Exercises 3

1. You are given the following points from a two dimensional distribution: $p_1 = (-1, 0),\ p_2 = (1, 0),\ p_3 = (1, 2),\ p_4 = (3, 2),\ p_5 = (4, 3),\ p_6 = (5, 3)$ Plot these points and notice that one direction seems more significant. Using PCA, determine this direction.

   Solution: The idea is to compute the covariance of the points, and pick out the direction that has the largest variance.

   Start by computing the mean:

   $$\overline{p} = \frac{1}{N}\sum_{i=1}^{N} p_i$$

   In our case $N = 6$, and we get

   $$\overline{p} = \frac{1}{6}\begin{pmatrix} 13 \\ 10 \end{pmatrix}$$

   When computing the covariance we need points with mean zero, why we subtract the mean from each point $p'_i = p_i - \overline{p}$

   Now we can compute the covariance as

   $$C = \frac{1}{N}\sum_{i=1}^{N} p'_i p'^T_i = \frac{1}{36}\begin{pmatrix} 149 & 80 \\ 80 & 56 \end{pmatrix}$$

   Finding the direction with the largest variance equals to finding the eigenvector with the largest eigenvalue. So we compute the eigenvectors x

   $$Cx = \lambda x \Leftrightarrow (C - I\lambda)x = 0$$

   which means that we first have to solve the characteristic equation

   $$|C - \lambda x| = 0 \Rightarrow \begin{vmatrix} \frac{149}{36} - \lambda & \frac{80}{36} \\ \frac{80}{36} & \frac{56}{36} - \lambda \end{vmatrix} = 0 \Leftrightarrow$$

   $$\begin{vmatrix} 149 - 36\lambda & 80 \\ 80 & 56 - 36\lambda \end{vmatrix} = 0 \Leftrightarrow$$

   $$(149 - 36\lambda)(56 - 36\lambda) - 80^2 = 0 \Leftrightarrow$$

   $$(36\lambda)^2 - (149 * 36 + 56 * 36)\lambda + 149 * 56 - 80^2 = 0 \Leftrightarrow$$

   $$\lambda^2 - \frac{7380}{1296}\lambda + \frac{1944}{1296} = 0 \Rightarrow$$

   $$\lambda = \frac{\sqrt{34249} \pm 205}{72}$$

which gives the largest eigenvalue $\lambda_1 = \frac{\sqrt{34249}+205}{72}$ for which we compute the largest eigenvector

$$\begin{pmatrix} \frac{149}{36} - \frac{\sqrt{34249}+205}{72} & \frac{80}{36} \\ \frac{80}{36} & \frac{56}{36} - \frac{\sqrt{34249}+205}{72} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow$$

$$\begin{cases} (\frac{149}{36} - \frac{\sqrt{34249}+205}{72})x_1 + \frac{80}{36}x_2 = & 0 \ (1) \\ \frac{80}{36}x_1 + (\frac{56}{36} - \frac{\sqrt{34249}+205}{72})x_2 = & 0 \ (2) \end{cases}$$

Wince we are looking for the direction, only the relation between x_1 and x_2 is relevant. Using Eq. (1) we conclude that

$$x_1 = \frac{-\frac{80}{36}}{\frac{149}{36} - \frac{\sqrt{34249}+205}{72}} x_2$$

Taking $x_2 = 1$, we get $x_1 \approx 1.74$. Alternatively, we could normalize the vector and get the answer

$$x = \begin{pmatrix} 0.867 \\ 0.499 \end{pmatrix}$$

2. A ball is moving with constant velocity straight towards a camera along the optical axis. At time $t_0 = 0$ it covers 500 pixels, and at time $t_1 = 3$ it covers 750 pixels. At what time does it cover 1000 pixels? (The camera is assumed to be of pinhole type.)

   Solution: A ball moving towards a pinhole camera will be projected as a circle on the image plane. We make the assumption that the entire intersection is visible in the image (see figure). At t_0 the ball covers 500 pixels, why the radius $r_0 = \sqrt{500/\pi}$. Similarly, at t_1, $r_1 = \sqrt{750/\pi} = \sqrt{1.5}r_0$ and at $t_2 = \sqrt{1000/\pi} = \sqrt{2}r_0$. Furthermore at t_i the ball is at distance Z_i. Let the ball have radius R. From the figure we see that

$$\frac{r_i}{f} = \frac{R}{Z_i} \Leftrightarrow r_i Z_i = Rf$$

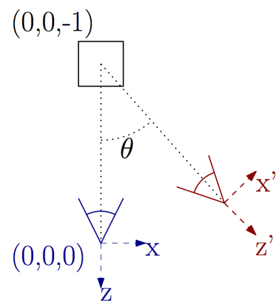
   We can now write the equalities

$$r_0 Z_0 = r_1 Z_1 = \sqrt{1.5} r_0 Z_1 \Leftrightarrow Z_0 = \sqrt{1.5} Z_1 \Leftrightarrow Z_1 = \frac{Z_0}{\sqrt{1.5}}$$
$$r_0 Z_0 = r_2 Z_2 = \sqrt{2} r_0 Z_2 \Leftrightarrow Z_0 = \sqrt{2} Z_2 \Leftrightarrow Z_2 = \frac{Z_0}{\sqrt{2}}$$

