Advanced graph theory: CS60047: Autumn 2022

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August 13, 2022

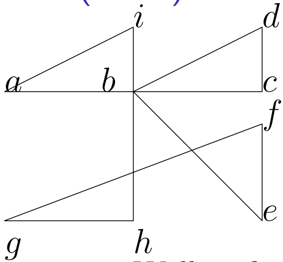
Existence of spanning bipartite subgraphs of high vertex degrees [GGL95]

Mantel's theorem

Walks and cycles

- A *walk* is just a finite sequence of vertices where consecutive vertices are connected by an edge. So, vertices and edges can repeat. It is not difficult to show by induction that any closed walk of odd length at least three must contain an odd cycle. See Lemma 1.6.1 of [Sur10]. See Figure 1.

Walks and cycles (cont.)



Walk $a, b, c, d, b, e, f, g, h, b, i, a$

Euler tour $b, i, a, b, c, d, b, e, f, g, h, b$

b has degree 6, all others degree 2

Figure: 1: Walks and cycles

Bipartite graph cycles

- We also note that a non-trivial graph is bipartite if and only if it has no odd cycles (Theorem 1.6.4 of [Sur10]).
- We can show the only if part by showing that every closed cycle $v_0 v_1 v_2 \ldots v_p = v_0$ will have an even size p. If V_1 and V_2 are the two partites then without loss of generality let us assume that $p_0 \in V_1$. Then $v_1, v_3, \ldots \in V_2$ and $v_0 = v_p, v_2, \ldots \in V_1$. Thus p is even.

Bipartite graph cycles (cont.)

- For the if-part, we assume that all cycles are even. From an arbitrary vertex $u \in V$, in the simply connected graph, we define sets V_1 (resp. V_2) of vertices of even (odd) distances from u. Now, suppose we have an edge connecting $v, w \in V_1$ then the shortest path from v to u appended by the shortest path from u to w and then the edge vw will form an odd cycle.

Vertex and edge connectivity

- *Edge connectivity* $\lambda(G)$ is at most the minimum degree $\delta(G)$ in a simple connected graph G, because, by simply deleting as few as $\delta(G)$ edges we can disconnect the graph.
- Disconnecting means creating at least two components.
- A graph G on at least two vertices is k-edge-connected if any two vertices are connected by at least k edge-disjoint paths, and k-connected if any two vertices are connected by at least k internally-disjoint paths.
- So, for a k-edge-connected graph G, $\lambda(G) \geq k$.

Vertex and edge connectivity (cont.)

- A graph on one vertex is defined to be both k-edge-connected and k-connected for $k = 0, 1$, but not for $k > 2$.
- Thus every graph is 0-connected, a graph is 1-connected if and only if it is connected.
- A graph G is m-connected if the vertex connectivity $\kappa(G) \geq m$.
- Also, since internally-disjoint paths are edge-disjoint, k-connected graphs are k-edge-connected.
- Do we need to delete more than $\delta(G)$ vertices to disconnect a simple connected graph?

Vertex and edge connectivity (cont.)

- We show that *vertex connectivity* $\kappa(G)$ is at most $\lambda(G)$. See Theorem 3.3.1 in [Sur10].
- For trivial or disconnected graphs both connectivities are zero.
- If G is connected but has a cut edge e then $\lambda(G) = 1$, and additionally if $G = K_2$ then $\kappa(G) = 1$. Otherwise, at least one end of e has degree at least 2 and thus will be a cut vertex yielding $\kappa(G) = 1$.
- Now if $\lambda(G) \geq 2$, then after removing some $\lambda(G) - 1$ edges we must get a graph H that must have a cut edge, say $e = uv$.

Vertex and edge connectivity (cont.)

- Since uv survives as a cut edge, in the connected graph H, we can now choose and delete one vertex (which is neither u nor v) from each of the $\lambda(G) - 1$ edges deleted.
- If the resulting graph is still connected then we can remove u or v additionally, thus disconnecting G with at most $\lambda(G)$ vertex deletions.

Paths and connectivity in trees

- Problem 6.24 in [Lov93] requires showing the existence of $n-k$ distinct paths of length k in a tree T with diameter $2k-3$.
- We can take the longest path P of $2k-2$ vertices $x_1, x_2, \ldots, x_{2k-2}$ in T, and consider distinct paths of length k from x_1 to x_{k+1}, x_2 to x_{k+2}, ..., and from x_{k-2} to x_{2k-2}. These are $k-2$ distinct paths in P.
- We can also find $n-2k-2$ distinct paths in T of length k, starting at each vertex outside P. This makes a total of $n-k$ paths.

