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CSE512 Fall 2018 - Machine Learning - Homework 1

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## HOMEWORK 1

Solution:

We have  $X = \max(X_1, X_2) - X_1$

$\therefore$  We have 3 cases:

i)  $X_1 = X_2$

In this case,  $\max(X_1, X_2) = X_1$  (or  $X_2$ ) [same]

$$\therefore X = X_1 - X_1 = \underline{0}$$

ii)  $X_1 > X_2$

In this case,  $\max(X_1, X_2) = X_1$

$$\therefore X = X_1 - X_1 = \underline{0}$$

iii)  $X_2 > X_1$

In this case,  $\max(X_2, X_1) = X_2$

$$\therefore X = \underline{X_2 - X_1}$$

number of  
Enumerating ~~all~~ possible outcomes  
of all 3 cases:

i)  $X_1 = X_2$  [range is  $(1, N)$  for  $X_1$  &  $X_2$ ]

we have the following possibilities  
[(1, 1), (2, 2), (3, 3) ... (N, N)]

$\therefore$  No. of possible outcomes =  $\underline{N}$

ii) We know that:

No. of possible outcomes of  $(X_1 > X_2) =$

No. of possible outcomes of  $(X_2 > X_1)$

Total no. of outcomes =  $(N) \cdot (N) = N^2$

$\therefore$  No. of outcomes of  $X_1 > X_2 =$  No. of outcomes  
of  $X_2 > X_1 = \underline{\frac{N^2 - N}{2}} = \underline{\frac{N(N-1)}{2}}$

a) Calculating expectation  $E(X)$ :

Expected value  $E(X) = \frac{\text{sum of all outcomes}}{\text{Total no. of outcomes}}$

$$\therefore E(X) = \frac{\left( N \cdot 0 + \frac{N(N-1)}{2} \cdot 0 + (\text{sum of event } X_2 > X_1) \right)}{N^2}$$

For sum of outcomes when  $X_2 > X_1$ :

(Proving by induction)

Let difference between  $X_2$  &  $X_1$  be 1  
when  $X_2 - X_1 = 1$

we have  $(N-1)$  pairs of  $(X_1, X_2)$

eg:  $[(1,2), (2,3), (3,4) \dots (N-1, N)]$

when  $X_2 - X_1 = 2$

we have  $(N-2)$  pairs of  $(X_1, X_2)$

eg:  $[(1,3), (2,4), (3,5) \dots (N-2, N)]$

similarly, for  $X_2 - X_1 = i$

we have  $(N-i)$  pairs

$\therefore$  sum of outcomes when  $X_2 > X_1 =$

$$1 \cdot (N-1) + 2 \cdot (N-2) + \dots + i(N-i) + \dots + (N-1) \cdot 1$$

$$= \sum_{i=1}^N i \cdot (N-i)$$

$$= \sum_{i=1}^{N-1} (Ni - i^2)$$

$$= N \cdot \sum_{j=1}^{N-1} j - \sum_{j=1}^{N-1} j^2$$

$$= N \cdot \left[ \frac{N(N-1)}{2} \right] - \frac{(N-1)(N)(2(N-1)+1)}{6}$$

$$= \frac{N^2(N-1)}{2} - \frac{N(N-1)(2N-1)}{6}$$

$$\begin{aligned}
 \text{Sum} &= \frac{N(N-1)}{2} \left[ N - \frac{(2N-1)}{3} \right] \\
 &= \frac{N(N-1)}{2} \left[ \frac{3N - 2N + 1}{3} \right] \\
 &= \frac{N(N-1)(N+1)}{6} \\
 &= \frac{N(N^2-1)}{6}
 \end{aligned}$$

∴ Expected value  $E(X) = \frac{[N(N^2-1)/6]}{N^2}$

$$E(X) = \frac{N^2-1}{6N}$$

b) Calculating Variance (X):  
 We know that  $\text{Var}(X) = E(X^2) - [E(X)]^2$   
 we have the value of  $E(X)$   
 calculating  $E(X^2)$ :

We have the same total no. of outcomes as in  $E(X)$ .

