CDM classes: notes

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April 7, 2018

Contents

Preface (Prefazione)

"Surely, two things are infinite: the knowledge and the ignorance."
– Anonimous

Questi appunti trattano gli argomenti del corso di Cinematica e Dinamica di Meccanismi (Laurea Magistrale in Ingegneria meccanica). Gli argomenti sono studiati dal punto di vista teorico tralasciando le soluzioni dei problemi svolti in aula.
Gli appunti sono redatti in lingua inglese per permettere agli studenti di prendere confidenza con un lessico tecnico nel quale potrebbero imbattersi in un prossimo futuro. D'altra parte, per natura degli argomenti trattati, il testo rappresenta un supporto minimo al corpo matematico degli argomenti.

1

Rigid-Body Displacements

1.1 Description of a position

Position of any point with respect to a fixed reference of system $\mathcal{F}_A$ is decribed by ${}^A\mathbf{p} = \begin{pmatrix} p_x & p_y & p_z \end{pmatrix}^T = p_x\hat{\mathbf{x}} + p_y\hat{\mathbf{y}} + p_z\hat{\mathbf{z}}$. The terms p_x, p_y and p_z represent the projections of the position vetctor onto the 3 coordinate axes of the reference frame.

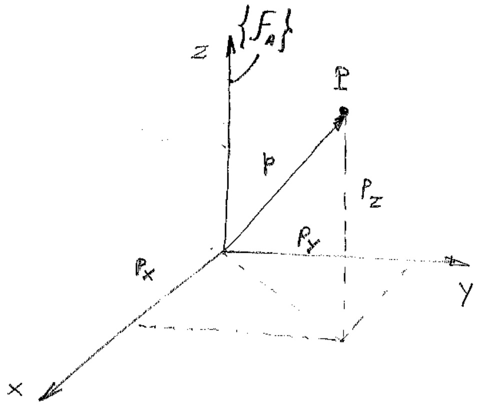

Figure 1.1: Position vector of a point P in $\mathbb{R}^3$.

1.2 Description of a orientation

In this section the orientation of a reference frame (and of a rigid body moving with it) will be shown according to the following representations (parametrizations):

- Direction cosine representation;

- Screw axis representation;
- Cayley-Hamilton formula;
- Cayley formula;
- Rodrigues parameters;

1.2.1 Direction cosine representation

Let consider a fixed reference frame A whose unit vectors pointing the coordinate axes are $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ and a moving frame B whose unit vectors pointing the coordinate axes are $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, $\hat{\mathbf{w}}$.
$\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, $\hat{\mathbf{w}}$ can be expressed in A as:

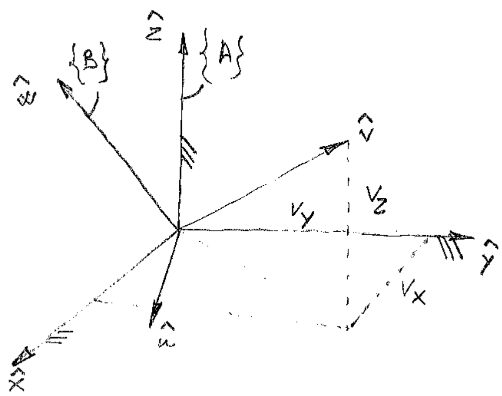

Figure 1.2: Orientation of B with respect to A.

$$\begin{aligned} {}^A\hat{\mathbf{u}} &= u_x\hat{\mathbf{x}} + u_y\hat{\mathbf{y}} + u_z\hat{\mathbf{z}} \\ {}^A\hat{\mathbf{v}} &= v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}} \\ {}^A\hat{\mathbf{w}} &= w_x\hat{\mathbf{x}} + w_y\hat{\mathbf{y}} + w_z\hat{\mathbf{z}} \end{aligned} \tag{1.1}$$

Eqs.1.1 allows us to obtain the representation sought via vector algebra (symbol $\hat{\bullet}$ is omitted):

$${}^A\mathbf{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}, {}^A\mathbf{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}, {}^A\mathbf{w} = \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix}$$

Then, casting them in a matrix we obtain:

$${}^A\mathbf{Q}_B = \begin{pmatrix} {}^A\mathbf{u} & {}^A\mathbf{v} & {}^A\mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{u}^T\mathbf{x} & \mathbf{v}^T\mathbf{x} & \mathbf{w}^T\mathbf{x} \\ \mathbf{u}^T\mathbf{y} & \mathbf{v}^T\mathbf{y} & \mathbf{w}^T\mathbf{y} \\ \mathbf{u}^T\mathbf{z} & \mathbf{v}^T\mathbf{z} & \mathbf{w}^T\mathbf{z} \end{pmatrix} = \begin{pmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{pmatrix} \tag{1.2}$$

${}^A\mathbf{Q}_B$ is the **orientation matrix** of B with respect to A. It represents the orientation of frame B with respect to frame A.

Properties of ${}^A\mathbf{Q}_B$

${}^A\mathbf{Q}_B$ is a special orthogonal matrix such that:

$$\begin{aligned} {}^A\mathbf{Q}_B^{-1} &= {}^A\mathbf{Q}_B^T \\ \det({}^A\mathbf{Q}_B) &= 1 \end{aligned} \tag{1.3}$$

Orthogonality properties can also be expressed as:

$$\begin{aligned} \|{}^A\mathbf{u}\| = \|{}^A\mathbf{v}\| = \|{}^A\mathbf{w}\| = 1; \\ \mathbf{u}^T\mathbf{v} = \mathbf{v}^T\mathbf{w} = \mathbf{w}^T\mathbf{u} = 0; \\ \mathbf{u}\times\mathbf{v} = \mathbf{w},\ \mathbf{v}\times\mathbf{w} = \mathbf{u},\ \mathbf{w}\times\mathbf{u} = \mathbf{v}. \end{aligned} \tag{1.4}$$

Because of the first two eqs. of Eqs. 1.4 there are only three independent parameters defining an orientation.
Furthermore, if we define ${}^B\mathbf{Q}_A$ as the orientation matrix of A with respect to B:

$${}^B\mathbf{Q}_A = \begin{pmatrix} {}^B\mathbf{x} & {}^B\mathbf{y} & {}^B\mathbf{z} \end{pmatrix}$$

because of the orthogonality properties:

$${}^B\mathbf{Q}_A{}^T = \begin{pmatrix} {}^B\mathbf{x}^T \\ {}^B\mathbf{y}^T \\ {}^B\mathbf{z}^T \end{pmatrix} = {}^A\mathbf{Q}_B = \begin{pmatrix} {}^A\mathbf{u} & {}^A\mathbf{v} & {}^A\mathbf{w} \end{pmatrix}.$$

Basic Rotation Matrices

1. Consider a fixed frame A and a moving frame B as depicted in Figure 1.3:

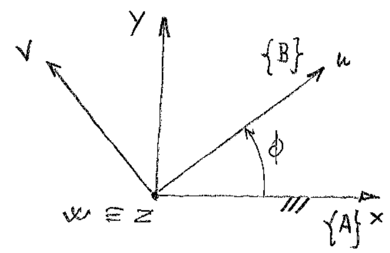

Figure 1.3: $\mathbf{R}_z(\phi)$.

$$
{}^{A}\mathbf{Q}_{B} = \mathbf{R}_{z}(\phi) = \begin{pmatrix} c_{\phi} & -s_{\phi} & 0 \\ s_{\phi} & c_{\phi} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.5}
$$

2. Consider a fixed frame A and a moving frame B as depicted in Figure 1.4:

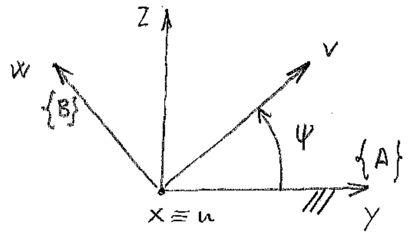

Figure 1.4: $\mathbf{R}_{x}(\psi)$.

$$
{}^{A}\mathbf{Q}_{B} = \mathbf{R}_{x}(\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{\psi} & -s_{\psi} \\ 0 & s_{\psi} & c_{\psi} \end{pmatrix} \tag{1.6}
$$

3. Consider a fixed frame A and a moving frame B as depicted in Figure 1.5:

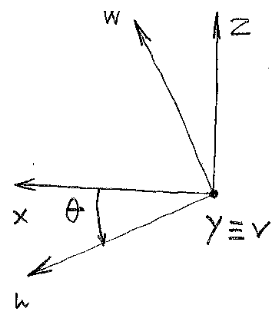

Figure 1.5: $\mathbf{R}_{y}(\theta)$.

$$
{}^{A}\mathbf{Q}_{B} = \mathbf{R}_{y}(\theta) = \begin{pmatrix} c_{\theta} & 0 & s_{\theta} \\ 0 & 1 & 0 \\ -s_{\theta} & 0 & c_{\theta} \end{pmatrix} \tag{1.7}
$$

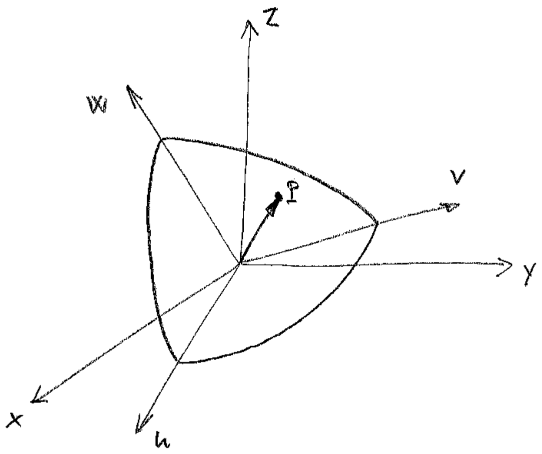

Figure 1.6: Position of a point.

Position of a point

Positon vector of P belonging to a rigid body moving together with the frame B can be expressed by using its projections along with the frame A axes:

$$^{A}\mathbf{p} = p_x\mathbf{x} + p_y\mathbf{y} + p_z\mathbf{z} \tag{1.8}$$

However, the same vector can be expressed by using its projections along with the frame B axes:

$$^{B}\mathbf{p} = p_u\mathbf{u} + p_v\mathbf{v} + p_w\mathbf{w} \tag{1.9}$$

Now, by substituting eqs.1.1 into eq.1.9 we obtain a position vector expressed in A, ($\|^{A}\mathbf{p}\| = \|^{B}\mathbf{p}\|$):

$$^{A}\mathbf{p} = p_u(u_x\mathbf{x} + u_y\mathbf{y} + u_z\mathbf{z}) + p_v(v_x\mathbf{x} + v_y\mathbf{y} + v_z\mathbf{z}) + p_w(w_x\mathbf{x} + w_y\mathbf{y} + w_z\mathbf{z});$$

and rearranging:

$$\begin{aligned} ^{A}\mathbf{p} = (p_u u_x + p_v v_x + p_w w_x)\mathbf{x} + \\ (p_u u_y + p_v v_y + p_w w_y)\mathbf{y} + \\ (p_u u_z + p_v v_z + p_w w_z)\mathbf{z} \end{aligned} \tag{1.10}$$

From eq.1.10 we obtain:

$$\begin{aligned} p_x &= u_x p_u + v_x p_v + w_x p_w \\ p_y &= u_y p_u + v_y p_v + w_y p_w \\ p_z &= u_z p_u + v_z p_v + w_z p_w \end{aligned} \tag{1.11}$$

Finally, eq.1.11 can be expressed in matrix form:

$$^{A}\mathbf{p} =^{A} \mathbf{Q}_{B} \, ^{B}\mathbf{p} \tag{1.12}$$

where $^{A}\mathbf{Q}_{B}$ is the rotation matrix defined above. Eq.1.12 states that *by multiplying the position vector expressed in B by $^{A}\mathbf{Q}_{B}$ the position vector expressed in A is obtained.*
Pre-multiply eq.1.12 by $^{A}\mathbf{Q}_{B}^{-1}$ leads to:

$$^{B}\mathbf{p} = {}^{A}\mathbf{Q}_{B}^{-1}{}^{A}\mathbf{p} = {}^{A}\mathbf{Q}_{B}^{T}{}^{A}\mathbf{p} = {}^{B}\mathbf{Q}_{A}{}^{A}\mathbf{p} \tag{1.13}$$

Passive and Active interpretations of the Rotation matrix

Consider Figure 1.7: ref. system A is fixed whereas ref. system B moves

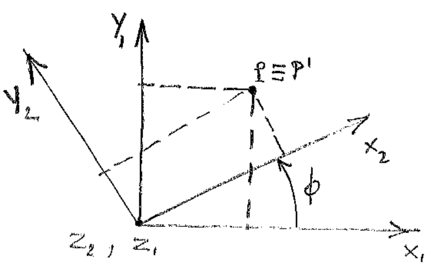

Figure 1.7: Passive interpretation of the Rotation matrix.

with the point P. According to the figure we can write:

$$\mathbf{p} = \mathbf{Q}\mathbf{p}'$$

with $\mathbf{p} = \begin{pmatrix} x_1 & y_1 & z_1 \end{pmatrix}^T$ and $\mathbf{p}' = \begin{pmatrix} x_2 & y_2 & z_2 \end{pmatrix}^T$. In the example, we obtain:

$$\begin{aligned} x_1 &= x_2 c_\phi - y_2 s_\phi; \\ y_1 &= x_2 s_\phi + y_2 c_\phi. \end{aligned} \tag{1.14}$$

Figure 1.7 shows the *Passive Interpretation* of equation $\mathbf{p} = \mathbf{Q}\mathbf{p}'$.
Now, consider Figure 1.8: here, both ref. systems A and B are fixed and the point moves from position $\mathbf{p}'$ to $\mathbf{p}$. It can be proved that eq. $\mathbf{p} = \mathbf{Q}\mathbf{p}'$ is still valid. If $l = \|\mathbf{OP}\| = \|\mathbf{OP'}\|$ and γ is the angle that the position vector forms with x_1 (x_2) then:

$$\begin{aligned} x_2 &= l c_\gamma; \quad y_2 = l s_\gamma; \\ x_1 = l c_{\gamma+\phi} &= l c_\gamma c\phi - l s_\gamma s\phi = x_2 c_\phi - y_2 s_\phi; \\ y_1 = l s_{\gamma+\phi} &= l s_\gamma c\phi + l c_\gamma s\phi = y_2 c_\phi + x_2 s_\phi. \end{aligned} \tag{1.15}$$

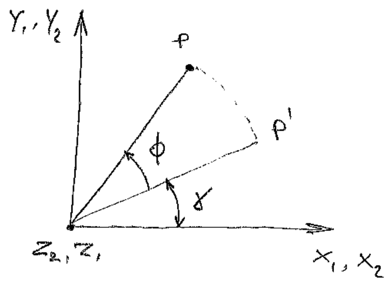

Figure 1.8: Active interpretation of the Rotation matrix.

Eqs. 1.15 are identical to eqs. 1.14 and thus in matrix form it can be written as $\mathbf{p} = \mathbf{Q}\mathbf{p}'$ closing the proof. Figure 1.8 shows the *Active Interpretation* of equation $\mathbf{p} = \mathbf{Q}\mathbf{p}'$. The active interpretation provides to the rotation matrix the algebraic meaning of *Linear Transformation*: $\mathbf{Q}$ maps a vector $\mathbf{p}'$ into a vector $\mathbf{p}$ wtih $\mathbf{Q}$ endowed with two properties, namely *additivity* and *scalar multiplication (homogeneity).*

1.2.2 Screw Axis Representation

The most general motion of a rigid body in $\mathbb{R}^3$ is elicoidal. Indeed, the rigid body may translate along an axis and at the same time may rotate along with the same axis. The axis is called the *screw axis* of the motion. The screw axis representation of the rotation matrix considers the rigid body that only rotates along with that axis.

According to the Figure 1.9, point P moves from the initial position to the final position P' because of the rotation ϕ along with the screw axis whose unit vetor is $\mathbf{e} = \begin{pmatrix} e_x & e_y & e_z \end{pmatrix}^T$.

$$\mathbf{p}' = \overrightarrow{OQ} + \overrightarrow{QP'} \tag{1.16}$$

where $\overrightarrow{OQ}$ is the axial component of $\mathbf{p}$ along $\mathbf{e}$:

$$\overrightarrow{OQ} = \mathbf{e}\mathbf{e}^T\mathbf{p} \tag{1.17}$$

According to the Figure 1.9:

$$\overrightarrow{QP'} = c_\phi\overrightarrow{QP} + s_\phi\overrightarrow{QP''} \tag{1.18}$$

However, eq. 1.18 can be easily proved by analysis as follows.
Consider $\|\overrightarrow{QP}\| = \|\overrightarrow{QP'}\| = \|\overrightarrow{QP''}\| = l$ and a ref. system B centred in Q

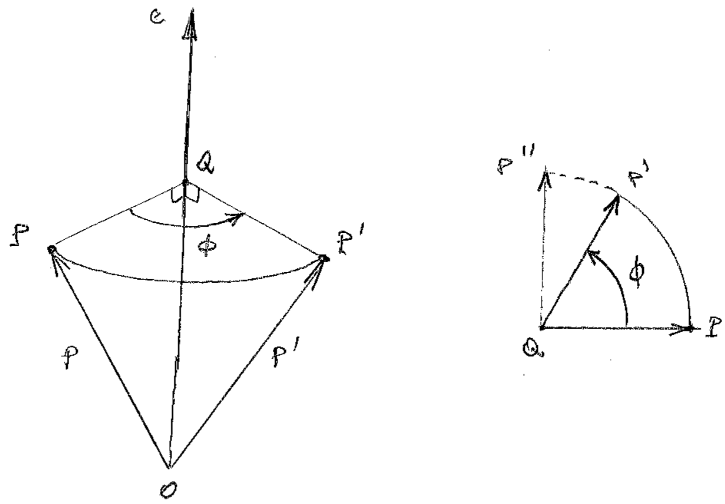

Figure 1.9: Rotation of a rigid body about a line.

and whose first axis is aligned with $\overrightarrow{QP'}$ and the third axis is aligned with $\mathbf{e}$ to express the vectors:

$$
\begin{aligned}
{}^{B}\mathbf{QP'} &= l \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T; \\
{}^{B}\mathbf{QP} &= l \begin{pmatrix} c_\phi & -s_\phi & 0 \end{pmatrix}^T; \\
{}^{B}\mathbf{QP''} &= l \begin{pmatrix} s_\phi & c_\phi & 0 \end{pmatrix}^T;
\end{aligned}
$$

Eq. 1.18 expressed by the ref. system B states:

$$
l \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = l c_\phi \begin{pmatrix} c_\phi \\ -s_\phi \\ 0 \end{pmatrix} + l s_\phi \begin{pmatrix} s_\phi \\ c_\phi \\ 0 \end{pmatrix}
$$

which proves eq. 1.18 element-wise.
It is apparent that $\overrightarrow{QP}$ is the normal component of $\mathbf{p}$ with respect to $\mathbf{e}$:

$$
\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ} = (\mathbf{I} - \mathbf{e}\mathbf{e}^T)\mathbf{p}
$$

and

$$
\overrightarrow{QP''} = \mathbf{e} \times \mathbf{p} = \mathbf{E}\mathbf{p} = \mathbf{e} \times (\overrightarrow{OQ} + \overrightarrow{QP}) = \mathbf{e} \times \overrightarrow{QP}
$$

Matrix $\mathbf{E}$ is a skew-symmetric matrix ($\mathbf{E} + \mathbf{E}^T = \mathbf{0}$) whose elements are the components of $\mathbf{e}$:

$$
\mathbf{E} = \begin{pmatrix} 0 & -e_z & e_y \\ e_z & 0 & -e_x \\ -e_y & e_x & 0 \end{pmatrix}
$$

If now eqs. 1.17 and 1.18 are substituted into eq. 1.16 one obtains:

$$\begin{aligned}\mathbf{p}' &= \mathbf{e}\mathbf{e}^T\mathbf{p} + c_\phi(\mathbf{I} - \mathbf{e}\mathbf{e}^T)\mathbf{p} + s_\phi\mathbf{E}\mathbf{p}\\ \mathbf{p}' &= [\mathbf{e}\mathbf{e}^T + c_\phi(\mathbf{I} - \mathbf{e}\mathbf{e}^T) + s_\phi\mathbf{E}]\mathbf{p}\\ \mathbf{Q} &= \mathbf{e}\mathbf{e}^T + c_\phi(\mathbf{I} - \mathbf{e}\mathbf{e}^T) + s_\phi\mathbf{E} \qquad (1.19)\\ \mathbf{p}' &= \mathbf{Q}\mathbf{p}\end{aligned}$$

According to eq. 1.19 the rotation matrix $\mathbf{Q}$ depends on 4 parameters: e_x, e_y, e_z, ϕ. Terms of $\mathbf{Q}$ can expliclty calculated and assume the following form, for example:

$Q_{11} = e_x^2 + c_\phi(1 - e_x^2); Q_{12} = e_xe_y(1 - c_\phi) - s_\phi e_z; Q_{13} = e_xe_z(1 - c_\phi) + s_\phi e_y.$

1.2.3 Cayley-Hamilton Theorem (Representation)

The Cayley-Hamilton theorem provides another representation of the rotation matrix in terms of ϕ and $\mathbf{e}$. When considering a fixed rotation axis then $\mathbf{Q}$ only depends on ϕ such that:

$$\mathbf{Q}(\phi) = \mathbf{Q}(0) + \mathbf{Q}'(0)\phi + \frac{1}{2!}\mathbf{Q}''(0)\phi^2 + \ldots + \frac{1}{k!}\mathbf{Q}^{(k)}(0)\phi^k + \ldots \qquad (1.20)$$

where $\mathbf{Q}(0) = \mathbf{Q}(\phi = 0)$ and $\mathbf{Q}^{(k)}(0) = (\frac{d^k\mathbf{Q}}{d\phi^k})_0$. First, we show that $\mathbf{Q}'(0)$ is a skew-symmetric matrix. Starting from the orthogonal matrix property $\mathbf{Q}(0)^T\mathbf{Q}(0) = \mathbf{I}$ we differentiate it with respect to the angle ϕ such that:

$$[(\frac{d\mathbf{Q}}{d\phi})_0]^T\mathbf{Q}(0) + \mathbf{Q}(0)^T(\frac{d\mathbf{Q}}{d\phi})_0 = \mathbf{0} \qquad (1.21)$$

and considering that $\mathbf{Q}(0)^T = \mathbf{Q}(0) = \mathbf{I}$ eq. 1.21 becomes:

$$[(\frac{d\mathbf{Q}}{d\phi})_0]^T + (\frac{d\mathbf{Q}}{d\phi})_0 = \mathbf{0} \qquad (1.22)$$

Eq. 1.22 claims that $\mathbf{Q}'(0)$ is skew-symmetric.
It is easy to show that $\mathbf{Q}'(0)$ is nothing else but $\mathbf{E}$. Indeed, by differentiating with respect to ϕ eq. 1.19 we obtain:

$$\mathbf{Q}'(\phi) = -(\mathbf{I} - \mathbf{e}\mathbf{e}^T)s_\phi + c_\phi\mathbf{E}$$

that with $\phi = 0$ becomes $\mathbf{Q}'(0) = \mathbf{E}$. Furthermore by successive differentations:

$$\begin{aligned}\mathbf{Q}''(\phi) &= -(\mathbf{I} - \mathbf{e}\mathbf{e}^T)c_\phi - s_\phi\mathbf{E};\\ \phi = 0 \ : \ \mathbf{Q}''(0) &= -(\mathbf{I} - \mathbf{e}\mathbf{e}^T) = \mathbf{E}^2.\end{aligned}$$

and

$$\begin{aligned}\mathbf{Q}'''(\phi) &= (\mathbf{I} - \mathbf{e}\mathbf{e}^T)s_\phi - c_\phi \mathbf{E};\\ \phi = 0 \ : \ \mathbf{Q}'''(0) &= -\mathbf{E} = \mathbf{E}^3.\end{aligned}$$

Thus, in general $\mathbf{Q}^{(k)}(0) = \mathbf{E}^k$, $(k = 1, \cdots, n.\ n \in \mathbb{Z})$. The previous equations take advantages of:

$$\begin{aligned}\mathbf{E}^{2k} &= (-1)^k(\mathbf{I} - \mathbf{e}\mathbf{e}^T)\\ \mathbf{E}^{2k+1} &= (-1)^k\mathbf{E}.\end{aligned}$$

which can be verfied numerically. Then, eq. 1.20 can be wrtitten:

$$\mathbf{Q}(\phi) = \mathbf{I} + \mathbf{E}\phi + \frac{1}{2!}\mathbf{E}^2\phi^2 + \frac{1}{3!}\mathbf{E}^3\phi^3 + \ldots + \frac{1}{k!}\mathbf{E}^k\phi^k = e^{\mathbf{E}\phi} \tag{1.23}$$

Eq. 1.23 is the exponential representation of the rotation matrix in terms of its *natural invariants* $\mathbf{e}$ (in $\mathbf{E}$) and ϕ. The foregoing parameters are termed *invariants* as they do not depend on the coordinate axes chosen to represent the rotation under study. And finally:

$$\begin{aligned}\mathbf{Q}(\phi) = \mathbf{I} + (-\frac{1}{2!}\phi^2 + \frac{1}{4!}\phi^4 + \ldots + \frac{1}{2k!}(-1)^k\phi^{2k})(\mathbf{I} - \mathbf{e}\mathbf{e}^T) +\\ \mathbf{E}(\phi - \frac{1}{3!}\phi^3 + \ldots + \frac{1}{(2k+1)!}(-1)^k\phi^{2k+1})\end{aligned}$$

The series inside the first pair of brackets is apparently $c_\phi - 1$ while that in the second pair is s_ϕ. Thus:

$$\begin{aligned}\mathbf{Q}(\phi) &= \mathbf{I} + (c_\phi - 1)(\mathbf{I} - \mathbf{e}\mathbf{e}^T) + s_\phi\mathbf{E}\\ \mathbf{Q}(\phi) &= \mathbf{I} + (1 - c_\phi)\mathbf{E}^2 + s_\phi\mathbf{E}\end{aligned} \tag{1.24}$$

Eq. 1.24 is the Cayley-Hamilton's representation of the rotation matrix.

1.2.4 Cayley Representation (Rodrigues parameters)

Consider again Figure 1.9. The geometrical equality $\|\mathbf{p}\|^2 = \|\mathbf{p}'\|^2$ can be written as:

$$\begin{aligned}\mathbf{p}'^T\mathbf{p}' = \mathbf{p}^T\mathbf{p} \ :\\ (\mathbf{p}' - \mathbf{p})^T(\mathbf{p}' + \mathbf{p}) = 0\end{aligned} \tag{1.25}$$

Eq. 1.25 can be written as $\mathbf{f}^T\mathbf{g} = \mathbf{f}\cdot\mathbf{g} = 0$ with $\mathbf{f} = \mathbf{p}' - \mathbf{p}$ and $\mathbf{g} = \mathbf{p}' + \mathbf{p}$. $\mathbf{f}$, $\mathbf{g}$ may be expressed in terms of $\mathbf{Q}$.

$$\mathbf{g} = \mathbf{p}' + \mathbf{p} = \mathbf{Q}\mathbf{p} + \mathbf{p} = (\mathbf{Q} + \mathbf{I})\mathbf{p} \tag{1.26}$$

$$\mathbf{f} = \mathbf{p}' - \mathbf{p} = \mathbf{Q}\mathbf{p} - \mathbf{p} = (\mathbf{Q} - \mathbf{I})\mathbf{p} \tag{1.27}$$

If one substitutes $\mathbf{p} = (\mathbf{Q}+\mathbf{I})^{-1}\mathbf{g}$ obtained from eq. 1.26 into eq. 1.27:

$$\mathbf{f} = (\mathbf{Q}-\mathbf{I})(\mathbf{Q}+\mathbf{I})^{-1}\mathbf{g} = \mathbf{B}\mathbf{g} \tag{1.28}$$

$\mathbf{B}$ is the skew-symmetric matrix that contains the Rodrigues parameters b_1, b_2, b_3. We now show that $\mathbf{B}$ is skew-symmetric. From eq. 1.28 then $\mathbf{B}\mathbf{g}\cdot\mathbf{g} = 0$, that is:

$$\mathbf{B}\mathbf{g} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{3} b_{1j}g_j \\ \sum_{j=1}^{3} b_{2j}g_j \\ \sum_{j=1}^{3} b_{3j}g_j \end{pmatrix}$$

Thus:

$$\begin{aligned} \mathbf{B}\mathbf{g}\cdot\mathbf{g} = \begin{pmatrix} \sum_{j=1}^{3} b_{1j}g_j & \sum_{j=1}^{3} b_{2j}g_j & \sum_{j=1}^{3} b_{3j}g_j \end{pmatrix} \cdot \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} &= 0 \\ \sum_{j=1}^{3} b_{1j}g_jg_1 + \sum_{j=1}^{3} b_{2j}g_jg_2 + \sum_{j=1}^{3} b_{3j}g_jg_3 &= 0 \\ (b_{11}g_1^2 + b_{12}g_2g_1 + b_{13}g_3g_1) + & \\ (b_{21}g_1g_2 + b_{22}g_2^2 + b_{23}g_3g_2) + (b_{31}g_3g_1 + b_{32}g_3g_2 + b_{33}g_3^2) &= 0 \end{aligned} \tag{1.29}$$

Eq. 1.29 is true if and only if $b_{jj} = 0$ and $b_{ij} + b_{ji} = 0$ $(i \neq j)$ proving that $\mathbf{B}$ is skew-symmetric:

$$\mathbf{B} = \begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix}.$$

It is worth noting that b_i $(i = 1, \cdots, 3)$ are numbers while g_1, g_2, g_3 are the $\mathbf{g}$'s components along the ref. axes. Now since $\mathbf{B} = (\mathbf{Q}-\mathbf{I})(\mathbf{Q}+\mathbf{I})^{-1}$ then we may obtain $\mathbf{Q}$.

$$\begin{aligned} \mathbf{B}(\mathbf{Q}+\mathbf{I}) &= (\mathbf{Q}-\mathbf{I}) \\ \mathbf{I}+\mathbf{B} = \mathbf{Q}-\mathbf{B}\mathbf{Q} &= (\mathbf{I}-\mathbf{B})\mathbf{Q} \\ \mathbf{Q} &= (\mathbf{I}-\mathbf{B})^{-1}(\mathbf{I}+\mathbf{B}) \end{aligned} \tag{1.30}$$

By expanding eq. 1.30 we obtain:

$$\mathbf{Q} = \Delta^{-1} \begin{pmatrix} 1+b_1^2-b_2^2-b_3^2 & 2(b_1b_2-b_3) & 2(b_1b_3+b_2) \\ 2(b_2b_1+b_3) & 1-b_1^2+b_2^2-b_3^2 & 2(b_2b_3-b_1) \\ 2(b_3b_1-b_2) & 2(b_3b_2+b_1) & 1-b_1^2-b_2^2+b_3^2 \end{pmatrix} \tag{1.31}$$

with $\Delta = 1 + b_1^2 + b_2^2 + b_3^2$. It has to be noticed that the rotation matrix expressed via b_i's leads to a polynomial expression with no trigonometric

formula.
Eq. 1.30 allows us to obtain $\mathbf{Q}$ given b_i $(i = 1, \cdots, 3)$ whereas $\mathbf{B} = (\mathbf{Q} - \mathbf{I})(\mathbf{Q} + \mathbf{I})^{-1}$ allows us to obtain b_i $(i = 1, \cdots, 3)$ when $\mathbf{Q}$ is given. However, to obtain b_i $(i = 1, \cdots, 3)$ when $\mathbf{Q}$ is given, we may also use the *linear invariants of* $\mathbf{Q}$ that will be defined in the next section. To show the polynomial (not trigonometric) nature of b_i let image to proceed as follows.
Denote with Q_{ii} $(i = 1, \cdots, 3)$, the terms of the diagonal of $\mathbf{Q}$. By comparing them with the terms in eq. 1.31 we have:

$$\begin{aligned} 1 + b_1^2 - b_2^2 - b_3^2 &= \Delta Q_{11}, \\ 1 - b_1^2 + b_2^2 - b_3^2 &= \Delta Q_{22}, \\ 1 - b_1^2 - b_2^2 + b_3^2 &= \Delta Q_{33} \end{aligned}$$

where Q_{ii}, $i = (1, \cdots, 3)$, is the ith. diagonal term of the rotation matrix. The three equations can also be written as:

$$\begin{aligned} A_1 b_1^2 + B_1 b_2^2 + C_1 b_3^2 + D_1 &= 0, \\ A_2 b_1^2 + B_2 b_2^2 + C_2 b_3^2 + D_2 &= 0, \\ A_3 b_1^2 + B_3 b_2^2 + C_3 b_3^2 + D_3 &= 0, \end{aligned} \tag{1.32}$$

where A_i, B_i, C_i, $i = (1, \cdots, 3)$, can be obtained by simple calculations. In order to solve eqs. 1.32 we use the *Sylvester dialytic elimination method.* We include one of the unknowns, namely b_3, into new terms: $K_i = C_i b_3^2 + D_i$, $i = (1, \cdots, 3)$, such that the system becomes:

$$\begin{pmatrix} A_1 & B_1 & K_1 \\ A_2 & B_2 & K_2 \\ A_3 & B_3 & K_3 \end{pmatrix} \begin{pmatrix} b_1^2 \\ b_2^2 \\ 1 \end{pmatrix} = \mathbf{0}_{3\times 1} \quad :$$

$$\mathbf{H} \begin{pmatrix} b_1^2 \\ b_2^2 \\ 1 \end{pmatrix} = \mathbf{0}_{3\times 1}. \tag{1.33}$$

Non-trivial solutions of the linear system of equations 1.33 are obtained whenever

$$det(\mathbf{H}) = 0 \tag{1.34}$$

Eq. 1.34 represents a polynomial with b_3 as unknown. Once values of b_3 are calculated then b_1 and b_2 can be obtained by backward substitutions. It is worth noting that this procedure introduces extraneous solutions that can be easily eliminated with the help of the off-diagonal terms of $\mathbf{Q}$. In some details, we can find b_3^2 from $det(\mathbf{H}) = 0$ and b_2^2, b_1^2 by back substitutions and then Δ, too. Eventually, by using the off-diagonal terms of $\mathbf{Q}$ we obtain:

$$b_1 = \frac{\Delta}{4}(Q_{32} - Q_{23}), \quad b_2 = \frac{\Delta}{4}(Q_{13} - Q_{31}), \quad b_3 = \frac{\Delta}{4}(Q_{21} - Q_{12}). \tag{1.35}$$

1.2.5 The linear invariants of a Rotation

Given any 3×3 matrix $\mathbf{A}$, its *Cartesian decomposition* consists of the sum of its symmetric part $\mathbf{A}_s$ and the skew-symmetric part $\mathbf{A}_{ss}$ defined as:

$$\mathbf{A}_s = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T), \quad \mathbf{A}_{ss} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T).$$

The linear invarinats of $\mathbf{A}$ are:

- *axial vector* or vector of $\mathbf{A}$ is a vector $\mathbf{a}$ such that $\mathbf{a} \times \mathbf{v} = \mathbf{A}_{ss}\mathbf{v}$, $\forall \mathbf{v}$
- *trace* of $\mathbf{A}$ is the sum of the eigenvalues of $\mathbf{A}_s$: $\sum_{i=1}^{3} \lambda_i$, which are real.

Since there are no coordinate frame involved in the definitions given they are invariants. However, a particular reference has to be chosen whenever the invariants are calculated. The above-defined invarinats can be calculated as:

$$\begin{aligned} vect(\mathbf{A}) = \mathbf{a} = \frac{1}{2}\begin{pmatrix} a_{32} - a_{23} \\ a_{13} - a_{31} \\ a_{21} - a_{12} \end{pmatrix}; \\ tr(\mathbf{A}) = a_{11} + a_{22} + a_{33}. \end{aligned}$$

Now, consider the rotation matrix $\mathbf{Q}$ instead, and according to the Cartesian decomposition we can write:

$$\begin{aligned} \mathbf{Q} = \mathbf{e}\mathbf{e}^T + c_\phi(\mathbf{I} - \mathbf{e}\mathbf{e}^T) + s_\phi \mathbf{E}; \\ \mathbf{Q}_s = \mathbf{e}\mathbf{e}^T + c_\phi(\mathbf{I} - \mathbf{e}\mathbf{e}^T); \\ \mathbf{Q}_{ss} = s_\phi \mathbf{E}. \end{aligned}$$

That is clear when considering that $\mathbf{E}$ is skew-symmetric. By using $\mathbf{Q}_s$ we obtain:

$$\mathbf{Q}_s = \begin{pmatrix} e_x^2 & * & * \\ * & e_y^2 & * \\ * & * & e_z^2 \end{pmatrix} + c_\phi \begin{pmatrix} 1 - e_x^2 & * & * \\ * & 1 - e_y^2 & * \\ * & * & 1 - e_z^2 \end{pmatrix}$$

such that

$$\begin{aligned} \sum_{i=1}^{3} \lambda_i = tr(\mathbf{Q}) = tr(\mathbf{Q}_s) = 1 + 2c_\phi : \\ c_\phi = \frac{tr(\mathbf{Q}) - 1}{2}, \quad \phi = acos(\frac{tr(\mathbf{Q}) - 1}{2}). \end{aligned} \tag{1.36}$$

By using $\mathbf{Q}_{ss}$ we obtain:

$$
\begin{aligned}
\mathbf{Q}_{ss} = s_\phi \mathbf{E} = s_\phi \begin{pmatrix} 0 & -e_z & e_y \\ e_z & 0 & -e_x \\ -e_y & e_x & 0 \end{pmatrix}; \\
vect(\mathbf{Q}) = vect(\mathbf{Q}_{ss}) = vect \begin{pmatrix} 0 & -s_\phi e_z & s_\phi e_y \\ s_\phi e_z & 0 & -s_\phi e_x \\ -s_\phi e_y & s_\phi e_x & 0 \end{pmatrix} = \\
\frac{1}{2} \begin{pmatrix} 2 s_\phi e_x \\ 2 s_\phi e_y \\ 2 s_\phi e_z \end{pmatrix} = s_\phi \mathbf{e} : \\
\mathbf{e} = \frac{vect(\mathbf{Q})}{s_\phi}
\end{aligned}
\tag{1.37}
$$

Eqs. 1.36 and 1.37 allows one to obtain ϕ and $\mathbf{e}$ when the natural invariants of the rotation matrix are known. It has to be noticed that eqs. 1.36 and 1.37

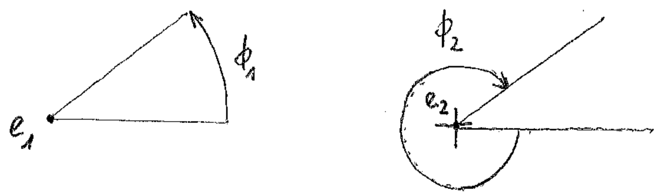

Figure 1.10: Couple of $\{\mathbf{e}, \phi\}$ detected by the linear invariants of a rotation.

select a couple of $\{\mathbf{e}, \phi\}$. Indeed, according to Figure 1.10, eq. 1.36 finds out ϕ_1 and ϕ_2 to which eq. 1.37 associates respectively $\mathbf{e}_1$ and $\mathbf{e}_2$.

1.2.6 Relationship between b_i's and $\{\mathbf{e}, \phi\}$.

Once the natural invariants of $\mathbf{Q}$ have been introduced it is straightforward to have the relationship bewteen the Rodrigues's parameters and the screw axis parameters. We calculate the natural invariants when $\mathbf{Q}$ is written as in eq. 1.30.

$$
\begin{aligned}
vect(\mathbf{Q}) = \frac{2}{\Delta} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}; \\
tr(\mathbf{Q}) = \frac{1}{\Delta}(3 - (b_1^2 + b_2^2 + b_3^2)) = \frac{1}{\Delta}(4 - (1 + b_1^2 + b_2^2 + b_3^2)) = \frac{4 - \Delta}{\Delta}.
\end{aligned}
$$

We note that the first of the foregoing equations is nothing else but eq. 1.35. Then, according to eqs. 1.36 and 1.37 we can write:

$$1 + 2c_\phi = \frac{4-\Delta}{\Delta}; \quad \frac{2}{\Delta}\mathbf{b} = s_\phi \mathbf{e}. \tag{1.38}$$

If the following equations are used in eqs. 1.38:

$$s_\phi = \frac{2T}{1+T^2}, \quad c_\phi = \frac{1-T^2}{1+T^2} \quad \text{with} \quad T = \tan(\phi/2)$$

the relationship searched is eventually found:

$$\mathbf{b} = \tan(\phi/2)\mathbf{e} \tag{1.39}$$

1.2.7 Euler's angles representation

So far, only the Cayley's representation via Rodrigues parameters is minimal. That is, it uses 3 parameters to express a rotation which is a motion with only 3 degrees of freedom. Instead, both the Screw and Cayley-Hamilton representations use 4 parameters and direction cosine representation uses even 9 parameters. Euler's angles representation is another miniminal parametrization of the rotation matrix as it uses 3 successive rotations about coordinate axes to describe the orientation of a rigid body.
There are innumerable Euler's representations, we will deal with two of them:

1. Roll-Pitch-Yaw
   With the reference to Figure 1.11, consider a fixed reference frame

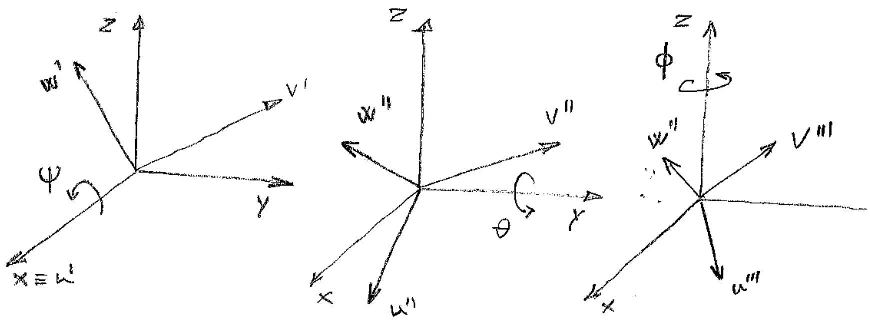

Figure 1.11: Successive rotations about the fixed coordinate axes (RPY).

   $A : \{Oxyz\}$ and a moving frame $B : \{Ouvw\}$. A, B initially coincide. First, rotate B about x-axis by ψ resulting in the $B' : \{Ou'v'w'\}$ system. Then it follows a second rotation of θ about y-axis carrying B' to $B'' : \{Ou''v''w''\}$. Finally a third rotation of ϕ follows about

z-axis carrying B'' to $B''' : \{Ou'''v'''w'''\}$.
Since all the rotations take place about coordinate axes of the fixed frame A the resulting rotation matrix is obtained by pre-multiplying three basic matrices:

$$^A\mathbf{R}_B = \mathbf{R}_z(\phi)\mathbf{R}_y(\theta)\mathbf{R}_x(\psi)\mathbf{I} = \begin{pmatrix} c_\phi c_\theta & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ s_\phi c_\theta & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{pmatrix} \quad (1.40)$$

If ψ, θ, ϕ are given from eq. 1.40 then $^A\mathbf{R}_B$ can be obtained. On the other hand, given $^A\mathbf{R}_B$ whose term are called r_{ij}, we may calculate the roll-pitch-yaw angles as:

$$\theta = asin(-r_{31}); \quad \psi = atan2(\frac{r_{32}}{c_\theta}, \frac{r_{33}}{c_\theta}); \quad \phi = atan2(\frac{r_{21}}{c_\theta}, \frac{r_{11}}{c_\theta}); \quad (1.41)$$

We observe that there are 2 solutions for θ from the first of the eqs. 1.41. Thus there are 2 set of three angles that lead to the same rotation matrix: $\{\theta_1, \psi_1, \phi_1\}$ and $\{\theta_2, \psi_2, \phi_2\}$ with $\theta_2 = \pi - \theta_1$. Furthermore, no Euler's angles can be calculated when $c_\theta = 0$, that is $\theta = (2k+1)\frac{\pi}{2}$, $(k = 0, \cdots, n)$. The rotation about the x-axis, ψ, is a *roll*, the rotation about the y-axis, θ, is a *pitch* and the rotation about the z-axis, ϕ, is a *raw*.

2. <u>w-v-w</u>
Consider a fixed reference frame $A : \{Oxyz\}$ and a moving frame $B : \{Ouvw\}$. A, B initially coincide. First, rotate B about w-axis by ϕ followed by a second rotation of θ about the rotated v-axis. Finally, a third rotation of ψ about the rotated w-axis.
Since all the rotations take place about coordinate axes of the mobile frame B the resulting rotation matrix is obtained by post-multiplying three basic matrices:

$$^A\mathbf{R}_B = \mathbf{I}\mathbf{R}_w(\phi)\mathbf{R}_v(\theta)\mathbf{R}_w(\psi) = \begin{pmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{pmatrix} \quad (1.42)$$

If ψ, θ, ϕ are given from eq. 1.42 then $^A\mathbf{R}_B$ can be obtained. On the other hand, given $^A\mathbf{R}_B$ whose term are called r_{ij}, we may calculate the Euler's angles as:

$$\theta = acos(r_{33}); \quad \phi = atan2(\frac{r_{23}}{s_\theta}, \frac{r_{13}}{s_\theta}); \quad \psi = atan2(\frac{r_{32}}{s_\theta}, -\frac{r_{31}}{s_\theta}); \qquad (1.43)$$

We observe that there are 2 solutions for θ from the first of the eqs. 1.43. Thus there are two sets of three angles that lead to the same rotation matrix: $\{\theta_1, \psi_1, \phi_1\}$ and $\{\theta_2, \psi_2, \phi_2\}$ with $\theta_2 = 2\pi - \theta_1$. Furthermore, no Euler's angles can be calculated when $s_\theta = 0$, that is $\theta = k\pi$, $(k = 0, \cdots, n)$.

