Probabilistic Algorithms: What, Why, and How

A Deep Dive into Randomness in Computing

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Outline

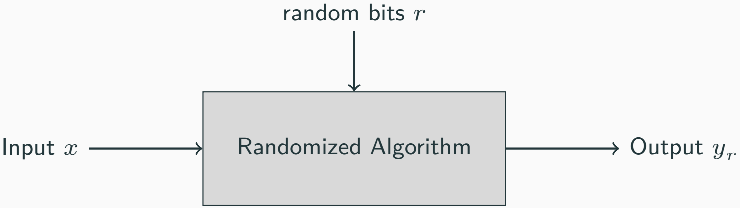
What are Probabilistic Algorithms?

What is a Probabilistic Algorithm?

Definition
An algorithm that makes random choices during execution to influence its behavior or output.

Deterministic vs Probabilistic Algorithm

Deterministic



Probabilistic

Types of Probabilistic Algorithms

- Las Vegas Algorithms (Babai, 1979)
- Monte Carlo Algorithms (Metropolis & and, 1949)

Las Vegas Algorithms

Definition: A Las Vegas algorithm always produces a correct result or reports failure, with the running time depending on random choices. (Gupta & Ramachandran, 1992)

Monte Carlo Algorithms

Definition: A Monte Carlo algorithm has a probability of producing an incorrect result, but its running time is bounded. (James, 1990)

Applications of Probabilistic Algorithms

Real-World Motivation

- Web search (PageRank)
- Load balancing (power of two choices)
- Hashing (universal hash functions)
- Primality testing (Miller-Rabin)

Why Probabilistic Algorithms?

Why Randomness?

Motivation

- Simpler algorithms
- Better expected performance
- Avoid worst-case scenarios
- Useful for large-scale and distributed systems

How do Probabilistic Algorithms Work?

How: Randomization in Algorithms

Key Idea
Use random choices to influence the algorithm's path or output.

- Random pivot in Quicksort
- Random walks in graphs
- Random sampling

Example: Randomized Quicksort

QuickSort vs Randomized QuickSort

QuickSort:

1. Pick a pivot element from the array (Hoare, 1962)
2. Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself
3. Recursively sort the subarrays, and concatenate them

Randomized QuickSort:

1. Pick a pivot element **uniformly at random** from the array (Motwani & Raghavan, 1995)
2. Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself
3. Recursively sort the subarrays, and concatenate them

Example: Randomized Quicksort

Recall: QuickSort can take $\Omega(n^2)$ time to sort an array of size n (Sedgewick, 1978)

Randomized QuickSort: Expected Runtime

Theorem

Randomized QuickSort sorts a given array of length n in $O(n \log n)$ expected time. (Sedgewick, 1977)

Note: On every input, randomized QuickSort takes $O(n \log n)$ time in expectation. On every input, it may take $\Omega(n^2)$ time with some small probability.

Randomized Quicksort: Step 1 (Initial Array)

Consider the array:

15	3	1	10	9	0	6	4

Randomized Quicksort: Step 1.1 (Pivot Chosen)

Suppose the random pivot chosen is 10 (at index 3):

15	3	1	10	9	0	6	4

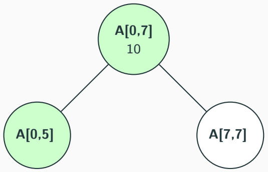
Randomized Quicksort: Step 2 (Partitioning Around Pivot 10)

After selecting pivot 10, we partition the array:

- Left: 4, 3, 1, 9, 0, 6 (elements before pivot position)
- Middle: 10 (pivot)
- Right: 15 (element after pivot position)

After partitioning:

4	3	1	9	0	6	10	15

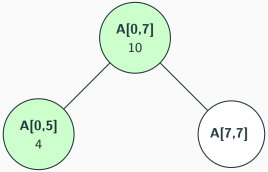


Randomized Quicksort: Step 3 (Recurse Left [A[0,5]], Pivot 4)

Recurse on the left subarray:

Let's choose a random pivot, say 4.

