5.13.74

EE25BTECH11049 - Sai Krishna Bakki

Question:
The trace of a square matrix is defined to be the sum of its diagonal entries. If $\mathbf{A}$ is a 2 x 2 matrix, such that the trace of $\mathbf{A}$ is 3 and the trace of $\mathbf{A}^3$ is -18, then the value of the determinant of $\mathbf{A}$ is
Solution:
Given:

$$\text{tr}(\mathbf{A}) = 3 \tag{0.1}$$
$$\text{tr}(\mathbf{A^3}) = -18 \tag{0.2}$$
$$\text{tr}(\mathbf{I}) = 2 \tag{0.3}$$

Using Cayley-Hamilton Theorem, we know that

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0, \text{for } \lambda = \mathbf{A} \tag{0.4}$$

For a 2 x 2 matrix $\mathbf{A}$, the characteristic equation is:

$$\lambda^2 - \text{tr}(\mathbf{A})\lambda + \left|\mathbf{A}\right| = 0 \tag{0.5}$$

According to the theorem, the matrix $\mathbf{A}$ itself will satisfy this equation:

$$\mathbf{A^2} - \text{tr}(\mathbf{A})\mathbf{A} + \left|\mathbf{A}\right|\mathbf{I} = 0 \tag{0.6}$$
$$\mathbf{A^2} - 3\mathbf{A} + \left|\mathbf{A}\right|\mathbf{I} = 0 \tag{0.7}$$
$$\mathbf{A^2} = 3\mathbf{A} - \left|\mathbf{A}\right|\mathbf{I} \tag{0.8}$$

To find $\mathbf{A^3}$, we multiply the equation by $\mathbf{A}$

$$\mathbf{A^3} = 3\mathbf{A^2} - \left|\mathbf{A}\right|\mathbf{A} \tag{0.9}$$

Now, substitute the expression for $\mathbf{A^2}$ into the equation for $\mathbf{A^3}$:

$$\mathbf{A^3} = 3\left(3\mathbf{A} - \left|\mathbf{A}\right|\mathbf{I}\right) - \left|\mathbf{A}\right|\mathbf{A} \tag{0.10}$$
$$\mathbf{A^3} = \left(9 - \left|\mathbf{A}\right|\right)\mathbf{A} - 3\left|\mathbf{A}\right|\mathbf{I} \tag{0.11}$$

Let's take the trace of both sides of this equation. Using the linearity properties of the trace

$$\text{tr}(\mathbf{X} + \mathbf{Y}) = \text{tr}(\mathbf{X}) + \text{tr}(\mathbf{Y}) \tag{0.12}$$
$$\text{tr}(k\mathbf{X}) = k\,\text{tr}(\mathbf{X}) \tag{0.13}$$
$$\text{tr}(\mathbf{A^3}) = \text{tr}\left(\left(9 - \left|\mathbf{A}\right|\right)\mathbf{A} - 3\left|\mathbf{A}\right|\mathbf{I}\right) \tag{0.14}$$
$$\text{tr}(\mathbf{A^3}) = \left(9 - \left|\mathbf{A}\right|\right)\text{tr}(\mathbf{A}) - 3\left|\mathbf{A}\right|\text{tr}(\mathbf{I}) \tag{0.15}$$

Substituting equations (0.1) and (0.2) in above equation, we get

$$-18 = \left(9 - \left|\mathbf{A}\right|\right)(3) - 3\left|\mathbf{A}\right|(2) \tag{0.16}$$

$$\therefore \left|\mathbf{A}\right| = 5 \tag{0.17}$$