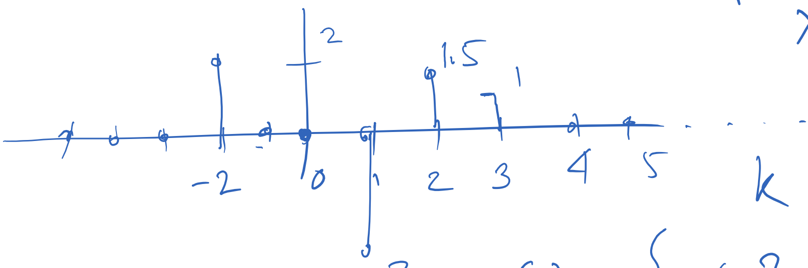
Z - Transform

22 September 2017 09:30

Definition seq : $\{x(k)\}$

$$X(z) = \sum_{k=-\infty}^{\infty} x(k)\, z^{-k}$$ $\longrightarrow$ ROC : find domain of z where $X(z)$ converges

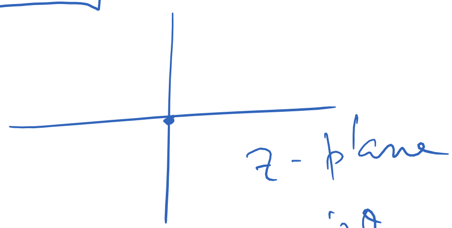


$$x(k) = \{\ldots 0, \underline{2}, 0, 0, -3, 1.5, 1, 0 \ldots\}$$

$\downarrow$ $k=0$

$$\boxed{X(z) = 2.z^{2} - 3z^{-1} + 1.5\, z^{-2} + z^{-3}}$$

ROC : excluding 0, ∞, Entire z - plane



If $x(n) \rightarrow$ finite length seq. $z = re^{j\theta}$

ROC :

Ex

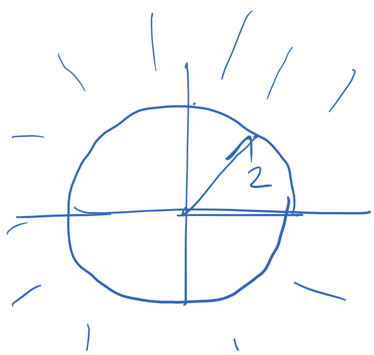
a) $x(n) = 2^{n}\, u(n)$

$$X(z) = \sum_{n} 2^{n} z^{-n} = \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^{-n}$$

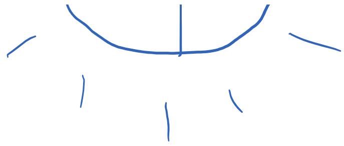
$$\sum (2z^{-1})^{n} = \frac{1}{1 - 2z^{-1}}$$

if $|2z^{-1}| < 1$

ROC : $|z| > 2$



b) $x(n) = \left(\frac{1}{\ }\right)^{n} u(n)$



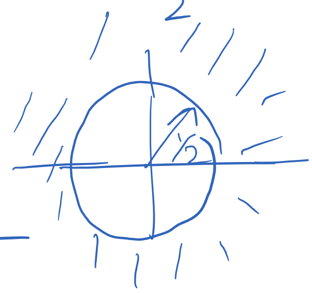
b) $x(n) = \left(\frac{1}{2}\right)^n u(n)$

$$X(z) = \sum_{n=0} \left(\frac{1}{2} z^{-1}\right)^n = \frac{1}{1-\frac{1}{2}z^{-1}}$$

Causal seq.

