Gate EC 36.2023

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Question 36.2023:
A random variable X, distributed normally as N(0,1), undergoes the transformation Y=h(X), given in Fig. 0. The form of probability density function of Y is (In the options given below, a,b,c are non-zero constants and g(y) is piece-wise continuous function).

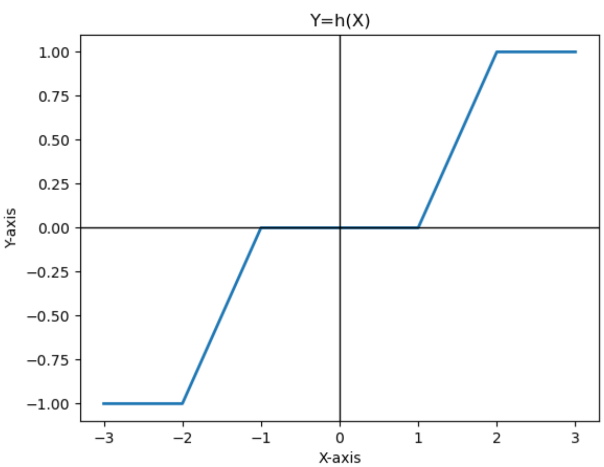

Fig. 0

1) $a\delta(y-1)+b\delta(y+1)+g(y)$
2) $a\delta(y+1)+b\delta(y)+c\delta(y-1)+g(y)$
3) $a\delta(y+2)+b\delta(y)+c\delta(y-2)+g(y)$
4) $a\delta(y+2)+b\delta(y-2)+g(y)$

Solution:

$$Y = h(X) \tag{1}$$

$$Y = \begin{cases} -1 & X < -2 \\ X+1 & -2 < X < -1 \\ 0 & -1 < X < 1 \\ X-1 & 1 < X < 2 \\ 1 & X > 2 \end{cases} \tag{2}$$

$$p_Y(y) = \begin{cases} \Pr(X \le -2) & Y = 1 \\ \Pr(-1 < X \le 1) & Y = 0 \\ \Pr(X > 2) & Y = -1 \end{cases} \tag{3}$$

$$= \begin{cases} 1 - Q(-2) = Q(2) & Y = 1 \\ Q(-1) - Q(1) = 1 - 2Q(1) & Y = 0 \\ Q(2) & Y = -1 \end{cases} \tag{4}$$

For $-1 < Y < 0$

$$Y = X + 1 \tag{5}$$
$$\Pr(Y \le y) = \Pr(X \le y - 1) \tag{6}$$
$$= 1 - Q(y-1) \tag{7}$$
$$= Q(1-y) \tag{8}$$
$$p_Y(y) = \frac{d}{dy} Q(1-y) \tag{9}$$
$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{(1-y)^2}{2}} \tag{10}$$

For $0 < Y < 1$

$$Y = X - 1 \tag{11}$$
$$\Pr(Y \le y) = \Pr(X \le y + 1) \tag{12}$$
$$= 1 - Q(y+1) \tag{13}$$
$$= Q(-1-y) \tag{14}$$
$$p_Y(y) = \frac{d}{dy} Q(-1-y) \tag{15}$$
$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{(1+y)^2}{2}} \tag{16}$$

Therefore,

$$p_Y(y) = \begin{cases} Q(2) & Y = -1 \\ \frac{1}{\sqrt{2\pi}} e^{-\frac{(1-y)^2}{2}} & -1 < Y < 0 \\ 1 - 2Q(1) & Y = 0 \\ \frac{1}{\sqrt{2\pi}} e^{-\frac{(1+y)^2}{2}} & 0 < Y < 1 \\ Q(2) & Y = 1 \end{cases} \tag{17}$$

$$p_Y(y) = \delta(y+1)Q(2) + \delta(y)\left(1 - 2Q(1)\right) + \delta(y-1)Q(2) + \frac{1}{\sqrt{2\pi}} e^{-\frac{(1-y)^2}{2}} \left(u(y+1) - u(y)\right) + \frac{1}{\sqrt{2\pi}} e^{-\frac{(1+y)^2}{2}} \left(u(y) - u(y-1)\right) \tag{18}$$

which is in the form,

$$p_Y(y) = a\delta(y+1) + b\delta(y) + c\delta(y-1) + g(y) \tag{19}$$

where,
a,b,c are non-zero constants
g(y) is a piece-wise continuous function