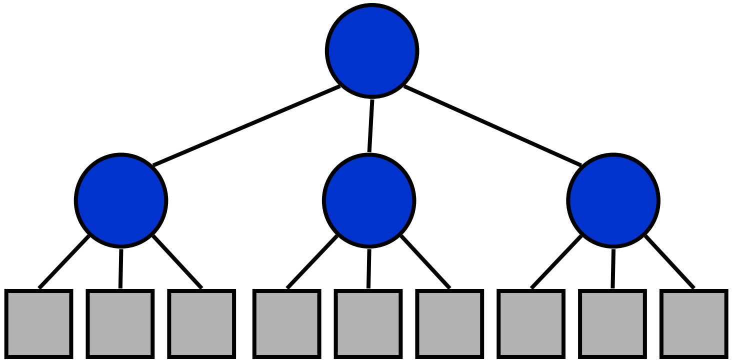
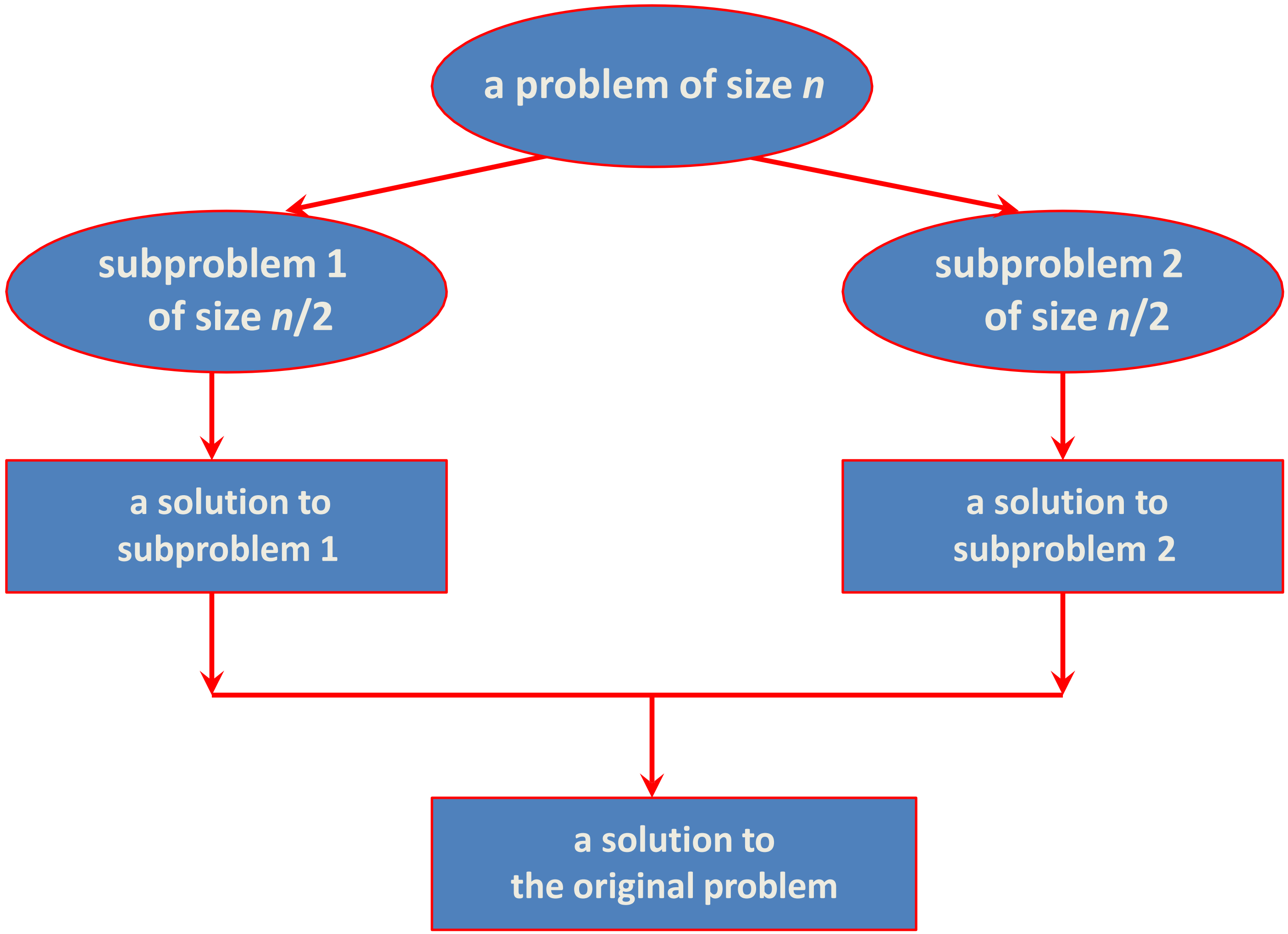
Divide-and-Conquer Technique: Merge Sort

