# Session – 2: Linear Programming Problem by Graphical Method

**Solution:**

A set of values of decision variables $x_j$, $j$=1,2,3,…, $n$ satisfying all the constraints of the problem is called a solution to that problem.

**Feasible solution:**

A set of values of the decision variables that satisfies all the constraints of the problem and non-negativity restrictions is called a feasible solution of the problem.

**Optimal solution:**

Any feasible solution which maximizes or minimizes the objective function is called an optimal solution.

**Feasible region:**

The common region determined by all the constraints including non-negative constraints $x_j \geq 0$ of a linear programming problem is called the feasible region (or solution region) for the problem.

**Solution of LPP by graphical method**

After formulating the linear programming problem, our aim is to determine the values of decision variables to find the optimum (maximum or minimum) value of the objective function. Linear programming problems which involve only two variables can be solved by graphical method. If the problem has three or more variables, the graphical method is impractical.

**The major steps involved in this method are as follows**

(i) **State the problem mathematically.**

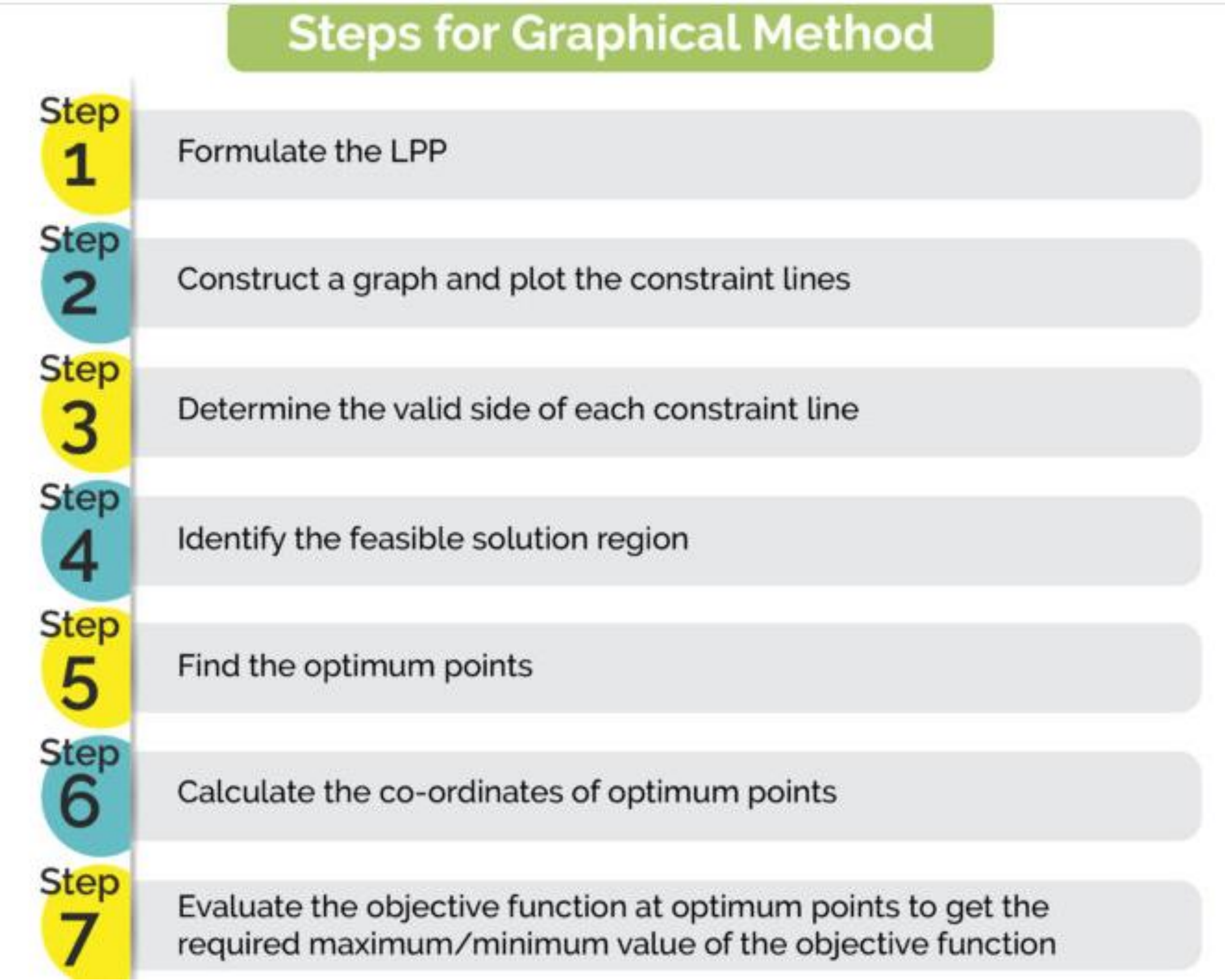
(ii) **Write all the constraints in the form of equations and draw the graph.**

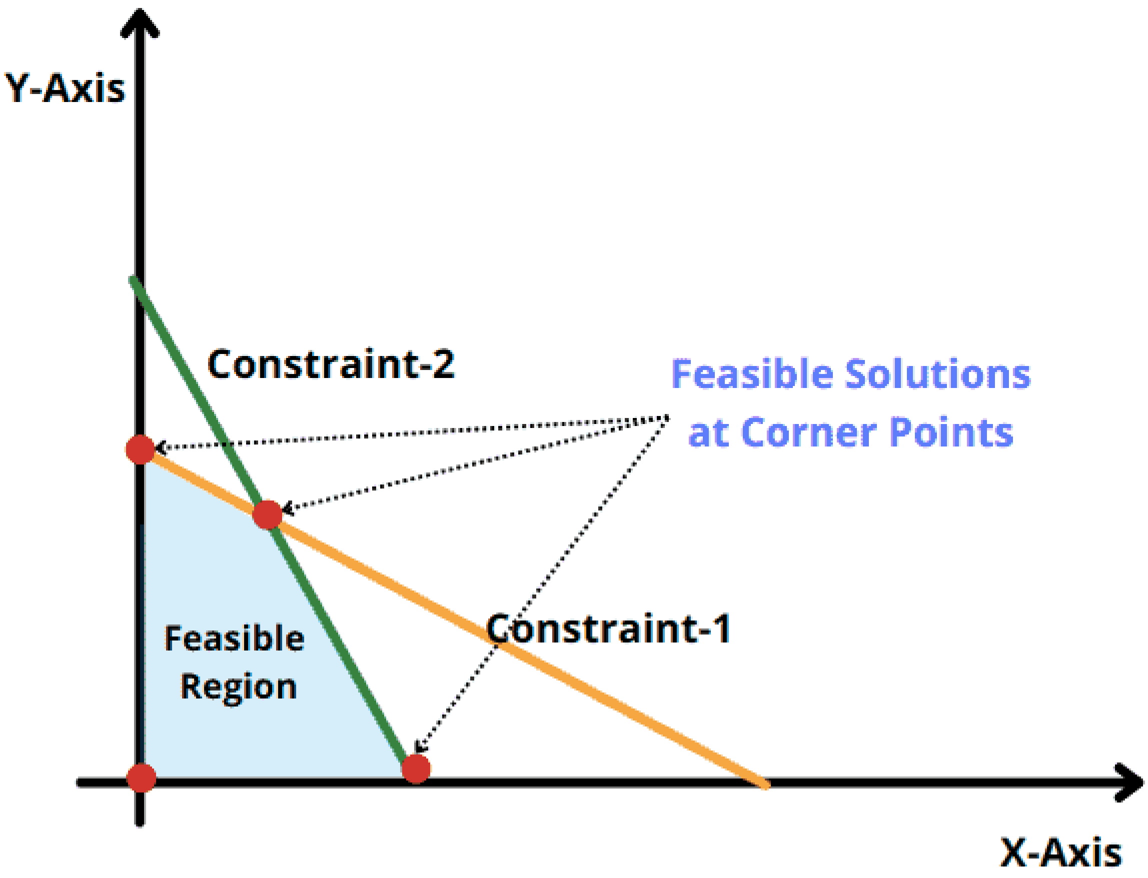
(iii) **Find the feasible region.**

**(iv)** **Find the coordinates of each vertex (corner points) of the feasible region. The coordinates of the vertex can be obtained either by inspection or by solving the two equations of the lines intersecting at the point.**

**(v)** **By substituting these corner points in the objective function, we can get the values of the objective function.**

**(vi)** **If the problem is maximization, then the maximum of the above values is the optimum value. If the problem is minimization, then the minimum of the above values is the optimum value.**






**CaseStudy1:**

Solve the following LPP

**Maximize $Z = 2x_1 + 5x_2$**

**subject to the conditions $x_1 + 4x_2 \leq 24$**

**$3x_1 + x_2 \leq 21$**

**$x_1 + x_2 \leq 9$ and $x_1, x_2 \geq 0$**

***Solution:***

First, we have to find the feasible region using the given conditions.