   We know that the ball moves the distance $Z_0 - \frac{Z_0}{\sqrt{1.5}}$ in three time steps, so the speed $v = \frac{\sqrt{1.5}Z_0 - Z_0}{3\sqrt{1.5}}$
   Finally we can compute time t_2

$$t_2 = \frac{Z_0 - Z_2}{v} = \frac{Z_0 - \frac{Z_0}{\sqrt{2}}}{\frac{\sqrt{1.5}Z_0 - Z_0}{3\sqrt{1.5}}} = \frac{1 - \frac{1}{\sqrt{2}}}{\frac{\sqrt{1.5}-1}{3\sqrt{1.5}}} = 3\frac{\sqrt{3} - \sqrt{1.5}}{\sqrt{3} - \sqrt{2}}$$

3. A robot is trying to gather 3D information from an object. Since it has only a single camera, it rotates around the object to obtain multiple views from it, as seen in the figure below.



a) What is the relation between points P in 3D space and their image projections p on the image camera at $(0,0)$? Consider unit focal length, $f = 1$, and centered image origin, with x increasing to the right and y increasing up.

b) What is the relation between 3D points $P = (x, y, z)$ in the original coordinate frame and $P' = (x', y', z')$ in the new coordinate frame after rotating θ radians?

c) What is the relation between image points p and p'?

d) At some point the encoders of the motors fail and the robot doesn't know how large θ is. Estimate θ given point correspondences between the two images. How many point do you need?

Solution:

a) Since z-axis points backwards, z-values are inverted in the projection.

$$P = (x, y, z) \Rightarrow p = \left(\frac{x}{-z}, \frac{y}{-z}\right)$$

b)

$$t = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \quad R = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

Transforming the 3D points look the same as transfoming the camera, but in opposite direction. We first move the camera forward towards the object (-t), then rotate (R) and finally move back backwards (t).

$$P' = R(P - t) + t = R(P - (t - R^{-1}t)) = R(P - e)$$

Here e corresponds to the final translation of the camera.

$$e = t - R^{-1}t = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \sin\theta \\ 0 \\ -1 + \cos\theta \end{pmatrix}$$

c) There is an essential matrix $E = RT_e$ for which $P'^T EP = 0$ holds.

$$T_e = \begin{pmatrix} 0 & 1-\cos\theta & 0 \\ -(1-\cos\theta) & 0 & -\sin\theta \\ 0 & \sin\theta & 0 \end{pmatrix}$$

$$E = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 & 1-\cos\theta & 0 \\ -(1-\cos\theta) & 0 & -\sin\theta \\ 0 & \sin\theta & 0 \end{pmatrix}$$

$$E = \begin{pmatrix} 0 & -(1-\cos\theta) & 0 \\ -(1-\cos\theta) & 0 & -\sin\theta \\ 0 & \sin\theta & 0 \end{pmatrix}$$

Then the projections can be related by

$$\begin{pmatrix} p'_x & p'_y & -1 \end{pmatrix} E \begin{pmatrix} p_x \\ p_y \\ -1 \end{pmatrix} = 0$$

d)

$$\begin{pmatrix} p'_x & p'_y & -1 \end{pmatrix} \begin{pmatrix} 0 & -(1-\cos\theta) & 0 \\ -(1-\cos\theta) & 0 & -\sin\theta \\ 0 & \sin\theta & 0 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ -1 \end{pmatrix} = 0$$

$$\begin{pmatrix} p'_x & p'_y & -1 \end{pmatrix} \begin{pmatrix} -(1-\cos\theta)p_y \\ -(1-\cos\theta)p_x + \sin\theta \\ \sin\theta p_y \end{pmatrix} = 0$$

$$-(1-\cos\theta)p_y p'_x - (1-\cos\theta)p_x p'_y + \sin\theta p'_y - \sin\theta p_y = 0$$

$$(p_y p'_x + p_x p'_y)\cos\theta + (p'_y - p_y)\sin\theta = (p_y p'_x + p_x p'_y)$$

Finally, solve for θ by removing the cosine and then using arcsin.

4. You are given the following binary image:

$$\begin{matrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{matrix}$$

Compute the following:

- Moments: $m_{00},\ m_{10},\ m_{01}$ and m_{20}
- Centers of gravity: x_0 and y_0
- Central moments: $\mu_{00},\ \mu_{01}$ and μ_{02}

Solution: Image moments for an image I are defined as:

$$m_{ij} = \sum_y \sum_x I(x, y) x^i y^j$$

Hence we get:

$$m_{00} = 17,\ m_{10} = 58,\ m_{01} = 53,\ m_{20} = 248$$

Centers of gravity, x_0 and y_0 are means in x- and y-directions:

$$x_0 = \frac{m_{10}}{m_{00}} \approx 3.41$$
$$y_0 = \frac{m_{01}}{m_{00}} \approx 3.12$$

Finally central moments are defined as

$$\mu_{ij} = \sum_y \sum_x I(x, y)(x - x_0)^i (y - y_0)^j$$

So we get:

$$\mu_{00} = 17,\ \mu_{01} = 0,\ \mu_{02} \approx 63.76$$