Divide-and-Conquer

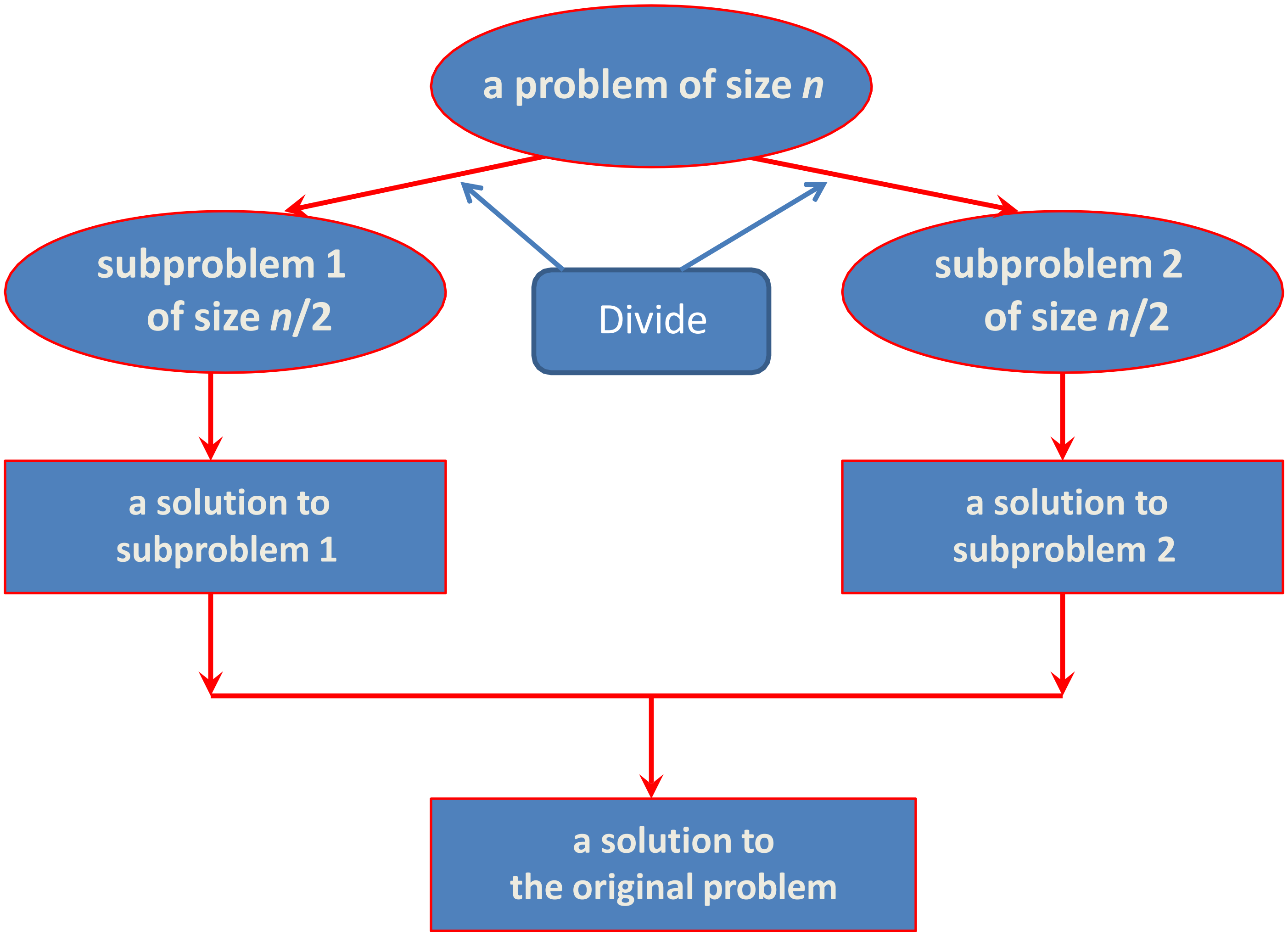
- Divide-and-Conquer is a general algorithm design paradigm:
  - Divide the problem into a number of subproblems that are smaller instances of the same problem
  - Conquer the subproblems by solving them recursively
  - Combine the solutions to the subproblems into the solution for the original problem
- The base case for the recursion are subproblems of constant size
- Analysis can be done using **recurrence equations**



Divide-and-Conquer



Divide-and-Conquer



Merge Sort and Quick Sort

Two well-known sorting algorithms adopt this divide-and-conquer strategy

- Merge sort
  - Divide step is trivial – just split the list into two equal parts
  - Work is carried out in the conquer step by merging two sorted lists
- Quick sort
  - Work is carried out in the divide step using a pivot element
  - Conquer step is trivial

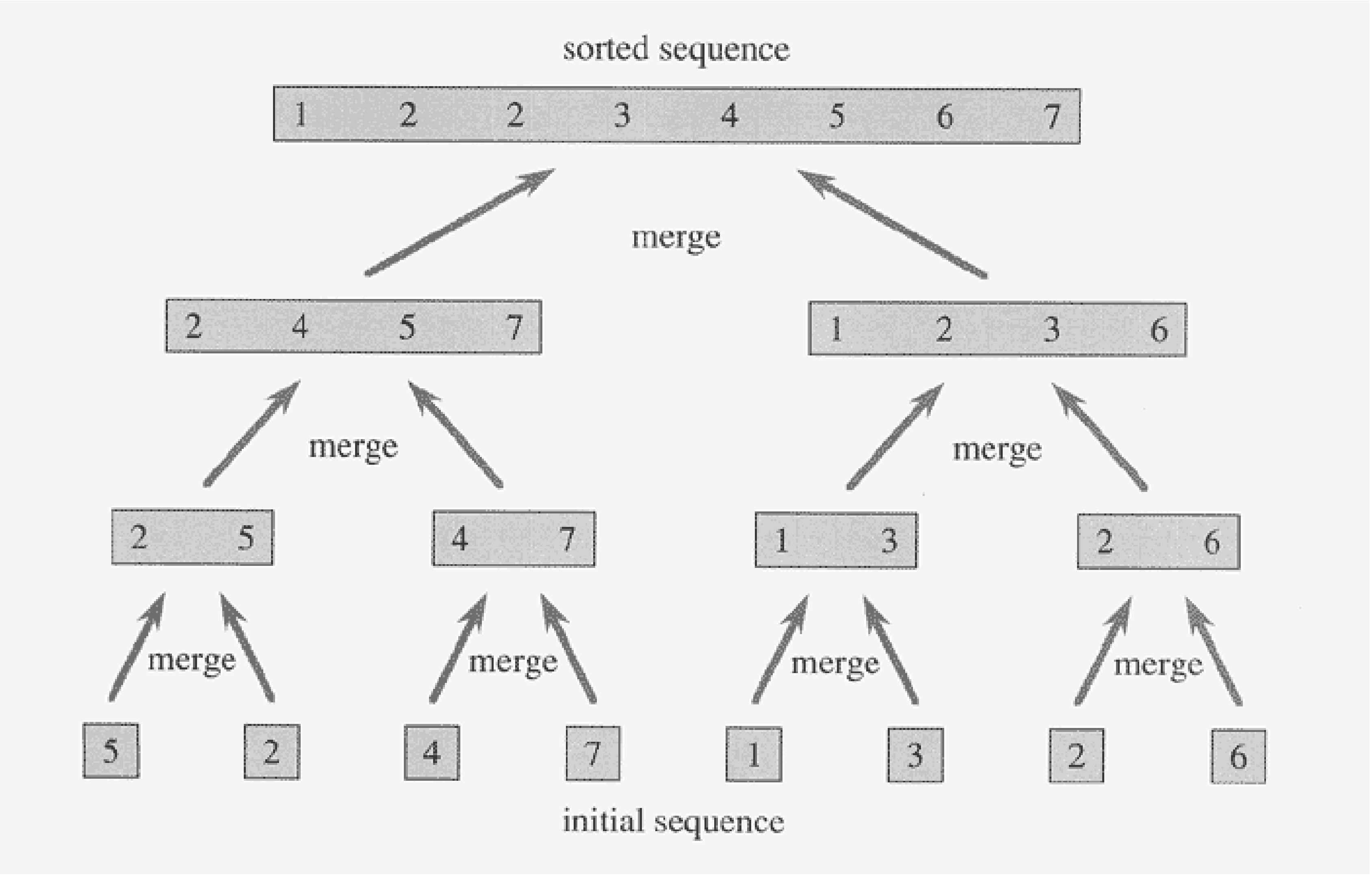
Merge Sort: Algorithm

```
MERGE-SORT(A, p, r)
    if p < r
        then q ← ⌊(p + r)/2⌋
             MERGE-SORT(A, p, q)
             MERGE-SORT(A, q + 1, r)
             MERGE(A, p, q, r)
```