ROC: $\left|\frac{1}{2}z^{-1}\right| < 1$

$$|z| > \frac{1}{2}$$

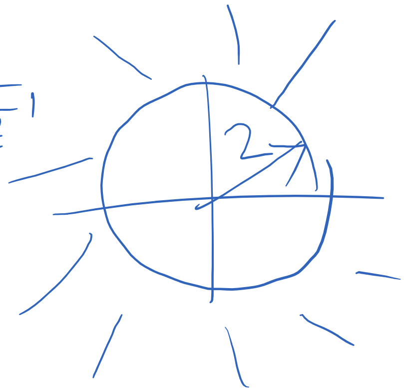


c) $x(n) = \sum\left[2^n + \left(\frac{1}{2}\right)^n\right] u(n)$

↳ prove linearity property

ROC: $ROC_1 \cap ROC_2$

$$X(z) = \frac{1}{1-2z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}$$



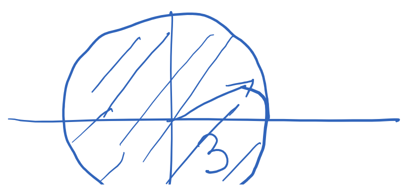
d) $x(n) = 3^{+n} u(-n)$

$$X(z) = \sum_{-\infty}^{n=0} 3^n z^{-n} \qquad k=-n$$

$$= \sum_{k=0}^{\infty} 3^{-k} z^{k} = \frac{1}{1-z/3}$$

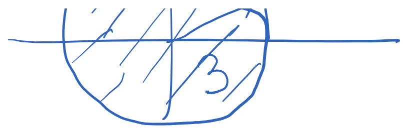
ROC: $|z| < 3$

anti-causal seq.



k

seq.



e) $x(k) = \left(-\frac{1}{3}\right)^k u(-n)$

$$X(z) = \frac{1}{1+3z}$$

ROC: $|z| < \frac{1}{3}$

$$x(k) = a^k u(-k)$$

$$X(z) = \sum_{-\infty}^{0} a^k z^{-k} = \sum_{0}^{\infty} a^{-k} z^k$$

$n = -k$

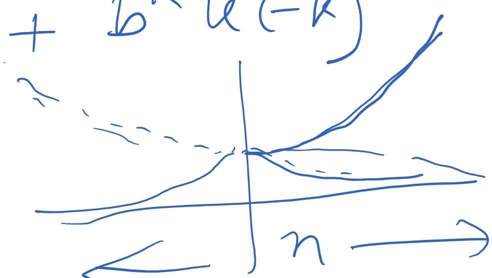
$$= \frac{1}{1 - a^{-1}z}$$

provided $|a^{-1}z| < 1$

$|z| < a$

$1, a, a^2, a^3 \ldots \rightarrow \frac{1}{1-a}$ $\quad |a| < 1$

f) $x(k) = \{ a^k u(k) + b^k u(-k)$



$$X(z) = \sum_{k=0}^{\infty} a^k z^{-k} + \sum_{-\infty}^{0} b^k z^{-k}$$

$$= \frac{1}{1 - az^{-1}} + \frac{1}{1 - b^{-1}z}$$

$$= \frac{1}{1-az^{-1}} + \frac{\cdot}{1-b^{-1}z}$$

$\downarrow$ ROC

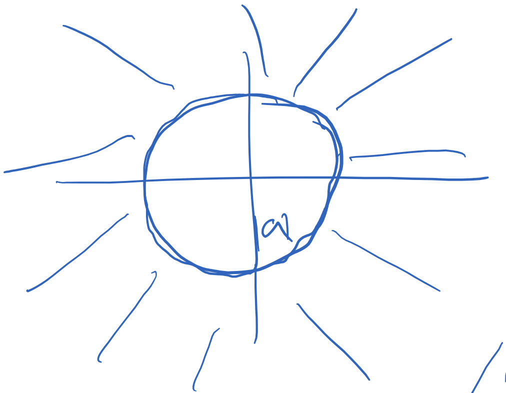
$|az^{-1}| < 1$

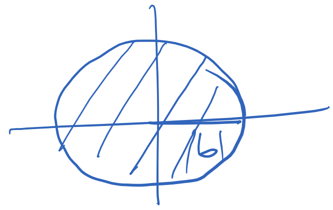
$|z| > |a|$

$\downarrow$ ROC

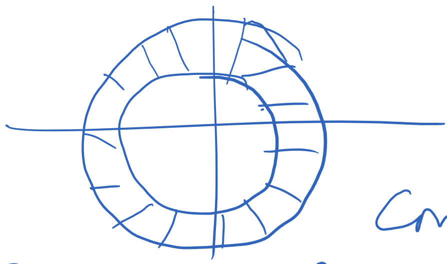
$|b^{-1}z| < 1$

$|z| < |b|$





ROC: $(|z| > |a|) \cap (|z| < |b|)$



Cond. $|b| > |a|$

Existence of Z-transform.

