Divide-and-Conquer Technique:
Maximum Subarray problem

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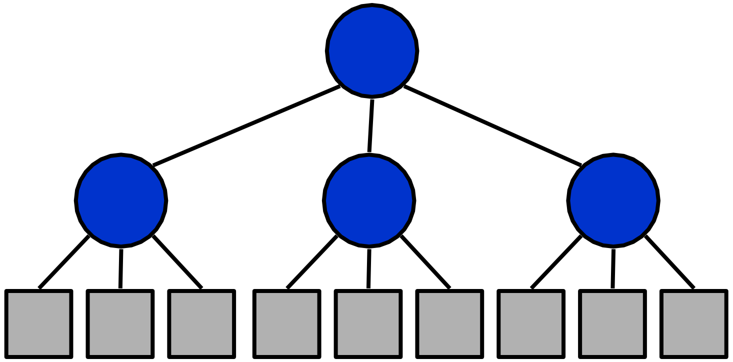
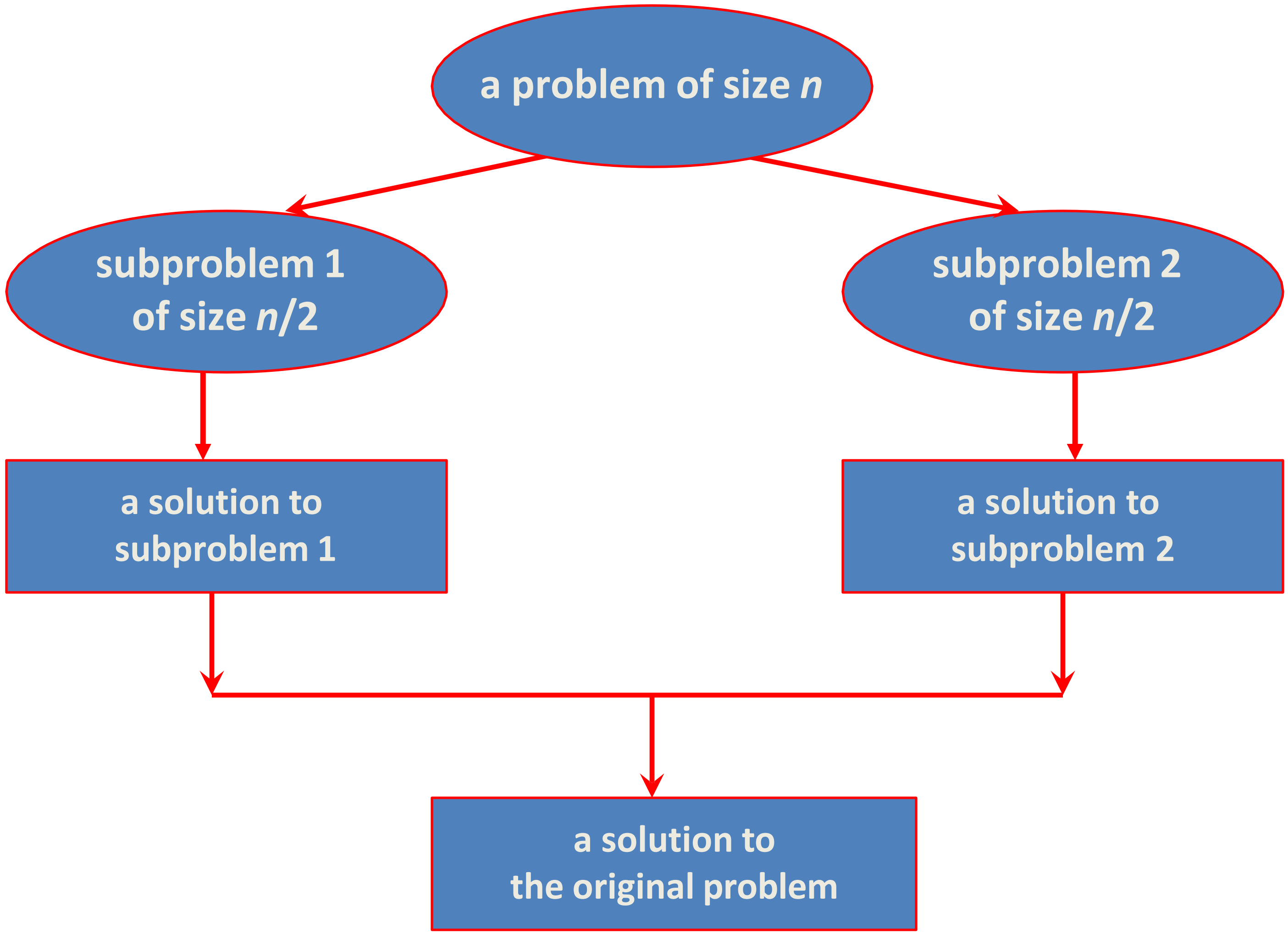
Divide-and-Conquer