Connectivity of the complement graph

- We know that undirected graphs have edges and therefore there may exist paths connecting vertices.
- In case there is no path connecting two arbitrary vertices u and v in an undirected simple graph G, the complement graph G' will contain the edge $\{u, v\}$ if u and v are not connected by an edge in G.
- However, if there is an edge between u and v in G, then these two vertices will not be directly connected in G'. Note that even in this case, will the two vertices be connected by a path in G'?

Connectivity of the complement ...

- So, we ask whether the complement of a simple disconnected graph must be connected.
- Let G be a simple disconnected graph and $u, v \in V(G)$. If u and v belong to different components of G, then clearly the edge $uv \in G'$, yielding a trivial path connecting the two vertices.

Connectivity of the complement ... (cont.)

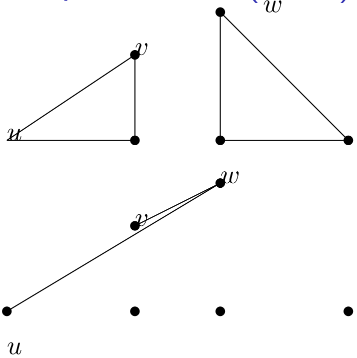


Figure: 2: Connectivity of the complement graph

- If u and v belong to the same component of G but are not

Degrees and connectivity

- It is interesting to see what happens if both the degree of each vertex as well as the girth of the graph, are both $k > 3$.
- In this case we show that there would be at least $2k$ vertices in the graph.
- For a vertex v, let K be the set of k neighbours of v.
- Take a vertex $w \in K$.
- If $x \neq v$ is a neighbour of w then x cannot be in the set K because that would yield a triangle, contradicting the fact that the girth is $k \geq 3$.

Degrees and connectivity (cont.)

- So, all the $k-1$ neighhbours of w (other than v) are none of the vertices in K.
- Therefore, we already have $|K|+1+(k-1)=2k$ vertices in the graph.

Girth four regular graphs

- If we have high vertex degrees then we will have at least a proportionate number of edges even with bounded girth like four.
- Suppose we have k-regular graph of girth four.
- Let u have the set $N(u)$ as its k neighbours.
- With the same reasoning as in the case of large girth, we can say that for $v, w \in N(u)$, vw is not an edge.
- So, for one $v \in N(u)$, its $k-1$ neighbours other than u are not in $N(u)$, already account for $|\{u\}| + |N(u)| + k - 1 = 2k$ vertices.

Girth four regular graphs (cont.)

- Suppose above $N(u) = \{v_1, v_2, ..., v_k\}$.
- Then we have edges $N(v_1) = \{u, w_1, w_2, ..., w_{k-1}\}$.
- We can connect each of $v_2, v_3, ..., v_k$ as well to $N(v_1) \setminus \{u\}$, whereas u is already connected to $N(u)$.
- This gives the complete bipartite graph with partites $N(u)$ and $N(v_1)$.

Distinct degrees

- If all the vertices have disinct degrees in a simple connected undirected graph then these must be 0, 1, 2, ..., and $n-1$, respectively.
- This implies that one vertex must be connected to all the $n-1$ other vertices, including the one with degree zero, a contradiction because we assumed the graph was connected.

High degree, longer path

- When there is much connectivity, like if each vertex has at least k neighbours, then we can also have long paths.
- If we take a maximal path starting at u, then all the k neighbours of u must be on this path because we cannot extend this maximal path by connecting a neighbour of u, thus requiring this maximal path to be of length at least k.
- See Proposition 1.2.28 in [Wes00].

Connectivity with edges

- Suppose we have an even graph. Can this graph have a cut edge?
- If it were so then dropping this edge would render two connected components in the disconnected graph to have odd degrees.
- However, no connected subgraph can have just a single odd degree vertex.
- Adding edges can increase connectivity; in other words, an edge added to a graph $G(V, E)$ may reduce the $c(G)$, that is, the number of connected components by at most 1.

Connectivity with edges (cont.)

- So, by induction we can show that $c(G)$ is lower bounded by $|V| - |E|$.
- A graph with two edges has exactly $|V| - 2$ connected components. A graph with one edge has exactly $|V| - 1$ components but a graph with three edges may have $|V| - 2$ or $|V| - 3$ components.