Since both the decision variables $x_1$ and $x_2$ are non-negative, the solution lies in the first quadrant.

Write all the inequalities of the constraints in the form of equations.

Therefore, we have the lines $x_1 + 4x_2 = 24$; $3x_1 + x_2 = 21$; $x_1 + x_2 = 9$ $x_1 + 4x_2 = 24$ is a line passing through the points (0, 6) and (24, 0). [(0,6) is obtained by taking $x_1=0$ in $x_1 + 4x_2 = 24$, (24, 0) is obtained by taking $x_2 = 0$ in $x_1 + 4x_2 = 24$].

Any point lying on or below the line $x_1 + 4x_2 = 24$ satisfies the constraint $x_1 + 4x_2 \leq 24$.

$3x_1 + x_2 = 21$ is a line passing through the points (0, 21) and (7, 0). Any point lying on or below the line $3x_1 + x_2 = 21$ satisfies the constraint $3x_1 + x_2 \leq 21$.

$x_1 + x_2 = 9$ is a line passing through the points (0, 9) and (9, 0). Any point lying on or below the line $x_1 + x_2 = 9$ satisfies the constraint $x_1 + x_2 \leq 9$.

Now we draw the graph.

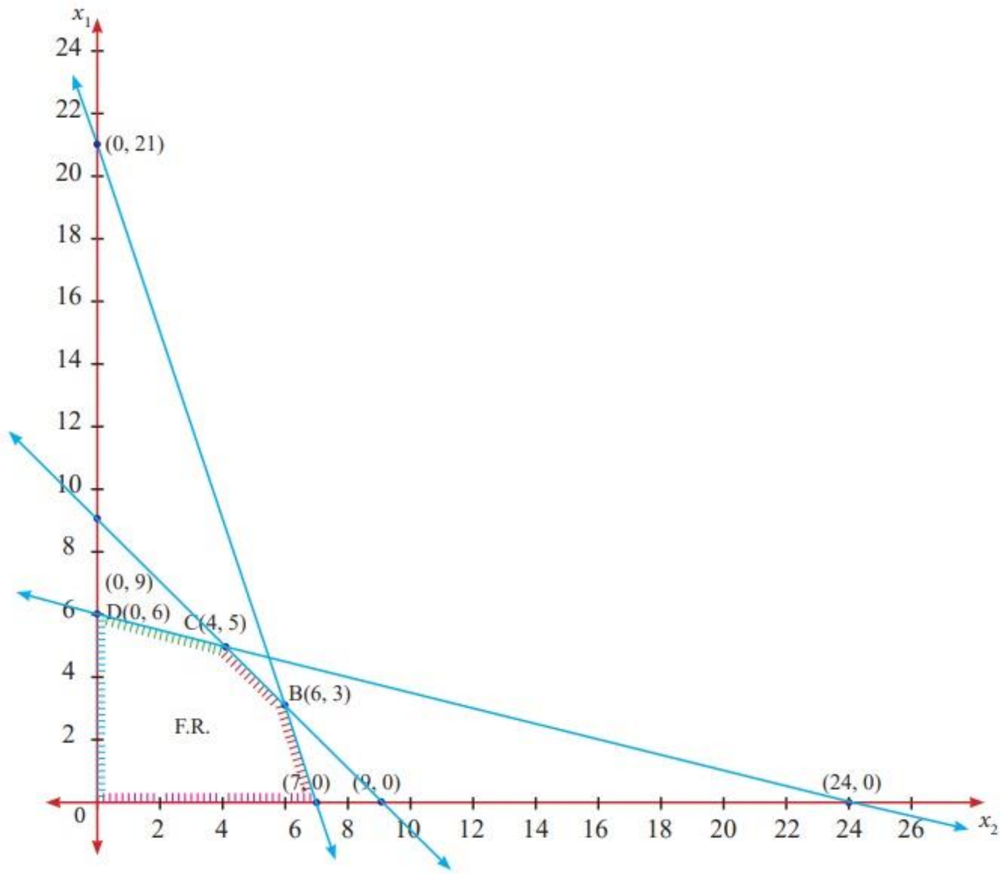



Fig 10.1

The feasible region satisfying all the conditions is OABCD. The co-ordinates of the points are O(0,0) A(7,0);B(6,3) [ the point B is the intersection of two lines $x_1 + x_2 = 9$ and $3x_1 + x_2 = 21$];C(4,5) [ the point C is the intersection of two lines

$x_1 + x_2 = 9$ and $x_1 + 4x_2 = 24$] and D(0,6).

| Corner points | $Z = 2x_1 + 5x_2$ |
|---|---|
| O(0,0) | 0 |
| A(7,0) | 14 |
| B(6,3) | 27 |
| C(4,5) | 33 |
| D(0,6) | 30 |

Table 10.2

**Maximum value of Z occurs at C. Therefore, the solution is $x_1 = 4$, $x_2 = 5$, $Z_{max} = 33$**

**CaseStudy2:**

Solve the following LPP by graphical method

**Minimize $z = 5x_1+4x_2$**

**Subject to constraints:**

**$4x_1+ x_2 \geq 40$;**

**$2x_1+3x_2 \geq 90$ and**

**$x_1, x_2 \geq 0$**

***Solution:***

Since both the decision variables $x_1$ and $x_2$ are non-negative, the solution lies in the first quadrant of the plane.

Consider the equations $4x_1+x_2 = 40$ and $2x_1+3x_2 = 90$

$4x_1+x_2 = 40$ is a line passing through the points (0,40) and (10,0). Any point lying on or above the line $4x_1+x_2= 40$ satisfies the constraint $4x_1+ x_2 \geq 40$.

$2x_1+3x_2 = 90$ is a line passing through the points (0,30) and (45,0). Any point lying on or above the line $2x_1+3x_2= 90$ satisfies the constraint $2x_1+3x_2 \geq 90$.

Draw the graph using the given constraints.