1.2.8 Sequence of Rotations

Consider a fixed ref. system A and moving ref. system B that initially coincide. Thus, in this case ${}^A\mathbf{Q}_B = \mathbf{I}$ where $\mathbf{I}$ is the identity matrix. Now take into consideration three cases:

. First rotate along with an axis of the fixed ref. system A and then again rotate along with another axis of the fixed ref. system. Call the rotation matrices $\mathbf{R}_1$ and $\mathbf{R}_2$, respectively. The rotation matrix that represents the total rotation is: ${}^A\mathbf{Q}_B = \mathbf{R}_2\mathbf{R}_1\mathbf{I}$. Note that each rotation made along with an axis of the fixed ref. system leads to a pre-multiplication of the rotation matrix with respect to the current matrix. Indeed, first $\mathbf{R}_1$ pre-multiplies $\mathbf{I}$ then $\mathbf{R}_2$ pre-multiplies $(\mathbf{R}_1\mathbf{I})$.

. First rotate along with an axis of the fixed ref. sysyem A and then rotate along with an axis of the moving ref. system B. Call the rotation matrices $\mathbf{R}_1$ and $\mathbf{R}_2$, respectively. The rotation matrix that represents the total rotation is: ${}^A\mathbf{Q}_B = \mathbf{R}_1\mathbf{I}\mathbf{R}_2$. Note that a rotation made along with an axis of the moving ref. system leads to a post-multiplication of the rotation matrix with respect to the current matrix. Indeed, first $\mathbf{R}_1$ pre-multiplies $\mathbf{I}$ then $\mathbf{R}_2$ post-multiplies $(\mathbf{R}_1\mathbf{I})$.

. First rotate along with an axis of the moving ref. system B and then again rotate along with an axis of the moving ref. system B. Call the rotation matrices $\mathbf{R}_1$ and $\mathbf{R}_2$, respectively. The rotation matrix that represents the total rotation is: ${}^A\mathbf{Q}_B = \mathbf{I}\mathbf{R}_1\mathbf{R}_2$. Note that each rotation made along with an axis of the moving ref. system leads to a post-multiplication of the rotation matrix with respect to the current matrix. Indeed, first $\mathbf{R}_1$ post-multiplies $\mathbf{I}$ then $\mathbf{R}_2$ post-multiplies $(\mathbf{I}\mathbf{R}_1)$.

Matrices in the cases 2, 3 represent the same rotations (A and B coincide initally). Case 1 is different from them as the matrix multiplication is not commutative.

1.3 An application: the universal (Cardano's) joint

The universal joint is used to connect two rotating shafts whose axes form an arbitrary angle and intersect each other in a point. A schematic geometry of the joint is shown in Figure 1.12 where the universal joint connects the shafts 1 and 2 which form an angle α. An intermediate cross shaft allows for the motion transmission between 1 and 2. The intermediate cross shaft is connected to the shafts 1 and 2 by revolute joints. Since two intersecting axes form a plane, we may draw the joint in that plane (the joint's plane) as in Figure 1.13. Therefore, with reference to Figure 1.13, we define two

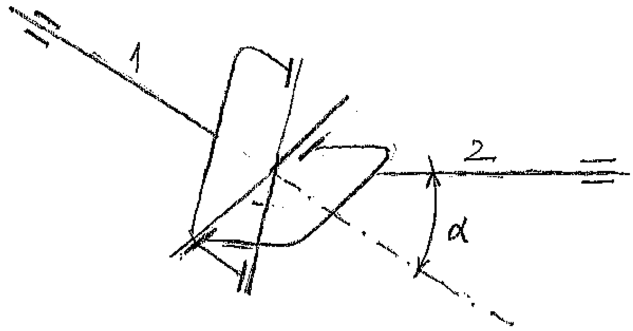

Figure 1.12: Geometry of the universal joint.

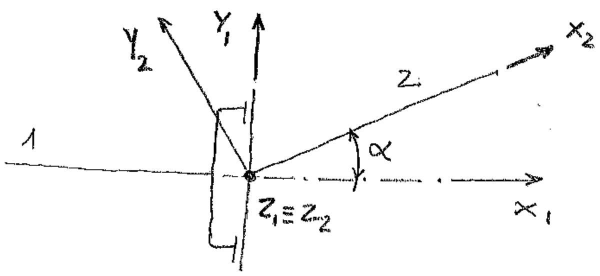

Figure 1.13: The joint on its plane with the fixed ref. systems {1}, {2}.

fixed ref. system. Ref. system {1} has $\mathbf{x}_1$ aligned with the axis of the shaft 1 whereas ref. system {2} has $\mathbf{x}_2$ aligned with the axis of the shaft 2. Both the ref. systems have $\mathbf{z}_1$ and $\mathbf{z}_2$ coincident with the normal to the joint's plane and are centred at the axes intersecting point. The orientation of {2} with respect to {1} is the orientation marix ${}^1\mathbf{R}_2$:

$$
{}^1\mathbf{R}_2 = \mathbf{R}_z(\alpha) = \begin{pmatrix} c_\alpha & -s_\alpha & 0 \\ s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix};
$$

${}^1\mathbf{R}_2$ is constant under the shafts rotations. Furthermore, define two ref. systems which are being in rotation with the shafts 1 and 2. Ref. system $\{m_1\}$ coincides with ref. system {1} except that it is rotating with the shaft 1. Similarly, ref. system $\{m_2\}$ coincides with ref. system {2} except that it is rotating with the shaft 2. Figure 1.14 can help to understand the moving ref. systems. The orientation of $\{m_1\}$ with respect to {1} is the orientation

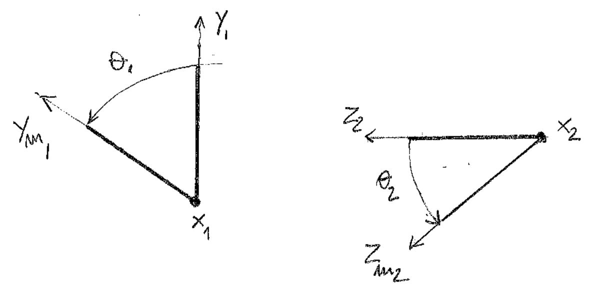

Figure 1.14: The moving ref. systems $\{m_1\}$, $\{m_2\}$.

marix ${}^1\mathbf{R}_{m_1}$:

$$
{}^1\mathbf{R}_{m_1} = \mathbf{R}_x(\theta_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{\theta_1} & -s_{\theta_1} \\ 0 & s_{\theta_1} & c_{\theta_1} \end{pmatrix};
$$

Identically, the orientation of $\{m_2\}$ with respect to $\{2\}$ is the orientation marix ${}^2\mathbf{R}_{m_2}$:

$$
{}^2\mathbf{R}_{m_2} = \mathbf{R}_x(\theta_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{\theta_2} & -s_{\theta_2} \\ 0 & s_{\theta_2} & c_{\theta_2} \end{pmatrix}.
$$

θ_i, $(i = 1, 2)$ represents the rotation of the shaft i. The relationship between θ_1 and θ_2 can be obtained by considering that the legs of the intermediate cross shaft remain perpendicular under the motion. This condition may be expressed as:

$$
({}^1\mathbf{y}_{m_1})^T({}^1\mathbf{z}_{m_2}) = 0. \tag{1.44}
$$

In eq. 1.44 we decided to express both the unit vectors into the ref. system $\{1\}$. Therefore:

$$
\begin{aligned}
{}^1\mathbf{y}_{m_1} &= {}^1\mathbf{R}_{m_1}{}^{m_1}\mathbf{y}_{m_1} = \begin{pmatrix} 0 & c_{\theta_1} & s_{\theta_1} \end{pmatrix}^T, \\
{}^1\mathbf{z}_{m_2} &= {}^1\mathbf{R}_2{}^2\mathbf{R}_{m_2}{}^{m_2}\mathbf{z}_{m_2} = \begin{pmatrix} s_\alpha s_{\theta_2} & -c_\alpha s_{\theta_2} & c_{\theta_2} \end{pmatrix}^T;
\end{aligned} \tag{1.45}
$$

with ${}^{m_1}\mathbf{y}_{m_1} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T$, ${}^{m_2}\mathbf{z}_{m_2} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$. Results form eqs. 1.45 are put into eq. 1.44 to obtain:

$$
s_{\theta_1}c_{\theta_2} - c_{\theta_1}s_{\theta_2}c_\alpha = 0 \tag{1.46}
$$

Eventually, the relationship searched is obtained by dividing eq.1.46 by $c_{\theta_1}c_{\theta_2} \neq 0$:

$$
\tan(\theta_1) = \tan(\theta_2)c_\alpha \tag{1.47}
$$

Now, we would like to prove that the universal joint is not a constant-velocity shaft coupling. To this end differentiate with the respect to the time eq. 1.47:

$$[1+\tan^2(\theta_1)]\dot{\theta}_1 = c_\alpha[1+\tan^2(\theta_2)]\dot{\theta}_2$$

and by simple mathematical manipulations we obtain:

$$\frac{\dot{\theta}_1}{\dot{\theta}_2} = \frac{c_\alpha^2 + \frac{s_{\theta_1}^2}{c_{\theta_1}^2}}{c_\alpha(1+\frac{s_{\theta_1}^2}{c_{\theta_1}^2})} \tag{1.48}$$

where in eq. 1.48 we use:

$$1+\tan^2(\theta_2) = 1+\frac{\tan^2(\theta_1)}{c_\alpha^2},$$

and finally from eq. 1.48 we obtain:

$$\frac{\dot{\theta}_1}{\dot{\theta}_2} = \frac{c_{\theta_1}^2 c_\alpha^2 + s_{\theta_1}^2}{c_\alpha} = \frac{c_{\theta_1}^2(c_\alpha^2-1)+1}{c_\alpha} = \frac{1-s_\alpha^2 c_{\theta_1}^2}{c_\alpha} \tag{1.49}$$

From eq. 1.49 it is apparent that $\dot{\theta}_1/\dot{\theta}_2$ is not constant. We have:

$$(\frac{\dot{\theta}_1}{\dot{\theta}_2})_{|max} = \frac{1}{c_\alpha} \quad \text{when} \quad \theta_1 = \frac{\pi}{2};$$
$$(\frac{\dot{\theta}_1}{\dot{\theta}_2})_{|min} = c_\alpha \quad \text{when} \quad \theta_1 = 0.$$

In order to have a constant-velocity shaft coupling we need to double the universal joint as shown in Figure 1.15. The proof is straighforward once

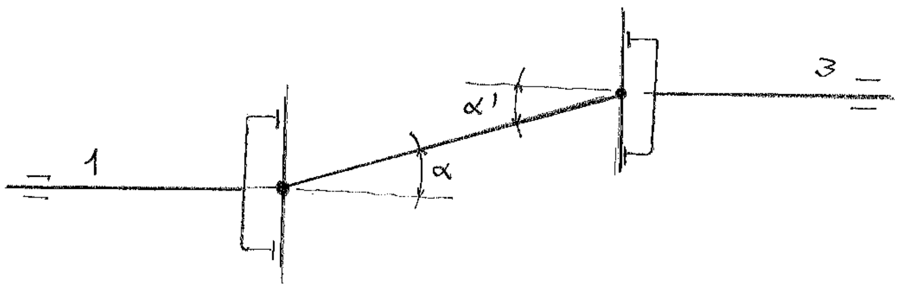

Figure 1.15: A constant-velocity shaft coupling.

the porcedure followed for the single universal joint is repeated twice. The double universal joint is a constant-velocity shaft coupling when $\alpha = \alpha'$ and the shafts are assembled correctly as shown in Figure 1.15, for example.

1.4 Orthogonal projections and reflections

So far, we dealt with rigid-body rotations, that are linear transformations of $\mathbb{R}^3$. In this context, we need to know two other linear transformations: orthogonal projection and reflection.

1.4.1 Orthogonal projections

First, a formal definition of the *orthogonal projection* $\mathbf{P}$ is given:
$\mathbf{P}$ is a linear tranformation of $\mathbb{R}^3$ onto a plane Π passing through the origin and having unit normal $\mathbf{n}$ with the properties:

$$\mathbf{P} = \mathbf{P}^2, \quad \mathbf{P}\mathbf{n} = \mathbf{0} \tag{1.50}$$

Matrix $\mathbf{P}$ can be easily obtained by considering the orthogonal projection $\mathbf{p}'$ of a position vector $\mathbf{p}$ onto Π as shown in Figure 1.16:

$$\mathbf{p}' = \mathbf{p} - \mathbf{n}(\mathbf{n}^T\mathbf{p}) = \mathbf{P}\mathbf{p} \quad : \quad \mathbf{P} = (\mathbf{1} - \mathbf{n}\mathbf{n}^T). \tag{1.51}$$

Thus, eqs. 1.50 can easily be proved by considering that $\mathbf{n}^T\mathbf{n} = 1$:

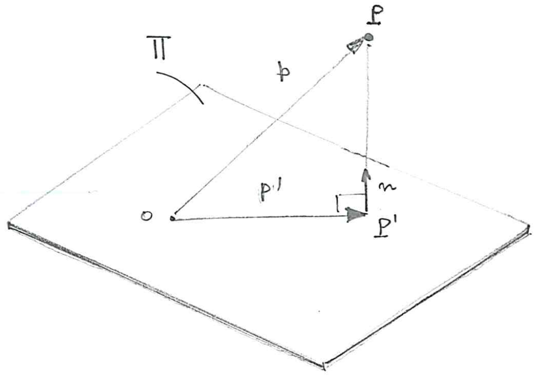

Figure 1.16: Orthogonal projection of $\mathbf{p}$ onto Π.

$$\begin{aligned} \mathbf{P}^2 &= (\mathbf{1} - \mathbf{n}\mathbf{n}^T)(\mathbf{1} - \mathbf{n}\mathbf{n}^T) = \\ &= \mathbf{1} - 2\mathbf{n}\mathbf{n}^T + \mathbf{n}(\mathbf{n}^T\mathbf{n})\mathbf{n}^T = \mathbf{1} - \mathbf{n}\mathbf{n}^T, \end{aligned}$$

and

$$\mathbf{P}\mathbf{n} = (\mathbf{1} - \mathbf{n}\mathbf{n}^T)\mathbf{n} = \mathbf{n} - (\mathbf{n}\mathbf{n}^T)\mathbf{n} = \mathbf{0}.$$

1.4.2 Reflections

A *reflection* $\mathbf{T}$ is a linear tranformation of $\mathbb{R}^3$ onto a plane Π passing through the origin and having unit normal $\mathbf{n}$ such that a vector $\mathbf{p}$ is mapped by $\mathbf{T}$

into a vector $\mathbf{p}'$ given by:

$$\mathbf{p}' = \mathbf{p} - 2\mathbf{n}\mathbf{n}^T\mathbf{p} = \mathbf{T}\mathbf{p} \quad : \quad \mathbf{T} = (\mathbf{1} - 2\mathbf{n}\mathbf{n}^T). \tag{1.52}$$

A reflection is shown in Figure 1.17. The properties of $\mathbf{T}$ are $\mathbf{T}^2 = \mathbf{1}$ and

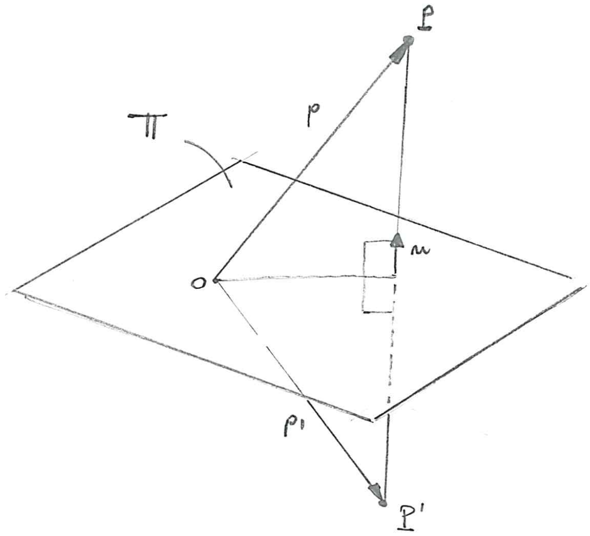

Figure 1.17: Reflection of $\mathbf{p}$ onto Π.

$\mathbf{T}^{-1} = \mathbf{T}$. They can promptly be proved as follows:

$$\mathbf{T}^2 = (\mathbf{1} - 2\mathbf{n}\mathbf{n}^T)(\mathbf{1} - 2\mathbf{n}\mathbf{n}^T) = \mathbf{1} - 4\mathbf{n}\mathbf{n}^T + 4\mathbf{n}(\mathbf{n}^T\mathbf{n})\mathbf{n}^T = \mathbf{1},$$

and

$$\mathbf{T}^2 = \mathbf{T}\mathbf{T} = \mathbf{1} \quad : \quad (\mathbf{T}^{-1}\mathbf{T})\mathbf{T} = \mathbf{T}^{-1}\mathbf{1} \quad : \quad \mathbf{T} = \mathbf{T}^{-1}.$$

1.5 Description of a Location (coordinate transformation)

As we said the most general motion of a rigid body in $\mathbb{R}^3$ is elicoidal. Indeed,

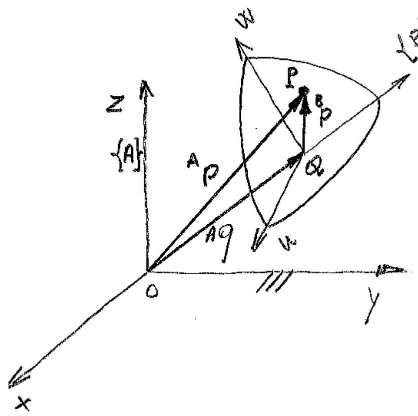

Figure 1.18: Location of a rigid body.

the rigid body may translate along an axis and at the same time may rotate along with the same axis. With reference to the Figure 1.18 position of point P can be axpressed as:

$$^A\mathbf{p} = {}^A\mathbf{q} +^A \mathbf{Q}_B{}^B\mathbf{p} \tag{1.53}$$

$^A\mathbf{q}$ is the position of the origin of the ref. system B moving together with the rigid body. This position of P is *measured* by A, that is $^A\mathbf{p} \equiv \overrightarrow{OP}$ and *expressed* in A as $^A\mathbf{p}$. $^B\mathbf{p}$ is the position of the point P *measured* by B, that is $^B\mathbf{p} \equiv \overrightarrow{QP}$ and *expressed* in B. $^A\mathbf{Q}_B{}^B\mathbf{p}$ represents the position of P *measured* by B but *expressed* in A. Thus, all vectors in eq. 1.53 are expressed in A. According to the *Active Interpretation* given previously, eq. 1.53 can represent the general and arbitrary motion of a rigid body due to rotation about a such axis and a translation along the direction of $^A\mathbf{q}$.
In order to obtain $^B\mathbf{p}$, eq. 1.53 can be written as:

$$^B\mathbf{p} = {}^A\mathbf{Q}_B{}^T({}^A\mathbf{p} - {}^A\mathbf{q}) \tag{1.54}$$

Eq. 1.53, that is the general coordinate transformation, is not a *Linear transformation* because of the orgin shift term which is independent on $^A\mathbf{p}$. It is a *Affine transformation*, indeed. In sum, eq. 1.53 can be interpreted as: *a)* a general coordinates tranformation according to the *passive interpretation*, *b)* an elicoidal motion, according to the *active interpretation*.
Nevertheless, this transformation can be represented in homogeneous form

if the *homogeneous coordinates* are introduced such that:

$$^{A}\tilde{\mathbf{p}} = {}^{A}\mathbf{T}_{B}{}^{B}\tilde{\mathbf{p}} \tag{1.55}$$

with $^{A}\tilde{\mathbf{p}} = \begin{pmatrix} \rho p_x & \rho p_y & \rho p_z & \rho \end{pmatrix}^T$ where ρ is a nonzero scalar factor. Now, if we set $\rho = 1$ then the first three homogeneous coordinates represent the actual coordinates of a vector in $\mathbb{R}^3$ and the (4×4) homogeneous transformation matrix $^{A}\mathbf{T}_{B}$ takes the form:

$$^{A}\mathbf{T}_{B} = \begin{pmatrix} ^{A}\mathbf{Q}_{B} & ^{A}\mathbf{q} \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix} \tag{1.56}$$

$^{A}\mathbf{T}_{B}$ is not a orthogonal matrix thus $^{A}\mathbf{T}_{B}^{T} \neq {}^{A}\mathbf{T}_{B}^{-1}$. $^{B}\tilde{\mathbf{p}}$ can be obtained as follows:

$$^{B}\mathbf{p} = {}^{A}\mathbf{Q}_{B}{}^{T}({}^{A}\mathbf{p} - {}^{A}\mathbf{q}) = {}^{A}\mathbf{Q}_{B}{}^{T\,A}\mathbf{p} - {}^{A}\mathbf{Q}_{B}{}^{T\,A}\mathbf{q} \tag{1.57}$$

Eq. 1.57 can be written in compact form:

$$^{B}\tilde{\mathbf{p}} = \begin{pmatrix} ^{B}\mathbf{p} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{H} & \mathbf{k} \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix} \begin{pmatrix} ^{A}\mathbf{p} \\ 1 \end{pmatrix} : \quad {}^{B}\mathbf{p} = \mathbf{H}{}^{A}\mathbf{p} + \mathbf{k} \tag{1.58}$$

By comparing eqs. 1.57 and. 1.58 we obtain $^{B}\tilde{\mathbf{p}}$.

$$\mathbf{H} = {}^{A}\mathbf{Q}_{B}{}^{T}; \quad \mathbf{k} = -{}^{A}\mathbf{Q}_{B}{}^{T\,A}\mathbf{q}.$$

and thus

$$^{B}\tilde{\mathbf{p}} = \begin{pmatrix} ^{A}\mathbf{Q}_{B}{}^{T} & -{}^{A}\mathbf{Q}_{B}{}^{T\,A}\mathbf{q} \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix} {}^{A}\tilde{\mathbf{p}} = {}^{B}\mathbf{T}_{A}{}^{A}\tilde{\mathbf{p}} \tag{1.59}$$

The homogeneous transformation matrix $^{A}\mathbf{T}_{B}$ takes a simplified form whenever the ref. system B either only translates or only rotates.

- B translates.

  $$^{A}\mathbf{T}_{B} = \begin{pmatrix} \mathbf{I} & \mathbf{s} \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix}$$

  with $\mathbf{s}$ representing the translation of the ref. system B.

- B rotates.

  $$^{A}\mathbf{T}_{B} = \begin{pmatrix} \mathbf{R}(\mathbf{e}, \phi) & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix}$$

  with $\mathbf{R}(\mathbf{e}, \phi)$ representing the rotation of B about $\mathbf{e}$ of ϕ.

1.5.1 Sequence of Roto-translations

1. Consider a fixed ref. system A and moving ref. system B that initially coincide.

   - ${}^A\mathbf{T}_B = \mathbf{T}_R\mathbf{T}_s\mathbf{I}$
     First consider a translation $\mathbf{s}$ of B expressed in the A ref. system followed by a rotation ϕ about an axis $\mathbf{e}$ expressed in the ref. system A. Since the transformations are about/along axes of the fixed ref. system we need to perform two successive pre-multiplications. The final location of the ref. system B is:

$$
\begin{aligned}
{}^A\mathbf{T}_B = \mathbf{T}_R\mathbf{T}_s\mathbf{I} &= \begin{pmatrix} \mathbf{R}(\mathbf{e},\phi) & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{s} \\ \mathbf{0}_{1\times3} & 1 \end{pmatrix} = \\
&= \begin{pmatrix} \mathbf{R}(\mathbf{e},\phi) & \mathbf{R}(\mathbf{e},\phi)\mathbf{s} \\ \mathbf{0}_{1\times3} & 1 \end{pmatrix}
\end{aligned} \tag{1.60}
$$

   - ${}^A\mathbf{T}_B = \mathbf{I}\mathbf{T}_R\mathbf{T}_s$
     First consider a rotation ϕ about an axis $\mathbf{e}$ expressed in the ref. system B followed by a translation $\mathbf{s}$ expressed in the rotated B ref. system. Since the transformations are about/along axes of the moving ref. system we need to perform two successive post-multiplications. The final location of the ref. system B is represented by a matrix identical to that in eq. 1.60. That means that the same location has been reached.

   Similar considerations can be drawn by comparing ${}^A\mathbf{T}_B = \mathbf{T}_s\mathbf{T}_R\mathbf{I} = \mathbf{I}\mathbf{T}_s\mathbf{T}_R$ which locate the ref. system B identically.

   - $\mathbf{T}_R\mathbf{I}\mathbf{T}_s \neq \mathbf{T}_s\mathbf{I}\mathbf{T}_R$
     First consider an opportune composition of a rotation and a translation to obtain $\mathbf{T}_R\mathbf{I}\mathbf{T}_s$. For example, consider a rotation ϕ about an axis $\mathbf{e}$ expressed in the ref. system A followed by a translation $\mathbf{s}$ expressed in the rotated B ref. system.
     Then, consider an opportune composition of a rotation and a translation to obtain $\mathbf{T}_s\mathbf{I}\mathbf{T}_R$. For example, consider a translation $\mathbf{s}$ expressed in the ref. system A followed by a rotation ϕ about an axis $\mathbf{e}$ expressed in the translated B ref. system.

$$
\mathbf{T}_R\mathbf{I}\mathbf{T}_s = \begin{pmatrix} \mathbf{R}(\mathbf{e},\phi) & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{s} \\ \mathbf{0}_{1\times3} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{R}(\mathbf{e},\phi) & \mathbf{R}(\mathbf{e},\phi)\mathbf{s} \\ \mathbf{0}_{1\times3} & 1 \end{pmatrix};
$$

$$
\mathbf{T}_s\mathbf{I}\mathbf{T}_R = \begin{pmatrix} \mathbf{I} & \mathbf{s} \\ \mathbf{0}_{1\times3} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R}(\mathbf{e},\phi) & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{R}(\mathbf{e},\phi) & \mathbf{s} \\ \mathbf{0}_{1\times3} & 1 \end{pmatrix}.
$$

2. Consider a fixed ref. system A and moving ref. system B that initially do not coincide. The location of B with to A is represented by ${}^A\mathbf{T}_B$:

$$ {}^A\mathbf{T}_B = \begin{pmatrix} {}^A\mathbf{R}_B & {}^A\mathbf{p}_o \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix}; $$

- ${}^A\mathbf{T}_{B''} = \mathbf{T}_s \mathbf{T}_R {}^A\mathbf{T}_B$

$$ {}^A\mathbf{T}_{B''} = \begin{pmatrix} \mathbf{I} & \mathbf{s} \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R} & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix} \begin{pmatrix} {}^A\mathbf{R}_B & {}^A\mathbf{p}_o \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix} = $$
$$ = \begin{pmatrix} \mathbf{R}{}^A\mathbf{R}_B & \mathbf{R}{}^A\mathbf{p}_o + \mathbf{s} \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix} $$

- ${}^A\mathbf{T}_{B''} = {}^A\mathbf{T}_B \mathbf{T}_s \mathbf{T}_R$

$$ {}^A\mathbf{T}_{B''} = \begin{pmatrix} {}^A\mathbf{R}_B & {}^A\mathbf{p}_o \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{s} \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R} & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix} = $$
$$ = \begin{pmatrix} {}^A\mathbf{R}_B\mathbf{R} & {}^A\mathbf{R}_B\mathbf{s} + {}^A\mathbf{p}_o \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix} $$

- ${}^A\mathbf{T}_{B''} = \mathbf{T}_R \mathbf{T}_s {}^A\mathbf{T}_B$

$$ {}^A\mathbf{T}_{B''} = \begin{pmatrix} \mathbf{R} & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{s} \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix} \begin{pmatrix} {}^A\mathbf{R}_B & {}^A\mathbf{p}_o \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix} = $$
$$ = \begin{pmatrix} \mathbf{R}{}^A\mathbf{R}_B & \mathbf{R}(\mathbf{s} + {}^A\mathbf{p}_o) \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix} $$

- ${}^A\mathbf{T}_{B''} = {}^A\mathbf{T}_B \mathbf{T}_R \mathbf{T}_s$

$$ {}^A\mathbf{T}_{B''} = \begin{pmatrix} {}^A\mathbf{R}_B & {}^A\mathbf{p}_o \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R} & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{s} \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix} = $$
$$ = \begin{pmatrix} {}^A\mathbf{R}_B\mathbf{R} & {}^A\mathbf{R}_B\mathbf{R}\mathbf{s} + {}^A\mathbf{p}_o \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix} $$

2

Kinematics of Rigid-Bodies

2.1 Angular velocity of a rigid-body

Figure 2.1 shows a rigid body $\mathcal{B}$ which is rotating with respect to a ref. system A about a fixed point O. Consider a moving ref. system B attached to the rigid body. At any istant, the orientation of the frame B with respect

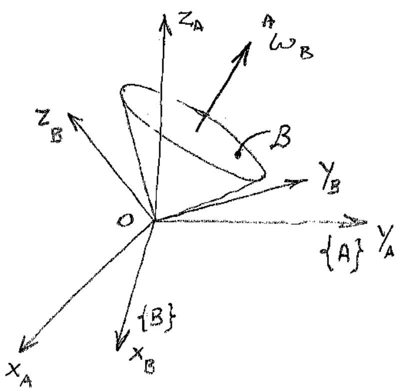

Figure 2.1: Instataneous rotation of B with respect to A.

to frame A is described by an orientation matrix ${}^A\mathbf{Q}_B$. As we know, because of the orthogonality, we have:

$$ {}^A\mathbf{Q}_B {}^A\mathbf{Q}_B^T = \mathbf{I} \tag{2.1} $$

Taking the derivative of eq. 2.1 with respect to time, we obtain:

$$ {}^A\dot{\mathbf{Q}}_B {}^A\mathbf{Q}_B^T + {}^A\mathbf{Q}_B {}^A\dot{\mathbf{Q}}_B^T = \mathbf{0} \tag{2.2} $$

In eq. 2.2 ${}^A\mathbf{Q}_B {}^A\dot{\mathbf{Q}}_B^T = ({}^A\dot{\mathbf{Q}}_B {}^A\mathbf{Q}_B^T)^T$ because of the rule in linear algebra $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T \quad \forall \mathbf{A}, \mathbf{B}$. Thus eq. 2.2 becomes:

$$ {}^A\dot{\mathbf{Q}}_B {}^A\mathbf{Q}_B^T + ({}^A\dot{\mathbf{Q}}_B {}^A\mathbf{Q}_B^T)^T = \mathbf{0} \tag{2.3} $$

Further, according to orthogonality, we have ${}^A\mathbf{Q}_B^T = {}^A\mathbf{Q}_B^{-1}$ such that

$$ {}^A\dot{\mathbf{Q}}_B {}^A\mathbf{Q}_B^{-1} + ({}^A\dot{\mathbf{Q}}_B {}^A\mathbf{Q}_B^{-1})^T = \mathbf{0} \tag{2.4}$$

Eq. 2.4 defines a 3×3 skew-symmetric matrix $\mathbf{\Omega}$:

$$\mathbf{\Omega} = {}^A\dot{\mathbf{Q}}_B {}^A\mathbf{Q}_B^{-1} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \tag{2.5}$$

$\mathbf{\Omega}$ is the *Angular Velocity matrix* of the ref. system B and of the rigid body attached to it. ω_x, ω_y, ω_z are three independent parameters specifying the angular velocity of a rigid body and represent the components in A of the angular velocity vector of the rigid body $\mathcal{B}$.

2.1.1 Instantaneous screw-axis

Consider a point P of the rigid body $\mathcal{B}$ and thus moving together with the ref. system B. Taking the derivative with respect to time of the position equation ${}^A\mathbf{p} = {}^A\mathbf{Q}_B {}^B\mathbf{p}$ we obtain the linear velocity of P:

$$ {}^A\mathbf{v}_P = {}^A\dot{\mathbf{Q}}_B {}^B\mathbf{p} \tag{2.6}$$

since the magnitude of ${}^B\mathbf{p}$ does not change. Eq. 2.6 can be written as:

$$ {}^A\mathbf{v}_P = ({}^A\dot{\mathbf{Q}}_B {}^A\mathbf{Q}_B^{-1}) {}^A\mathbf{p} = \mathbf{\Omega} {}^A\mathbf{p}. \tag{2.7}$$

Incidentally, we notice that eqs. 2.6 and 2.7 can be interpreted as the *Lagrangian* and the *Eulerian* descriptions of the point velocity in the context of the active interpretation of the position.
Now, we look for a point $\tilde{P}$ such that its instantaneous velocity is zero. Then:

$$ {}^A\mathbf{v}_{\tilde{P}} = \mathbf{\Omega} {}^A\tilde{\mathbf{p}} = \mathbf{0} \tag{2.8}$$

Eq. 2.8 is a system of three linear equations in three unknowns $\tilde{p}_x$, $\tilde{p}_y$, $\tilde{p}_z$. To have non trivial solutions we need that $det(\mathbf{\Omega}) = 0$. That is always true since $\mathbf{\Omega}$ is a skew-symmetric matrix. Thus, only two of the three equations are independent and we may obtain the ratios as:

$$\frac{\tilde{p}_x}{\tilde{p}_y} = \frac{\omega_x}{\omega_y}, \quad \frac{\tilde{p}_y}{\tilde{p}_z} = \frac{\omega_y}{\omega_z}, \quad \frac{\tilde{p}_z}{\tilde{p}_x} = \frac{\omega_z}{\omega_x}. \tag{2.9}$$

We can conclude that there are infinite stazionary points (${}^A\mathbf{v}_{\tilde{P}} = \mathbf{0}$) and they lie on a line passing through the origin and parallel to a vector ${}^A\boldsymbol{\omega}_B = \begin{pmatrix} \omega_x & \omega_y & \omega_z \end{pmatrix}^T$. We call the vector ${}^A\boldsymbol{\omega}_B$ the *angular velocity vector* and the line (whose $\tilde{P}$ is a point) the *instantaneous screw*. Eq. 2.7 can be written in vector notation as:

$$ {}^A\mathbf{v}_P = {}^A\boldsymbol{\omega}_B \times {}^A\mathbf{p} \tag{2.10}$$

2.1.2 Linear velocity and acceleration of a point

Figure 2.2 shows a rigid body $\mathcal{B}$ which is moving by an elicoidal motion with respect to ref. system A together with the moving ref. system B. Then,

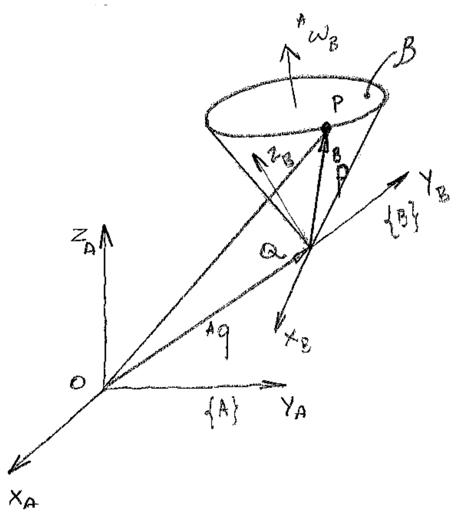

Figure 2.2: Instataneous motion of a rigid body $\mathcal{B}$ with respect to A.

consider a point P which is not necessarily fixed in B. Position of P can be written as:

$$^A\mathbf{p} =^A \mathbf{q} +^A \mathbf{Q}_B{}^B\mathbf{p} \tag{2.11}$$

Taking the derivative of eq. 2.11 with respect to time we have:

$$^A\mathbf{v}_P =^A \mathbf{v}_Q +^A \dot{\mathbf{Q}}_B{}^B\mathbf{p} +^A \mathbf{Q}_B{}^B\mathbf{v}_P \tag{2.12}$$

$^B\mathbf{v}_P$ denotes the velocity of P with respect to B (namely, the velocity measured by the ref. system B). According to eq. 2.5 the time derivative of the rotation matrix can be written as $^A\dot{\mathbf{Q}}_B = \mathbf{\Omega}^A\mathbf{Q}_B$ such that eq. 2.12 takes the following form:

$$^A\mathbf{v}_P =^A \mathbf{v}_Q + \mathbf{\Omega}^A\mathbf{Q}_B{}^B\mathbf{p} +^A \mathbf{Q}_B{}^B\mathbf{v}_P \tag{2.13}$$

$$^A\mathbf{v}_P =^A \mathbf{v}_Q + \mathbf{\Omega}(^A\mathbf{p} - {}^A\mathbf{q}) +^A \mathbf{Q}_B{}^B\mathbf{v}_P \tag{2.14}$$

In eq. 2.14 the term $^A\mathbf{v}_Q + \mathbf{\Omega}(^A\mathbf{p} - {}^A\mathbf{q})$ represent the velocity of P when considered to be rigidly connected to B whereas the term $^A\mathbf{Q}_B{}^B\mathbf{v}_P$ is the relative velocity as mentioned. For the sake of completeness eq. 2.14 can also be written as:

$$^A\mathbf{v}_P = {}^A\mathbf{v}_Q +^A \boldsymbol{\omega}_B \times (^A\mathbf{p} - {}^A\mathbf{q}) +^A \mathbf{Q}_B{}^B\mathbf{v}_P \tag{2.15}$$

Now consider the case of ${}^B\mathbf{v}_P = \mathbf{0}$, that is P belongs to $\mathcal{B}$. In this case we have:

$$({}^A\mathbf{v}_P - {}^A\mathbf{v}_Q) = {}^A\boldsymbol{\omega}_B \times ({}^A\mathbf{p} - {}^A\mathbf{q}) \tag{2.16}$$

- Dot-multiplying eq. 2.16 by ${}^A\boldsymbol{\omega}_B$ we obtain

$$\begin{aligned}({}^A\mathbf{v}_P - {}^A\mathbf{v}_Q)\cdot {}^A\boldsymbol{\omega}_B &= {}^A\boldsymbol{\omega}_B \times ({}^A\mathbf{p} - {}^A\mathbf{q})\cdot {}^A\boldsymbol{\omega}_B;\\ ({}^A\mathbf{v}_P - {}^A\mathbf{v}_Q)\cdot {}^A\boldsymbol{\omega}_B &= -({}^A\mathbf{p} - {}^A\mathbf{q})\times {}^A\boldsymbol{\omega}_B \cdot {}^A\boldsymbol{\omega}_B;\\ ({}^A\mathbf{v}_P - {}^A\mathbf{v}_Q)\cdot {}^A\boldsymbol{\omega}_B &= -({}^A\mathbf{p} - {}^A\mathbf{q})\cdot {}^A\boldsymbol{\omega}_B \times {}^A\boldsymbol{\omega}_B = 0. \end{aligned} \tag{2.17}$$

  Eq. 2.17 states that all the points of a rigid body (P and Q are two of them arbirarly chosen) have equal velocities along the angular velocity vector.

- Dot-multiplying eq. 2.16 by $({}^A\mathbf{p} - {}^A\mathbf{q})$ we obtain

$$\begin{aligned}({}^A\mathbf{v}_P - {}^A\mathbf{v}_Q)\cdot({}^A\mathbf{p} - {}^A\mathbf{q}) &= {}^A\boldsymbol{\omega}_B \times ({}^A\mathbf{p} - {}^A\mathbf{q})\cdot({}^A\mathbf{p} - {}^A\mathbf{q});\\ ({}^A\mathbf{v}_P - {}^A\mathbf{v}_Q)\cdot({}^A\mathbf{p} - {}^A\mathbf{q}) &= {}^A\boldsymbol{\omega}_B \cdot ({}^A\mathbf{p} - {}^A\mathbf{q})\times({}^A\mathbf{p} - {}^A\mathbf{q}) = 0. \end{aligned} \tag{2.18}$$

  Eq. 2.18 states that the relative velocity between two points of the rigid body is perpendicular to the vector joining them.

Now, we take the derivative of eq. 2.13 with respect the time:

$$\begin{aligned}{}^A\mathbf{a}_P &= {}^A\mathbf{a}_Q + \dot{\boldsymbol{\Omega}}{}^A\mathbf{Q}_B{}^B\mathbf{p} + \boldsymbol{\Omega}{}^A\dot{\mathbf{Q}}_B{}^B\mathbf{p} + \boldsymbol{\Omega}{}^A\mathbf{Q}_B{}^B\mathbf{v}_P + {}^A\dot{\mathbf{Q}}_B{}^B\mathbf{v}_P + {}^A\mathbf{Q}_B{}^B\mathbf{a}_P;\\ {}^A\mathbf{a}_P &= {}^A\mathbf{a}_Q + \dot{\boldsymbol{\Omega}}{}^A\mathbf{Q}_B{}^B\mathbf{p} + \boldsymbol{\Omega}^{2A}\mathbf{Q}_B{}^B\mathbf{p} + \boldsymbol{\Omega}{}^A\mathbf{Q}_B{}^B\mathbf{v}_P + \boldsymbol{\Omega}{}^A\mathbf{Q}_B{}^B\mathbf{v}_P + {}^A\mathbf{Q}_B{}^B\mathbf{a}_P;\\ {}^A\mathbf{a}_P &= {}^A\mathbf{a}_Q + \dot{\boldsymbol{\Omega}}{}^A\mathbf{Q}_B{}^B\mathbf{p} + \boldsymbol{\Omega}^{2A}\mathbf{Q}_B{}^B\mathbf{p} + 2\boldsymbol{\Omega}{}^A\mathbf{Q}_B{}^B\mathbf{v}_P + {}^A\mathbf{Q}_B{}^B\mathbf{a}_P \end{aligned} \tag{2.19}$$

where:
${}^A\mathbf{a}_Q + \dot{\boldsymbol{\Omega}}{}^A\mathbf{Q}_B{}^B\mathbf{p} + \boldsymbol{\Omega}^{2A}\mathbf{Q}_B{}^B\mathbf{p}$ is the acceleration of P when considered to be rigidly connected to B, $2\boldsymbol{\Omega}{}^A\mathbf{Q}_B{}^B\mathbf{v}_P$ is the Coriolis acceleration, ${}^A\mathbf{Q}_B{}^B\mathbf{a}_P$ is the acceleration with respect to B. It is worth noting that all vectors in eq. 2.19 are expressed in the ref. system A. Now, as before, consider the case of P belonging to the rigid body $\mathcal{B}$. Eq. 2.19 takes the simplified form:

$${}^A\mathbf{a}_P = {}^A\mathbf{a}_Q + (\dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega}^2){}^A\mathbf{Q}_B{}^B\mathbf{p} = {}^A\mathbf{a}_Q + (\dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega}^2)({}^A\mathbf{p} - {}^A\mathbf{q}) \tag{2.20}$$

Matrix $(\dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega}^2) = \mathbf{W}$ is called *acceleration matrix*. $\mathbf{W}$ is composed by a symmetrix part $\boldsymbol{\Omega}^2$ and a skew-symmetric part $\dot{\boldsymbol{\Omega}}$. Thus,

$$\dot{\boldsymbol{\Omega}} = \begin{pmatrix} 0 & -\dot{\omega}_z & \dot{\omega}_y \\ \dot{\omega}_z & 0 & -\dot{\omega}_x \\ -\dot{\omega}_y & \dot{\omega}_x & 0 \end{pmatrix};$$

$$vect(\mathbf{W}) = vect(\dot{\boldsymbol{\Omega}}) = \frac{1}{2}\begin{pmatrix} \dot{\Omega}_{32} - \dot{\Omega}_{23} \\ \dot{\Omega}_{13} - \dot{\Omega}_{31} \\ \dot{\Omega}_{21} - \dot{\Omega}_{12} \end{pmatrix} = \begin{pmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{pmatrix}. \tag{2.21}$$

${}^A\dot{\boldsymbol{\omega}}_B = \begin{pmatrix} \dot{\omega}_x & \dot{\omega}_y & \dot{\omega}_z \end{pmatrix}^T$ is the *angular acceleration vector* of the rigid body.