4	3	1	9	0	6



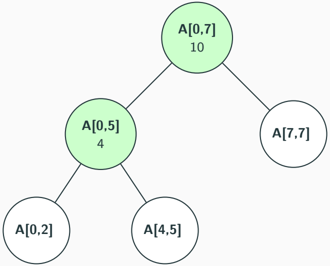
Randomized Quicksort: Step 3.1 (Partition Left [A[0,5]] Around 4)

After partitioning the left subarray:

0	3	1	4	9	6

Partition:

- Left: 0, 3, 1 (elements before pivot)
- Middle: 4 (pivot)
- Right: 9, 6 (elements after pivot)

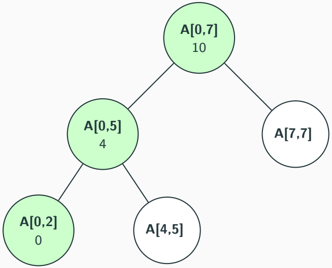


Randomized Quicksort: Step 3.1.1 (Recurse Left [A[0,2]], Pivot 0)

Recurse on the left subarray:

Let's choose a random pivot, say 0.

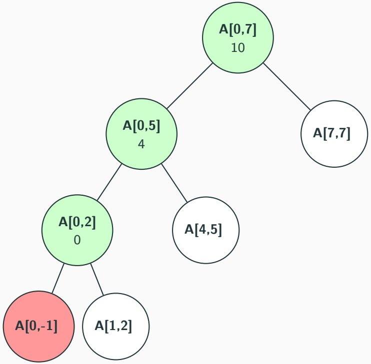
0	3	1



Randomized Quicksort: Step 3.1.1.1 (Partition Left [A[0,2]] Around 0)

After partitioning the left subarray:

0	3	1



Partition:

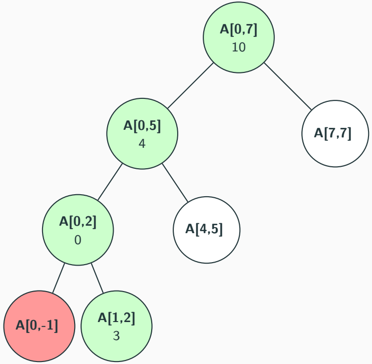
- Left: (empty)
- Middle: 0 (pivot)
- Right: 3, 1 (elements after pivot)

Randomized Quicksort: Step 3.1.1.2 (Recurse Right [A[1,2]], Pivot 3)

Recurse on the right subarray:

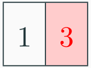
Let's choose a random pivot, say 3.

3	1



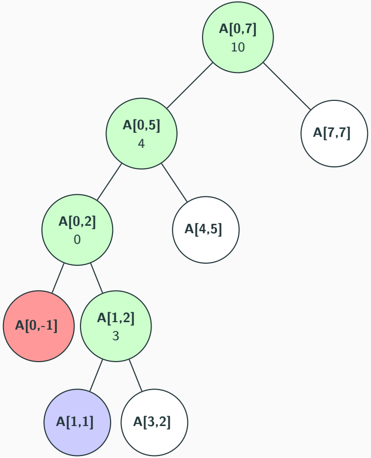
Randomized Quicksort: Step 3.1.1.2.1 (Partition [A[1,2]] Around 3)

After partitioning the left subarray:



Partition:

- Left: 1 (element before pivot)
- Middle: 3 (pivot)
- Right: (empty)



Randomized Quicksort: Step 3.1.1.2.1.1 (Recurse Left [A[1,1]], Done)

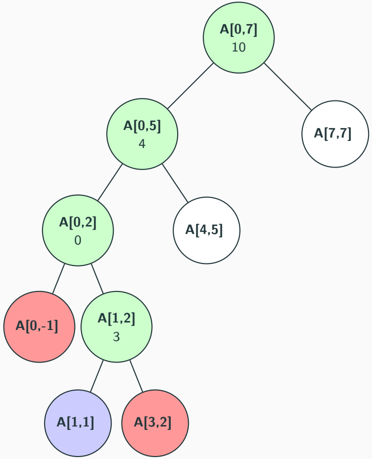
After partitioning the left subarray:

1

Partition:

- Left: (empty)
- Middle: 1 (pivot)
- Right: (empty)

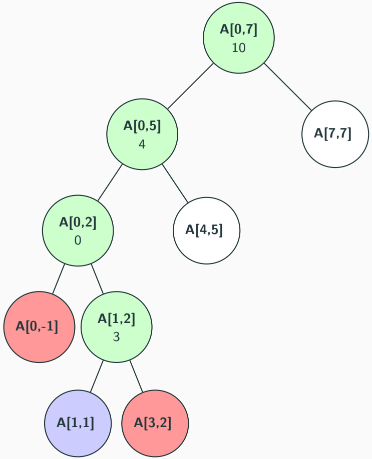
Single element subarray, done, return.



Randomized Quicksort: Step 3.1.1.2.1.2 (Recurse Right [A[3,2]], Done)

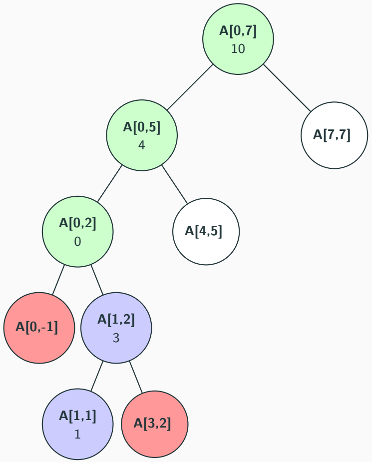
Recurse on the right subarray $A[3, 2]$ (empty, done).

Return to parent call $A[1, 2]$



Randomized Quicksort: Step 3.1.2 (Return to [A[0,2]])

Return to parent call $A[0,2]$

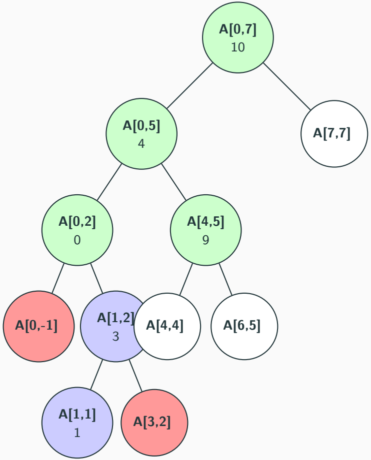


Randomized Quicksort: Step 3.2 (Recurse Right [A[4,5]], Pivot 9)

Recurse on the right subarray $A[4,5]$:

Let's choose a random pivot, say 9.

9	6



Randomized Quicksort: Step 3.2.1 (Partition [A[4,5]] Around 9)

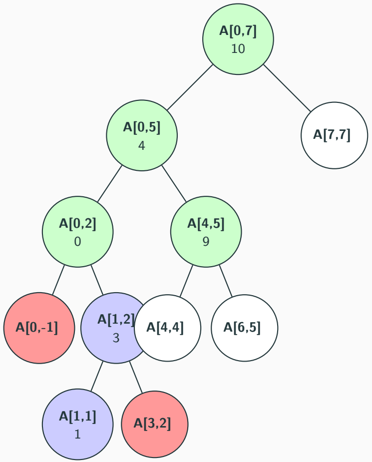
Recurse on the right subarray $A[4,5]$:

Suppose the random pivot is 9:

6	9

Partition:

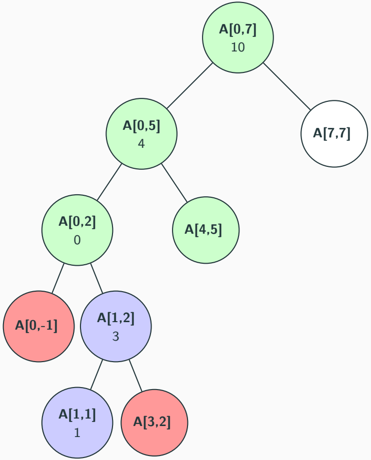
- Left: 6 (element before pivot)
- Middle: 9 (pivot)
- Right: (empty)