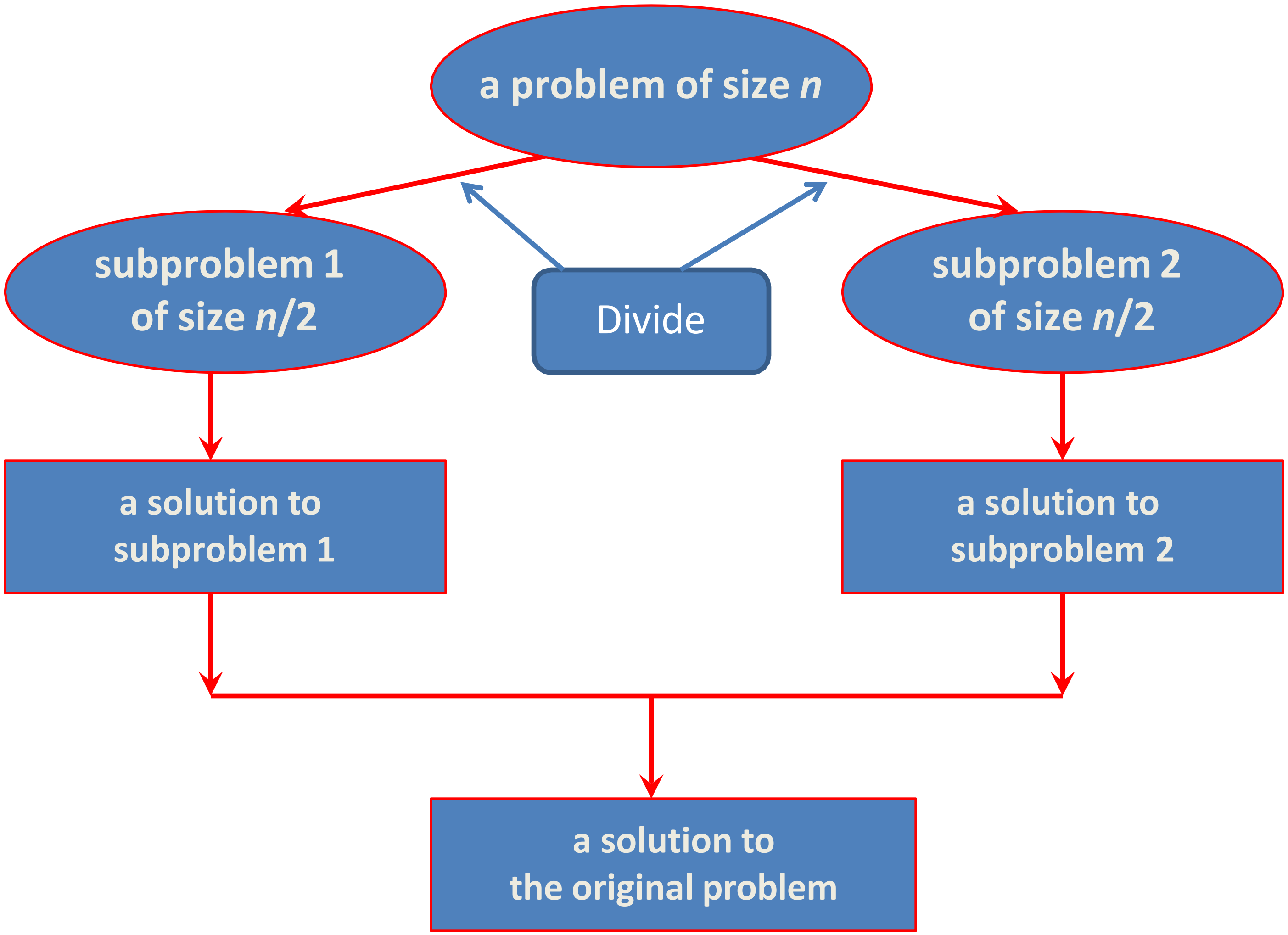
- Divide-and-Conquer is a general algorithm design paradigm:
  - Divide the problem into a number of subproblems that are smaller instances of the same problem
  - Conquer the subproblems by solving them recursively
  - Combine the solutions to the subproblems into the solution for the original problem
- The base case for the recursion are subproblems of constant size
- Analysis can be done using **recurrence equations**



