Exercises 2

1. A one-dimensional classification problem for three classes, A, B and C and pdf-s $f_A(x)$, $f_B(x)$, $f_C(x)$ is defined as:

   $p_A = 1/2, f_A(x) = 1/8$ for $x \in [-4, 4]$

   $p_B = 1/3, f_B(x) = 3(1 - x^2)/4$ for $x \in [-1, 1]$

   $p_C = 1/6, f_C(x) = x/8$ for $x \in [0, 4]$

   Estimate optimal classification boundaries and decision rules for the system. Explain the general estimation steps and draw a figure that describes the estimation process and the results.

   Answer: In general in a classification problem with a number of classes C_k, with *a priori* probabilities p_{C_k} and pdf's $f_{C_k}(x)$, the optimal decision rule for a given outcome x is to choose the class C_{opt} that maximizes the expression:

   $$C_{opt} = \arg\max_{C_k} p_{C_k} \cdot f_{C_k}(x)$$

   Here we set:

   $g_A(x) = p_A \cdot f_A(x) = \frac{1}{2} \cdot \frac{1}{8} = \frac{1}{16}$ when $x \in [-4, 4]$

   $g_B(x) = p_B \cdot f_B(x) = \frac{1}{3} \cdot \frac{3(1-x^2)}{4} = \frac{1-x^2}{4}$ when $x \in [-1, 1]$

   $g_C(x) = p_C \cdot f_C(x) = \frac{1}{6} \cdot \frac{x}{8} = \frac{x}{48}$ when $x \in [0, 4]$

   In order to find the classification borders we need to solve the following system of inequalities:

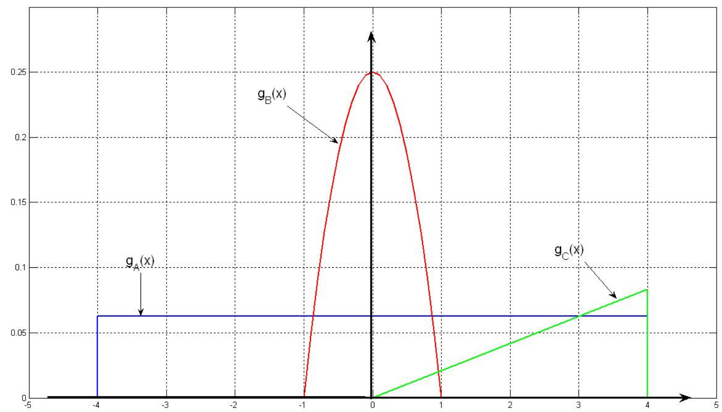
   $$\begin{cases} g_A(x) > g_B(x) & (1) \\ g_A(x) > g_C(x) & (2) \\ g_B(x) > g_C(x) & (3) \end{cases}$$

   so that $x \in C_k$, where

   $$C_k = \begin{cases} A & if & (1) \wedge (2) \\ B & if & (3) \wedge \neg(1) \\ C & if & \neg(2) \wedge \neg(3) \end{cases}$$

   Now, let us solve this system. Equation (1) yields:

   $$\frac{1}{16} > \frac{1-x^2}{4} \Leftrightarrow x^2 > \frac{3}{4} \Rightarrow \begin{cases} x > \frac{\sqrt{3}}{2} \vee x < -\frac{\sqrt{3}}{2} \end{cases}$$



Figur 1: Illustration of the classification problem

Likewise, equation (2) yields:

$$\frac{1}{16} > \frac{x}{48} \Leftrightarrow x < 3$$

Before we calculate the solution to equation (3), let us plot the three functions $g_{C_k}(x)$.See Figure 1

We see that the solution to equation (3) is obsolete, since it's equivalent with the negated solution to equation (1) and the solution to equation (2). (Reason is that $3 > \sqrt{3}/2$). In other words, the classification boundaries are:

$$C_k = \begin{cases} A \quad if \quad -4 < x < -\frac{\sqrt{3}}{2} \quad \vee \quad \frac{\sqrt{3}}{2} < x < 3 \\ B \quad if \quad -\frac{\sqrt{3}}{2} < x < -\frac{\sqrt{3}}{2} \\ C \quad if \quad 3 < x < 4 \end{cases}$$

2. For a classification problem with two classes C_A and C_B, are the a priori probabilities $p_A = 3/4$ and $p_B = 1/4$. Assume the following pdf-s

$$p(\bar{z}|C_k) = \frac{1}{2\pi|det\Sigma_k|^{1/2}} e^{-(\bar{z}-m_k)^T\Sigma_k^{-1}(\bar{z}-m_k)/2}$$

with

$$m_A = m_B = 0,$$

and

$$\Sigma_A = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \Sigma_B = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}.$$

Estimate decision boudaries for the problem!

