

Gate EC 36.2023

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Question 36.2023:

A random variable X , distributed normally as $N(0,1)$, undergoes the transformation $Y=h(X)$, given in Fig. ???. The form of probability density function of Y is (In the options given below, a, b, c are non-zero constants and $g(y)$ is piece-wise continuous function).

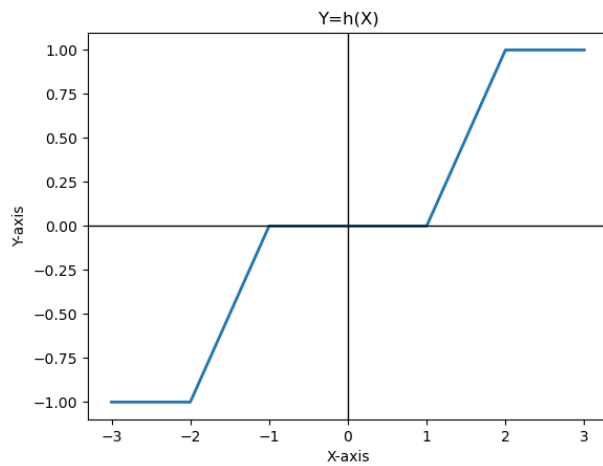


Fig. 0

Solution:

$$Y = h(X) \quad (1)$$

$$Y = \begin{cases} -1 & X < -2 \\ X + 1 & -2 < X < -1 \\ 0 & -1 < X < 1 \\ X - 1 & 1 < X < 2 \\ 1 & X > 2 \end{cases} \quad (2)$$

$$p_Y(y) = \begin{cases} \Pr(X \leq -2) & Y = 1 \\ \Pr(-1 < X \leq 1) & Y = 0 \\ \Pr(X > 2) & Y = -1 \end{cases} \quad (3)$$

$$= \begin{cases} 1 - Q(-2) = Q(2) & Y = 1 \\ Q(-1) - Q(1) = 1 - 2Q(1) & Y = 0 \\ Q(2) & Y = -1 \end{cases} \quad (4)$$

For $-1 < Y < 0$

$$Y = X + 1 \quad (5)$$

$$\Pr(Y \leq y) = \Pr(X \leq y - 1) \quad (6)$$

$$= 1 - Q(y - 1) \quad (7)$$

$$= Q(1 - y) \quad (8)$$

$$p_Y(y) = \frac{d}{dy} Q(1 - y) \quad (9)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{(1-y)^2}{2}} \quad (10)$$

For $0 < Y < 1$

$$Y = X - 1 \quad (11)$$

$$\Pr(Y \leq y) = \Pr(X \leq y + 1) \quad (12)$$

$$= 1 - Q(y + 1) \quad (13)$$

$$= Q(-1 - y) \quad (14)$$

$$p_Y(y) = \frac{d}{dy} Q(-1 - y) \quad (15)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{(1+y)^2}{2}} \quad (16)$$

1) $a\delta(y - 1) + b\delta(y + 1) + g(y)$

2) $a\delta(y + 1) + b\delta(y) + c\delta(y - 1) + g(y)$

3) $a\delta(y + 2) + b\delta(y) + c\delta(y - 2) + g(y)$

4) $a\delta(y + 2) + b\delta(y - 2) + g(y)$

Therefore,

$$p_Y(y) = \begin{cases} Q(2) & Y = -1 \\ \frac{1}{\sqrt{2\pi}} e^{-\frac{(1-y)^2}{2}} & -1 < Y < 0 \\ 1 - 2Q(1) & Y = 0 \\ \frac{1}{\sqrt{2\pi}} e^{-\frac{(1+y)^2}{2}} & 0 < Y < 1 \\ Q(2) & Y = 1 \end{cases} \quad (17)$$

$$p_Y(y) = \delta(y+1)Q(2) + \delta(y)(1-2Q(1)) + \delta(y-1)Q(2) + \frac{1}{\sqrt{2\pi}} e^{-\frac{(1-y)^2}{2}} (u(y+1) - u(y)) + \frac{1}{\sqrt{2\pi}} e^{-\frac{(1+y)^2}{2}} (u(y) - u(y-1)) \quad (18)$$

which is in the form,

$$p_Y(y) = a\delta(y+1) + b\delta(y) + c\delta(y-1) + g(y) \quad (19)$$

where,

a,b,c are non-zero constants

g(y) is a piece-wise continuous function