2.1.3 Computation of angular velocity from point-velocity data

Consider three noncollinear points of a rigid body whose positions are detected by $\mathbf{p}_1,\ \mathbf{p}_2,\ \mathbf{p}_3$. The geometric centroid of the three points has a position vector $\mathbf{c} = 1/3\sum_{i=1}^{3}\mathbf{p}_i$. Likewise the velocities of the three points are denoted by $\dot{\mathbf{p}}_i$ and the velocity of the centroid by $\dot{\mathbf{c}} = 1/3\sum_{i=1}^{3}\dot{\mathbf{p}}_i$. From eq. 2.16 we have:

$$\dot{\mathbf{p}}_i = \dot{\mathbf{c}} + \mathbf{\Omega}(\mathbf{p}_i - \mathbf{c}), \quad i = 1, 2, 3$$

and then

$$\dot{\mathbf{p}}_i - \dot{\mathbf{c}} = \mathbf{\Omega}(\mathbf{p}_i - \mathbf{c}), \quad i = 1, 2, 3 \tag{2.22}$$

Now we define a 3×3 matrix $\mathbf{P}$ as:

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 - \mathbf{c} & \mathbf{p}_2 - \mathbf{c} & \mathbf{p}_3 - \mathbf{c} \end{pmatrix};$$

and by differentation with respect to time:

$$\dot{\mathbf{P}} = \begin{pmatrix} \dot{\mathbf{p}}_1 - \dot{\mathbf{c}} & \dot{\mathbf{p}}_2 - \dot{\mathbf{c}} & \dot{\mathbf{p}}_3 - \dot{\mathbf{c}} \end{pmatrix}$$

such that eq. 2.22 may be written in matrix form as:

$$\dot{\mathbf{P}} = \mathbf{\Omega}\mathbf{P} \tag{2.23}$$

We would like to obtain $\mathbf{\Omega}$, or equivalently $\boldsymbol{\omega}$, directly from eq. 2.23 but unfortunately this is not possible as $\mathbf{P}$ is singular. This can be proved by substituting $\mathbf{c}$ in terms of $\mathbf{p}_i$ and then calculating $det(\mathbf{P})$ which will result to be null. However we can still use eq. 2.23 by performing the vector of both sides of the equation:

$$vect(\dot{\mathbf{P}}) = vect(\mathbf{\Omega}\mathbf{P}). \tag{2.24}$$

where:

$$vect(\mathbf{\Omega}\mathbf{P}) = \frac{1}{2}[tr(\mathbf{P})\mathbf{I} - \mathbf{P}]vect(\mathbf{\Omega}) = \mathbf{D}\boldsymbol{\omega}. \tag{2.25}$$

with:

$$\mathbf{D} = \frac{1}{2}[tr(\mathbf{P})\mathbf{I} - \mathbf{P}],$$
$$\boldsymbol{\omega} = vect(\mathbf{\Omega}).$$

Eq. 2.25 is a theorem in kinematics which is valid as $\mathbf{P}$ is an arbitrary 3×3 matrix and $\mathbf{\Omega}$ is even skew-symmetric. Its proof may be conducted by calculating both sides of eq. 2.25 and eventually comparing them. Therefore, according to the theorem given in eq. 2.25, eq. 2.24 leads to:

$$\mathbf{D}\boldsymbol{\omega} = vect(\dot{\mathbf{P}}) \tag{2.26}$$

Eq. 2.26 can be solved for $\boldsymbol{\omega}$ as long as $\mathbf{D}$ is invertible. It may be proved that $\mathbf{D}$ is not invertible whenever the three points are collinear but also when $tr(\mathbf{P}) = 0$ although the three points are not collinear.
However, in order to have a formulation that works without depending on the value assumed by $tr(\mathbf{P})$, we can use an alternative calculation for $\boldsymbol{\omega}$. Post-multiply both sides of eq. 2.23 by $\mathbf{P}^T$ such that we obtain:

$$\dot{\mathbf{P}}\mathbf{P}^T = \boldsymbol{\Omega}\mathbf{R}, \quad \mathbf{R} = \mathbf{P}\mathbf{P}^T \tag{2.27}$$

and then take the vector of both sides of eq. 2.27:

$$vect(\dot{\mathbf{P}}\mathbf{P}^T) = vect(\boldsymbol{\Omega}\mathbf{R}) = \frac{1}{2}\mathbf{J}\boldsymbol{\omega} \tag{2.28}$$

where, by the application of the kinematics theorem mentioned above, $\mathbf{J}$ is defined as:

$$\mathbf{J} = tr(\mathbf{R})\mathbf{I} - \mathbf{R}.$$

It may be proved that $\mathbf{J}$ is, in general, positive-definite, becoming semi-positive definite only in special cases when the three points are collinear. The *singularity formulation* due to vanishing of $tr(\mathbf{P})$ is thus eliminated. As long as the three points are not collinear we can obtain $\boldsymbol{\omega}$ as:

$$\boldsymbol{\omega} = 2\mathbf{J}^{-1}vect(\dot{\mathbf{P}}\mathbf{P}^T). \tag{2.29}$$

Finally, we have to notice that positions $\mathbf{p}_i$ and velocities $\dot{\mathbf{p}}_i$, $i = 1, 2, 3$, has to satisfy the *velocity compatibily condition.* This condition is simply obtained by considering that the angles between any two of the vectors $(\mathbf{p}_i - \mathbf{c})$ must be preserved throughout the motion:

$$(\mathbf{p}_i - \mathbf{c})^T(\mathbf{p}_j - \mathbf{c}) = c_{ij}, \quad i, j = 1, 2, 3$$

or in compact form:

$$\mathbf{P}^T\mathbf{P} = \mathbf{C} \tag{2.30}$$

where the (i, j) entry of the constant matrix $\mathbf{C}$ is c_{ij}. The velocity compatibily condition is thus obtained upon differentiation of eq. 2.30 with respect to time:

$$\dot{\mathbf{P}}^T\mathbf{P} + \mathbf{P}^T\dot{\mathbf{P}} = \mathbf{0} \tag{2.31}$$

Eq. 2.31 states that for the given velocities of three points of a rigid body to be compatible, the product $\mathbf{P}^T\dot{\mathbf{P}}$ must be skew-symmetric. Thus, eq. 2.31 represents six ($\mathbf{P}^T\dot{\mathbf{P}}$ is skew-symmetric) equations that the data of the problem have to satisfy.

2.1.4 Computation of angular acceleration from point-acceleration data

We derive the angular acceleration of a rigid body under general motion from knowledge of the position, velocity and accelerations of three noncollinear points of the body, respectively: $\mathbf{p}_i$, $\dot{\mathbf{p}}_i$, $\ddot{\mathbf{p}}_i$, $i = 1, 2, 3$. The notation and the procedure for the angular acceleration follows those used for the angular velocity calculation. From eq. 2.20:

$$\ddot{\mathbf{p}}_i = \ddot{\mathbf{c}} + (\dot{\mathbf{\Omega}} + \mathbf{\Omega}^2)(\mathbf{p}_i - \mathbf{c}), \quad i = 1, 2, 3$$

and then

$$\ddot{\mathbf{p}}_i - \ddot{\mathbf{c}} = (\dot{\mathbf{\Omega}} + \mathbf{\Omega}^2)(\mathbf{p}_i - \mathbf{c}), \quad i = 1, 2, 3 \tag{2.32}$$

where $\ddot{\mathbf{c}} = 1/3 \sum_{i=1}^{3} \ddot{\mathbf{p}}_i$. Furthermore, we define:

$$\ddot{\mathbf{P}} = \begin{pmatrix} \ddot{\mathbf{p}}_1 - \ddot{\mathbf{c}} & \ddot{\mathbf{p}}_2 - \ddot{\mathbf{c}} & \ddot{\mathbf{p}}_3 - \ddot{\mathbf{c}} \end{pmatrix}$$

such that eq. 2.32 may be written in matrix form as:

$$\ddot{\mathbf{P}} = (\dot{\mathbf{\Omega}} + \mathbf{\Omega}^2)\mathbf{P} \tag{2.33}$$

from which:

$$\dot{\mathbf{\Omega}}\mathbf{P} = \mathbf{H} \tag{2.34}$$

with $\mathbf{H} = \ddot{\mathbf{P}} - \mathbf{\Omega}^2\mathbf{P}$. Now, similarly to what has been done for the angular velocity calculation we take the vector of both sides of eq. 2.34 and then we use the same kinematics theorem:

$$\begin{aligned} \mathbf{D}\dot{\boldsymbol{\omega}} &= vect(\mathbf{H}) : \\ \dot{\boldsymbol{\omega}} &= \mathbf{D}^{-1} vect(\mathbf{H}) \end{aligned} \tag{2.35}$$

where $\mathbf{D}$ is defined as in the previous section. It is apparent that eq. 2.35 holds the same limitations that eq. 2.26 has. Therefore, according to the procedure followed for the angular velocity, an alternative approach is proposed to overcome the *singularity formulation* of eq. 2.35. To this end post-multiply both sides of eq. 2.34 by $\mathbf{P}^T$:

$$\dot{\mathbf{\Omega}}\mathbf{R} = \mathbf{H}\mathbf{P}^T = \ddot{\mathbf{P}}\mathbf{P}^T - \mathbf{\Omega}^2\mathbf{R}. \tag{2.36}$$

where $\mathbf{R}$ is defined as in the previous section. Now, we take the vector of both sides of eq. 2.36 and we use the kinematics theorem:

$$\begin{aligned} vect(\dot{\mathbf{\Omega}}\mathbf{R}) = \frac{1}{2}\mathbf{J}\dot{\boldsymbol{\omega}} &= vect(\ddot{\mathbf{P}}\mathbf{P}^T - \mathbf{\Omega}^2\mathbf{R}) : \\ \dot{\boldsymbol{\omega}} &= 2\mathbf{J}^{-1} vect(\ddot{\mathbf{P}}\mathbf{P}^T - \mathbf{\Omega}^2\mathbf{R}) \end{aligned} \tag{2.37}$$

where $\mathbf{J}$ is defined as in the previous section.
Finally, we have to notice that positions $\mathbf{p}_i$, velocities $\dot{\mathbf{p}}_i$ and accelerations $\ddot{\mathbf{p}}_i$, $i = 1, 2, 3$, has to satisfy the *acceleration compatibily condition.* This condition is obtained directly from eq. 2.31 upon differentation with respect to time:

$$\ddot{\mathbf{P}}^T\mathbf{P} + 2\dot{\mathbf{P}}^T\dot{\mathbf{P}} + \mathbf{P}^T\ddot{\mathbf{P}} = \mathbf{0}.$$

2.2 Position analysis of serial manipulators

A serial manipulator consists of several links connected in series by various types of joints, typically revolute and prismatic joints. One end (*base*) of the manipulator is attached to the ground and the other end (*end-effector*) is free to move. A serial manipulator is called sometimes *open-loop* manipulator. In order to deal with the position analysis of the serial manipulator we follow the procedure summarized below and depicted in Figure 2.3:

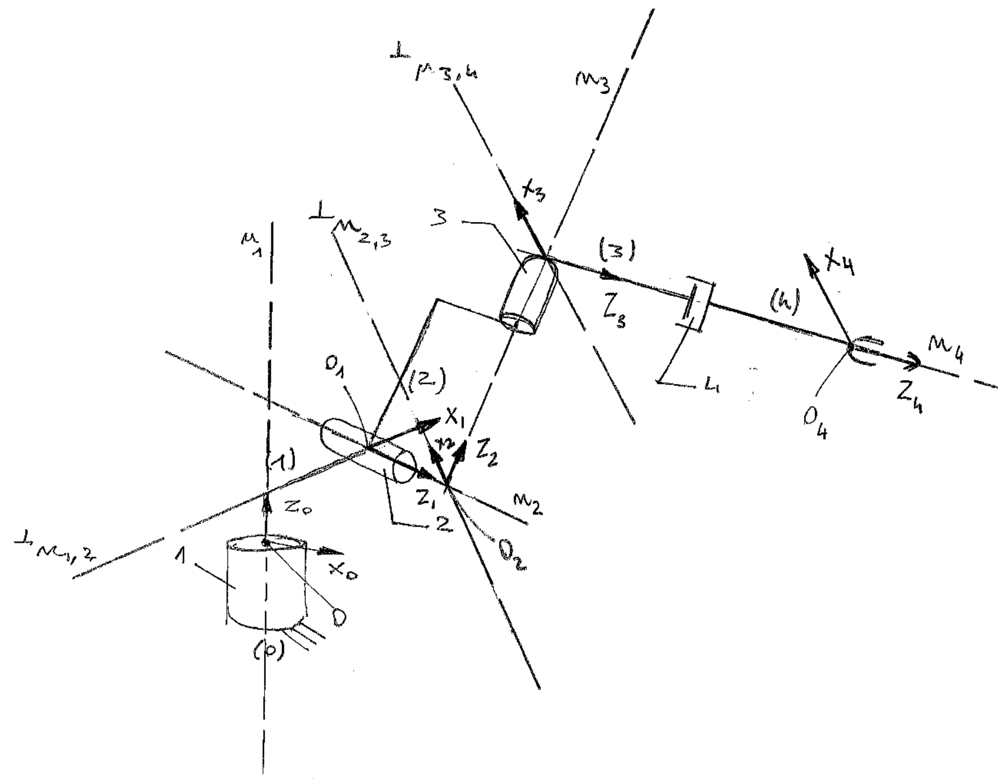

Figure 2.3: Procedure to locate the ref. systems in a serial manipulator.

1. Starting from the base numerate the links and the joints. In the serial manipulator with no redundancy we have the same number of links

and joints. The base is link (0), the end-effector is link (n).

2. For each joint detect and draw the joint's axis. We call n_i the axis of the joint i.

3. Detect and draw the common normal between two successive joints axes. We call ${}^{\perp}n_{i,i+1}$ the common normal between the axis of the joint i and the axis of the joint $i+1$.

4. Attach the ref. system $\{Ox_0y_0z_0\}$ to the link (0) such that z_0 coincides with n_1. Positive direction for z_0 is arbitrary.

5. Attach the ref. system $\{O_nx_ny_nz_n\}$ to the link (n) with the origin coinciding with the end-effector referring point such that x_n was normal to the last joint axis, namely n_n. Positive direction for z_n is arbitrary.

6. Attach the ref. system $\{O_ix_iy_iz_i\}$ to the link (i) according to the **Denavit-Hartenberg convention**:

   - the z_i axis coincides with n_{i+1};
   - the x_i axis coincides with ${}^{\perp}n_{i,i+1}$ pointing from the ith. to the $(i+1)$th. joint axis. In the case of intersecting joint axes, the x_i axis can be defined as $\pm(\mathbf{z}_{i-1} \times \mathbf{z}_i)$ and the O_i origin is at the intersection point.
   - the y_i axis is defined as $\mathbf{y}_i = \mathbf{z}_i \times \mathbf{x}_i$.

7. Determine the link parameters and joints variables a_i, α_i, θ_i and d_i.

2.2.1 D-H. homogenous transformation matrices

We have estabilished a ref. system for each link of the manipulator. Now, we define a (4×4) transformation matrix relating two successive coordinate systems. Consider the location of the ref. system $\{O_ix_iy_iz_i\}$ with respect to the ref. system $\{O_{i-1}x_{i-1}y_{i-1}z_{i-1}\}$. This location can be thought as being obtained by four successive displacements (rotations and translations) starting from the ith. ref. system initially coincident with the $(i-1)$th. ref. system. See Figure 2.4 for reference.

1. The $(i-1)$th. ref. system is translated along the z_{i-1} axis of d_i. This brings the origin O_{i-1} into H_{i-1}. $d_i = \|\mathbf{O}_{i-1}\mathbf{H}_{i-1}\|$ which is taken positive when pointing as $\mathbf{z}_{i-1}$. The corresponding transformation matrix is:

$$\mathbf{T}(z,d) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

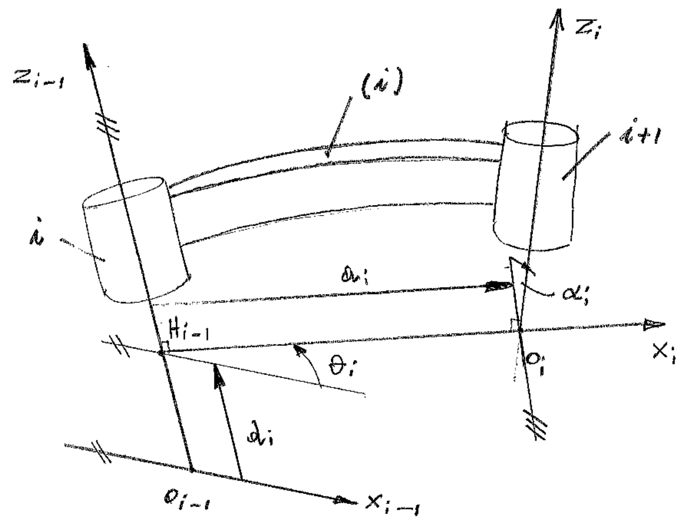

Figure 2.4: Definition of link parameters according to the *D.H. convention.*

2. The displaced $(i-1)$th. ref. system is rotated about z_{i-1} axis of θ_i, which brings the x_{i-1} axis into alignment with the x_i axis. θ_i is taken positive according to the righ hand rule (thumb pointing as $\mathbf{z}_{i-1}$). The corresponding transformation matrix is:

$$\mathbf{T}(z,\theta) = \begin{pmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

3. The displaced $(i-1)$th. ref. system is translated along the x_i axis of a_i. This brings the origin O_{i-1} into O_i. $a_i = \|\mathbf{H}_{i-1}\mathbf{O}_i\|$ which is taken positive when poiniting as $\mathbf{x}_i$. The corresponding transformation matrix is:

$$\mathbf{T}(x,a) = \begin{pmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

4. The displaced $(i-1)$th. ref. system is rotated about x_i axis of α_i, which brings the ref. system $i-1$ into coincidence with the ref. system i. α_i is taken positive according to the righ hand rule (thumb pointing as $\mathbf{x_i}$). The corresponding transformation matrix is:

$$\mathbf{T}(x,\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

Eventually, the resulting tranformation matrix which locates the ref. system i with respect to the ref. system $i-1$ is:

$$^{i-1}\mathbf{A}_i = \mathbf{I}\mathbf{T}(z,d)\mathbf{T}(z,\theta)\mathbf{T}(x,a)\mathbf{T}(x,\alpha)$$

Expanding the matrices multiplication we obtain:

$$^{i-1}\mathbf{A}_i = \begin{pmatrix} c\theta_i & -c\alpha_i s\theta_i & s\alpha_i s\theta_i & a_i c\theta_i \\ s\theta_i & c\alpha_i c\theta_i & -s\alpha_i c\theta_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.38}$$

Eq. 2.38 is called the *D-H. transformation matrix.* $^{i-1}\mathbf{A}_i$ may always be seen as:

$$^{i-1}\mathbf{A}_i = \begin{pmatrix} ^{i-1}\mathbf{R}_i & ^{i-1}\mathbf{p}_{i-1,i} \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix} \tag{2.39}$$

with

- $^{i-1}\mathbf{R}_i = {}^{i-1}\mathbf{A}_i(1:3,1:3)$: the orientation matrix of the ref. system i with respect to the ref. system $i-1$;
- $^{i-1}\mathbf{p}_{i-1,i} = {}^{i-1}\mathbf{A}_i(1:3,4)$: the position vector of O_i with respect to O_{i-1} expressed in the $(i-1)$ ref. system.

Therefore, we are able to locate the n ref. system attached at the end-effector and whose origin O_n is the end-effector referring point according to the *loop-closure equation* of a serial manipulator:

$$^{0}\mathbf{A}_n = {}^{0}\mathbf{A}_1{}^{1}\mathbf{A}_2 \cdots {}^{i-1}\mathbf{A}_i \cdots {}^{n-1}\mathbf{A}_n. \tag{2.40}$$

The geometric parameters a_i, α_i and the joint variables d_i and θ_i are the *D.H. parameters* of the manipulator.

2.2.2 Direct and inverse position problems

First we define the joint variables vector as $\mathbf{q} = \begin{pmatrix} q_1 \cdots & q_i \cdots & q_n \end{pmatrix}^T$ with $q_i = \theta_i$ if the ith. joint is a revolute joint or $q_i = d_i$ if the ith. joint is a prismatic joint and $\mathbf{\Pi}$ as a vector containing the geometrical parameters α_i and a_i. Then, consider an independent ref. system (*tool ref. system*) $\{u, v, w\}$ attached at the end effector with origin, detected by the position vector $\mathbf{r}$ at the end effector referring point such that its location with respect to the base frame is given by the homogeneous matrix $^{0}\mathbf{T}_t$:

$$^{0}\mathbf{T}_t \equiv \begin{pmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} & \mathbf{r} \\ 0 & 0 & 0 & 1 \end{pmatrix} = {}^{0}\mathbf{A}_n \tag{2.41}$$

We know that $^{0}\mathbf{A}_n = \mathcal{F}(\mathbf{q}, \mathbf{\Pi})$, thus eq. 2.41 represents 16 non-linear equations whose 4 are trivial and of the remaining 12 equations only 6 are independent. These equations have 3 Euler's angles (included in $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$), r_x, r_y, r_z, and n $q_1, \cdots, q_i, \cdots, q_n$ as variables.

Direct position problem

The direct position problem claims that given the joints variables $q_1, \cdots, q_i, \cdots, q_n$, we wish to find the location of the tool ref. system: 3 Euler's angles and $r_x,\ r_y,\ r_z$. Thus, we have 6 equations available for 6 unknowns. This problem is usually straightforward in the serial manipulators.

Inverse position problem

The inverse position problem claims that given the location of the tool ref. system, that is 3 Euler's angles and $r_x,\ r_y,\ r_z$, we wish to find $q_1, \cdots, q_i, \cdots, q_n$. Thus, we have 6 equations available for n unknowns. For a fully defined motion of a rigid body in $\mathbb{R}^3$ we need only 6 actuated joints such that the problem can be solved at hand, $(n = 6)$. However the inverse position problem in the serial manipulator can lead to more than one solution for the same tool ref. system location and it is more challenging than the direct counterpart.

2.2.3 Inverse position problem of decoupled manipulators

Figure 2.5 shows a general decoupled serial manipulator. The manipulator

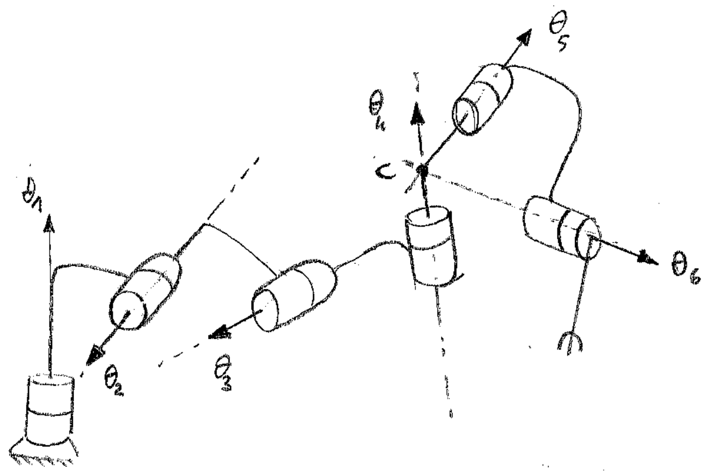

Figure 2.5: A general $6R$ manipulator with decoupled architecture.

is costitued by 6 revolute joints whose the last three form the robot *wrist* having a common axes intersection point C. The manipulator is said decoupled as the rotations of the first three joints allow point C to be located, *i.e., positioning problem*, whereas rotations of the the last three joints allow the end-effector ref. system to be orientated, *i.e.,orientating problem.* This kind of architecture is widely used in industrial applications.

According to the D.H. convention we have that the geometrical parameters $a_4 = a_5 = d_5 = \alpha_6 = 0$, while the joint variables are θ_i, $(i = 1, \cdots, 6)$.

The positioning problem

Apparently the position of the wrist centre C is independent from θ_4, θ_5 and θ_6. Therefore, we need to find θ_1, θ_2 and θ_3 when positon of C is given. According to the arm structure depicted in Figure 2.6 we have:

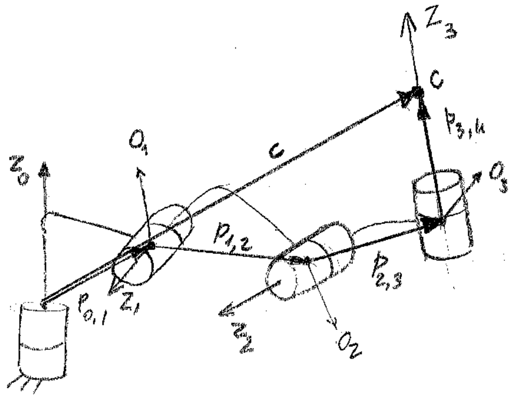

Figure 2.6: The positioning manipulator from the general $6R$ architecture.

$$\mathbf{c} = {}^0\mathbf{R}_1{}^1\mathbf{p}_{0,1} + {}^0\mathbf{R}_2{}^2\mathbf{p}_{1,2} + {}^0\mathbf{R}_3{}^3\mathbf{p}_{2,3} + {}^0\mathbf{R}_4{}^4\mathbf{p}_{3,4} \quad :$$
$$\mathbf{c} - {}^0\mathbf{R}_1{}^1\mathbf{p}_{0,1} = {}^0\mathbf{R}_1{}^1\mathbf{p}_{1,2} + {}^0\mathbf{R}_1{}^1\mathbf{R}_2{}^2\mathbf{p}_{2,3} + {}^0\mathbf{R}_1{}^1\mathbf{R}_2{}^2\mathbf{R}_3{}^3\mathbf{p}_{3,4},$$

with

$${}^{i-1}\mathbf{p}_{i-1,i} = {}^{i-1}\mathbf{R}_i{}^i\mathbf{p}_{i-1,i}\ ,\ {}^{i-1}\mathbf{R}_i = {}^{i-1}\mathbf{R}_i(\theta_i)\ ,\ {}^{i-1}\mathbf{p}_{i-1,i} = {}^{i-1}\mathbf{p}_{i-1,i}(\theta_i), \quad i = 1, \cdots, 4.$$

Then, by pre-multiplying both sides of the equation by ${}^0\mathbf{R}_1^T$:

$${}^0\mathbf{R}_1^T\mathbf{c} - {}^1\mathbf{p}_{0,1} = {}^1\mathbf{p}_{1,2} + {}^1\mathbf{R}_2({}^2\mathbf{p}_{2,3} + {}^2\mathbf{R}_3{}^3\mathbf{p}_{3,4}). \tag{2.42}$$

The left-hand side of the eq. 2.42 depends on θ_1 in ${}^0\mathbf{R}_1^T$ while ${}^1\mathbf{p}_{0,1}$ is constant as the following equation readily shows for the general case:

$${}^i\mathbf{p}_{i-1,i} = {}^i\mathbf{R}_{i-1}{}^{i-1}\mathbf{p}_{i-1,i} = \begin{pmatrix} c_{\theta_i} & s_{\theta_i} & 0 \\ -c_{\alpha_i}s_{\theta_i} & c_{\alpha_i}c_{\theta_i} & s_{\alpha_i} \\ s_{\alpha_i}s_{\theta_i} & -s_{\alpha_i}c_{\theta_i} & c_{\alpha_i} \end{pmatrix} \begin{pmatrix} a_i c_{\theta_i} \\ a_i s_{\theta_i} \\ d_i \end{pmatrix} = \begin{pmatrix} a_i \\ d_i s_{\alpha_i} \\ d_i c_{\alpha_i} \end{pmatrix}.$$

The right-hand side of the eq. 2.42 depends on θ_2 and θ_3 since ${}^3\mathbf{p}_{3,4} = \begin{pmatrix} 0 & 0 & d_4 \end{pmatrix}^T$.

In order to obtain θ_i, $(i = 1, \cdots, 3)$ when $\mathbf{c}$ is given, we use the squared norms of the vectors in the eq. 2.42, namely l^2, r^2 and the third equation of the same eq. 2.42:

$$l^2(\theta_1) = r^2(\theta_3),$$
$$\eta_1 c_z + \eta_2 c_y c_{\theta_1} + \eta_3 c_x s_{\theta_1} = \eta_4 + \eta_5 s_{\theta_3} + \eta_6 c_{\theta_3}.$$

where η_i, $(i = 1, \cdots, 6)$ are geometrical parameters. These two equations only depend on θ_1 and θ_3 (and they are linear in c_{θ_1}, s_{θ_1}, c_{θ_3}, s_{θ_3}) and thus, they can be solved. The remaining rotation θ_2 can be found by back substitution into one of the eq. 2.42.

The orientating problem

The problem consists in determing the wrists angles θ_4, θ_5 and θ_6 that will produce a prescribed orientation parametrized by $\mathbf{Q} = \begin{pmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{pmatrix}$. Let consider Figure 2.7 for reference. We note that $\alpha_{3,4} = \alpha_4$ and $\alpha_{4,5} = \alpha_5$ are

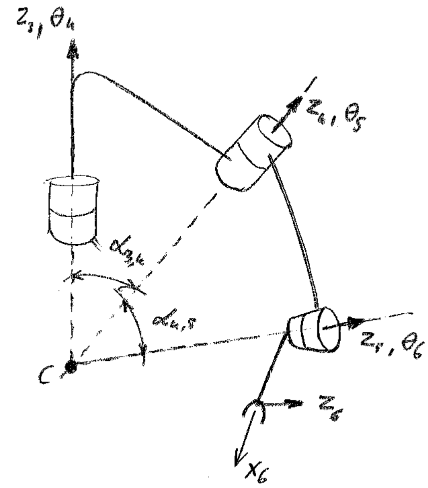

Figure 2.7: The orientating manipulator (*wrist*) from the general $6R$ architecture.

given geometrical parameters of the wrist. Fisrt, we calculate θ_4 according to the following scalar equation:

$$^3\mathbf{z}_5^{T3}\mathbf{z}_4 = \cos(\alpha_{4,5}) \tag{2.43}$$

where $^0\mathbf{z}_5 = {}^0\mathbf{z}_6 = \mathbf{w}$ is given as well as $^3\mathbf{z}_5 = {}^0\mathbf{R}_3^{T0}\mathbf{z}_5$ since θ_i, $(i = 1, \cdots, 3)$ are known from the positioning problem. Thus, 2.43 represents an equation in θ_4 and it can be easily solved. The remaining angles can be obtained as follows. Let consider the relationship:

$$^3\mathbf{R}_4{}^4\mathbf{R}_5{}^5\mathbf{R}_6 = {}^3\mathbf{R}_6 = {}^0\mathbf{R}_3^T\mathbf{Q} \quad :$$
$$^4\mathbf{R}_5(\theta_5) = [{}^3\mathbf{R}_4^T(\theta_4){}^0\mathbf{R}_3^T(\theta_1,\theta_2,\theta_3)\mathbf{Q}]{}^5\mathbf{R}_6^T(\theta_6). \tag{2.44}$$

By virtue of the form of ${}^4\mathbf{R}_5(\theta_5)$ and ${}^5\mathbf{R}_6^T(\theta_6)$, eq. 2.44 leads to 2 equations which depend only on θ_5, namely from the third column of the matrices:

$$\begin{pmatrix} c_{\theta_5} & -c_{\alpha_{4,5}}s_{\theta_5} & s_{\alpha_{4,5}}s_{\theta_5} \\ s_{\theta_5} & c_{\alpha_{4,5}}c_{\theta_5} & -s_{\alpha_{4,5}}c_{\theta_5} \\ 0 & s_{\alpha_{4,5}} & c_{\alpha_{4,5}} \end{pmatrix} = \begin{pmatrix} h_{11}c_{\theta_6} - h_{12}s_{\theta_6} & h_{11}s_{\theta_6} + h_{12}c_{\theta_6} & h_{13} \\ h_{21}c_{\theta_6} - h_{22}s_{\theta_6} & h_{21}s_{\theta_6} + h_{22}c_{\theta_6} & h_{23} \\ h_{31}c_{\theta_6} - h_{32}s_{\theta_6} & h_{31}s_{\theta_6} + h_{32}c_{\theta_6} & h_{33} \end{pmatrix}$$

with h_{ij} element of the ith. row, jth. column of $\mathbf{H}(\theta_1, \theta_2, \theta_3, \theta_4) = {}^3\mathbf{R}_4^{T\,0}\mathbf{R}_3^T\mathbf{Q}$. θ_5 is thus calculated by solving these two equations. The last angle θ_6 can be obtained once again by:

$$\begin{aligned} {}^3\mathbf{R}_4{}^4\mathbf{R}_5{}^5\mathbf{R}_6 = {}^3\mathbf{R}_6 = {}^0\mathbf{R}_3^T\mathbf{Q} \quad &: \\ {}^5\mathbf{R}_6(\theta_6) = {}^4\mathbf{R}_5^T(\theta_5){}^3\mathbf{R}_4^T(\theta_4){}^0\mathbf{R}_3^T(\theta_1, \theta_2, \theta_3)\mathbf{Q}. \end{aligned} \tag{2.45}$$

The righ-hand side of eq 2.45 is known from the previous calculations.

2.3 Velocity analysis of serial manipulators

We would like to find a relationship between the velocity of the end effector $\boldsymbol{\omega}_n$, $\mathbf{v}_n$ and the joints rates $\dot{\theta}_i$, $\dot{d}_i$.

2.3.1 Recursive Formulas

First we calculate ${}^{i-1}\mathbf{A}_i^{-1}$ that always exists.

$$\begin{aligned} {}^{i-1}\mathbf{A}_i^{-1\,i-1}\mathbf{A}_i = \tilde{\mathbf{I}} \quad &: \\ \begin{pmatrix} \mathbf{H} & \mathbf{s} \\ \mathbf{0}_{1\times3} & 1 \end{pmatrix} \begin{pmatrix} {}^{i-1}\mathbf{R}_i & {}^{i-1}\mathbf{p}_{i-1,i} \\ \mathbf{0}_{1\times3} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}_{1\times3} & 1 \end{pmatrix} \end{aligned} \tag{2.46}$$

Expanding eq. 2.46 we obtain:

$$\begin{aligned} \mathbf{H}^{i-1}\mathbf{R}_i = \mathbf{I} \ &: \mathbf{H} = {}^{i-1}\mathbf{R}_i^{-1} = {}^{i-1}\mathbf{R}_i^T \\ \mathbf{s} = -\mathbf{H}^{i-1}\mathbf{p}_{i-1,i} &= -{}^{i-1}\mathbf{R}_i^{T\,i-1}\mathbf{p}_{i-1,i} \end{aligned}$$

such that

$${}^{i-1}\mathbf{A}_i^{-1} = \begin{pmatrix} {}^{i-1}\mathbf{R}_i^T & -{}^{i-1}\mathbf{R}_i^{T\,i-1}\mathbf{p}_{i-1,i} \\ \mathbf{0}_{1\times3} & 1 \end{pmatrix} \tag{2.47}$$

Then we calculate ${}^{i-1}\dot{\mathbf{A}}_i$ simply by taking the derivative of ${}^{i-1}\mathbf{A}_i$ with respect to time:

$${}^{i-1}\dot{\mathbf{A}}_i = \begin{pmatrix} -s\theta_i\dot{\theta}_i & -c\alpha_i c\theta_i\dot{\theta}_i & s\alpha_i c\theta_i\dot{\theta}_i & -a_i s\theta_i\dot{\theta}_i \\ c\theta_i\dot{\theta}_i & -c\alpha_i s\theta_i\dot{\theta}_i & s\alpha_i s\theta_i\dot{\theta}_i & a_i c\theta_i\dot{\theta}_i \\ 0 & 0 & 0 & \dot{d}_i \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} {}^{i-1}\dot{\mathbf{R}}_i & {}^{i-1}\dot{\mathbf{p}}_{i-1,i} \\ \mathbf{0}_{1\times3} & 0 \end{pmatrix} \tag{2.48}$$

Now, we calculate:

$$
{}^{i-1}\dot{\mathbf{A}}_i{}^{i-1}\mathbf{A}_i^{-1} = \begin{pmatrix} {}^{i-1}\dot{\mathbf{R}}_i & {}^{i-1}\dot{\mathbf{p}}_{i-1,i} \\ \mathbf{0}_{1\times 3} & 0 \end{pmatrix} \begin{pmatrix} {}^{i-1}\mathbf{R}_i^T & -{}^{i-1}\mathbf{R}_i^{T\,i-1}\mathbf{p}_{i-1,i} \\ \mathbf{0}_{1\times 3} & 1 \end{pmatrix} =
$$

$$
\begin{pmatrix} {}^{i-1}\dot{\mathbf{R}}_i{}^{i-1}\mathbf{R}_i^T & -{}^{i-1}\dot{\mathbf{R}}_i{}^{i-1}\mathbf{R}_i^{T\,i-1}\mathbf{p}_{i-1,i} + {}^{i-1}\dot{\mathbf{p}}_{i-1,i} \\ \mathbf{0}_{1\times 3} & 0 \end{pmatrix} \tag{2.49}
$$

We now look at the terms of ${}^{i-1}\dot{\mathbf{A}}_i{}^{i-1}\mathbf{A}_i^{-1}$ in some details.

- ${}^{i-1}\dot{\mathbf{R}}_i{}^{i-1}\mathbf{R}_i^T = {}^{i-1}\mathbf{\Omega}_i$ is the angular velocity matrix of the ref. system (link) i with respect to the ref. system (link) $i-1$ according to the definition given in eq. 2.5. Its explicit calculation leads to:

$$
{}^{i-1}\mathbf{\Omega}_i = \dot{\theta}_i \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \dot{\theta}_i{}^{i-1}\mathbf{Z}_{i-1}.
$$

- $-{}^{i-1}\dot{\mathbf{R}}_i{}^{i-1}\mathbf{R}_i^{T\,i-1}\mathbf{p}_{i-1,i} + {}^{i-1}\dot{\mathbf{p}}_{i-1,i} = \dot{d}_i{}^{i-1}\mathbf{z}_{i-1}$ is the velocity of the origin of the ref. system i which instantaneously coincides with the origin of the ref. system $i-1$. This statement may easily be proved according to the definition given for the D.H. transformation matrix:

$$
-{}^{i-1}\dot{\mathbf{R}}_i{}^{i-1}\mathbf{R}_i^{T\,i-1}\mathbf{p}_{i-1,i} + {}^{i-1}\dot{\mathbf{p}}_{i-1,i} =
$$

$$
-\dot{\theta}_i \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_i\cos\theta_i \\ a_i\sin\theta_i \\ d_i \end{pmatrix} + \begin{pmatrix} -a_i\sin\theta_i\dot{\theta}_i \\ a_i\cos\theta_i\dot{\theta}_i \\ \dot{d}_i \end{pmatrix} = \dot{d}_i{}^{i-1}\mathbf{z}_{i-1}.
$$

  with ${}^{i-1}\mathbf{z}_{i-1} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$.

Eventually we may write eq. 2.49 as:

$$
{}^{i-1}\dot{\mathbf{A}}_i{}^{i-1}\mathbf{A}_i^{-1} = \begin{pmatrix} \dot{\theta}_i{}^{i-1}\mathbf{Z}_{i-1} & \dot{d}_i{}^{i-1}\mathbf{z}_{i-1} \\ \mathbf{0}_{1\times 3} & 0 \end{pmatrix}. \tag{2.50}
$$

Now, we wish to calculate ${}^0\dot{\mathbf{A}}_n{}^0\mathbf{A}_n^{-1}$.
To start the calculation, take the derivative of ${}^0\mathbf{A}_n$ with respect time:

$$
{}^0\dot{\mathbf{A}}_n = {}^0\dot{\mathbf{A}}_1{}^1\mathbf{A}_2\cdots{}^{i-1}\mathbf{A}_i\cdots{}^{n-1}\mathbf{A}_n + {}^0\mathbf{A}_1{}^1\dot{\mathbf{A}}_2\cdots{}^{i-1}\mathbf{A}_i\cdots{}^{n-1}\mathbf{A}_n + \cdots +
$$
$$
+{}^0\mathbf{A}_1{}^1\mathbf{A}_2\cdots{}^{i-1}\dot{\mathbf{A}}_i\cdots{}^{n-1}\mathbf{A}_n + \cdots + {}^0\mathbf{A}_1{}^1\mathbf{A}_2\cdots{}^{i-1}\mathbf{A}_i\cdots{}^{n-1}\dot{\mathbf{A}}_n.
$$

then calculate the inverse of ${}^0\mathbf{A}_n$:

$$
{}^0\mathbf{A}_n^{-1} = ({}^0\mathbf{A}_1\cdots{}^{i-1}\mathbf{A}_i\cdots{}^{n-1}\mathbf{A}_n)^{-1} = {}^{n-1}\mathbf{A}_n^{-1}\cdots{}^{i-1}\mathbf{A}_i^{-1}\cdots{}^1\mathbf{A}_2^{-1\,0}\mathbf{A}_1^{-1}
$$

and finally calculate their product:

$$
\begin{aligned}
{}^0\dot{\mathbf{A}}_n{}^0\mathbf{A}_n^{-1} &= {}^0\dot{\mathbf{A}}_1{}^1\mathbf{A}_2\cdots{}^{i-1}\mathbf{A}_i\cdots{}^{n-1}\mathbf{A}_n{}^{n-1}\mathbf{A}_n^{-1}\cdots{}^{i-1}\mathbf{A}_i^{-1}\cdots{}^1\mathbf{A}_2^{-1}{}^0\mathbf{A}_1^{-1}+\\
&+{}^0\mathbf{A}_1{}^1\dot{\mathbf{A}}_2\cdots{}^{i-1}\mathbf{A}_i\cdots{}^{n-1}\mathbf{A}_n{}^{n-1}\mathbf{A}_n^{-1}\cdots{}^{i-1}\mathbf{A}_i^{-1}\cdots{}^1\mathbf{A}_2^{-1}{}^0\mathbf{A}_1^{-1}+\\
&+\cdots+{}^0\mathbf{A}_1{}^1\mathbf{A}_2\cdots{}^{i-1}\dot{\mathbf{A}}_i\cdots{}^{n-1}\mathbf{A}_n{}^{n-1}\mathbf{A}_n^{-1}\cdots{}^{i-1}\mathbf{A}_i^{-1}\cdots{}^1\mathbf{A}_2^{-1}{}^0\mathbf{A}_1^{-1}+\\
&+\cdots+{}^0\mathbf{A}_1{}^1\mathbf{A}_2\cdots{}^{i-1}\mathbf{A}_i\cdots{}^{n-1}\dot{\mathbf{A}}_n{}^{n-1}\mathbf{A}_n^{-1}\cdots{}^{i-1}\mathbf{A}_i^{-1}\cdots{}^1\mathbf{A}_2^{-1}{}^0\mathbf{A}_1^{-1}=\\
&= {}^0\dot{\mathbf{A}}_1{}^0\mathbf{A}_1^{-1}+{}^0\mathbf{A}_1{}^1\dot{\mathbf{A}}_2{}^1\mathbf{A}_2^{-1}{}^0\mathbf{A}_1^{-1}+\cdots+{}^0\mathbf{A}_1{}^1\mathbf{A}_2\cdots{}^{i-1}\dot{\mathbf{A}}_i{}^{i-1}\mathbf{A}_i^{-1}\cdots{}^1\mathbf{A}_2^{-1}{}^0\mathbf{A}_1^{-1}+\\
&+{}^0\mathbf{A}_1{}^1\mathbf{A}_2\cdots{}^{i-1}\mathbf{A}_i\cdots{}^{n-1}\dot{\mathbf{A}}_n{}^{n-1}\mathbf{A}_n^{-1}\cdots{}^{i-1}\mathbf{A}_i^{-1}\cdots{}^0\mathbf{A}_1^{-1}.
\end{aligned}
$$

and thus:

$$
\begin{aligned}
{}^0\dot{\mathbf{A}}_n{}^0\mathbf{A}_n^{-1} &= \begin{pmatrix} \dot{\theta}_1{}^0\mathbf{Z}_0 & \dot{d}_1{}^0\mathbf{z}_0 \\ \mathbf{0}_{1\times3} & 0 \end{pmatrix} + {}^0\mathbf{A}_1 \begin{pmatrix} \dot{\theta}_2{}^1\mathbf{Z}_1 & \dot{d}_2{}^1\mathbf{z}_1 \\ \mathbf{0}_{1\times3} & 0 \end{pmatrix} {}^0\mathbf{A}_1^{-1}+\\
&+{}^0\mathbf{A}_1{}^1\mathbf{A}_2\cdots \begin{pmatrix} \dot{\theta}_i{}^{i-1}\mathbf{Z}_{i-1} & \dot{d}_i{}^{i-1}\mathbf{z}_{i-1} \\ \mathbf{0}_{1\times3} & 0 \end{pmatrix}\cdots{}^1\mathbf{A}_2^{-1}{}^0\mathbf{A}_1^{-1}+\\
&+{}^0\mathbf{A}_1{}^1\mathbf{A}_2\cdots{}^{i-1}\mathbf{A}_i\cdots \begin{pmatrix} \dot{\theta}_n{}^{i-1}\mathbf{Z}_{n-1} & \dot{d}_n{}^{n-1}\mathbf{z}_{n-1} \\ \mathbf{0}_{1\times3} & 0 \end{pmatrix}\cdots{}^{i-1}\mathbf{A}_i^{-1}\cdots{}^1\mathbf{A}_2^{-1}{}^0\mathbf{A}_1^{-1}. \quad (2.51)
\end{aligned}
$$

Eq. 2.51 can be finally expressed as:

$$
\begin{aligned}
{}^0\dot{\mathbf{A}}_n{}^0\mathbf{A}_n^{-1} &= \sum_{i=1}^{n} \begin{pmatrix} {}^0\mathbf{R}_{i-1} & {}^0\mathbf{p}_{i-1} \\ \mathbf{0}_{1\times3} & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta}_i{}^{i-1}\mathbf{Z}_{i-1} & \dot{d}_i{}^{i-1}\mathbf{z}_{i-1} \\ \mathbf{0}_{1\times3} & 0 \end{pmatrix} \begin{pmatrix} {}^0\mathbf{R}_{i-1}^T & -{}^0\mathbf{R}_{i-1}^T{}^0\mathbf{p}_{i-1} \\ \mathbf{0}_{1\times3} & 1 \end{pmatrix} =\\
&= \sum_{i=1}^{n} \begin{pmatrix} \dot{\theta}_i{}^0\mathbf{Z}_{i-1} & -\dot{\theta}_i{}^0\mathbf{Z}_{i-1}{}^0\mathbf{p}_{i-1}+\dot{d}_i{}^0\mathbf{z}_{i-1} \\ \mathbf{0}_{1\times3} & 0 \end{pmatrix} \qquad (2.52)
\end{aligned}
$$