Randomized Quicksort: Step 3.2.1.1 (Recurse Left [A[4,4]], Done)

After partitioning the left subarray:

6



Partition:

- Left: (empty)
- Middle: 6 (pivot)
- Right: (empty)

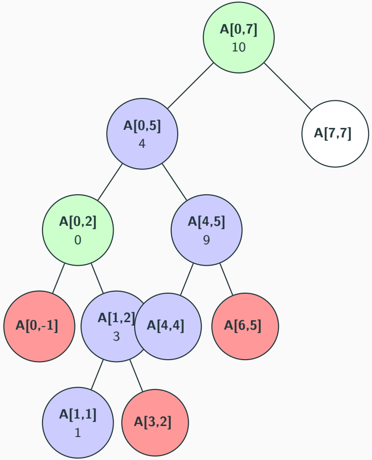
Single element subarray, done, return.

Randomized Quicksort: Step 3.2.1.2 (Recurse Right [A[6,5]], Done)

Recurse on the right subarray $A[6,5]$ (empty, done).

Return to parent call $A[4,5]$

Return to parent call $A[0,5]$



Randomized Quicksort: Step 3.3 (Recurse Right [A[7,7]], Done)

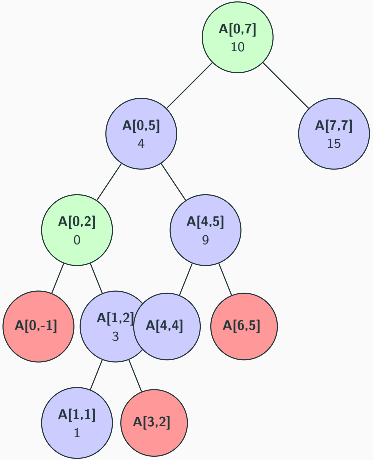
Recurse on the right subarray $A[7,7]$ After partitioning the right subarray:

$$\boxed{1}$$

Partition:

- Left: (empty)
- Middle: 1 (pivot)
- Right: (empty)

Single element subarray, done, return.



Randomized Quicksort: Step 4 (Final Sorted Array)

The final sorted array is:

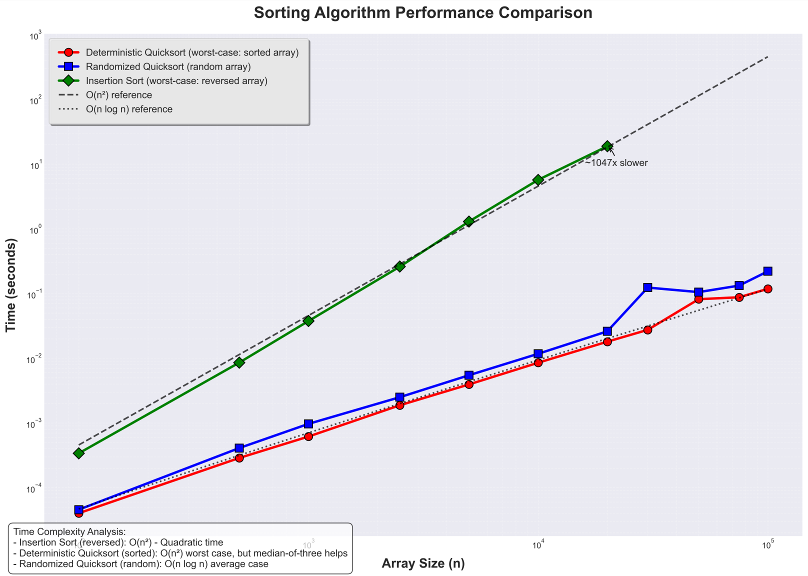
0	1	3	4	6	9	10	15

Quicksort Time Complexity

- **Worst-case:** $O(n^2)$
- **Best-case:** $O(n \log n)$
- **Expected:** $O(n \log n)$ (randomized)

Time Complexity Analysis of Randomized Quicksort

Sorting Algorithms Benchmark



Quicksort Recurrence

Expected Comparisons
Let $T(n)$ be the expected number of comparisons to sort n distinct elements using randomized quicksort:

$$T(n) \leq n + \frac{1}{n} \sum_{i=1}^{n} (T(i-1) + T(n-i))$$

- n comparisons in partitioning: each element compared to the pivot.
- Pivot is chosen uniformly at random.
- For pivot at position i, recursive calls sort subarrays of size $i-1$ and $n-i$.
- We average over all n possible pivot positions.

