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Symmetry protected topological phases in statistical mechanics models

The aim of this work is to determine if the recent notion of symmetry protected topological has a meaning for statistical models. It presents a first attempt based on an Ising model that leads to non-physical model. The second try is more fruitfull, based on the Trotter-Suzuki decomposition, we present a statistical models associated with a spin chain that presents SPT order. This report ends on the description of the tensor renormalisation group, which is a tool already used in the investigation on SPT phase and that can be applied to the stat-mech model we obtained.

Key words : *SPT, stat-mech, Topological order, edge state, tensor network, duality.*

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Contents

Introduction

For a long time, continuous phase transitions have been believed to be described by symmetry breaking theory [1]. Each phase can be described with the tools of this theory : order parameter, long range correlation and group theory. However, it has appeared that this paradigm was not enought. For instance, fractional quantum Hall states [2] are actually topologically ordered states [3] and cannot be described with the breaking symmetry theory. The name topological order [4] was coined since those new phases can be characterized by topological features. For a review on topological order notion, see Wen's review [5].
The main interest of this report lays in a recent aspect in topological phase where symmetry (time reversal or spin inversion for instance) can protect the topological distinction, the so-called Symmetry protected topological (SPT) phases[6]. The ground states of SPT phases contain only short-range entanglement and can be smoothly deformed into a totally trivial product state if the symmetry requirement is not enforced explicitely [1] in the system. However, with symmetry, the non-trivial SPT order is manifested in the existence of gapless edge states on the boundary of the system which cannot be removed as long as symmetry is not broken. Many SPT phases have been discovered over the past decades and a general structure of the theory of SPT phases is emerging. In 1d SPT phases have been completely classified for general interacting bosonic/fermionic systems which carry non-trivial projective representations of the symmetry in their degenerate edge states [7–9]. In fact, the Haldane/AKLT phase of spin-one quantum antiferromagnet, is a physical realization of a 1D SPT phase, which is protected by spin rotation or time reversal symmetries [10]. There have been consequent advances in higher dimension, see for example [11, 12].
It is worth noting that SPT phases have always been seen and thought with quantum toolkit: wavefunction, ground state, Berry phase, Matrix product... However, a recent paper, Geraedts and Motrunich[13] displays a 3D classical statistical-mechanics model that displays SPT order. This paper raises a question that is the main question of this report: what SPT order means for classical 2D stat-mech models? This work tries to find a stat-model that displays SPT order and to answer this question.

In this report, we first introduce the concept of SPT order by giving two relevant examples for the physics of this concept. The AKLT chain is used to show edges states and short range entanglement and a Haldane chain is used to underline the symmetry aspect of SPT order. The goal is to give a physical idea of what is SPT order.
The second part is the description of the first attempt to have a stat-mech model, similar to the one given by Geraedts and Motrunich[13] in two dimensions with Ising spin. The idea was to use the well-known Ashkin-Teller model which have partially ordered phases, apply duality on only one spin to make the parameter order non-local. It appears that the dual model have a sign problem. Which means negative Boltzman weight, so the model is non-physical.
The third part exposes mapping of quatum system to classical one. The main idea is to map a quantum chain to a 2D stat-mech model with some usual tools like Suzuki-Trotter decomposition or exact mapping and find the intersection of the parameter space where the quantum chain has SPT order and the stat-mech model has positive Boltzman weight.
The fourth part describes a vertex model using a recent approach namely the tensor-network formalism. Tensor network renormalistion is explained in order to apply it to vertex model.

[1]In the following, we will make an important distinction between explicitely broken symmetry which are symmetry broken by the model and spontaneously broken symmetries that are the keystone of Landau's theory. Our interest goes to the first.

1 Symmetry protected topological order

Symmetry protected topological (SPT) phases are bulk-gapped quantum phases with symmetries. If the system is on a closed manifold, the ground state is unique and assumed to not spontaneously break the symmetry. However, if the system has a boundary, there are gapless or degenerate edge states as long as the symmetries are not broken. Therefore SPT phases represent a non-trivial type of order beyond Landau's symmetry breaking theory as there can be several phases with the same symmetries with phase transition between these phases as long as symmetry is preserved. For more details about quantum phase classification, see [7]. I propose here to describe the simplest example of SPT phase *i.e.* the so-called AKLT chain, and then to expose one of the feature of the SPT order using the Haldane chain.