Degrees and independence

- If $\alpha(G)$ is the maximum size of an independent set in a simple graph $G(V, E)$ then each of the $|V| - \alpha(G)$ vertices have some edges coming out, such that the sum of degrees of these vertices is at least the number $|E|$ of edges of G.
- Thus $\Delta(G)(|V| - \alpha(G)) \geq |E|$, where $\Delta(G)$ is the maximum degree of a vertex in G.
- Thus we have $\alpha(G) \leq |V| - \frac{|E|}{\Delta(G)}$.
- In a regular graph $|E| = \frac{\Delta(G)}{2}|V|$, whence $\alpha(G) \leq \frac{|V|}{2}$.

Proof of Hall's theorem using the Konig-Egervary theorem [Die17]

Matchings and factors

- A *matching* M is an independent set of edges in a graph $G(V, E)$ [Die17].
- So, no vertex in the graph will be in more than one edge of a matching.
- A k-factor of G is a k-regular spanning subgraph of G.
- So, a subgraph is a 1-factor if and only if it is a matching for the entire set of vertices in the graph, or in other words, it is a *perfect matching*.
- Such perfectly matched graphs must therefore have an even number of vertices.

Proof of Hall's theorem using the Konig-Egervary theorem [Die17]

Matchings and factors (cont.)

- Note that even non-bipartite graphs may have perfect matchings.
- We can characterize general graphs that have a perfect matching by Tutte's condition, as per Tutte's theorem (Theorem 3.3.3 in [Wes00]).

Proof of Hall's theorem using the Konig-Egervary theorem [Die17]

Berge's theorem

- A matching M in a graph G is a maximum matching in G if and only if G has no M-augmenting path. This is a result from 1957.
- Let p denote the statement "M is a maximum matching", and q denote the statement "there is no M-augmenting path".
- Then the statement p if and only if q has two implications.
- The if-part is $q \implies p$, and the only-if part is $p \implies q$.
- To show the only-if part we show therefore that $\neg p \vee q$ holds or $p \wedge \neg q$ does not hold.
- Suppose a maximum matching M has an M-augmenting path.

Proof of Hall's theorem using the Konig-Egervary theorem [Die17]

Berge's theorem (cont.)

- Then we can demonstrate a larger matching, contradicting the assumption that M is a maximum matching. This completes the only-if part.
- For the if-part we show $\neg q \vee p$ holds or $q \wedge \neg p$ does not hold.
- So, to show the if-part we show the impossibility of M being not a maximum matching as well as that there are no M-augmenting paths.
- Assume that there is no M-augmenting path, but M is not a maximum matching. We show that this is impossible as follows.

Proof of Hall's theorem using the Konig-Egervary theorem [Die17]

Berge's theorem (cont.)

- Let the maximum matching be M' and F be the symmetric difference betweem M and M'.
- Since $|M'| > |M|$, at least one component C of F must have more edges from M'.
- Since all cycles are of even length, and edges alternate between M and M' in F, C must therefore be a path and not a cycle in F.
- The two extreme edges in C must thus be from M', yielding an M-augmenting path.
- This completes the proof of the if-part of Berge's theorem

Proof of Hall's theorem using the Konig-Egervary theorem [Die17]

Berge's theorem (cont.)

- Now we formally state the definition of the symmetric difference of two matchings and study its properties.
- If M and M' are matchings, then $M \Delta M' = (M \cup M') \setminus (M \cap M')$.
- We show that every component of this symmetric difference of is a path or an even cycle.
- At most one edge of M and at most one edge of M' is incident on any vertex v.
- So maximum degree of any node in F is 2. So, components of F must be paths or cycles.

Proof of Hall's theorem using the Konig-Egervary theorem [Die17]

Berge's theorem (cont.)

- Also, edges in a path or cycle will alternate between edges of $M \setminus M'$ and $M' \setminus M$.
- So all cycles must be even.

Proof of Hall's theorem using the Konig-Egervary theorem [Die17]

Proof of Hall's theorem using alternating paths

- For a bipartite graph $G(X \cup Y, E)$, suppose the neighbourhood $N(S)$ of any subset of the partite X is at least as larger as S, then we must show that a matching that covers the whole of X. This is known as Hall's theorem. (See Theorem 3.1.11 in [Wes00])
- Equivalently, we can establish sufficiency by demostrating the contrapositive; if the maximum matching M fails to match a vertex say $u \in X$ then we must demonstrate a subset S of X whose neighbourhood $N(S)$ is smaller than $|S|$.