In eq. 2.52 all the terms are expressed in the fixed ref. system since ${}^0\mathbf{R}_{i-1}{}^{i-1}\mathbf{z}_{i-1} = {}^0\mathbf{z}_{i-1}$ and ${}^0\mathbf{R}_{i-1}{}^{i-1}\mathbf{Z}_{i-1}{}^0\mathbf{R}_{i-1}^T = {}^0\mathbf{Z}_{i-1}$. This second equation is nothing but an example of *similarity transformation*, that is a change of basis for a linear operator (isomorphism). Instead of pursuing a general proof, the equation can be explained as follows:

$$
\begin{aligned}
{}^0\mathbf{R}_{i-1}{}^{i-1}\mathbf{Z}_{i-1}{}^0\mathbf{R}_{i-1}^T &= ({}^0\mathbf{R}_1{}^1\mathbf{R}_2\cdots{}^{i-2}\mathbf{R}_{i-1}){}^{i-1}\mathbf{Z}_{i-1}({}^0\mathbf{R}_1{}^1\mathbf{R}_2\cdots{}^{i-2}\mathbf{R}_{i-1})^T =\\
&= {}^0\mathbf{R}_1{}^1\mathbf{R}_2\cdots({}^{i-2}\mathbf{R}_{i-1}{}^{i-1}\mathbf{Z}_{i-1}{}^{i-2}\mathbf{R}_{i-1}^T)\cdots{}^1\mathbf{R}_2^T{}^0\mathbf{R}_1^T. \quad (2.53)
\end{aligned}
$$

Now, given that ${}^{i-2}\mathbf{R}_{i-1}{}^{i-1}\mathbf{Z}_{i-1}{}^{i-2}\mathbf{R}_{i-1}^T = {}^{i-2}\mathbf{Z}_{i-1}$ then eq. 2.53 takes the form:

$$
\begin{aligned}
{}^0\mathbf{R}_{i-1}{}^{i-1}\mathbf{Z}_{i-1}{}^0\mathbf{R}_{i-1}^T &= {}^0\mathbf{R}_1{}^1\mathbf{R}_2\cdots({}^{i-3}\mathbf{R}_{i-2}{}^{i-2}\mathbf{Z}_{i-1}{}^{i-3}\mathbf{R}_{i-2}^T)\cdots{}^1\mathbf{R}_2^T{}^0\mathbf{R}_1^T =\\
&= {}^0\mathbf{R}_1{}^1\mathbf{R}_2\cdots({}^{i-4}\mathbf{R}_{i-3}{}^{i-3}\mathbf{Z}_{i-1}{}^{i-4}\mathbf{R}_{i-3}^T)\cdots{}^1\mathbf{R}_2^T{}^0\mathbf{R}_1^T = \cdots\\
&\cdots = ({}^0\mathbf{R}_1{}^1\mathbf{Z}_{i-1}{}^0\mathbf{R}_1^T) = {}^0\mathbf{Z}_{i-1}.
\end{aligned}
$$

The previous statement is based on ${}^{i-2}\mathbf{R}_{i-1}{}^{i-1}\mathbf{Z}_{i-1}{}^{i-2}\mathbf{R}_{i-1}^T = {}^{i-2}\mathbf{Z}_{i-1}$:

$$
{}^{i-2}\mathbf{R}_{i-1}{}^{i-1}\mathbf{Z}_{i-1}{}^{i-2}\mathbf{R}_{i-1}^T = \begin{pmatrix} 0 & -c\alpha_{i-1} & -s\alpha_{i-1}c\theta_{i-1} \\ c\alpha_{i-1} & 0 & -s\alpha_{i-1}s\theta_{i-1} \\ s\alpha_{i-1}c\theta_{i-1} & s\alpha_{i-1}s\theta_{i-1} & 0 \end{pmatrix}. \quad (2.54)
$$

Terms in the skew-symmetric matrix in eq. 2.54 are the components of ${}^{i-2}\mathbf{z}_{i-1} = \begin{pmatrix} s\alpha_{i-1}s\theta_{i-1} & -s\alpha_{i-1}c\theta_{i-1} & c\alpha_{i-1} \end{pmatrix}^T : vect({}^{i-2}\mathbf{Z}_{i-1}) = {}^{i-2}\mathbf{z}_{i-1}$. ${}^{i-2}\mathbf{z}_{i-1}$ can be readily calculated if we consider the rotations that bring ref. system $i-1$, being initally coincident with the ref. system $i-2$, to the final location : ${}^{i-2}\mathbf{R}_{i-1} = \mathbf{IR}(z_{i-2}, \theta_{i-1})\mathbf{R}(x_{i-1}, \alpha_{i-1})$ and finally obtain:

$$ {}^{i-2}\mathbf{z}_{i-1} = {}^{i-2}\mathbf{R}_{i-1}{}^{i-1}\mathbf{z}_{i-1} = \mathbf{IR}(z_{i-2}, \theta_{i-1})\mathbf{R}(x_{i-1}, \alpha_{i-1})^{i-1}\mathbf{z}_{i-1} = \begin{pmatrix} s\alpha_{i-1}s\theta_{i-1} \\ -s\alpha_{i-1}c\theta_{i-1} \\ c\alpha_{i-1} \end{pmatrix} $$

to complete the proof.
Back to eq. 2.52 it may be noted that ${}^0\dot{\mathbf{A}}_n{}^0\mathbf{A}_n^{-1}$ is nothing else but:

$$ {}^0\dot{\mathbf{A}}_n{}^0\mathbf{A}_n^{-1} = \begin{pmatrix} {}^0\mathbf{\Omega}_n & {}^0\mathbf{v}_O \\ \mathbf{0}_{1\times 3} & 0 \end{pmatrix} $$

where ${}^0\mathbf{\Omega}_n$ is the angular velocity matrix of ref. system (link) n with respect to the fixed frame and $\mathbf{v}_O$ is the velocity of a point belonging to link n which instantaneously coincides with the origin O of the fixed ref. system. Therefore:

$$ {}^0\boldsymbol{\omega}_n = \sum_{i=1}^{n} \dot{\theta}_i{}^0\mathbf{z}_{i-1}; $$

$$ {}^0\mathbf{v}_O = \sum_{i=1}^{n} (-\dot{\theta}_i{}^0\mathbf{z}_{i-1} \times {}^0\mathbf{p}_{i-1} + \dot{d}_i{}^0\mathbf{z}_{i-1}). \tag{2.55} $$

An by using eqs. 2.55 to obtain the velocity of the origin of the ref. system n we have:

$$ {}^0\mathbf{v}_n ={}^0 \mathbf{v}_O +{}^0 \boldsymbol{\omega}_n \times {}^0\mathbf{p}_n ={}^0 \mathbf{v}_O +{}^0 \boldsymbol{\omega}_n \times ({}^0\mathbf{p}_{i-1} +{}^0 \mathbf{p}_{i-1,n}) = $$
$$ = \sum_{i=1}^{n} (-\dot{\theta}_i{}^0\mathbf{z}_{i-1} \times {}^0\mathbf{p}_{i-1} + \dot{d}_i{}^0\mathbf{z}_{i-1} +{}^0 \dot{\theta}_i{}^0\mathbf{z}_{i-1} \times {}^0\mathbf{p}_{i-1} + \dot{\theta}_i{}^0\mathbf{z}_{i-1} \times {}^0\mathbf{p}_{i-1,n}) = $$
$$ = \sum_{i=1}^{n} (\dot{\theta}_i{}^0\mathbf{z}_{i-1} \times {}^0\mathbf{p}_{i-1,n} + \dot{d}_i{}^0\mathbf{z}_{i-1}). $$

The velocities of the end effector can thus be calculated by the *recursive formulas* (Figure 2.8):

$$ {}^0\boldsymbol{\omega}_n = \sum_{i=1}^{n} \dot{\theta}_i{}^0\mathbf{z}_{i-1}; $$

$$ {}^0\mathbf{v}_n = \sum_{i=1}^{n} (\dot{\theta}_i{}^0\mathbf{z}_{i-1} \times {}^0\mathbf{p}_{i-1,n} + \dot{d}_i{}^0\mathbf{z}_{i-1}). \tag{2.56} $$

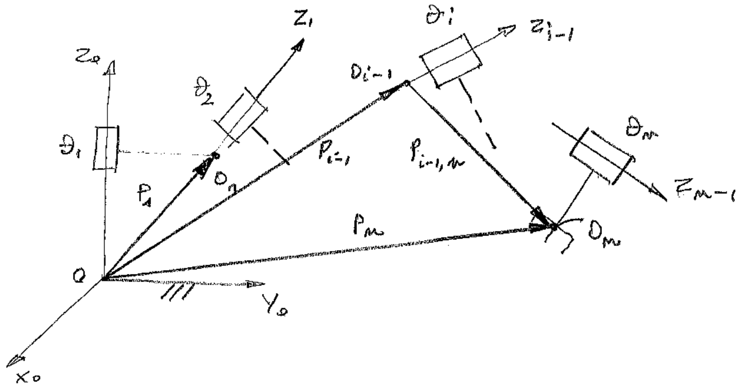

Figure 2.8: Link parameters of a serial manipulator.

2.3.2 Recursive formula-forward computation: a comparison

In this section we would like to show that the recursive formula for linear velocities provide the same result of the classical forward computation.

- Recursive formula:

$$\mathbf{v}_n = \sum_{i=1}^{n} (\dot{\theta}_i \mathbf{z}_{i-1} \times \mathbf{p}_{i-1,n} + \dot{d}_i \mathbf{z}_{i-1});$$

- Forward computation:

$$\mathbf{v}'_n = \sum_{i=1}^{n} (\boldsymbol{\omega}_i \times \mathbf{p}_{i-1,i} + \dot{d}_i \mathbf{z}_{i-1}).$$

Now, considering that:

$$\boldsymbol{\omega}_i = \dot{\theta}_1 \mathbf{z}_0 + \dot{\theta}_2 \mathbf{z}_1 + \cdots + \dot{\theta}_i \mathbf{z}_{i-1} = \sum_{j=1}^{i} \dot{\theta}_j \mathbf{z}_{j-1},$$

then

$$\mathbf{v}'_n = \sum_{i=1}^{n} [(\sum_{j=1}^{i} \dot{\theta}_j \mathbf{z}_{j-1}) \times \mathbf{p}_{i-1,i} + \dot{d}_i \mathbf{z}_{i-1}] \equiv \mathbf{v}_n.$$

To gain insight of the proof, let consider $n = 3$.

$$\mathbf{v}'_n = (\dot\theta_1\mathbf{z}_0 \times \mathbf{p}_{0,1} + \dot d_1\mathbf{z}_0) + (\dot\theta_1\mathbf{z}_0 \times \mathbf{p}_{1,2} + \dot\theta_2\mathbf{z}_1 \times \mathbf{p}_{1,2} + \dot d_2\mathbf{z}_1) +$$
$$+(\dot\theta_1\mathbf{z}_0 \times \mathbf{p}_{2,3} + \dot\theta_2\mathbf{z}_1 \times \mathbf{p}_{2,3} + \dot\theta_3\mathbf{z}_2 \times \mathbf{p}_{2,3} + \dot d_3\mathbf{z}_2) =$$
$$= \dot\theta_1\mathbf{z}_0 \times (\mathbf{p}_{0,1} + \mathbf{p}_{1,2} + \mathbf{p}_{2,3}) + \dot\theta_2\mathbf{z}_1 \times (\mathbf{p}_{1,2} + \mathbf{p}_{2,3}) + \dot\theta_3\mathbf{z}_2 \times \mathbf{p}_{2,3} +$$
$$+\dot d_1\mathbf{z}_0 + \dot d_2\mathbf{z}_1 + \dot d_3\mathbf{z}_2 = \dot\theta_1\mathbf{z}_0 \times \mathbf{p}_{0,3} + \dot\theta_2\mathbf{z}_1 \times \mathbf{p}_{1,3} + \dot\theta_3\mathbf{z}_2 \times \mathbf{p}_{2,3} + \sum_{i=1}^{3} \dot d_i\mathbf{z}_{i-1} =$$
$$= \sum_{i=1}^{3}(\dot\theta_i\mathbf{z}_{i-1} \times \mathbf{p}_{i-1,n} + \dot d_i\mathbf{z}_{i-1}) = \mathbf{v}_n.$$

2.3.3 Manipulator jacobian matrix

Eqs. 2.56 can be written in matrix form as:

$$\dot{\mathbf{x}} = \mathbf{J}\dot{\mathbf{q}} \tag{2.57}$$

with $\dot{\mathbf{x}} = \begin{pmatrix} {}^0\mathbf{v}_n & {}^0\boldsymbol{\omega}_n \end{pmatrix}^T$ and $\dot{\mathbf{q}} = \begin{pmatrix} \dot q_1 \cdots & \dot q_i & \cdots \dot q_n \end{pmatrix}^T$ and $\dot q_i$ being either $\dot\theta_i$ or $\dot d_i$ depending on which is the actuated joint. Matrix $\mathbf{J}$ is the manipulator jacobian. $\mathbf{J}$ is configuration dependent, that is $\mathbf{J} = \mathbf{J}(\theta_i, d_i)$ and of course it depends on the manipulator geometry as well. From eqs. 2.56 we can write:

$$\mathbf{J} = \begin{pmatrix} \mathbf{J}_1 & \cdots & \mathbf{J}_i & \cdots & \mathbf{J}_n \end{pmatrix};$$
$$\mathbf{J}_i = \begin{pmatrix} \mathbf{z}_{i-1} \times \mathbf{p}_{i-1,n} \\ \mathbf{z}_{i-1} \end{pmatrix} \text{ for a revolute joint,} \tag{2.58}$$
$$\mathbf{J}_i = \begin{pmatrix} \mathbf{z}_{i-1} \\ \mathbf{0}_{3\times 1} \end{pmatrix} \text{ for a prismatic joint.}$$

It is worth noting that we dropped the leading superscript (^{0}) on the left to shorter the notation.
A more general procedure can be followed to define the manipulator jacobian. Consider a set of m equations. Each equation is expressed as function of n independent variables: $x_i = \mathcal{F}_i(q_1, \cdots, q_j, \cdots, q_n)$, such that:

$$\begin{aligned} x_1 &= \mathcal{F}_1(q_1, \cdots, q_j, \cdots, q_n); \\ &\vdots \\ x_i &= \mathcal{F}_i(q_1, \cdots, q_j, \cdots, q_n); \\ &\vdots \\ x_m &= \mathcal{F}_m(q_1, \cdots, q_j, \cdots, q_n) \end{aligned}$$

with $i = 1, \cdots, m$ and $j = 1, \cdots, n$. By taking the derivative of x_i with respect time we have:

$$\dot{x}_i = \frac{\partial \mathcal{F}_i}{\partial q_1}\dot{q}_1 + \frac{\partial \mathcal{F}_i}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial \mathcal{F}_i}{\partial q_n}\dot{q}_n$$

which in matrix form leads to:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_m \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathcal{F}_1}{\partial q_1} & \frac{\partial \mathcal{F}_1}{\partial q_2} & \cdots & \frac{\partial \mathcal{F}_1}{\partial q_n} \\ \frac{\partial \mathcal{F}_2}{\partial q_1} & \frac{\partial \mathcal{F}_2}{\partial q_2} & \cdots & \frac{\partial \mathcal{F}_2}{\partial q_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial \mathcal{F}_m}{\partial q_1} & \frac{\partial \mathcal{F}_m}{\partial q_2} & \cdots & \frac{\partial \mathcal{F}_m}{\partial q_n} \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{pmatrix}$$

or simply $\dot{\mathbf{x}} = \mathbf{J}\dot{\mathbf{q}}$. The jacobian $\mathbf{J}$ $(m \times n)$, is a linear transformation matrix that maps a n-dimensional velocity vector $\dot{\mathbf{q}}$, $\dot{\mathbf{q}} \in \mathbb{V}^n$, into a a m-dimensional velocity vector $\dot{\mathbf{x}}$, $\dot{\mathbf{x}} \in \mathbb{V}^m$. The $(i,\ j)$ element in $\mathbf{J}$ describes how a differential change in q_j influences the differencial change in x_i. In general $\mathcal{F}_i$ is a non linear function of $\mathbf{q}$, therefore, as we mentioned, $\mathbf{J}$ is configuration dependent.

Sometimes, it can be useful to partitionate $\mathbf{J}$ as:

$$\mathbf{J} = \begin{pmatrix} \mathbf{J}_v \\ \mathbf{J}_\omega \end{pmatrix}$$

$\mathbf{J}_v$ and $\mathbf{J}_\omega$ are the translational and rotational jacobians, respectively. In particular $\mathbf{J}_v$ can be readily written as:

$$\mathbf{J}_v = \begin{pmatrix} \frac{\partial x_n}{\partial q_1} & \frac{\partial x_n}{\partial q_2} & \cdots & \frac{\partial x_n}{\partial q_n} \\ \frac{\partial y_n}{\partial q_1} & \frac{\partial y_n}{\partial q_2} & \cdots & \frac{\partial y_n}{\partial q_n} \\ \frac{\partial z_n}{\partial q_1} & \frac{\partial z_n}{\partial q_2} & \cdots & \frac{\partial z_n}{\partial q_n} \end{pmatrix}$$

with $\mathbf{p}_n = \begin{pmatrix} x_n & y_n & z_n \end{pmatrix}^T$.

2.3.4 Singularity analysis

First of all we only consider the case whenever $n = m$ such that the jacobian matrix becomes square.

A manipulator is said to be at a *singular configuration* when $\mathbf{J}$ looses its full rank, namely $\det(\mathbf{J}) = 0$. When it happens we have:

- The inverse velocity problem, $\dot{\mathbf{q}} = \mathbf{J}^{-1}\dot{\mathbf{x}}$ has ∞ solutions. That means that given $\dot{\mathbf{x}}$ we cannot find only one solution for $\dot{q}_i$ since not all the equations are linearly independent.

- When approaching the singularity, small $\dot{\mathbf{x}}$ may cause large $\dot{\mathbf{q}}$. That can be seen if we consider the one degree of freedom relationship $\dot{x} = j\dot{q}$. If $j \to 0$ then to have $\dot{x} \neq 0$, but finite, we need to have $\dot{q} \to \infty$.

- At sigularities the manipulator has its mobility reduced. There are some directions along with the motion is instantneously forbidden. Again, to explain it better, consider the one degree of freedom relationship $\dot{x} = j\dot{q}$. If $j \to 0$ then to have $\dot{q} \neq 0$, but finite, we need to have $\dot{x} \to 0$.

There are two types of singularities for serial manipulator: *boundary* and *interior* singularity. The boundary singularity occurs when the end effector is on the boundary of the manipulator workspace (locus of points that the end effector can reach). Instead, interior singularity occurs when the end effector is inside the manipulator workspace. In Figure 2.9 the locus of singular (boundary) points for the $3R$ planar manipulator are depicted.

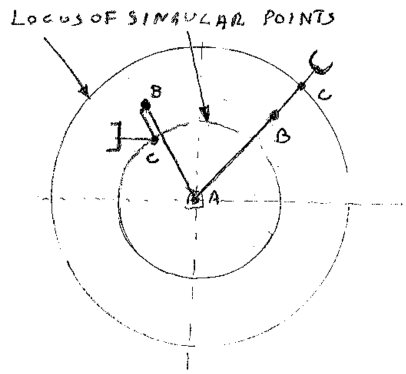

Figure 2.9: Locus of singular points for a $3R$ planar manipulator.

2.3.5 Performance indices

Performance indices help us to gain insight into the nature of the mapping between the operational space and the joint space. They, somehow, provide a *level of quality* of the mapping. However, we have to define the performance indices for those manipulators with only one type of joint and for one type of task, namely, either point positioning or body orienting but not both. Indeed, one has to be careful in manipulating $\mathbf{J}$ as its elements may not necessary have uniform dimensions. For example, if all the joints are revoulte joints, the elements of $\mathbf{J}_v$ have dimensions of lengths while the elements of $\mathbf{J}_\omega$ are numbers.

Condition number

The condition number of a dimensionally uniform Jacobian $\mathbf{J}$ is defined as:

$$k(\mathbf{J}) = \|\mathbf{J}\|\|\mathbf{J}^{-1}\| \tag{2.59}$$

where $\|\bullet\|$ stands for the matrix norm. If the matrix 2-norm is used in the difinition of eq. 2.59 then:

$$\|\mathbf{J}\|_2 = \max\{\sigma_i\} = \sigma_M;$$
$$\|\mathbf{J}^{-1}\|_2 = \max\{\frac{1}{\sigma_i}\} = \frac{1}{\sigma_m}.$$

where $\sigma_m = \min\{\sigma_i\}$ and σ_i is the ith. singular value of $\mathbf{J}$. Thus the 2-norm condition number is:

$$k_2(\mathbf{J}) = \frac{\sigma_M}{\sigma_m}. \tag{2.60}$$

It is noteworthy that $k_2(\mathbf{J})$ is frame-invariant since it is given in terms of the matrix singular values.
The singular values of $\mathbf{J}$ can be calulcated by its *singular values decomposition*, SVD, such that:

$$\mathbf{J} = \mathbf{U}\mathbf{S}\mathbf{V}^T$$

with $\mathbf{U}$ and $\mathbf{V}$ orthogonal matrices. If $\mathbf{J} \in \mathbb{V}^{m,n}$ then $\mathbf{U} \in \mathbb{V}^{m,m}$, $\mathbf{V} \in \mathbb{V}^{n,n}$ and $\mathbf{S} \in \mathbb{V}^{m,n}$. $\mathbf{S}$ is the diagonal matrix containing the singular values of $\mathbf{J}$. The eigenvectors of $\mathbf{J}^T\mathbf{J}$ make up the columns of $\mathbf{V}$, the eigenvectors of $\mathbf{J}\mathbf{J}^T$ make up the columns of $\mathbf{U}$. Also, the singular values in $\mathbf{S}$ are square roots of eigenvalues from $\mathbf{J}\mathbf{J}^T$ or $\mathbf{J}^T\mathbf{J}$. A geometrical interpretation of SVD can be provided if we consider that $\mathbf{V}^T$ acts as a rotation, plus an optional reflection, followed by a scaling performed by $\mathbf{S}$ and by $\mathbf{U}$ acting as new rotation. Therefore $\mathbf{J}$ maps a point, $\dot{\mathbf{q}}$, in the joint rates space into a point, $\dot{\mathbf{x}}$, in the end effector velocity space performing the transformations mentioned.
Since $k_2(\mathbf{J}) \geq 1$ for convenience the inverse of the condition number is used:

$$c_2(\mathbf{J}) = \frac{1}{k_2(\mathbf{J})} = \frac{\sigma_m}{\sigma_M}, \quad 0 \leq c_2(\mathbf{J}) \leq 1, \tag{2.61}$$
$$\text{with}$$
$$c_2(\mathbf{J}) = 0: \quad \text{singularity point;}$$
$$c_2(\mathbf{J}) = 1: \quad \text{isotropic point.}$$

In other words, at the singular point $\mathbf{J}$ is singular, that is not invertible. At the isotropic point $\mathbf{J}$ is perfectly conditioned.

Velocity ellipsoid

We wish to compare the joint rates contributions required to produce an unity end effector veclocity in all possible directions. We confine the end effector velocity vector on a m-dimensional unit hyper-sphere:

$$\dot{\mathbf{x}}^T\dot{\mathbf{x}} = 1 \tag{2.62}$$

Eq. 2.62 represents a hyper-sphere in $\mathbb{V}^m$. And we compare the corresponding joint rates in $\mathbb{V}^n$ joint space:

$$\dot{\mathbf{x}}^T\dot{\mathbf{x}} = (\mathbf{J}\dot{\mathbf{q}})^T(\mathbf{J}\dot{\mathbf{q}}) = \dot{\mathbf{q}}^T\mathbf{J}^T\mathbf{J}\dot{\mathbf{q}} = 1 \tag{2.63}$$

Eq. 2.63 represents an hyper-ellipsoid in $\mathbb{V}^n$. The nature of the transformation depends on $\mathbf{J}^T\mathbf{J}$ which is configuration dependent as $\mathbf{J}$ is. $\mathbf{J}^T\mathbf{J}$ is a symmetric semipositive definite matrix, namely all the eigenvalues $\lambda_i \geq 0$, $(i = 1, \cdots, n)$, and the eigenvectors are orthogonal. The pricipal axes of the ellipsoid coincide with the eigenvectors of $\mathbf{J}^T\mathbf{J}$ and the lengths l_i of the principal axes are $l_i = 1/\sqrt{\lambda_i}$.
At an isotropic point, a unit sphere in $\mathbb{V}^m$ maps onto a sphere in $\mathbb{V}^n$ with $\mathbf{J}^T\mathbf{J} = \sigma^2\mathbf{1}$, where σ denotes the unique singular value of $\mathbf{J}$ at the isotropic point. In a singular point at least one eigenvalue becomes zero thus the corresponding axis become infinitely long and the ellipsoid degenerates into a cilynder. In all the other points it will be possible to calculate the joint rates values for a given end effector velocity.

2.4 Kinematic redundancy

A manipulator is kinematically redundant when the dimension r of the task space is smaller than the dimension of the joints space n: $r < n$. A such manipulator has $(n - r)$ redundant degrees of mobility. Redundancy is relative to the task assigned to the manipulator. r can be smaller or equal to the dimension of the cartesian space m: $r \leq m$. For example a $3R$-planar manipulator is redundant if the task assigned concerns to locate the end-effector point with no interest on the final body orientation. Conversely, the same manipulator is not longer redundant whenever the task assigned deals with a complete location of the end-effector. The kinematic redundancy allows the manipulator to gain in dexterity and versatility during its motion.
Let consider the differential kinematics equation

$$\mathbf{v} = \mathbf{J}\dot{\mathbf{q}}, \tag{2.64}$$

with $\mathbf{v} \in \mathbb{V}^r$, $r \leq m$, $\dot{\mathbf{q}} \in \mathbb{V}^n$ and $\mathbf{J} \in \mathbb{V}^{(r\times n)}$. The mapping is schematically illustrated in Figure 2.10.

- The range of $\mathbf{J}$ is the subspace $\mathcal{R}(\mathbf{J}) \in \mathbb{V}^r$ of the end-effector velocity that can be generated by the jonits rates in a given posture;
- The null of $\mathbf{J}$ is the subspace $\mathcal{N}(\mathbf{J}) \in \mathbb{V}^n$ of the joints rates that do not produce any end-effector velocity ($\mathbf{v} = \mathbf{0}$) in a given posture.

If $\mathbf{J}$ has full rank then:

$$dim(\mathcal{R}(\mathbf{J})) = r, \qquad dim(\mathcal{N}(\mathbf{J})) = n - r.$$

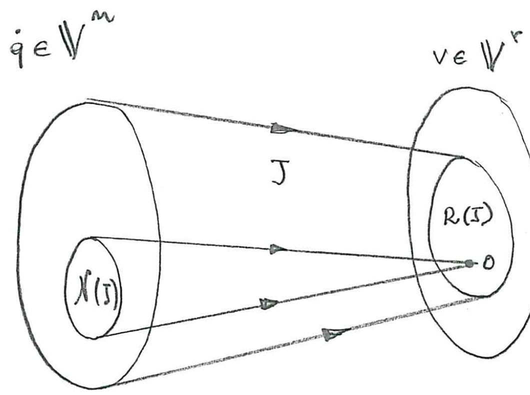

Figure 2.10: Mapping between joints rates space and task velocity space.

In this case the range of $\mathbf{J}$ spans the entire $\mathbb{V}^r$. According to the well known theorem in linear algebra, independently fo the rank of $\mathbf{J}$:

$$dim(\mathcal{R}(\mathbf{J})) + dim(\mathcal{N}(\mathbf{J})) = n.$$

The point is the exsistence of the subspace $\mathcal{N}(\mathbf{J})$ for a redundant manipulator. In this case, if $\dot{\mathbf{q}}^*$ is a solution of eq. (2.64) then $\dot{\mathbf{q}}$ is also a solution of that equation, where:

$$\dot{\mathbf{q}} = \dot{\mathbf{q}}^* + (\mathbf{1} - \mathbf{J}^\dagger \mathbf{J})\dot{\mathbf{q}}_0 \tag{2.65}$$

In the foregoing equation $\dot{\mathbf{q}}_0$ is arbitrary, $\mathbf{J}^\dagger$ is the right-pseudo-inverse of $\mathbf{J}$ and $(\mathbf{1} - \mathbf{J}^\dagger \mathbf{J}) \in \mathbb{V}^{(n\times n)}$ projects $\dot{\mathbf{q}}_0$ to the null space of $\mathbf{J}$ such that:

$$\mathbf{J}(\mathbf{1} - \mathbf{J}^\dagger \mathbf{J})\dot{\mathbf{q}}_0 = \mathbf{0}. \tag{2.66}$$

In essence, in a redundant manipulator it is possible to generate internal motion described by $(\mathbf{1} - \mathbf{J}^\dagger \mathbf{J})\dot{\mathbf{q}}_0$ that reconfigure the manipulator structure with no variations in the end-effector position and orientation.
To prove that $\dot{\mathbf{q}}$ as well as $\dot{\mathbf{q}}^*$ are both solutions of eq. (2.64) we premultiply eq. (2.65) by $\mathbf{J}$ and we consider that $\mathbf{J}^\dagger = \mathbf{J}^T(\mathbf{J}\mathbf{J}^T)^{-1}$:

$$\mathbf{J}\dot{\mathbf{q}} = \mathbf{J}\dot{\mathbf{q}}^* + \mathbf{J}\dot{\mathbf{q}}_0 - \mathbf{J}(\mathbf{J}^T\mathbf{J}^{-T}\mathbf{J}^{-1}\mathbf{J})\dot{\mathbf{q}}_0 = \mathbf{J}\dot{\mathbf{q}}^* = \mathbf{v} \tag{2.67}$$

From eq. (2.67) it is proved eq. (2.66) $\forall \dot{\mathbf{q}}_0$. There is, thus, the possibility to choose $\dot{\mathbf{q}}_0$ to advantageously exploit the redundant degrees of mobility. Indeed, the only effect of $\dot{\mathbf{q}}_0$ is to generate internal motions of the structure that do not change either the orientation or the position of the end-effector but that allows the manipulator to have better configurations (more dexterous postures) for the execution of a given task.

2.4.1 Solution procedure

In a redundant manipulator $r < n$ therefore eq. (2.64) has infinite solutions. The goal is to find the solution $\dot{\mathbf{q}}$ that satisfies the eq. (2.64) and *minimize* the square of the Euclidean norm of $\dot{\mathbf{q}}$.
We thus have:

$$\underset{\dot{\mathbf{q}}}{\text{minimize}} \;\; \mathcal{F} = \frac{1}{2}\|\dot{\mathbf{q}}\|^2 \tag{2.68}$$

subjected to the constraint $\mathbf{v} = \mathbf{J}\dot{\mathbf{q}}$. This is a *constained minimization problem* that may be solved by the *Lagrangian multipliers.* We have to consider a modified objective function defined as:

$$\underset{\dot{\mathbf{q}},\boldsymbol{\lambda}}{\text{minimize}} \;\; \mathcal{G} = \frac{1}{2}\|\dot{\mathbf{q}}\|^2 + \boldsymbol{\lambda}^T(\mathbf{J}\dot{\mathbf{q}} - \mathbf{v}) \tag{2.69}$$

subjected to no constraints. $\boldsymbol{\lambda} \in \mathbb{V}^r$ is a vector of unknowns. The normality conditions of the foregoing problem are:

$$\frac{\partial \mathcal{G}}{\partial \dot{\mathbf{q}}} = \dot{\mathbf{q}} + \mathbf{J}^T\boldsymbol{\lambda} = \mathbf{0}_n, \quad \frac{\partial \mathcal{G}}{\partial \boldsymbol{\lambda}} = \mathbf{J}\dot{\mathbf{q}} - \mathbf{v} = \mathbf{0}_r. \tag{2.70}$$

Upon elimination of $\boldsymbol{\lambda}$ from eqs. (2.70), we obtain:

$$\dot{\mathbf{q}} = \mathbf{J}^T(\mathbf{J}\mathbf{J}^T)^{-1}\mathbf{v}. \tag{2.71}$$

Eq. (2.71) is the *minimum-norm solution* of the proposed problem and $\mathbf{J}^T(\mathbf{J}\mathbf{J}^T)^{-1} = \mathbf{J}^\dagger$ is the *right pseudo-inverse* of $\mathbf{J}$.
It was pointed out that if eq. (2.71) is a solution of eq. (2.64) then $\dot{\mathbf{q}} + (\mathbf{1} - \mathbf{J}^\dagger\mathbf{J})\dot{\mathbf{q}}_0$ is also a solution. As it was said, $\dot{\mathbf{q}}_0$ is an arbitrary vector that can be specified in order to satisfy additional constraint to the problem. Conveniently, $\dot{\mathbf{q}}_0$ can be chosen in the direction of the antigradient of a scalar configuration performance criteria $w(\mathbf{q})$ which must be minimized:

$$\dot{\mathbf{q}}_0 = -k_w \nabla w(\mathbf{q}) \tag{2.72}$$

where k_w is a scalar step size and $\nabla w(\mathbf{q})$ denotes the gradient of $w(\mathbf{q})$ at the current configuration. The complete inverse kinematic problem is obtained by minimizing the objective function $\mathcal{I}$ defined as:

$$\underset{\dot{\mathbf{q}},\boldsymbol{\lambda}}{\text{minimize}} \;\; \mathcal{I} = \frac{1}{2}\dot{\mathbf{q}}^T\dot{\mathbf{q}} + k_w\dot{\mathbf{q}}^T\nabla w(\mathbf{q}) + \boldsymbol{\lambda}^T(\mathbf{J}\dot{\mathbf{q}} - \mathbf{v}) \tag{2.73}$$

subjected to no constraints. The normality conditions of the foregoing problem are:

$$\frac{\partial \mathcal{I}}{\partial \dot{\mathbf{q}}} = \dot{\mathbf{q}} + k_w\nabla w(\mathbf{q}) + \mathbf{J}^T\boldsymbol{\lambda} = \mathbf{0}_n, \quad \frac{\partial \mathcal{I}}{\partial \boldsymbol{\lambda}} = \mathbf{J}\dot{\mathbf{q}} - \mathbf{v} = \mathbf{0}_r. \tag{2.74}$$

And eventually, upon elimination of $\boldsymbol{\lambda}$ from eqs. (2.74), we obtain:

$$\dot{\mathbf{q}} = \mathbf{J}^{\dagger}\mathbf{v} - k_w(\mathbf{1} - \mathbf{J}^{\dagger}\mathbf{J})\nabla w(\mathbf{q}). \tag{2.75}$$

It is worth noting that the solution has two terms whose the former is the minimum norm solution, the latter (*homogeneous solution:* $\mathbf{v} = \mathbf{0}$) satisfies the additional constraint to specify via $\dot{\mathbf{q}}_0$.
In other words $w(\mathbf{q})$ is a secondary objective function that has to be minimized. Amongst others $w(\mathbf{q})$ can be connected to the kinematic performance of the manipulator:

- The *manipulability measure* defined as:

$$w(\mathbf{q}) = \sqrt{det(\mathbf{J}\mathbf{J}^T)}. \tag{2.76}$$

  The manipulability measure vanishes at a singular configuration, thus maximizing it (minimizing it with sign reversed) uses the redundancy to move away form singualrities.

3

Dynamics of Rigid-Bodies

3.1 Basic definitions

First of all we need to define some mass properties for a rigid body as shown in Figure 3.1.

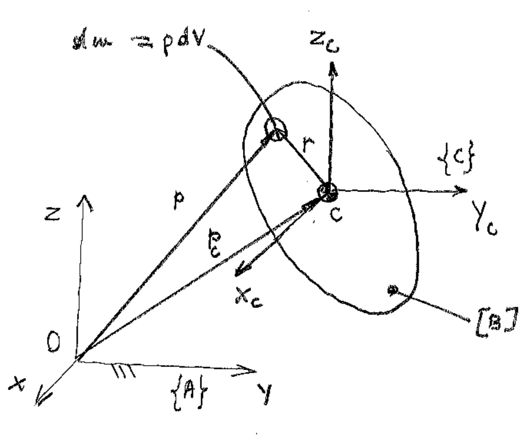

Figure 3.1: Mass properties of the rigid body $\mathcal{B}$.

3.1.1 Center of mass

The *center of mass* of a rigid body $\mathcal{B}$ is defined as the point C, whose position vector $\mathbf{p}_c$, satisfies the following condition:

$$\mathbf{p}_c = \frac{1}{m}\int_V \mathbf{p}\rho dV \tag{3.1}$$

where $m = \int_V \rho dV$ is the total mass of the body. Furthermore C is a point of $\mathcal{B}$ such that

$$\int_V \mathbf{r}\rho dV = \mathbf{0} \tag{3.2}$$

3.1.2 Inertia matrix (Tensor of Inertia)

The inertia matrix of the body $\mathcal{B}$ about point O is defined as:

$$\mathbf{I}_B^O = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \tag{3.3}$$

with:

$$I_{xx} = \int_V (y^2 + z^2)\rho dV, \quad I_{yy} = \int_V (z^2 + x^2)\rho dV, \quad I_{zz} = \int_V (x^2 + y^2)\rho dV$$

that are called moments of inertia and

$$\begin{aligned} I_{xy} = I_{yx} &= -\int_V xy\rho dV, \\ I_{yz} = I_{zy} &= -\int_V yz\rho dV, \\ I_{zx} = I_{xz} &= -\int_V zx\rho dV. \end{aligned}$$

that are called products of inertia. As it is evident $\mathbf{I}_B^O$ is symmetric. Furthermore, note that $\mathbf{I}_B^O$ is expressed in the ref. system A.

3.1.3 Parellel Axis Theorem

According to Figure 3.1, consider a ref. system C: $\{C, x_c, y_c, z_c\}$ with its coordinate axes parallel to those of ref. system A: $\{O, x, y, z\}$. Then, the parellel axis theorem is satisfied when:

$$I_{xx}^O = I_{xx}^C + m(y_c^2 + z_c^2), \quad I_{yy}^O = I_{yy}^C + m(z_c^2 + x_c^2), \quad I_{zz}^O = I_{zz}^C + m(x_c^2 + y_c^2),$$
$$I_{xy}^O = I_{xy}^C + mx_c y_c, \quad I_{yz}^O = I_{yz}^C + my_c z_c, \quad I_{zx}^O = I_{zx}^C + mz_c x_c.$$

with (x_c, y_c, z_c) coordinates of the point C expressed in A.

3.1.4 Principal Moments of Inertia

Inertia matrix depends on the choice of the ref. system and its orientation. For a certain orientation of the ref. system the products of inertia will vanish. In this case the coordinate axes of such ref. system are called *principal axes* and the corresponding moments of inertia are the *principal moments of inertia*.

3.1.5 Linear Momentum

$$d\mathbf{l}^O = \frac{d\mathbf{p}}{dt}\rho dV \tag{3.4}$$

Eq. 3.4 represents the linear momentum of a $dm = \rho dV$ about O. The reference to the point O is due to the position vector $\mathbf{p}$ in the equation. Hence the linear momentum of the body $\mathcal{B}$ is given as:

$$\mathbf{l}^O = \int_V \frac{d\mathbf{p}}{dt}\rho dV \tag{3.5}$$

According to Figure 3.1 we can write $\mathbf{p} = \mathbf{p}_c + \mathbf{r}$ such that

$$\mathbf{l}^O = \int_V \frac{d\mathbf{p}_c}{dt}\rho dV + \int_V \frac{d\mathbf{r}}{dt}\rho dV \tag{3.6}$$

where

$$\begin{aligned} \frac{d\mathbf{p}_c}{dt} &= \mathbf{v}_c, \\ \int_V \frac{d\mathbf{r}}{dt}\rho dV &= \frac{d}{dt}\int_V \mathbf{r}\rho dV = \mathbf{0}. \end{aligned} \tag{3.7}$$

$\mathbf{v}_c$ is the velocity of the center of mass C with respect to ref. system A. Second equation of eqs. 3.7 is guaranteed by the definition of the center of mass given in eq. 3.2. Therefore eq. 3.6 becomes

$$\mathbf{l}^O = m\mathbf{v}_c \tag{3.8}$$

3.1.6 Angular Momentum

$$d\mathbf{h}^O = (\mathbf{p} \times \frac{d\mathbf{p}}{dt})\rho dV \tag{3.9}$$

Eq. 3.9 represents the moment of a linear momentum of a $dm = \rho dV$ about O expressed in A. Hence the angular momentum of the body $\mathcal{B}$ about O is given as:

$$\mathbf{h}^O = \int_V (\mathbf{p} \times \frac{d\mathbf{p}}{dt})\rho dV \tag{3.10}$$

By using $\mathbf{p} = \mathbf{p}_c + \mathbf{r}$ we obtain:

$$\begin{aligned} \mathbf{h}^O &= \int_V (\mathbf{p}_c + \mathbf{r}) \times (\frac{d\mathbf{p}_c}{dt} + \frac{d\mathbf{r}}{dt})\rho dV = \\ = \int_V (\mathbf{p}_c \times \frac{d\mathbf{p}_c}{dt})\rho dV &+ \int_V (\mathbf{p}_c \times \frac{d\mathbf{r}}{dt})\rho dV + \int_V (\mathbf{r} \times \frac{d\mathbf{p}_c}{dt})\rho dV + \int_V (\mathbf{r} \times \frac{d\mathbf{r}}{dt})\rho dV = \\ = \int_V (\mathbf{p}_c \times \frac{d\mathbf{p}_c}{dt})\rho dV &+ \mathbf{p}_c \times \frac{d}{dt}\int_V \mathbf{r}\rho dV + (\int_V \mathbf{r}\rho dV) \times \frac{d\mathbf{p}_c}{dt} + \int_V (\mathbf{r} \times \frac{d\mathbf{r}}{dt})\rho dV. \end{aligned}$$

where

$$\begin{aligned} \mathbf{p}_c \times \frac{d}{dt}\int_V \mathbf{r}\rho dV &= \mathbf{0}, \\ (\int_V \mathbf{r}\rho dV) \times \frac{d\mathbf{p}_c}{dt} &= \mathbf{0}. \end{aligned}$$

because of eq. 3.2.
Eventually, eq. 3.10 becomes:

$$\mathbf{h}^O = m(\mathbf{p}_c \times \mathbf{v}_c) + \mathbf{h}^C \tag{3.11}$$

with $\mathbf{h}^C$ is the angular momentum of $\mathcal{B}$ about C. Eq. 3.11 asserts that the total angular momentum of the body $\mathcal{B}$ about O is equivalent to the angular momentum of a concentrated mass located in C, (namely the term $m(\mathbf{p}_c \times \mathbf{v}_c)$ in the equation) summed by the angular momentum of the body about C.
From the previous equations $\mathbf{h}^C$ is given as:

$$\mathbf{h}^C = \int_V (\mathbf{r} \times \frac{d\mathbf{r}}{dt})\rho dV$$

and since for a rigid body $\frac{d\mathbf{r}}{dt} = \boldsymbol{\omega}_B \times \mathbf{r}$ we have

$$\mathbf{h}^C = \int_V \mathbf{r} \times (\boldsymbol{\omega}_B \times \mathbf{r})\rho dV \tag{3.12}$$

Without providing any further detail the form of $\mathbf{h}^C$ in eq. 3.12 allows one to write:

$$\mathbf{h}^C = \mathbf{I}_B^C \boldsymbol{\omega_B} \tag{3.13}$$

$\mathbf{I}_B^C$ is the inertia matrix of $\mathcal{B}$ about C and expressed in A (the superscript on the left is omitted).

3.1.7 Transformation of inertia matrix

All the terms in eq. 3.13 are expressed in A. However the angular momentum can be expressed in any ref. system. In the ref. system C: $\{C, x_c, y_c, z_c\}$ it takes the following form:

$$^C\mathbf{h}^C = {}^C\mathbf{I}_B^{C\,C}\boldsymbol{\omega_B} \tag{3.14}$$

where ${}^C\mathbf{I}_B^C$ is the the inertia matrix of $\mathcal{B}$ about C and expressed in C, ${}^C\boldsymbol{\omega_B}$ is the absolute angular velocity (with respect to the fixed ref. system A) of $\mathcal{B}$ but expressed in C:

$$^C\boldsymbol{\omega_B} = {}^A\mathbf{R}_C^{T\,A}\boldsymbol{\omega_B}. \tag{3.15}$$

Now we need to find the relationship between ${}^C\mathbf{I}_B^C$ and ${}^A\mathbf{I}_B^C$.
Similarly to the angular velocity vector transformation we may write:

$$^C\mathbf{h}^C = {}^A\mathbf{R}_C^{T\,A}\mathbf{h}^C \tag{3.16}$$

with ${}^{C}\mathbf{h}^{C}$ given by eq. 3.14 and ${}^{A}\mathbf{h}^{C}$ given by eq. 3.13. By substitution we obtain:

$${}^{C}\mathbf{I}_{B}^{C}{}^{C}\boldsymbol{\omega_B} = {}^{A}\mathbf{R}_{C}^{T}{}^{A}\mathbf{I}_{B}^{C}{}^{A}\boldsymbol{\omega_B}$$

and by substituting eq. 3.15 leads:

$${}^{C}\mathbf{I}_{B}^{C}{}^{A}\mathbf{R}_{C}^{T}{}^{A}\boldsymbol{\omega_B} = {}^{A}\mathbf{R}_{C}^{T}{}^{A}\mathbf{I}_{B}^{C}{}^{A}\boldsymbol{\omega_B} \tag{3.17}$$

Finally we pre-multiply eq. 3.17 by ${}^{A}\mathbf{R}_{C}$ to obtain the relationship pursued:

$${}^{A}\mathbf{R}_{C}{}^{C}\mathbf{I}_{B}^{C}{}^{A}\mathbf{R}_{C}^{T} = {}^{A}\mathbf{I}_{B}^{C} \tag{3.18}$$

It is worth noting that terms of ${}^{C}\mathbf{I}_{B}^{C}$ are *constant* while terms of ${}^{A}\mathbf{I}_{B}^{C}$ are not but they depend on the ref. system orientation as claimed by eq. 3.18.