Base Case

$$T(1) = 0 \quad \text{(single element requires no comparisons)}$$

Solving the Recurrence Step-by-Step

Step 1: Write the Recurrence

$$T(n) \leq n + \frac{1}{n}\sum_{i=1}^{n}(T(i-1) + T(n-i))$$

Step 2: Guess the Solution Assume $T(n) \leq cn \log n$ for some constant c.

Solving the Recurrence Step-by-Step

Step 3: Plug the Guess

$$T(n) \leq n + \frac{2c}{n} \sum_{k=1}^{n-1} k \log k$$

Use:

$$\sum_{k=1}^{n-1} k \log k \leq \frac{1}{2} n^2 \log n - \frac{1}{8} n^2$$

Then:

$$T(n) \leq n + cn \log n$$

Step 4: Conclusion

$$T(n) = O(n \log n)$$

N-Queens problem with Las Vegas Approach

What is the N-Queens Problem?

- Place N queens on an $N \times N$ chessboard so that no two queens threaten each other.
- Each queen must be in a different row, column, and diagonal.
- Number of possible arrangements grows rapidly with N ($N!$ for rows/columns). (Bell & Stevens, 2009)

Why is N-Queens Interesting?

- **Computational Challenge:** Classic example of a constraint satisfaction problem. (Motwani & Raghavan, 1995)
- **Applications:**
  - Constraint satisfaction problems
  - Testing algorithms for search and optimization
  - AI and backtracking benchmarks
- **Why solve efficiently?** For large N, brute-force and naive methods become infeasible.

How: Traditional Backtracking

- Systematically tries to place queens row by row
- Backtracks when a conflict is detected
- **Time complexity:** $O(N!)$ in the worst case (Motwani & Raghavan, 1995)

Traditional Backtracking: Pseudocode

```
Function solve(row):
    if row = N then
        return true
    for col ← 0 to N-1 do
        if isSafe(row, col) then
            placeQueen(row, col);
            if solve(row + 1) then
                return true
            removeQueen(row, col);
    return false
```

Why Consider Randomization?

- Deterministic search (backtracking) can be slow for large N (Motwani & Raghavan, 1995)
- Randomized (Las Vegas) algorithms can find solutions much faster on average (Babai, 1979)
- Demonstrates the power of probabilistic algorithms in combinatorial search