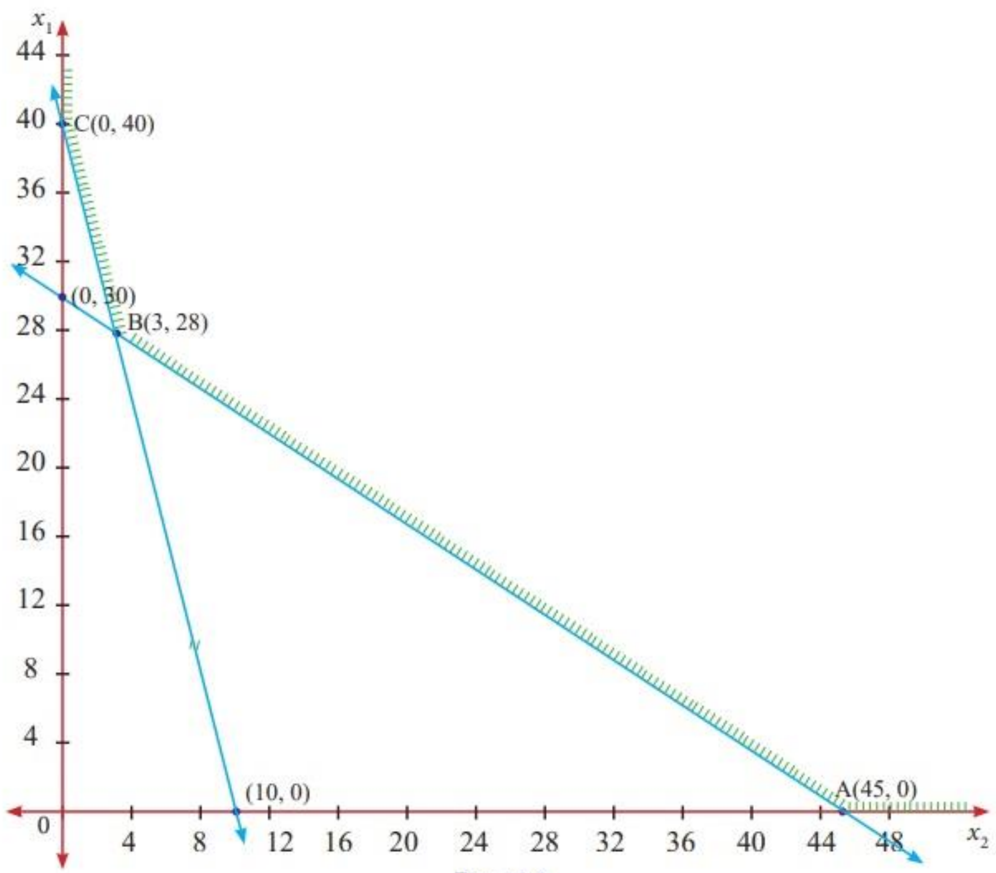



Fig 10.2

The feasible region is ABC (since the problem is of minimization type we are moving towards the origin.

| Corner points | $z = 5x_1+4x_2$ |
|---|---|
| A(45,0) | 225 |
| B(3,28) | 127 |
| C(0,40) | 160 |

Table 10.4

The minimum value of Z occurs at B (3,28).

Hence the optimal solution is $x_1 = 3$, $x_2 = 28$ and $Z_{min}=127$

**CaseStudy3:**

Solve the following LPP.

**Maximize $Z= 2x_1 + 3x_2$**

**subject to constraints:**

$x_1 + x_2 \leq 30$;

$x_2 \leq 12$;

$x_1 \leq 20$ **and**

$x_1, x_2 \geq 0$

***Solution:***

We find the feasible region using the given conditions.

Since both the decision variables $x_1$ and $x_2$ are non-negative, the solution lies in the first quadrant of the plane.

Write all the inequalities of the constraints in the form of equations.

Therefore, we have the lines

$x_1+x_2=30$; $x_2 =12$; $x_1= 20$

$x_1+x_2 =30$ is a line passing through the points (0,30) and (30,0)

$x_2 = 12$ is a line parallel to $x_1$–axis

$x_1 = 20$ is a line parallel to $x_2$–axis.

The feasible region satisfying all the conditions $x_1 + x_2 \leq 30$; $x_2 \leq 12$; $x_1 \leq 20$ and $x_1, x_2 \geq 0$ is shown in the following graph.

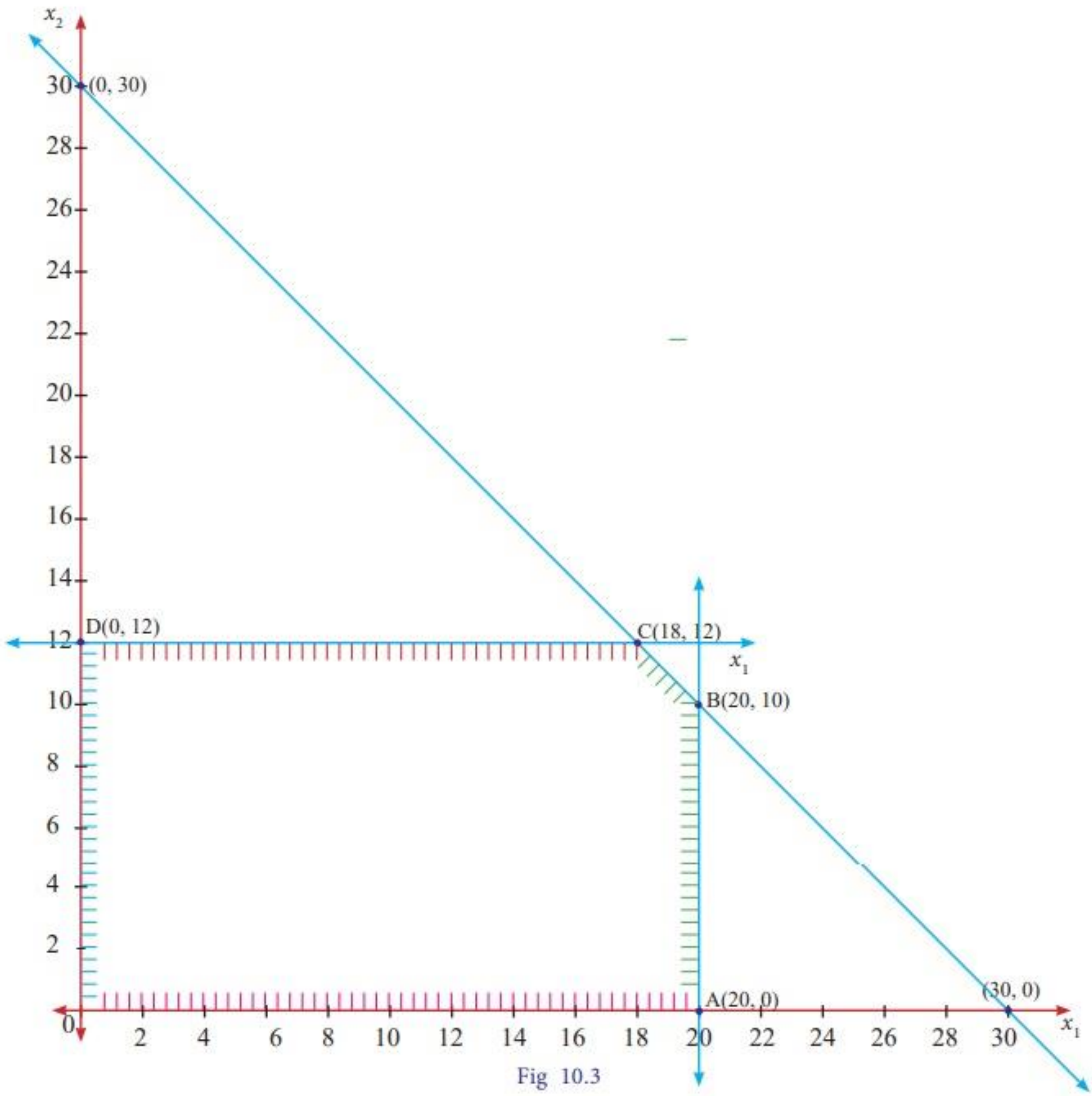



Fig 10.3

The feasible region satisfying all the conditions is OABCD.

The co-ordinates of the points are O (0,0); A (20,0); B (20,10); C (18,12) and D (0,12).

| Corner points | $Z = 2x_1 + 3x_2$ |
|---|---|
| O(0,0) | 0 |
| A(20,0) | 40 |
| B(20,10) | 70 |
| C(18,12) | 72 |
| D(0,12) | 36 |

Table 10.3

**Maximum value of $Z$ occurs at C. Therefore, the solution is $x_1 = 18$, $x_2 = 12$, $Z_{max} = 72$**

**CaseStudy4:**

**Maximize Z = $3x_1 + 4x_2$**

**subject to $x_1 - x_2 \leq -1$;**

**$-x_1 + x_2 \leq 0$ and**

**$x_1, x_2 \geq 0$**

***Solution:***

Since both the decision variables $x_1$, $x_2$ are non-negative, the solution lies in the first quadrant of the plane.

