10.7.97

EE25BTECH11049 - Sai Krishna Bakki

Question:
Let $\mathbf{P}\left(a\sec\theta, b\tan\theta\right)$ and $\mathbf{Q}\left(a\sec\phi, b\tan\phi\right)$, where $\theta+\phi=\pi/2$, be two points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If (h,k) is the point of intersection of the normals at $\mathbf{P}$ and $\mathbf{Q}$, then k is equal to?

Solution:
The equation of the hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Following the general form for a conic $g(\mathbf{x})=\mathbf{x}^T\mathbf{V}\mathbf{x}+2\mathbf{u}^T\mathbf{x}+f=0$, we can identify the corresponding matrices and vectors for our hyperbola:

$$\mathbf{V}=\begin{pmatrix} b^2 & 0 \\ 0 & -a^2 \end{pmatrix}, \mathbf{u}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}, f=-a^2b^2 \tag{1}$$

The equation of the normal to the conic at a point of contact $\mathbf{q}$ is given by:

$$\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)^T\mathbf{R}\left(\mathbf{x}-\mathbf{q}\right)=0 \tag{2}$$

where $\mathbf{R}$ is the 90-degree rotation matrix, $\mathbf{R}=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

The coordinates are $\mathbf{P}=\begin{pmatrix} a\sec\theta \\ b\tan\theta \end{pmatrix}$. The equation of the normal at $\mathbf{P}$ is:

$$\left(\mathbf{V}\mathbf{P}+\mathbf{u}\right)^T\mathbf{R}\left(\mathbf{x}-\mathbf{P}\right)=0 \tag{3}$$

$$\begin{pmatrix} ab^2\sec\theta & -a^2b\tan\theta \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x-a\sec\theta \\ y-b\tan\theta \end{pmatrix}=0 \tag{4}$$

$$\begin{pmatrix} -a^2b\tan\theta & -ab^2\sec\theta \end{pmatrix}\begin{pmatrix} x-a\sec\theta \\ y-b\tan\theta \end{pmatrix}=0 \tag{5}$$

$$\begin{pmatrix} a\tan\theta & b\sec\theta \end{pmatrix}\mathbf{x}=\left(a^2+b^2\right)\tan\theta\sec\theta \tag{6}$$

The coordinates are $\mathbf{Q}=\begin{pmatrix} a\sec\phi \\ b\tan\theta \end{pmatrix}$. The equation of the normal at $\mathbf{Q}$ is:

$$\begin{pmatrix} -a^2b\tan\phi & -ab^2\sec\phi \end{pmatrix}\begin{pmatrix} x-a\sec\phi \\ y-b\tan\phi \end{pmatrix}=0 \tag{7}$$

$$\begin{pmatrix} a\tan\phi & b\sec\phi \end{pmatrix}\mathbf{x}=\left(a^2+b^2\right)\tan\phi\sec\phi \tag{8}$$

We are given the condition $\theta+\phi=\pi/2$. We can use this to simplify the second equation. The intersection point (h,k) must satisfy the normal equations for both P and Q.

$$\begin{pmatrix} a\tan\theta & b\sec\theta \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix}=\left(a^2+b^2\right)\tan\theta\sec\theta \tag{9}$$

$$\begin{pmatrix} a\cot\theta & b\csc\theta \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix}=\left(a^2+b^2\right)\cot\theta\csc\theta \tag{10}$$

We can solve this linear system for the variables h and k by setting up an augmented matrix.

$$\left(\begin{array}{cc|c} a\tan\theta & b\sec\theta & (a^2+b^2)\tan\theta\sec\theta \\ a\cot\theta & b\csc\theta & (a^2+b^2)\cot\theta\csc\theta \end{array}\right) \tag{11}$$

Simplifying to $\sin\theta$ and $\cos\theta$:

$$\left(\begin{array}{cc|c} a\sin\theta\cos\theta & b\cos\theta & (a^2+b^2)\sin\theta \\ a\sin\theta\cos\theta & b\sin\theta & (a^2+b^2)\cos\theta \end{array}\right)$$

$$\xrightarrow{R_2\to R_2-R_1} \left(\begin{array}{cc|c} a\sin\theta\cos\theta & b\cos\theta & (a^2+b^2)\sin\theta \\ 0 & b(\sin\theta-\cos\theta) & (a^2+b^2)(\cos\theta-\sin\theta) \end{array}\right) \tag{12}$$

We get the value of k:

$$k = \frac{(a^2+b^2)(\cos\theta-\sin\theta)}{b(\sin\theta-\cos\theta)} \tag{13}$$

Assuming $\theta \neq \pi/4$,

$$k = -\frac{a^2+b^2}{b} \tag{14}$$

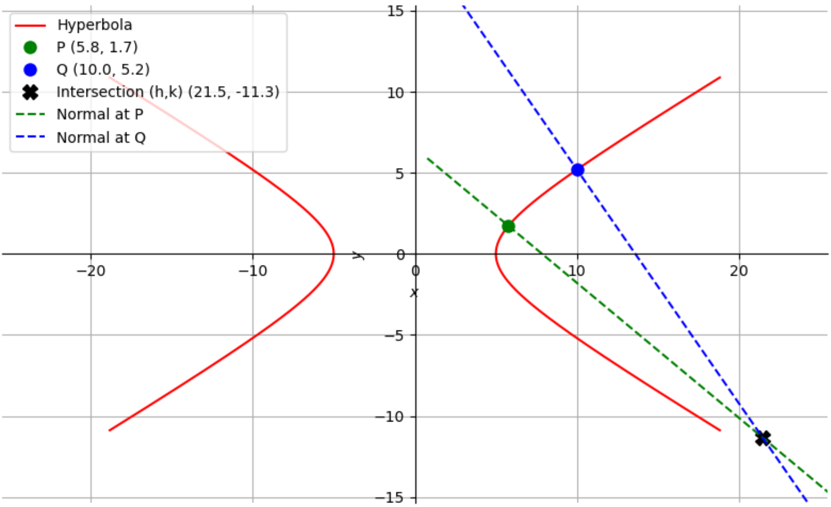
Hyperbola
P (5.8, 1.7)
Q (10.0, 5.2)
Intersection (h,k) (21.5, -11.3)
Normal at P
Normal at Q
15
10
5
0
−5
−10
−15
−20
−10
0
10
20
x
y

Fig. 1