Projections and Transformations

DD2423 Image Analysis and Computer Vision

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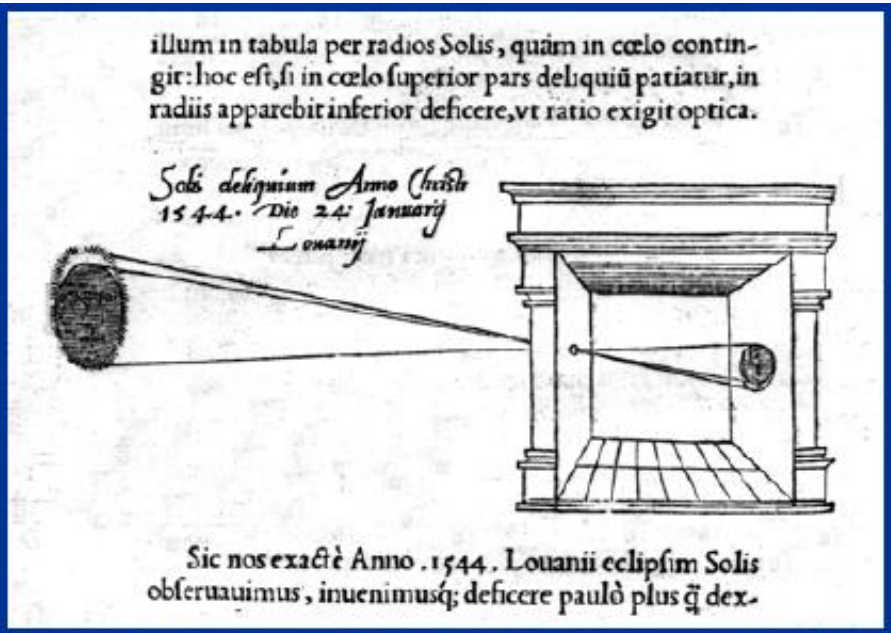
Topics for today

- Perspective projection
  - properties
  - approximations
- Homogeneous coordinates
- Image transformations
- Neighborhood concepts
- Connectivity, connected components
- Distance measures and transforms

Pinhole camera or “Camera Obscura”



"When images of illuminated objects ... penetrate through a small hole into a very dark room ... you will see [on the opposite wall] these objects in their proper form and color, reduced in size ... in a reversed position, owing to the intersection of the rays".

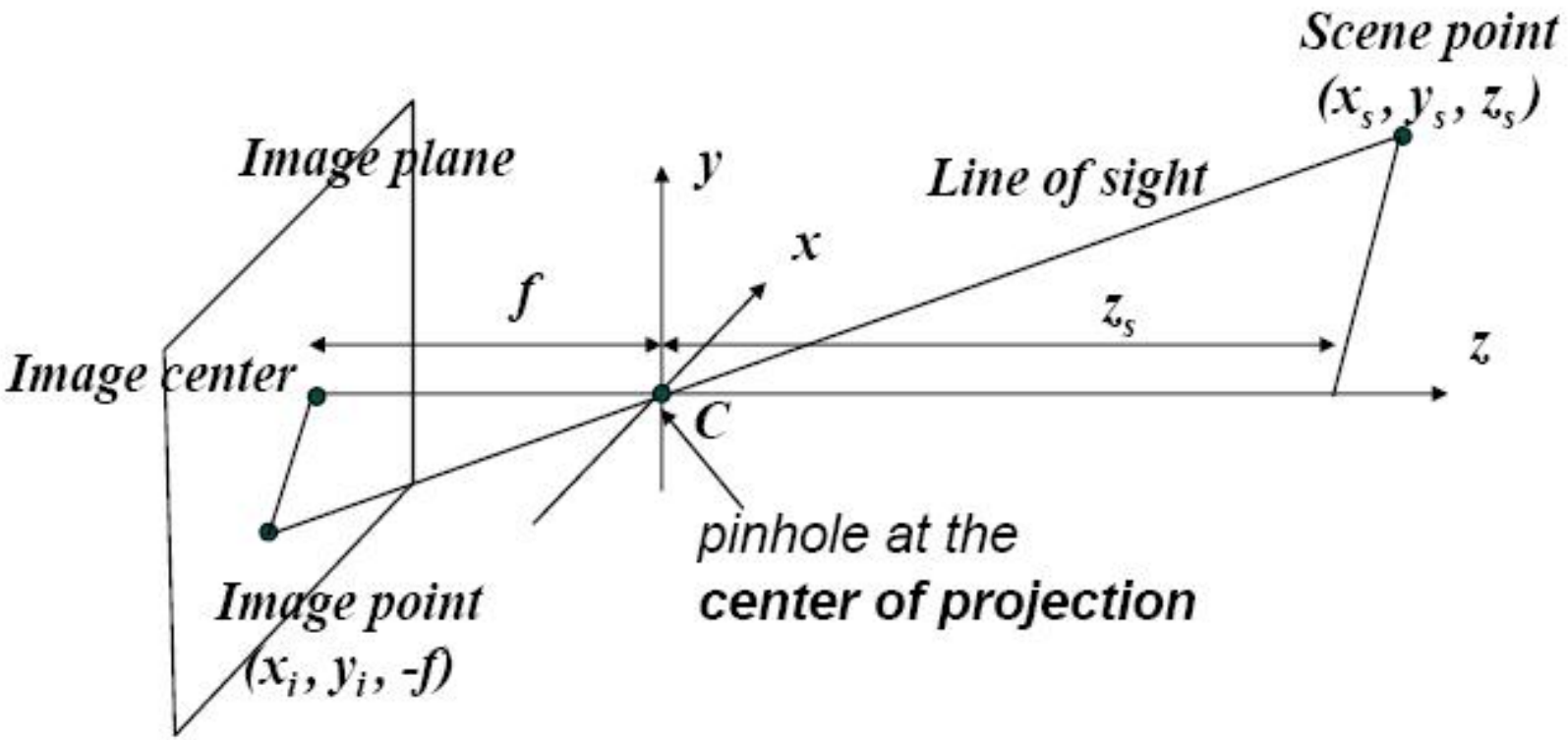
Leonardo Da Vinci

http://www.acmi.net.au/AIC/CAMERA_OBSCURA.html (Russell Naughton)

Pinhole camera and perspective projection

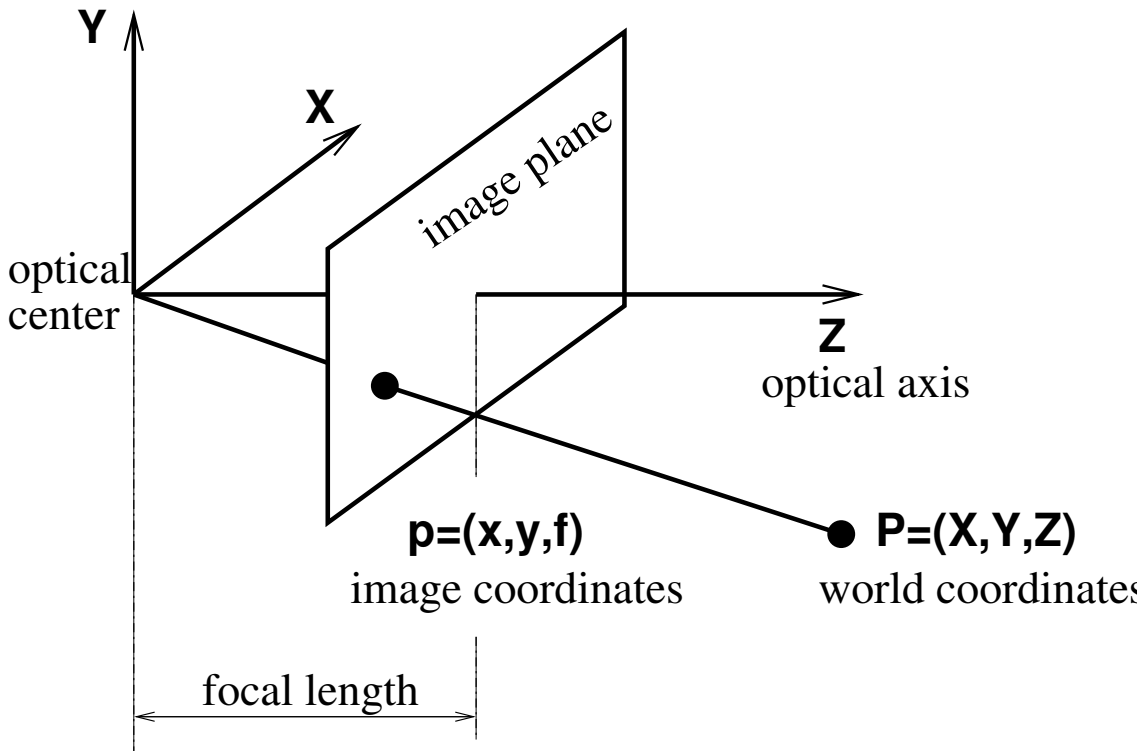
- A mapping from a three dimensionsal (3D) world onto a two dimensional (2D) plane in the previous example is called perspective projection.
- A pinhole camera is the simplest imaging device which captures the geometry of perspective projection.
- Rays of light enter the camera through an infinitesimally small aperture.
- The intersection of light rays with the image plane form the image of the object.

