4.11.26

EE25BTECH11049 - Sai Krishna Bakki

Question:
Find the area bounded by the curves $y = |x - 1|$ and $y = 1$.
Solution

1. Representing Lines in Matrix Form

We express the three boundary lines in the vector form $\mathbf{n}^T\mathbf{x} = c$, where $\mathbf{n}$ is the normal vector and $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$.

$$\mathbf{n_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, c_1 = 1 \implies \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 1 \tag{0.1}$$

$$\mathbf{n_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, c_1 = 1 \implies \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 1 \tag{0.2}$$

$$\mathbf{n_3} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, c_1 = 1 \implies \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 1 \tag{0.3}$$

2. Finding Vertices with Matrix Inversion

The intersection of any two lines is the solution to a system of linear equations, which we solve using matrix inversion.

Vertex A (Intersection of L_1 and L_2): The system is $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The solution is $\mathbf{x} = \mathbf{N}_{12}^{-1}\mathbf{c}_{12}$.

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{0.4}$$

$$= \frac{1}{1(1) - (-1)(1)} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{0.5}$$

$$= \frac{1}{2} \begin{pmatrix} 1(1) + 1(1) \\ -1(1) + 1(1) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{0.6}$$

Vertex B (Intersection of L_1 and L_3): The system is $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The solution is $\mathbf{x} = \mathbf{N}_{13}^{-1}\mathbf{c}_{13}$.

$$\mathbf{B} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{0.7}$$

$$= \frac{1}{1(1) - (-1)(0)} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{0.8}$$

$$= \begin{pmatrix} 1(1) + 1(1) \\ 0(1) + 1(1) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{0.9}$$

Vertex C (Intersection of L_2 and L_3): The system is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The solution is $\mathbf{x} = \mathbf{N}_{23}^{-1}\mathbf{c}_{23}$.

$$\mathbf{C} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{0.10}$$

$$= \frac{1}{1(1) - 1(0)} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{0.11}$$

$$= \begin{pmatrix} 1(1) - 1(1) \\ 0(1) + 1(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{0.12}$$

The vertices are $\mathbf{A} = (1, 0)$, $\mathbf{B} = (2, 1)$, and $\mathbf{C} = (0, 1)$.

3. Calculating Area with Vector Determinant

We form two vectors representing two sides of the triangle, **AB** and **AC**.

$$\mathbf{AB} = \mathbf{B} - \mathbf{A} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{0.13}$$

$$\mathbf{AC} = \mathbf{C} - \mathbf{A} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{0.14}$$

The area is half the absolute value of the determinant of the matrix formed by these two vectors.

$$\text{Area} = \frac{1}{2} \left\| \left(\mathbf{B} - \mathbf{A}\right) \times \mathbf{C} - \mathbf{A} \right\| \tag{0.15}$$

$$\text{Area} = \frac{1}{2} \left\| \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \right\| \tag{0.16}$$

$$= \frac{1}{2} \left(1(1) - (-1)(1)\right) \tag{0.17}$$

$$= \frac{1}{2} (1 + 1) = \frac{1}{2} (2) = 1 \text{ square unit.} \tag{0.18}$$

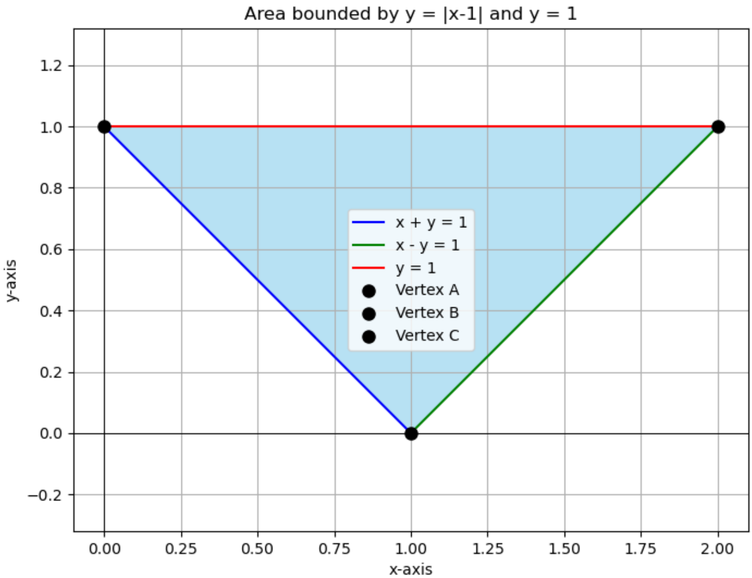
Area bounded by y = |x-1| and y = 1
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
y-axis
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
x-axis
x + y = 1
x - y = 1
y = 1
Vertex A
Vertex B
Vertex C

Fig. 0.1