Consider the equations $x_1 - x_2 = -1$ and $-x_1 + x_2 = 0$

$x_1 - x_2 = -1$ is a line passing through the points (0,1) and (–1,0)

$-x_1 + x_2 = 0$ is a line passing through the point (0,0)

Now we draw the graph satisfying the conditions $x_1 - x_2 \leq -1$; $-x_1 + x_2 \leq 0$ and $x_1, x_2 \geq 0$

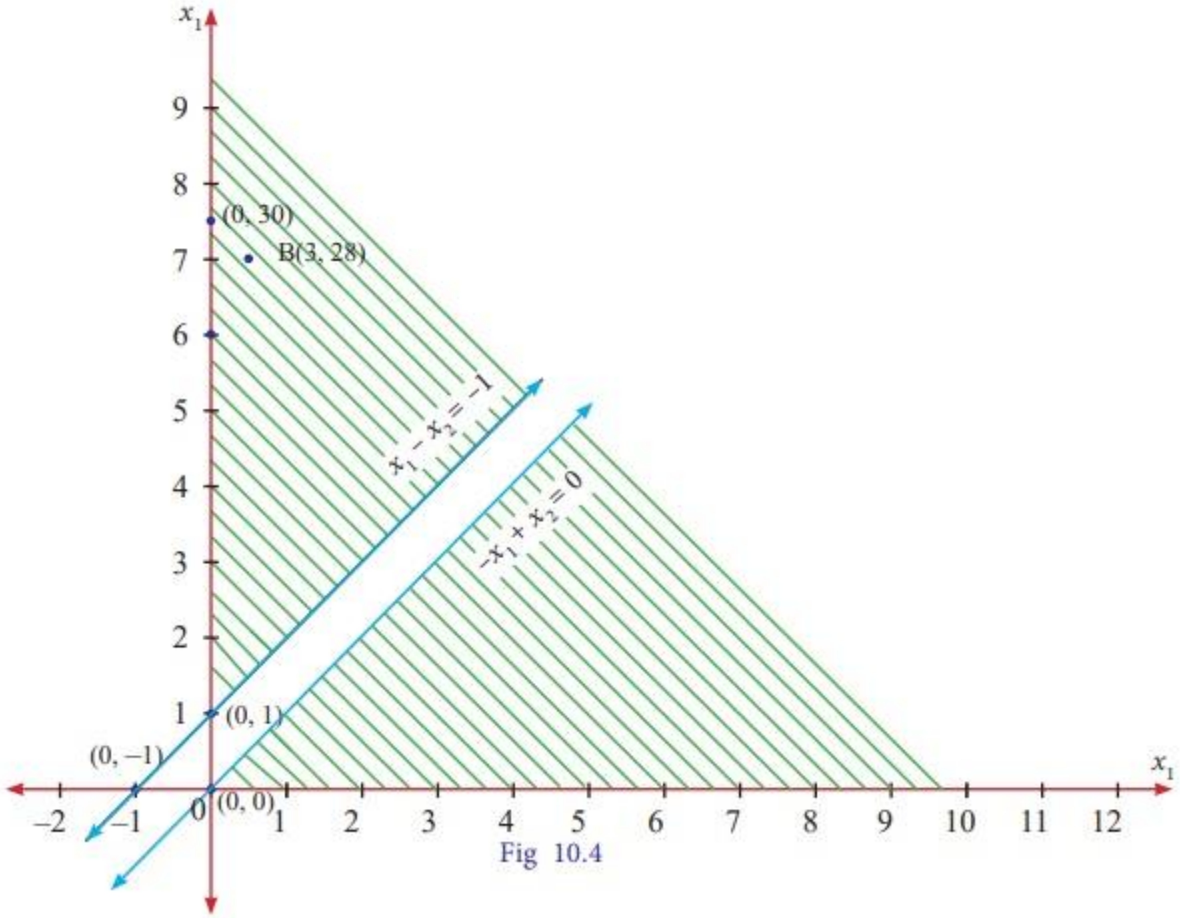



Fig 10.4

There is no common region(feasible region) satisfying all the given conditions.

**Hence the given LPP has no solution.**

# FAQs on Graphical Method

**Q1: What is the purpose of a graphical method?**

**Answer:** We use a graphical method of linear programming for solving the problems by finding out the maximum or lowermost point of the intersection on a graph between the objective function line and the feasible region.

**Q2: How do you solve the LPP with the help of a graphical method?**

**Answer:** We can solve the LPP with the graphical method by following these steps:

**1st Step:** First of all, formulate the LP problem.

**2nd Step:** Then, make a graph and plot the constraint lines over there.

**3rd Step:** Determine the valid part of each constraint line.

**4th Step:** Recognize the possible solution area.

**5thStep:** Place the objective function in the graph.

**6th Step:** Finally, find out the optimum point.

**Q3: Define the graphical method for the simultaneous equations?**

**Answer:** For graphically solving a pair of simultaneous equations, firstly we have to draw a graph of both the equations simultaneously. We have 2 straight lines crossing each other at a common point which provides the solution of this pair of equations.

**Q4: What is a graphical interpretation?**

**Answer:** A graphical interpretation proposes a number of valuable problem-solving methods. For example, finding the greatest value of a nonstop differentiable function ‘f(x)’ defined in some interval ‘an ≤ x ≤ b’.

## Exercise:

1. Solve the following linear programming problems by graphical method.

(i) Maximize Z = $6x_1 + 8x_2$ subject to constraints $30x_1+20x_2 \leq 300; 5x_1+10x_2 \leq 110$; and $x_1, x_2 \geq 0$ .

(ii) Maximize $Z = 22x_1 + 18x_2$ subject to constraints $960x_1 + 640x_2 \leq 15360$ ; $x_1 + x_2 \leq 20$ and $x_1 , x_2 \geq 0$ .

(iii) Minimize $Z = 3x_1 + 2x_2$ subject to the constraints $5x_1+ x_2 \geq 10$; $x_1+ x_2 \geq 6$; $x_1+ 4 x_2 \geq 12$ and $x_1, x_2 \geq 0$.

(iv) Maximize $Z = 40x1 + 50x2$ subject to constraints $30x1 + x2 \leq 9$ ; $x1 + 2x2 \leq 8$ and $x1 , x2 \geq 0$

(v) Maximize $Z = 20x_1 + 30x_2$ subject to constraints $3x_1 + 3x_2 \leq 36$ ; $5x_1 + 2x_2 \leq 50$ ; $2x_1 + 6x_2 \leq 60$ and $x_1 , x_2 \geq 0$

(vi) Minimize $Z = 20x_1 + 40x_2$ subject to the constraints $36x_1 + 6x_2 \geq 108$, $3x_1 + 12x_2 \geq 36$, $20x_1 + 10x_2 \geq 100$ and $x_1 , x_2 \geq 0$

Thus, the mathematical formulation of the LPP is
maximize $z = 100x_1 + 150x_2$
subject to the constraints

$$\begin{aligned} 0.8x_1 + 1.2x_2 &\leq 720 \\ x_1 &\leq 600 \\ x_2 &\leq 400 \\ x_1, x_2 &\geq 0 \end{aligned}$$

**5. Solve the following linear programming problems by graphical method.**

**(i) Maximize $Z = 6x_1 + 8x_2$ subject to constraints $30x_1 + 20x_2 \leq 300$; $5x_1 + 10x_2 \leq 110$; and $x_1, x_2 \geq 0$.** *(GMQP - 2019)*