Merge Sort: Algorithm

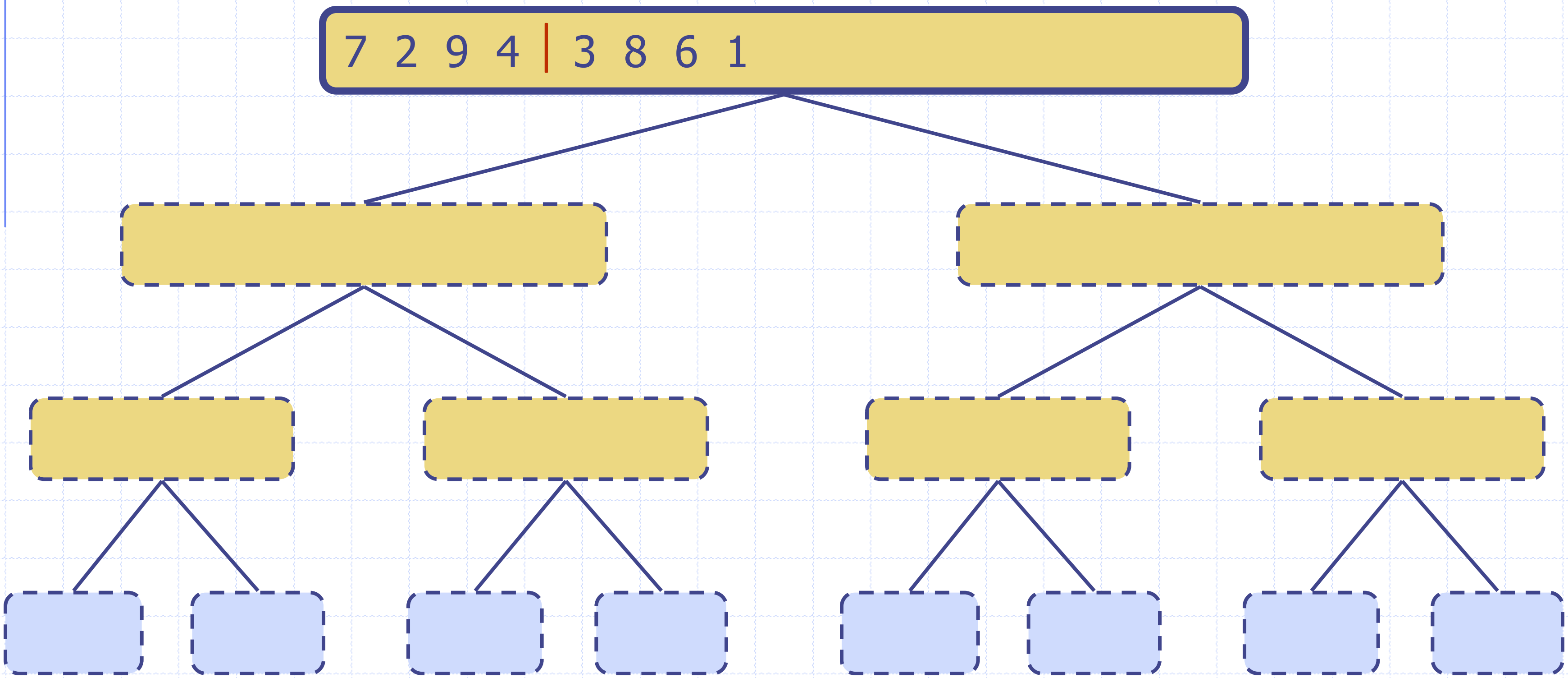
```
MERGE(A, p, q, r)
  n1 ← q − p + 1
  n2 ← r − q
  create arrays L[1 . . n1 + 1] and R[1 . . n2 + 1]
  for i ← 1 to n1
       do L[i] ← A[p + i − 1]
  for j ← 1 to n2
       do R[j] ← A[q + j]
  L[n1 + 1] ← ∞
  R[n2 + 1] ← ∞
  i ← 1
  j ← 1
  for k ← p to r
       do if L[i] ≤ R[j]
             then A[k] ← L[i]
                  i ← i + 1
             else A[k] ← R[j]
                  j ← j + 1
```