1.1 AKLT Chain

The first known example of SPT phase is the antiferromagnetic Heisenberg spin 1 chain:

$$\mathcal{H} = \sum_i \mathbf{S}_i \mathbf{S}_{i+1}, \tag{1.1}$$

Haldane first conjectured and it was demonstrated that the bulk is gapped, as opposed to the spin 1/2 cases which is gapless. However, the ground state of the Heisenberg antiferromagnet is four-fold degenerate for an open boundary. To understand these properties, the Affleck, Lieb, Kennedy and Tasaki (AKLT) Hamiltonian [14], which is in the same phase as the Heisenberg antiferromagnet offers a better insight:

$$\mathcal{H}_{AKLT} = \sum_i \mathbf{S}_i \mathbf{S}_{i+1} + \frac{1}{3}(\mathbf{S}_i \mathbf{S}_{i+1})^2, \tag{1.2}$$

The ground state wavefunction was shown to be described by the figure 1 where each spin 1 decomposes into two spin 1/2 (the sum of the two sppin 1/2 of the same site are always in a spin 1 state) and those form the singlet $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$ between nearest neighbor sites. The total wavefunction is a product of thoses singlets between site i and $i+1$. It's clear that the bulk of the system does not break symmetry and is short range entangled via the singlet bond.

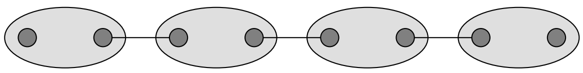


Figure 1: Valence bond structure of AKLT ground state. Each small circle represents a spin 1/2, each oval represent a spin 1 (triplet). Connected circles are singlet of spin 1/2.

If we put this chain on open boundary condition, there will be two spins 1/2 which cannot form a singlet. Therefore a four-fold degeneracy appears due to two-fold degeneracy for each end spin. This example exposes clearly two major features of the SPT Phase *i.e.* edge states and Short Range Entanglement (in contrast to the gapless $S = 1/2$ chain does not present such features).

1.2 The Haldane Chain

One of the most striking aspect of SPT order is that if some symmetries are enforced in the Hamiltonian, the system has to go throught a phase transition to go from the non-trivial SPT phase to the SPT phase. Whereas if one relax the symmetry condition on the Hamiltonian, one can find a path to connect every SPT phase to the trivial one. In order to make this clear, let's consider a Haldane chain:

$$H_0 = J \sum_j \mathbf{S}_j \cdot \mathbf{S}_{j+1} + B_x \sum_j S_j^x + U_{zz} \sum_j (S_j^z)^2. \tag{1.3}$$

This model had been studied by [15]. Its symmetries include translation, spatial inversion, a rotation by π around the x-axis, and a combination of a rotation by π around the y-axis and time-reversal $e^{-i\pi S^y}\times$TR (which takes $S_j^{x,z} \to S_j^{x,z}$, $S_j^y \to -S_j^y$). At large U_{zz}, the system is a trivial insulator which can be approximated by the wavefunction where all the sites are in the $|S^z = 0\rangle$ state. There is two antiferromagnetic phases Z_2^y and Z_2^z with spontaneous non-zero expectation values of $\langle S^y\rangle$ and $\langle S^z\rangle$, respectively. These phases are well described by the Landau's theory of symmetry breaking. $U_{zz} = B_x = 0$ is the Heisenberg point, for which one finds the gapped Haldane phase discussed above. The Haldane phase is still separated from the non-degenerate phases for nonzero U_{zz} and B_x values by phase transitions *i.e.* this model exhibit SPT phases is shown in FIG. 2(a). In order to effectively see the "symmetry protection", Pollmann et al.[10] added two perturbations to this Hamiltonian which break differents symmetries :

- a perturbation which is translation invariant and symmetric under spatial inversion, but breaks the $e^{-i\pi S^x}$ and the $e^{i\pi S^y}\times$TR symmetry.

$$H_1 = B_z \sum_j S_j^z + U_{xy} \sum_j \left(S_j^x S_j^y + S_j^y S_j^x\right). \tag{1.4}$$

  The phase diagram for fixed $B_z = 0.1J$ and $U_{xy} = 0.1J$ as a function of B_x and U_{zz} is shown in FIG. 2(b) and presents a Haldane phase as the precedent Hamiltonian.

- a perturbation which breaks the inversion symmetry but is invariant under $e^{i\pi S^y}\times$TR :

$$H_2 = R \sum_j [S_j^z (S_j^x S_{i+1}^x + S_j^y S_{j+1}^y) - S_{j+1}^z (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) + \text{H.c.}] \tag{1.5}$$

  The phase diagram for the parameter $R = 0.1J$ is shown in FIG. 2(c). Here, there is no Haldane phase. More precisely the Haldane phase is no more distinct from the trivial insulator state. According to [10], the same scenario appears for very small R. It appears that the breaking of the inversion symmetry is one of the protecting symmetry.

