HOMEWORK 1

Solution:

We have $X = \max(X_1, X_2) - X_1$

$\therefore$ We have 3 cases:

i) $X_1 = X_2$

In this case, $\max(X_1, X_2) = X_1$ (or X_2) [same]

$\therefore X = X_1 - X_1 = \underline{\underline{0}}$

ii) $X_1 > X_2$

In this case, $\max(X_1, X_2) = X_1$

$\therefore X = X_1 - X_1 = 0$ //

iii) $X_2 > X_1$

In this case, $\max(X_2, X_1) = X_2$

$\therefore X = \underline{\underline{X_2 - X_1}}$

Enumerating ~~all~~ number of possible outcomes of all 3 cases:

i) $X_1 = X_2$ [range is $(1, N)$ for X_1 & X_2]

we have the following possibilities

$[(1,1), (2,2), (3,3) \dots (N,N)]$

$\therefore$ No. of possible outcomes $= N$ //

ii) We know that:

No. of possible outcomes of $(X_1 > X_2)$ =

No. of possible outcomes of $(X_2 > X_1)$

Total no. of outcomes $= (N) \cdot (N) = N^2$

$\therefore$ No. of outcomes of $X_1 > X_2$ = No. of outcomes of $X_2 > X_1 = \dfrac{N^2 - N}{2} = \dfrac{N(N-1)}{2}$ //

a) Calculating expectation $E(X)$:

Expected value $E(X) = \dfrac{\text{sum of all outcomes}}{\text{Total no. of outcomes}}$

$\therefore$ $E(X) = \dfrac{\left(N \cdot 0 + \frac{N(N-1)}{2} \cdot 0 + (\text{sum of event } X_2 > X_1) \right)}{N^2}$

For sum of outcomes when $X_2 > X_1$:
(Proving by induction)
Let difference between X_2 & X_1 be i
when $X_2 - X_1 = 1$
we have $(N-1)$ pairs of (X_1, X_2)
eg: $[(1,2), (2,3), (3,4) \ldots (N-1, N)]$
when $X_2 - X_1 = 2$
we have $(N-2)$ pairs of (X_1, X_2)
eg: $[(1,3), (2,4), (3,5) \ldots (N-2, N)]$
similarly, for $X_2 - X_1 = i$
we have $(N-i)$ pairs

$\therefore$ sum of outcomes when $X_2 > X_1 =$

$$1 \cdot (N-1) + 2 \cdot (N-2) + \ldots i(N-i) + \ldots (N-1) \cdot 1$$

$$= \sum_{i=1}^{N} i \cdot (N-i)$$

$$= \sum_{i=1}^{N-1} (Ni - i^2)$$

$$= N \cdot \sum_{i=1}^{N-1} i - \sum_{i=1}^{N-1} i^2$$

$$= N \cdot \left[\frac{N \cdot (N-1)}{2} \right] - \frac{(N-1)(N)[2(N-1)+1]}{6}$$

$$= \frac{N^2(N-1)}{2} - \frac{N(N-1)(2N-1)}{6}$$

$$\therefore \text{Sum} = \frac{N(N-1)}{2}\left[N - \frac{(2N-1)}{3}\right]$$

$$= \frac{N(N-1)}{2}\left[\frac{3N-2N+1}{3}\right]$$

$$= \frac{N(N-1)(N+1)}{6}$$

$$= \frac{N(N^2-1)}{6}$$

$\therefore$ Expected value $E(X) = \frac{\left[N(N^2-1)/6\right]}{N^2}$

$$\boxed{E(X) = \frac{N^2-1}{6N}}$$

b) Calculating Variance (X):

We know that $Var(X) = E(X^2) - [E(X)]^2$

we have the value of $E(X)$

Calculating $E(X^2)$:

We have the same total no. of outcomes as in $E(X)$.