Proof of Hall's theorem using the Konig-Egervary theorem [Die17]

Proof of Hall's theorem using alternating paths (cont.)

- Towards this goal, we find the vertex subsets S(*resp.*, T) of X(*resp.*, Y) that are in M-alternating paths starting at the unmatched vertex u of X. Here $u \in S$ and $T = N(S)$.
- The unmatched vertex u cannot reach out to opposite side vertices outside M, as in that case u would match a vertex outside M and create a matching larger than M.
- Suppose we show that M matches $S \setminus \{u\}$ to T. Then we would have shown that $N(S) = T$ has only $|S| - 1$ elements, violating Hall's condition (given that we had started with the assumption that the maximum matching M failed to match $u \in X$).

Proof of Hall's theorem using the Konig-Egervary theorem [Die17]

Proof of Hall's theorem using alternating paths (cont.)

- Now we show how M matches the whole of $S \setminus \{u\}$ to T.
- Each vertex of $S \setminus \{u\}$ must be reached from a vertex of T in some M-alternating path via an edge of M.
- Also, M being a maximum matching, by Berge's theorem we do not have an M-augmenting path.
- So, the whole of T is saturated.
- Thus $T = N(S)$, with M defining the bijective mapping.
- Now as an application of Hall's theorem we can show that a k-regular bipartite graph has a perfect matching i.e., a 1-factor.

Proof of Hall's theorem using the Konig-Egervary theorem [Die17]

Proof of Hall's theorem using the Konig-Egervary theorem [Die17]

- The theorem of Konig-Egervary is a well-known duality result stating that the size of the maximum matching is the same as the size of the minimum vertex cover in a bipartite graph.
- Let $A' \subseteq A$ and $B' \subseteq B$ be the two mutually disjoint subsets of V constituting the minimum vertex cover U for $G(V, E)$.
- Consider $A \setminus A'$ and $B \setminus B'$.
- These sets do not induce any edges in G and therefore constitute a maximum independent set, because $A' \cup B$' is the minimum vertex cover.

Proof of Hall's theorem using the Konig-Egervary theorem [Die17]

Proof of Hall's theorem using the Konig-Egervary theorem [Die17] (cont.)

- So, $|N(A \setminus A')| \leq B'$.
- Now let us now assume that G does not have a matching for the whole of A, implying $|A'| + |B'| = |U| < |A|$, or $|A| - |A'| > |B'|$, and thus $|A \setminus A'| > |B'| \geq |N(A \setminus A')|$.
- This establishes the contrapositive for the sufficiency condition for Hall's theorem with the subset $A \setminus A'$ as witness.

Proof of Hall's theorem using the Konig-Egervary theorem [Die17]

Proof of Hall's theorem using the Konig-Egervary theorem [Die17] (cont.)

- Here, the strict inequality $|U| < |A|$ holds because the maximum matching size is the same as the size $|U|$ of the minimum vertex cover by the Konig-Egervary theorem, and at least one vertex in A is not matched in any maximum matching.

Proof of Hall's theorem using the Konig-Egervary theorem [Die17]

Notations and definitions about independence and covering

- For the sake of some notation, let us use
  $\alpha(G)$ to denote the size of the maximum independent (stable) set in a simple connected graph $G(V, E)$,
  $\beta(G)$ to denote the size of the minimum vertex cover,
  $\alpha'(G)$ for the size of the maximum matching, and
  $\beta'(G)$ for the size of the minimum edge cover.
- We know that $\alpha(G) + \beta(G) = |V| = n$ for any graph.
- For bipartite graphs we know by the Konig-Egervary theorem that $\beta(G) = \alpha'(G)$.

Proof of Hall's theorem using the Konig-Egervary theorem [Die17]

Notations and definitions about independence and covering (cont.)

- For general graphs $\beta(G) \geq \alpha'(G)$ because we need to cover each edge of a matching by at least one vertex.
- We also know that for any graph, no edge can cover two vertices of an independent set.
- So, we can write $\beta'(G) \geq \alpha(G)$.
- Further, note that by Gallai's theorem we know that $\alpha'(G) + \beta'(G) = |V| = n$ for any connected graph.
- To show that $\alpha(G) + \beta(G) = |V| = n$ for any connected graph, we argue as follows.

Proof of Hall's theorem using the Konig-Egervary theorem [Die17]

Notations and definitions about independence and covering (cont.)