Divide-and-Conquer



Divide-and-Conquer



Maximum Subarray Problem

- *Input:* an array $A[1..n]$ of n numbers
  - Assume that some of the numbers are negative, because this problem is trivial when all numbers are nonnegative
- *Output:* a nonempty subarray $A[i..j]$ having the largest sum $S[i, j] = a_i + a_{i+1} + ... + a_j$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
A	13	-3	-25	20	-3	-16	-23	18	20	-7	12	-5	-22	15	-4	7

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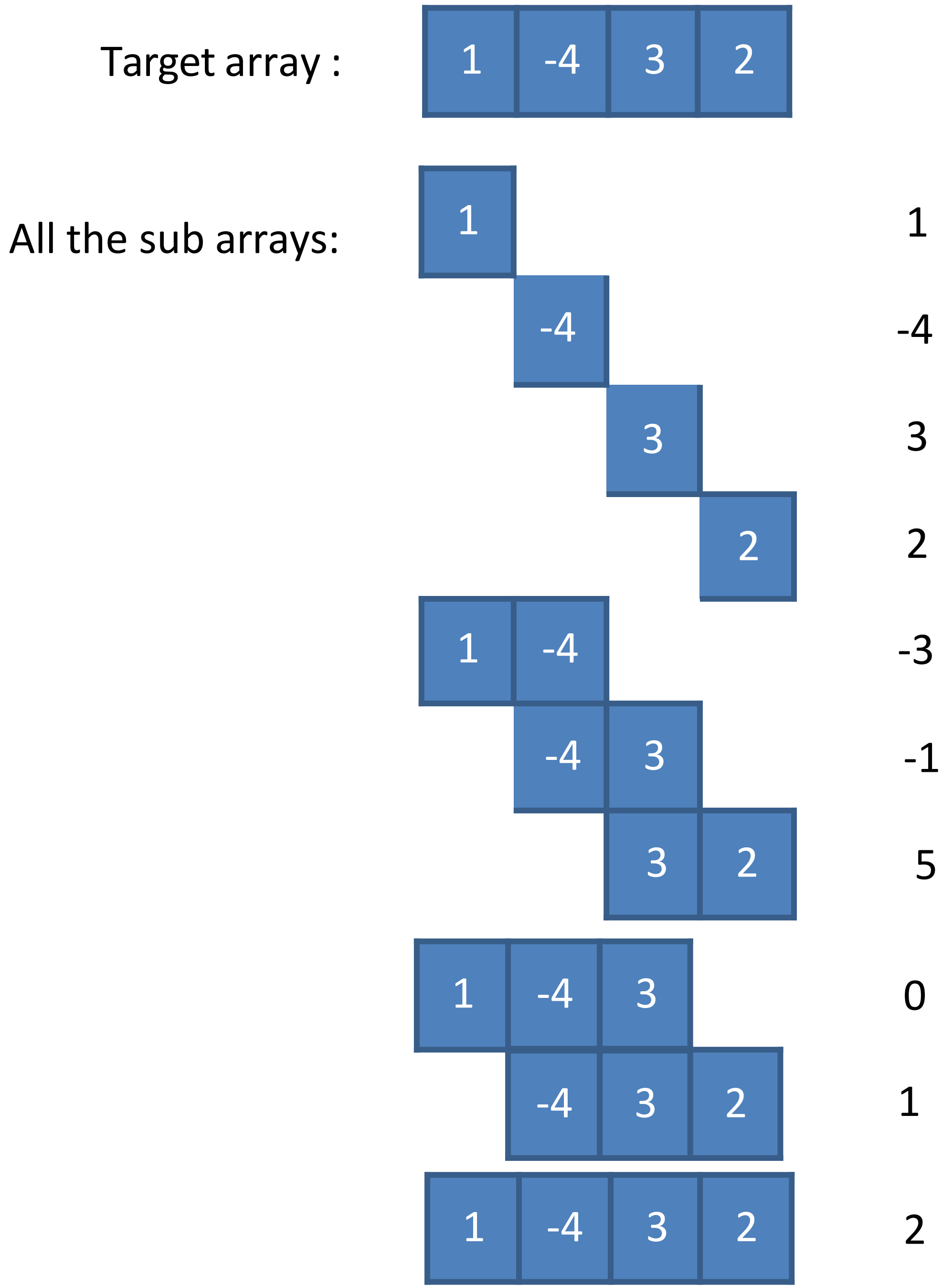
maximum subarray

