9.2.11

EE25BTECH11049 - Sai Krishna Bakki

Question:
Find the area of the region bounded by the curve $y^2 = 9x$ and the lines $x = 2$ and $x = 4$ and the x-axis in the first quadrant.

Solution:
The general equation of a conic section is given by $\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0$, where $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$.
The parameters of the conic are

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} -9/2 \\ 0 \end{pmatrix}, \quad f = 0 \tag{0.1}$$

For the line x - 2 = 0, the parameters are

$$\mathbf{h}_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad \mathbf{m}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{0.2}$$

The parameter κ for the points of intersection is found using the formula:

$$\kappa = \frac{1}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \left(-\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})]^2 - g(\mathbf{h})(\mathbf{m}^\top \mathbf{V}\mathbf{m})} \right) \tag{0.3}$$

where $g(\mathbf{h}) = \mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f$.

Substituting the values into the formula for κ:

$$\kappa = \left(-0 \pm \sqrt{0^2 - (-18)(1)} \right) = 3\sqrt{2}, -3\sqrt{2} \tag{0.4}$$

yielding the points of intersection

$$\mathbf{a}_0 = \begin{pmatrix} 2 \\ 3\sqrt{2} \end{pmatrix}, \mathbf{a}_1 = \begin{pmatrix} 2 \\ -3\sqrt{2} \end{pmatrix} \tag{0.5}$$

For the line x - 4 = 0, the parameters are:

$$\mathbf{h}_2 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \quad \mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{0.6}$$

$$\kappa = \frac{1}{1} \left(-0 \pm \sqrt{0^2 - (-36)(1)} \right) = 6, -6 \tag{0.7}$$

yielding the points of intersection

$$\mathbf{a}_2 = \begin{pmatrix} 4 \\ 6 \end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix} 4 \\ -6 \end{pmatrix} \tag{0.8}$$

Thus, the area of the parabola in between the lines x = 2 and x = 4 is given by

$$A = \int_0^4 3\sqrt{x}\,dx - \int_0^2 3\sqrt{x}\,dx \tag{0.9}$$

$$= 16 - 4\sqrt{2} \tag{0.10}$$

Thus, the area of the specified region is 16 - 4 $\sqrt{2}$ square units.

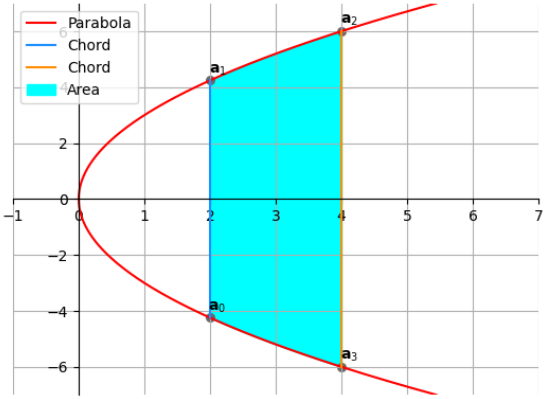

Fig. 0.1