Answer: The decision boundary is given by the "equal probability solution": $Pr(A|\bar{x}) = Pr(B|\bar{x})$. Now multiply each side with $Pr(\bar{x})$, giving $Pr(\bar{x}) \cdot Pr(A|\bar{x}) = Pr(\bar{x}) \cdot Pr(B|\bar{x})$. However, according to Bayes' rule, $Pr(\bar{x}) \cdot Pr(A|\bar{x}) = Pr(A) \cdot Pr(\bar{x}|A)$ which gives us tbar the decision boundary is given by $Pr(A) \cdot Pr(\bar{x}|A) = Pr(B) \cdot Pr(\bar{x}|B)$. Thus we need to solve the equation

$$p_A \cdot p(\bar{x}|A) = p_B \cdot p(\bar{x}|B)$$

for $\bar{x} = [xy]$. Since we have the expression for $p(\bar{x}|C_k)$ and p_{C_k}, this is trivial. We will solve the equation as an inequality for the case $\bar{x} \in A$, i.e. $p_A \cdot p(\bar{x}|A) > p_B \cdot p(\bar{x}|B)$ (this is to have correct boundary conditions):

$$\frac{p_A}{2\pi\,|\det\Sigma_A|^{1/2}} e^{\frac{-(\bar{x}-m_A)^T\Sigma_A^{-1}(\bar{x}-m_A)}{2}} > \frac{p_B}{2\pi\,|\det\Sigma_B|^{1/2}} e^{\frac{-(\bar{x}-m_B)^T\Sigma_B^{-1}(\bar{x}-m_B)}{2}} \Leftrightarrow$$

$$\frac{3}{4}\frac{2}{2\pi\sqrt{4}} e^{\frac{-\left[\begin{array}{cc} x & y \end{array}\right]\left[\begin{array}{cc} 1/4 & 0 \\ 0 & 1 \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right]}{2}} > \frac{1}{4}\frac{2}{2\pi\sqrt{4}} e^{\frac{-\left[\begin{array}{cc} x & y \end{array}\right]\left[\begin{array}{cc} 1 & 0 \\ 0 & 1/4 \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right]}{2}} \Leftrightarrow$$

$$\ln 3 - \left[\begin{array}{cc} x/8 & y/2 \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right] > -\left[\begin{array}{cc} x/2 & y/8 \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right] \Leftrightarrow$$

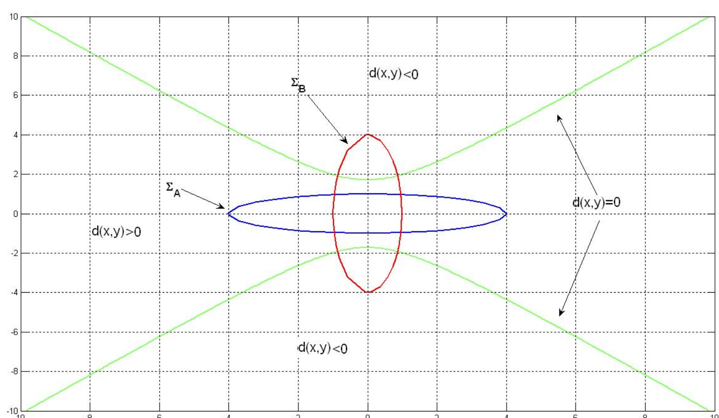
$$\ln 3 - \left(\frac{x^2}{8} + \frac{y^2}{2}\right) > -\left(\frac{x^2}{2} + \frac{y^2}{8}\right) \Leftrightarrow$$

$$x^2 - y^2 + \frac{8}{3}\ln 3 > 0$$

Now define $d(\bar{x}) \equiv d(x,y) \equiv x^2 - y^2 + \frac{8}{3}\ln 3$, then our classification rule becomes:

$$C_k = \begin{cases} A & if \quad d(\bar{x}) > 0 \\ B & if \quad d(\bar{x}) < 0 \end{cases}$$