Perspective projection



- The point on the image plane that corresponds to a particular point in the scene is found by following the line that passes through the scene point and the center of projection

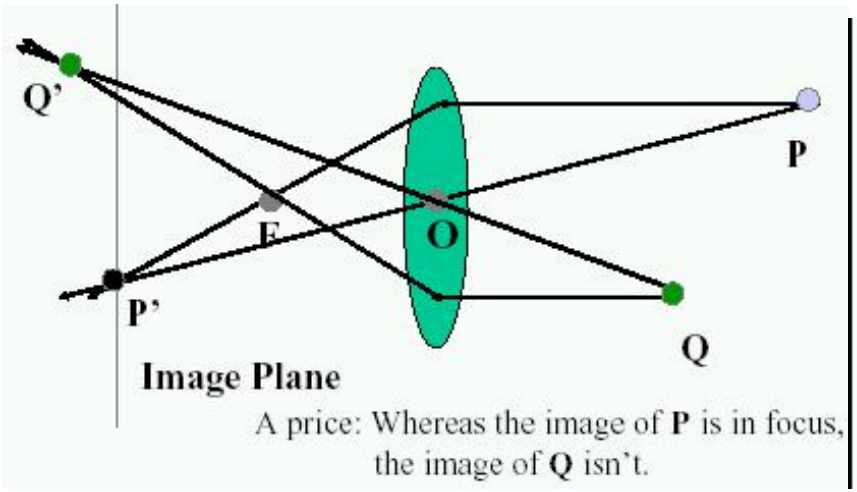
Pinhole camera - Perspective geometry

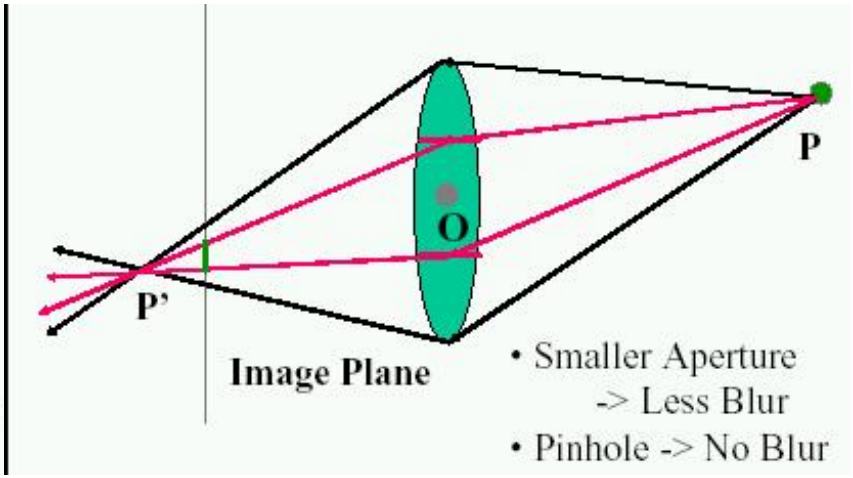


- The image plane is usually modeled in front of the optical center.
- The coordinate systems in the world and in the image domain are parallel. The optical axis is $\perp$ image plane.

Lenses

- Purpose: gather light from from larger opening (aperture)
- Problem: only light rays from points on the focal plane intersect the same point on the image plane
- Result: blurring in-front or behind the focal plane
- Focal depth: the range of distances with acceptable blurring



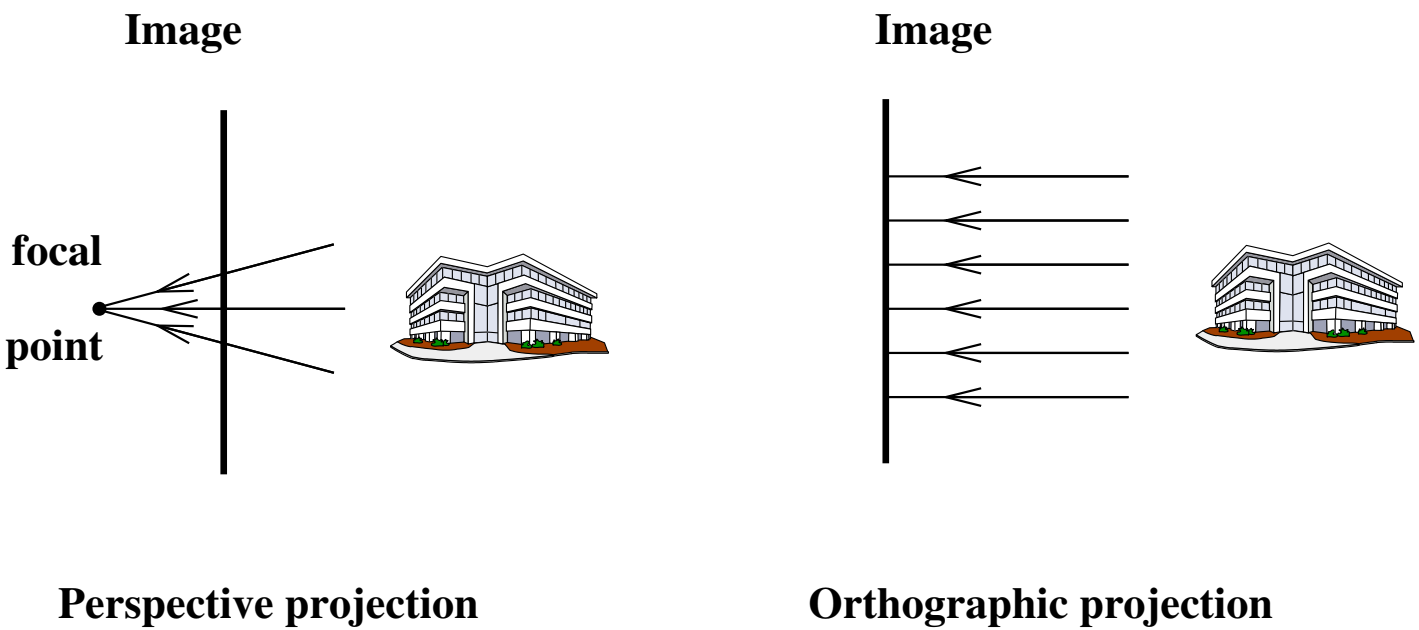


Imaging geometry - Basic camera models

- **Perspective projection** (general camera model)

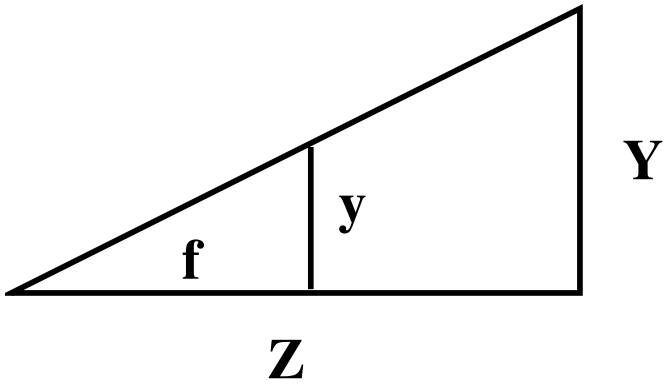
  All visual rays converge to a common point - the focal point
- **Orthographic projection** (approximation: distant objects, center of view)