Merge Sort: Example



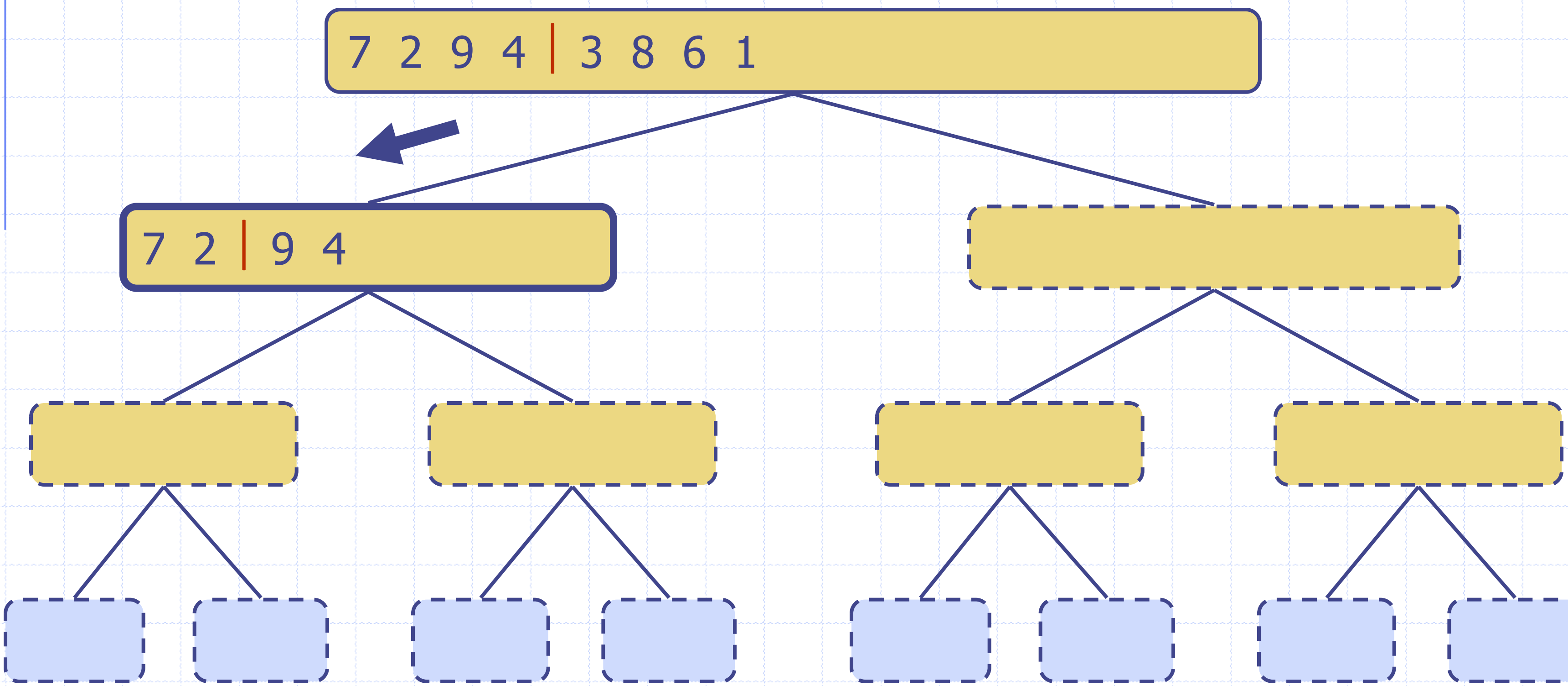
Execution Example

- Partition



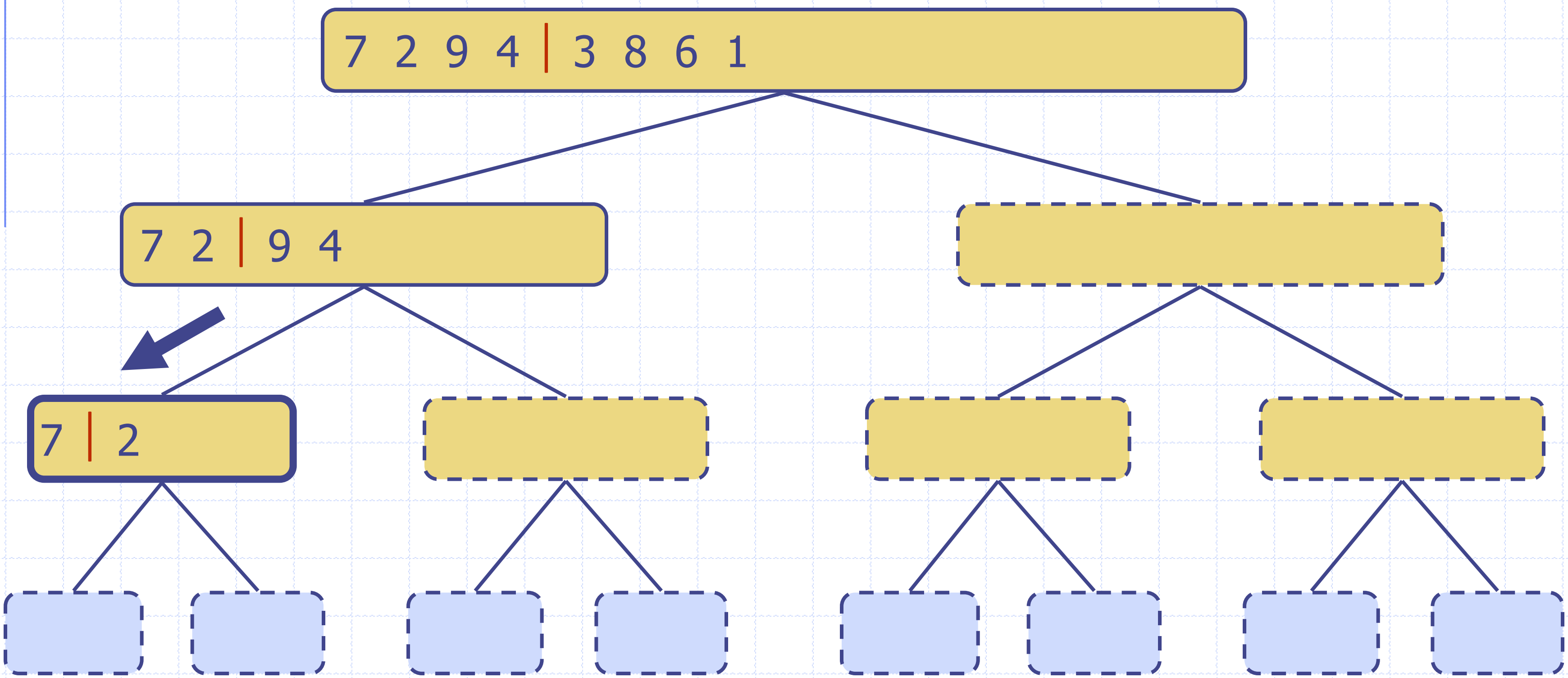
Execution Example

- Recursive call, partition



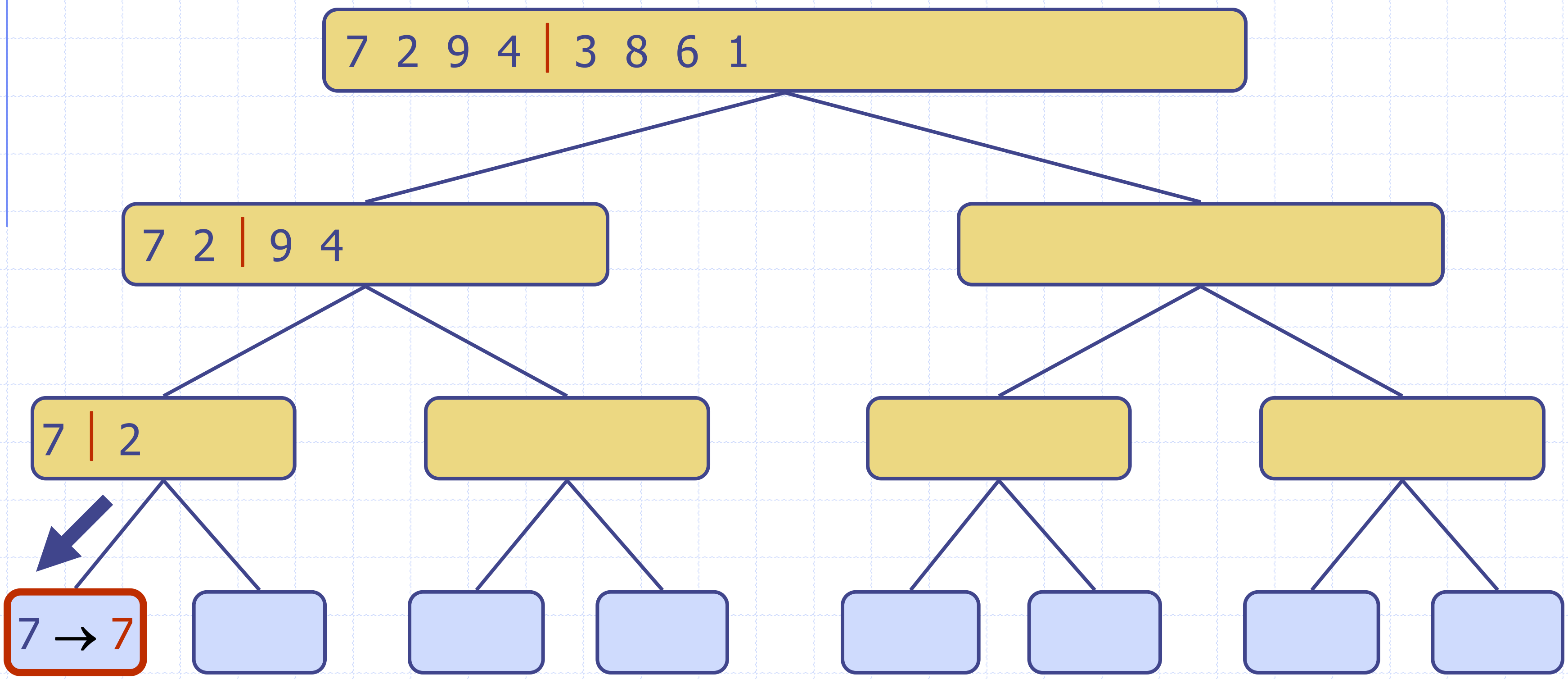
Execution Example

- Recursive call, partition



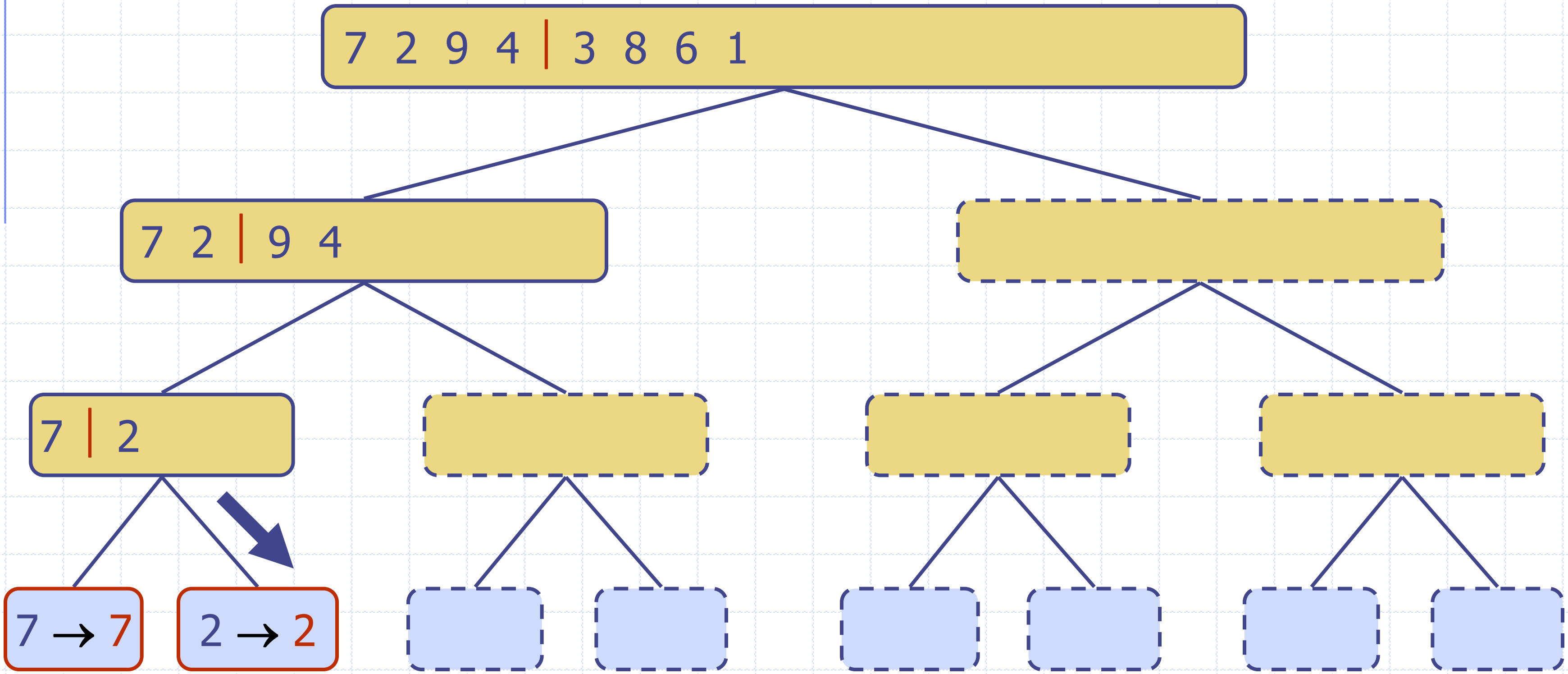
Execution Example

- Recursive call, base case



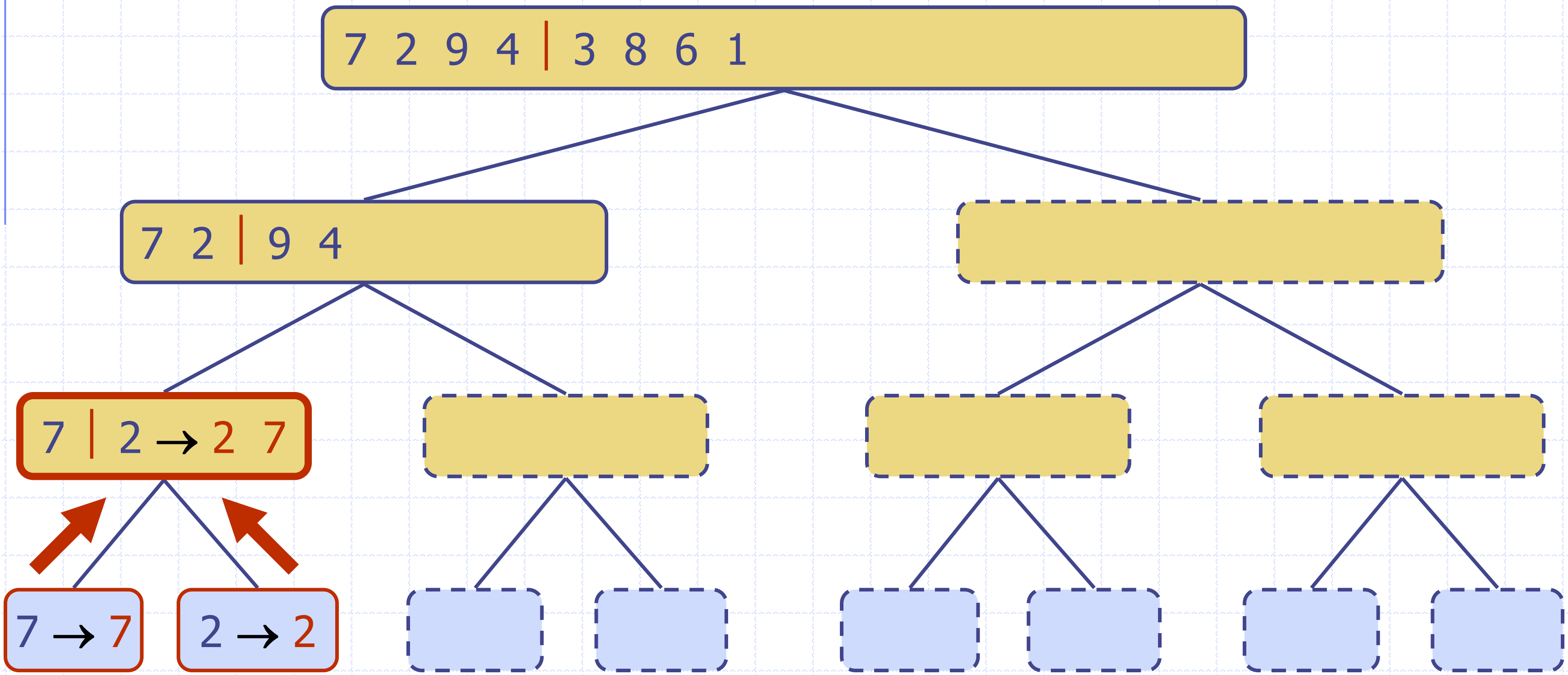