**(ii) Maximize $Z = 22x_1 + 18x_2$ subject to constraints $960x_1 + 640x_2 \leq 15360$; $x_1 + x_2 \leq 20$ and $x_1, x_2 \geq 0$.**

**(iii) Minimize $Z = 3x_1 + 2x_2$ subject to the constraints $5x_1 + x_2 \geq 10$; $x_1 + x_2 \geq 6$; $x_1 + 4x_2 \geq 12$ and $x_1, x_2 \geq 0$.**

**(iv) Maximize $Z = 40x_1 + 50x_2$ subject to constraints $30x_1 + x_2 \leq 9$; $x_1 + 2x_2 \leq 8$ and $x_1, x_2 \geq 0$**

**(v) Maximize $Z = 20x_1 + 30x_2$ subject to constraints $3x_1 + 3x_2 \leq 36$, $5x_1 + 2x_2 \leq 50$; $2x_1 + 6x_2 \leq 60$ and $x_1, x_2 \geq 0$**

**(vi) Minimize $Z = 20x_1 + 40x_2$ subject to constraints $36x_1 + 6x_2 \geq 108$, $3x_1 + 12x_2 \geq 36$, $20x_1 + 10x_2 \geq 100$ and $x_1, x_2 \geq 0$**

***Solution :***

(i) Maximize $Z = 6x_1 + 8x_2$
subject to the constraints

$$\begin{aligned} 30x_1 + 20x_2 &\leq 300 \\ 5x_1 + 10x_2 &\leq 110 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Since both the decision variables $x_1$ and $x_2$ are non-negative, the solution lies in the first quadrant of the plane.

Consider the equations,

$30x_1 + 20x_2 = 300$

| $x_1$ | 0 | 10 |
|---|---|---|
| $x_2$ | 15 | 0 |

$5x_1 + 10x_2 = 110$

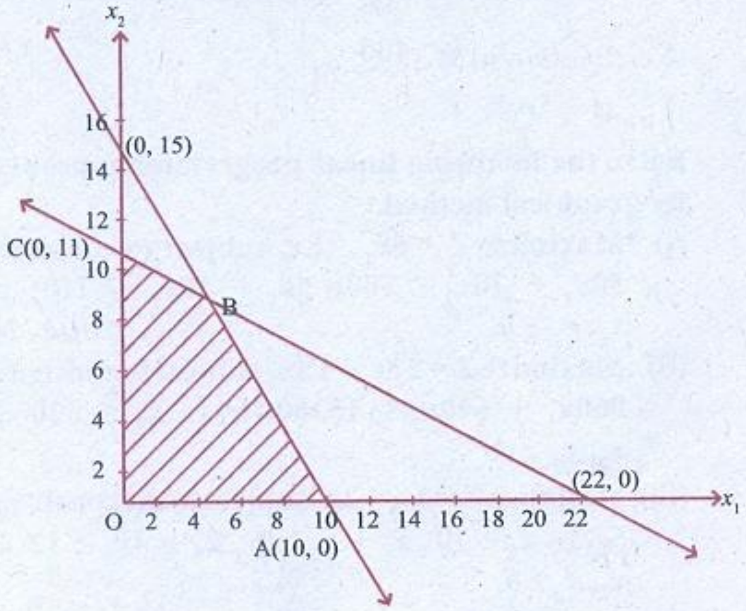
| $x_1$ | 0 | 22 |
|---|---|---|
| $x_2$ | 11 | 0 |

$30x_1 + 20x_2 = 300$ is a line passing through the points (0, 15) and (10, 0).

Any point lying on or below the line $30x_1 + 20x_2 = 300$ satisfies the constraint $30x_1 + 20x_2 \leq 300$.

$5x_1 + 10x_2 = 110$ is a line passing through the points (0, 11) and (22, 0)

Any point lying on or below the line $5x_1 + 10x_2 = 110$ satisfies the constraint $5x_1 + 10x_2 \le 110$




The feasible region satisfying all the conditions is on BC.

The coordinates of the points are O(0, 0), A(10, 0), C(0, 11) and B is the point of intersection of two lines.

$$30x_1 + 20x_2 = 300 \quad \text{... (1)}$$
$$5x_1 + 10x_2 = 110 \quad \text{... (2)}$$

$$(1) \Rightarrow 30x_1 + \cancel{20x_2} = 300$$
$$(-) \quad (-) \quad (-)$$
$$(2)\times 2 \Rightarrow 10x_1 + \cancel{20x_2} = 220$$
$$\Rightarrow \quad 20x_1 = 80$$
$$\Rightarrow \quad x_1 = 4$$

Substituting $x_1 = 4$ in (1) we get,

$$\Rightarrow \quad 30(4) + 20x_2 = 300$$
$$\Rightarrow \quad 120 + 20x_2 = 300$$
$$\Rightarrow \quad 20x_2 = 300 - 120 = 180$$
$$\Rightarrow \quad x_2 = 9$$

$\therefore$ B is (4, 9)

| Corner Points | $Z = 6x_1 + 8x_2$ |
| --- | --- |
| O(0, 0) | 0 |
| A(10, 0) | 60 |
| B(4, 9) | 24 + 72 = 96 |
| C(0, 11) | 88 |

Maximum value of Z occurs at B(4, 9)

$\therefore$ The solution is $x_1 = 4, x_2 = 9$ and $Z_{max} = 96$.

(ii) Maximize $Z = 22x_1 + 18x_2$

Subject to the constraints

$$960\,x_1 + 640\,x_2 \leq 15360$$
$$x_1 + x_2 \leq 20$$
$$x_1, x_2 \geq 0$$

Since both the decision variables $x_1$ and $x_2$ are non-negative, the solution lies in the first quadrant of the plane.

Consider the equations

$960x_1 + 640x_2 = 15360$

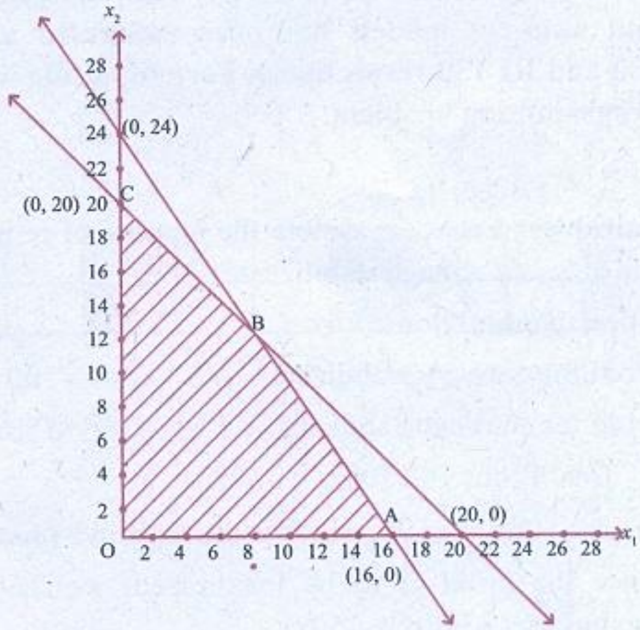
| $x_1$ | 0 | 16 |
|---|---|---|
| $x_2$ | 24 | 0 |

$x_1 + x_2 = 20$

| $x_1$ | 0 | 20 |
|---|---|---|
| $x_2$ | 20 | 0 |

Any point lying on or below the line $960x_1 + 640x_2 = 15360$ satisfies the constraint $960x_1 + 640x_2 \leq 15360$

Any point lying on or below the line $x_1 + x_2 = 20$, satisfies the constraint $x_1 + x_2 \leq 20$




The feasible region satisfying all the constraints OABC.

Its co-ordinates are O(0, 0), A(16, 0), C(0, 20) and B is the point of intersection of the lines.