  All visual rays are perpendicular to the image plane



Perspective projection **Orthographic projection**

Projection equations



- Perspective mapping

$$\frac{x}{f} = \frac{X}{Z}, \qquad \frac{y}{f} = \frac{Y}{Z}$$

- Orthographic projection

$$x = X, \qquad y = Y$$

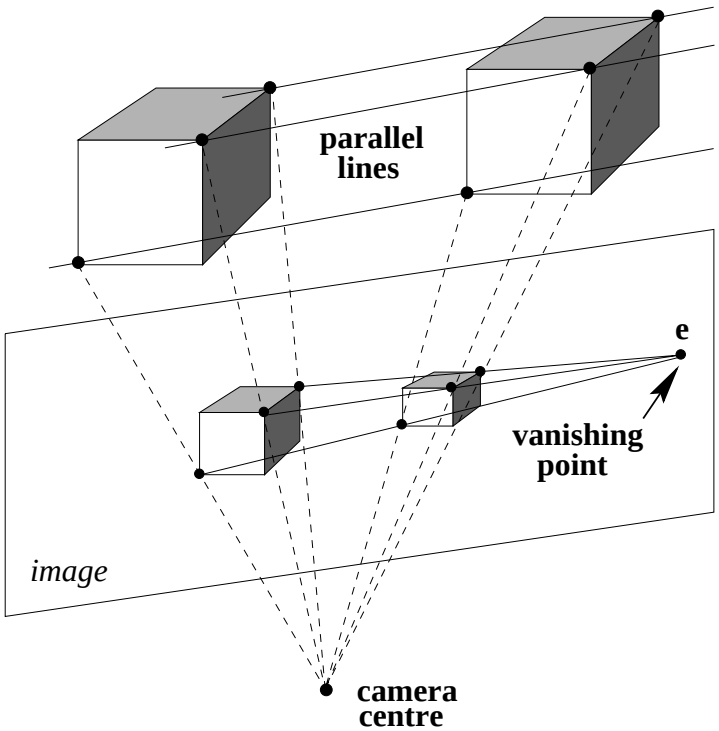
- Scaled orthography - Z_0 constant (representative depth)

$$\frac{x}{f} = \frac{X}{Z_0}, \qquad \frac{y}{f} = \frac{Y}{Z_0}$$

Perspective transformation

- A perspective transformation has three components:
  - Rotation - from world to camera coordinate system
  - Translation - from world to camera coordinate system
  - Perspective projection - from camera to image coordinates
- Basic properties which are preserved:
  - lines project to lines,
  - collinear features remain collinear,
  - tangencies,
  - intersections.

Perspective transformation (cont)

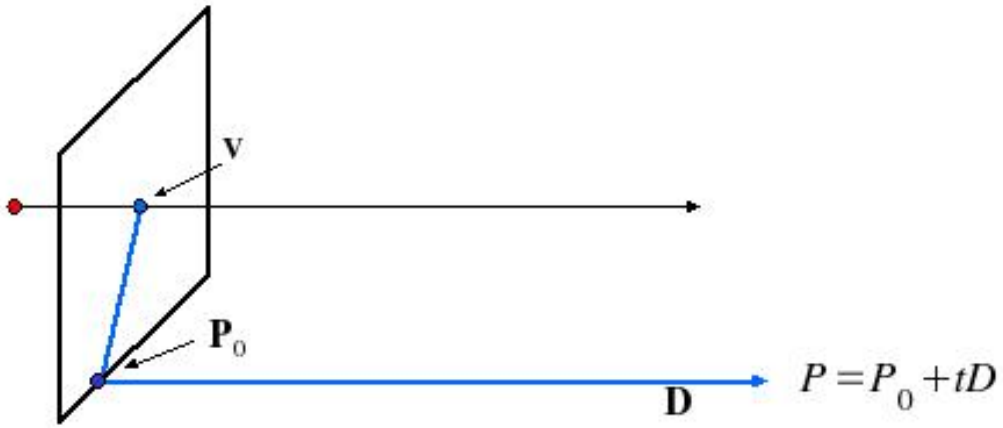


Each set of parallel lines meet at a different vanishing point - vanishing point associated to this direction. Sets of parallel lines on the same plane lead to collinear vanishing points - the line is called the horizon for that plane.

Homogeneous coordinates

- Model points (X, Y, Z) in $\mathcal{R}^3$ world by (kX, kY, kZ, k) where k is arbitrary $\neq 0$, and points (x, y) in $\mathcal{R}^2$ image domain by (cx, cy, c) where c is arbitrary $\neq 0$.
- Equivalence relation: (k_1X, k_1Y, k_1Z, k_1) is same as (k_2X, k_2Y, k_2Z, k_2).
- Homogeneous coordinates imply that we regard all points on a ray (cx, cy, c) as equivalent (if we only know the image projection, we do not know the depth).
- Possible to represent "points in infinity" with homogeneous coordinates $(X, Y, Z, 0)$ - intersections of parallel lines.

Computing vanishing points



$$P_t = \begin{bmatrix} P_X + tD_X \\ P_Y + tD_Y \\ P_Z + tD_Z \\ 1 \end{bmatrix} \simeq \begin{bmatrix} P_X/t + D_X \\ P_Y/t + D_Y \\ P_Z/t + D_Z \\ 1/t \end{bmatrix} \quad t \to \infty \quad P_\infty \simeq \begin{bmatrix} D_X \\ D_Y \\ D_Z \\ 0 \end{bmatrix}$$

Properties $v = \mathbf{P}_\infty$

- $\mathbf{P}_\infty$ is a point at *infinity*, **v** is its projection
- They depend only on line *direction*
- Parallel lines $\mathbf{P}_0 + t\mathbf{D}$, $\mathbf{P}_1 + t\mathbf{D}$ intersect at $\mathbf{P}_\infty$

Homogeneous coordinates (cont)

In homogeneous coordinates the projection equations can be written

$$\begin{pmatrix} cx \\ cy \\ c \end{pmatrix} = \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} kX \\ kY \\ kZ \\ k \end{pmatrix} = \begin{pmatrix} fkX \\ fkY \\ kZ \end{pmatrix}$$

Image coordinates obtained by normalizing the third component to one (divide by $c = kZ$).

$$x = \frac{xc}{c} = \frac{fkX}{kZ} = f\frac{X}{Z}, \qquad y = \frac{yc}{c} = \frac{fkY}{kZ} = f\frac{Y}{Z}$$

Transformations in homogeneous coordinates

- Translation

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \rightarrow \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \begin{pmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{pmatrix}$$

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & \Delta X \\ 0 & 1 & 0 & \Delta Y \\ 0 & 0 & 1 & \Delta Z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

- Scaling

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} S_X & 0 & 0 & 0 \\ 0 & S_Y & 0 & 0 \\ 0 & 0 & S_Z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

Transformations in homogeneous coordinates II

- Rotation around the Z axis

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

- Mirroring in the XY plane

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

Transformations in homogeneous coordinates III

Common case: Rigid body transformations (Euclidean)

$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} \rightarrow R \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \begin{pmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{pmatrix}$$

where R is a rotation matrix ($R^{-1} = R^T$) is written

$$\begin{pmatrix} X' \\ Y' \\ Z' \\ 1 \end{pmatrix} = \begin{pmatrix} & R & & \begin{matrix} \Delta X \\ \Delta Y \\ \Delta Z \end{matrix} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

Perspective projection - Extrinsic parameters

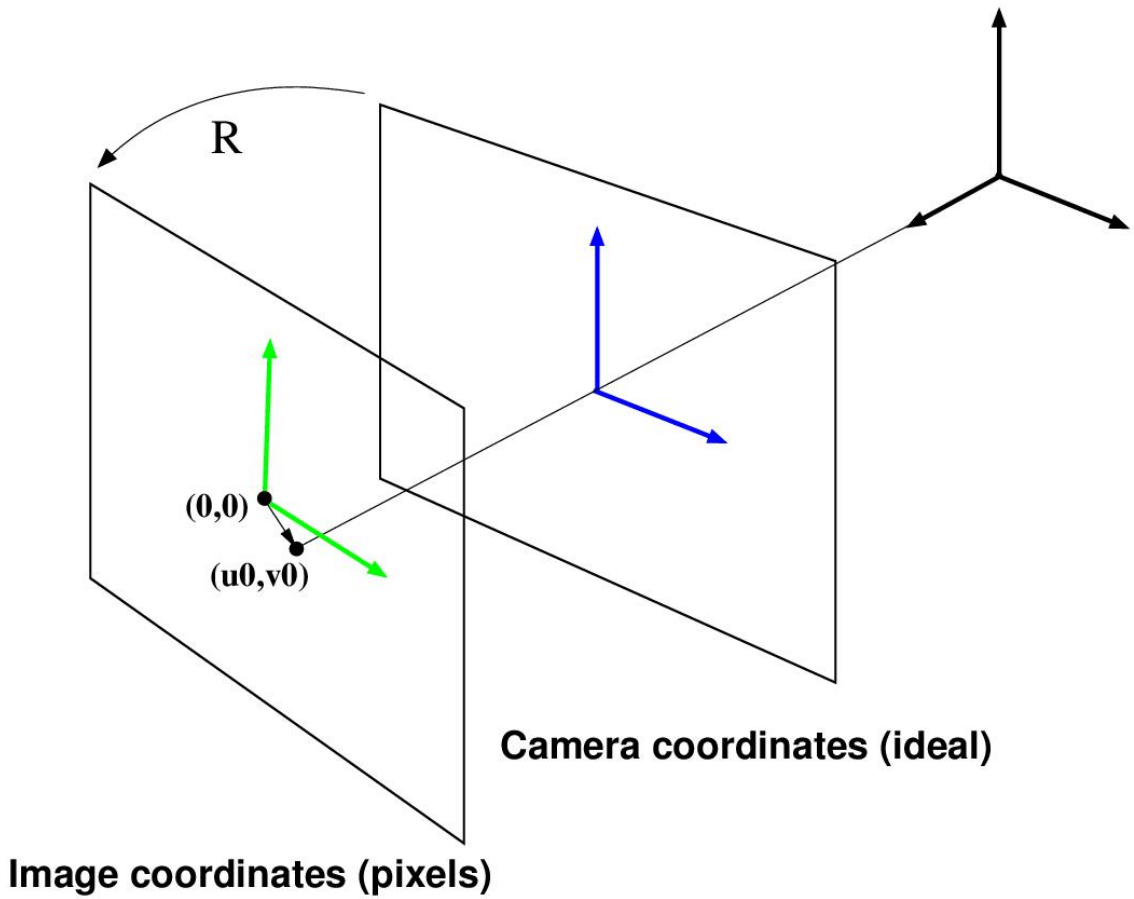
Consider world coordinates $(X', Y', Z', 1)$ expressed in a coordinate system not aligned with the camera coordinate system