Target array :

1	-4	3	2

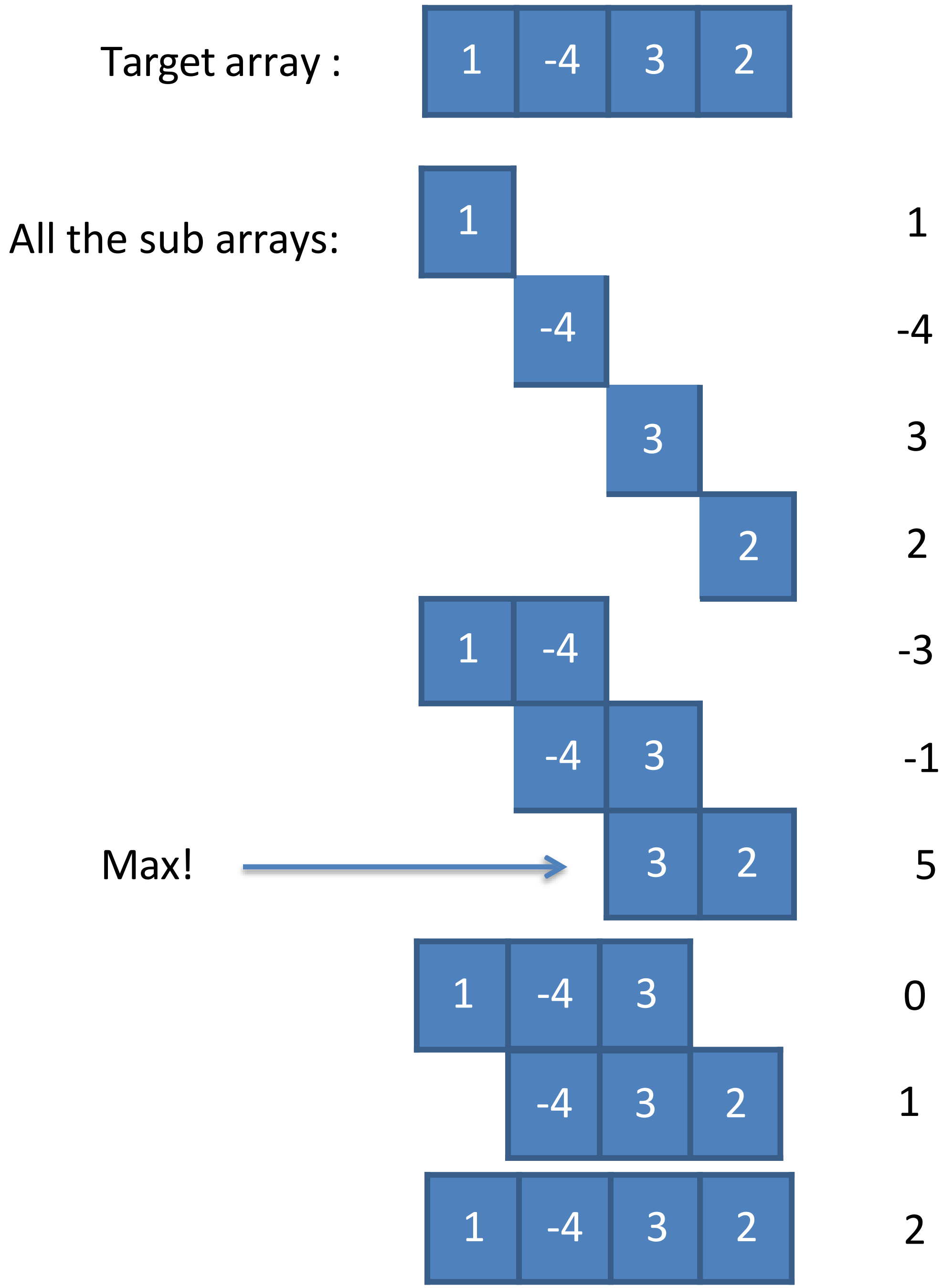
What is a maximum subarray?