$\therefore$ B(8, 12) [From the graph]

**Verification for B(8, 12):**

$$960x_1 + 640x_2 = 15360 \quad \text{... (1)}$$
$$x_1 + x_2 = 20 \quad \text{... (2)}$$

$$(1) \Rightarrow \quad 960x_1 + 640x_2 = 15360$$
$$(-) \quad (-) \quad (-)$$
$$(2) \times 640 \Rightarrow \quad 640x_1 + 640x_2 = 12800$$
$$\Rightarrow \quad 320x_1 = 2560$$
$$x_1 = \frac{2560}{320} = 8$$

Substituting $x_1 = 8$ in (2) we get,

$$8 + x_2 = 20$$

$$\Rightarrow \quad x_2 = 12$$

$\therefore$ B(8, 12)

| **Corner Points** | $\mathbf{Z = 22x_1 + 18x_2}$ |
|---|---|
| O(0, 0) | 0 |
| A(16, 0) | 352 |
| B(8, 12) | 392 |
| C(0, 20) | 360 |

Maximum value of Z occurs at B(8, 12)

$\therefore$ The solution is $x_1 = 8, x_2 = 12$ and $Z_{max} = 392$.

(iii) Minimize $Z = 3x_1 + 2x_2$

Subject to the constraints

$$5x_1 + x_2 \geq 10, \qquad x_1 + x_2 \geq 6$$

$$x_1 + 4x_2 \geq 12$$

$$x_1, x_2 \geq 0$$

Since both the decision variables $x_1$ and $x_2$ are non-negative the solution lies in the I quadrant. Consider the equations.

$5x_1 + x_2 = 10$

| | | |
|---|---|---|
| $x_1$ | 0 | 2 |
| $x_2$ | 10 | 0 |

$x_1 + x_2 = 6$

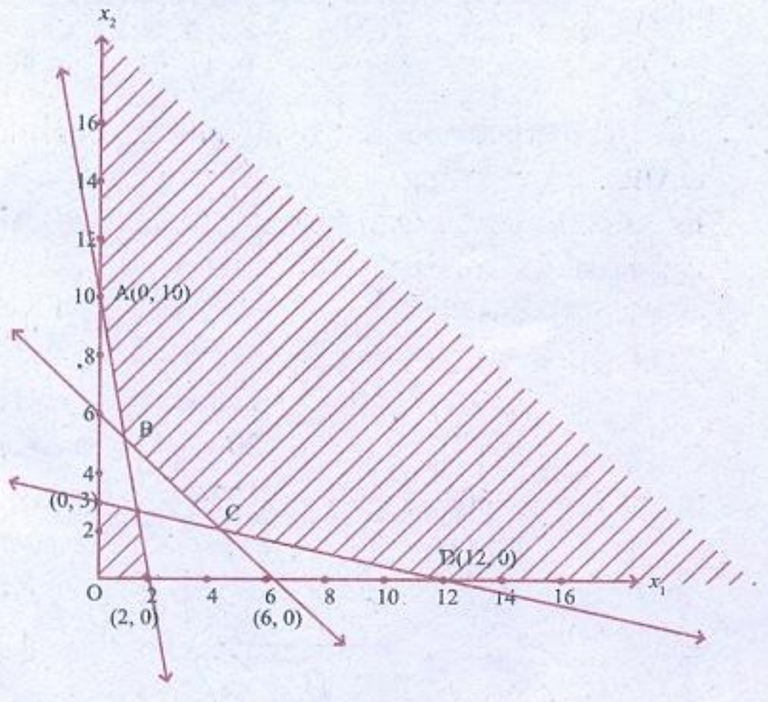
| | | |
|---|---|---|
| $x_1$ | 0 | 6 |
| $x_2$ | 6 | 0 |

$x_1 + 4x_2 = 12$

| | | |
|---|---|---|
| $x_1$ | 0 | 12 |
| $x_2$ | 3 | 0 |

Any point lying on or above the lines

$5x_1 + x_2 = 10$, $x_1 + x_2 = 6$, $x_1 + 4x_2 = 12$ satisfy the constriants $5x_1 + x_2 \geq 10$, $x_1 + x_2 \geq 6$ and $x_1 + 4x_2 \geq 12$ respectively.



The feasible region is ABCD and its co-ordinates as A(0, 10), D(12, 0) and B is the point of the intersection of the lines $5x_1 + x_2 = 10$ and $x_1 + x_2 = 6$.

Also C is the point of intersection of the lines $x_1 + x_2 = 6$ and $x_1 + 4x_2 = 12$.

**Verification of B and C:**

**For B**

$$
\begin{array}{rcll}
5x_1 + \cancel{x_2} & = & 10 & \quad \text{... (1)} \\
(-) \quad (-) & & (-) & \\
x_1 + \cancel{x_2} & = & 6 & \quad \text{... (2)} \\
\hline
4x_1 & = & 4 & \\
\Rightarrow \quad x_1 & = & 1 &
\end{array}
$$

Substituting $x_1 = 1$ in (2) we get,

$$
\begin{aligned}
1 + x_2 &= 6 \\
\Rightarrow \quad x_2 &= 5
\end{aligned}
$$

$\therefore$ B is (1, 5)

**For C**

$$
\begin{array}{rcl}
\cancel{x_1} + x_2 & = & 6 \\
(-) \quad (-) & & (-) \\
\cancel{x_1} + 4x_2 & = & 12 \\
\hline
-3x_2 & = & -6 \\
\Rightarrow \quad x_2 & = & 2
\end{array}
$$

Substituting $x_2 = 2$ in (3) we get,

$$
\begin{aligned}
x_1 + 2 &= 6 \\
\Rightarrow \quad x_1 &= 4
\end{aligned}
$$

$\therefore$ C is (4, 2)

| **Corner Points** | $z = 3x_1 + 2x_2$ |
|---|---|
| A(0, 10) | 20 |
| B(1, 5) | 13 |
| C(4, 2) | 16 |
| D(12, 0) | 36 |

Minimum value of Z occurs at B(1, 5).

$\therefore$ The solution is $x_1 = 1$, $x_2 = 5$ and $Z_{min} = 13$.

(iv) Maximize $Z = 40x_1 + 50x_2$ subject to the constraints $3x_1 + x_2 \leq 9$, $x_1 + 2x_2 \leq 8$ and $x_1, x_2 \geq 0$

Since both the decision variables $x_1$ and $x_2$ are non-negative, the solution lies in the first quadrant of the plane.

Consider the equations

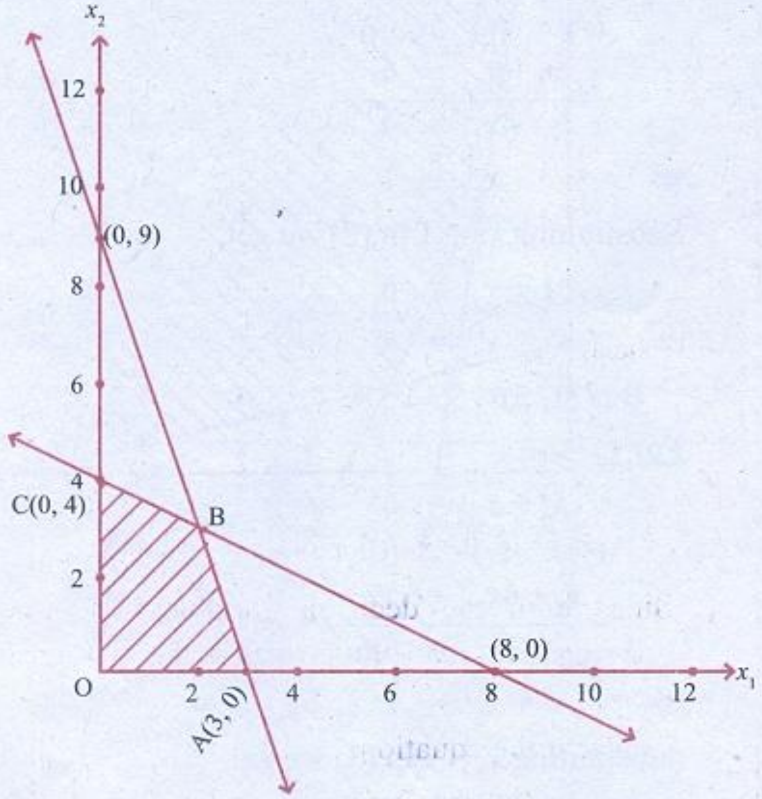
$3x_1 + x_2 = 9$

| $x_1$ | 0 | 3 |
|---|---|---|
| $x_2$ | 9 | 0 |

$x_1 + 2x_2 = 8$

| $x_1$ | 0 | 8 |
|---|---|---|
| $x_2$ | 4 | 0 |

Any point lying on or below the lines $3x_1 + x_2 = 9$ and $x_1 + 2x_2 = 8$ satisfies the constraints $3x_1 + x_2 \leq 9$ and $x_1 + 2x_2 \leq 8$.