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} & R & & \begin{matrix} \Delta X \\ \Delta Y \\ \Delta Z \end{matrix} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X' \\ Y' \\ Z' \\ 1 \end{pmatrix} = A \begin{pmatrix} X' \\ Y' \\ Z' \\ 1 \end{pmatrix}$$

Perspective projection (more general later)

$$c \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = PA \begin{pmatrix} X' \\ Y' \\ Z' \\ 1 \end{pmatrix} = M \begin{pmatrix} X' \\ Y' \\ Z' \\ 1 \end{pmatrix}$$

Intrinsic camera parameters



Due to imperfect placement of the camera chip relative to the lens system, there is always a small relative rotation and shift of center position.

Intrinsic camera parameters

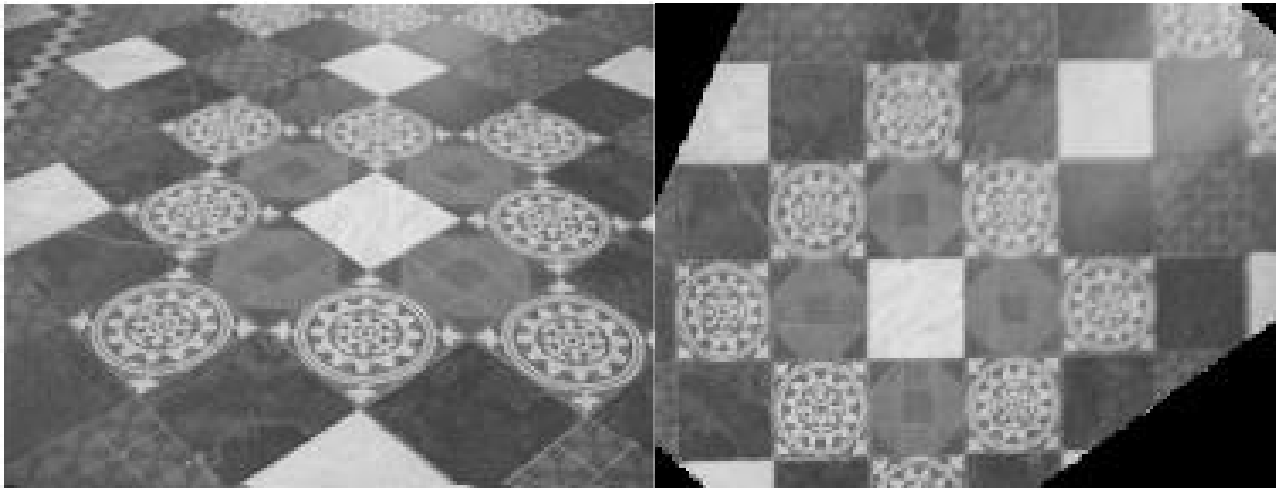
A more general projection matrix allows:

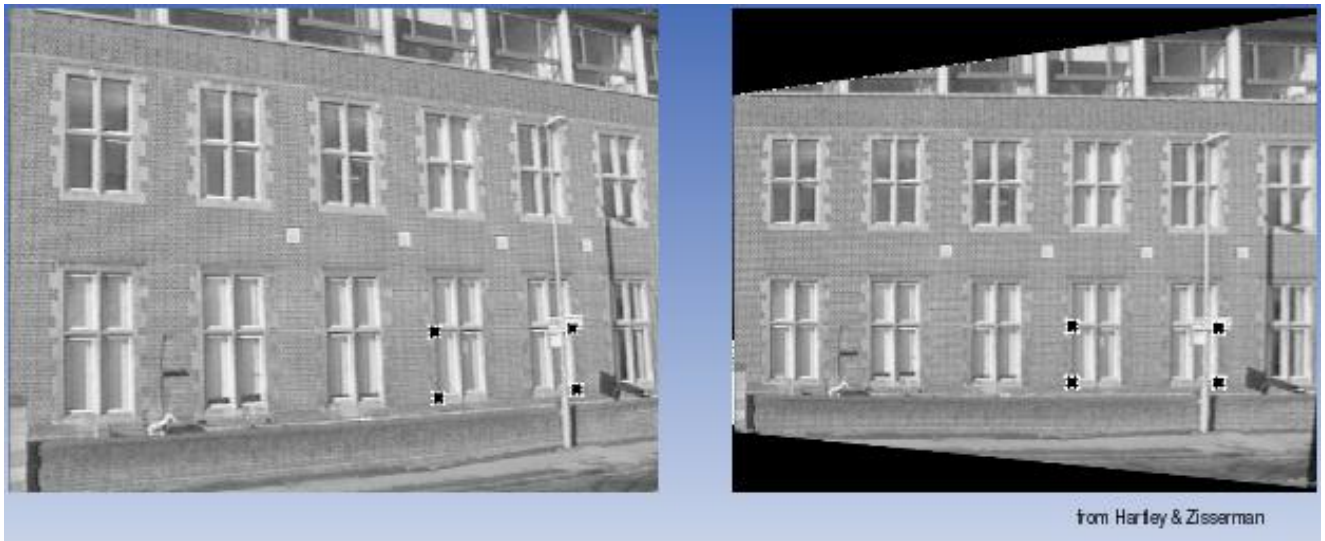
- Image coordinates with an offset origin
- Non-square pixels
- Skewed coordinate axes
- Five variables below are known as the camera's intrinsic parameters

$$K = \begin{pmatrix} f_u & \gamma & u_0 \\ 0 & f_v & v_0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} K & 0 \end{pmatrix} = \begin{pmatrix} f_u & \gamma & u_0 & 0 \\ 0 & f_v & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Most important is the focal length (f_u, f_v). Normally, f_u and f_v are assumed equal and the parameters γ, u_o and v_o close to zero.

