1					
δt	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
error					
error red.					

Adams-Moulton – linearisation 2					
δt	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
error					
error red.					

Stable cases						
	explicit Euler	Heun	implicit Euler	Adams-Moulton	Adams-Moulton l1	Adams-Moulton l2
$\delta t = \frac{1}{2}$						
$\delta t = \frac{1}{4}$						
$\delta t = \frac{1}{8}$						
$\delta t = \frac{1}{16}$						
$\delta t = \frac{1}{32}$						

Questions:

1) For which integer q can you conclude that the accuracy of the

   a) explicit Euler method,
   b) method of Heun,
   c) implicit Euler method,
   d) Adams-Moulton method,
   e) Adams-Moulton method – linearisation 1,
   f) Adams-Moulton method – linearisation 2

   behaves like $O(\delta t^q)$?

2) In the lecture, we saw that both the implicit Euler and the Adams-Moulton method are unconditionally stable and, thus, give us stable solutions for every choice of δt. For the example of this worksheet we stated in **c)** and **d)** that the resulting equation for each timestep is sometimes not solvable and, thus, the method cannot be applied for certain δt. Can you explain this apparent discrepancy?

3) Can you give a reason why the linearisation 1 of the Adams-Moulton method works better?

   Hint: Write both linearisations in explicit form (y^{n+1} as a function of y^n) and investigate their behavior around the critical point(s) of the ODE.

4) Which type of methods (explicit/implicit) would you choose for

   – 1) the initial value problem from worksheet 2,
   – 2) the initial value problem (1,2) from this worksheet? Give a reason why you choose a certain type of methods and why you do not choose the other type in each case?

5) Can you give a real world example, where you need an explicit time stepping scheme? Can you give a real world example, where you need an implicit time stepping scheme?