Note that $d(x,y) = 0$ defines a hyperbolic curve, see Figure 2



Figur 2: Illustration of the 2D classification problem

3. Describe how to estimate the equation of a line that minimizes the sum of squared orthogonal distances for a set of points. Apply it on a following set of points:

$$p_1 = (-6,-2), p_2 = (-3,-1), p_3 = (0,0), p_4 = (1,1), p_5 = (3,2)$$

Answer:

First we normalize the coordinates with respect to the mean position.

$$\bar{p} = \frac{1}{N}\sum p_i = \frac{1}{5}(-6-3+0+1+3, -2-1+0+1+2) = (-1,0)$$

$$p_i' = p_i - \bar{p} = (x_i', y_i')$$

$$p_1 = (-5,-2),\ p_2 = (-2,-1),\ p_3 = (1,0),\ p_4 = (2,1),\ p_5 = (4,2)$$

Then we compute the elements of the covariance matrix.

$$C_{xx} = \frac{1}{N}\sum x_i'^2 = \frac{1}{5}(5^2+2^2+1^2+2^2+4^2) = \frac{1}{5}50$$

$$C_{xy} = \frac{1}{N}\sum x_i' y_i' = \frac{1}{5}(5\cdot 2+2\cdot 1+1\cdot 0+2\cdot 1+4\cdot 2) = \frac{1}{5}22$$

$$C_{yy} = \frac{1}{N}\sum y_i'^2 = \frac{1}{5}(2^2+1^2+0^2+1^2+2^2) = \frac{1}{5}10$$

We can ignore the $N=5$, since we are only interested in the normal direction given by the eigenvector corresponding to the smallest eigenvalue of

$$C = \begin{pmatrix} C_{xx} & C_{xy} \\ C_{xy} & C_{yy} \end{pmatrix}$$

$$|\lambda I - 5C| = \begin{vmatrix} \lambda-50 & -22 \\ -22 & \lambda-10 \end{vmatrix} = \lambda^2 - 60\lambda + 16 = 0$$

The smallest eigenvector is thus

$$\lambda = 30 - \sqrt{30^2-16} = 30 - \sqrt{884} \approx 0.26786$$

and the corresponding eigenvector

$$u = (30-\sqrt{884}-10, 22) = (20-\sqrt{884}, 22)$$

The mean $\bar{p} = (-1,0)$ has to be on this line. This the line is given by

$$(20-\sqrt{884})\cdot x + 22\cdot y + (20-\sqrt{884}) = 0.$$

4. An image has been smoothed with the following kernel:

$$h = k \cdot [1, 5, 10, 10, 5, 1]$$

Can repeated convolutions of an image with the kernel

$$g = \frac{1}{2}[1, 1]$$

be used to obtain the same result as with the first kernel? If yes, how many convolutions are needed? If no, explain the reasons why.

What should the constant k be so that the filter gain is equal to 1?

Answer: We see that $g * g = \frac{1}{4}[1, 2, 1]$, thus $g * g * g * g = \frac{1}{16}[1, 4, 6, 4, 1]$ and $g_*^5 = \frac{1}{32}[1, 5, 10, 10, 5, 1]$. Therefore, if $k = \frac{1}{32}$ we have $h = g_*^5$, i.e. five convolutions with the g kernel yields the h kernel for the given k-value.

5. (a) Derive a mask that approximates the first partial derivative in the x-direction when convolved with an image.

Answer: $d_x = \frac{1}{2}[1, -1]$.

(b) Derive a mask, d_{xxx} for generating the third order derivative using the masks $d_x = 1/2(1, 0, -1)$, and $d_{xx} = (1, -2, 1)$ corresponding to the first and second order derivatives.

Answer: $d_{xxx} = d_x * d_{xx} = \frac{1}{2}[1, -2, 0, 2, -1]$.