$\therefore$ Total no. of outcomes $= N^2$

$X^2 = [\max(X_1, X_2) - X_1]^2$

when $X_1 = X_2$, we have $X = 0$ [N cases]

when $X_1 > X_2$, we have $X = 0$ $\left[\frac{N(N-1)}{2} \text{ cases}\right]$

when $X_2 > X_1$:

we have $X^2 = (X_2 - X_1)^2$

$$\therefore E(X^2) = \frac{N \cdot 0 + \frac{N(N-1)}{2} \cdot 0 + \sum (X_2 - X_1)^2}{N^2}$$

To calculate $\sum (X_2 - X_1)^2$: (By induction)

let's take $X_2 = N$ & $X_1 = 1$

difference $X_2 - X_1 = N-1$

$\therefore (X_2 - X_1)^2 = (N-1)^2$

No. of possible cases for the above scenario = 1.

Take ~~X_2~~ difference $(X_2 - X_1) = N-2$

No. of cases = 2 $[(1, N-1), (2, N)]$

sum = $2 \cdot (N-2)^2$

when difference $(X_2 - X_1) = N-3$

No. of cases = 3 $[(1, N-2), (2, N-1), (3, N)]$

sum = $3 \cdot (N-3)^2$

$\therefore$ when difference $(X_2 - X_1) = N - i$

No. of cases = i [By induction]

sum = $i \cdot (N-i)^2$

$$\therefore \sum (X_2 - X_1)^2 = \sum_{i=1}^{N-1} i (N-i)^2$$

$$= \sum_{i=1}^{N-1} i [N^2 - 2Ni + i^2]$$

$$= \sum_{i=1}^{N-1} N^2 i - 2Ni^2 + i^3$$

$$= N^2 \cdot \sum_{i=1}^{N-1} i - 2N \sum_{i=1}^{N-1} i^2 + \sum_{i=1}^{N-1} i^3$$

$$= N^2 \cdot \frac{N(N-1)}{2} - 2N \left[\frac{N(N-1)(2N-1)}{6}\right] + \frac{N^2(N-1)^2}{4}$$

$$\therefore E(X^2) = \frac{1}{\cancel{N^2}} \left[\frac{\cancel{N^2} \cdot N(N-1)}{2} - \frac{\cancel{N^2}(N-1)(2N-1)}{3} + \frac{\cancel{N^2}(N-1)^2}{4}\right]$$

$$= \frac{N(N-1)}{2} - \frac{(N-1)(2N-1)}{3} + \frac{(N-1)^2}{4}$$

$$= (N-1)\left[\frac{N}{2} - \frac{2N-1}{3} + \frac{N-1}{4}\right]$$

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$$\therefore E(X^2) = \frac{(N-1)}{12}\left[6N - 4(2N-1) + 3(N-1)\right]$$

$$= \frac{(N-1)}{12}\left[6N - 8N + 4 + 3N - 3\right]$$

$$= \frac{(N-1)}{12}\left[N+1\right]$$

$$\therefore E(X^2) = \frac{N^2-1}{12}$$

$$\therefore Var(X) = E(X^2) - \left[E(X)\right]^2$$

$$= \frac{N^2-1}{12} - \left[\frac{N^2-1}{6N}\right]^2$$

$$= \frac{N^2-1}{12} - \frac{(N^2-1)^2}{36N^2}$$

$$= \frac{N^2-1}{12}\left[1 - \frac{N^2-1}{3N^2}\right]$$

$$= \frac{N^2-1}{12}\left[\frac{3N^2 - N^2 + 1}{3N^2}\right]$$

$$\boxed{Var(X) = \frac{(N^2-1)(2N^2+1)}{36N^2}}$$

c) Calculating Covariance (XX_1) :

X	0	1	2	...	$N-1$	0	0	1	2	...	$N-2$	...	0	0	...	$N-i$	
X_1	1	1	1	...	1	2	2	2	2 ...	...	2	...	i	i	...	i	

when $X_1 = 1$ we have 1 case when $X = 0$ $[X_1 = X_2 \; (1,1)]$ otherwise, difference increases ~~to~~ from 1 to $N-1$

when $X_1 = 2$, we have 2 cases when $X = 0$ $[(2,1), (2,2)]$ else, difference increases from 1 to $N-2$

similarly, when $X_1 = i$, we have i cases when $X = 0$ $[(i,1), (i,2) \cdots (i,i)]$ else, difference increases from 1 to $N-i$