3.1.8 Kinetic Energy

The *kinetic energy* dK of a $dm = \rho dV$ with respect to A is defined as:

$$dK = \frac{1}{2}\mathbf{v}^T\mathbf{v}\rho dV$$

with $\mathbf{v}$ the velocity of dm. The kinetic energy of the body $\mathcal{B}$ is:

$$K = \frac{1}{2}\int_V \mathbf{v}^T\mathbf{v}\rho dV \tag{3.19}$$

Eq. 3.19 may be expressed in terms of $\mathbf{v}_c$ and $\boldsymbol{\omega}_B$ as $\mathbf{v} = \mathbf{v}_c + \boldsymbol{\omega}_B \times \mathbf{r}$:

$$\begin{aligned}\mathbf{v}^T\mathbf{v} = \mathbf{v}\cdot\mathbf{v} &= (\mathbf{v}_c + \boldsymbol{\omega}_B \times \mathbf{r})\cdot(\mathbf{v}_c + \boldsymbol{\omega}_B \times \mathbf{r}) = \\ &= \mathbf{v}_c\cdot\mathbf{v}_c + 2\mathbf{v}_c\cdot(\boldsymbol{\omega}_B \times \mathbf{r}) + (\boldsymbol{\omega}_B \times \mathbf{r})\cdot(\boldsymbol{\omega}_B \times \mathbf{r})\end{aligned}$$

Now taking advantage of the triple scalar product property:

$$\begin{aligned}\mathbf{v}_c\cdot(\boldsymbol{\omega}_B \times \mathbf{r}) &= (\mathbf{v}_c \times \boldsymbol{\omega}_B)\cdot\mathbf{r}, \\ (\boldsymbol{\omega}_B \times \mathbf{r})\cdot(\boldsymbol{\omega}_B \times \mathbf{r}) &= \boldsymbol{\omega}_B\cdot\mathbf{r}\times(\boldsymbol{\omega}_B \times \mathbf{r}).\end{aligned}$$

we can write:

$$K = \frac{1}{2}\mathbf{v}_c^T\mathbf{v}_c\int_V \rho dV + (\mathbf{v}_c \times \boldsymbol{\omega}_B)^T\int_V \mathbf{r}\rho dV + \frac{1}{2}\boldsymbol{\omega}_B^T\int_V \mathbf{r}\times(\boldsymbol{\omega}_B \times \mathbf{r})\rho dV$$

and considering that:

$$(\mathbf{v}_c \times \boldsymbol{\omega}_B)^T\int_V \mathbf{r}\rho dV = 0$$

because of eq. 3.2 and

$$\int_V \mathbf{r}\times(\boldsymbol{\omega}_B \times \mathbf{r})\rho dV = \mathbf{I}_B^C\boldsymbol{\omega}_B,$$

then we obtain K as:

$$K = \frac{1}{2}m\mathbf{v}_c^T\mathbf{v}_c + \frac{1}{2}\boldsymbol{\omega}_B^T\mathbf{I}_B^C\boldsymbol{\omega}_B \tag{3.20}$$

3.2 *Newton-Euler* laws for a rigid body

Consider the rigid body $\mathcal{B}$ as in Figure 3.2.

Let $\mathbf{f}$ be the resultant of the forces exerted on the rigid body $\mathcal{B}$ and $\mathbf{n}^O$ be

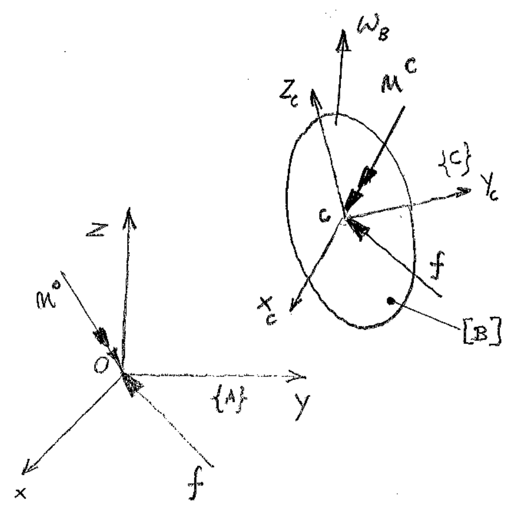

Figure 3.2: Resultant force and moment acting on a rigid body.

the resultant of the moments exerted on the rigid body $\mathcal{B}$ about O. Then the Newton-Euler laws can be stated as:

$$\mathbf{f} = \frac{d\mathbf{l}^O}{dt}; \tag{3.21}$$

$$\mathbf{n}^O = \frac{d\mathbf{h}^O}{dt}. \tag{3.22}$$

The previous equations are written considering O, the origin of the fixed ref. system A, as reference point. However, eq. 3.21 can be easily written in terms of the motion of the center of mass C. Indeed, by considering eq. 3.8 we obtain:

$$\mathbf{f} = \frac{d(m\mathbf{v}_c)}{dt} = m\frac{d\mathbf{v}_c}{dt} \tag{3.23}$$

Eq. 3.23 is called the *Newton's equation of motion* for the center of mass. Now, we differentiate with respect time $\mathbf{h}^O$, given by eq. 3.11, to use the result into eq 3.22.

$$\frac{d\mathbf{h}^O}{dt} = \frac{d\mathbf{h}^C}{dt} + m\frac{d\mathbf{p}_c}{dt} \times \mathbf{v}_c + m\mathbf{p}_c \times \frac{d\mathbf{v}_c}{dt}$$

and since $\frac{d\mathbf{p}_c}{dt} \times \mathbf{v}_c = \mathbf{v}_c \times \mathbf{v}_c = \mathbf{0}$ we obtain:

$$\frac{d\mathbf{h}^O}{dt} = \frac{d\mathbf{h}^C}{dt} + m\mathbf{p}_c \times \frac{d\mathbf{v}_c}{dt} \tag{3.24}$$

Eq. 3.24 finally becomes:

$$\mathbf{n}^O = \mathbf{n}^C + \mathbf{p}_c \times \mathbf{f} \tag{3.25}$$

because of eqs. 3.22, 3.23 and

$$\mathbf{n}^C = \frac{d\mathbf{h}^C}{dt} \tag{3.26}$$

Eq. 3.26 claims that the rate of change of the angular momentum of $\mathcal{B}$ about its center of mass C is equal to the resultant of the moments exerted on the rigid body $\mathcal{B}$ about C.
Now, we expand eq. 3.26:

$$\begin{aligned}\mathbf{n}^C = \frac{d\mathbf{h}^C}{dt} = \frac{d}{dt}(\mathbf{I}_B^{CA}\boldsymbol{\omega}_B) = \frac{d}{dt}[({}^A\mathbf{R}_C{}^C\mathbf{I}_B^{CA}\mathbf{R}_C^T){}^A\boldsymbol{\omega}_B] = \\ = {}^A\dot{\mathbf{R}}_C{}^C\mathbf{I}_B^{CA}\mathbf{R}_C^{TA}\boldsymbol{\omega}_B +{}^A\mathbf{R}_C{}^C\mathbf{I}_B^{CA}\dot{\mathbf{R}}_C^{TA}\boldsymbol{\omega}_B + {}^A\mathbf{R}_C{}^C\mathbf{I}_B^{CA}\mathbf{R}_C^{TA}\dot{\boldsymbol{\omega}}_B = \\ = {}^A\boldsymbol{\omega}_B \times \mathbf{I}_B^{CA}\boldsymbol{\omega}_B + \mathbf{I}_B^{CA}\dot{\boldsymbol{\omega}}_B\end{aligned} \tag{3.27}$$

where

$$\dot{\mathbf{R}}_C^{TA}\boldsymbol{\omega}_B = {}^A\mathbf{R}_C^{TA}\boldsymbol{\omega}_B \times {}^A\boldsymbol{\omega}_B = \mathbf{0},$$

and from eq. 3.18: ${}^A\mathbf{R}_C{}^C\mathbf{I}_B^{CA}\mathbf{R}_C^T = \mathbf{I}_B^C$.
Eq. 3.27 is called *Euler's equation of motion* for the fixed ref. system whose origin is located at the center of mass and with its coordinate axes parallel to those of A. Indeed, we have to point out that we are dealing with eq. 3.26 which is only a part of the eq. 3.25 that instead takes into account the resultant of the moments exerted on $\mathcal{B}$ about O, origin of A. Eq. 3.27 can be written in the center of mass ref. system as well by pre-multiplying both sides by ${}^A\mathbf{R}_C^T$.

$${}^A\mathbf{R}_C^T\mathbf{n}^C = {}^A\mathbf{R}_C^T({}^A\boldsymbol{\omega}_B \times \mathbf{I}_B^{CA}\boldsymbol{\omega}_B) + {}^A\mathbf{R}_C^T\mathbf{I}_B^{CA}\dot{\boldsymbol{\omega}}_B$$

and taking advantage of ${}^A\boldsymbol{\omega}_B = {}^A\mathbf{R}_C{}^C\boldsymbol{\omega}_B$ from eq. 3.15 and of ${}^A\dot{\boldsymbol{\omega}}_B = {}^A\mathbf{R}_C{}^C\dot{\boldsymbol{\omega}}_B$, we have:

$${}^C\mathbf{n}^C = {}^A\mathbf{R}_C^T\mathbf{I}_B^{CA}\mathbf{R}_C{}^C\dot{\boldsymbol{\omega}}_B + {}^A\mathbf{R}_C^{TA}\mathbf{R}_C{}^C\boldsymbol{\omega}_B \times {}^A\mathbf{R}_C^T\mathbf{I}_B^{CA}\mathbf{R}_C{}^C\boldsymbol{\omega}_B$$

that leads to:

$${}^C\mathbf{n}^C = {}^C\boldsymbol{\omega}_B \times {}^C\mathbf{I}_B^{CC}\boldsymbol{\omega}_B + {}^C\mathbf{I}_B^{CC}\dot{\boldsymbol{\omega}}_B \tag{3.28}$$

Eq. 3.28 is called *Euler's equation of motion* for the center of mass ref. system.

3.2.1 Direct and Inverse dynamics problems

1. *Direct problem*:
   Given $\mathbf{f}$ and ${}^C\mathbf{n}^C$ then the motion of $\mathcal{B}$ is obtained by integranting eqs. 3.23 and 3.28.

2. *Inverse problem*:
   Given the motion of $\mathcal{B}$ then $\mathbf{f}$ and ${}^C\mathbf{n}^C$ are obtained by eqs. 3.23 and 3.28.

3.2.2 Special Case

When the axes of the center of mass ref. system coincide with the principal axes of $\mathcal{B}$ then ${}^C\mathbf{I}_B^C$ becomes:

$$ {}^C\mathbf{I}_B^C = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix} \tag{3.29} $$

with I_{xx}, I_{yy}, I_{zz} principal moments of inertia of $\mathcal{B}$ about the center of mass ref. system. Eq. 3.28 takes the simplified form:

$$ \begin{aligned} n_x &= I_{xx}\dot{\omega}_x - \omega_y\omega_z(I_{yy} - I_{zz}); \\ n_y &= I_{yy}\dot{\omega}_y - \omega_z\omega_x(I_{zz} - I_{xx}); \\ n_z &= I_{zz}\dot{\omega}_z - \omega_z\omega_x(I_{xx} - I_{yy}); \end{aligned} $$

with ${}^C\mathbf{n}^C = \begin{pmatrix} n_x & n_y & n_z \end{pmatrix}^T$, ${}^C\dot{\boldsymbol{\omega}}_B = \begin{pmatrix} \dot{\omega}_x & \dot{\omega}_y & \dot{\omega}_z \end{pmatrix}^T$ and ${}^C\boldsymbol{\omega}_B = \begin{pmatrix} \omega_x & \omega_y & \omega_z \end{pmatrix}^T$.

3.2.3 Gyroscopic motion

Let consider the gyroscopic body shown in Figure 3.3. The orientation of the on-board ref. system $\{Cuvw\}$ with respect to the fixed ref. system $\{Oxyz\}$ is parametrized by ${}^A\mathbf{R}_C$. The orientation matrix can be handly obtained by the *zxz Euler's* angles convention such that:

$$ {}^A\mathbf{R}^C = \mathbf{R}_z(\psi)\mathbf{R}_x(\theta)\mathbf{R}_z(\phi) \quad \text{with } \phi = 0. $$

Angle θ is known as *nutation*, ψ as *precession* and ϕ as *revolution* of the gyro body. Now, we deal with the terms of eq. 3.27 which is the equation of motion of the gyroscopic body.

$$ \boldsymbol{\omega} = \dot{\psi}\mathbf{z} + \dot{\theta}\mathbf{u}' + \dot{\phi}\mathbf{w}'' \tag{3.30} $$

with

$$ \mathbf{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{u}' = \mathbf{R}_z(\psi)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{w}'' = \mathbf{R}_z(\psi)\mathbf{R}_x(\theta)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} $$

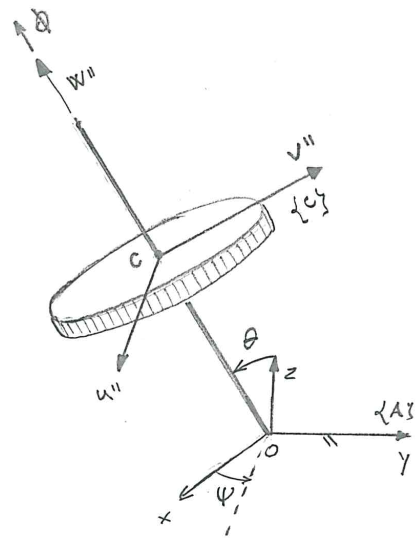

Figure 3.3: The gyro body.

we obtain:

$$\begin{aligned} \omega_x &= \dot{\theta} c_\psi + \dot{\phi} s_\psi s_\theta; \\ \omega_y &= \dot{\theta} s_\psi - \dot{\phi} c_\psi s_\theta; \\ \omega_z &= \dot{\psi} + \dot{\phi} c_\theta. \end{aligned} \tag{3.31}$$

By differentiating with the respect the time eqs. 3.31 we obtain $\dot{\boldsymbol{\omega}} = \begin{pmatrix} \dot{\omega}_x & \dot{\omega}_y & \dot{\omega}_z \end{pmatrix}^T$.
The inertia tensor $\mathbf{I}^C$ can be calculated according to eq. 3.18 with ${}^C I^C_{xx} = {}^C I^C_{yy} = I_0$ and ${}^C I^C_{zz} = I$, because of the axial symmetry of the gyro body. Eventually we can calculate $\mathbf{n}^C$ and then ${}^C\mathbf{n}^C = {}^A\mathbf{R}^T_C \mathbf{n}^C$:

$$\begin{aligned} n^C_{u''} &= I_0(\ddot{\theta} - \dot{\psi}^2 s_\theta c_\theta) + I\dot{\psi}(\dot{\psi} c_\theta + \dot{\phi}); \\ n^C_{v''} &= I_0(\ddot{\psi} s_\theta + 2\dot{\psi}\dot{\theta} c_\theta) - I\dot{\theta}(\dot{\psi} c_\theta + \dot{\phi}); \\ n^C_{w''} &= I(\ddot{\psi} c_\theta - \dot{\psi}\dot{\theta} s_\theta + \ddot{\phi}). \end{aligned} \tag{3.32}$$

Eqs. 3.32 are the general equations of rotation of a gyro body (*axial symmetrical body*) with fixed point O about its center of mass expressed in the body ref. system.
In the case of steady-state precession $\dot{\psi} = constant : \ddot{\psi} = 0, \theta = constant : \dot{\theta} = \ddot{\theta} = 0$, $\phi = constant : \dot{\phi} = \ddot{\phi} = 0$, then eqs. 3.32 take the simplified form:

$$\begin{aligned} n^C_{u''} &= \dot{\psi} s_\theta [I(\dot{\psi} c_\theta + \dot{\phi}) - I_0 \dot{\psi} c_\theta]; \\ n^C_{v''} &= 0; \\ n^C_{w''} &= 0. \end{aligned} \tag{3.33}$$

3.3 Recursive *Newton-Euler* formulation for serial manipulators

The recursive Newton-Euler formulation allows one to write the dynamic balance equations for each link of the serial manipulator. These equations include the forces of constraints between two adjacent links. The method consists of a *forward computation* of the velocities and accelerations of each link, followed by a *backward computation* of the forces and moments in each joints. For its own recursive nature the method is very well suited for automatic computation.
For the purpose of analysis we refer to Figure 3.4.

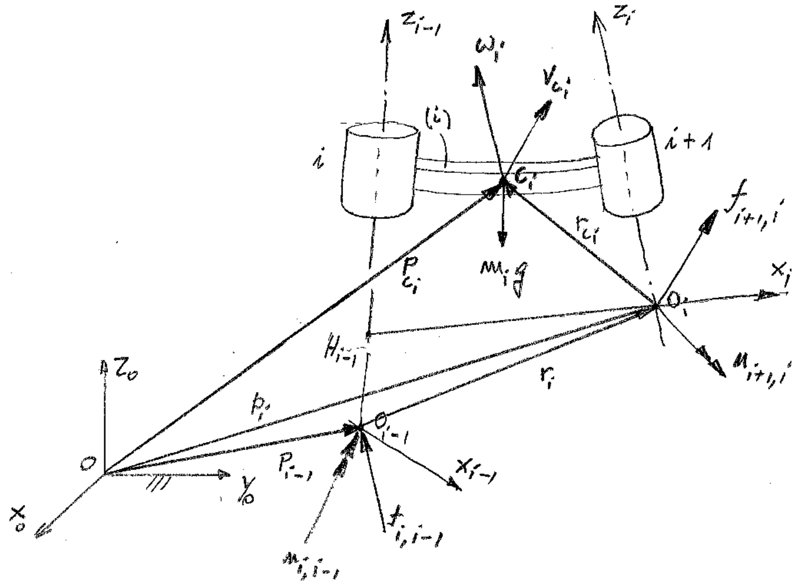

Figure 3.4: Forces and moments exerted on link i.

3.3.1 Forward computation

We calculate angular velocity, angular acceleration, linear velocity and linear acceleration of link i starting from the base, (link 0) which has: $\mathbf{v}_o = \dot{\mathbf{v}}_o = \boldsymbol{\omega}_0 = \dot{\boldsymbol{\omega}}_0 = \mathbf{0}$.

1. angular velocity propagation:

- revolute joint i

$$\begin{aligned}\boldsymbol{\omega}_i &= \boldsymbol{\omega}_{i-1} + \dot{\theta}_i \mathbf{z}_{i-1},\\ {}^i\boldsymbol{\omega}_i &=^i \mathbf{R}_{i-1}({}^{i-1}\boldsymbol{\omega}_{i-1} + \dot{\theta}_i{}^{i-1}\mathbf{z}_{i-1}),\\ {}^{i-1}\mathbf{z}_{i-1} &= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T.\end{aligned}$$

- primatic joint i

$$\begin{aligned}\boldsymbol{\omega}_i &= \boldsymbol{\omega}_{i-1},\\ {}^i\boldsymbol{\omega}_i &=^i \mathbf{R}_{i-1}{}^{i-1}\boldsymbol{\omega}_{i-1}.\end{aligned}$$

2. <u>angular acceleration propagation</u>:

- revolute joint i

$$\dot{\boldsymbol{\omega}}_i = \dot{\boldsymbol{\omega}}_{i-1} + \ddot{\theta}_i \mathbf{z}_{i-1} + \dot{\theta}_i \frac{d\mathbf{z}_{i-1}}{dt},$$

but $\frac{d\mathbf{z}_{i-1}}{dt} = \boldsymbol{\omega}_{i-1} \times \mathbf{z}_{i-1}$ then

$$\begin{aligned}\dot{\boldsymbol{\omega}}_i &= \dot{\boldsymbol{\omega}}_{i-1} + \ddot{\theta}_i \mathbf{z}_{i-1} + \boldsymbol{\omega}_{i-1} \times \dot{\theta}_i \mathbf{z}_{i-1}\\ {}^i\dot{\boldsymbol{\omega}}_i &=^i \mathbf{R}_{i-1}({}^{i-1}\dot{\boldsymbol{\omega}}_{i-1} + \ddot{\theta}_i^{i-1}\mathbf{z}_{i-1} + {}^{i-1}\boldsymbol{\omega}_{i-1} \times \dot{\theta}_i{}^{i-1}\mathbf{z}_{i-1})\end{aligned}$$

- prismatic joint i

$$\begin{aligned}\dot{\boldsymbol{\omega}}_i &= \dot{\boldsymbol{\omega}}_{i-1},\\ {}^i\dot{\boldsymbol{\omega}}_i &=^i \mathbf{R}_{i-1}{}^{i-1}\dot{\boldsymbol{\omega}}_{i-1}.\end{aligned}$$

3. <u>linear velocity propagation</u>:

- revolute joint i

$$\begin{aligned}\mathbf{v}_{Oi} \equiv \mathbf{v}_i &= \mathbf{v}_{i-1} + \boldsymbol{\omega}_i \times \mathbf{r}_i,\\ {}^i\mathbf{v}_i &=^i \mathbf{R}_{i-1}{}^{i-1}\mathbf{v}_{i-1} +^i \boldsymbol{\omega}_i \times {}^i\mathbf{r}_i.\end{aligned}$$

- prismatic joint i

$$\begin{aligned}\mathbf{v}_i &= \mathbf{v}_{i-1} + \boldsymbol{\omega}_i \times \mathbf{r}_i + \dot{d}_i \mathbf{z}_{i-1},\\ {}^i\mathbf{v}_i &=^i \mathbf{R}_{i-1}{}^{i-1}\mathbf{v}_{i-1} +^i \boldsymbol{\omega}_i \times {}^i\mathbf{r}_i + \dot{d}_i{}^i\mathbf{R}_{i-1}{}^{i-1}\mathbf{z}_{i-1},\\ {}^i\mathbf{r}_i =^i \mathbf{R}_{i-1}{}^{i-1}\mathbf{r}_i, \quad {}^i\mathbf{r}_i &= \begin{pmatrix} a_i & d_i \sin\alpha_i & d_i \cos\alpha_i \end{pmatrix}^T.\end{aligned}$$

4. <u>linear acceleration propagation</u>:
Firstly, we need $\frac{d\mathbf{r}_i}{dt}$ for further calculation:

$$\frac{d\mathbf{r}_i}{dt} = \frac{d(\mathbf{R}_i{}^i\mathbf{r}_i)}{dt} = \dot{\mathbf{R}}_i{}^i\mathbf{r}_i + \mathbf{R}_i{}^i\dot{\mathbf{r}}_i = \boldsymbol{\Omega}_i\mathbf{R}_i{}^i\mathbf{r}_i + \mathbf{R}_i{}^i\dot{\mathbf{r}}_i = \boldsymbol{\omega}_i \times \mathbf{r}_i + \mathbf{R}_i{}^i\dot{\mathbf{r}}_i.$$

and with

$$\mathbf{R}_i{}^i\dot{\mathbf{r}}_i = \mathbf{R}_{i-1}{}^{i-1}\mathbf{R}_i{}^i\dot{\mathbf{r}}_i = \mathbf{R}_{i-1}\begin{pmatrix} c_{\theta_i} & -c_{\alpha_i}s_{\theta_i} & s_{\alpha_i}s_{\theta_i} \\ s_{\theta_i} & c_{\alpha_i}c_{\theta_i} & -s_{\alpha_i}c_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} \end{pmatrix}\begin{pmatrix} 0 \\ \dot{d}_i s_{\alpha_i} \\ \dot{d}_i c_{\alpha_i} \end{pmatrix} = $$
$$= \dot{d}_i \mathbf{R}_{i-1}{}^{i-1}\mathbf{z}_{i-1} = \dot{d}_i \mathbf{z}_{i-1}.$$

we obtain:

$$\frac{d\mathbf{r}_i}{dt} = \boldsymbol{\omega}_i \times \mathbf{r}_i + \dot{d}_i \mathbf{z}_{i-1}.$$

- revolute joint i

$$\begin{aligned} \dot{\mathbf{v}}_i &= \dot{\mathbf{v}}_{i-1} + \dot{\boldsymbol{\omega}}_i \times \mathbf{r}_i + \boldsymbol{\omega}_i \times \frac{d\mathbf{r}_i}{dt} = \\ &= \dot{\mathbf{v}}_{i-1} + \dot{\boldsymbol{\omega}}_i \times \mathbf{r}_i + \boldsymbol{\omega}_i \times (\boldsymbol{\omega}_i \times \mathbf{r}_i), \\ {}^i\dot{\mathbf{v}}_i &= {}^i\mathbf{R}_{i-1}{}^{i-1}\dot{\mathbf{v}}_{i-1} + {}^i\dot{\boldsymbol{\omega}}_i \times {}^i\mathbf{r}_i + {}^i\boldsymbol{\omega}_i \times ({}^i\boldsymbol{\omega}_i \times {}^i\mathbf{r}_i). \end{aligned}$$

- prismatic joint i

$$\dot{\mathbf{v}}_i = \dot{\mathbf{v}}_{i-1} + \dot{\boldsymbol{\omega}}_i \times \mathbf{r}_i + \boldsymbol{\omega}_i \times \frac{d\mathbf{r}_i}{dt} + \ddot{d}_i \mathbf{z}_{i-1} + \dot{d}_i \frac{d\mathbf{z}_{i-1}}{dt} =$$
$$= \dot{\mathbf{v}}_{i-1} + \dot{\boldsymbol{\omega}}_i \times \mathbf{r}_i + \boldsymbol{\omega}_i \times (\boldsymbol{\omega}_i \times \mathbf{r}_i) + \boldsymbol{\omega}_i \times \dot{d}_i \mathbf{z}_{i-1} + \ddot{d}_i \mathbf{z}_{i-1} + \boldsymbol{\omega}_i \times \dot{d}_i \mathbf{z}_{i-1} =$$
$$= \dot{\mathbf{v}}_{i-1} + \dot{\boldsymbol{\omega}}_i \times \mathbf{r}_i + \boldsymbol{\omega}_i \times (\boldsymbol{\omega}_i \times \mathbf{r}_i) + 2\boldsymbol{\omega}_i \times \dot{d}_i \mathbf{z}_{i-1} + \ddot{d}_i \mathbf{z}_{i-1}.$$

in the last equation we use $\frac{d\mathbf{z}_{i-1}}{dt} = \boldsymbol{\omega}_{i-1} \times \mathbf{z}_{i-1} = \boldsymbol{\omega}_i \times \mathbf{z}_{i-1}$ as for a prismatic joint i: $\boldsymbol{\omega}_{i-1} = \boldsymbol{\omega}_i$.

$${}^i\dot{\mathbf{v}}_i = {}^i\mathbf{R}_{i-1}{}^{i-1}\dot{\mathbf{v}}_{i-1} + {}^i\dot{\boldsymbol{\omega}}_i \times {}^i\mathbf{r}_i +$$
$${}^i\boldsymbol{\omega}_i \times ({}^i\boldsymbol{\omega}_i \times {}^i\mathbf{r}_i) + 2{}^i\boldsymbol{\omega}_i \times \dot{d}_i{}^i\mathbf{R}_{i-1}{}^{i-1}\mathbf{z}_{i-1} + \ddot{d}_i{}^i\mathbf{R}_{i-1}{}^{i-1}\mathbf{z}_{i-1}.$$

5. linear acceleration of the center of mass:

$${}^i\dot{\mathbf{v}}_{c_i} = {}^i\dot{\mathbf{v}}_i + {}^i\dot{\boldsymbol{\omega}}_i \times {}^i\mathbf{r}_{c_i} + {}^i\boldsymbol{\omega}_i \times ({}^i\boldsymbol{\omega}_i \times {}^i\mathbf{r}_{c_i})$$

6. acceleration of gravity:

$${}^i\mathbf{g} = {}^i\mathbf{R}_{i-1}{}^{i-1}\mathbf{g}$$

3.3.2 Backward computation

We write down the force and moment balance equations of link i starting from the end effector (link n). We call ${}^i\mathbf{f}_{i,i-1}$ the force that link $i-1$ exerts on link i at O_{i-1} expressed into the ith. ref. system. The same meaning is for ${}^i\mathbf{n}_{i,i-1}$ denoting moment. We call ${}^i\mathbf{f}_i^*$ and ${}^i\mathbf{n}_i^*$, respectively, the inertia force and couple at C_i expressed into the ith. ref. system. We consider to be known the output force ${}^n\mathbf{f}_{n+1,n}$ and moment ${}^n\mathbf{n}_{n+1,n}$ that link n exerts onto environment.

1. force and moment balance equations about C_i:

$$
\begin{aligned}
{}^i\mathbf{f}_i^* + {}^i\mathbf{f}_{i,i-1} - {}^i\mathbf{f}_{i+1,i} + m_i{}^i\mathbf{g} = \mathbf{0} \\
{}^i\mathbf{n}_i^* + {}^i\mathbf{n}_{i,i-1} - {}^i\mathbf{n}_{i+1,i} + {}^i\mathbf{r}_{c_i} \times {}^i\mathbf{f}_{i+1,i} - ({}^i\mathbf{r}_i + {}^i\mathbf{r}_{c_i}) \times {}^i\mathbf{f}_{i,i-1} = \mathbf{0}.
\end{aligned} \tag{3.34}
$$

Refer to Figure 3.5 to understand the signs in the moment equation.

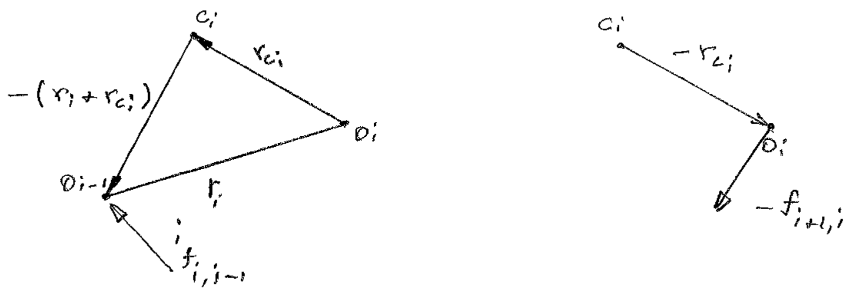

Figure 3.5: Signs of the moments in eq. 3.34.

Explicitly:

$$ {}^i\mathbf{f}_i^* = -m_i{}^i\dot{\mathbf{v}}_{c_i} \tag{3.35} $$

$$ {}^i\mathbf{n}_i^* = -{}^i\mathbf{I}_i{}^i\dot{\boldsymbol{\omega}}_i - {}^i\boldsymbol{\omega}_i \times ({}^i\mathbf{I}_i{}^i\boldsymbol{\omega}_i) \tag{3.36} $$

Eq. 3.36 can be easily obtained by following the same calculation performed to obtain eq. 3.27. The procedure is briefly reported here for clarity:

$$
\begin{aligned}
\mathbf{n}_i^* = -\frac{d\mathbf{h}^i}{dt} = -\frac{d(\mathbf{I}_i\boldsymbol{\omega}_i)}{dt} = -\frac{d(\mathbf{R}_i{}^i\mathbf{I}_i\mathbf{R}_i^T\boldsymbol{\omega}_i)}{dt} = \\
= -(\dot{\mathbf{R}}_i{}^i\mathbf{I}_i\mathbf{R}_i^T\boldsymbol{\omega}_i + \mathbf{R}_i{}^i\mathbf{I}_i\dot{\mathbf{R}}_i^T\boldsymbol{\omega}_i + \mathbf{R}_i{}^i\mathbf{I}_i\mathbf{R}_i^T\dot{\boldsymbol{\omega}}_i) = \\
= -(\boldsymbol{\omega}_i \times \mathbf{I}_i\boldsymbol{\omega}_i + \mathbf{I}_i\dot{\boldsymbol{\omega}}_i),
\end{aligned} \tag{3.37}
$$

where

$$ \dot{\mathbf{R}}_i^T\boldsymbol{\omega}_i = \mathbf{R}_i^T\boldsymbol{\Omega}_i^T\boldsymbol{\omega}_i = \mathbf{0}. $$

Eventually ${}^i\mathbf{n}_i^* = \mathbf{R}_i{}^T\mathbf{n}_i^*$.

2. force and moment in the recursive procedure:
From eqs. 3.34 we obtain ${}^i\mathbf{f}_{i,i-1}$ and ${}^i\mathbf{n}_{i,i-1}$ in recursive form (that is, starting from the end effector backward to the base):

$$
\begin{aligned}
{}^i\mathbf{f}_{i,i-1} = {}^i\mathbf{f}_{i+1,i} - m_i{}^i\mathbf{g} - {}^i\mathbf{f}_i^* \\
{}^i\mathbf{n}_{i,i-1} = {}^i\mathbf{n}_{i+1,i} + ({}^i\mathbf{r}_i + {}^i\mathbf{r}_{c_i}) \times {}^i\mathbf{f}_{i,i-1} - {}^i\mathbf{r}_{c_i} \times {}^i\mathbf{f}_{i+1,i} - {}^i\mathbf{n}_i^*
\end{aligned}
$$

and eventually express them into $i-1$ ref. system:

$$
\begin{aligned}
{}^{i-1}\mathbf{f}_{i,i-1} &= {}^{i-1}\mathbf{R}_i {}^{i}\mathbf{f}_{i,i-1} \\
{}^{i-1}\mathbf{n}_{i,i-1} &= {}^{i-1}\mathbf{R}_i {}^{i}\mathbf{n}_{i,i-1}
\end{aligned}
$$

3. Joint torque equations:

- revolute joint i

$$\tau_i = {}^{i-1}\mathbf{n}_{i,i-1}^T \, {}^{i-1}\mathbf{z}_{i-1} \tag{3.38}$$

- prismatic joint i

$$\tau_i = {}^{i-1}\mathbf{f}_{i,i-1}^T \, {}^{i-1}\mathbf{z}_{i-1} \tag{3.39}$$

3.4 Lagrange Formulation

We wish to write the equations of motion for a mechanical system. The proof is carried out for a mechanical system represented by a mass point. However, the same proof is valid for a mechanical system under linear motion once the mass point kinematics parameters are substituted by those of the center of mass of the mechanical system.

3.4.1 Lagrange's equations for effective displacements

Consider a mass point in a mechanical system whose vector position is $\mathbf{p} = \mathbf{p}(t)$. The work balance equation for that system states:

$$d\mathcal{W} + d\mathcal{L}_{cons} + d\mathcal{L}^* = 0 \tag{3.40}$$

with $d\mathcal{W}$ the work done by the external forces, $d\mathcal{L}_{cons}$ the work done by the conservative forces and $d\mathcal{L}^*$ the work done by the inertial forces onto the mass point to move it by a infinitesimal displacement. Eq. 3.40 may be written in terms of energy. Indeed:

$$
\begin{aligned}
d\mathcal{L}_{cons} &= -dU; \\
d\mathcal{L}^* = -m\mathbf{a} \cdot d\mathbf{p} &= -m\frac{d\mathbf{v}}{dt} \cdot \mathbf{v}dt = -dK.
\end{aligned}
$$

with U and K which denote the potential and kinetic energy of the system, respectively. Therefore eq. 3.40 becomes:

$$d(U + K) = d\mathcal{W} \tag{3.41}$$

In the robotic system, however, any

$$\mathbf{p} = \mathbf{p}(\mathbf{q}(t)) \tag{3.42}$$

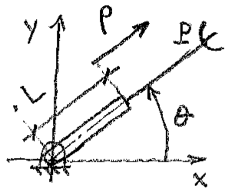

Figure 3.6: Lagrangian coordinates for a rotating link with variable length.

with $\mathbf{q} = \begin{pmatrix} q_1 & \cdots & q_i & \cdots & q_n \end{pmatrix}^T$ being the vector of *independent generalized coordinates*. For example for the rotating link with variable length in Figure 3.6:
$\mathbf{p} = \begin{pmatrix} (L+\rho)c_\theta & (L+\rho)s_\theta \end{pmatrix}^T$, with $\mathbf{q}(t) = \begin{pmatrix} \rho(t) & \theta(t) \end{pmatrix}^T$.
The independent generalized coordinates are also called *Lagrangian coordinates*. They represent the minimum number of parameters required to define the position of any point in the system. It is worth noting that the Lagrangian coordinates coincide with the joint variables in the serial manipulators (with no redundant actuation neither under-actuated).
Because of eq. 3.42 we obtain a new equation from eq. 3.40:

$$d\mathbf{p} = \mathbf{J}\dot{\mathbf{q}}dt, \quad \text{with } \mathbf{J} = \begin{pmatrix} \frac{\partial \mathbf{p}}{\partial q_1} & \cdots & \frac{\partial \mathbf{p}}{\partial q_n} \end{pmatrix},$$
$$\mathbf{v} = \frac{d\mathbf{p}}{dt} = \mathbf{J}\dot{\mathbf{q}}, \quad \text{and} \quad \frac{\partial \mathbf{v}}{\partial \dot{\mathbf{q}}} = \mathbf{J}.$$

We know $\mathbf{J}$ as the jacobian of the system.
Then:

$$d\mathcal{L}^* = -m\mathbf{a}\cdot d\mathbf{p} = -m(\frac{d\mathbf{v}}{dt})^T(\mathbf{J}d\mathbf{q}) = -m[(\frac{d\mathbf{v}}{dt})^T\frac{\partial \mathbf{v}}{\partial \dot{\mathbf{q}}}]d\mathbf{q}.$$

It is worth noting here that the term in the bracket is $(1 \times n)$. Then:

$$(\frac{d\mathbf{v}}{dt})^T\frac{\partial \mathbf{v}}{\partial \dot{\mathbf{q}}} = (\frac{d\mathbf{v}}{dt})\cdot\frac{\partial \mathbf{v}}{\partial \dot{\mathbf{q}}} = \frac{d}{dt}(\mathbf{v}\cdot\frac{\partial \mathbf{v}}{\partial \dot{\mathbf{q}}}) - \mathbf{v}\cdot\frac{d}{dt}(\frac{\partial \mathbf{v}}{\partial \dot{\mathbf{q}}})$$

and thus:

$$d\mathcal{L}^* = -m[\frac{d}{dt}(\mathbf{v}\cdot\frac{\partial \mathbf{v}}{\partial \dot{\mathbf{q}}}) - \mathbf{v}\cdot\frac{d}{dt}(\frac{\partial \mathbf{v}}{\partial \dot{\mathbf{q}}})]d\mathbf{q}$$

The terms in the bracket can be written as:

$$\mathbf{v}\cdot\frac{\partial \mathbf{v}}{\partial \dot{\mathbf{q}}} = \frac{\partial}{\partial \dot{\mathbf{q}}}(\frac{1}{2}\mathbf{v}\cdot\mathbf{v})$$
$$\mathbf{v}\cdot\frac{d}{dt}(\frac{\partial \mathbf{v}}{\partial \dot{\mathbf{q}}}) = \mathbf{v}\cdot\frac{d}{dt}(\frac{\partial \mathbf{p}}{\partial \mathbf{q}}) = \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{q}}(\frac{d\mathbf{p}}{dt}) = \mathbf{v}\cdot\frac{\partial \mathbf{v}}{\partial \mathbf{q}} = \frac{\partial}{\partial \mathbf{q}}(\frac{1}{2}\mathbf{v}\cdot\mathbf{v})$$

and eventually by using column vector $(n \times 1)$ in the bracket:

$$d\mathcal{L}^* = -[\frac{d}{dt}(\frac{\partial K}{\partial \dot{\mathbf{q}}}) - (\frac{\partial K}{\partial \mathbf{q}})]^T d\mathbf{q}, \tag{3.43}$$

$$d\mathcal{L}_{cons} = \mathbf{F}_{cons}^T d\mathbf{p} = \mathbf{F}_{cons}^T \frac{\partial \mathbf{p}}{\partial \mathbf{q}} d\mathbf{q} = (\frac{\partial \Phi}{\partial \mathbf{q}})^T d\mathbf{q} = -(\frac{\partial U}{\partial \mathbf{q}})^T d\mathbf{q}, \tag{3.44}$$

$$d\mathcal{W} = \mathbf{F}_e^T d\mathbf{p} = \mathbf{F}_e^T \frac{\partial \mathbf{p}}{\partial \mathbf{q}} d\mathbf{q} = \mathbf{F}_e^T \mathbf{J} d\mathbf{q} \tag{3.45}$$

By using eqs. 3.43, 3.44 and 3.45 we obtain eq. 3.40 as a set of n equations:

$$\frac{d}{dt}(\frac{\partial K}{\partial \dot{q}_i}) - \frac{\partial K}{\partial q_i} + \frac{\partial U}{\partial q_i} = \frac{d\mathcal{W}}{dq_i} = Q_i \quad (i = 1, \cdots, n). \tag{3.46}$$

Eq. 3.46 is the *Lagrange's equation of motion* written for effective displacements.

3.4.2 Lagrange's equations for virtual displacements

In a more general case position of any mass point in a mechanical system may depend on $\mathbf{q}(t)$ and on the time t such that:

$$\mathbf{p} = \mathbf{p}(\mathbf{q}(t), t). \tag{3.47}$$

In this case it may not be able to obtain the equations of motions starting from eq. 3.40. The only way to proceed is to consider a *virtual displacement* $\delta\mathbf{p}$ instead of the effective one:

$$\delta\mathbf{p} = \mathbf{p}(\mathbf{q} + \delta\mathbf{q}, t) - \mathbf{p}(\mathbf{q}, t) = \frac{\partial \mathbf{p}}{\partial \mathbf{q}} \delta\mathbf{q}. \tag{3.48}$$

The virtual displacement is a geometric tool. It represents an arbitrary infinitesimal change in the system configuration due to any change of the lagrangianan coordinates. The virtual displacement has to be consistent with the constrained imposed at the time t. The word *displacement* is misleading as there is no change of the position vector in time into eq. 3.48. If eq. 3.47 is used to calculate the terms in eq. 3.40 we obtain the *Lagrange's equation of motion* written for virtual displacements:

$$\frac{d}{dt}(\frac{\partial K}{\partial \dot{q}_i}) - \frac{\partial K}{\partial q_i} + \frac{\partial U}{\partial q_i} = \frac{\delta\mathcal{W}}{\delta q_i} = Q_i \quad (i = 1, \cdots, n). \tag{3.49}$$

Eq. 3.49 (as well as eq. 3.46) can be expressed in terms of the *Lagrange's Function* $\mathcal{L} = K - U$ as:

$$\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}) - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i \quad (i = 1, \cdots, n). \tag{3.50}$$

3.4.3 Nature of the static equilibrium

Consider the static equilibrium for a such mechanical system. Then:

$$\frac{\partial K}{\partial \dot{q}_i} = \frac{\partial K}{\partial q_i} = 0,$$

Furthermore consider no external forces exerted onto the system:

$$\frac{\delta \mathcal{W}}{\delta q_i} = 0,$$

then the eq. 3.49 takes the simplified form:

$$\begin{aligned} \frac{\partial U}{\partial q_1} &= 0 \\ &\vdots \\ \frac{\partial U}{\partial q_i} &= 0 \\ &\vdots \\ \frac{\partial U}{\partial q_n} &= 0 \end{aligned} \tag{3.51}$$

Eq. 3.51 provides the equilibrium configurations for the system. In order to gain more details on the nature of the equilibrium configurations one may analyze the *Hessian* matrix defined as:

$$\mathbf{H} = \begin{pmatrix} U''_{1,1} & \cdots & U''_{1,n} \\ \vdots & \cdots & \vdots \\ U''_{n,1} & \cdots & U''_{n,n} \end{pmatrix},$$

where

$$U''_{i,j} = \frac{\partial}{\partial q_i}\left(\frac{\partial U}{\partial q_j}\right).$$

H is a symmetric and positive definite matrix whose discriminant analysis allows one to know if the equilibrium configurations were stable.

We, now, restrict the study to a mechanical system with only one degree of freedom, namely with only one lagrangian coordinate q needed to define any configuration of the system. Then, the equilibrium equation becomes:

$$U' = \frac{dU}{dq} = 0. \tag{3.52}$$

Solution of eq. 3.52 is obtained when $q = \overline{q}$, being $\overline{q}$ the value assumed by the lagrangian coordinate at the equilibrium configuration. By taking the second derivative of the potential energy of the system calculated at the equilibrium configuration ($q = \overline{q}$) allows us to gain insight into the nature of the equilibrium:

$$U''(\overline{q}) = \frac{d^2U}{dq^2}_{|q=\overline{q}} > 0 \quad \text{stable equilibrium},$$
$$U''(\overline{q}) = \frac{d^2U}{dq^2}_{|q=\overline{q}} < 0 \quad \text{instable equilibrium},$$
$$U''(\overline{q}) = \frac{d^2U}{dq^2}_{|q=\overline{q}} = 0 \quad \text{indefferent equilibrium}.$$

3.4.4 General form of dynamical equations in serial manipulators

We wish to write for serial manipulators eq. 3.49 explicitly.