All the sub arrays:



What is a maximum subarray?

Ans: The subarray with the largest sum



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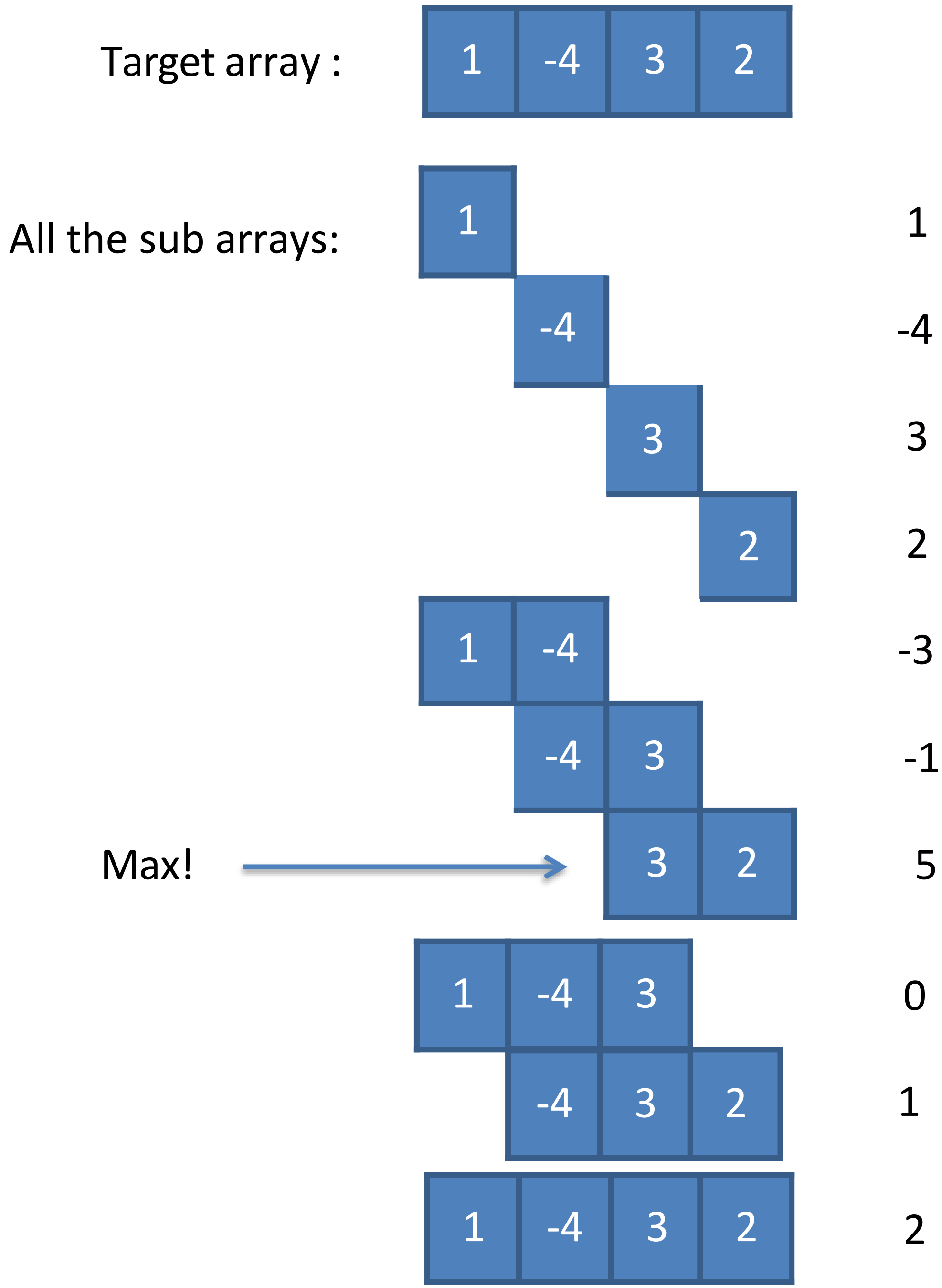