Execution Example

- Recursive call, base case



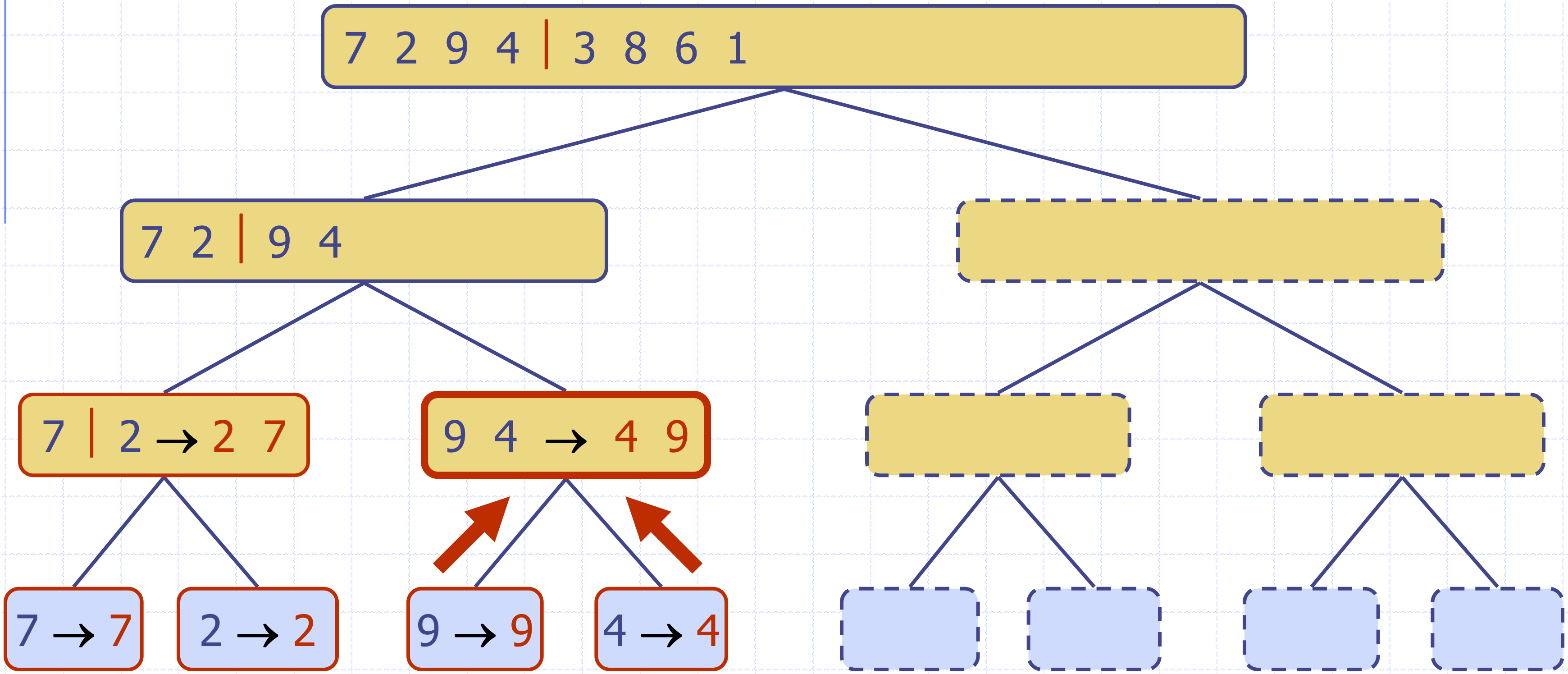
Execution Example

- Merge



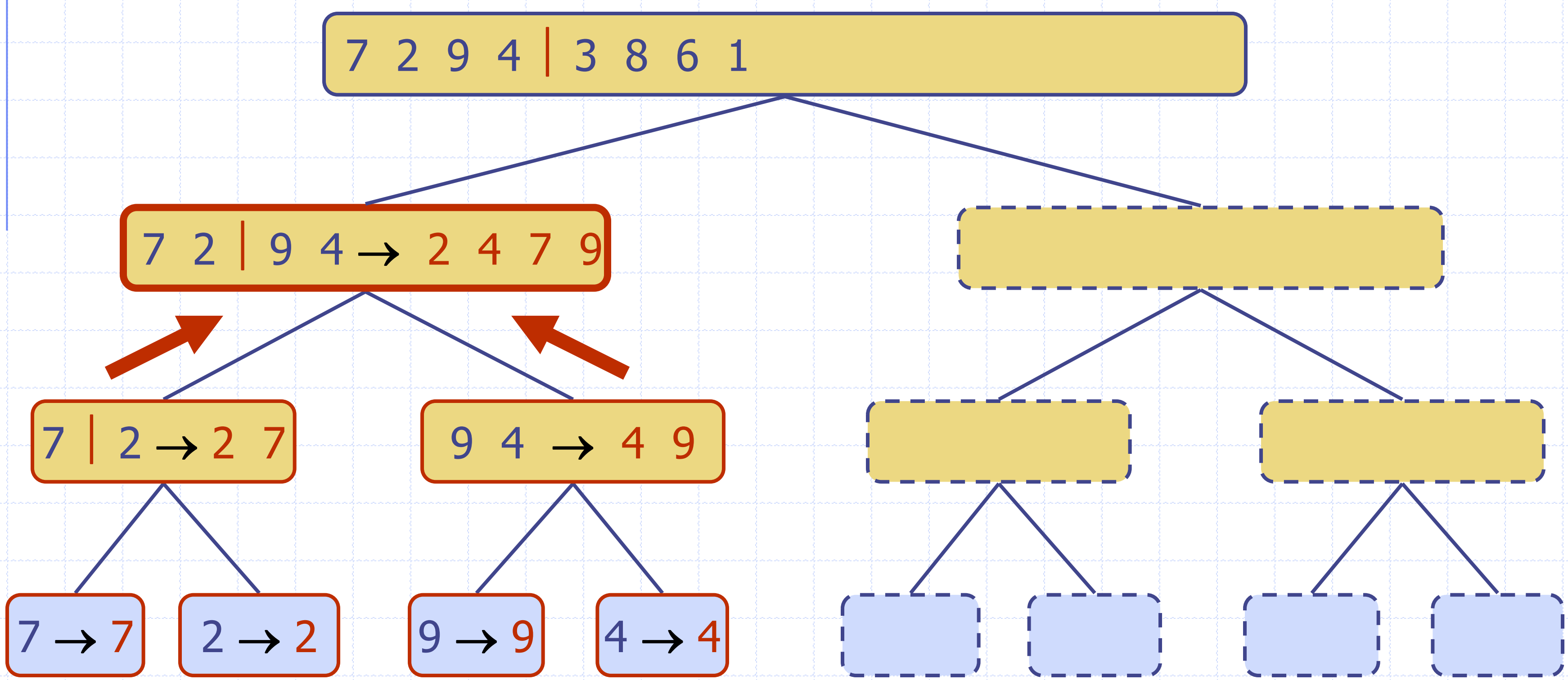
Execution Example

- Recursive call, …, base case, merge



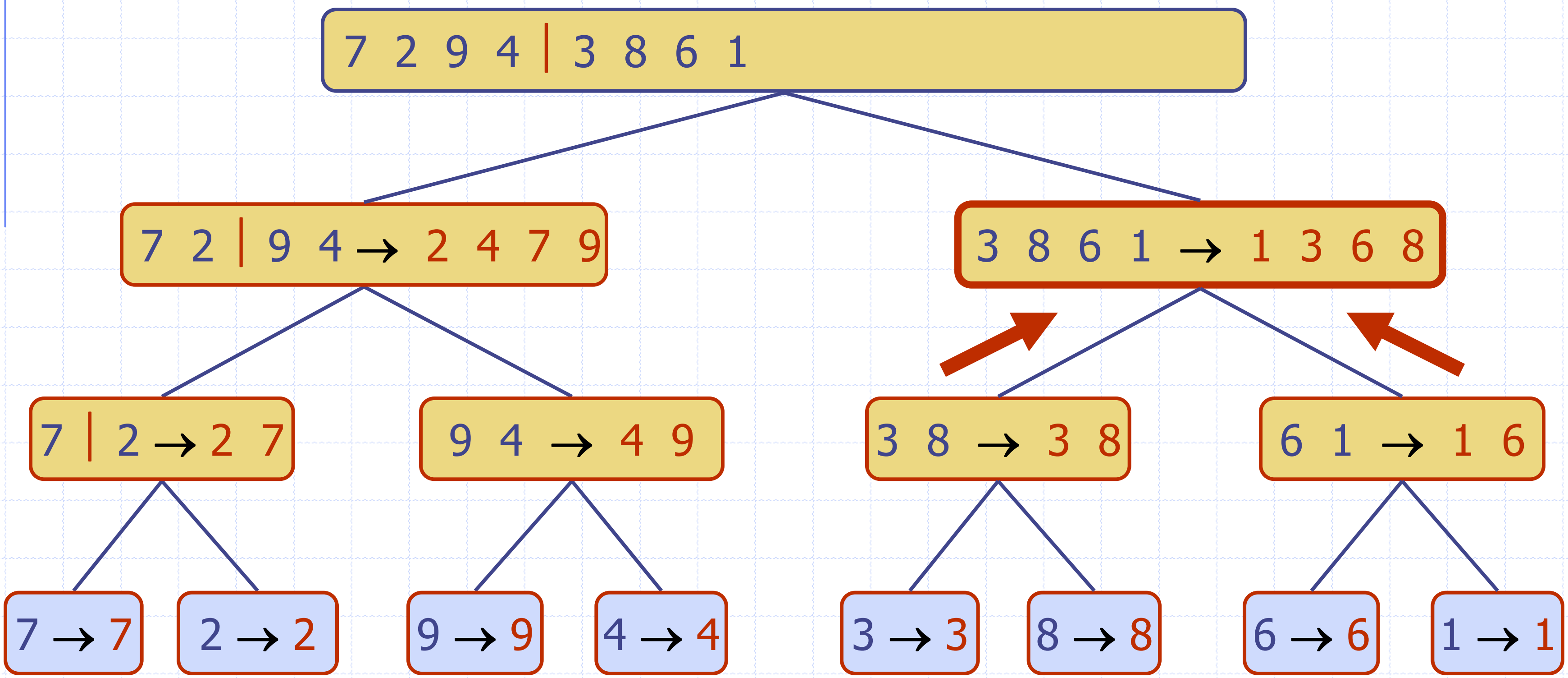
Execution Example

- Merge



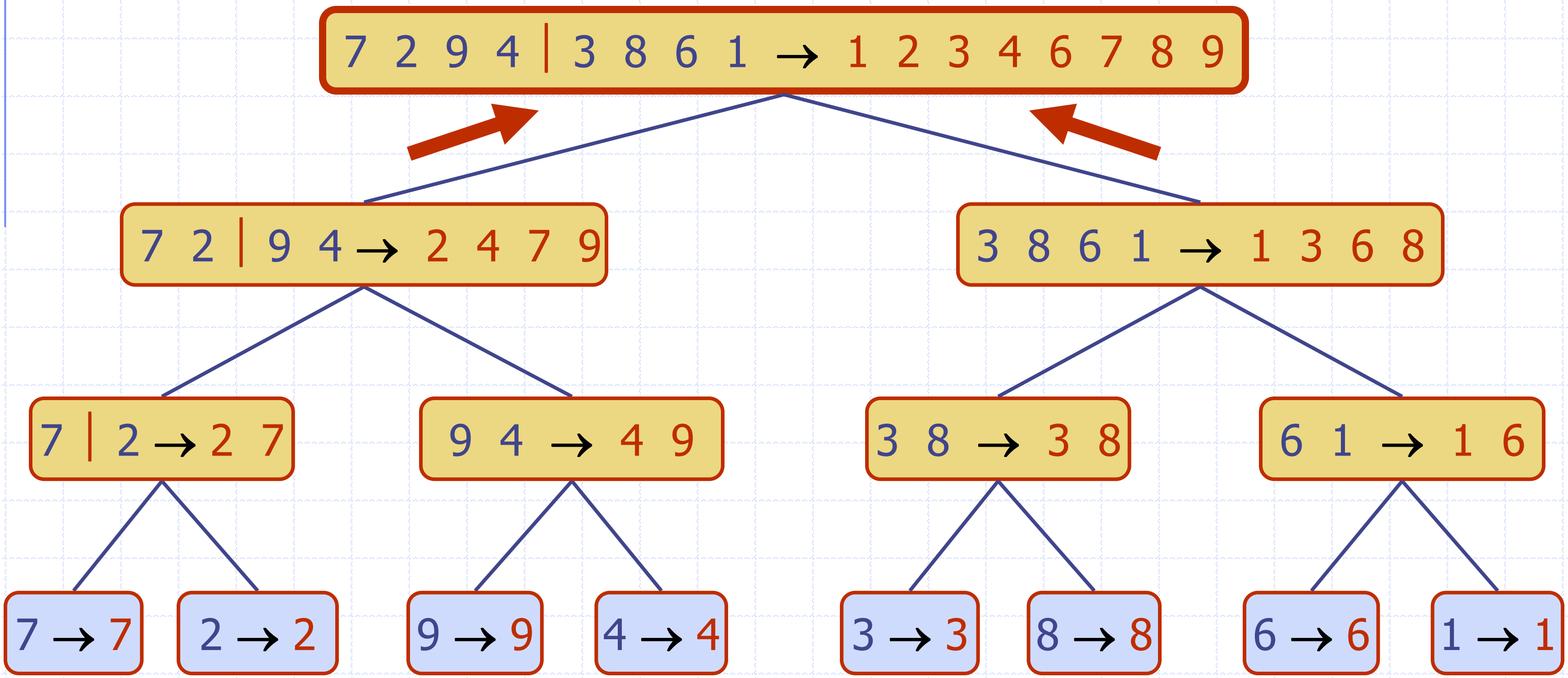
Execution Example

- Recursive call, …, merge, merge



