2.8.39

EE25BTECH11049 - Sai Krishna Bakki

Question:
Find the angle between the lines whose direction cosines are given by the equations $l + m + n = 0,\ l^2 + m^2 - n^2 = 0$.
Solution:

1 Setting Up the Equations in Matrix Form

We begin by representing the given system of equations using vectors and matrices. Let the direction cosines be represented by the column vector $\mathbf{x}$:

$$\mathbf{x} = \begin{pmatrix} l \\ m \\ n \end{pmatrix} \tag{0.1}$$

The three conditions on the direction cosines can be written in matrix form:

$$l + m + n = 0 \implies \mathbf{C}^T\mathbf{x} = 0, \mathbf{C} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{0.2}$$

$$l^2 + m^2 - n^2 = 0 \implies \mathbf{x}^T\mathbf{A}\mathbf{x} = 0, \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{0.3}$$

$$l^2 + m^2 + n^2 = 1 \implies \mathbf{x}^T\mathbf{I}\mathbf{x} = 1, \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{0.4}$$

2 Solving for the Direction Cosine Vectors

Our goal is to find the two vectors, $\mathbf{D}_1$ and $\mathbf{D}_2$, that satisfy these matrix equations.

2.1 Finding the value of n using Matrix Theory

We can efficiently find the value of n by subtracting the two quadratic form equations:

$$\mathbf{x}^T\mathbf{I}\mathbf{x} - \mathbf{x}^T\mathbf{A}\mathbf{x} = 1 \implies \mathbf{x}^T(\mathbf{I} - \mathbf{A})\mathbf{x} = 1 \tag{0.5}$$

The matrix $(\mathbf{I} - \mathbf{A})$ is:

$$\mathbf{I} - \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{0.6}$$

Substituting this back into the equation (0.5) gives:

$$\begin{pmatrix} l & m & n \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} = 1 \tag{0.7}$$

$$2n^2 = 1 \implies n = \pm \frac{1}{\sqrt{2}} \tag{0.8}$$

2.2 *Finding the values of l and m*

Let's choose $n = -\frac{1}{\sqrt{2}}$. Substituting this into the remaining linear and normalization equations gives a system for l and m:

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} l \\ m \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = 0 \implies l + m = \frac{1}{\sqrt{2}} \tag{0.9}$$

$$\begin{pmatrix} l & m & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} l & m & \frac{-1}{\sqrt{2}} \end{pmatrix} \implies l^2 + m^2 = \frac{1}{2} \tag{0.10}$$

Squaring the first part gives

$$(l + m)^2 = \left(\frac{1}{\sqrt{2}} \right)^2 \implies l^2 + 2lm + m^2 = \frac{1}{2} \tag{0.11}$$

subsituting equation (0.10) in equation (0.11),we get

$$2lm = 0 \tag{0.12}$$

so we get l=0 and m=0,

$$l = 0, m = \frac{1}{\sqrt{2}} \tag{0.13}$$

$$m = 0, l = \frac{1}{\sqrt{2}} \tag{0.14}$$

2.3 *Assembling the Direction Cosine Vectors*

Combining our results, the two direction cosine vectors are:

$$\mathbf{D}_1 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \quad \text{and} \quad \mathbf{D}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \tag{0.15}$$

3 Calculating the Angle using Matrix Theory

The cosine of the angle θ between the lines is the dot product of their direction cosine vectors. Using matrix multiplication, this is

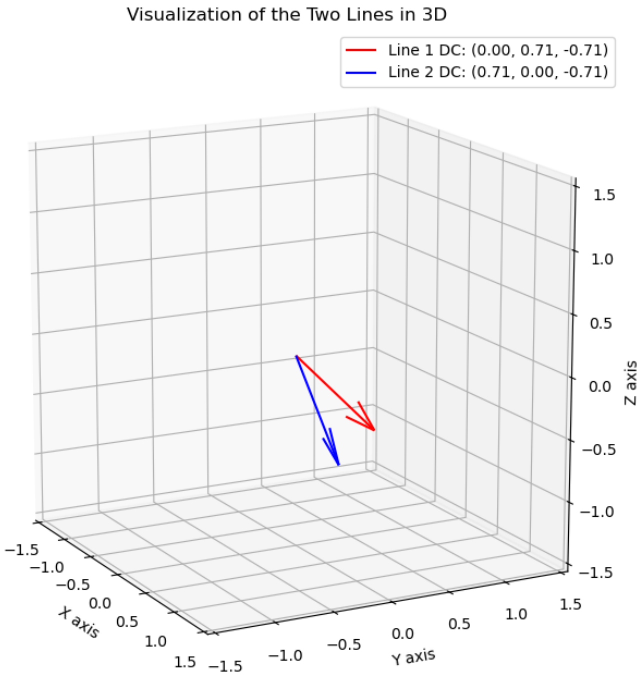
$$\cos\theta = \mathbf{D}_1^T\mathbf{D}_2 \tag{0.16}$$

$$\cos\theta = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \tag{0.17}$$

$$\cos\theta = 0 + 0 + \frac{1}{2} = \frac{1}{2} \tag{0.18}$$

Therefore, the angle is:

$$\theta = cos^{-1}\left(\frac{1}{2}\right) = 60^\circ \quad \text{or} \quad \frac{\pi}{3} \text{ radians} \tag{0.19}$$



Visualization of the Two Lines in 3D
Line 1 DC: (0.00, 0.71, -0.71)
Line 2 DC: (0.71, 0.00, -0.71)
X axis
Y axis
Z axis
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5