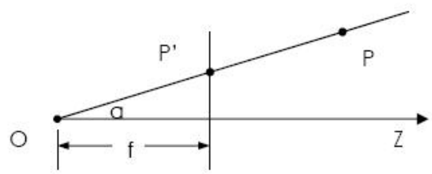
EXAM SOLUTIONS
Tuesday, 10^{th} of February 2009, 14.00–19.00

Exercise 1:

(a) Field of View (FOV)

i. The following diagram shows the perspective projection of a camera



If the width of the image is *L*, The Field of View

FOV = 2a = 2atan(L/2f)

ii. For the given camera,

FOV-vertical = 2atan(8/24) = 36.9°

FOV-horizontal = 2atan(6/24) = 28.1°

iii. There are only finite numbers of pixels in an image. The larger the FOV, the more scene is projected to the image. Hence the resolution for every pixel is decreased.

(b) Application

i. Let the camera-frame coordinates of a point be (X_c, Y_c, Z_c) and the image plane coordinates be (x, y), then

$$\begin{bmatrix} sx \\ sy \\ s \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix}$$

Let the pixel coordinates **w** = (*u*, *v*), then

$$\begin{bmatrix} su \\ sv \\ s \end{bmatrix} = \begin{bmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sx \\ sy \\ s \end{bmatrix}$$

Combine together:

$$\begin{bmatrix} su \\ sv \\ s \end{bmatrix} = \begin{bmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix}$$

ii. Assume the origin of camera frame is at the center of the image, then

$(u_0, v_0) = (250, 250)$.

$(k_u, k_v) = (500/12, -500/16) = (41.67, -31.25)$

It can be computed that **w** = (*u*, *v*) = (367, 199)

Exercise 2:

The solution is according to the lecture "Projections, image sampling" (pages 29-34).

Exercise 3:

Moments are defined by

$$m_{ij} = \sum_x \sum_y x^i y^j f(x,y).$$

A center of gravity is expressed by moments:

$$x_0 = \frac{\sum\sum x f(x,y)}{\sum\sum f(x,y)} = \frac{m_{10}}{m_{00}}$$

$$y_0 = \frac{\sum\sum y f(x,y)}{\sum\sum f(x,y)} = \frac{m_{01}}{m_{00}}$$

Central moments are defined by

$$\mu_{ij} = \sum_x \sum_y (x-x_0)^i (y-y_0)^j f(x,y).$$

They can be expressed as moments as shown in the following examples:

$$\begin{aligned}
\mu_{00} &= \sum\sum f(x,y) = m_{00} \\
\mu_{01} &= \sum\sum y f(x,y) - \sum\sum y_0 f(x,y) = m_{01} - (m_{01}/m_{00}) m_{00} = 0 \\
\mu_{10} &= \sum\sum x f(x,y) - \sum\sum x_0 f(x,y) = m_{10} - (m_{10}/m_{00}) m_{00} = 0 \\
\mu_{11} &= m_{11} - m_{01} m_{10}/m_{00} \\
\mu_{20} &= m_{20} - m_{10}^2/m_{00} \\
\mu_{02} &= m_{02} - m_{01}^2/m_{00}.
\end{aligned}$$

Image 1

i	j	m_{ij}
0	0	$\sum\sum f(x,y) = 14$
0	1	$\sum\sum y f(x,y) = 38$
1	0	$\sum\sum x f(x,y) = 32$
1	1	$\sum\sum xy f(x,y) = 88$
2	0	$\sum\sum x^2 f(x,y) = 98$
0	2	$\sum\sum y^2 f(x,y) = 188$

Image 2

i	j	m_{ij}
0	0	$\sum\sum f(x,y) = 14$
0	1	$\sum\sum y f(x,y) = 43$
1	0	$\sum\sum x f(x,y) = 31$
1	1	$\sum\sum xy f(x,y) = 95$
2	0	$\sum\sum x^2 f(x,y) = 89$
0	2	$\sum\sum y^2 f(x,y) = 169$

Image 1	Image 2
$x_0 = 2\frac{2}{7}$	$x_0 = 2\frac{3}{14}$
$y_0 = 2\frac{5}{7}$	$y_0 = 3\frac{1}{14}$

Image 1

i	j	μ_{ij}
0	0	14
0	1	0
1	0	0
1	1	8/7
2	0	174/7
0	2	594/7

Image 2

i	j	μ_{ij}
0	0	14
0	1	0
1	0	0
1	1	-3/14
2	0	285/14
0	2	517/14

Exercise 4:

PCA (principal component analysis) -transform is given by

$$\mathbf{y} = \mathbf{A}(\mathbf{x} - \mathbf{m_x}) \quad \text{(the rows of } \mathbf{A} \text{ are the eigenvectors of } \mathbf{C_x}\text{)}$$

Mean:

$$\mathbf{m_x} = \frac{1}{M}\sum_{k=1}^{M} \mathbf{x}_k$$

Covariance matrix:

$$\mathbf{C_x} = \frac{1}{M}\sum_{k=1}^{M} \mathbf{x}_k\mathbf{x}_k^T - \mathbf{m_x}\mathbf{m_x}^T$$

