Take-home Final Exam (3 days)

1) **Finite element (FE) method** **(10 points)**

   Proof that the following shape/basis functions are true for 2D linear triangle elements.

$$[N(x, y)] = [N_i(x, y)\ N_j(x, y)\ N_k(x, y)],$$

$$N_i(x, y) = \frac{1}{2A}(a_i + b_i x + c_i y)$$

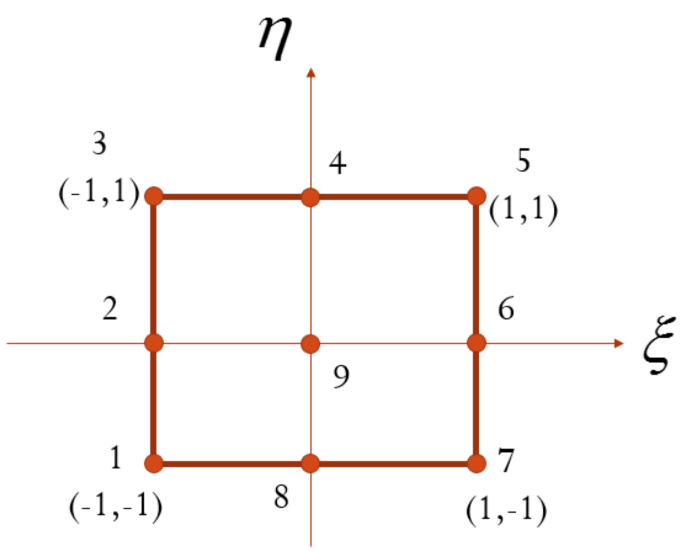
$$N_j(x, y) = \frac{1}{2A}(a_j + b_j x + c_j y)$$

$$N_k(x, y) = \frac{1}{2A}(a_k + b_k x + c_k y)$$

where A is the area of an element, and a, b and c are x and y dependent coefficients (you will need to find A, and a/b/c in terms of x and y).

2) **Finite element (FE) method** **(10 points)**

   For 9-node quadrilateral 2D triangle element, demonstrate all the 9 shape functions.



3) Finite element (FE) method (20 points)

Given the following partial differential equation and boundary conditions:

$$k_x \frac{\partial^2 T}{\partial x^2} + k_y \frac{\partial^2 T}{\partial y^2} = Q$$

$$T = T_0(x, y) \qquad on\,\Gamma_1 \quad (bounary)$$

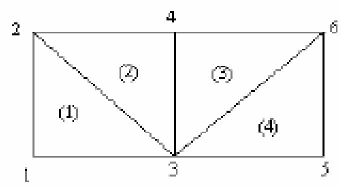
$$k_x \frac{\partial T}{\partial x} n_x + k_y \frac{\partial T}{\partial y} n_y = q \qquad on\,\Gamma_2$$

(a) Derive the FE solution equation for the above Laplacian equation using the Galerkin's method (10 points).

(b) Demonstrate how to assemble the element stiffness matrix *S* in order to generate the global system matrix *K* based on the following uniform mesh. Also give the NBW (half bandwidth) value of this global system matrix. (10 points).

- Example

Element number	Nodal points		
1	1	3	2
2	4	2	3
3	4	3	6
4	5	6	3



Element stiffness

$$\begin{Bmatrix} F_i \\ F_j \\ F_k \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{Bmatrix} \delta_i \\ \delta_j \\ \delta_k \end{Bmatrix}$$

4) Finite element (FE) method (25 points)

Write a Matlab, C++ or FORTRAN code (or other computer language of your choice) to demonstrate how to solve the following photoacoustic wave equation and its boundary conditions:

$$\nabla^2 P(\mathbf{r}, t) - \frac{1}{v_0^2} \frac{\partial^2 P(\mathbf{r}, t)}{\partial t^2} = -\frac{\beta \Phi(\mathbf{r})}{C_p} \frac{\partial J(t)}{\partial t},$$

$$\nabla P \cdot \hat{n} = -\frac{1}{v_0} \frac{\partial P}{\partial t} - \frac{P}{2r}$$

Example model:

The geometry of test case is shown below where a circular background region (15mm in radius) contains three circular targets (2mm in radius each) positioned at 2, 9 and 6 o'clock, respectively. A pulsed laser $J(t) = Ae^{-\left(\frac{t-t_0}{\tau}\right)^2}$ illuminates the whole domain and generates the photoacoustic wave $P(r,t)$. You can use $T = 3.0 \times 10^4 ns$ as the time range and $\Delta t = 50ns$ as the time interval in this calculation. You will need to generate a finite element mesh with appropriate size and all the preprocessing files to obtain a solution with good accuracy. The optical and acoustic properties to be used are listed in Table 1. For the pulsed laser, $A = 1.0, t_0 = 200ns, \tau = 10ns$.

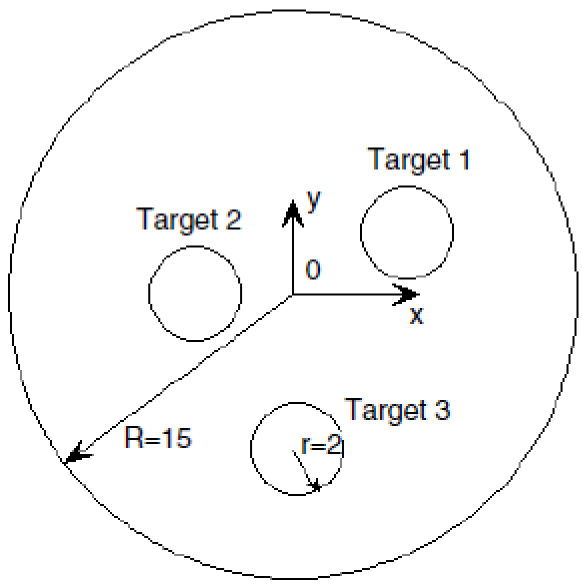


Table 1. Optical properties, acoustic velocity and size of targets used in the simulation study. (Note: $C_p = 1.0$ cal g^{-1} °C^{-1}, $\beta = 10^{-5}$ °C^{-1} and $v_0 = 1495.0$ m s^{-1} for all tests.)

Simulation cases	Background c (m s^{-1})	Background Φ (mJ mm^{-3})	Targets c (m s^{-1})	Targets Φ (mJ mm^{-3})	Diameter (mm)
1	1487.5	1.0	1487.5	2.0	4.0

5) Inverse problem (35 points)

Temperature imaging/monitoring in RF (radio frequency), microwave and laser ablation for treating cancer is critical. The goal of such a treatment is to use high temperature generated by the radiation to kill cancer cells. It is very important, however, that no or minimal healthy cells will be destroyed. Hence, knowing the spatial distribution or imaging of the temperature becomes important. This is a typical inverse problem where you can use the measured data at the tissue boundary to reconstruct the tissue temperature image. Assuming you will use microwave or laser to treat cancer, and you are asked to formulize the inverse problem in two-dimension.

(1) Find a mathematical model (i.e., a partial differential equation) that describes the heat transfer in tissue when microwave or laser is used (15 points).
(2) Use Tikhonov regularization based Newton method (i.e., Taylor expansion method) to obtain all the equations needed for solving the inverse problem of temperature reconstruction (20 points).