What is a maximum subarray?

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What is the brute-force time?

Brute-Force Algorithm

All possible contiguous subarrays

- A[1..1], A[1..2], A[1..3], ..., A[1..(n-1)], A[1..n]
- A[2..2], A[2..3], ..., A[2..(n-1)], A[2..n]
- ...
- A[(n-1)..(n-1)], A[(n-1)..n]
- A[n..n]

Brute-Force Algorithm

All possible contiguous subarrays

- A[1..1], A[1..2], A[1..3], ..., A[1..(n-1)], A[1..n]
- A[2..2], A[2..3], ..., A[2..(n-1)], A[2..n]
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- A[(n-1)..(n-1)], A[(n-1)..n]
- A[n..n]

How many of them in total?

Brute-Force Algorithm

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- A[1..1], A[1..2], A[1..3], ..., A[1..(n-1)], A[1..n]
- A[2..2], A[2..3], ..., A[2..(n-1)], A[2..n]
- ...
- A[(n-1)..(n-1)], A[(n-1)..n]
- A[n..n]

How many of them in total?

$O(n^2)$

Brute-Force Algorithm

All possible contiguous subarrays

- A[1..1], A[1..2], A[1..3], ..., A[1..(n-1)], A[1..n]
- A[2..2], A[2..3], ..., A[2..(n-1)], A[2..n]
- ...
- A[(n-1)..(n-1)], A[(n-1)..n]
- A[n..n]

How many of them in total?

$O(n^2)$

Algorithm: For each subarray, compute the sum.
Find the subarray that has the maximum sum.

Brute-Force Algorithm

Example:	2	-6	-1	3	-1	2	-2
sum from A[1]:	2	-4	-5	-2	-3	-1	-3
sum from A[2]:		-6	-7	-4	-5	-3	-5
sum from A[3]:			-1	2	1	3	1
sum from A[4]:				3	2	4	2
sum from A[5]:					-1	1	-1
sum from A[6]:						2	0
sum from A[7]:							-2

Brute-Force Algorithm

Outer loop: index variable i to indicate start of subarray, for $1 \le i \le n$, i.e., A[1], A[2], ..., A[n]

- for $i = 1$ to n do ...

Inner loop: for each start index i, we need to go through A[i..i], A[i..(i+1)], ..., A[i..n]

- use an index j for $i \le j \le n$, i.e., consider A[i..j]
- for $j = i$ to n do ...

