4.7.46

EE25BTECH11049 - Sai Krishna Bakki

Question:
The equations of the lines passing through the point (1, 0) and at a distance $\frac{\sqrt{3}}{2}$ from the origin, are

Solution:

1. Represent the Line with Matrices

We can represent the equation of a line in its normal form using matrix notation:

$$\mathbf{n}^T\mathbf{x} - p = 0 \tag{0.1}$$

Where:

- $\mathbf{n}$ is the **unit normal vector** to the line.

$$\mathbf{n} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{0.2}$$

- $\mathbf{x}$ is a vector to any point (x, y) on the line.

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{0.3}$$

- p is the perpendicular distance from the origin to the line.

From the problem, we are given:

- The distance from the origin,

$$p = \frac{\sqrt{3}}{2} \tag{0.4}$$

- A point on the line, $(1, 0)$, which we can represent as a vector

$$\mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{0.5}$$

2. Formulate and Solve the Matrix Equation

Since the point $\mathbf{p_1}$ lies on the line, it must satisfy the line's equation:

$$\mathbf{n}^T\mathbf{p_1} - p = 0 \tag{0.6}$$

Now, we substitute the known matrices and scalars into this equation:

$$\begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{\sqrt{3}}{2} = 0 \tag{0.7}$$

Performing the matrix multiplication gives a scalar equation:

$$\cos\theta = \frac{\sqrt{3}}{2} \tag{0.8}$$

3. Determine the Normal Vectors

Using the identity,

$$\begin{pmatrix}\cos\theta & \sin\theta\end{pmatrix}\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} = 1 \tag{0.9}$$

we can find the possible values for $\sin\theta$:

$$\begin{pmatrix}\frac{\sqrt{3}}{2} & \sin\theta\end{pmatrix}\begin{pmatrix}\frac{\sqrt{3}}{2} \\ \sin\theta\end{pmatrix} = 1 \tag{0.10}$$

$$\sin^2\theta = \frac{1}{4} \implies \sin\theta = \pm\frac{1}{2} \tag{0.11}$$

This gives us two possible unit normal vectors for our two lines:

$$\mathbf{n_1} = \begin{pmatrix}\frac{\sqrt{3}}{2} \\ \frac{1}{2}\end{pmatrix} \tag{0.12}$$

$$\mathbf{n_2} = \begin{pmatrix}\frac{\sqrt{3}}{2} \\ -\frac{1}{2}\end{pmatrix} \tag{0.13}$$

4. Find the Equations of the Lines

We can now find the equation for each line by substituting its normal vector back into the general matrix equation $\mathbf{n}^T\mathbf{x} - p = 0$.

Line 1: Using $\mathbf{n_1}$

$$\begin{pmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix} - \frac{\sqrt{3}}{2} = 0 \tag{0.14}$$

Multiplying by 2, we get the first equation:

$$\begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\mathbf{x} = \sqrt{3} \tag{0.15}$$

Line 2: Using $\mathbf{n_2}$

$$\begin{pmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix} - \frac{\sqrt{3}}{2} = 0 \tag{0.16}$$

Multiplying by 2, we get the second equation:

$$\begin{pmatrix}\sqrt{3} & -1\end{pmatrix}\mathbf{x} = \sqrt{3} \tag{0.17}$$

The equations of the lines passing through the point (1, 0) and at a distance $\frac{\sqrt{3}}{2}$ from the origin, are

$$\begin{pmatrix}\sqrt{3} & -1\end{pmatrix}\mathbf{x} = \sqrt{3}, \begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\mathbf{x} = \sqrt{3} \tag{0.18}$$

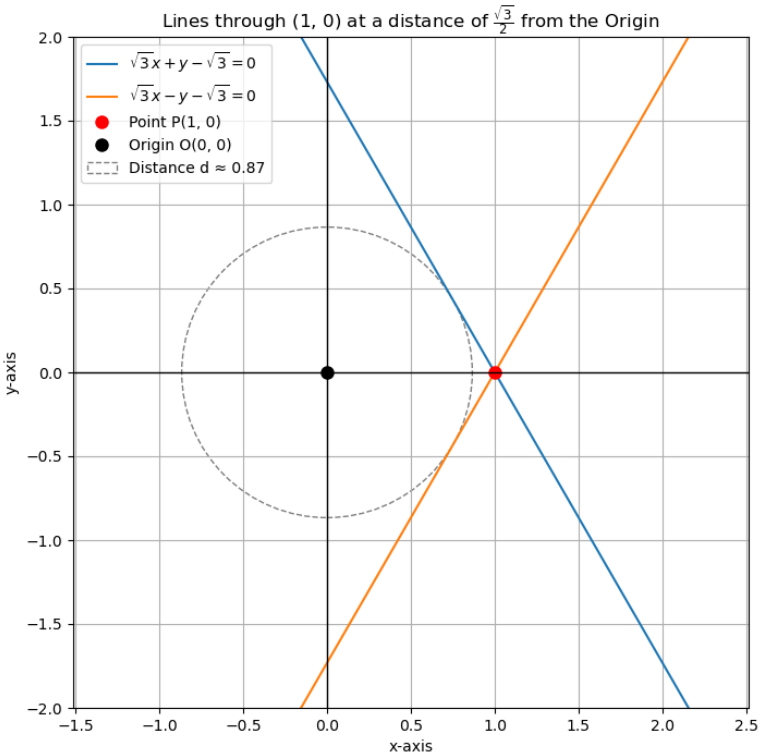
Lines through (1, 0) at a distance of $\frac{\sqrt{3}}{2}$ from the Origin
$\sqrt{3}x + y - \sqrt{3} = 0$
$\sqrt{3}x - y - \sqrt{3} = 0$
Point P(1, 0)
Origin O(0, 0)
Distance d ≈ 0.87
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
2.5
x-axis
y-axis

Fig. 0.1