Execution Example

- Merge



Merge Sort: Running Time

The recurrence for the worst-case running time T(n) is

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 2T(n/2) + \Theta(n) & \text{if } n > 1 \end{cases}$$

equivalently

$$T(n) = \begin{cases} b & \text{if } n = 1 \\ 2T(n/2) + bn & \text{if } n > 1 \end{cases}$$

Solve this recurrence by

(1) iteratively expansion

(2) using the recursion tree

Merge Sort: Running Time (Iterative Expansion)

$$
\begin{aligned}
T(n) &= 2T(n/2) + bn \\
&= 2(2T(n/2^2)) + b(n/2)) + bn \\
&= 2^2 T(n/2^2) + 2bn \\
&= 2^3 T(n/2^3) + 3bn \\
&= 2^4 T(n/2^4) + 4bn \\
&= \ldots \\
&= 2^i T(n/2^i) + ibn
\end{aligned}
$$

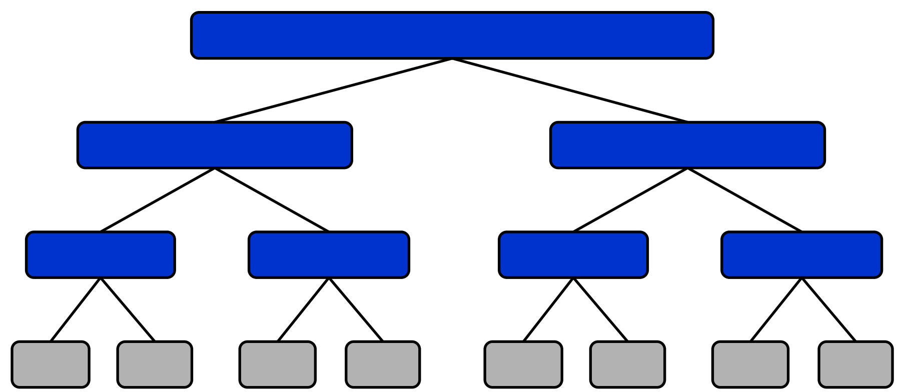
- Note that base, $T(n) = b$, case occurs when $2^i = n$.
  That is, $i = \log n$.
- So, $T(n) = bn + bn\log n$
- Thus, $T(n)$ is $O(n \log n)$.

Merge Sort: Running Time (Recursion Tree)

- Draw the recursion tree for the recurrence relation and look for a pattern:

$$T(n) = \begin{cases} b & \text{if } n = 1 \\ 2T(n/2) + bn & \text{if } n \geq 2 \end{cases}$$

depth	T's	size	time
0	1	n	bn
1	2	$n/2$	bn
i	2^i	$n/2^i$	bn
...	...	...	...



Total time = $bn + bn \log n$

(last level plus all previous levels)