Brute-Force Algorithm

```
max = -∞
for i = 1 to n do
begin
    sum = 0
    for j = i to n do
    begin
        sum = sum + A[j]
        if sum > max
        then max = sum
    end
end
```

Brute-Force Algorithm

```
max = -∞
for i = 1 to n do
begin
    sum = 0
    for j = i to n do
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        sum = sum + A[j]
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        then max = sum
    end
end
```
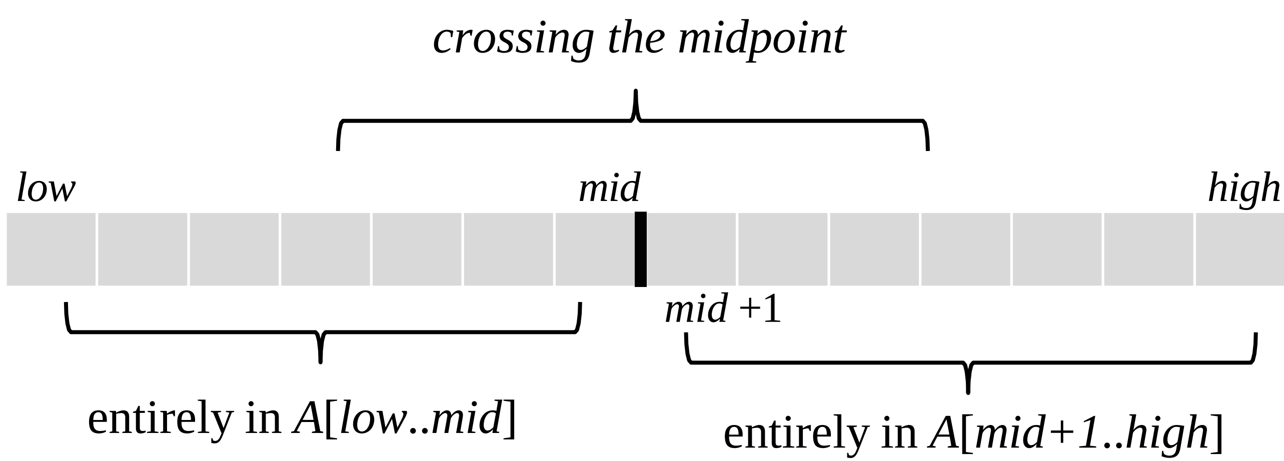

Time complexity? $O(n^2)$

Divide-and-Conquer Algorithm

Possible locations of a maximum subarray $A[i..j]$ of $A[low..high]$, where $mid = \lfloor (low + high)/2 \rfloor$

- entirely in $A[low..mid]$ $(low \le i \le j \le mid)$
- entirely in $A[mid+1..high]$ $(mid < i \le j \le high)$
- crossing the midpoint $(low \le i \le mid < j \le high)$



Possible locations of subarrays of A[low..high]

Divide-and-Conquer Algorithm

FIND-MAXIMUM-SUBARRAY (*A*, *low*, *high*)

if *high* == *low*

return (*low*, *high*, *A*[*low*]) *// base case: only one element*

else *mid* = $\lfloor low + high / 2 \rfloor$

(*left-low*, *left-high*, *left-sum*) =

FIND-MAXIMUM-SUBARRAY(*A*, *low*, *mid*)

(*right-low*, *right-high*, *right-sum*) **=**

FIND-MAXIMUM-SUBARRAY(*A*, *mid* + *1*, *high*)

(*cross-low*, *cross-high*, *cross-sum*) =

FIND-MAX-CROSSING-SUBARRAY(*A*, *low*, *mid*, *high*)

if *left-sum* ≧ *right-sum* ***and*** *left-sum* ≧ *cross-sum*

return (*left-low*, *left-high*, *left-sum*)

elseif *right-sum* ≧ *left-sum* ***and*** *right-sum* ≧ *cross-sum*

return (*right-low*, *right-high*, *right-sum*)

else return (*cross-low*, *cross-high*, *cross-sum*)

