Propositions, predicates, proofs.

1. Consider the set of n propositions in which the ith proposition is " Exactly i of these propositions are false". What can you conclude about the truth values of these propositions. What happens if "Exactly i " is replaced by " at least i"?

2. Express the following statement as a formal proposition:
   " For any two numbers x, y, there exists a number gcd(x,y) such that gcd(x,y) divides x, gcd(x,y) divides y, and for any number d such that d divides x and d divides y, d also divides gcd(x,y)".
   Prove this proposition assuming that for any two numbers a, b there exists a unique q and r such that a = qb + r, where 0 <= r < b.

3. Consider the statement x is a friend of y, where x, y are variables whose values are students in this class. Two students a, b in the class can communicate with each other if there exists a finite sequence of students $a = s_1, s_2, ..., s_k = b$ such that s_i is a friend of s_{i+1} for all 1 <= i < k. Express the statement "Any two students in the class can communicate with each other" formally. Hint: For some statements, using only fixed predicates is not enough. We may need to allow predicates that are variable, and use the existential and/or universal quantifiers for them. Such statements are said to be of the second order. Can you express the statement in this problem without using variable predicates? Note that a predicate can only involve a fixed number of variables.

4. Prove that for any odd number x, $x^2 - 1$ is divisible by 8.

5. Prove that a number is divisible by 9 if and only if the sum of the digits in the decimal representation of the number is divisible by 9. How would you generalize this statement for divisibility by any number k?

6. Prove that if a prime p divides ab, then either p divides a or p divides b. What is the converse of this statement? Is it true?

7. Prove that if n is prime then n divides n_C_k, for all 1 <= k < n. Is the converse of this statement true? (n_C_k is the binomial coefficient n choose k).