4.13.53

EE25BTECH11049 - Sai Krishna Bakki

Question

Lines $L_1 \equiv ax + by + c = 0$ and $L_2 \equiv lx + my + n = 0$ intersect at the point P and make an angle θ with each other. Find the equation of a line L different from L_2 which passes through P and makes the same angle θ with L_1.

Solution

The intersection of lines is given as

$$L \equiv L_1 + kL_2 = 0 \tag{1}$$

If L is the reflection of L_2 in L_1, then for any point $\mathbf{Q}$ that lies on L_2, its reflection $\mathbf{R}$ across the line L_1 must lie on L.

$$L_1 \equiv ax + by + c = 0 \implies \mathbf{n}_1^T\mathbf{x} + c = 0, \mathbf{n}_1 = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{2}$$

$$L_2 \equiv lx + my + n = 0 \implies \mathbf{n}_2^T\mathbf{x} + n = 0, \mathbf{n}_2 = \begin{pmatrix} l \\ m \end{pmatrix}. \tag{3}$$

$$L \equiv (ax + by + c) + k(lx + my + n) = 0 \implies (\mathbf{n}_1^T\mathbf{x} + c) + k(\mathbf{n}_2^T\mathbf{x} + n) = 0. \tag{4}$$

Let us choose an arbitrary point Q, with position vector $\mathbf{q}$, that lies on the line L_2. The condition that Q is on L_2 is:

$$\mathbf{n}_2^T\mathbf{q} + n = 0 \tag{5}$$

Next, we find the position vector $\mathbf{r}$ for the point R, which is the reflection of Q in the line L_1. The standard vector formula for this reflection is:

$$\mathbf{r} = \mathbf{q} - 2\left(\frac{\mathbf{n}_1^T\mathbf{q} + c}{\mathbf{n}_1^T\mathbf{n}_1}\right)\mathbf{n}_1 \tag{6}$$

According to our principle, the reflected point R must lie on the line L. We substitute the expression for its position vector $\mathbf{r}$ from (6) directly into the equation for L.

$$(\mathbf{n}_1^T\mathbf{r} + c) + k(\mathbf{n}_2^T\mathbf{r} + n) = 0 \tag{7}$$

$$\left[\mathbf{n}_1^T\left(\mathbf{q} - 2\frac{\mathbf{n}_1^T\mathbf{q} + c}{\mathbf{n}_1^T\mathbf{n}_1}\mathbf{n}_1\right) + c\right] + k\left[\mathbf{n}_2^T\left(\mathbf{q} - 2\frac{\mathbf{n}_1^T\mathbf{q} + c}{\mathbf{n}_1^T\mathbf{n}_1}\mathbf{n}_1\right) + n\right] = 0 \tag{8}$$

$$\implies \left[\mathbf{n}_1^T\mathbf{q} - 2\frac{\mathbf{n}_1^T\mathbf{q} + c}{\cancel{\mathbf{n}_1^T\mathbf{n}_1}}(\cancel{\mathbf{n}_1^T\mathbf{n}_1}) + c\right] + k\left[(\mathbf{n}_2^T\mathbf{q} + n) - 2\frac{\mathbf{n}_1^T\mathbf{q} + c}{\mathbf{n}_1^T\mathbf{n}_1}(\mathbf{n}_2^T\mathbf{n}_1)\right] = 0 \tag{9}$$

$$\implies \left[\mathbf{n}_1^T\mathbf{q} - 2(\mathbf{n}_1^T\mathbf{q} + c) + c\right] + k\left[0 - 2\frac{(\mathbf{n}_1^T\mathbf{q} + c)(\mathbf{n}_1^T\mathbf{n}_2)}{\mathbf{n}_1^T\mathbf{n}_1}\right] = 0 \quad \text{(using (5))} \tag{10}$$

$$\implies -(\mathbf{n}_1^T\mathbf{q} + c) - k\left[2\frac{(\mathbf{n}_1^T\mathbf{q} + c)(\mathbf{n}_1^T\mathbf{n}_2)}{\mathbf{n}_1^T\mathbf{n}_1}\right] = 0 \tag{11}$$

