Induction.

1. The natural numbers are defined formally by the following 3 properties, called Peano's axioms.

a) 0 is a natural number.
b) If x is a natural number then x+1 is a natural number.
c) If P is any predicate such that P(0) is true, and for all numbers x, if P(x) is true then P(x+1) is true, then P is true for all numbers x. (Induction).

All properties of numbers follow from this.
Consider the definition of the addition operation on numbers.

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add(x,0) := x for all x
add(x,y+1) := add(x,y) + 1  for all x and y.
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This defines add for all x, y by induction.

Prove that addition is commutative and associative, that is

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add(x,y) = add(y,x) for all x, y.
add(add(x,y),z)  = add(x, add(y,z)) for all x, y, z.
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Do the same for multiplication, after defining it properly.
Justify each step in your proof.

Consider the predicate x <= y defined as follows:

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0 <= y is true for all y.
x+1 <= 0 is false for all x.
x+1 <= y+1 if and only if x <= y.
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Show that it is reflexive, that is, x <= x for all x.

Show that <= is transitive, that is (x <= y) and (y <= z) implies (x <= z).

Show that it is antisymmetric, that is (x <= y) and (y <= x) implies x = y.

2. Prove that if n > 0 then 133 divides 11^(n+1) + 12^(2n-1).

3. Suppose there are n lines in the plane such that no two are parallel and no 3 are concurrent. Show that the lines divide the plane into (n^2 + n + 2)/2. What is this number if you have n circles such that any two circles intersect in two distinct points?

4. Consider a round robin tennis tournament with n players, in which each player plays everybody else. Prove that there always exists a player x such that for any other player y, either x defeated y, or there is a player z such that x defeated z and z defeated y. Write this property down in formal notation. Also show that we can arrange the players in order, p_1, p_2, .., p_n, such that p_i defeated p_{i+1} for all 1 <= i < n.

5. The game of chomp is defined as follows. Suppose you have a n by m matrix with all entries 0. Two players play the game by making moves alternately. At each move a player selects a 0 entry from the matrix, say from position (i,j), and converts all entries (k,l) to 1, where k >= i, and l >= j. In other words, all entries below and to the right of the selected entry are converted to 1. Note that some of them may be already 1. The player who is forced to select the (1,1) entry loses the game.

   Prove by contradiction that the first player can always ensure a win, for all n and m. However, an explicit winning strategy is not known in general. Show that for a 2 by m matrix, player 1 has a winning

strategy by explicitly describing the first move. Can you
generalize for a 3 by m matrix?