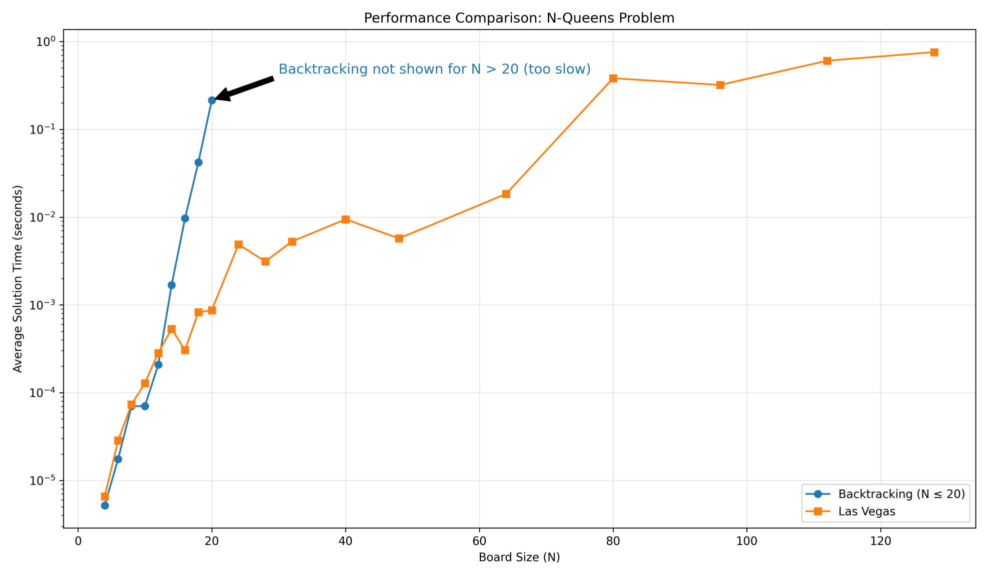
How: Las Vegas (Randomized) Approach

- For each row, randomly select a safe column
- If stuck (no safe columns), restart from scratch
- Always finds a correct solution (if one exists), but runtime is random
- **Expected time:** Much faster than backtracking for large N (Babai, 1979)

Las Vegas (Randomized) Approach: Pseudocode

```
Function lasVegasNQueens(N, maxAttempts):
    for attempt ← 1 to maxAttempts do
        board ← empty_board(N) success ← true for row ← 0 to N-1 do
            safeCols ← get_safe_columns(row, board) if safeCols is empty then
                success ← false; break
            col ← random choice from safeCols place_queen(row, col, board)
        if success then
            return board
    return None
```

Performance Comparison (Visualization)



Performance Comparison (Interpretation)

- Average solution time vs. N for Backtracking and Las Vegas approaches.
- Las Vegas (randomized) is much faster for large N.
- Backtracking becomes infeasible as N grows.

Performance Comparison

- **Backtracking:**
  - Deterministic, exhaustive search
  - Predictable but slow for large N
- **Las Vegas:**
  - Randomized, may restart
  - Runtime varies, but much faster on average for large N

Key Insights

- Randomization can dramatically improve performance for some combinatorial problems (Motwani & Raghavan, 1995)
- Las Vegas algorithms always produce correct results, but runtime is a random variable (Babai, 1979)
- For N-Queens, Las Vegas approach is practical for very large N where backtracking is infeasible
- Illustrates the power of probabilistic algorithms in search and optimization

Example: Monte Carlo Estimation of pi

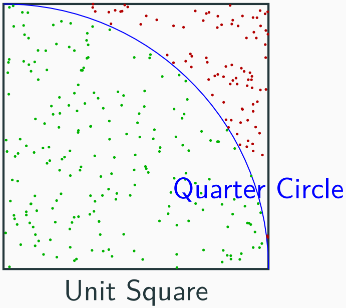
Monte Carlo Method - Estimating π

The Monte Carlo method (Metropolis & and, 1949) estimates π by simulating random points in a unit square and counting how many fall inside a quarter circle of radius 1. The ratio of points inside the circle to the total points, multiplied by 4, approximates π (Beckmann, 1971).

Monte Carlo Algorithm

1. Generate N random points (x, y) where $0 \leq x \leq 1$ and $0 \leq y \leq 1$.
2. For each point, check if it lies inside the quarter circle: $x^2 + y^2 \leq 1$.
3. Count the number of points M that satisfy the condition.
4. Estimate π as: $\pi \approx 4 \times \frac{M}{N}$ (Hammersley & Handscomb, 1964).

Visual Illustration



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[1]This slide was generated using a probabilistic algorithm

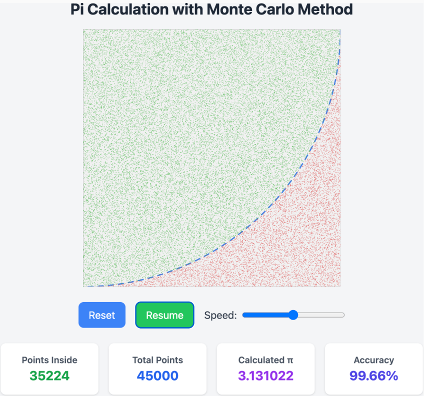
Example Calculation

- Suppose we generate $N = 1000$ random points in the unit square
- After simulation, we count $M = 785$ points inside the quarter circle
- We estimate π as:
$$\pi \approx 4 \times \frac{M}{N} = 4 \times \frac{785}{1000} = 3.14$$
- The true value of π is approximately 3.14159 (Beckmann, 1971)