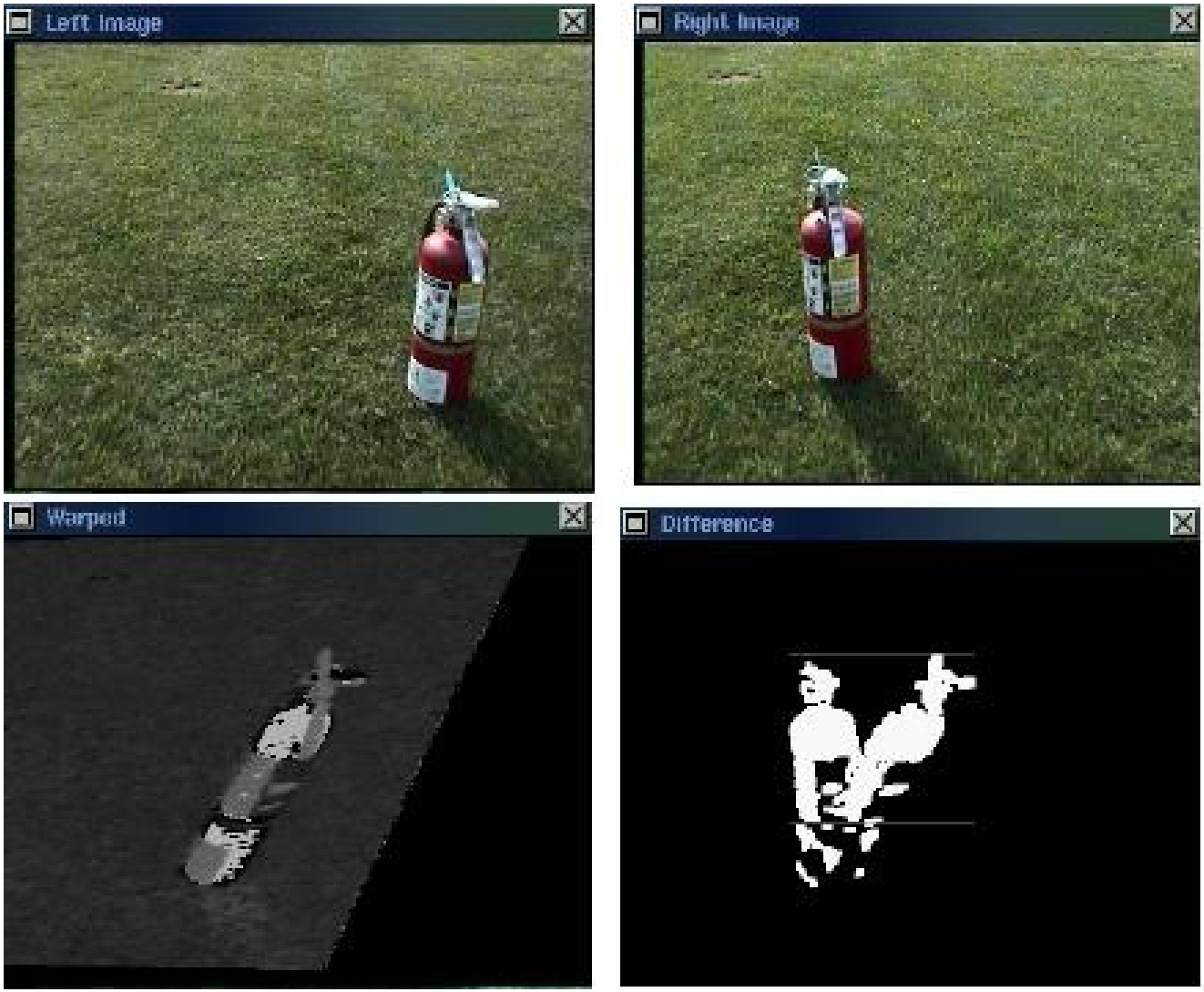
Example: Perspective mapping





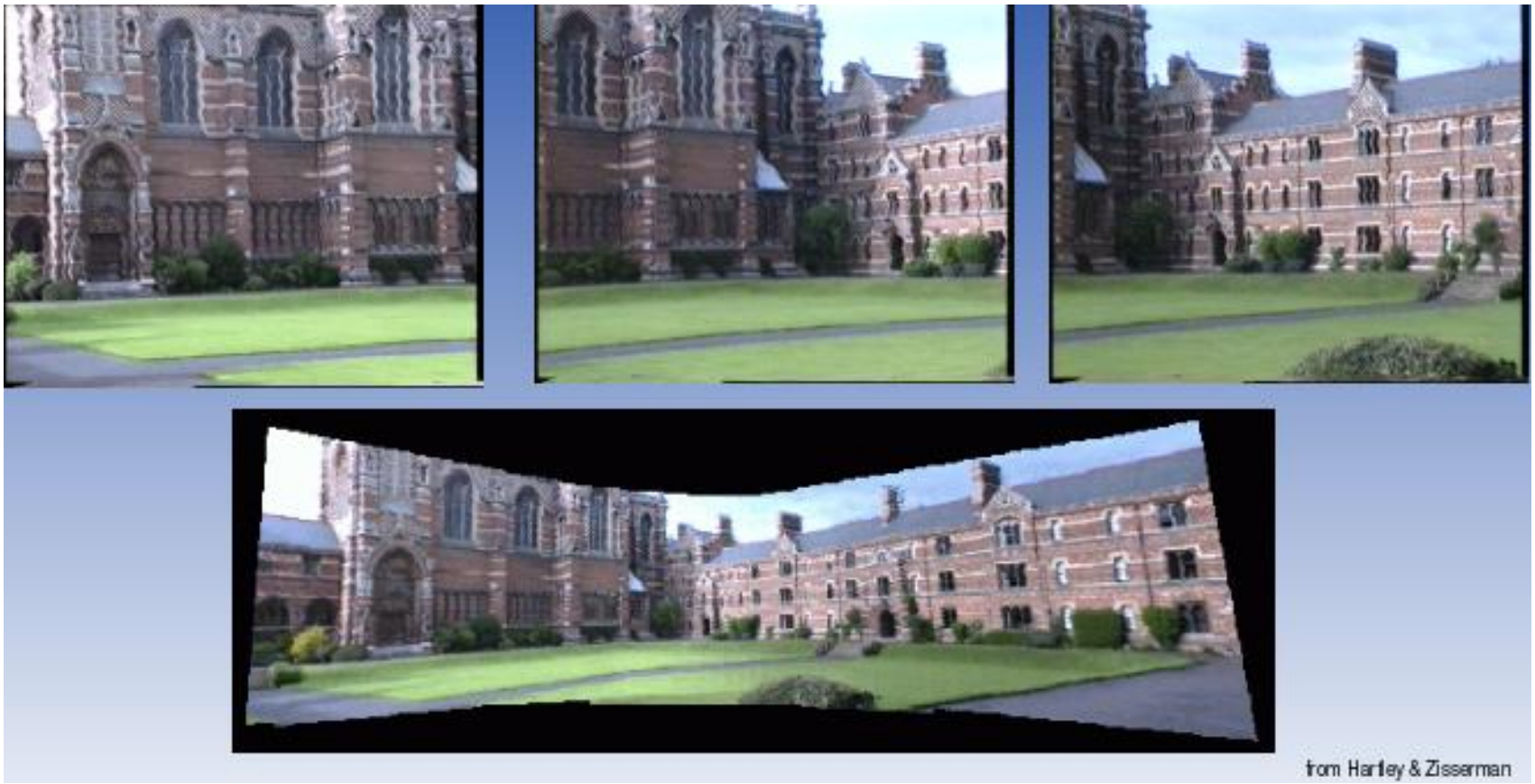
Planes can be mapped from one image to another.

Example: Perspective mapping in stereo



After image subtraction, difference are things not on the plane.

Mosaicing



Since the world is not a plane, you will get small artifacts.

Exercise

Assume you have a point at $(3, -2, 8)$ with respect to the cameras coordinate system. What are the image coordinates, if the image has a size $(w, h) = (640, 480)$ and origin in the upper-left corner, and the focal length is $f = 480$?

Exercise

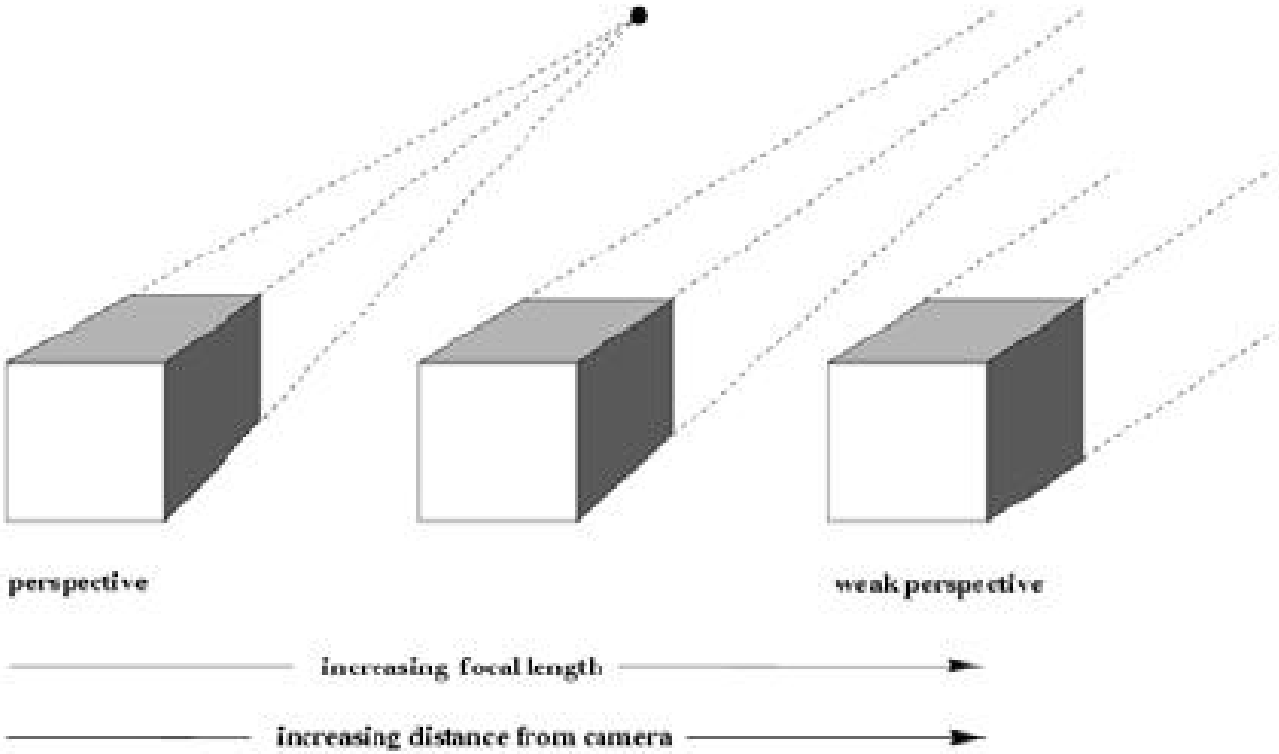
Assume you have a point at $(3, -2, 8)$ with respect to the cameras coordinate system. What are the image coordinates, if the image has a size $(w, h) = (640, 480)$ and origin in the upper-left corner, and the focal length is $f = 480$?