The mean of the six sample points is now

$$\mathbf{m_x} = \frac{1}{6}\begin{pmatrix} -2-1+0+0+1+2 \\ 0+2+3+1+2+4 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

and the covariance matrix is

$$\mathbf{C_x} = \frac{1}{6}\begin{pmatrix} 4+1+0+0+1+4 & 0-2+0+0+2+8 \\ 0-2+0+0+2+8 & 0+4+9+1+4+16 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 5/3 & 4/3 \\ 4/3 & 5/3 \end{pmatrix}$$

Then we calculate the eigenvalues and eigenvectors of $\mathbf{C_x}$:

$$\mathbf{C_x}\mathbf{e}_i = \lambda_i\mathbf{e}_i \quad \Leftrightarrow \quad \begin{pmatrix} 5/3 & 4/3 \\ 4/3 & 5/3 \end{pmatrix}\begin{pmatrix} \mathbf{e}_{i1} \\ \mathbf{e}_{i2} \end{pmatrix} = \lambda_i\begin{pmatrix} \mathbf{e}_{i1} \\ \mathbf{e}_{i2} \end{pmatrix}$$

The solution is obtained when $|\mathbf{C_x} - \lambda\mathbf{I}| = 0$,

$$\begin{vmatrix} 5/3-\lambda & 4/3 \\ 4/3 & 5/3-\lambda \end{vmatrix} = \left(\frac{5}{3}\right)^2 - 2\cdot\frac{5}{3}\cdot\lambda + \lambda^2 - \left(\frac{4}{3}\right)^2 = \lambda^2 - \frac{10}{3}\lambda + 1 = 0 \Rightarrow \lambda_1 = 3 \text{ and } \lambda_2 = 1/3.$$

Next we calculate the eigenvectors

$$\Rightarrow \quad \mathbf{e}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

The transform matrix $\mathbf{A}$ is thus

$$\mathbf{A} = \begin{pmatrix} \mathbf{e}_1^T \\ \mathbf{e}_2^T \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

a) Transform 2-dim $\Rightarrow$ 1-dim: the transform matrix is the eigenvector that corresponds to the largest eigenvalue:

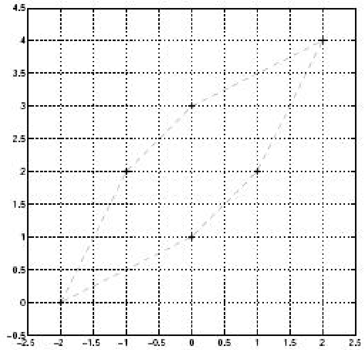
$$\mathbf{A} = \mathbf{e}_1^T = (1/\sqrt{2} \quad 1/\sqrt{2})$$

The average error is $R = \lambda_2 = 1/3$.
The transformed points are:

$$\mathbf{y}_1 = \mathbf{A}(\mathbf{x} - \mathbf{m_x}) = \mathbf{e}_1^T(\mathbf{x}_1 - \mathbf{m_x}) = (1/\sqrt{2} \quad 1/\sqrt{2})\begin{pmatrix} -2-0 \\ 0-2 \end{pmatrix} = -2\sqrt{2}$$

$\mathbf{y}_2 = -\sqrt{2}/2$, $\mathbf{y}_3 = \sqrt{2}/2$, $\mathbf{y}_4 = -\sqrt{2}/2$, $\mathbf{y}_5 = \sqrt{2}/2$, $\mathbf{y}_6 = 2\sqrt{2}$.

b) The six sample points and the region spanned by them is shown below.



The elongatedness of the region can be obtained straight from the eigenvalues λ_1 and λ_2. Since they correspond to variances of the sample points along the principal components, the elongatedness is obtained from their ratio:

$$E = \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_2}} = \sqrt{\frac{3}{1/3}} = 3.$$

So the elongatedness is now 3:1. (The square roots are needed to obtain the standard deviations from the variances.)

Exercise 5:

All answers available from lectures or labs.

Exercise 6:

Solution:

The columns of the rotation matrix ${}^{E}_{C}R$ are the unit vectors of {C} with respect to {E}.
The X axis of the camera is $[\,1, 0, 0\,]^T$ with respect to {E}.
The Y axis of the camera is $[\,0, -0.707, -0.707\,]^T$ with respect to {E}.
The Z axis of the camera is $[\,0, 0.707, -0.707\,]^T$ with respect to {E}.

So

$$ {}^{E}_{C}T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -0.707 & 0.707 & 10 \\ 0 & -0.707 & -0.707 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

It is also possible to do this one by plugging into the formula for the rotation matrix for XYZ angles … we have a single rotation of 135 degrees about the x axis.

Exercise 7 (based on Exercise lecture 2):

Answer: We see that $g * g = \frac{1}{4}[1, 2, 1]$, thus $g * g * g * g = \frac{1}{16}[1, 4, 6, 4, 1]$ and $g_*^5 = \frac{1}{32}[1, 5, 10, 10, 5, 1]$. Therefore, if $k = \frac{1}{32}$ we have $h = g_*^5$, i.e. five convolutions with the g kernel yields the h kernel for the given k-value.

Preparation for lab 3

(a) Derive a mask that approximates the first partial derivative in the x-direction when convolved with an image. **Answer:** $d_x = \frac{1}{2}[1, -1]$.

(b) Derive a mask, d_{xxx} for generating the third order derivative using the masks $d_x = 1/2(1, 0, -1)$, and $d_{xx} = (1, -2, 1)$ corresponding to the first and second order derivatives.

Answer: $d_{xxx} = d_x * d_{xx} = \frac{1}{2}[1, -2, 0, 2, -1]$.