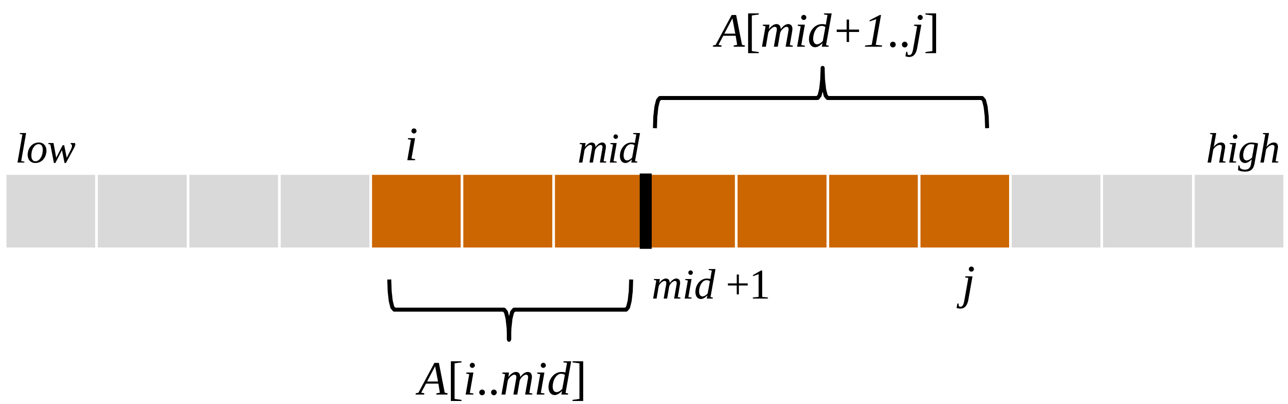
Initial call: FIND-MAXIMUM-SUBARRAY (A, 1, n)

Divide-and-Conquer Algorithm

FIND-MAX-CROSSING-SUBARRAY (A, *low*, *mid*, *high*)

left-sum = -∞ // Find a maximum subarray of the form A[*i*..*mid*]
sum = *0*
for *i* = *mid* **downto** *low*
    sum = *sum* + *A*[*i*]
    if *sum* > *left-sum*
        left-sum = *sum*
        max-left = *i*
right-sum = -∞ // Find a maximum subarray of the form A[*mid* + *1* .. *j*]
sum =*0*
for *j* = *mid* +*1* **to** *high*
    sum = *sum* + *A*[*j*]
    if *sum* > *right-sum*
        right-sum = *sum*
        max-right = *j*
// Return the indices and the sum of the two subarrays
return (*max-left*, *max-right*, *left-sum* + *right-sum*)

Divide-and-Conquer Algorithm



$A[i..j]$ comprises two subarrays $A[i..mid]$ and $A[mid+1..j]$

Divide-and-Conquer Algorithm

mid =5

	1	2	3	4	5	6	7	8	9	10
A	**13**	**-3**	**-25**	**20**	**-3**	**-16**	**-23**	**18**	**20**	**-7**

$S[5 .. 5]$ = -3

$S[4 .. 5]$ = 17 ⇐ (max-left = 4)

$S[3 .. 5]$ = -8

$S[2 .. 5]$ = -11

$S[1 .. 5]$ = 2

mid =5

	1	2	3	4	5	6	7	8	9	10
A	**13**	**-3**	**-25**	**20**	**-3**	**-16**	**-23**	**18**	**20**	**-7**

$S[6 .. 6]$ = -16

$S[6 .. 7]$ = -39

$S[6 .. 8]$ = -21

$S[6 .. 9]$ = (max-right = 9) ⇒ -1

$S[6..10]$ = -8

⇒ maximum subarray crossing *mid* is $S[4..9]$ = 16

Divide-and-Conquer Algorithm

Analyzing time complexity

FIND-MAX-CROSSING-SUBARRAY : $\Theta(n)$,

where $n = high - low + 1$

FIND-MAXIMUM-SUBARRAY

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 2T(n/2) + \Theta(n) & \text{if } n > 1. \end{cases}$$

$$T(n) = 2T(n/2) + \Theta(n)$$

$= \Theta(n \lg n)$ (similar to merge-sort)

Conclusion: Divide-and-Conquer

- This Divide and conquer algorithm is clearly substantially faster than any of the brute-force methods. It required some cleverness, and the programming is a little more complicated – but the payoff is large.

- Divide and conquer is just one of several powerful techniques for algorithm design
- Divide-and-conquer algorithms can be analyzed using recurrences
- Can lead to more efficient algorithms