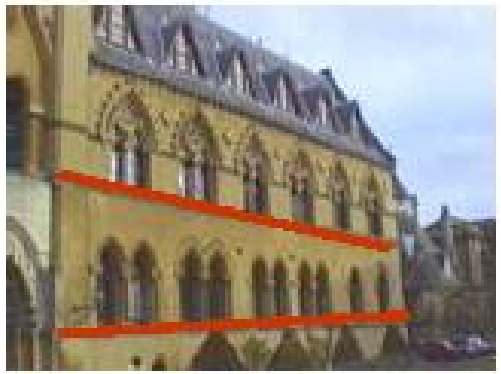
Answer:

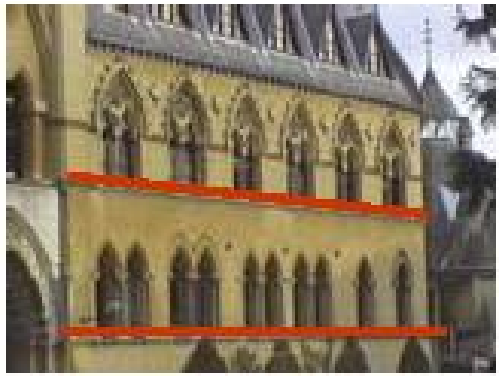
$$x = f\frac{X}{Z} + \frac{w}{2} = (480 * 3/8 + 640/2) = 500$$

$$y = f\frac{Y}{Z} + \frac{h}{2} = (-480 * 2/8 + 480/2) = 120$$

Approximation: affine camera







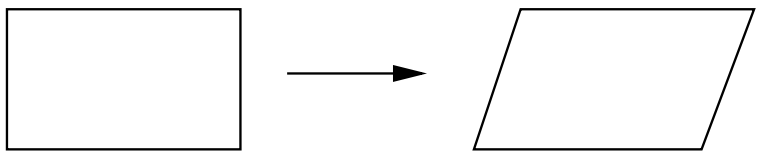
Approximation: affine camera

- A linear approximation of perspective projection

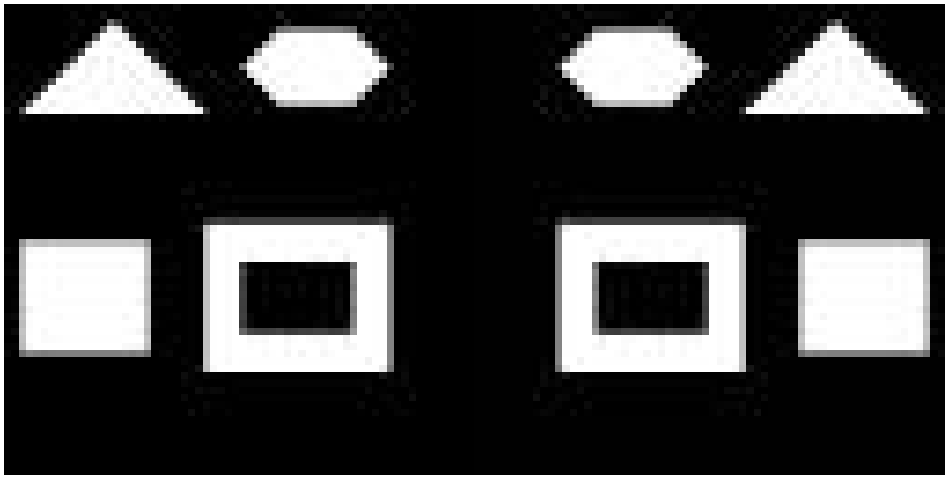
$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

- Basic properties
  - linear transformation (no need to divide at the end)
  - parallel lines in 3D mapped to parallel lines in 2D

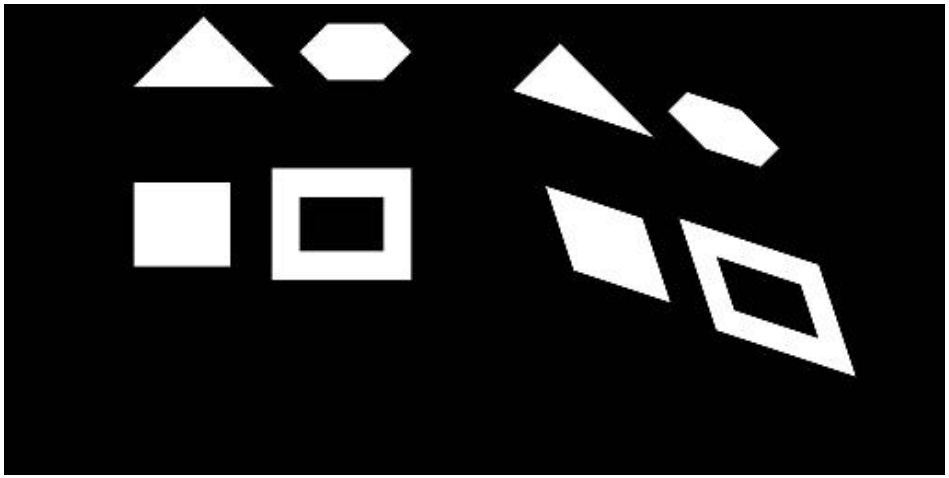
Angles are not preserved!



Planar Affine Transformation



Original Flipped x-size



Shifted and scaled Sheared

Summary of models

Projective (11 degrees of freedom):

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{pmatrix}$$

Affine (8 degrees of freedom):

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Scaled orthographic (6 degrees of freedom):

$$M = \begin{pmatrix} r_{11} & r_{12} & r_{13} & \Delta X \\ r_{21} & r_{22} & r_{23} & \Delta Y \\ 0 & 0 & 0 & Z_0 \end{pmatrix}$$

Orthographic (5 degrees of freedom):

$$M = \begin{pmatrix} r_{11} & r_{12} & r_{13} & \Delta X \\ r_{21} & r_{22} & r_{23} & \Delta Y \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

All these are just approximations, since they all assume a pin-hole, which is supposed to be infinitesimally small.

Image warping

Resample image $f(x, y)$ to get a new image $g(u, v)$, using a coordinate transformation: $u = u(x, y),\ v = v(x, y)$.

Examples of transformations:

translation

rotation

aspect

affine

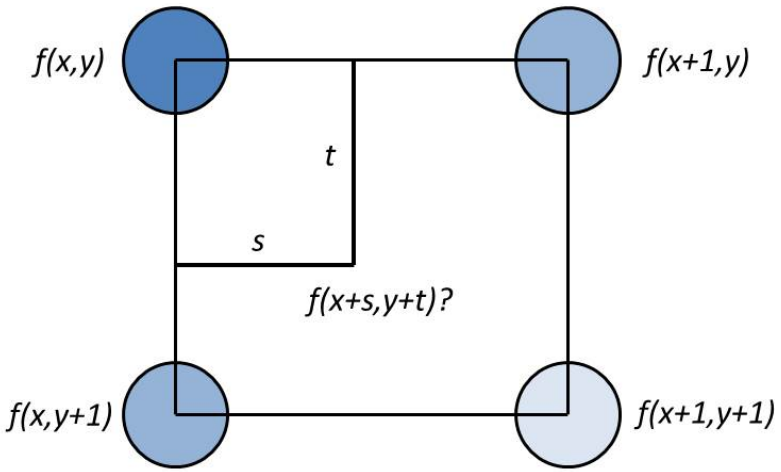
perspective

cylindrical

Image Warping

- For each grid point in (u, v) domain compute corresponding (x, y) values.