1. Kinetic Energy
   For the body i:

$$K_i = \frac{1}{2}\mathbf{v}_{c_i}^T m_i \mathbf{v}_{c_i} + \frac{1}{2}\boldsymbol{\omega}_i^T \mathbf{I}_i \boldsymbol{\omega}_i$$

   with $\mathbf{I}_i \equiv {}^0\mathbf{I}_i^{C_i} = {}^0\mathbf{R}_i {}^i\mathbf{I}_i^{C_i} {}^0\mathbf{R}_i^T$. The velocity of C_i and the angular velocity of the link i can be expressed in matrix form:

$$\dot{\mathbf{x}}_{c_i} = \begin{pmatrix} \mathbf{v}_{c_i} \\ \boldsymbol{\omega}_i \end{pmatrix} = \mathbf{J}_i \dot{\mathbf{q}} = \begin{pmatrix} \mathbf{J}_{v_i} \\ \mathbf{J}_{\omega_i} \end{pmatrix} \dot{\mathbf{q}}$$

   $\mathbf{J}_i$ is $(6 \times n)$, while $\mathbf{J}_{v_i}$ and $\mathbf{J}_{\omega_i}$ are $(3 \times n)$.

$$\mathbf{J}_{v_i} = \begin{pmatrix} \mathbf{J}_{v_i}^1 & \cdots & \mathbf{J}_{v_i}^{j=i} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix},$$
$$\mathbf{J}_{\omega_i} = \begin{pmatrix} \mathbf{J}_{\omega_i}^1 & \cdots & \mathbf{J}_{\omega_i}^{j=i} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix},$$

   - revolute joint $j \leq i$ (link i).

$$\mathbf{J}_{v_i}^j = \mathbf{z}_{j-1} \times \mathbf{p}_{o_{j-1},c_i},$$
$$\mathbf{J}_{\omega_i}^j = \mathbf{z}_{j-1},$$

   - prismatic joint $j \leq i$ (link i).

$$\mathbf{J}_{v_i}^j = \mathbf{z}_{j-1},$$
$$\mathbf{J}_{\omega_i}^j = \mathbf{0}_{3\times 1}.$$

Substituting the $\mathbf{J}_{v_i}$ and $\mathbf{J}_{\omega_i}$ expressions and summing over all links, we obtain an expression of the kinetic energy of the manipulator as:

$$K = \sum_{i=1}^{n} K_i = \frac{1}{2}\sum_{i=1}^{n}(\mathbf{v}_{c_i}^T m_i \mathbf{v}_{c_i} + \frac{1}{2}\boldsymbol{\omega}_i^T \mathbf{I}_i \boldsymbol{\omega}_i) =$$
$$= \frac{1}{2}\sum_{i=1}^{n}[(\mathbf{J}_{v_i}\dot{\mathbf{q}})^T m_i(\mathbf{J}_{v_i}\dot{\mathbf{q}}) + (\mathbf{J}_{\omega_i}\dot{\mathbf{q}})^T \mathbf{I}_i(\mathbf{J}_{\omega_i}\dot{\mathbf{q}})] =$$
$$= \frac{1}{2}\dot{\mathbf{q}}^T[\sum_{i=1}^{n}(\mathbf{J}_{v_i}^T m_i \mathbf{J}_{v_i} + \mathbf{J}_{\omega_i}^T \mathbf{I}_i \mathbf{J}_{\omega_i})]\dot{\mathbf{q}} = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M}\dot{\mathbf{q}} \tag{3.53}$$

The matrix $\mathbf{M}$:

$$\mathbf{M} = \sum_{i=1}^{n}(\mathbf{J}_{v_i}^T m_i \mathbf{J}_{v_i} + \mathbf{J}_{\omega_i}^T \mathbf{I}_i \mathbf{J}_{\omega_i}) \tag{3.54}$$

is called *manipulator inertia matrix.* $\mathbf{M}$ is $(n \times n)$, it depends on the configuration $\mathbf{M} = \mathbf{M}(\mathbf{q})$, it is symmetric and positive definite.

2. <u>Potential Energy</u>
   Work required to raise the center of mass C_i (link i) from horizontal plane ($U = 0$) to the actual position under the presence of gravity:

$$U = -\sum_{i=1}^{n} m_i \mathbf{g}^T \mathbf{p}_{c_i} \tag{3.55}$$

   where $\mathbf{g}$ denotes the gravity acceleration.

3. <u>Generalized forces</u>
   Consider external generalized forces ${}^n\mathbf{F}_{env}$ applied at the ref. point of the end effector and motor generalized torques $\boldsymbol{\tau}$ applied at the joints. Furthermore, consider the external generalized forces expressed by the n ref. system.

$${}^n\mathbf{F}_{env} = \begin{pmatrix} {}^n\mathbf{f}^T & {}^n\mathbf{n}^T \end{pmatrix}^T,$$
$$\boldsymbol{\tau} = \begin{pmatrix} \tau_1 & \cdots & \tau_n \end{pmatrix}^T.$$

   The external generalized forces are usually expressed by the n ref. system because of the force sensors mounted aboard the end effector. First we express the external generalized forces by the base ref. system:

$$\mathbf{F}_{env} = {}^0\tilde{\mathbf{R}}_n {}^n\mathbf{F}_{env}$$

   with:

$${}^0\tilde{\mathbf{R}}_n = \begin{pmatrix} {}^0\mathbf{R}_n & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & {}^0\mathbf{R}_n \end{pmatrix},$$

then, the virtual work done by $\mathbf{F}_{env}$ and $\boldsymbol{\tau}$ is:

$$\delta\mathcal{W} = \mathbf{Q}^T\delta\mathbf{q} = \boldsymbol{\tau}^T\delta\mathbf{q} + \mathbf{F}_{env}^T\delta\mathbf{x}_{o_n} = \boldsymbol{\tau}^T\delta\mathbf{q} + \mathbf{F}_{env}^T\mathbf{J}\delta\mathbf{q} = (\boldsymbol{\tau}^T + \mathbf{F}_{env}^T\mathbf{J})\delta\mathbf{q}$$

Thus:

$$\frac{\delta\mathcal{W}}{\delta\mathbf{q}} = \mathbf{Q} = \boldsymbol{\tau} + \mathbf{J}^T\mathbf{F}_{env} \tag{3.56}$$

The same result may be obtained whenever the n ref. system was used to write the work done by the external generalized forces. Thus:

$$\mathbf{Q} = \boldsymbol{\tau} + {}^n\mathbf{J}^{T\,n}\mathbf{F}_{env} \tag{3.57}$$

$$\text{with} \quad {}^n\mathbf{J} = {}^n\tilde{\mathbf{R}}_0\mathbf{J}, \quad {}^n\tilde{\mathbf{R}}_0 = \begin{pmatrix} {}^0\mathbf{R}_n^T & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & {}^0\mathbf{R}_n^T \end{pmatrix}.$$

Eqs. 3.56, 3.57 lead to the same result since:

$${}^n\mathbf{J}^{T\,n}\mathbf{F}_{env} = ({}^n\tilde{\mathbf{R}}_0\mathbf{J})^{T\,n}\tilde{\mathbf{R}}_0\mathbf{F}_{env} = \mathbf{J}^T({}^n\tilde{\mathbf{R}}_0^{T\,n}\tilde{\mathbf{R}}_0)\mathbf{F}_{env} = \mathbf{J}^T\mathbf{F}_{env}.$$

Now, we derive the general form of the dynamical equations. First we write the Lagrange's function:

$$\mathcal{L} = K - U = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}} + \sum_{i=1}^{n} m_i\mathbf{g}^T\mathbf{p}_{c_i} =$$
$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}\dot{q}_i\dot{q}_j + \sum_{i=1}^{n} m_i\mathbf{g}^T\mathbf{p}_{c_i}$$

then we take the opportune partial derivatives:

$$\frac{\partial\mathcal{L}}{\partial\dot{q}_i} = \sum_{j=1}^{n} M_{ij}\dot{q}_j,$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right) = \sum_{j=1}^{n} M_{ij}\ddot{q}_j + \sum_{j=1}^{n}\left(\frac{dM_{ij}}{dt}\dot{q}_j\right) = \sum_{j=1}^{n} M_{ij}\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial M_{ij}}{\partial q_k}\dot{q}_k\dot{q}_j,$$

$$\frac{\partial\mathcal{L}}{\partial q_i} = \frac{1}{2}\frac{\partial}{\partial q_i}\left(\sum_{j=1}^{n}\sum_{k=1}^{n} M_{jk}\dot{q}_j\dot{q}_k\right) + \sum_{j=1}^{n} m_j\mathbf{g}^T\left(\frac{\partial\mathbf{p}_{c_j}}{\partial q_i}\right).$$

The term $(\frac{\partial\mathbf{p}_{c_j}}{\partial q_i})$ is the column i of $\mathbf{J}_{v_j}$, that is:

$$\left(\frac{\partial\mathbf{p}_{c_j}}{\partial q_i}\right) = \mathbf{J}_{v_j}^i,$$

thus:

$$\frac{\partial\mathcal{L}}{\partial q_i} = \frac{1}{2}\frac{\partial}{\partial q_i}\left(\sum_{j=1}^{n}\sum_{k=1}^{n} M_{jk}\dot{q}_j\dot{q}_k\right) + \sum_{j=1}^{n} m_j\mathbf{g}^T\mathbf{J}_{v_j}^i =$$
$$= \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial M_{jk}}{\partial q_i}\dot{q}_j\dot{q}_k + \sum_{j=1}^{n} m_j\mathbf{g}^T\mathbf{J}_{v_j}^i.$$

Eventually, we can substitute the equations obtained into eq. 3.50:

$$\sum_{j=1}^{n} M_{ij}\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial M_{ij}}{\partial q_k}\dot{q}_k\dot{q}_j - \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial M_{jk}}{\partial q_i}\dot{q}_j\dot{q}_k - \sum_{j=1}^{n} m_j \mathbf{g}^T \mathbf{J}_{v_j}^i = Q_i$$

that we may write as:

$$\sum_{j=1}^{n} M_{ij}\ddot{q}_j + V_i + G_i = Q_i \quad i = 1, \cdots, n. \tag{3.58}$$

where

$$V_i = \sum_{j=1}^{n}\sum_{k=1}^{n} \left(\frac{\partial M_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial M_{jk}}{\partial q_i}\right)\dot{q}_k\dot{q}_j,$$

$$G_i = -\sum_{j=1}^{n} m_j \mathbf{g}^T \mathbf{J}_{v_j}^i.$$

The first term $\sum_{j=1}^{n} M_{ij}\ddot{q}_j$ in eq. 3.58 accounts for inertia forces, the second term V_i represents the *Coriolis* and *centrifugal* forces, and the third term G_i gives gravitational effects. The n scalar equations given in eq. 3.58 can be written in matrix form as:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{V} + \mathbf{G} = \mathbf{Q}, \tag{3.59}$$

where $\mathbf{V} = \begin{pmatrix} V_1 & \cdots & V_n \end{pmatrix}^T$, $\mathbf{G} = \begin{pmatrix} G_1 & \cdots & G_n \end{pmatrix}^T$, $\mathbf{Q} = \begin{pmatrix} Q_1 & \cdots & Q_n \end{pmatrix}^T$. Eq. 3.59 is called the *general form of dynamical equations*. $\mathbf{V}$ is called the *velocity coupling vector*, $\mathbf{G}$ is called the vector of gravitational forces. The velocity coupling between joints may be of two types: *a*) velocity-squared terms correspond to centrifugal forces, *b*) velocity product term correspond to the Coriolis forces. As mentioned, $\mathbf{M}$ is symmetric, positive definite and thus always invertible. The off-diagonal terms of $\mathbf{M}$ represent the accelerarion coupling effect between joints.

1. *Direct problem*:
   Given $\mathbf{Q}$, $\mathbf{q}$ and $\dot{\mathbf{q}}$ at a given instant then the later motion of the manipulator links is obtained by integrating $\ddot{\mathbf{q}}$ obtained from eq. 3.59.

2. *Inverse problem*:
   Given the joints trajectories: $\mathbf{q}(t)$, $\dot{\mathbf{q}}(t)$ and $\ddot{\mathbf{q}}(t)$ then $\mathbf{Q}(t)$ is obtained from eq. 3.59.

3.4.5 Force ellipsoid

Consider the static equilibrium for a serial manipulator whose links have all negligible mass. In this case eq. 3.49 takes the simplified form:

$$\frac{\delta \mathcal{W}}{\delta q_i} = Q_i = 0 \quad (i = 1, \cdots, n) \tag{3.60}$$

which in compact form becomes:

$$\frac{\delta \mathcal{W}}{\delta \mathbf{q}} = \mathbf{Q} = \mathbf{0} \tag{3.61}$$

Furthermore, eq. 3.61 can be expressed according to the following substitutions:

$$\delta \mathcal{W} = \mathbf{F}_{env}^T \delta \mathbf{x}_{o_n} + \boldsymbol{\tau}^T \delta \mathbf{q} = \mathbf{F}_{env}^T \mathbf{J} \delta \mathbf{q} + \boldsymbol{\tau}^T \delta \mathbf{q}$$

such that:

$$\frac{\delta \mathcal{W}}{\delta \mathbf{q}} = \mathbf{J}^T \mathbf{F}_{env} + \boldsymbol{\tau} = \mathbf{0} \tag{3.62}$$

Finally we consider $\mathbf{F}_r = -\mathbf{F}_{env}$ as the end-effector output generalized force such that eq. 3.62 becomes:

$$\boldsymbol{\tau} = \mathbf{J}^T \mathbf{F}_r \tag{3.63}$$

Eq. 3.63 maps an m-dimensional end-effector output generalized force ($\mathbf{F}_r \in \mathbb{V}^m$) into an n-dimensional joint torques ($\boldsymbol{\tau} \in \mathbb{V}^n$). Since $\mathbf{J}$ is configuration dependent then the mapping is configuration dependent too.
Similar to the transformation of velocities, for a manipulator with only one type of joints and one type of tasks we may compare the end-effector generalized forces produced by an unity joint torque. Thus, we confine the joint torque vector on a n-dimensional unit hyper-sphere:

$$\boldsymbol{\tau}^T \boldsymbol{\tau} = 1 \tag{3.64}$$

And, then we compare the corresponding end-effector output generalized forces in $\mathbb{V}^m$:

$$\mathbf{F}_r^T \mathbf{J} \mathbf{J}^T \mathbf{F}_r = 1 \tag{3.65}$$

Eq. 3.65 represents an hyper-ellipsoid in $\mathbb{V}^m$. $\mathbf{J}\mathbf{J}^T$ is a symmetric semipositive definite matrix, namely all the eigenvalues $\lambda_k \geq 0$, $(k = 1, \cdots, m)$, and the eigenvectors are orthogonal. The pricipal axes of the ellipsoid coincide with the eigenvectors of $\mathbf{J}\mathbf{J}^T$ and the lengths l_k of the principal axes are $l_k = 1/\sqrt{\lambda_k}$. The closer the force ellipsoid to a sphere, the better the transmission characteristics are. The transformation is said to be *isotropic* when the principal axes are of equal lengths. At an isotropic point, an n-dimensional unit sphere in the joint torque space maps onto an m-dimensional sphere in the end effector output generalized force space. On the other hand, at a singular point, an n-dimensional unit sphere in the joint torque space maps onto an m-dimensional cylinder in the end effector output generalized force space. Thus the mechanical advantage of the manipulator becomes infinitely large in some direction (namely, we obtain output force in some directions with no joint torques, that is, there exist some generalized forces that may be balanced by no joint torques).

4

Analysis of Alternative Robotic Mechanical Systems

In this chapter we deal with:

1. Kinematics of parallel manipulators;
2. Dynamics of constraint mechanisms (parallel manipulators);
3. Kinematics and dynamics of rolling robots;
4. Kinematic synthesis.

4.1 Parallel manipulators: basic definitions

A fairly general parallel manipulator is shown in Figure 4.1. It is formed by a fixed base $\mathcal{B}$, a moving platform $\mathcal{P}$ and some serial kinematic chains, *i.e., legs*, connecting $\mathcal{B}$ and $\mathcal{P}$. The base, moving platform and legs form closed subchains. At least one of the joints in each leg has to be actuated

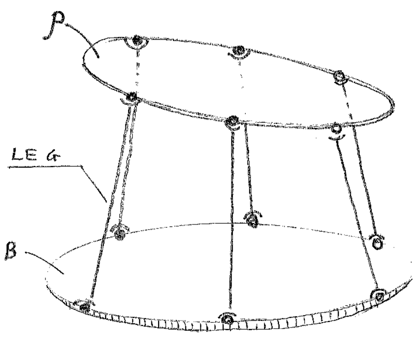

Figure 4.1: A general 6-dof parallel manipulator (*Stewart-Gough platform*).

in order to have the number, m, of the actuated joints equal to the degrees of freedom of the moving platform, n. In the case of $m > n$ the parallel manipulator is redundandly actuated, if $m < n$ it is underactuated.

4.2 Kinematics of a planar parallel manipulator

The planar parallel manipulator $3-\underline{R}RR$ is shown in Figure 4.2. The mov-

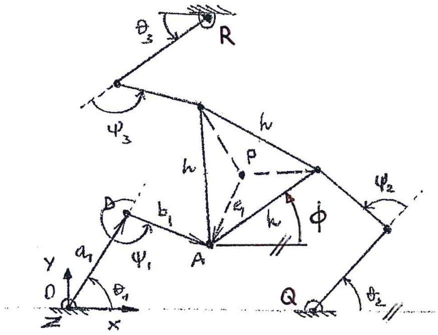

Figure 4.2: Geometry of the $3-\underline{R}RR$ manipulator.

ing platform is an equilateral triangle with sides length h and it has $n = 3$ dof, namely x_P, y_P, ϕ and the $m = 3$ revolute joints at the base are actuated. The inverse position kinematics consists of finding θ_i, $(i = 1, 2, 3)$ when the moving platform position is given, whereas the direct kinematics consists of finding x_P, y_P, ϕ when joints rotations θ_i, $(i = 1, 2, 3)$ are given. However, the main problem is to find out the relationship between the position of a joint pivot at the vertex of the moving platform, for example A, and the joints rotations. Indeed, according to the geometry:

$$x_P = x_A + \frac{h}{\sqrt{3}}\cos(\frac{\pi}{6} + \phi),$$
$$y_P = y_A + \frac{h}{\sqrt{3}}\sin(\frac{\pi}{6} + \phi). \tag{4.1}$$

In the inverse position problem, eqs. 4.1 allow us to obtain x_A and y_A at the beginning of the computation to proceed with the joints rotations calculation. On the contrary, in the direct position problem, eqs. 4.1 allow us to obtain x_P and y_P at the end of the computation once x_A and y_A and ϕ have been calculated according to the procedure presented below.
The equations allowing us to solve the position analysis are the *loop-closure*

equations for each leg:

$$\begin{aligned} \mathbf{OA} &= \mathbf{OD} + \mathbf{DA}; \\ x_A = a_1c_1 + b_1c(\theta_1+\psi_1): \ (x_A - a_1c_1) &= b_1c(\theta_1+\psi_1); \\ y_A = a_1s_1 + b_1s(\theta_1+\psi_1): \ (y_A - a_1s_1) &= b_1s(\theta_1+\psi_1). \end{aligned} \tag{4.2}$$

By squaring eqs. 4.2 and summing them up we obtain:

$$x_A^2 + y_A^2 - 2x_Aa_1c_1 - 2y_Aa_1s_1 + a_1^2 - b_1^2 = 0. \tag{4.3}$$

Similarly, two additional equations can be derived for legs 2 and 3:

$$\begin{gathered} x_A^2 + y_A^2 - 2x_Ax_Q - 2y_Ay_Q + x_Q^2 + y_Q^2 + h^2 + a_2^2 - b_2^2 + \\ +2x_Ahc_\phi + 2y_Ahs_\phi - 2x_Aa_2c_2 - 2y_Aa_2s_2 - 2a_2hc_\phi c_2 - 2x_Qhc_\phi - 2y_Qhs_\phi + \\ +2x_Qa_2c_2 + 2y_Qa_2s_2 - 2a_2hs_\phi s_2 = 0. \end{gathered} \tag{4.4}$$

$$\begin{gathered} x_A^2 + y_A^2 - 2x_Ax_R - 2y_Ay_R + x_R^2 + y_R^2 + h^2 + a_3^2 - b_3^2 + \\ +2x_Ahc_{\phi+\pi/3} + 2y_Ahs_{\phi+\pi/3} - 2x_Aa_3c_3 - 2y_Aa_3s_3 - 2a_3hc_{\phi+\pi/3}c_3 - \\ -2x_Rhc_{\phi+\pi/3} - 2y_Rhs_{\phi+\pi/3} + 2x_Ra_3c_3 + 2y_Ra_3s_3 - 2a_3hs_{\phi+\pi/3}s_3 = 0. \end{gathered} \tag{4.5}$$

4.2.1 Inverse position kinematics

For the inverse kinematics x_P, y_P and ϕ are given, and the joint angles θ_i, $(i = 1, 2, 3)$ are to be found. First we use eqs. 4.1 to obtain x_A, y_A. Eq. 4.2 can be written as:

$$\begin{aligned} e_1s_1 + e_2c_1 + e_3 &= 0; \\ e_1 = -2y_Aa_1, \ e_2 = -2x_Aa_1, \ e_3 &= x_A^2 + y_A^2 + a_1^2 - b_1^2. \end{aligned} \tag{4.6}$$

Substituting the following trigonometric identities:

$$s_1 = \frac{2t_1}{1+t_1^2}, \quad c_1 = \frac{1-t_1^2}{1+t_1^2}, \quad t_1 = \tan\frac{\theta_1}{2}$$

into eq. 4.6, we obtain:

$$(e_3 - e_2)t_1^2 + 2e_1t_1 + (e_3 + e_2) = 0. \tag{4.7}$$

Solving eq. 4.7 for t_1 leads to:

$$\theta_1 = 2\tan^{-1}\frac{-e_1 \pm \sqrt{e_1^2 + e_2^2 - e_3^2}}{e_3 - e_2}$$

Therefore, there are generally 2 solutions of θ_1 when the position of the moving platform is given. This statement needs few details. Indeed, eq. 4.7

has 2 roots for t_1 and thus 4 roots for θ_1 according to the tan function periodicity. By calculating ψ_1 from eqs. 4.2 it will be apparent that only two of them are compatible with the geometric constraints. Alternatively, according to the trigonometric identities above, θ_1 can be calculated directly as $\theta_1 = atan2(\frac{2t_1}{1+t_1^2}, \frac{1-t_1^2}{1+t_1^2})$ discarding the extraneous solutions at hand. The configurations obtained for the leg are exactly the same as the upper/lower-elbow solutions of a planar $2R$ manipulator.
If eq. 4.6 has no real roots then the specified moving platform location cannot be reached. Joints angles θ_2 and θ_3 can be obtained by following the same procedure. Thus, there are 2^3 possible configurations to a given moving platform location.

4.2.2 Direct position kinematics

For the direct kinematics θ_i, $(i = 1, 2, 3)$ are given, and the location of the moving platform, namely, x_P, y_P and ϕ are to be found. Eqs. 4.3, 4.4 and 4.5 can be written as:

$$\begin{aligned} x_A^2 + y_A^2 + e_{11}x_A + e_{12}y_A + e_{13} = 0; \\ x_A^2 + y_A^2 + e_{21}x_A + e_{22}y_A + e_{23} = 0; \\ x_A^2 + y_A^2 + e_{31}x_A + e_{32}y_A + e_{33} = 0; \end{aligned} \tag{4.8}$$

In eqs. 4.8 e_{11}, e_{12} and e_{13} are constants while e_{21}, e_{22}, e_{23} and e_{31}, e_{32}, e_{33} are linear functions of $\sin(\phi)$ and $\cos(\phi)$. The system may be simplified by substrating the second and the third equation from the first one. The 2 equations obtained are:

$$\begin{aligned} e'_{11}x_A + e'_{12}y_A + e'_{13} = 0, \\ e'_{21}x_A + e'_{22}y_A + e'_{23} = 0, \end{aligned} \tag{4.9}$$

with $e'_{11} = e_{11} - e_{21}$, $e'_{12} = e_{12} - e_{22}$, $e'_{13} = e_{13} - e_{23}$, $e'_{21} = e_{11} - e_{31}$, $e'_{22} = e_{12} - e_{32}$, $e'_{23} = e_{13} - e_{33}$. Eqs. 4.9 together with the fisrt equation of eqs. 4.8 form a new system of equations that may be solved. By solving x_A and y_A from eqs. 4.9 and then substituting into the other equation we obtain:

$$\delta_1^2 + \delta_2^2 + e_{11}\delta\delta_1 + e_{12}\delta\delta_2 + e_{13}\delta^2 = 0 \tag{4.10}$$

where:

$$\begin{aligned} \delta = e'_{11}e'_{22} - e'_{12}e'_{21}, \\ \delta_1 = e'_{12}e'_{23} - e'_{13}e'_{22}, \\ \delta_2 = e'_{13}e'_{21} - e'_{11}e'_{23}. \end{aligned}$$

Eq. 4.10 can be converted into a 8th. degree polynomial equation with ϕ as unknown by the half-tangent expressions. Hence, there are at most 8 possible poses to each given set of input joint angles. At the end of the procedure, as previously mentioned, eqs. 4.1 allow us to obtain the position of the moving platform centroid P.

4.2.3 Differential kinematics

We wish to write a relationship between the velocity of the moving platform: $\dot{\mathbf{x}} = \begin{pmatrix} v_{P_x} & v_{P_y} & \dot{\phi} \end{pmatrix}^T$ and the actuated joints rates: $\dot{\mathbf{q}} = \begin{pmatrix} \dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_3 \end{pmatrix}^T$. We write the velocity of the moving pivots at the platform vertices through point P on the moving platform and through the leg chain. For example for the first leg:

$$\begin{gathered} \mathbf{v}_A = \mathbf{v}_P + \mathbf{v}_{A/P} = \mathbf{v}_D + \mathbf{v}_{A/D} \ : \\ \mathbf{v}_P + \dot{\phi}\mathbf{z} \times \mathbf{e}_1 = \dot{\theta}_1 \mathbf{z} \times \mathbf{a}_1 + (\dot{\theta}_1 + \dot{\psi}_1)\mathbf{z} \times \mathbf{b}_1 \end{gathered}$$

Thus, for the ith. leg we have:

$$\mathbf{v}_P + \dot{\phi}(\mathbf{z} \times \mathbf{e}_i) = \dot{\theta}_i(\mathbf{z} \times \mathbf{a}_i) + (\dot{\theta}_i + \dot{\psi}_i)(\mathbf{z} \times \mathbf{b}_i), \quad i = 1, 2, 3, \tag{4.11}$$

$\dot{\psi}_i$ is passive in eq. 4.11 and it must be eliminated. To achive this goal, we dot-multiply both sides of eq. 4.11 by $\mathbf{b}_i$:

$$\mathbf{b}_i \cdot \mathbf{v}_P + \dot{\phi}\mathbf{z} \times \mathbf{e}_i \cdot \mathbf{b}_i = \dot{\theta}_i \mathbf{z} \times \mathbf{a}_i \cdot \mathbf{b}_i + \dot{\theta}_i \mathbf{z} \times \mathbf{b}_i \cdot \mathbf{b}_i + \dot{\psi}_i \mathbf{z} \times \mathbf{b}_i \cdot \mathbf{b}_i, \quad i = 1, 2, 3$$

which, after simple manipulations, becomes:

$$\mathbf{b}_i \cdot \mathbf{v}_P + \dot{\phi}\mathbf{z} \cdot \mathbf{e}_i \times \mathbf{b}_i = \dot{\theta}_i \mathbf{z} \cdot \mathbf{a}_i \times \mathbf{b}_i, \quad i = 1, 2, 3. \tag{4.12}$$

Eq. 4.12 can be arranged in matrix form as:

$$\mathbf{J}_x \dot{\mathbf{x}} = \mathbf{J}_q \dot{\mathbf{q}} \tag{4.13}$$

with

$$\mathbf{J}_x = \begin{pmatrix} b_{1_x} & b_{1_y} & e_{1_x} b_{1_y} - e_{1_y} b_{1_x} \\ b_{2_x} & b_{2_y} & e_{2_x} b_{2_y} - e_{2_y} b_{2_x} \\ b_{3_x} & b_{3_y} & e_{3_x} b_{3_y} - e_{3_y} b_{3_x} \end{pmatrix}$$

$$\mathbf{J}_q = \begin{pmatrix} a_{1_x} b_{1_y} - a_{1_y} b_{1_x} & 0 & 0 \\ 0 & a_{2_x} b_{2_y} - a_{2_y} b_{2_x} & 0 \\ 0 & 0 & a_{3_x} b_{3_y} - a_{3_y} b_{3_x} \end{pmatrix}.$$

(a) Inverse kinematic singularities
An inverse kinematic singularity occurs when $det(\mathbf{J}_q) = 0$. When assuming that the null space of $\mathbf{J}_q$ is not empty, this means that there exists some $\dot{\mathbf{q}}$ that transform in zero $\dot{\mathbf{x}}$. Thus, even if the motors are actuated, infinitesimal motion along certain directions cannot be

accomplished. The manipulators loses one or more degree of freedom. This type of singularity is similar to that of a serial manipulator. $det(\mathbf{J}_q) = 0$ when at least one of the diagonal terms of $\mathbf{J}_q$ becomes zero, namely:

$$a_{i_x} b_{i_y} - a_{i_y} b_{i_x} = 0 \quad : \quad \|\mathbf{a}_i \times \mathbf{b}_i\| = 0, \quad i = 1, 2, 3 \tag{4.14}$$

Eq. 4.14 claims that if, at least one of the 3 legs is fully stretched-out or fully folded-back then the manipulator is in a singular configuration. The manipulator loses one, two or three degrees of freedom depending on the number of legs holding that configuration.

(b) Direct kinematic singularities
A direct kinematic singularity occurs when $det(\mathbf{J}_x) = 0$. When assuming that the null space of $\mathbf{J}_x$ is not empty, this means that there exists some $\dot{\mathbf{x}}$ that transform in zero $\dot{\mathbf{q}}$. Thus, the moving platform can possess infinitesimal motion in some directions when all the actuators are locked. The moving platform gains one or more degrees of freedom, or, in other words, the manipulator cannot resist forces or moments in some directions.
We limit the direct singularities analysis to two cases which can be found by inspection of $\mathbf{J}_x$.

b.1 The first 2 columns of $\mathbf{J}_x$ represent the x and y components of $\mathbf{b}_i$, for $i = 1, 2, 3$. $det(\mathbf{J}_x) = 0$ when these columns become linearly dependent, that is when $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ are parallel. At this configuration, the moving platform can make an infinitesimal translation along a direction being perpendicular to $\mathbf{b}_i$ with all the actuators locked.

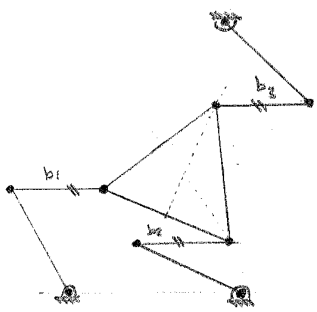

Figure 4.3: Direct kinematic singularity for the $3 - RRR$ manipulator.

b.2 $det(\mathbf{J}_x) = 0$ when the third column of $\mathbf{J}_x$ is zero. That means $\|\mathbf{e}_i \times \mathbf{b}_i\| = 0$, for $i = 1, 2, 3$. Thus, at this configuration, $\mathbf{e}_i$ and $\mathbf{b}_i$ are aligned for all the legs. The moving platform can make

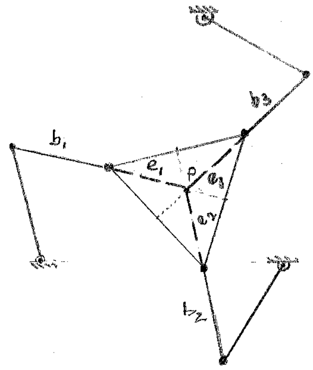

Figure 4.4: Another direct kinematic singularity for the $3 - RRR$ manipulator.

infinitesimal rotation about P while the actuators are locked. Hence, the moving platform gains one degree of freedom (a rotation) and it cannot resist any external moment about P.

(c) Combined kinematic singularities
In this case both $det(\mathbf{J}_x) = det(\mathbf{J}_q) = 0$ and eq. 4.13 degenerates. This type of singularity is not only configuration dependent but also architecture dependent. This analysis is not carried out in this notes.

4.3 Dynamics of constraint mechanisms (parallel manipulators)

In this section we deal with mechanisms where the *Lagrangian (generalized) coordinates* q_i, $(i = 1, \cdots, n)$ are not all independent but they have m constraint equations to guarantee. These equations may assume the following form:

$$\mathbf{\Psi}(\mathbf{q}) = \begin{pmatrix} \Psi_1(\mathbf{q}) & \dots & \Psi_m(\mathbf{q}) \end{pmatrix}^T = \mathbf{0}. \tag{4.15}$$

If the geometrical constraint equation 4.15 can be differentiate with respect to the time, *holonomic*, as it happens for the parallel manipulators, then it leads to:

$$(\frac{\partial \mathbf{\Psi}}{\partial \mathbf{q}})\dot{\mathbf{q}} = \mathbf{A}\dot{\mathbf{q}} = \mathbf{0}. \tag{4.16}$$

where $\mathbf{A} \in \mathbb{V}^{m,n}$. Therefore, the work balance for a constraint mechanical system subjected to time-differentiable geometrical constraints becomes:

$$[[\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}) - (\frac{\partial \mathcal{L}}{\partial \mathbf{q}})]^T - \mathbf{Q}^T + \boldsymbol{\lambda}^T \mathbf{A}]d\mathbf{q} = 0, \tag{4.17}$$

with $\mathbf{Q}^T = \boldsymbol{\tau}^T + \mathbf{F}_{env}^T \mathbf{J}$.
In the eq. 4.17, the constraint equation $\mathbf{A}d\mathbf{q} = \mathbf{0}$ was added to the work balance equation, and opportunely multiplied by the vector of lagrangian multipliers $\boldsymbol{\lambda} \in \mathbb{V}^m$. In doing that, it turns out that it is possible to choose any value for $d\mathbf{q} \neq \mathbf{0}$ since $\boldsymbol{\lambda}$ will guarantee that $\boldsymbol{\lambda}^T \mathbf{A}d\mathbf{q} = 0$.
Thus, in the column vector form we obtain:

$$\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}) - (\frac{\partial \mathcal{L}}{\partial \mathbf{q}}) - \mathbf{Q} + \mathbf{A}^T \boldsymbol{\lambda} = \mathbf{0}. \tag{4.18}$$

which, according to the procedure followed to obtain eq. 3.59 becomes:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{V}(\dot{\mathbf{q}}, \mathbf{q}) + \mathbf{G}(\mathbf{q}) - \mathbf{Q} + \mathbf{A}^T \boldsymbol{\lambda} = \mathbf{0}. \tag{4.19}$$

Now, we treat briefly two common procedures to solve the eq. 4.19.

- *Augmented Formulation - forward dynamics solution.*
  Acording to this formulation eq. 4.19 is gathered with $\mathbf{A}\ddot{\mathbf{q}} + \dot{\mathbf{A}}\dot{\mathbf{q}} = \mathbf{0}$ to obtain:

  $$\begin{pmatrix} \mathbf{M} & \mathbf{A}^T \\ \mathbf{A} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} -\mathbf{V} - \mathbf{G} + \mathbf{Q} \\ -\dot{\mathbf{A}}\dot{\mathbf{q}} \end{pmatrix}. \tag{4.20}$$

  Eq. 4.20 can be solved for $\begin{pmatrix} \ddot{\mathbf{q}} & \boldsymbol{\lambda} \end{pmatrix}^T$ whenever the block matrix was non-singular. This formulation allows one to calculate both $\ddot{\mathbf{q}}$ and $\boldsymbol{\lambda}$ at the same time. Conversely, the number of unknowns is increased and the solution algorithm has to be able to solve differential/algebraic equations *(DAE)*. Here, we first present a procedure where $\boldsymbol{\lambda}$ is calculated directly and then $\ddot{\mathbf{q}}$ will be obtained by substitutions.

  1. a) *Direct calculation of* $\boldsymbol{\lambda}$*:*
     First, for sake of brevity, we call:

     $$\begin{pmatrix} \mathbf{Q}_E \\ \mathbf{Q}_D \end{pmatrix} = \begin{pmatrix} -\mathbf{V} - \mathbf{G} + \mathbf{Q} \\ -\dot{\mathbf{A}}\dot{\mathbf{q}} \end{pmatrix},$$

     thus, from eq. 4.19 we have:

     $$\ddot{\mathbf{q}} = \mathbf{M}^{-1}(\mathbf{Q}_E - \mathbf{A}^T \boldsymbol{\lambda}). \tag{4.21}$$

     Now, let put eq. 4.21 into $\mathbf{A}\ddot{\mathbf{q}} = \mathbf{Q}_D$ such that by simple calculations we obtain $\boldsymbol{\lambda}$ as:

     $$\boldsymbol{\lambda} = (\mathbf{A}\mathbf{M}^{-1}\mathbf{A}^T)^{-1}(\mathbf{A}\mathbf{M}^{-1}\mathbf{Q}_E - \mathbf{Q}_D). \tag{4.22}$$

     Then, $\boldsymbol{\lambda}$, obtained by eq. 4.22, is substituted into eq. 4.21 to obtain $\ddot{\mathbf{q}}$. Eventually, numerical integration technique allows one to get $\dot{\mathbf{q}}(t)$ and $\mathbf{q}(t)$. It can be noted that $\mathbf{M}$ is symmetric thus its inversion is not computationally expensive. However, according to the references it seems that calculation procedures dealing with eq. 4.20 are more convenient than the calculation proposed.

2. a) *Calculation by block matrix inversion:*
   Let consider eq. 4.20, according to the procedure of block matrix inversion, we obtain:

$$\ddot{\mathbf{q}} = \mathbf{C}_{11}\mathbf{Q}_E + \mathbf{C}_{12}\mathbf{Q}_D,$$
$$\boldsymbol{\lambda} = \mathbf{C}_{21}\mathbf{Q}_E + \mathbf{C}_{22}\mathbf{Q}_D,$$

   with:

$$\begin{pmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{M} & \mathbf{A}^T \\ \mathbf{A} & \mathbf{0} \end{pmatrix}^{-1},$$
$$\mathbf{C}_{22} = (\mathbf{A}\mathbf{M}^{-1}\mathbf{A}^T)^{-1},\ \mathbf{C}_{11} = \mathbf{M}^{-1} - \mathbf{M}^{-1}\mathbf{A}^T\mathbf{C}_{22}\mathbf{A}\mathbf{M}^{-1},$$
$$\mathbf{C}_{12} = \mathbf{C}_{21}^T = -\mathbf{M}^{-1}\mathbf{A}^T\mathbf{C}_{22}.$$

- *Formulation with the minimum number of coordinates - forward dynamics solution*
  Let define a vector containing the minimum number of coordinates, the independent variables, for the mechanism: $\mathbf{p} \in \mathbb{V}^{(n-m)}$. The relationships between $\mathbf{p}$ and $\mathbf{q}$ is $\boldsymbol{\Phi}(\mathbf{p},\mathbf{q}) = \mathbf{0}$ such that we can define:

$$\mathbf{H}(\mathbf{p},\mathbf{q}) = \begin{pmatrix} \boldsymbol{\Psi}(\mathbf{q}) \\ \boldsymbol{\Phi}(\mathbf{p},\mathbf{q}) \end{pmatrix} = \mathbf{0}. \quad (4.23)$$

  Now, the time derivative of eq. 4.23 leads to:

$$\mathbf{J}_p\dot{\mathbf{p}} + \mathbf{J}_q\dot{\mathbf{q}} = \mathbf{0} \quad : \quad \dot{\mathbf{q}} = -\mathbf{J}_q^{-1}\mathbf{J}_p\dot{\mathbf{p}} = \mathbf{V}(\mathbf{p},\mathbf{q})\dot{\mathbf{p}} \quad (4.24)$$

  where the jacobians $\mathbf{J}_p$ and $\mathbf{J}_q$ are:

$$\mathbf{J}_p = (\frac{\partial \mathbf{H}}{\partial \mathbf{p}}), \quad \mathbf{J}_q = (\frac{\partial \mathbf{H}}{\partial \mathbf{q}}).$$

  Eq. 4.24 can be substituted into eq. 4.16 to obtain:

$$\mathbf{A}\mathbf{V}\dot{\mathbf{p}} = \mathbf{0}, \quad (4.25)$$

  such that:

$$\forall \dot{\mathbf{p}} \neq \mathbf{0} \quad \mathbf{A}\mathbf{V} = \mathbf{0} \ \text{ and } \ \mathbf{V}^T\mathbf{A}^T = \mathbf{0}^T. \quad (4.26)$$

  Further, eq. 4.24 can be differentiated with respect to the time:

$$\ddot{\mathbf{q}} = \dot{\mathbf{V}}\dot{\mathbf{p}} + \mathbf{V}\ddot{\mathbf{p}} \quad (4.27)$$

and then put it into eq. 4.19 to have:

$$\mathbf{M}(\dot{\mathbf{V}}\dot{\mathbf{p}} + \mathbf{V}\ddot{\mathbf{p}}) + \mathbf{A}^T\boldsymbol{\lambda} = (\mathbf{Q} - \mathbf{V} - \mathbf{G}) = \mathbf{Q}_E. \tag{4.28}$$

Eventually, in order to eliminate $\boldsymbol{\lambda}$ from eq. 4.28, we can pre-multiply it by $\mathbf{V}^T$ exploiting eq. 4.26:

$$\mathbf{V}^T\mathbf{M}\mathbf{V}\ddot{\mathbf{p}} = \mathbf{V}^T\mathbf{Q}_E - \mathbf{V}^T\mathbf{M}\dot{\mathbf{V}}\dot{\mathbf{p}}. \tag{4.29}$$

Eq. 4.29 represent $(n - m)$ ordinary differential equations to be integrated by numerical techniques.
The procedure to solve the forward dynamics consists to finding the acceleration of the minimum coordinates at the initial time t_0, namely $\ddot{\mathbf{p}}(t_0)$, with given initial condition $\mathbf{q}(t_0)$ and $\dot{\mathbf{q}}(t_0)$ then to integrating eq. 4.26 to obtain $\dot{\mathbf{p}}(t_0 + \Delta t)$ and $\mathbf{p}(t_0 + \Delta t)$. In doing so, eq. 4.27 can be solved at the time $t_0 + \Delta t$ to use the results as input at the next time step. It may be of interest here to look at how the matrix $\mathbf{V}$ can be calculated. For example, by using the *Orthogonal-triangular decomposition* of the matrix $\mathbf{A}^T$ we have:

$$\mathbf{A}^T = \mathbf{Q}\mathbf{R} = \begin{pmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \end{pmatrix} \begin{pmatrix} \mathbf{R}_1 \\ \mathbf{0} \end{pmatrix} \quad : \quad \mathbf{A}^T = \mathbf{Q}_1\mathbf{R}_1.$$

Now, since $\mathbf{Q}^T\mathbf{Q} = \mathbf{1}$ then it is straightforward to show that $\mathbf{Q}_2^T\mathbf{Q}_1 = \mathbf{0}$ and:

$$\mathbf{Q}_2^T\mathbf{A}^T = \mathbf{Q}_2^T\mathbf{Q}_1\mathbf{R}_1 = \mathbf{0}. \tag{4.30}$$

By comparing eq. 4.30 with eq. 4.26 we obtain $\mathbf{V} = \mathbf{Q}_2$.
Similarly we can use the *Singular value decomposition* of the matrix $\mathbf{A}^T$:

$$\mathbf{A}^T = \mathbf{W}\mathbf{S}\mathbf{L}^T = \begin{pmatrix} \mathbf{W}_1 & \mathbf{W}_2 \end{pmatrix} \begin{pmatrix} \mathbf{S}_1 \\ \mathbf{0} \end{pmatrix} \mathbf{L}^T \quad : \quad \mathbf{A}^T = \mathbf{W}_1\mathbf{S}_1\mathbf{L}^T.$$

Now, since $\mathbf{W}^T\mathbf{W} = \mathbf{1}$ then it is straightforward to show that $\mathbf{W}_2^T\mathbf{W}_1 = \mathbf{0}$ and:

$$\mathbf{W}_2^T\mathbf{A}^T = \mathbf{W}_2^T\mathbf{W}_1\mathbf{S}_1\mathbf{L}^T = \mathbf{0}. \tag{4.31}$$

And finally, by comparing eq. 4.31 with eq. 4.26 we obtain $\mathbf{V} = \mathbf{W}_2$.

4.3.1 Notes on the dynamics of a slider-crank mechanism

In order to gain insight of the foregoing procedures let consider the dynamics of the slider-crank mechanism in Figure 4.5
It is well known that the mechanism has 1 dof. The mechanism have masses m_i with centres of mass c_i for the moving rigid bodies (i), $(i = 1, 2, 3)$.

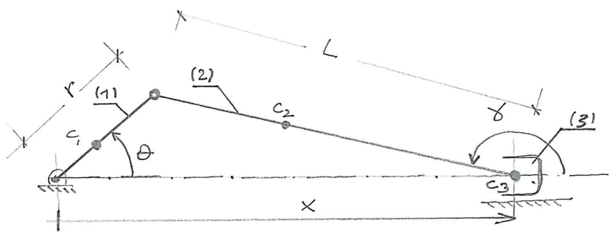

Figure 4.5: The slider-crank mechanism.

- *Augmented Formulation - direct calculation for the forward dynamics solution.*
  We choose $\mathbf{q} = \begin{pmatrix} \theta & \gamma & x \end{pmatrix}^T$ such that:

$$\mathbf{\Psi}(\mathbf{q}) = \begin{pmatrix} rs_\theta - Ls_\gamma \\ rc_\theta - Lc_\gamma - x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

  Therefore we have $n - m = 1$ independent parameter to describe the mechanism motion as expected.
  Then:

$$\mathbf{A} = (\frac{\partial \mathbf{\Psi}}{\partial \mathbf{q}}) = \begin{pmatrix} rc_\theta & -Lc_\gamma & 0 \\ -rs_\theta & Ls_\gamma & -1 \end{pmatrix}.$$

  $\mathbf{M}$, $\mathbf{Q}_E$ and $\mathbf{Q}_D$ can be easily calculated to obtain $\boldsymbol{\lambda}$ and $\ddot{\mathbf{q}}$ by eq. 4.22 and eq. 4.21.