Assuming Q is not on L_1, the term $(\mathbf{n}_1^T\mathbf{q} + c)$ is non-zero, allowing us to divide the entire equation by it:

$$-1 - k\left[\frac{2(\mathbf{n}_1^T\mathbf{n}_2)}{\mathbf{n}_1^T\mathbf{n}_1}\right] = 0 \tag{12}$$

$$k = -\frac{\mathbf{n}_1^T\mathbf{n}_1}{2(\mathbf{n}_1^T\mathbf{n}_2)} \tag{13}$$

Final Equation

Substitute this value of k back into the equation $L_1 + kL_2 = 0$

$$L_1 - \frac{\mathbf{n}_1^T\mathbf{n}_1}{2(\mathbf{n}_1^T\mathbf{n}_2)}L_2 = 0 \tag{14}$$

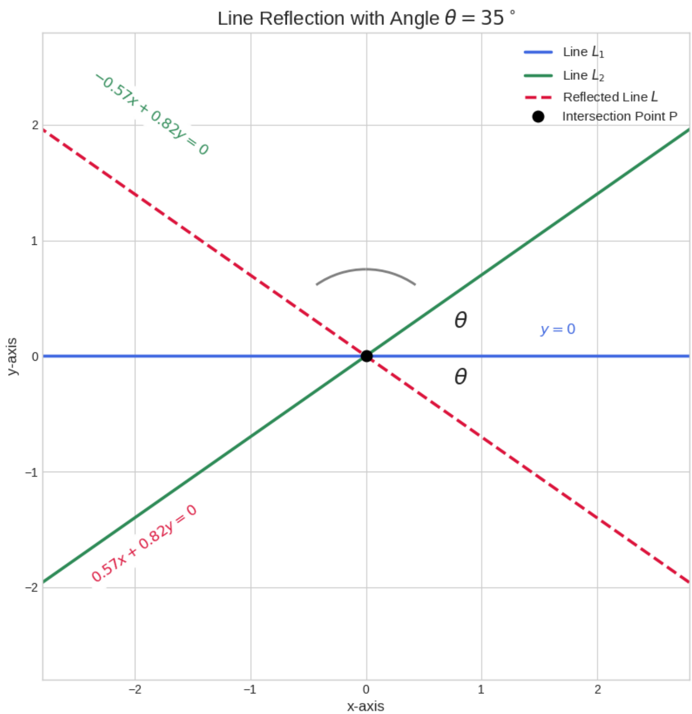
$$2(\mathbf{n}_1^T\mathbf{n}_2)L_1 - (\mathbf{n}_1^T\mathbf{n}_1)L_2 = 0 \tag{15}$$

Finally, substituting the algebraic forms for the scalar products:

- $\mathbf{n}_1^T\mathbf{n}_2 = al + bm$
- $\mathbf{n}_1^T\mathbf{n}_1 = a^2 + b^2$

We arrive at the final equation for the line L:

$$\boxed{2\begin{pmatrix} a & b \end{pmatrix}\begin{pmatrix} l \\ m \end{pmatrix}\left(\begin{pmatrix} a & b \end{pmatrix}\mathbf{x} + c\right) - \begin{pmatrix} a & b \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}\left(\begin{pmatrix} l & m \end{pmatrix}\mathbf{x} + n\right) = 0} \tag{16}$$



Line Reflection with Angle $\theta = 35^\circ$
Line L_1
Line L_2
Reflected Line L
Intersection Point P
$-0.57x + 0.82y = 0$
$0.57x + 0.82y = 0$
$y = 0$
θ
θ
y-axis
x-axis
2
1
0
−1
−2
−2
−1
0
1
2

Fig. 1