- If T is an independent set, then edges can have at most one endpoint in T.
- So each edge has at least one endpoint in $V \setminus T$, making it a vertex cover.
- Also, if $V \setminus T$ is a vertex cover, T will not have both endpoints of any edge.
- Study exercise: Proof of Gallai's theorem.

Proof of the Konig-Egervary theorem

- It is sufficient to show that for any minimum cardinality vertex cover Q of $G(X \cup Y, E)$, we can demonstrate a matching M of size $\beta(G) = |U|$. Why?
- (We know that $\beta(G) \geq \alpha'(G)$. We need at least as many vertices as the number of edges in the maximum matching in order to cover all edges.)
- Consider the partition of any minimum cardinality vertex cover Q into $R = Q \cap X$ and $T = Q \cap Y$.
- Consider (edge-disjoint) subgraphs H and H' induced by $R \cup (Y \setminus T)$ and $T \cup (X \setminus R)$.

Proof of the Konig-Egervary theorem (cont.)

- Using Hall's theorem we show that H has a matching for R into $Y \setminus T$ and H' has a matching for T into $X \setminus R$.
- So, a matching of size $|Q|$ from H and H' for the whole of G can be demonstrated.
- Since $R \cup T$ is a vertex cover for G, no edges exist between $Y \setminus T$ and $X \setminus R$.
- For any $S \subseteq R$, consider $N_H(S) \subseteq Y \setminus T$. Can the vertex cover $R \cup T$ be replaced by $(R \setminus S) \cup N_H(S) \cup T$?
- Since this can never shrink the minimum vertex cover Q, we have Hall's condition $|N_H(S)| \geq |S|$ for any $S \subseteq R$.
- So, R matches into $Y \setminus T$ by Hall's theorem. See [Wes00].

Proof of the Konig-Egervary theorem (cont.)

- Similarly we can show that T matches into $X \setminus R$.

Large number of edges lead to subgraphs with proportionate minimum vertex degree

- We observe that if G is a graph on n vertices with more than $(c-1)n$ edges, where c is a positive integer, then G has a subgraph H of minimum degree at least c [GGL95].
- This is so as any minimal subgraph H with more than $(c-1)v(H)$ edges has the necessary property of minimum vertex degree at least c.

Large number of edges lead to subgraphs with proportionate minimum vertex degree (cont.)

- If H had a vertex v of degree at most $c-1$, then subgraph $H \setminus \{v\}$ would contradict the choice of H because in that case $H \setminus \{v\}$, and not H would be the minimal subgraph with the required property.

Extremal results: Spanning subgraphs of high vertex degrees

- In similar vein, we can show that every graph G has a bipartite spanning subgraph B such that $degree_B(v) \geq \frac{degree_G(v)}{2}$ for all vertices v [GGL95].
- We note that any bipartite spanning subgraph $B(X, Y)$ with the maximum number of edges has this property.

Extremal results: Spanning subgraphs of high vertex degrees (cont.)

- Suppose B had a vertex v of degree less than $\frac{degree_G(v)}{2}$, and without loss of generality $v \in X$, then the bipartite spanning subgraph with bi-partition $(X \setminus \{v\}, Y \cup \{v\})$ would contradict the choice of B because this modified graph would have more edges.
- Such results may be required in the proofs of extremal properties where the number of edges is only of some modest smaller magnitude, serving a required purpose, even by restricting the class of graphs under consideration to bipartite graphs of large degree.

Extremal results: Spanning subgraphs of high vertex degrees (cont.)

- In the breadth-first search trees of bipartite graphs of large degree, the sets reachable grow rapidly.

Extremal results: Mantel's theorem

- Consider the maximum independent set A of $G(V, E)$.
- Note that the neighbourhood $N(x) \subseteq V$ of each vertex $x \in V$ in a triangle-free graph G is an independent set, each of which is at most as big as A.
- Also note that all edges of G land in the set $B = V \setminus A$, which is the minimum vertex cover for G.
- Therefore, $m = |E| = \Sigma_{x \in B} d(x) \leq |B||A| \leq (\frac{|A|+|B|}{2})^2 = \frac{n^2}{4}$.
- This is because geometric mean never exceeds arithmetic mean.
- So, triangle-free graphs would have at most $\frac{n^2}{4}$ edges.

Extremal results: Mantel's theorem (cont.)

- Therefore, a graph with more than $\frac{n^2}{4}$ edges must admit a triangle.
- This bound is attained by $K_{\frac{n}{2},\frac{n}{2}}$.

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