- *Formulation with the minimum number of coordinates - forward dynamics solution*
  We choose $\mathbf{q} = \begin{pmatrix} \theta & \gamma & x \end{pmatrix}^T$ and $p = \theta$ such that:

$$\mathbf{H}(\mathbf{q}, p) = \begin{pmatrix} rs_\theta - Ls_\gamma \\ rc_\theta - Lc_\gamma - x \\ p - \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

  Then:

$$\mathbf{J}_q = (\frac{\partial \mathbf{H}}{\partial \mathbf{q}}) = \begin{pmatrix} rc_\theta & -Lc_\gamma & 0 \\ -rs_\theta & Ls_\gamma & -1 \\ -1 & 0 & 0 \end{pmatrix}, \quad \mathbf{J}_p = (\frac{\partial \mathbf{H}}{\partial p}) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix},$$

  and:

$$\mathbf{V} = -\mathbf{J}_q^{-1}\mathbf{J}_p = \begin{pmatrix} 1 \\ \frac{rc_\theta}{Lc_\gamma} \\ \frac{rs_{\gamma-\theta}}{c_\gamma} \end{pmatrix}.$$

4.4 Analysis of a parallel manipulator for translational motion

In this section we treat the kinematics (position problems, singularities) and the dynamics of a parallel manipulator whose moving platform undergoes translation motion only. The type of architecture analyzed finds common applications in robots suited for pick and place operations.

4.4.1 Geometry of the manipulator

Figure 4.6 shows the manipulator under study. It consists of a fixed base,

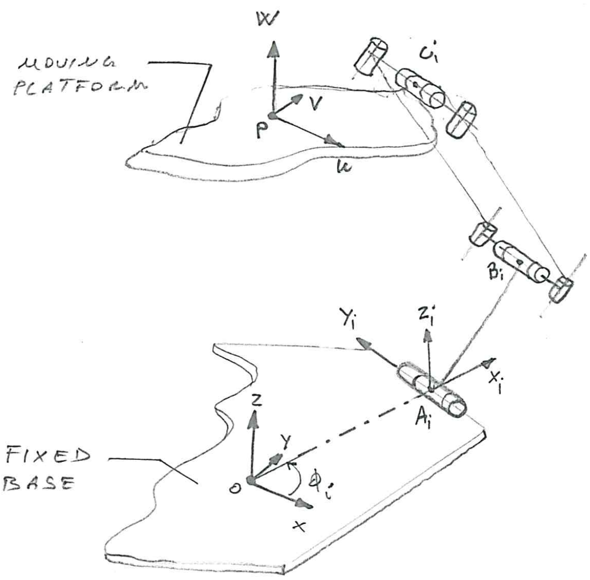

Figure 4.6: The Translational Parallel Mechanism (TPM).

three legs and a moving platform. Each leg consists of a upper arm and a lower arm. The upper arm is made up of a planar four-bar parallelogram connected to the moving platform by a R joint whereas the lower arm is made up of a link connected to the base by a R joint. Another R is connecting the upper arm an the lower arm. The three revolute joints defined have parallel axes. The R joints belonging to the parallelogram have their axes perpendicular to the other R joints.
In order to proceed with the analysis we define a ref. system at the base $\{O\mathbf{xyz}\}$ and a moving ref. system at the moving platform $\{P\mathbf{uvw}\}$. Further, for the leg i we define the ref. system $\{A_i\mathbf{x}_i\mathbf{y}_i\mathbf{z}_i\}$ according to the Figure 4.6. Let note that $\mathbf{R}_i = \mathbf{R}_z(\phi_i)$.
For this manipulator θ_{1i}, $i = 1, 2, 3$ are considered as the actuated joints. A

loop-closure equation can be written for the leg i as:

$$\overrightarrow{A_iB_i} + \overrightarrow{B_iC_i} = \overrightarrow{OP} + \overrightarrow{PC_i} - \overrightarrow{OA_i} \tag{4.32}$$

According to the Figure 4.7, eq. 4.32 can be expressed in the i-leg ref. system:

$$\begin{pmatrix} ac\theta_{1i} + bs\theta_{3i}c(\theta_{1i} + \theta_{2i}) \\ bc\theta_{3i} \\ as\theta_{1i} + bs\theta_{3i}s(\theta_{1i} + \theta_{2i}) \end{pmatrix} = \begin{pmatrix} c_{xi} \\ c_{yi} \\ c_{zi} \end{pmatrix} \tag{4.33}$$

where:

$$\begin{pmatrix} c_{xi} \\ c_{yi} \\ c_{zi} \end{pmatrix} = \begin{pmatrix} c\phi_i & s\phi_i & 0 \\ -s\phi_i & c\phi_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} + \begin{pmatrix} h-r \\ 0 \\ 0 \end{pmatrix}. \tag{4.34}$$

The meaning of the symbols may be easily deduced from Figure 4.7.

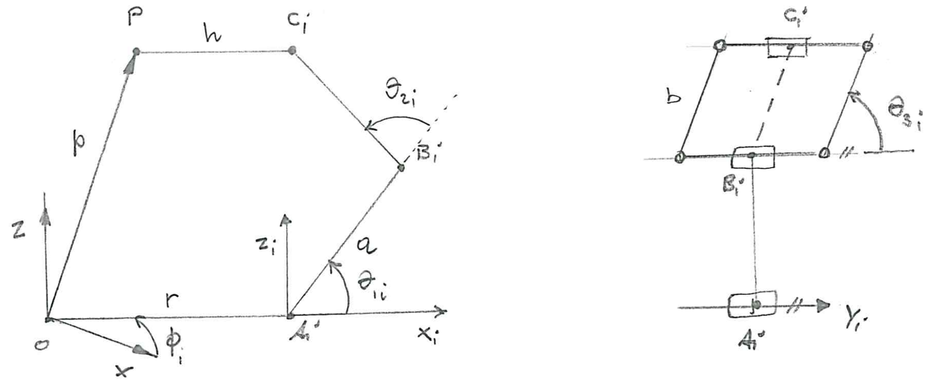

Figure 4.7: Front and side views of the TPM $i^{th.}$ leg.

4.4.2 Inverse position problem

The problem consists to finding θ_{11}, θ_{12}, θ_{13} given $\mathbf{p} = \begin{pmatrix} p_x & p_y & p_z \end{pmatrix}^T$. First, we note that if the position of P is given then position of C_i is known as well. Thus, from the second of eqs. 4.33 we obtain two solutions for θ_{3i}:

$$\theta_{3i} = \cos^{-1} \frac{c_{yi}}{b}.$$

Once θ_{3i} is known we obtain an equation with θ_{2i} as the only unknown by summing the squares of c_{xi}, c_{yi}, c_{zi} in eq. 4.33:

$$2abs\theta_{3i}c\theta_{2i} + a^2 + b^2 = c_{xi}^2 + c_{yi}^2 + c_{zi}^2.$$

Hence:

$$\theta_{2i} = \cos^{-1}\mu$$

with $\mu = (c_{xi}^2 + c_{yi}^2 + c_{zi}^2 - a^2 - b^2)/2abs\theta_{3i}$. Therefore for each value of θ_{3i} we obtain two values of θ_{2i} thus resulting four solution sets for θ_{2i} and θ_{3i}. Eventually, only one solution for θ_{1i} is obtained from eq. 4.33 with the four solution sets for θ_{2i} and θ_{3i}. θ_{1i} can be obtained by manipulating the first and the last of eqs. 4.33:

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}\begin{pmatrix} c\theta_{1i} \\ s\theta_{1i} \end{pmatrix} = \begin{pmatrix} c_{x_i} \\ c_{z_i} \end{pmatrix} \quad : \quad \theta_{1i} = atan2(s\theta_{1i}, c\theta_{1i})$$

with $A = a + bs\theta_{3i}c\theta_{2i}$, $B = bs\theta_{3i}s\theta_{2i}$.

4.4.3 Direct position problem

The problem consists to finding $\mathbf{p} = \begin{pmatrix} p_x & p_y & p_z \end{pmatrix}^T$ given θ_{11}, θ_{12}, θ_{13}. The solution procedure starts from eq. 4.33 that is rearranged as:

$$\begin{pmatrix} bs\theta_{3i}c(\theta_{1i}+\theta_{2i}) \\ bc\theta_{3i} \\ bs\theta_{3i}s(\theta_{1i}+\theta_{2i}) \end{pmatrix} = \begin{pmatrix} c\phi_i & s\phi_i & 0 \\ -s\phi_i & c\phi_i & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} + \begin{pmatrix} -ac\theta_{1i} - r + h \\ 0 \\ -as\theta_{1i} \end{pmatrix} \tag{4.35}$$

By summing the squares of the three equations in eq. 4.35 leads to:

$$\begin{aligned} b^2 &= p_x^2 + p_y^2 + p_z^2 - 2(p_xc\phi_i + p_ys\phi_i)(ac\theta_{1i} + r - h) - \\ &+ 2p_zas\theta_{1i} + (ac\theta_{1i} + r - h)^2 + a^2s^2\theta_{1i}, \quad i = 1, 2, 3. \end{aligned} \tag{4.36}$$

Eq. 4.36 represents a sphere. The intersections of the three spheres, ($i = 1, 2, 3$), provides the solutions of the problem at hand. However, an alternative procedure is presented here. We substract eq. 4.36 for $i = 1$ from eq. 4.36 for $i = j$ to obtain the plane containing the intersection circle between the spheres with $i = 1$ and $i = j$. We repeat the procedure for $j = 2, 3$:

$$e_{1j}p_x + e_{2j}p_y + e_{3j}p_z + e_{4j} = 0 \quad j = 2, 3, \tag{4.37}$$

where:

$$\begin{aligned} e_{1j} &= 2c\phi_j(ac\theta_{1j} + r - h) - 2c\phi_1(ac\theta_{11} + r - h), \\ e_{2j} &= 2s\phi_j(ac\theta_{1j} + r - h) - 2s\phi_1(ac\theta_{11} + r - h), \\ e_{3j} &= 2as\theta_{1j} - 2as\theta_{11}, \\ e_{4j} &= (ac\theta_{11} + r - h)^2 + a^2s^2\theta_{11} - (ac\theta_{1j} + r - h)^2 - a^2s^2\theta_{1j}. \end{aligned}$$

The solutions of the system formed by the eqs. 4.37 represent a line containing P (if real solutions exsist) between the foregoing planes. Finally, the intersections of this line with one of the spheres defined in eq. 4.36 solve the direct position problem. By analysis, the problem can be solved by writing p_y and p_z in terms of p_x from eqs. 4.37 and then by substituing them in the first, for example, of eq. 4.36. With no details on the nature of the coefficients, the form of the final equation is $k_0 p_x^2 + k_1 p_x + k_2 = 0$.

4.4.4 Jacobian and singularities

The goal here is to find the relationship between $\mathbf{v}_p$, velocity of of the moving platform point P, and the rates of the actuated joints $\dot{\mathbf{q}} = \begin{pmatrix} \dot\theta_{11} & \dot\theta_{12} & \dot\theta_{13} \end{pmatrix}^T$. First, we differentiate with respect to time eq. 4.32 for leg i:

$$\boldsymbol{\omega}_{1i} \times \mathbf{a}_i + \boldsymbol{\omega}_{2i} \times \mathbf{b}_i = \mathbf{v}_p. \tag{4.38}$$

To eliminate the passive joint rates in $\boldsymbol{\omega}_{2i}$ we dot-multiply eq. 4.38 by b_i:

$$\boldsymbol{\omega}_{1i} \cdot (\mathbf{a}_i \times \mathbf{b}_i) = \mathbf{b}_i \cdot \mathbf{v}_p. \tag{4.39}$$

Eq. 4.39 is a scalar equation. Vectors are conveniently expressed in the i ref. system:

$$^i\mathbf{a}_i = a\begin{pmatrix} c\theta_{1i} \\ 0 \\ s\theta_{1i} \end{pmatrix}, \quad ^i\mathbf{b}_i = b\begin{pmatrix} s\theta_{3i}c(\theta_{1i}+\theta_{2i}) \\ c\theta_{3i} \\ s\theta_{3i}s(\theta_{1i}+\theta_{2i}) \end{pmatrix}, \quad ^i\boldsymbol{\omega}_{1i} = \begin{pmatrix} 0 \\ -\dot\theta_{1i} \\ 0 \end{pmatrix},$$
$$^i\mathbf{v}_p = \mathbf{R}_i^T\mathbf{v}_p = \begin{pmatrix} v_{p,x}c\phi_i + v_{p,y}s\phi_i \\ -v_{p,x}s\phi_i + v_{p,y}c\phi_i \\ v_{p,z} \end{pmatrix}$$

such that eq. 4.39 becomes:

$$j_{ix}v_{p,x} + j_{iy}v_{p,y} + j_{iz}v_{p,z} = as\theta_{2i}s\theta_{3i}\dot\theta_{1i} \tag{4.40}$$

with

$$\begin{aligned} j_{ix} &= c(\theta_{1i}+\theta_{2i})s\theta_{3i}c\phi_i - c\theta_{3i}s\phi_i, \\ j_{iy} &= c(\theta_{1i}+\theta_{2i})s\theta_{3i}s\phi_i + c\theta_{3i}c\phi_i, \\ j_{iz} &= s(\theta_{1i}+\theta_{2i})s\theta_{3i}. \end{aligned}$$

Let note that $\mathbf{j}_i = \begin{pmatrix} j_{ix} & j_{iy} & j_{iz} \end{pmatrix}^T$ is the unit vector of $\overrightarrow{B_iC_i}$ in $\{O\mathbf{xyz}\}$. By writing eq. 4.40 three times and casting them in matrix form leads to:

$$\mathbf{J}_x\mathbf{v}_p = \mathbf{J}_q\dot{\mathbf{q}}$$

where:

$$\mathbf{J}_x = \begin{pmatrix} j_{1x} & j_{1y} & j_{1z} \\ j_{2x} & j_{2y} & j_{2z} \\ j_{3x} & j_{3y} & j_{3z} \end{pmatrix},$$

$$\mathbf{J}_q = a \begin{pmatrix} s\theta_{21}s\theta_{31} & 0 & 0 \\ 0 & s\theta_{22}s\theta_{32} & 0 \\ 0 & 0 & s\theta_{23}s\theta_{33} \end{pmatrix}.$$

(a) Inverse kinematic singularities
An inverse kinematic singularity occurs when $det(\mathbf{J}_q) = 0$. It may happen whenever $\theta_{2i} = k\pi$ or $\theta_{3i} = k\pi$, $k = 0, 1, \cdots, n$. $\theta_{2i} = k\pi$ occurs when lower and upper arm belong to the same plane. $\theta_{3i} = k\pi$ occurs when all the parallelograms are collinear.

(b) Direct kinematic singularities
A direct kinematic singularity occurs when $det(\mathbf{J}_x) = 0$. For example, whenever the matrix rows, which are nothing but the unit vectors $\mathbf{j}_i$, are linearly dependent then $\mathbf{J}_x$ becomes singular, that is:

$$h_1\mathbf{j}_1 + h_2\mathbf{j}_2 + h_3\mathbf{j}_3 = \mathbf{0} \tag{4.41}$$

for some real values of h_i, $i = 1, 2, 3$, where not all of them are zero. Eq. 4.41 is satisfied when $\mathbf{j}_1$, $\mathbf{j}_2$ and $\mathbf{j}_3$ lie on a plane. For example when the three upper arms lie on the $x - y$ plane then $j_{iz} = 0$. In this case the third column of $\mathbf{J}_x$ is null:

$$s(\theta_{11} + \theta_{21})s\theta_{31} = s(\theta_{12} + \theta_{22})s\theta_{32} = s(\theta_{13} + \theta_{23})s\theta_{33} = 0$$

that is: $\theta_{1i} + \theta_{2i} = k\pi$ or $\theta_{3i} = k\pi$, $k = 0, 1, \cdots, n$. Figure 4.8 shows this singular configuration. Another direct singular configuration is

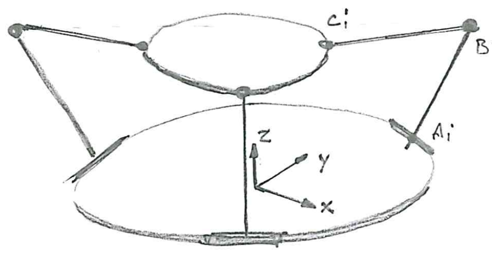

Figure 4.8: Direct singularity: $j_{iz} = 0,\ i = 1, 2, 3$.

reached whenever at least two upper arms are parallel: $\mathbf{j}_i = \pm\zeta\mathbf{j}_k$ with $i \neq k$ and $\forall\zeta \in \mathbb{R}$ as shown in Figure 4.9.

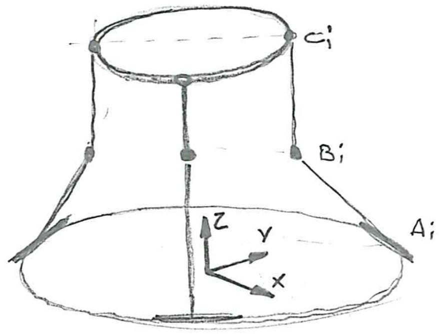

Figure 4.9: Direct singularity: $\mathbf{j}_i \times \mathbf{j}_k = \mathbf{0},\ i \neq k$.

4.4.5 Lagrangian dynamics

We deal with the manipulator under study as a constraint mechanism and we take $\mathbf{q} = \begin{pmatrix} p_x & p_y & p_z & \theta_{11} & \theta_{12} & \theta_{13} \end{pmatrix}^T$ although we have only three independent variables. The goal here is to write six equations expressed in the form of eq. 4.18.
The constraint equations are $\Psi_i = \|\overrightarrow{B_iC_i}\|^2 - b^2 = 0,\ i = 1, 2, 3$:

$$\Psi_i = (p_x + hc\phi_i - rc\phi_i - ac\phi_i c\theta_{1i})^2 + \\ +(p_y + hs\phi_i - rs\phi_i - as\phi_i c\theta_{1i})^2 + (p_z - as\theta_{1i})^2 - b^2 = 0.$$

Thus:

$$\mathbf{A} = \begin{pmatrix} \frac{\partial\Psi_1}{\partial p_x} & \frac{\partial\Psi_1}{\partial p_y} & \frac{\partial\Psi_1}{\partial p_z} & \frac{\partial\Psi_1}{\partial\theta_{11}} & \frac{\partial\Psi_1}{\partial\theta_{12}} & \frac{\partial\Psi_1}{\partial\theta_{13}} \\ \frac{\partial\Psi_2}{\partial p_x} & \frac{\partial\Psi_2}{\partial p_y} & \frac{\partial\Psi_2}{\partial p_z} & \frac{\partial\Psi_2}{\partial\theta_{11}} & \frac{\partial\Psi_1}{\partial\theta_{12}} & \frac{\partial\Psi_2}{\partial\theta_{13}} \\ \frac{\partial\Psi_3}{\partial p_x} & \frac{\partial\Psi_3}{\partial p_y} & \frac{\partial\Psi_3}{\partial p_z} & \frac{\partial\Psi_3}{\partial\theta_{11}} & \frac{\partial\Psi_3}{\partial\theta_{12}} & \frac{\partial\Psi_3}{\partial\theta_{13}} \end{pmatrix}.$$

$\mathbf{A} \in \mathbb{V}^{3,6}$ such that $\mathbf{A}^T\boldsymbol{\lambda} \in \mathbb{V}^{6,1}$ with $\boldsymbol{\lambda} \in \mathbb{V}^{3,1}$.
In order to calculate the *Lagrange's function* $\mathcal{L} = K - U$, we consider the mass m_u of the upper arm concentrated evenly at B_i and C_i. That leads to an *approximate/exact equivalent dynamic system* of the upper arm. Therefore:

$$K = K_p + \sum_{i=1}^{3} K_{li} + K_{ui}, \quad U = U_p + \sum_{i=1}^{3} U_{li} + U_{ui},$$

with K_p, U_p the kinetic and potential energy of the moving platform, K_{li}, K_{li}, the kinetic and potential energy of the $i^{th.}$ lower arm and K_{ui}, K_{ui}, the kinetic and potential energy of the $i^{th.}$ upper arm, respectively.

Specifically:

$$K_p = \frac{1}{2}(\dot{p}_x^2 + \dot{p}_y^2 + \dot{p}_z^2),$$
$$K_{li} = \frac{1}{2}(I_m + \frac{1}{3}m_l a^2)\dot{\theta}_{1i}^2,$$
$$K_{ui} = \frac{1}{2}m_u(\dot{p}_x^2 + \dot{p}_y^2 + \dot{p}_z^2) + \frac{1}{2}m_u a^2\dot{\theta}_{1i}^2,$$

where m_p is the mass of the moving platform, m_l is the mass of the lower arm, I_m is the moment of inertia of the lower arm. Then, by assuming that the gravity acceleration $\mathbf{g}$ points along the $-z$ direction, we have:

$$U_p = m_p g p_z,$$
$$U_{li} = \frac{1}{2}m_l g a s\theta_{1i},$$
$$K_{ui} = m_u g(p_z + a s\theta_{1i}).$$

4.5 Rolling robots

In this section we deal with robots that are moving on horizontal surfaces, therefore the moving platform undergoes planar motion. Despite of the robot manipulators, rolling robots constraints are *nonholonomic*. That means that the minimum number of generalized coordinates m which define uniquely a posture of the robot is greater than the number of their independent generalized speed n, i.e., $m > n$. In other words, not all the contraint functions are integrable in time, i.e. $f(\dot{\mathbf{q}}, t) \neq 0$. For example, the pure rolling motion with no splippage is a nonholonomic constraint that cannot be integrated in time because of the arbitrary nature of the contact point trajectory. It is worth noting that this constraint becomes holonomic whenever the motion was planar.

4.5.1 Kinematics of the rolling robots

Let consider the 2-dof (in terms of speed coordinates) rolling robot depicted in Figure 4.10. The robot is constituted by a chassis (0), two motorized wheels (1), (2) and a third passive wheel (3) mounted on a bracket (4). An alternative solution can be obtained by actuating whells (1) and (2) with only one motor and then take the control of the steering wheel (3) by another motor. In this case, however, it has to be considered that propulsion motor calls for velocity control whereas steering motor calls for position control, thereby giving rise to two independent control systems.
As shown in Figure 4.11 we define a moving ref. system on the rolling robot with the operating point C as centre and a fixed ref. system with centre O. Positions of the centres of the motorized wheels are $\mathbf{o}_i$. Position of the

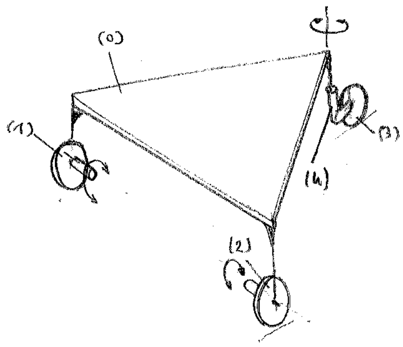

Figure 4.10: A 2-dof rolling robot.

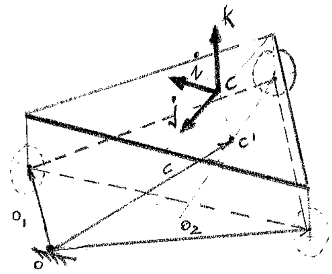

Figure 4.11: Ref. systems for the 2-dof rolling robot.

operating point is $\mathbf{c}$ according to the fact that $C' \equiv C$ with C' on the plane defined by the wheels centres. Then, we define ω as the angular velocity of the chassis about a vertical axis. By properly specifying the two joint rates $\dot{\theta}_1$, $\dot{\theta}_2$ we can control either $\dot{\mathbf{c}}$ or a combination of ω and a scalar function of $\dot{\mathbf{c}}$.
As reference for the geometrical parameters of the rolling robot under study we refer to Figure 4.12.
The velocities of the centres of the motorized wheels are given as:

$$\dot{\mathbf{o}}_i = r\dot{\theta}_i\mathbf{j}, \quad i = 1, 2, \tag{4.42}$$

with r the radius of the wheels.
The velocity of the operating point C is given as:

$$\dot{\mathbf{c}} = \dot{\mathbf{o}}_i + \omega\mathbf{E}(\mathbf{c} - \mathbf{o}_i), \quad i = 1, 2. \tag{4.43}$$

with $\mathbf{E} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Vectors in eqs. 4.42, 4.43 are 2-dimensional. In order to find ω in terms of $\dot{\theta}_1$ and $\dot{\theta}_2$ we write eq. 4.43 by using eq. 4.42 for $i = 1$ and $i = 2$ and then we substract each to the other:

$$r(\dot{\theta}_1 - \dot{\theta}_2)\mathbf{j} - \omega\mathbf{E}(\mathbf{o}_1 - \mathbf{o}_2) = \mathbf{0};$$

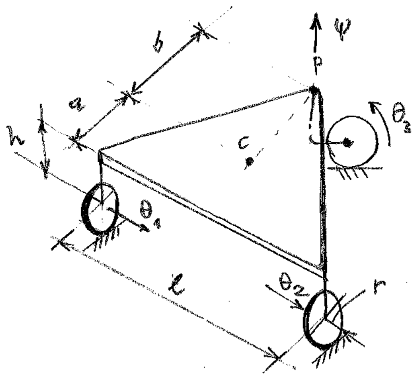

Figure 4.12: Geometrical parameters of the 2-dof rolling robot.

Furthermore, since $\mathbf{o}_1 = l/2\mathbf{i}$ and $\mathbf{o}_2 = -l/2\mathbf{i}$:

$$\mathbf{E}(\mathbf{o}_1 - \mathbf{o}_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} l \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ l \end{pmatrix} = l\mathbf{j};$$

and eventually:

$$\omega = \frac{r}{l}(\dot{\theta}_1 - \dot{\theta}_2). \tag{4.44}$$

In order to find $\dot{\mathbf{c}}$ in terms of $\dot{\theta}_1$ and $\dot{\theta}_2$ we write eq. 4.43 by using eq. 4.42 and eq. 4.44 for $i = 1$ and $i = 2$ and then we sum each to the other:

$$2\dot{\mathbf{c}} = r(\dot{\theta}_1 + \dot{\theta}_2)\mathbf{j} + \frac{r}{l}(\dot{\theta}_1 - \dot{\theta}_2)\mathbf{E}[(\mathbf{c} - \mathbf{o}_1) + (\mathbf{c} - \mathbf{o}_2)];$$

Thus, considering that:

$$(\mathbf{c} - \mathbf{o}_1) = -\frac{l}{2}\mathbf{i} - a\mathbf{j}, \quad (\mathbf{c} - \mathbf{o}_2) = \frac{l}{2}\mathbf{i} - a\mathbf{j},$$

we obtain

$$\mathbf{E}[(\mathbf{c} - \mathbf{o}_1) - (\mathbf{c} - \mathbf{o}_2)]\omega = 2a\mathbf{i}$$

and finally:

$$\dot{\mathbf{c}} = \frac{r}{2}(\dot{\theta}_1 + \dot{\theta}_2)\mathbf{j} + \frac{ar}{l}(\dot{\theta}_1 - \dot{\theta}_2)\mathbf{i} \tag{4.45}$$

Eqs. 4.44, 4.45 express the differential direct kinematics relations of the robot under study. We can write the equations in compact form as:

$$\mathbf{t} = \mathbf{L}\dot{\boldsymbol{\theta}}_a;$$

with

$$\mathbf{t} = \begin{pmatrix} \omega \\ \dot{\mathbf{c}} \end{pmatrix}, \qquad \dot{\boldsymbol{\theta}}_a = \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix},$$
$$\mathbf{L} = \begin{pmatrix} \frac{r}{l} & -\frac{r}{l} \\ \frac{ar}{l}\mathbf{i} + \frac{r}{2}\mathbf{j} & -\frac{ar}{l}\mathbf{i} + \frac{r}{2}\mathbf{j} \end{pmatrix}.$$

We now deal with the inverse kinematic of the rolling robot.
First, let dot-multiply by $\mathbf{j}$ eq. 4.45 to obtain:

$$\dot{\mathbf{c}} \cdot \mathbf{j} = \frac{r}{2}(\dot{\theta}_1 + \dot{\theta}_2), \tag{4.46}$$

and then let use eqs. 4.44, 4.46 to solve for $\dot{\theta}_1$ and $\dot{\theta}_2$:

$$\dot{\theta}_1 + \dot{\theta}_2 = (\frac{2}{r})\mathbf{j} \cdot \dot{\mathbf{c}} = (\frac{2}{r})\dot{y};$$
$$\dot{\theta}_1 - \dot{\theta}_2 = \frac{l}{r}\omega.$$

The previous equations can be cast as:

$$\mathbf{J}\dot{\boldsymbol{\theta}}_a = \mathbf{K}\mathbf{t}, \tag{4.47}$$

with

$$\mathbf{J} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} \frac{l}{r} & \mathbf{0}^T \\ 0 & \frac{2}{r}\mathbf{j}^T \end{pmatrix}.$$

Solving eq. 4.47 we obtain:

$$\dot{\theta}_1 = \frac{1}{2}(\frac{2\dot{y}}{r} + \frac{l}{r}\omega),$$
$$\dot{\theta}_2 = \frac{1}{2}(\frac{2\dot{y}}{r} - \frac{l}{r}\omega).$$

In order to complete the kinematic analysis we need to calculate the rates of the passive joints $\dot{\theta}_3$, $\dot{\psi}$ in terms of $\dot{\boldsymbol{\theta}}_a$.
To this end let calculate the 3-dimensional angular velocity vector for the actuated wheels, *i.e.*, $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, and for the caster wheel, *i.e.*, $\boldsymbol{\omega}_3$. Then, let calculate the scalar angular velocity of the bracket, *i.e.*, ω_4.
Thus:

$$\boldsymbol{\omega}_1 = -\dot{\theta}_1\mathbf{i} + \omega\mathbf{k} = -\dot{\theta}_1\mathbf{i} + \frac{r}{l}(\dot{\theta}_1 - \dot{\theta}_2)\mathbf{k} =$$
$$= \begin{pmatrix} -\mathbf{i} + (r/l)\mathbf{k} & -(r/l)\mathbf{k} \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}, \tag{4.48}$$

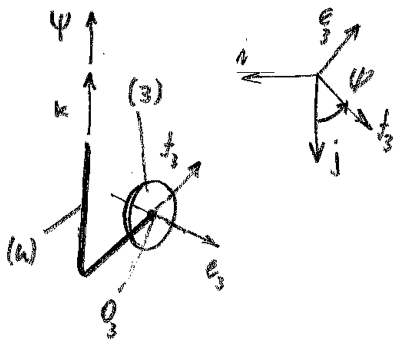

Figure 4.13: Ref. system for the caster wheel.

$$\boldsymbol{\omega}_2 = -\dot{\theta}_2\mathbf{i} + \omega\mathbf{k} = -\dot{\theta}_2\mathbf{i} + \frac{r}{l}(\dot{\theta}_1 - \dot{\theta}_2)\mathbf{k} =$$
$$= \begin{pmatrix} (r/l)\mathbf{k} & -\mathbf{i} - (r/l)\mathbf{k} \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} \tag{4.49}$$

and, according to Figure 4.13:

$$\boldsymbol{\omega}_3 = \dot{\theta}_3\mathbf{e}_3 + (\omega + \dot{\psi})\mathbf{k}, \tag{4.50}$$

$$\dot{\mathbf{o}}_3 = \boldsymbol{\omega}_3 \times r\mathbf{k} = -\dot{\theta}_3 r\mathbf{f}_3, \tag{4.51}$$

with

$$\begin{pmatrix} \mathbf{e}_3 \\ \mathbf{f}_3 \end{pmatrix} = \begin{pmatrix} -s_\psi & c_\psi \\ -c_\psi & -s_\psi \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \end{pmatrix}.$$

Finally:

$$\omega_4 = \omega + \dot{\psi} = \frac{r}{l}(\dot{\theta}_1 - \dot{\theta}_2) + \dot{\psi} \tag{4.52}$$

We calculate now the velocity of point P. Whenever P was considered part of the bracket (4) we have:

$$\dot{\mathbf{p}} = \dot{\mathbf{o}}_3 + \omega_4\mathbf{k} \times (\mathbf{p} - \mathbf{o}_3)$$

otherwise if P was considered part of the chassis (0) we have:

$$\dot{\mathbf{p}} = \dot{\mathbf{c}} + \omega\mathbf{k} \times (\mathbf{b}(-\mathbf{j}))$$

Upon equating the right-hand of the above equations we obtain a 3-dimensional vector equation relating $\boldsymbol{\theta}_u = \begin{pmatrix} \dot{\theta}_3 & \dot{\psi} \end{pmatrix}^T$ with $\boldsymbol{\theta}_a$:

$$-\dot{\theta}_3 r\mathbf{f}_3 + (\omega + \dot{\psi})\mathbf{k} \times (\mathbf{p} - \mathbf{o}_3) = \dot{\mathbf{c}} + b\omega\mathbf{i} \quad :$$
$$-\dot{\theta}_3 r\mathbf{f}_3 + \dot{\psi}\mathbf{k} \times (\mathbf{p} - \mathbf{o}_3) = \dot{\mathbf{c}} + b\omega\mathbf{i} - \omega\mathbf{k} \times (\mathbf{p} - \mathbf{o}_3).$$

Now:

$$(\mathbf{p}-\mathbf{o}_3)=d(-\mathbf{f}_3)+(h-r)\mathbf{k} \quad : \quad \mathbf{k}\times(\mathbf{p}-\mathbf{o}_3)=d\mathbf{e}_3.$$

Thus we have:

$$-\dot{\theta}_3 r\mathbf{f}_3+\dot{\psi}d\mathbf{e}_3=\dot{\mathbf{c}}+\omega(b\mathbf{i}-d\mathbf{e}_3).$$

Now, we dot-multiply the previous equation, respectively, by $\mathbf{f}_3$ and $\mathbf{e}_3$ to obtain:

$$\begin{aligned}-\dot{\theta}_3 r&=\dot{\mathbf{c}}\cdot\mathbf{f}_3+\omega b\mathbf{i}\cdot\mathbf{f}_3,\\ \dot{\psi}d&=\dot{\mathbf{c}}\cdot\mathbf{e}_3+\omega(b\mathbf{i}\cdot\mathbf{e}_3-d).\end{aligned}$$

By recalling eq. 4.45 we obtain $\dot{\mathbf{c}}$ expressed into $\{\mathbf{e}_3,\mathbf{f}_3,\mathbf{k}\}$ ref. system:

$$\dot{\mathbf{c}}=\frac{r}{2}(\dot{\theta}_1+\dot{\theta}_2)(c_\psi\mathbf{e}_3-s_\psi\mathbf{f}_3)+\frac{ar}{l}(\dot{\theta}_1-\dot{\theta}_2)(-s_\psi\mathbf{e}_3-c_\psi\mathbf{f}_3)$$

from which we obtain the terms for the dot-multiplied equations:

$$\begin{aligned}\dot{\mathbf{c}}\cdot\mathbf{f}_3&=-\frac{r}{2}(\dot{\theta}_1+\dot{\theta}_2)s_\psi-\frac{ar}{l}(\dot{\theta}_1-\dot{\theta}_2)c_\psi,\\ \dot{\mathbf{c}}\cdot\mathbf{e}_3&=\frac{r}{2}(\dot{\theta}_1+\dot{\theta}_2)c_\psi-\frac{ar}{l}(\dot{\theta}_1-\dot{\theta}_2)s_\psi.\end{aligned}$$

By using the foregoing equations and eq. 4.44 $\dot{\theta}_3=\dot{\theta}_3(\dot{\theta}_1,\dot{\theta}_2)$ and $\dot{\psi}=\dot{\psi}(\dot{\theta}_1,\dot{\theta}_2)$ become:

$$\begin{aligned}\dot{\theta}_3&=(\frac{s_\psi}{2}+\alpha c_\psi)\dot{\theta}_1+(\frac{s_\psi}{2}-\alpha c_\psi)\dot{\theta}_2,\\ \dot{\psi}&=\rho(-\alpha s_\psi+\frac{c_\psi}{2}-\delta)\dot{\theta}_1+\rho(\alpha s_\psi+\frac{c_\psi}{2}+\delta)\dot{\theta}_2,\end{aligned}\tag{4.53}$$

where:

$$\alpha\equiv\frac{a+b}{l},\quad \delta\equiv\frac{d}{l},\quad \rho\equiv\frac{r}{d}.$$

Eventually, eqs. 4.53 can be written in compact form:

$$\dot{\boldsymbol{\theta}}_u=\mathbf{N}\dot{\boldsymbol{\theta}}_a,$$

with $\mathbf{N}$ defined as

$$\mathbf{N}=\begin{pmatrix}\frac{s_\psi}{2}+\alpha c_\psi & \frac{s_\psi}{2}-\alpha c_\psi\\ \rho(-\alpha s_\psi+\frac{c_\psi}{2}-\delta) & \rho(\alpha s_\psi+\frac{c_\psi}{2}+\delta)\end{pmatrix}.$$

4.5.2 Dynamics of the rolling robots

The dynamics of the rolling robots, similar to that of other mechanical system, deals with direct and inverse problems. We derive the mathematical model of the robot which will be useful for both of the dynamics problems. It turns out that, although the rolling robot is an nonholomic mechanical system, its mathematical model is formally identical to that of the holonomic system. Because the system is nonholonomic, however, it will be required to calculate the relations between the dependent and independent variables. The approach followed is of multibody dynamics. For this reason 5 bodies will be considered: bodies 1, 2 are the motorized wheels, body 3 is the caster wheel, body 4 is the bracket and body 4 is the chassis (moving platform). In the model the gravity effects are neglected. The calculation of the kinetic energy K_i, $(i = 1, \cdots, 5)$ is presented whereas its derivations to obtain explicitly the equations of motions are not be preformed.

1. Body 1 (Motorized wheel)

$$K_1 = \frac{1}{2}(\mathbf{v}_1^T m_1 \mathbf{v}_1 + \boldsymbol{\omega}_1^T \mathbf{I}_1 \boldsymbol{\omega}_1) = \frac{1}{2}(m_1 \dot{\boldsymbol{\theta}}_a^T \mathbf{h}_{1_v}^T \mathbf{h}_{1_v} \dot{\boldsymbol{\theta}}_a + \dot{\boldsymbol{\theta}}_a^T \mathbf{h}_{1_\omega}^T \mathbf{I}_1 \mathbf{h}_{1_\omega} \dot{\boldsymbol{\theta}}_a)$$

where, according to the kinematic analysis presented:

$$\mathbf{v}_1 \equiv \dot{\mathbf{o}}_1 = \mathbf{h}_{1_v} \dot{\boldsymbol{\theta}}_a, \quad \mathbf{h}_{1_v} = \begin{pmatrix} 0 & 0 \\ r & 0 \\ 0 & 0 \end{pmatrix},$$

$$\boldsymbol{\omega}_1 = \mathbf{h}_{1_\omega} \dot{\boldsymbol{\theta}}_a, \quad \mathbf{h}_{1_\omega} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ \rho\delta & -\rho\delta \end{pmatrix},$$

Vectors $\mathbf{v}_1$ and $\boldsymbol{\omega}_1$ are expressed in the ref. system $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ as well as $\mathbf{I}_1$ which is equal to:

$$\mathbf{I}_1 = \frac{1}{4} m_1 r^2 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

2. Body 2 (Motorized wheel)

$$K_2 = \frac{1}{2}(\mathbf{v}_2^T m_2 \mathbf{v}_2 + \boldsymbol{\omega}_2^T \mathbf{I}_2 \boldsymbol{\omega}_2) = \frac{1}{2}(m_2 \dot{\boldsymbol{\theta}}_a^T \mathbf{h}_{2_v}^T \mathbf{h}_{2_v} \dot{\boldsymbol{\theta}}_a + \dot{\boldsymbol{\theta}}_a^T \mathbf{h}_{2_\omega}^T \mathbf{I}_2 \mathbf{h}_{2_\omega} \dot{\boldsymbol{\theta}}_a)$$

where, according to the kinematic analysis presented:

$$\mathbf{v}_2 \equiv \dot{\mathbf{o}}_2 = \mathbf{h}_{2_v} \dot{\boldsymbol{\theta}}_a, \quad \mathbf{h}_{2_v} = \mathbf{h}_{1_v},$$

$$\boldsymbol{\omega}_2 = \mathbf{h}_{2_\omega}\dot{\boldsymbol{\theta}}_a, \quad \mathbf{h}_{2_\omega} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \\ \rho\delta & -\rho\delta \end{pmatrix},$$

Vectors $\mathbf{v}_2$ and $\boldsymbol{\omega}_2$ are expressed in the ref. system $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ as well as $\mathbf{I}_2$ which is equal to:

$$\mathbf{I}_2 = \frac{1}{4}m_2 r^2 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

3. Body 3 (Caster wheel)

$$K_3 = \frac{1}{2}(\mathbf{v}_3^T m_3 \mathbf{v}_3 + \boldsymbol{\omega}_3^T \mathbf{I}_3 \boldsymbol{\omega}_3) = \frac{1}{2}(m_3 \dot{\boldsymbol{\theta}}_a^T \mathbf{h}_{3_v}^T \mathbf{h}_{3_v} \dot{\boldsymbol{\theta}}_a + \dot{\boldsymbol{\theta}}_a^T \mathbf{h}_{3_\omega}^T \mathbf{I}_3 \mathbf{h}_{3_\omega} \dot{\boldsymbol{\theta}}_a)$$

where, according to the kinematic analysis presented:

$$\dot{\boldsymbol{\theta}}_u = \mathbf{N}\dot{\boldsymbol{\theta}}_a \quad : \quad \dot{\theta}_3 = n_{11}\dot{\theta}_1 + n_{12}\dot{\theta}_2, \quad \dot{\psi} = n_{21}\dot{\theta}_1 + n_{22}\dot{\theta}_2,$$

thus:

$$\mathbf{v}_3 \equiv \dot{\mathbf{o}}_3 = \mathbf{h}_{3_v}\dot{\boldsymbol{\theta}}_a, \quad \mathbf{h}_{3_v} = -r \begin{pmatrix} 0 & 0 \\ n_{11} & n_{12} \\ 0 & 0 \end{pmatrix},$$

$$\boldsymbol{\omega}_3 = \mathbf{h}_{3_\omega}\dot{\boldsymbol{\theta}}_a, \quad \mathbf{h}_{3_\omega} = \begin{pmatrix} n_{11} & n_{12} \\ 0 & 0 \\ \rho\delta + n_{21} & -\rho\delta + n_{22} \end{pmatrix},$$

Vectors $\mathbf{v}_3$ and $\boldsymbol{\omega}_3$ are expressed in the ref. system $\{\mathbf{e}_3, \mathbf{f}_3, \mathbf{k}\}$ as well as $\mathbf{I}_3$ which is equal to:

$$\mathbf{I}_3 = \frac{1}{4}m_3 r^2 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

4. Body 4 (Bracket)

$$K_4 = \frac{1}{2}(\mathbf{v}_4^T m_4 \mathbf{v}_4 + \boldsymbol{\omega}_4^T \mathbf{I}_4 \boldsymbol{\omega}_4) = \frac{1}{2}(m_4 \dot{\boldsymbol{\theta}}_a^T \mathbf{h}_{4_v}^T \mathbf{h}_{4_v} \dot{\boldsymbol{\theta}}_a + \dot{\boldsymbol{\theta}}_a^T \mathbf{h}_{4_\omega}^T \mathbf{I}_4 \mathbf{h}_{4_\omega} \dot{\boldsymbol{\theta}}_a)$$

where, according to the kinematic analysis presented:

$$\boldsymbol{\omega}_4 = (\omega + \dot{\psi})\mathbf{k} = [\rho\delta(\dot{\theta}_1 - \dot{\theta}_2) + n_{21}\dot{\theta}_1 + n_{22}\dot{\theta}_2]\mathbf{k} =$$
$$= [(n_{21} + \rho\delta)\dot{\theta}_1 + (n_{22} - \rho\delta)\dot{\theta}_2]\mathbf{k} = [\bar{n}_{21}\dot{\theta}_1 + \bar{n}_{22}\dot{\theta}_2]\mathbf{k} = \mathbf{h}_{4_\omega}\dot{\boldsymbol{\theta}}_a,$$

with

$$\mathbf{h}_{4_\omega} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \bar{n}_{21} & \bar{n}_{22} \end{pmatrix}.$$

$$\mathbf{v}_4 \equiv \dot{\mathbf{c}}_4 = \dot{\mathbf{o}}_3 + \boldsymbol{\omega}_4 \times \mathbf{o}_3\mathbf{c}_4,$$

and, according to Figure 4.14:

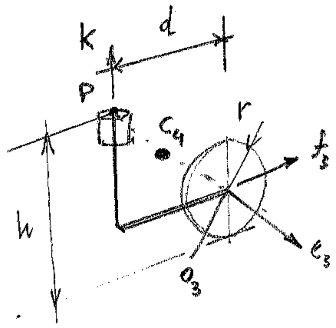

Figure 4.14: Geometrical parameters for the caster wheel.

$$\mathbf{o}_3\mathbf{c}_4 = \frac{d}{2}(-\mathbf{f}_3) + (\frac{h-r}{2})\mathbf{k}.$$

Thus, we have:

$$\mathbf{v}_4 = [-r \begin{pmatrix} 0 & 0 \\ n_{11} & n_{12} \\ 0 & 0 \end{pmatrix} + \frac{d}{2} \begin{pmatrix} \bar{n}_{21} & \bar{n}_{22} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}]\dot{\boldsymbol{\theta}}_a = \mathbf{h}_{4_v}\dot{\boldsymbol{\theta}}_a$$

Vectors $\mathbf{v}_4$ and $\boldsymbol{\omega}_4$ are expressed in the ref. system $\{\mathbf{e}_3, \mathbf{f}_3, \mathbf{k}\}$ as well as $\mathbf{I}_4$ which is equal to:

$$\mathbf{I}_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_b \end{pmatrix}.$$

5. Body 5 (Chassis)

$$K_5 = \frac{1}{2}(\dot{\mathbf{c}}^T m_5 \dot{\mathbf{c}} + \boldsymbol{\omega}^T \mathbf{I}_5 \boldsymbol{\omega}) = \frac{1}{2}(m_5 \dot{\boldsymbol{\theta}}_a^T \mathbf{h}_{5_v}^T \mathbf{h}_{5_v} \dot{\boldsymbol{\theta}}_a + \dot{\boldsymbol{\theta}}_a^T \mathbf{h}_{5_\omega}^T \mathbf{I}_5 \mathbf{h}_{5_\omega} \dot{\boldsymbol{\theta}}_a)$$

where, according to the kinematic analysis presented:

$$\boldsymbol{\omega} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \rho\delta & -\rho\delta \end{pmatrix} \dot{\boldsymbol{\theta}}_a$$

and

$$\dot{\mathbf{c}} = \begin{pmatrix} a\rho\delta & -a\rho\delta \\ \frac{r}{2} & \frac{r}{2} \\ 0 & 0 \end{pmatrix} \dot{\boldsymbol{\theta}}_a.$$

Vectors $\dot{\mathbf{c}}$ and $\boldsymbol{\omega}$ are expressed in the ref. system $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ as well as $\mathbf{I}_5$ which is equal to:

$$\mathbf{I}_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_c \end{pmatrix}.$$

The calculation of the virtual work done by the motors is straightforward and leads to:

$$\delta\mathcal{W}_m = \boldsymbol{\tau}^T \boldsymbol{\delta\theta}_a = \begin{pmatrix} \tau_1 & \tau_2 \end{pmatrix} \begin{pmatrix} \delta\theta_1 \\ \delta\theta_2 \end{pmatrix} = \sum_{i=1}^{2} \tau_i \delta\theta_i.$$

According to the eq. 3.49 the equation of motion can be written in the following form:

$$\mathbf{I}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}}_a + \mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}_a)\dot{\boldsymbol{\theta}}_a = \boldsymbol{\tau} \tag{4.54}$$

In the eq. 4.54 $\boldsymbol{\theta} = \begin{pmatrix} \boldsymbol{\theta}_a{}^T & \boldsymbol{\theta}_u{}^T \end{pmatrix}$, $\mathbf{I}(\boldsymbol{\theta})$ is the (2×2) matrix of generalized inertia and $\mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}_a)$ is the (2×2) matrix of Coriolis and centrifugal forces.