The feasible region is OABC and its co-ordinates are O(0, 0), A(3, 0), C(0, 4) and B is the point of intersection of the lines

$$3x_1 + x_2 = 9 \quad \text{... (1)}$$

$$\text{and } x_1 + 2x_2 = 8 \quad \text{... (2)}$$

**Verification of B**

$$(1) \Rightarrow \quad \cancel{3x_1} + x_2 = 9$$

$$\quad (-) \quad (-) \quad (-)$$

$$(2) \times 3 \Rightarrow \quad \cancel{3x_1} + 6x_2 = 24$$

$$-5x_2 = -15 \quad \Rightarrow x_2 = 3$$

Substituting $x_2 = 3$ in (1) we get

$$3x_1 + 3 = 9$$

$$\Rightarrow \quad 3x_1 = 6$$

$$\Rightarrow \quad x_1 = 2$$

$\therefore$ B is (2, 3)

| Corner Points | $Z = 40x_1 + 50x_2$ |
|---|---|
| O(0, 0) | 0 |
| A(3, 0) | 120 |
| B(2, 3) | 230 |
| C(0, 4) | 200 |

Maximum value occurs at B(2, 3).

$\therefore$ The solution is $x_1 = 2, x_2 = 3$ and $Z_{max} = 230$

(v) Maximize $Z = 20x_1 + 30x_2$ subject to the constraints $3x_1 + 3x_2 \leq 36$, $5x_1 + 2x_2 \leq 50$, $2x_1 + 6x_2 \leq 60$ and $x_1, x_2 \geq 0$.

Since both the decision variables $x_1$, $x_2$ are non-negative, the solution lies in the first quadrant of the plane. Consider the equations.

$3x_1 + 3x_2 = 36$

| | | |
|---|---|---|
| $x_1$ | 0 | 12 |
| $x_2$ | 12 | 0 |

$5x_1 + 2x_2 = 50$

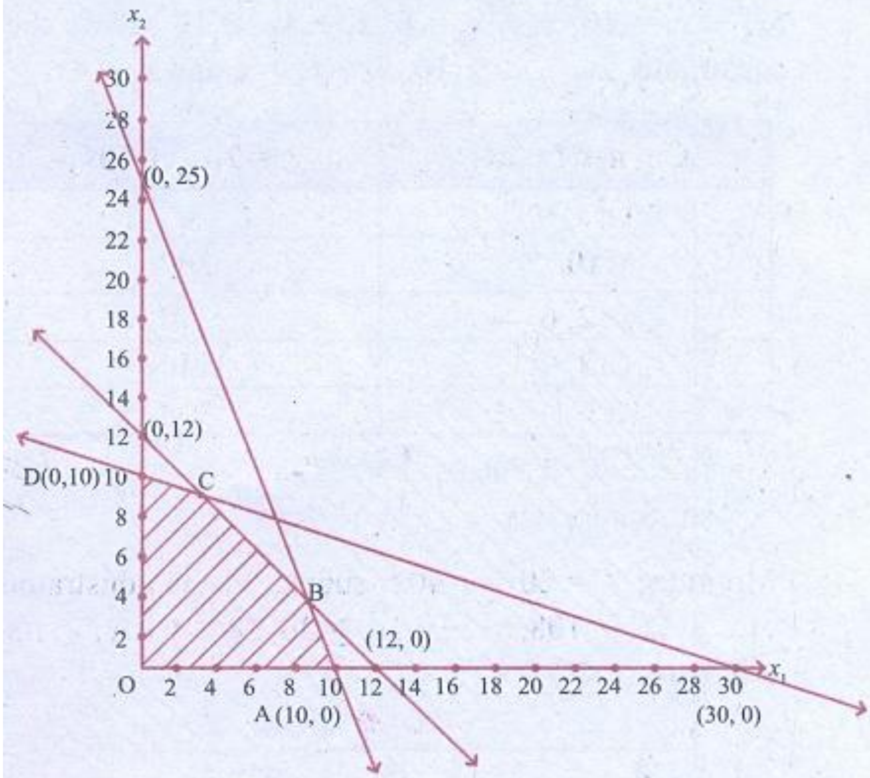
| | | |
|---|---|---|
| $x_1$ | 0 | 10 |
| $x_2$ | 25 | 0 |

$2x_1 + 6x_2 = 60$

| | | |
|---|---|---|
| $x_1$ | 0 | 30 |
| $x_2$ | 10 | 0 |

The feasible region is OABCD and its co-ordinates are O(0, 0), A(10, 0), D(0, 10), B is the point of intersection of the lines $3x_1 + 3x_2 = 36$ and $5x_1 + 2x_2 = 50$.

Also C is the point of intersection of the lines $2x_1 + 6x_2 = 60$ and $3x_1 + 3x_2 = 36$.



**Verification of the points B and C:**

**For B**

$$\begin{aligned}
 3x_1 + 3x_2 &= 36 \\
\Rightarrow \quad x_1 + x_2 &= 12 && \text{... (1)} \\
 5x_1 + 2x_2 &= 50 && \text{... (2)} \\
(1) \times 2 \Rightarrow 2x_1 + \cancel{2x_2} &= 24 \\
(-) \quad (-) \quad &\quad (-) \\
(2) \Rightarrow \quad 5x_1 + \cancel{2x_2} &= 50 \\
\hline
-3x_1 &= -6 \\
\Rightarrow \quad x_1 &= 2
\end{aligned}$$

Substituting $x_1 = 2$ in (1) we get

$$\begin{aligned}
2 + x_2 &= 12 \\
\Rightarrow \quad x_2 &= 6
\end{aligned}$$

$\therefore$ B is (2, 6)

**For C**

$$\begin{aligned}
 3x_1 + 3x_2 &= 36 \\
\Rightarrow \quad x_1 + x_2 &= 12 && \text{... (3)} \\
 2x_1 + 6x_2 &= 60 \\
\Rightarrow \quad x_1 + 3x_2 &= 30 && \text{... (4)} \\
(3) \Rightarrow \quad \cancel{x_1} + x_2 &= 12 \\
(-) \quad (-) \quad &\quad (-) \\
(4) \Rightarrow \quad \cancel{x_1} + 3x_2 &= 30 \\
\hline
-2x_2 &= -18 && \Rightarrow x_2 = 9
\end{aligned}$$