∴ Total no. of outcomes =  $N^2$   
 $X^2 = [\max(X_1, X_2) - X_1]^2$

∴ when  $X_1 = X_2$ , we have  $X = 0$  [N cases]  
 ∴ when  $X_1 > X_2$ , we have  $X = 0$   $\left[ \frac{N(N-1)}{2} \text{ cases} \right]$

when  $X_2 > X_1$ :

we have  $X^2 = (X_2 - X_1)^2$

∴  $E(X^2) = \frac{N \cdot 0 + N(N-1) \cdot 0 + \sum (X_2 - X_1)^2}{N^2}$

To calculate  $\sum (X_2 - X_1)^2$  : (By induction)  
 let's take  $X_2 = N$  &  $X_1 = 1$

difference  $X_2 - X_1 = N - 1$

$$\therefore (X_2 - X_1)^2 = (N - 1)^2$$

No. of possible cases for the above scenario = 1 ;

Take  ~~$X_2$~~  = difference  $(X_2 - X_1) = N - 2$

No. of cases = 2  $[(1, N-1), (2, N)]$

$$\text{sum} = 2 \cdot (N - 2)^2$$

when difference  $(X_2 - X_1) = N - 3$

No. of cases = 3  $[(1, N-2), (2, N-1), (3, N)]$

$$\text{sum} = 3 \cdot (N - 3)^2$$

$\therefore$  when difference  $(X_2 - X_1) = N - i$

No. of cases =  $i$  [By induction]

$$\text{sum} = i \cdot (N - i)^2$$

$$\therefore \sum (X_2 - X_1)^2 = \sum_{i=1}^{N-1} i (N - i)^2$$

$$= \sum_{i=1}^{N-1} i [N^2 - 2Ni + i^2]$$

$$= \sum_{i=1}^{N-1} N^2 i - 2N \sum_{i=1}^{N-1} i^2 + \sum_{i=1}^{N-1} i^3$$

$$= N^2 \sum_{i=1}^{N-1} i - 2N \sum_{i=1}^{N-1} i^2 + \sum_{i=1}^{N-1} i^3$$

$$= N^2 \cdot \frac{N(N-1)}{2} - 2N \left[ \frac{N(N-1)(2N-1)}{6} \right] + \frac{N^2(N-1)^2}{4}$$

$$\therefore E(X^2) = \frac{1}{N^2} \left[ \frac{N^2 \cdot N(N-1)}{2} - \frac{N^2(N-1)(2N-1)}{3} + \frac{N^2(N-1)^2}{4} \right]$$

$$= \frac{N(N-1)}{2} - \frac{(N-1)(2N-1)}{3} + \frac{(N-1)^2}{4}$$

$$= \frac{(N-1)}{12} \left[ \frac{6N}{2} - \frac{2(2N-1)}{3} + \frac{3(N-1)}{4} \right]$$



$$\begin{aligned} \therefore E(X^2) &= \frac{(N-1)}{12} [6N - 4(2N-1) + 3(N-1)] \\ &= \frac{(N-1)}{12} [6N - 8N + 4 + 3N - 3] \\ &= \frac{(N-1)}{12} [N+1] \end{aligned}$$

$$\therefore E(X^2) = \frac{N^2-1}{12}$$

$$\begin{aligned} \therefore \text{Var}(X) &= E(X^2) - [E(X)]^2 \\ &= \frac{N^2-1}{12} - \left[ \frac{N^2-1}{6N} \right]^2 \\ &= \frac{N^2-1}{12} - \frac{(N^2-1)^2}{36N^2} \\ &= \frac{N^2-1}{12} \left[ 1 - \frac{N^2-1}{3N^2} \right] \\ &= \frac{N^2-1}{12} \left[ \frac{3N^2 - N^2 + 1}{3N^2} \right] \end{aligned}$$

$$\text{Var}(X) = \frac{(N^2-1)(2N^2+1)}{36N^2}$$

c) Calculating Covariance ( $XX_1$ ) :