$\therefore$ As $\operatorname{cov}(XX_1) = E(XX_1) - E(X)\cdot E(X_1)$

For:

$$E(XX_1) = \frac{1}{N^2}\left[1\cdot\sum_{j=1}^{N-1} j + 2\cdot\sum_{j=1}^{N-2} j + \dots\dots j\sum_{j=1}^{N-j} j\right]$$

$$= \frac{1}{N^2}\left[1\cdot\frac{N(N-1)}{2} + 2\cdot\frac{(N-1)(N-2)}{2} + 3\cdot\frac{(N-2)(N-3)}{2}\dots\right]$$

$$= \frac{1}{2N^2}\left[\sum_{j=1}^{N} j(N-j)(N-j+1)\right]$$

$$= \frac{1}{2N^2}\left[\sum_{j=1}^{N} j\,[N^2 - Nj + N - Nj + j^2 - j]\right]$$

$$= \frac{1}{2N^2}\left[\sum_{j=1}^{N} N^2 j - 2Nj^2 + j^3 + Nj - j^2\right]$$

$$= \frac{1}{2N^2}\left[N^2\sum_{j=1}^{N} j - 2N\sum_{j=1}^{N} j^2 + \sum_{j=1}^{N} j^3 + N\sum_{j=1}^{N} j - \sum_{j=1}^{N} j^2\right]$$

$$= \frac{1}{2N^2}\left[N^2\cdot\frac{N(N+1)}{2} - \frac{2N\cdot N(N+1)(2N+1)}{6} + \frac{N^2(N+1)^2}{4} + N\cdot\frac{N(N+1)}{2} - \frac{N(N+1)(2N+1)}{6}\right]$$

$$= \frac{1}{4N^2}\left[N^3(N+1) - \frac{2N^2(N+1)(2N+1)}{3} + \frac{N^2(N+1)^2}{2} + N^2(N+1) - (N(N+1)(2N+1))/3\right]$$

$$= \frac{(N+1)}{4N^2}\left[N^3 - \frac{2N^2(2N+1)}{3} + \frac{N^2(N+1)}{2} + N^2 - \frac{N(2N+1)}{3}\right]$$

$$= \frac{(N+1)}{24N^2}\left[6N^3 - 4N^2(2N+1) + 3N^2(N+1) + 6N^2 - 2N(N+1)\right]$$

$$= \frac{(N+1)}{24N^2}\left[6N^3 - 8N^3 - 4N^2 + 3N^3 + 3N^2 + 6N^2 - 4N^2 - 2N\right]$$

$$= \frac{(N+1)}{24N^2}\left[N^3 + N^2 - 2N\right]$$

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$$\Rightarrow \frac{(N+1)}{24N}\left[N^2+N-2\right]$$

$$= \frac{(N+1)}{24N}(N-1)(N+2)$$

$$= \frac{(N^2-1)(N+2)}{24N}$$

$$E(X_1) = \frac{\sum_{i=1}^{N} x_i}{N}$$

$$= \frac{1}{\cancel{N}} \cdot \frac{\cancel{N}\cdot(N+1)}{2}$$

$$E(X_1) = \frac{N+1}{2}$$

$$\therefore \quad Cov(XX_1) = E(XX_1) - E(X)\cdot E(X_1)$$

$$= \frac{(N^2-1)(N+2)}{24N} - \frac{(N^2-1)}{6N}\cdot\frac{(N+1)}{2}$$

$$= \frac{(N^2-1)}{12N}\left[\frac{N+2}{2} - N+1\right]$$

$$= \frac{(N^2-1)}{12N}\left[\frac{N+\cancel{2}-2N-\cancel{2}}{2}\right]$$

$$= \frac{(N^2-1)}{24\cancel{N}}\left[-\cancel{N}\right]$$

$$\boxed{Cov(XX_1) = \frac{1-N^2}{24}}$$