Substituting $x_2 = 9$ in (3), we get

$$\begin{aligned}
x_1 + 9 &= 12 \\
\Rightarrow \quad x_1 &= 3
\end{aligned}$$

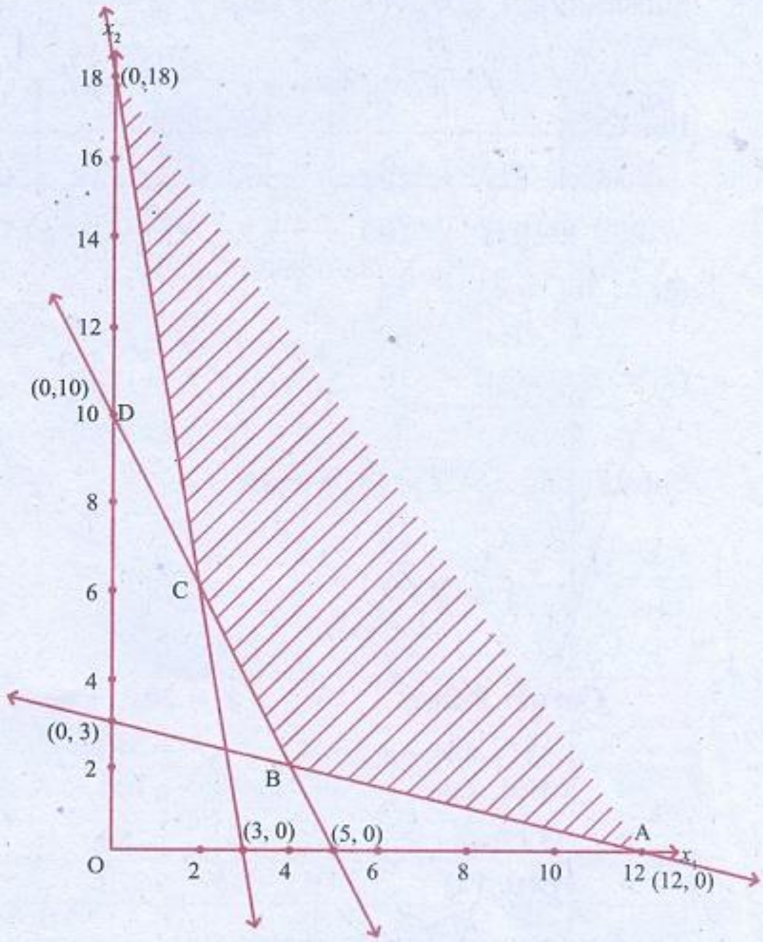
$\therefore$ C is (3,9)

| Corner Points | $Z = 20x_1 + 30x_2$ |
|---|---|
| O(0, 0) | 0 |
| A(10, 0) | 200 |
| B(2, 6) | 220 |
| C(3, 9) | 330 |
| D(0, 10) | 300 |

The maximum value of Z occurs at C(3, 9).

$\therefore$ The solution is $x_1 = 3, x_2 = 9$ and $Z_{max} = 330$.

(vi) Minimize $Z = 20x_1 + 40x_2$ subject to the constraints $36x_1 + 6x_2 \geq 108$, $3x_1 + 12x_2 \geq 36$, $20x_1 + 10x_2 \geq 100$ and $x_1, x_2 \geq 0$.




Since both the decision variables $x_1$ and $x_2$ are non-negative, the solution lies in the first quadrant of the plane.

Consider the equations.

$36x_1+6x_2=108$

| $x_1$ | 0 | 3 |
|---|---|---|
| $x_2$ | 18 | 0 |

$3x_1+12x_2=36$

| $x_1$ | 0 | 12 |
|---|---|---|
| $x_2$ | 3 | 0 |

$20x_1+10x_2=100$

| $x_1$ | 0 | 5 |
|---|---|---|
| $x_2$ | 10 | 0 |

Any point lying on or above the lines $36x_1+6x_2=108$, $3x_1+12x_2=36$ and $20x_1+10x_2=100$ satisfy the constraints $36x_1+6x_2 \geq 108$, $3x_1+12x_2 \geq 36$ and $20x_1+10x_2 \geq 100$.

The feasible region is ABCD and its co-ordinates are A(12, 0) D(0, 10), B is the point of intersection of the lines $3x_1+12x_2=36$, $20x_1+10x_2=100$.

Also, C is the point of the intersection of the lines $36x_1+6x_2=108$ and $20x_1+10x_2=100$.

**Verification of B and C: For B:**

$$3x_1+12x_2 = 36 \Rightarrow x_1+4x_2=12 \quad \ldots (1)$$
$$20x_1+10x_2 = 100 \Rightarrow 2x_1+x_2=10 \quad \ldots (2)$$

$$\begin{array}{lrcl}
(1) \Rightarrow & x_1+\cancel{4x_2} & = & 12 \\
 & (-) \quad (-) & & (-) \\
(2)\times 4 \Rightarrow & 8x_1+\cancel{4x_2} & = & 40 \\
\hline
 & -7x_1 & = & -28
\end{array} \quad \Rightarrow x_1=4$$

Substituting $x_1 = 4$ in (2), we get $8 + x_2 = 10 \Rightarrow x_2 = 2$

$\therefore$ B is (4, 2)

**For C:**

$$
\begin{aligned}
36x_1 + 6x_2 &= 108 \qquad \Rightarrow 6x_1 + x_2 = 18 \ldots (3) \\
20x_1 + 10x_2 &= 100 \qquad \Rightarrow 2x_1 + x_2 = 10 \ldots (4)
\end{aligned}
$$

$$
\begin{aligned}
(3) \Rightarrow 6x_1 + \cancel{x_2} &= 18 \\
(-) \quad (-) \quad & \quad (-) \\
(4) \Rightarrow 2x_1 + \cancel{x_2} &= 10 \\
\hline
4x_1 &= 8 \qquad \Rightarrow x_1 = 2
\end{aligned}
$$

Substituting $x_1 = 2$ in (4) we get,

$$
\begin{aligned}
4 + x_2 &= 10 \\
\Rightarrow \quad x_2 &= 6
\end{aligned}
$$

$\therefore$ C is (2, 6)

| **Corner Points** | $\mathbf{Z = 20x_1 + 40x_2}$ |
|---|---|
| A(12, 0) | 240 |
| B(4, 2) | 160 |
| C(2, 6) | 280 |
| D(0, 18) | 720 |

The minimum of Z value occurs at B.

$\therefore$ The solution is $x_1 = 4, x_2 = 2$ and $Z_{min} = 160$.

## Solving Linear Programming Graphical method using Python PuLP

```
from pulp import *

model = pulp.LpProblem('linear_programming', LpMaximize)

# get solver
solver = getSolver('PULP_CBC_CMD')

# declare decision variables
x1 = LpVariable('x1', lowBound = 0, cat = 'continuous')
x2 = LpVariable('x2', lowBound = 0, cat = 'continuous')

# declare objective
model += 10*x1 + 5*x2

# declare constraints
model += x1 + x2 <= 24
model += 10*x1 <= 100
model += 5*x2 <= 100

# solve
results = model.solve(solver=solver)

# print results
if LpStatus[results] == 'Optimal': print('The solution is optimal.')
print(f'Objective value: z* = {value(model.objective)}')
print(f'Solution: x1* = {value(x1)}, x2* = {value(x2)}')

The solution is optimal.
Objective value: z* = 170.0
Solution: x1* = 10.0, x2* = 14.0
```

```python
from pulp import *

model = pulp.LpProblem('linear_programming', LpMaximize)

# get solver
solver = getSolver('PULP_CBC_CMD')

# declare decision variables
x1 = LpVariable('x1', lowBound = 0, cat = 'continuous')
x2 = LpVariable('x2', lowBound = 0, cat = 'continuous')

# declare objective
model += 10*x1 + 5*x2

# declare constraints
model += x1 + x2 <= 24
model += 10*x1 <= 100
model += 5*x2 <= 100

# solve
results = model.solve(solver=solver)

# print results
if LpStatus[results] == 'Optimal': print('The solution is optimal.')
print(f'Objective value: z* = {value(model.objective)}')
print(f'Solution: x1* = {value(x1)}, x2* = {value(x2)}')
```

```
The solution is optimal.
Objective value: z* = 170.0
Solution: x1* = 10.0, x2* = 14.0
```