4.6 Kinematic synthesis

In this section we deal with the synthesis of linkages. In detail, we will treat the kinematic dimensional synthesis of a planar 4-bar linkage to generate a proper function between the input and the output angles. The problem can be stated as follows:
The goal is to find the link lengths $\{a_i\}_1^4$ of the planar 4-bar linkage so as to obtain a prescribed set of input-output pairs between the input angle ψ and the output angle ϕ: $\{\psi_k, \phi_k\}_1^m$.

Now, we wish to find the implicit input-output equation $F(\psi, \phi) = 0$. The mechanism closure equation is:

$$\mathbf{r}_1 + \mathbf{r}_2 = \mathbf{r}_3 + \mathbf{r}_4, \tag{4.55}$$

with

$$\mathbf{r}_1 = a_2 \begin{pmatrix} c_\psi \\ s_\psi \end{pmatrix}, \quad \mathbf{r}_2 = a_3 \begin{pmatrix} c_\theta \\ s_\theta \end{pmatrix}, \quad \mathbf{r}_3 = a_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{r}_4 = a_4 \begin{pmatrix} c_\phi \\ s_\phi \end{pmatrix}.$$

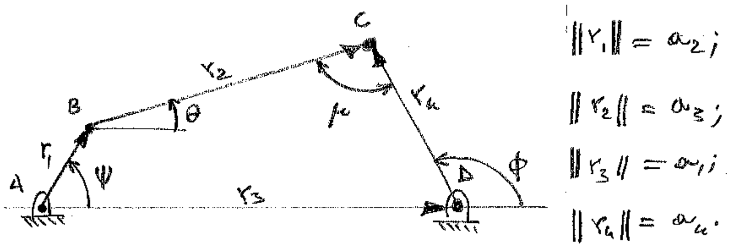

Figure 4.15: A 4-bar linkage for function generation.

From eq. 4.55 we have:

$$\mathbf{r}_2 = \mathbf{r}_3 + \mathbf{r}_4 - \mathbf{r}_1. \tag{4.56}$$

Now, we take the *Euclidean norm* of both sides of the eq. 4.56. In doing so, the angle θ is eliminated:

$$\|\mathbf{r}_2\|^2 = \|\mathbf{r}_3 + \mathbf{r}_4 - \mathbf{r}_1\|^2 = \|\mathbf{r}_3\|^2 + \|\mathbf{r}_4\|^2 + \|\mathbf{r}_1\|^2 + 2\mathbf{r}_3^T\mathbf{r}_4 - 2\mathbf{r}_3^T\mathbf{r}_1 - 2\mathbf{r}_4^T\mathbf{r}_1 \tag{4.57}$$

where

$$\|\mathbf{r}_1\|^2 = a_2^2, \quad \|\mathbf{r}_2\|^2 = a_3^2, \quad \|\mathbf{r}_3\|^2 = a_1^2 \quad \|\mathbf{r}_4\|^2 = a_4^2,$$
$$\mathbf{r}_3^T\mathbf{r}_4 = a_1 a_4 \cos\phi, \quad \mathbf{r}_3^T\mathbf{r}_1 = a_1 a_2 \cos\psi, \quad \mathbf{r}_4^T\mathbf{r}_1 = a_2 a_4 \cos(\phi - \psi).$$

Putting the foregoing equations into eq. 4.57 we obtain:

$$a_3^2 = a_1^2 + a_4^2 + a_2^2 + 2a_1a_4\cos\phi - 2a_1a_2\cos\psi - 2a_2a_4\cos(\phi - \psi) \tag{4.58}$$

Now, according to *Freudenstein* we provide a non-linear mapping from link lengths into nondimensional parameteters. Therefore, we define the *Freudenstein parameters*:

$$k_1 = \frac{a_1^2 + a_2^2 - a_3^2 + a_4^2}{2a_2a_4}, \quad k_2 = \frac{a_1}{a_2}, \quad k_3 = \frac{a_1}{a_4} \tag{4.59}$$

From eqs. 4.59 we obtain a_2, a_3 and a_4 in terms of a_1:

$$a_2 = \frac{1}{k_2}a_1, \quad a_4 = \frac{1}{k_3}a_1, \quad a_3 = \sqrt{a_1^2 + a_2^2 + a_4^2 - 2k_1a_2a_4},$$

and, eventually eq. 4.58 becomes:

$$k_1 + k_2\cos\phi - k_3\cos\psi = \cos(\phi - \psi). \tag{4.60}$$

Now, we write eq.4.60 for $\{\psi_k, \phi_k\}_1^m$ to obtain m linear equations with k_1, k_2 and k_3 as unknowns:

$$\mathbf{T}\mathbf{k} = \mathbf{b}, \tag{4.61}$$

with

$$\mathbf{T} = \begin{pmatrix} 1 & \cos\phi_1 & -\cos\psi_1 \\ 1 & \cos\phi_2 & -\cos\psi_2 \\ \vdots & \vdots & \vdots \\ 1 & \cos\phi_m & -\cos\psi_m \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} \cos(\phi_1-\psi_1) \\ \cos(\phi_2-\psi_2) \\ \vdots \\ \cos(\phi_3-\psi_3) \end{pmatrix},$$

where $\mathbf{T}$ is known as *synthesis matrix*.
Three cases are possible:

m<3 : When m=1 we have one equation and three unknowns. There are infinitely many solutions. When m=2, there are still infinitely many solutions. This case entails the synthesis of quick return mechanisms.

m=3 : This is the case of *exact synthesis*. The problem admits one unique solution unless the synthesis matrix $\mathbf{T}$ was singular.

m>3 : Numbers of equations exceeds number of unknowns leading to a overdetermined system. Therefore, no solution is possible, in general, but an optimum solution can be found that best approximates the synthesis equations in the least-square sense. Problem falls in the category of *approximate synthesis*.

4.6.1 Exact syntesis

In this case $m = 3$ therefore eq. 4.61 takes the form:

$$\begin{pmatrix} 1 & \cos\phi_1 & -\cos\psi_1 \\ 1 & \cos\phi_2 & -\cos\psi_2 \\ 1 & \cos\phi_3 & -\cos\psi_3 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} \cos(\phi_1-\psi_1) \\ \cos(\phi_2-\psi_2) \\ \cos(\phi_3-\psi_3) \end{pmatrix}. \tag{4.62}$$

To solve eq. 4.62 numerically is easy. Because of the simple structure of the system a *closed-form* solution can be obtained as well. To this end, we substract the first equation from the second and third ones, such that:

$$\begin{pmatrix} 1 & c_{\phi_1} & -c_{\psi_1} \\ 0 & c_{\phi_2}-c_{\phi_1} & -c_{\psi_2}+c_{\psi_1} \\ 0 & c_{\phi_3}-c_{\phi_1} & -c_{\psi_3}+c_{\psi_1} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} c_{(\phi_1-\psi_1)} \\ c_{(\phi_2-\psi_2)} - c_{(\phi_1-\psi_1)} \\ c_{(\phi_3-\psi_3)} - c_{(\phi_1-\psi_1)} \end{pmatrix}. \tag{4.63}$$

The second and third of eq. 4.63 are free of k_1 and thus they can be solved for k_2 and k_3:

$$\begin{pmatrix} k_2 \\ k_3 \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} c_{\psi_1}-c_{\psi_3} & c_{\psi_2}-c_{\psi_1} \\ c_{\phi_1}-c_{\phi_3} & c_{\phi_2}-c_{\phi_1} \end{pmatrix} \begin{pmatrix} c_{(\phi_2-\psi_2)} - c_{(\phi_1-\psi_1)} \\ c_{(\phi_3-\psi_3)} - c_{(\phi_1-\psi_1)} \end{pmatrix} \tag{4.64}$$

with

$$\Delta = det \begin{pmatrix} c_{\phi_2}-c_{\phi_1} & -c_{\psi_2}+c_{\psi_1} \\ c_{\phi_3}-c_{\phi_1} & -c_{\psi_3}+c_{\psi_1} \end{pmatrix} =$$
$$= (c_{\psi_2}-c_{\psi_1})(-c_{\psi_3}+c_{\psi_1}) + (c_{\phi_2}-c_{\phi_1})(c_{\phi_3}-c_{\phi_1}).$$

Solution of the system of eqs. 4.63 uses the formula available for inversion of a general (2×2) matrix $\mathbf{A}$:

$$\mathbf{A}^{-1} = \frac{1}{det(\mathbf{A})} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

with a_{ij} the matrix entry at $i^{th.}$ row and $j^{th.}$ column. With k_2 and k_3 obtained from eqs. 4.64 and k_1 from the first equation of eqs. 4.62, we eventually obtain:

$$k_i = \frac{H_i}{\Delta}, \quad i = 1, 2, 3 \tag{4.65}$$

with:

$$H_2 = (-c_{\psi_3} + c_{\psi_1})[c_{(\phi_2 - \psi_2)} - c_{(\phi_1 - \psi_1)}] + (c_{\psi_2} - c_{\psi_1})[c_{(\phi_3 - \psi_3)} - c_{(\phi_1 - \psi_1)}],$$
$$H_3 = (-c_{\phi_3} + c_{\phi_1})[c_{(\phi_2 - \psi_2)} - c_{(\phi_1 - \psi_1)}] + (c_{\phi_2} - c_{\phi_1})[c_{(\phi_3 - \psi_3)} - c_{(\phi_1 - \psi_1)}],$$
$$H_1 = c_{(\phi_1 - \psi_1)}\Delta - c_{\phi_2} H_2 + c_{\psi_1} H_3.$$

It is worth noticing that, by definition, k_2 and k_3 has to be positive while k_1 may take any positive or negative real values. However, the previous formulation can lead to negative values for k_2 and k_3. These values are valid and thay have to be interpreted geometrically. We notice that eq. 4.60 can be written with $\phi + \pi$ instead of ϕ as well as with $\psi + \pi$ instead of ψ. In doing that the signs of the second or third term of the left side of the eq. 4.60 is reversed.
Whenever the synthesis problem leads to $k_2 = 0$ then $a_2 \longrightarrow \infty$, which means that the input link is infinite length. Thus, the first joint of the linkage is prismatic leading to a $PRRR$ linkage. Likewise, if $k_3 = 0$, then $a_4 \longrightarrow \infty$, leading to a $RRRP$ linkage. Finally, although all the k_i, ($i = 1, 2, 3$) parameters are different from zero nothing may guarantee that the link lengths derived from them will lead to a feasible linkage. Indeed, they need to satisfy the *four-bar feasibility condition: any link length must be smaller than the sum of the three other link lengths.*

4.6.2 Approximate synthesis

We have no solutions of the system at hand, therefore we have a *design-error vector* $\mathbf{e} = \mathbf{Tk} - \mathbf{b}$. A positive number derived from $\mathbf{e}$ is the *design error* that we define as the Euclidean norm of the design-error vector $\|\mathbf{e}\|^2$. Therefore we can define the *objective function* $\mathcal{F}$ as:

$$\mathcal{F} = \frac{1}{2}\|\mathbf{e}\|^2 \tag{4.66}$$

The goal is to find the values of the Freudenstein parameters, $\mathbf{k}_0$, which minimimize $\mathcal{F}$:

$$\underset{k}{\text{minimize}} \;\; \mathcal{F} = \frac{1}{2}\|\mathbf{e}\|^2 \tag{4.67}$$

The normality condition of the problem in eq. 4.67 is:

$$\frac{\partial \mathcal{F}}{\partial \mathbf{k}} = \mathbf{0} \tag{4.68}$$

where

$$\frac{\partial \mathcal{F}}{\partial \mathbf{k}} = (\frac{\partial \mathbf{e}}{\partial \mathbf{k}})^T (\frac{\partial \mathcal{F}}{\partial \mathbf{e}}). \tag{4.69}$$

From the definitions of $\mathcal{F}$ and $\mathbf{e}$ we have:

$$\frac{\partial \mathcal{F}}{\partial \mathbf{e}} = \mathbf{Tk} - \mathbf{b}, \quad \frac{\partial \mathbf{e}}{\partial \mathbf{k}} = \mathbf{T}. \tag{4.70}$$

Thus, by introducing eqs. 4.70 into eq. 4.68 we obtain:

$$\mathbf{T}^T \mathbf{Tk} = \mathbf{T}^T \mathbf{b}. \tag{4.71}$$

Eq. 4.71 is a system of 3 linear equations with k_1, k_2, k_3 as unknowns. If $\mathbf{T}$ has full rank then this equation admits one unique solution which is the *least-square solution* of the given system:

$$\mathbf{k}_0 = \mathbf{T}^\dagger \mathbf{b}. \tag{4.72}$$

with $\mathbf{T}^\dagger$ termed left Moore-Penrose generalized inverse:

$$\mathbf{T}^\dagger = (\mathbf{T}^T \mathbf{T})^{-1} \mathbf{T}^T. \tag{4.73}$$

Hence, $\mathbf{e}_0 = \mathbf{Tk}_0 - \mathbf{b}$ is the *least-square error vector* and $e_{d0} = \sqrt{\frac{1}{m}} \|e_0\|$ is the *least-square design error*.
It is worth noting that e_{d0} measures $\mathbf{e}$ but not the positoning error (structural error). The structural error $\mathbf{s}$ produced by the synthesized linkage must be measured with respect to the task as:

$$\mathbf{s} = \begin{pmatrix} \phi_1 - \overline{\phi}_1 & \phi_2 - \overline{\phi}_2 & \cdots & \phi_m - \overline{\phi}_m \end{pmatrix}^T,$$

where ϕ_i is the output angle generated from ψ_i and $\overline{\phi}_i$ denotes its prescribed value.
We now derive an algorithm to obtain the output values ϕ corresponding to *(i)* the feasible link lengths $\{a_i\}_1^4$ of the synthesized linkage and *(ii)* the given input value ψ. Eq. 4.60 can be written as:

$$A(\psi)u + B(\psi)v + C(\psi) = 0 \tag{4.74}$$

with $u = \cos\phi$, $v = \sin\phi$, $A(\psi) = k_2 - \cos\psi$, $B(\psi) = -\sin\psi$, $C(\psi) = k_1 - k_3 \cos\psi$. Eq. 4.74 represents a line $\mathcal{L}$ in the $u - v$ plane. Unknowns u and v are subjected to the constraint $\mathcal{C} : \; u^2 + v^2 = 1$. Thus, the output

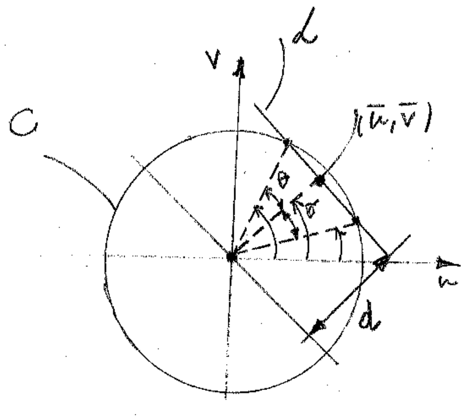

Figure 4.16: $\mathcal{L}$, $\mathcal{C}$ in the $u - v$ plane.

angle ϕ sought is obtained as intersections of $\mathcal{L}$ and $\mathcal{C}$.
According to Figure 4.16 we have:

$$\overline{u} = -\frac{C(\psi)}{S(\psi)}A(\psi), \quad \overline{v} = -\frac{C(\psi)}{S(\psi)}B(\psi), \quad d = \frac{|C(\psi)|}{\sqrt{S(\psi)}},$$

with $S(\psi) = A(\psi)^2 + B(\psi)^2$. And the angles:

$$\sigma = atan2(v, u), \quad \theta_{1,2} = acos(d),$$

and, eventually, with $d > 0$:

$$\phi_1 = \sigma \pm \theta_1, \quad \phi_2 = \sigma \pm \theta_2.$$

5

Trajectory Planning

The motions undergone by robotic mechanical system should be, as a rule, as smooth as possible. Indeed, abrupt changes in position, velocity and acceleration should be avoided, as they require unlimited amounts of power to be implemented, which the motor cannot supply because of their physical limitations. Under ideal conditions, a flexible manufacturing cell (structured environment) is a a work environment in which all the objects, machines and workpieces alike, move with programmed motions that by their nature, can be predicted at any instant. We devote this chapter to the planning of robot motions in structured environments without accounting any unpredictable situation (collisions, for example).
Two typical tasks call for trajectory planning techniques, namely:

- pick-and-place operations (PPO);
- continuous path (CP).

We deal with PPO technique for serial manipulators and briefly on the continous-path tracking.

5.1 Background on PPO

In PPO, we have a manipulator which takes a workpiece from a given *initial pose*, specified by the position of the end effector ref. point and its orientation with respect to a certain coordinate ref. system, to a *final pose*, specified likewise. The initial and final pose are prescribed in the Cartesian space but robot motions are implemented in the joint space by controlling the motors at the joints. Hence, the planning of PPO has to be conducted in the joint space and then mapped into the Cartesian space to check if any collision occurs with the motion planned. The mapping task from joint space to Cartesian space is far from being simple since it involves rendering of the motions of all the moving links of the robot, each of which has a particular geometry. A pragmatic approach consists of two steps: *a*) planning a

preliminary trajectory in the joint space, disregarding the obstacles, and *b*) visually verifying if collisions occur with the aid of solid modelling software able to animate the robot motion in presence of obstacles.
It is mandatory to notice that, in absence of singularities, the mapping of joint to Cartesian variables, and vice versa, is continuous. Therefore, a smooth trajectory planned in the joint space is guaranteed to be smooth in the Cartesian space, as long as the trajectory does not encounter a singularity.
In order to sinthesize the joint trajcectory we need first map the initial (I) and final (F) poses of the workpiece (which is assumed to be rigidly attached to the end effector ref. point) into manipulator configurations described in the joint space. This task is iplemented by the inverse kinematics.
We consider given the position of the end effector ref. point at the initial pose which is reached by the manipulator at time $t = 0$: $\mathbf{p}(0) = \mathbf{p}_I$. We consider given the orientation matrix of the end effector ref. system with respect to a base ref. system at the initial pose which is reached by the manipulator at time $t = 0$: $\mathbf{Q}(0) = \mathbf{Q}_I$. Identically, position and orientation matrix are defined at time $t = T$ when the manipulator reaches the final pose: $\mathbf{p}(T) = \mathbf{p}_F$, $\mathbf{Q}(T) = \mathbf{Q}_F$. Moreover we consider to be known $\dot{\mathbf{p}}$, $\ddot{\mathbf{p}}$ respectively, velocity and acceleration of the end effector ref. point at the initial and final poses and $\boldsymbol{\omega}$, $\dot{\boldsymbol{\omega}}$ respectively, angular velocity and angular acceleration of the end effector ref. system at the initial and final poses:

$$\begin{aligned} \mathbf{p}(0) = \mathbf{p}_I \quad & \dot{\mathbf{p}}(0) = \mathbf{0} \quad & \ddot{\mathbf{p}}(0) = \mathbf{0} \\ \mathbf{Q}(0) = \mathbf{Q}_I \quad & \boldsymbol{\omega}(0) = \mathbf{0} \quad & \dot{\boldsymbol{\omega}}(0) = \mathbf{0} \\ \mathbf{p}(T) = \mathbf{p}_F \quad & \dot{\mathbf{p}}(T) = \mathbf{0} \quad & \ddot{\mathbf{p}}(T) = \mathbf{0} \\ \mathbf{Q}(T) = \mathbf{Q}_F \quad & \boldsymbol{\omega}(T) = \mathbf{0} \quad & \dot{\boldsymbol{\omega}}(T) = \mathbf{0} \end{aligned}$$

In absence of singularities, by the inverse kinematics and considering that conditions of zero velocity and accelerations imply zero joint velocity and accelerations we have:

$$\begin{aligned} \mathbf{q}(0) = \mathbf{q}_I \quad & \dot{\mathbf{q}}(0) = \mathbf{0} \quad & \ddot{\mathbf{q}}(0) = \mathbf{0} \\ \mathbf{q}(T) = \mathbf{q}_F \quad & \dot{\mathbf{q}}(T) = \mathbf{0} \quad & \ddot{\mathbf{q}}(T) = \mathbf{0} \end{aligned} \tag{5.1}$$

where $\mathbf{q}$ denotes the vector of joints variables.

5.2 Polynomial Interpolation

To satisfy the six conditions given in eqs. 5.1 we may not use neither the linear or the quadratic interpolations. A higher-order interpolation is needed. These six conditions imply, in turn, six conditions for every joint trajectory, which means that, if a ploynomial is to be used to represent the motion of every joint, then this polynomial should be at least of the fifth-degree.

5.2.1 A 3-4-5 interpolating polynomial

In order to represent each joint motion, we use the *normal* polynomial $s(\tau)$ given as:

$$s(\tau) = a\tau^5 + b\tau^4 + c\tau^3 + d\tau^2 + e\tau + f, \tag{5.2}$$

with

$$0 \leq s \leq 1, \quad 0 \leq \tau \leq 1 \quad \text{and } \tau = \frac{t}{T}.$$

Hence, each joint variable q_j, throughout its range of motion, can be expressed as:

$$q_j(t) = q_j^I + (q_j^F - q_j^I)s(\tau)$$

where q_j^I and q_j^F are given in eqs. 5.1. In vector form we have:

$$\mathbf{q}(t) = \mathbf{q}_I + (\mathbf{q}_F - \mathbf{q}_I)s(\tau),$$

then

$$\dot{\mathbf{q}}(t) = (\mathbf{q}_F - \mathbf{q}_I)\frac{ds(\tau)}{dt},$$

$$\frac{ds(\tau)}{dt} = \frac{ds(\tau)}{d\tau}\frac{d\tau}{dt} = s'(\tau)\dot{\tau} = \frac{s'(\tau)}{T},$$

Therefore:

$$\dot{\mathbf{q}}(t) = (\mathbf{q}_F - \mathbf{q}_I)\frac{s'(\tau)}{T}.$$

Likewise:

$$\ddot{\mathbf{q}}(t) = (\mathbf{q}_F - \mathbf{q}_I)\frac{s''(\tau)}{T^2}.$$

Now, we need the values of the coefficients of $s(\tau)$ appearing in eq. 5.2:

$$\begin{aligned}
\mathbf{q}(0) = \mathbf{q}_I = \mathbf{q}_I + (\mathbf{q}_F - \mathbf{q}_I)s(0) &\ : \ s(0) = 0;\\
\dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_I = 0 = (\mathbf{q}_F - \mathbf{q}_I)\frac{s'(0)}{T} &\ : \ s'(0) = 0;\\
\ddot{\mathbf{q}}(0) = \ddot{\mathbf{q}}_I = 0 = (\mathbf{q}_F - \mathbf{q}_I)\frac{s''(0)}{T^2} &\ : \ s''(0) = 0;
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{q}(T) = \mathbf{q}_F = \mathbf{q}_I + (\mathbf{q}_F - \mathbf{q}_I)s(1) &\ : \ s(1) = 1;\\
\dot{\mathbf{q}}(T) = \dot{\mathbf{q}}_F = 0 = (\mathbf{q}_F - \mathbf{q}_I)\frac{s'(1)}{T} &\ : \ s'(1) = 0;\\
\ddot{\mathbf{q}}(T) = \ddot{\mathbf{q}}_F = 0 = (\mathbf{q}_F - \mathbf{q}_I)\frac{s''(1)}{T^2} &\ : \ s''(1) = 0;
\end{aligned}$$

On the other hand, from eq. 5.2 we have:

$$s'(\tau) = 5a\tau^4 + 4b\tau^3 + 3c\tau^2 + 2d\tau + e,$$
$$s''(\tau) = 20a\tau^3 + 12b\tau^2 + 6c\tau + 2d$$

and substituting the six conditions we obtain:

$$f = e = d = 0;$$
$$a + b + c = 1;$$
$$5a + 4b + 3c = 0;$$
$$30a + 12b + 6c = 0$$

and eventuallly:

$$a = 6;\ b = -15;\ c = 10;\ f = e = d = 0.$$

The interpolating 3-4-5 normal polynomial and its derivatives are thus obtained as:

$$s(\tau) = 6\tau^5 - 15\tau^4 + 10\tau^3,$$
$$s'(\tau) = 30\tau^4 - 60\tau^3 + 30\tau^2,$$
$$s''(\tau) = 120\tau^3 - 180\tau^2 + 60\tau.$$

It is, thus, possible to determine the evolution of each joint variable if we know both its end values and the time T to complete the motion. If no extra conditons are imposed, we have the freedom to perform the desired motion in as a short time T as possible. However, T cannot be given arbitrarily small as the motors specifications (maximum velocity, maximum torque) must be respected.
Now, we want to determine T for each joint so that the joint rates and accelerations lie within allowed limits.
By studying $s''(\tau)$, it is apparent that we find a maximum for $s'(\tau)$ at $\tau = 1/2$:

$$s'_{max} = s'(\frac{1}{2}) = \frac{15}{8},$$

and hence:

$$(\dot{q}_j)_{max} = \frac{15(q_j^F - q_j^I)}{8T}. \tag{5.3}$$

By studying $s'''(\tau) = 360\tau^2 - 360\tau + 60$, it is apparent that we find a maximum for $s''(\tau)$ at $\tau = (1/2-\sqrt{3}/6)$ and a minimum at $\tau = (1/2+\sqrt{3}/6)$:

$$s''_{max} = s''(\frac{1}{2} - \frac{\sqrt{3}}{6}) = \frac{10\sqrt{3}}{3},$$
$$s''_{min} = s''(\frac{1}{2} + \frac{\sqrt{3}}{6}) = -\frac{10\sqrt{3}}{3},$$

and hence:

$$(\ddot{q}_j)_{max} = \frac{10\sqrt{3}}{3} \frac{(q_j^F - q_j^I)}{T^2}. \tag{5.4}$$

Eqs. 5.3 and 5.4 allow us to find T such that the joints rates and accelerations lie within allowed limits. Obviously, since in general, motors at different joints are different, one has to select the largest T of those calculated.

5.3 Cycloidal Motion

An alternative motion that produces zero velocity and acceleration at the ends of a finite interval is the *cycloidal* motion. In normal form:

$$s(\tau) = \tau - \frac{1}{2\pi} \sin 2\pi\tau$$

and its derivatives are:

$$s'(\tau) = 1 - \cos 2\pi\tau,$$
$$s''(\tau) = 2\pi \sin 2\pi\tau.$$

By studying $s''(\tau)$, it is apparent that we find a maximum for $s'(\tau)$ at $\tau = 1/2$:

$$s'_{max} = s'(\frac{1}{2}) = 2,$$

and hence:

$$(\dot{q}_j)_{max} = \frac{2}{T}(q_j^F - q_j^I). \tag{5.5}$$

By studying $s'''(\tau) = 4\pi^2 \cos 2\pi\tau$, it is apparent that we find a maximum for $s''(\tau)$ at $\tau = 1/4$ and a minimum at $\tau = 3/4$:

$$s''_{max} = s''(\frac{1}{4}) = s''(\frac{1}{4}) = 2\pi,$$

and hence:

$$(\ddot{q}_j)_{max} = \frac{2\pi}{T^2}(q_j^F - q_j^I). \tag{5.6}$$

Eqs. 5.5 and 5.6 allow us to find T such that the joints rates and accelerations lie within allowed limits.

5.4 Trajectories with Via Poses

The polynomial trajectory discussed above does not allow the specification of intermediate Cartesian poses of the end effector. However, it may be possible to define intermediate poses in the Cartesian space, called *via poses*, that lie between the initial and final poses. Then by inverse kinematics, the values of the joint variables can be determined that correspond to the aforementioned via poses. These values, therefore, are nothing else but points on the joint-space trajectory and they are called *via points*.

The introduction of the via points in the joint-space trajectories increases the number of conditions to be satisfied by the desired trajectory. For example, in the case of the polynomial trajectory synthesized for continuity up to the second derivatives, we can introduce two via points by requiring that:

$$s(\tau_1) = s_1, \quad s(\tau_2) = s_2$$

where τ_1, τ_2 are arbitrary instants usually chosen just after the start and just before the end of the trajectory, while s_1, s_2 are the corresponding values. These values differ from joint to joint whereas τ_1 and τ_2 are the same for all the joints. Thus, we need to determine one normal polynomial for each joint.

We have to imagine to have the Cartesian poses at the via poses and by the inverse kinematics to get the corresponding joint variables and evenutally the normal polynomial coordinates s_1, s_2. By summing up, we have eight conditions to satisfy:

$$\begin{aligned} s(0) = 0, \quad s'(0) = 0, \quad s''(0) = 0; \\ s(1) = 1, \quad s'(1) = 0, \quad s''(1) = 0; \\ s(\tau_1) = s_1, \quad s(\tau_2) = s_2. \end{aligned}$$

In a very straightforward way we can satisfy the conditions above by using a seventh-order polynomial.

5.5 Synthesis of PPO using Cubic Splines

As we have seen at the end of the previous section an increase in the number of conditions to be met by the normal polynomial causes an increase in the degree of this polynomial. Now, finding the coefficients of the interpolating polynomial requires solving a system of linear equations. Unfortunately, the roundoff error quickly increases as the polynomial degree increases. As an alternative to higher order polynomials, *splines functions* have been found to offer a more robust interpolation schemes. Splines are piecewise polinomials with continuity properties imposed at the *supporting points* (points at which two neighboring polynomials join). The splines are defined as a set of rather lower-degree polynomials joined at a number of supporting points. We will deal with the periodic cubic spline which is especially suited for path planning robotics.

A cubic spline function $s(x)$ connecting N points $P_k(x_k, y_k)$, $k = 1, \cdots, N$, is a function defined *piecewise* by $N-1$ cubic polynomials joined at the points P_k such that $s(x_k) = y_k$. Furthermore, the spline function is twice differentiable everywhere in $x_1 \leq x \leq x_N$. This means that the spline has continuous derivatives up to the second order.

Let $P_k(x_k, y_k)$ and $P_{k+1}(x_{k+1}, y_{k+1})$ to be two consecutive supporting points. The kth. cubic polynomial $s_k(x)$ between those points is assumed to be:

$$s_k(x) = A_k(x - x_k)^3 + B_k(x - x_k)^2 + C_k(x - x_k) + D_k, \tag{5.7}$$

for $x_k \leq x \leq x_{k+1}$. Thus for the spline $s(x)$, $4(N-1)$ coefficients A_k, B_k, C_k, D_k, for $k = 1, \cdots, N-1$, are to be determined. The goal is now, to calculate the $s_k(x)$ coefficients in terms of $s_k \equiv s_k(x_k)$ and $s''_k \equiv s''_k(x_k)$. First we take the derivatives of $s_k(x)$:

$$\begin{aligned} s'_k(x) &= 3A_k(x - x_k)^2 + 2B_k(x - x_k) + C_k, \\ s''_k(x) &= 6A_k(x - x_k) + 2B_k, \end{aligned} \tag{5.8}$$

and if $x \equiv x_k$ then:

$$B_k = \frac{1}{2}s''_k, \quad C_k = s'_k, \quad D_k = s_k, \tag{5.9}$$

where also $s'_k \equiv s'_k(x_k)$. From the equations above we have that B_k and D_k are in terms of s_k and s''_k. We would like to have the same for A_k and C_k. First, we define:

$$\Delta x_k = x_{k+1} - x_k,$$

then we find the relations sought by imposing the continuity conditions on the spline function and its first and second derivatives with respect to x at the supporting points ($k = 1, \cdots, N-2$):

$$\begin{aligned} s_k(x_{k+1}) &= s_{k+1}, \\ s'_k(x_{k+1}) &= s'_{k+1}, \\ s''_k(x_{k+1}) &= s''_{k+1}. \end{aligned}$$

$s''_k(x_{k+1})$ may be computed by eq. 5.8 such that:

$$6A_k\Delta x_k + 2B_k = s''_{k+1} = 2B_{k+1},$$

where $s''_{k+1} = 2B_{k+1}$ because of eq. 5.9. Thus:

$$A_k = \frac{(2B_{k+1} - 2B_k)}{6\Delta x_k} = \frac{1}{6\Delta x_k}(s''_{k+1} - s''_k). \tag{5.10}$$

$s_k(x_{k+1})$ may be computed by eq. 5.7 such that:

$$A_k(\Delta x_k)^3 + B_k(\Delta x_k)^2 + C_k\Delta x_k + D_k = s_{k+1}$$

Now, we may obtain C_k by substituting B_k and D_k from eq. 5.9 and A_k from eq. 5.10 into the equation above:

$$C_k = \frac{1}{\Delta x_k}[s_{k+1} - \frac{1}{6\Delta x_k}(s''_{k+1} - s''_k)\Delta x_k^3 - \frac{1}{2}s''_k\Delta x_k - s_k] :$$
$$C_k = \frac{\Delta s_k}{\Delta x_k} - \frac{1}{6}\Delta x_k(s''_{k+1} - 2s''_k) \qquad (5.11)$$

with $\Delta s_k = s_{k+1} - s_k$.
Thus, B_k and D_k from eq. 5.9, A_k from eq. 5.10 and C_k from eq. 5.11 provide the constants in terms of s_k and s''_k. In order to find the coefficients, all we need is to set the values of the second derivatives $\{s''_k\}_1^N$ (we know $\{s_k\}_1^N$) at the supporting points. To compute these values we use the continuity condition imposed on the first derivative:

$$s'_k(x_{k+1}) = s'_{k+1} :$$
$$3A_k(\Delta x_k)^2 + 2B_k\Delta x_k + C_k = C_{k+1}$$

and shifting to the previous polynomial we have:

$$s'_{k-1}(x_k) = s'_k :$$
$$3A_{k-1}(\Delta x_{k-1})^2 + 2B_{k-1}\Delta x_{k-1} + C_{k-1} = C_k$$

Now, we substitute A_k, B_k, C_k and D_k in the equation above in order to obtain a linear system of $N-2$ equations for the N unknowns $\{s''_k\}_1^N$:

$$(\Delta x_k)s''_{k+1} + 2(\Delta x_{k-1} + \Delta x_k)s''_k + \Delta x_{k-1}s''_{k-1} =$$
$$6(\frac{\Delta s_k}{\Delta x_k} - \frac{\Delta s_{k-1}}{\Delta x_{k-1}}), \quad k = 2, \cdots, N-1. \qquad (5.12)$$

Further, if we define:

$$\mathbf{s} = \begin{pmatrix} s_1 & \cdots & s_k & \cdots & s_N \end{pmatrix}^T,$$
$$\mathbf{s}'' = \begin{pmatrix} s''_1 & \cdots & s''_k & \cdots & s''_N \end{pmatrix}^T,$$

then eq. 5.12 can be written in vector form as:

$$\mathbf{A}\mathbf{s}'' = 6\mathbf{C}\mathbf{s} \qquad (5.13)$$

where $\mathbf{A} \in \mathbb{V}^{(N-2)\times N}$ as well as $\mathbf{C}$:

$$\mathbf{A} = \begin{pmatrix} \alpha_1 & 2\alpha_{1,2} & \alpha_2 & 0 & \cdots & 0 & 0 \\ 0 & \alpha_2 & 2\alpha_{2,3} & \alpha_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \alpha_{N'''} & 2\alpha_{N''',N''} & \alpha_{N''} & 0 \\ 0 & 0 & 0 & \cdots & \alpha_{N''} & 2\alpha_{N'',N'} & \alpha_{N'} \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} \beta_1 & -\beta_{1,2} & \beta_2 & 0 & \cdots & 0 & 0 \\ 0 & \beta_2 & -\beta_{2,3} & \beta_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta_{N'''} & -\beta_{N''',N''} & \beta_{N''} & 0 \\ 0 & 0 & 0 & \cdots & \beta_{N''} & -\beta_{N'',N'} & \beta_{N'} \end{pmatrix}.$$

For $i = j = k = 1, \cdots, N-1$:

$$\alpha_k \equiv \Delta x_k, \quad \alpha_{i,j} = \alpha_i + \alpha_j,$$
$$\beta_k \equiv \frac{1}{\alpha_k}, \quad \beta_{i,j} = \beta_i + \beta_j,$$

and

$$N' \equiv N-1, \quad N'' \equiv N-2, \quad N''' \equiv N-3.$$

Thus, two additional equations are needed to render eq. 5.13 a determined system. The additional equations are derived, in turn, depending on the class of functions one is dealing with, which thus gives various types of splines. For example if $s''_1 = s''_N = 0$ then one obtains the *natural cubic splines*. In this case $\mathbf{s}''$ is of dimension $N-2$ and hence $\mathbf{A} \in \mathbb{V}^{(N-2)\times(N-2)}$ such that:

$$\mathbf{s}'' = 6\mathbf{A}^{-1}\mathbf{C}\mathbf{s}.$$

Once $\{s''_k\}_1^N$ are calculated we may obtain the constants A_k, B_k, C_k and D_k, $(k = 1, \cdots, N)$, to have finally the spline function sought.

5.6 Continuous path (CP)

In CP, we have a continous trajectory that has to be tracked by the end-effector. In practice, the trajectory is sampled by a discrete set of close-enough poses. For each pose, the robot configuration has to be found by calculating the joints variables. Thus, an *inverse kinematics position problem* has to be solved at each point of the discrete set of poses, *i.e.,* at each istant of the discretized time. In the following section we deal with few numerical algorithms dedicated to solve the inverse kinematics position problem numerically.

5.6.1 Inverse kinematics algorithms

The closed solutions for the inverse kinematics position problem can be found only in few cases. In general, the highly nonlinear nature of the relationship between the operational variables (end-effector motion variables) and the joints variables does not allow for a closed form solution. Nevertheless, a numerical solution can be found in any case exploiting the linear mapping in the differential kinematics: $\mathbf{v}(t) = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}}(t)$.
Let consider that the end-effector motion trajectory is given in terms of $\mathbf{v}$. The goal is to determine a feasible joint trajectory $\mathbf{q}(t)$, $\dot{\mathbf{q}}(t)$ the reproduces the given trajectory when $\mathbf{q}(0)$, the initial configuration of the robot, is known:

$$\dot{\mathbf{q}}(t) = \mathbf{J}(\mathbf{q})^{-1}\mathbf{v}(t),$$
$$\mathbf{q}(t) = \int_0^t \dot{\mathbf{q}}(\zeta)d\zeta + \mathbf{q}(0).$$

Integration may be performed by numerical techniques. For example, by using the technique based on the *Euler* integration method, the solution can be obtained at time $t_k + \Delta t \equiv t_{k+1}$ once the solution at time t_k is calculated:

$$\mathbf{q}(t_{k+1}) = \mathbf{q}(t_k) + \dot{\mathbf{q}}(t_k)\Delta t = \mathbf{q}(t_k) + \mathbf{J}(\mathbf{q}(t_k))^{-1}\mathbf{v}(t_k)\Delta t.$$

The method proposed is available whenever $\mathbf{J}(\mathbf{q})$ is invertible. If $\mathbf{J}(\mathbf{q})$ was not square, still its pseudo-inverse can be calculated, otherwise if the kinematics chain reaches a singularity configuration some more sophisticated techniques have to be employed. However, any numerical technique provides numerical errors that, by means of the forward kinematics, lead to a difference between the calculated position/orientation, $\mathbf{x}$, and those desired, $\mathbf{x}_d$: $\mathbf{e} = \mathbf{x}_d - \mathbf{x}$. Numerous control techniques are designed to ensure the convergence of $\mathbf{e}$ to zero.

1. Jacobian Inverse algorithm
   Let consider the desired end-effector trajectory given in terms of position/orientation and velocities, namely $\mathbf{x}_d$, $\dot{\mathbf{x}}_d$. Let define the position/orientation error as: $\mathbf{e} = \mathbf{x}_d - \mathcal{F}(\mathbf{q})$ where $\mathbf{x} = \mathcal{F}(\mathbf{q})$ represent the direct kinematics relationships. Thus, the error rate can be obtained as:

   $$\dot{\mathbf{e}} = \dot{\mathbf{x}}_d - \dot{\mathbf{x}} = \dot{\mathbf{x}}_d - \mathbf{J}(\mathbf{q})\dot{\mathbf{q}}. \tag{5.14}$$

   By eq. 5.14 we can obtain $\dot{\mathbf{q}}$ as:

   $$\dot{\mathbf{q}} = \mathbf{J}^{-1}(\mathbf{q})(\dot{\mathbf{x}}_d - \dot{\mathbf{e}}).$$

Now, we need to ensure the convergence of the error $\mathbf{e}$ to zero, for examble, by setting $\dot{\mathbf{e}} + \mathbf{G}\mathbf{e} = \mathbf{0}$ we have:

$$\dot{\mathbf{q}} = \mathbf{J}^{-1}(\mathbf{q})(\dot{\mathbf{x}}_d + \mathbf{G}\mathbf{e}). \tag{5.15}$$

If $\mathbf{G}$ is a positive definite (diagonal) matrix then eq. 5.15 is asymptotically stable. The error tends to zero along the trajectory with a convergence rate that depends on the eigenvalues of $\mathbf{G}$. Figure 5.1 shows a block scheme of the algorithm presented. As it can be expected the method does not work when the robot is approaching a singular configuration.

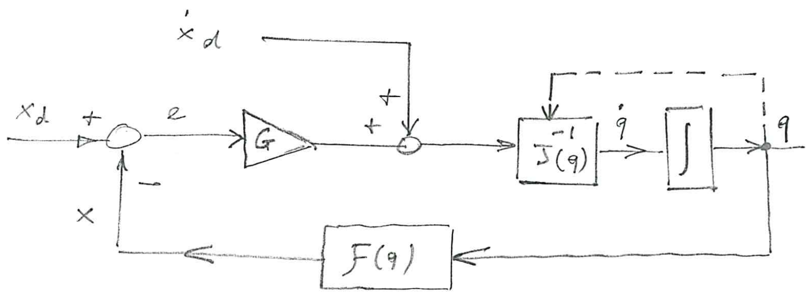

Figure 5.1: Scheme of the inverse kinematics algorithm with Jacobian inverse.

2. Damped least-squares (DLS) inverse
   In this algorithm $\mathbf{J}$ has still full rank and $\mathbf{J} \in \mathbb{V}^{m\times n}$ with $m \geq n$. Eq. 5.15 can be substituted by:

$$\dot{\mathbf{q}} = \mathbf{J}^{*}(\mathbf{q})(\dot{\mathbf{x}}_d + \mathbf{G}\mathbf{e}), \tag{5.16}$$

   where the DLS inverse is defined as:

$$\mathbf{J}^{*} = (\mathbf{J}^T\mathbf{J} + k^2\mathbf{1})^{-1}\mathbf{J}^T, \tag{5.17}$$

   where k is a damping factor which makes the inversion better conditionated numerically. The DLS inverse is nothing but the *special left inverse* corrected by the damping factor. Eq. 5.17 comes from the minimization of the objective function $\mathcal{L}$ defined as:

$$\mathcal{L} = \frac{1}{2}(\dot{\mathbf{x}} - \mathbf{J}\dot{\mathbf{q}})^T(\dot{\mathbf{x}} - \mathbf{J}\dot{\mathbf{q}}) + \frac{1}{2}k^2\dot{\mathbf{q}}^T\dot{\mathbf{q}}$$

   The normality condition leads to the foregoing definition of $\mathbf{J}^{*}$:

$$\nabla\mathcal{L} = \mathbf{J}^T(\mathbf{J}\dot{\mathbf{q}} - \dot{\mathbf{x}}) + k^2\dot{\mathbf{q}} = \mathbf{0}. \tag{5.18}$$

   Eventually, from eq. 5.18 we obtain:

$$\dot{\mathbf{q}} = (\mathbf{J}^T\mathbf{J} + k^2\mathbf{1})^{-1}\mathbf{J}^T\dot{\mathbf{x}}.$$