$$X(z) = \sum_{k=0}^{\infty} x(k) z^{-k} + \sum_{k=-1}^{-\infty} x(k) z^{-k}$$

for existence

for convergence

$$M > \sum |x(k) z^{-k}| + \sum |x(k) z^{-k}|$$

if $|x(k)| < M_1 R_1^k$ for $k > 0$

R^k else

if $|x(k)| < M_1 R_1^k$,

$< M_2 R_2^k$ else

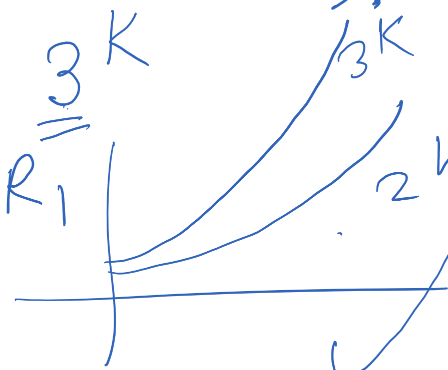
for convergence: $= R_1, R_2 \to +ve$ Real

$$\underbrace{M_1 \sum R_1^k z^{-k}} < \infty$$

Suppose $x(k) = 2^k$ $\quad k \geq 0$

$< \underline{3}^k$

R_1



$$\frac{1}{1 - R_1 z^{-1}}$$

ROC: $|R_1 z^{-1}| < 1$

$|z| > R_1$

$$\frac{1}{1 - z/R_2}$$

ROC.

$|z| < R_2$

General condition.

if you find a region $R_1 < |z| < R_2$
$\neq$ NULL

then Z-Tx exists.

Quiz: $x(k) = \{a_k\}$ for $k \geq 0$

Would Z-Tx always exist?

Sup: $x(k) = (a^a)^n$.

$= a^{a^n} = \beta^n$

a^{n^2}, a^{n^n}

~~Properties:~~

Special case:

$$\boxed{x(k) = a^{k}\, u(k) \quad \hookrightarrow X(z) = \frac{1}{1-az^{-1}}}$$

ROC: $|az^{-1}| < 1$

~~$x(k) = -a^{-k}\, u(-k)$~~

$$x(k) = \begin{cases} -\cancel{a}^{-k} & \text{for } k<0 \\ 0 & k \geq 0 \end{cases} \qquad \frac{1}{a}$$

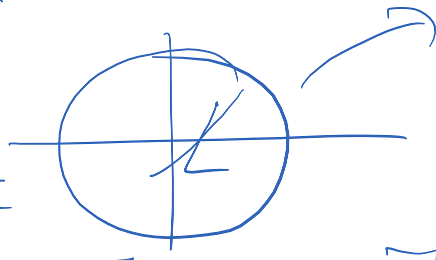
$$X(z) = -\sum_{-\infty}^{-1} a^{-k} z^{-k} =$$

$$= -\cancel{\frac{1}{1}}\; \frac{az}{az-1}$$

$$= \frac{1}{1-{}^{1}/_{a} z^{-1}}$$

$$\boxed{\begin{aligned} x(k) &= \begin{cases} -a^{k} & \text{for } k<0 \\ 0 & \text{for } k \geq 0 \end{cases} \\ X(z) &= \frac{1}{1-az^{-1}} \end{aligned}}$$

ROC: $|az^{-1}| > 1$



Z-Tx UNIQUE

the ROC is specified

Z - IX unique when ROC is specified

HW.

$$X(z) = \frac{1}{1-az^{-1}}$$

only for $|az^{-1}| < 1$

$$= (1-az^{-1})^{-1} = 1 + az^{-1} + a^2z^{-2} + \cdots$$

$$x(k) = \{1, a, a^2, a^3, \ldots\}$$

$$= a^k u(k)$$

$$X(z) = \frac{1}{1-az^{-1}} =$$

$|az^{-1}| > 1$ ← expand with this condition