  Note: transformation is inverted to avoid holes in result.
- Create $g(u, v)$ by sampling from $f(x, y)$ either by:
  - Nearest neighbour look-up (noisy result)
  - Bilinear interpolation (blurry result)



$$\begin{aligned} f(x+s, y+t) &= (1-t) \cdot ((1-s) \cdot f(x, y) + s \cdot f(x+1, y)) + \\ &+ t \cdot ((1-s) \cdot f(x, y+1) + s \cdot f(x+1, y+1)) \end{aligned}$$

Nearest Neighbor vs. Bilinear Interpolation

Reasoning about shape in binary images

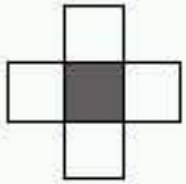
- Images with two colours, black (0) and white (1 or 255).
  - Commonly referred to as 'background' and 'foreground'.
- Typically obtained from thresholding or image segmentation.

$$g(x,y) = \begin{cases} 1 & \text{if } f(x,y) > T \\ 0 & \text{otherwise} \end{cases}$$

Neighbourhood concepts

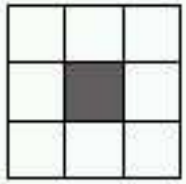
Many image processing operations are based on local neighborhood operations.

Pixels are 4-neighbours if their distance is $D_4 = 1$



all 4-neighbours of center pixel

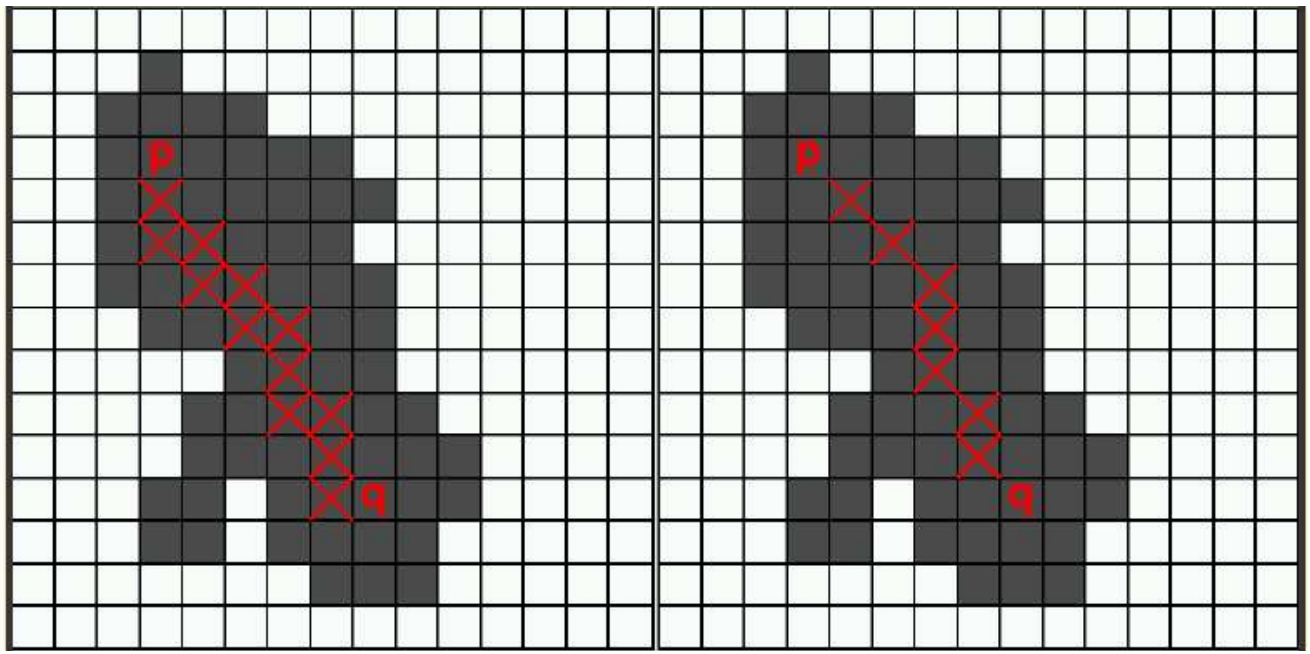
Pixels are 8-neighbours if their distance is $D_8 = 1$



all 8-neighbours of center pixel

Connectivity

- Path: A path from p to q is a set of points $p_0 \ldots p_n$, such that each point p_i is a neighbor of p_{i-1}.
- Connectivity: p is connected to q in S, if there is a path from p to q completely in S.

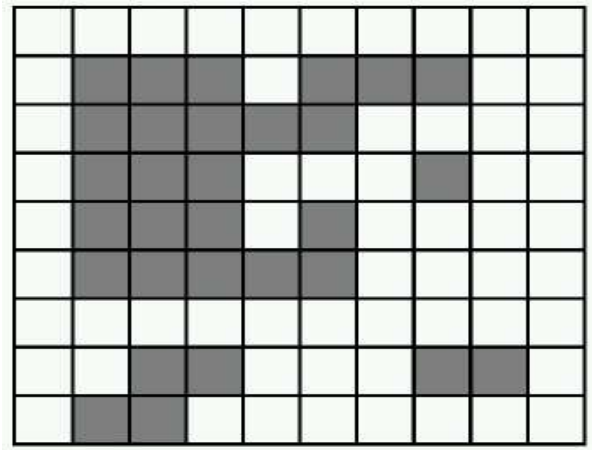


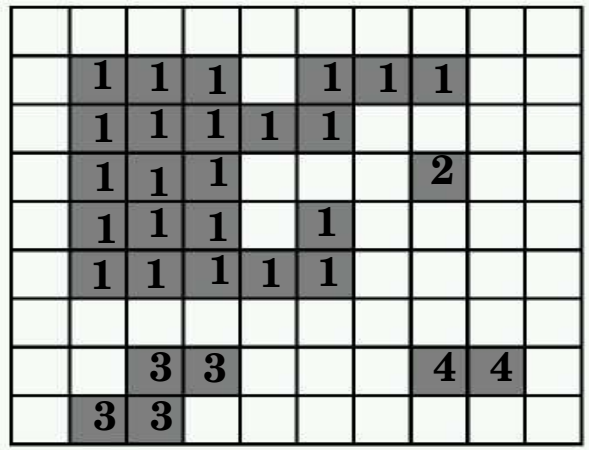
Connected components

- For every p, the set of all points q connected to p is said to be its connected component.

Recursive procedure that scans entire image:

1. for each unlabeled foreground pixel, assign it a new label L
2. assign label L to all neighboring foreground pixels
3. stop if there is no unlabeled foreground pixels

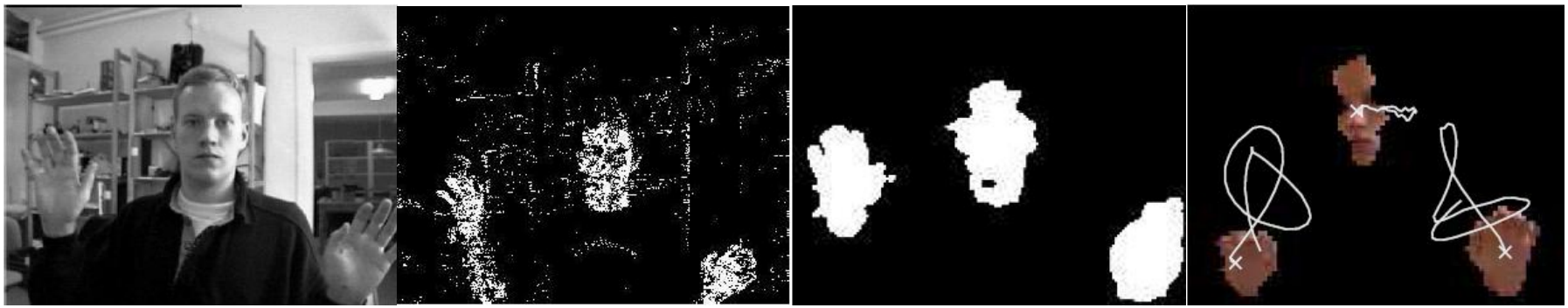




Connected component labeling

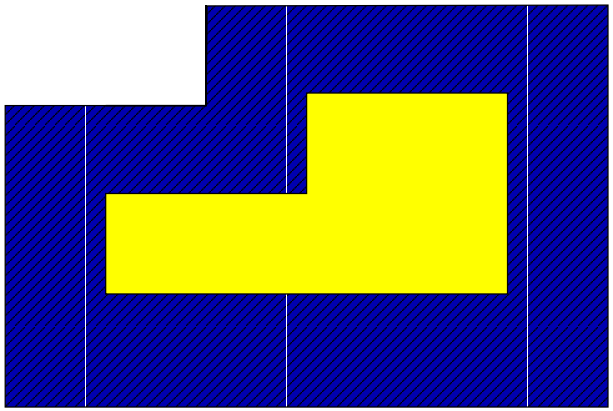
Regions (connected components) are often denoted by labels.