X	0	1	2	...	N-1	0	0	1	2	...	N-2	...	0	0	...	N-1
X <sub>1</sub>	1	1	1	...	1	2	2	2	2	...	2	...	i	i	...	i

when  $X_1 = 1$   
we have 1 case  
when  $X = 0$  [ $X_1 = X_2$ ]  
(1,1)

otherwise, difference  
increases to  
from 1 to N-1

when  $X_1 = 2$ , we  
have 2 cases when  
 $X = 0$  [(2,1), (2,2)]

else, difference  
increases from  
1 to N-2

similarly,  
when  $X_1 = i$ ,  
we have i cases  
when  $X = 0$   
[(i,1), (i,2), ..., (i,i)]  
else, difference  
increases from  
1 to N-i

$$\therefore \text{As } \text{cov}(X, X_1) = E(X X_1) - E(X) \cdot E(X_1)$$

For  $\sigma$

$$E(X X_1) = \frac{1}{N^2} \left[ 1 \cdot \sum_{j=1}^{N-1} j + 2 \cdot \sum_{j=1}^{N-2} j + \dots + j \sum_{j=1}^{N-j} j \right]$$

$$= \frac{1}{N^2} \left[ 1 \cdot \frac{N(N-1)}{2} + 2 \cdot \frac{(N-1)(N-2)}{2} + 3 \cdot \frac{(N-2)(N-3)}{2} \dots \right]$$

$$= \frac{1}{2N^2} \left[ \sum_{j=1}^N j (N-j)(N-j+1) \right]$$

$$= \frac{1}{2N^2} \left[ \sum_{j=1}^N j [N^2 - Nj + N - Nj + j^2 - j] \right]$$

$$= \frac{1}{2N^2} \left[ \sum_{j=1}^N N^2 j - 2Nj^2 + j^3 + Nj - j^2 \right]$$

$$= \frac{1}{2N^2} \left[ N^2 \sum_{j=1}^N j - 2N \sum_{j=1}^N j^2 + \sum_{j=1}^N j^3 + N \sum_{j=1}^N j - \sum_{j=1}^N j^2 \right]$$

$$= \frac{1}{2N^2} \left[ \frac{N^2 \cdot N(N+1)}{2} - \frac{2N \cdot N(N+1)(2N+1)}{6} + \frac{N^2(N+1)^2}{4} + \frac{N \cdot N(N+1)}{2} - \frac{N(N+1)(2N+1)}{6} \right]$$

$$= \frac{1}{4N^2} \left[ \frac{N^3(N+1)}{3} - \frac{2N^2(N+1)(2N+1)}{3} + \frac{N^2(N+1)^2}{2} + N^2(N+1) - \frac{N(N+1)(2N+1)}{3} \right]$$

$$= \frac{(N+1)}{4N^2} \left[ \frac{N^3 - 2N^2(2N+1) + N^2(N+1)}{3} + N^2 - \frac{N(2N+1)}{3} \right]$$

$$= \frac{(N+1)}{24N^2} \left[ 6N^3 - 4N^2(2N+1) + 3N^2(N+1) + 6N^2 - 2N(N+1) \right]$$

$$= \frac{(N+1)}{24N^2} \left[ 6N^3 - 8N^3 - 4N^2 + 3N^3 + 3N^2 + 6N^2 - 4N^2 - 2N \right]$$

$$= \frac{(N+1)}{24N^2} [N^3 + N^2 - 2N]$$

$$\begin{aligned}
 &\Rightarrow \frac{(N+1)}{24N} [N^2 + N - 2] & E(X_1) &= \frac{\sum_{i=1}^N X_i}{N} \\
 &= \frac{(N+1)}{24N} (N-1)(N+2) & &= \frac{1}{N} \cdot \frac{N \cdot (N+1)}{2} \\
 &= \frac{(N^2-1)(N+2)}{24N} & E(X_1) &= \frac{N+1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{Cov}(XX_1) &= E(XX_1) - E(X) \cdot E(X_1) \\
 &= \frac{(N^2-1)(N+2)}{24N} - \frac{(N^2-1)}{6N} \cdot \frac{(N+1)}{2} \\
 &= \frac{(N^2-1)}{12N} \left[ \frac{N+2}{2} - N+1 \right] \\
 &= \frac{(N^2-1)}{12N} \left[ \frac{N+2 - 2N - 2}{2} \right] \\
 &= \frac{(N^2-1)}{24N} [-N] \\
 \text{Cov}(XX_1) &= \frac{1-N^2}{24}
 \end{aligned}$$