Convergence and Error Analysis

- The error in our estimate decreases as $O(1/\sqrt{N})$ (Kalos & Whitlock, 2008)
- This means:
  - $N = 100$ points gives roughly 10% error
  - $N = 10,000$ points gives roughly 1% error
  - $N = 1,000,000$ points gives roughly 0.1% error
- The Monte Carlo method is especially useful for calculating multidimensional integrals (Cookson, 2005)
- For π calculation, there are more efficient methods, but this one is visually intuitive

Demo Visualization



Open Interactive Demo

Probabilistic Data Structures

Deterministic vs. Probabilistic Data Structures

Deterministic (e.g., Hash Set, List):

- Always provide exact answers.
- Can be space-intensive (store all elements).
- Operations might be slower for large datasets (e.g., disk I/O).
- **Guarantee:** No errors (false positives or negatives).

Probabilistic (e.g., Bloom Filter):

- Provide approximate answers with controlled error.
- Very space-efficient (use bits, not full elements).
- Operations are typically very fast (constant time).
- **Trade-off:** Small error probability for huge efficiency gains.

Key Idea
Use PDS when approximate answers are acceptable and space/speed are critical.

Example: Why PDS? Username Availability

The Problem
A website with millions of users needs to instantly check if a username is available during registration. How? (Bloom, 1970)

Deterministic Approach (Database Query):

- Store all usernames in a database.
- Query DB: ‘SELECT 1 FROM users WHERE username = ¿
- **Accurate? Yes.**
- **Fast? No.** Requires disk I/O, network latency.
- **Scalable? Poorly.** High load on DB servers.

Probabilistic Approach (Bloom Filter):

- Keep a compact Bloom filter in memory (Bloom, 1970).
- Check filter: Is ‘username‘ possibly present?
- **Accurate? Mostly.** Small chance of false positive (saying taken when available), needs DB check then.
- **Fast? Yes.** In-memory check is O(k).
- **Scalable? Excellently.** Drastically reduces DB load

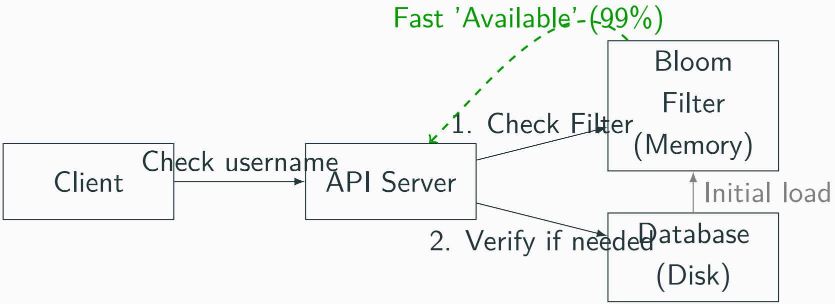
Username Checking: Implementation Details

1. **Initialization:** Load all existing usernames into Bloom filter at service startup (only infrequent DB reads).
2. **New registrations:** Add username to both database and Bloom filter.
3. **Availability check process:**
   - Check username against Bloom filter first (Fast, in-memory)
   - If Bloom filter says "definitely not in set" $\rightarrow$ Username is available (99% case for 1% error rate)
   - If Bloom filter says "possibly in set" $\rightarrow$ Verify with database query (Slow, but rare)

Performance Impact (10M users, 1% error)

- Memory: ≈ 18 MB Bloom Filter vs. hundreds of MB for DB index/cache.
- Speed: 99% of availability checks avoid slow database queries. (Wang & Reiter, 2012)

Username Checking: System Architecture



- Bloom filter acts as a fast, efficient preliminary check.
- Deterministic check (DB) used only as a fallback.
- Massively reduces load on the expensive resource (Database).