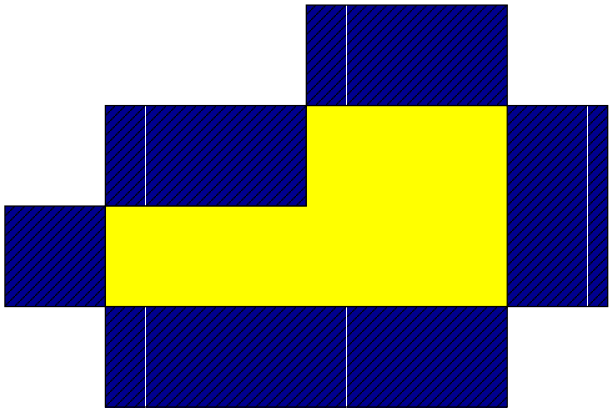
- statistics of regions (size, shape, gray-level statistics)
- size filtering (suppress objects of size $<$ threshold)



Duality of 4-connectivity and 8-connectivity

Outer boundary: background points with a neighbour on the object.





(left) based on 8-connectivity (right) based on 4-connectivity

- Jordan curve theorem (continuous case):
  Each closed curve divides plane into one region inside and one region outside.
- Note: Many region based methods, only store the boundary.

Distance measures

How to define distance between two points p and q?

Common distance measures:

- Euclidean distance $d(p,q) = \sqrt{(x-u)^2 + (y-v)^2}$
- City block distance $d(p,q) = \mid x-u \mid + \mid y-v \mid$
- Chessboard distance $d(p,q) = \max(\mid x-u \mid, \mid y-v \mid)$

All three measure satisfy metric axioms

- $d(p,q) \geq 0$
- $d(p,q) = d(q,p)$
- $d(p,r) \leq d(p,q) + d(q,r)$

Distance measures

Euclidean distance

$$\begin{matrix} \sqrt{8} & \sqrt{5} & 2 & \sqrt{5} & \sqrt{8} \\ \sqrt{5} & \sqrt{2} & 1 & \sqrt{2} & \sqrt{5} \\ 2 & 1 & 0 & 1 & 2 \\ \sqrt{5} & \sqrt{2} & 1 & \sqrt{2} & \sqrt{5} \\ \sqrt{8} & \sqrt{5} & 2 & \sqrt{5} & \sqrt{8} \end{matrix}$$

City block distance

$$\begin{matrix} 4 & 3 & 2 & 3 & 2 \\ 3 & 2 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 2 & 3 \\ 4 & 3 & 2 & 3 & 4 \end{matrix}$$

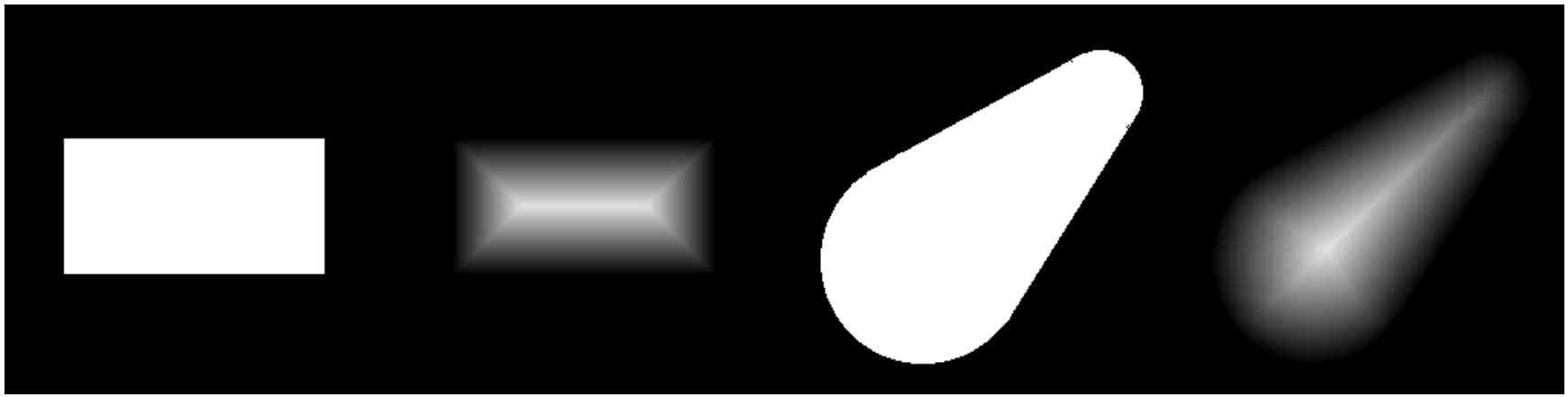
Chessboard distance

$$\begin{matrix} 2 & 2 & 2 & 2 & 2 \\ 2 & 1 & 1 & 1 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 2 & 2 \end{matrix}$$

The two last measures are usually faster, but equally useful.

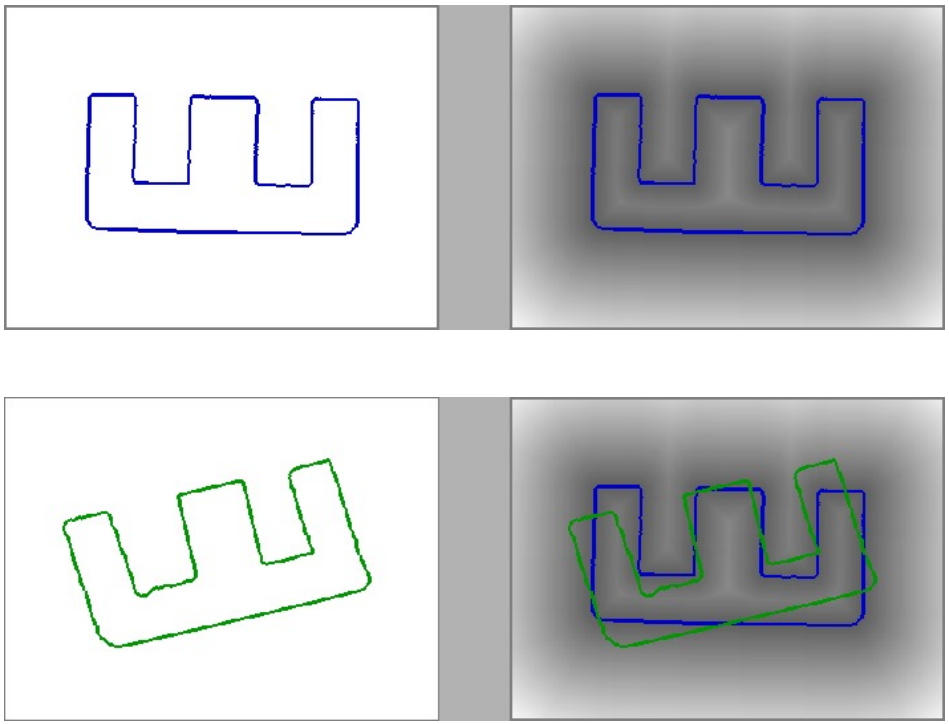
Distance transform

- The result is an image that shows the distance to the closest boundary from each point
- Useful for shape description, matching, skeletonization, etc



Distance transform for matching shapes

- Create distance transform from model shape S_{model} represented by edges.
- Extract new shape S_{image} from an image.
- Sum values in distance transform over edge points from S_{image}.
- Iteratively transform S_{image} until sum is mimimized.



Summary of good questions

- What is a pinhole camera model?
- What is the difference between intrinsic and extrinsic camera parameters?
- How does a 3D point get projected to a pixel with a perspective projection?
- What are homogeneous coordinates and what are they good for?
- What is a vanishing point and how do you find it?
- What is an affine camera model?
- What is a 4-neighbour and how is related to connectiveness?
- What kind of distance measures exist?

Readings

- Gonzalez and Woods: Chapters 2.4 - 2.5
- Szeliski: Chapters 2.1 and 3.3.3 - 3.3.4