Bloom Filters: The Theory

- Space-efficient probabilistic data structure (Bloom, 1970)
- Tests if an element is a member of a set
- Possible false positives, never false negatives
- Components:
  - Bit array of m bits (initially all 0)
  - k independent hash functions

0	0	0	0	0	0	0	0

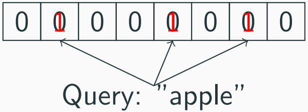
Bloom Filter Operations

Add element:

1. Hash element with k functions
2. Set bits at these k positions to 1

Query element:

1. Hash element with k functions
2. Check bits at these k positions
3. If **any** bit is 0: Definitely not in set
4. If **all** bits are 1: Probably in set



The Math Behind Bloom Filters

- **False positive probability (p):**

$$p \approx \left(1 - e^{-kn/m}\right)^k \tag{1}$$

(Broder & Mitzenmacher, 2003)

- **Optimal size (m bits) for n items, error p:**

$$m = -\frac{n \ln p}{(\ln 2)^2} \tag{2}$$

- **Optimal hash functions (k):**

$$k = \frac{m}{n} \ln 2 \approx 0.7 \cdot \frac{m}{n} \tag{3}$$

Time and Space Complexity

Structure	Space	Lookup	Insert	Error Type
Hash Set	$O(n)$	$O(1)$ avg	$O(1)$ avg	None
Bloom Filter	$O(m)$	$O(k)$	$O(k)$	False Positives
Sorted List	$O(n)$	$O(\log n)$	$O(n)$	None
Trie	$O(N)$	$O(L)$	$O(L)$	None

n=items, m=bits ($m \ll n \times item\_size$), k=hashes, N=total chars, L=key length

Bloom Filter Simulation

Bloom Filter Simulation

Check if your username or password has been compromised

Check Value

Enter username or password to check:

admin

Check | Add to Database

Possibly compromised! This value may exist in the database.

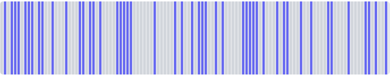
Confidence: 93.91%

Real match probability: 93.91%
False positive probability: 6.09%

A Bloom filter can have false positives but no false negatives.

Bloom Filter Visualization

Size: 384 bits | Hash Functions: 3



Settings

Filter Size

Hash Functions

Reset Filter

Known Compromised Values

3 | admin | qwerty | 123456 | letmein | baseball | dragon | football | monke

2 | zaq1zaq1 | hello123 | charlie | jesus | ninja | mustang | chocolate | starw

About Bloom Filters

A Bloom filter is a space-efficient probabilistic data structure designed to quickly test whether an element is present in a set. It can have false positives (incorrectly reporting an element is in the set) but no false negatives (if it reports an element is not in the set, it definitely is not).

How It Works

1. Multiple hash functions are applied to the input element
2. Each hash function gives a position in the bit array
3. When adding an element, all corresponding bits are set to 1
4. When checking, if all corresponding bits are 1, the element might be in the set
5. If any bit is 0, the element is definitely not in the set

Use Cases

- Checking if a username is taken before querying a database
- "Have I Been Pwned" password checking
- Spell checkers
- Web cache sharing
- Network packet routing

Other Applications of Bloom Filters

Web/Database:

- Cache hit/miss optimization (e.g., CDNs)
- Avoid unnecessary DB lookups (like username example)
- Recommendation systems (seen items) (Broder & Mitzenmacher, 2003)

Network:

- Web crawler URL deduplication (avoid re-crawling)
- Network packet routing (track flows efficiently)
- P2P network resource discovery

Security:

- Malware signature detection
- Spam filtering (known bad IPs/domains)
- Password breach checking (HaveIBeenPwned)

Big Data:

- Stream deduplication (unique visitors/events)
- Distributed data sync (approximate differences)
- Genomics (k-mer counting)

When to Use Bloom Filters

Bloom filters are ideal when:

- Memory is a critical constraint (Big Data, embedded systems)
- False positives are acceptable (can be handled by a secondary check)
- False negatives are unacceptable (must find all true positives)
- Elements are expensive to store or compare
- Lookup speed is crucial (real-time systems)
- Deletions are not needed (or use variants like Counting Bloom Filters)

References

Babai, L. (1979). **Monte-carlo algorithms in graph isomorphism testing.** *Technical Report DMS*, *79-10*.

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