This shows that some symmetries can protect topological features. This makes all the richness of the SPT order. Therefore, SPT phases are present when some specific symmetries protect the topological features. But when thoses symmetries are explicitly broken in the Hamiltonian, one can go smoothly (*i.e.* without phase transition) from one SPT phase to the Trivial Insulating phase.

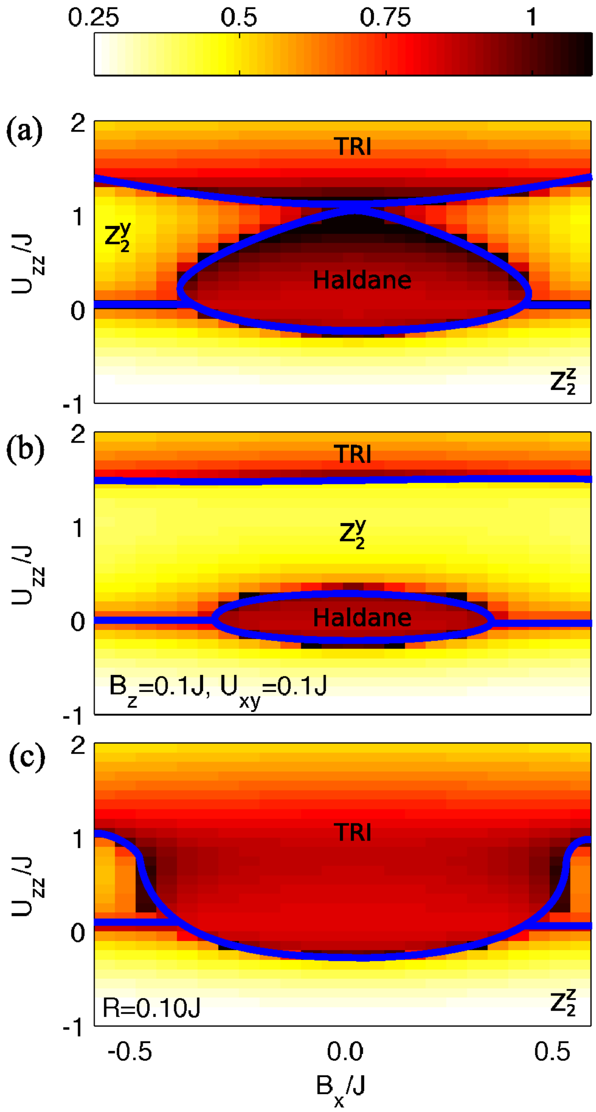


Figure 2: The colormaps show the entanglement entropy S for different spin-1 models: Panel (a) shows the data for Hamiltonian H_0 in (1.3), panel (b) for H_0 plus a term which breaks the $e^{-i\pi S^y}\times$TR symmetry [Eq. (1.4)], and panel (c) for H_0 plus a term which breaks the $e^{-i\pi S^y}\times$TR and inversion symmetry [Eq. (1.5)]. The blue lines indicate a diverging entanglement entropy as a signature of a continuous phase transition. The phase diagrams contain four different phases: A trivial insulating phase (TRI) for large U_{zz}, two symmetry breaking antiferromagnetic phases Z_2^z and Z_2^y, and a Haldane phase (which is absent in the last panel). Thoses results where obtained by Pollmann et al. [10].

2 First approch : Ising model

Ashkin-Teller model [16] is a stat-mech model similar to the Ising model, on each site, there are two Ising spins: s_i and t_i which couple via the Hamiltonian :

$$\mathcal{H}_{AT} = K_s s_i s_j + K_t t_i t_j + L s_i s_j t_i t_j \tag{2.1}$$

This model has been studied a lot as it presents interesting features such as a line of continous critical exponent and partially ordered phases [17]. The ferromagnetic partially ordered phase is identified by a non-zero value of the quantitie $\langle st\rangle \neq 0$ and $\langle s\rangle = \langle t\rangle = 0$. One of the usual tool of Ising-like problem is the so-called Kramers-Wannier duality [18] which consists in considering the dual of the spin variable. In two dimensions, the dual variable is the wall domains. Wall are not local anymore. The main idea of this section is to apply the Kramers-Wannier duality to only one of the Ising spin variable [19]. The order parameter will not be local anymore and we expect that some topological features will emerge front this. The only question is whether this approach will give physical model *ie* positive Boltzman weight. Unfortunately, the answer to this question is negative.

2.1 The Kramers-Wannier duality

We propose here to derive the Kramers-Wannier duality for inhomogenous Ising system in order to apply this duality to the Ashkin-Teller model. We describe each step as we want to know the condition which gives physical system when we apply the duality.

2.1.1 The Kramers-Wannier duality for the ferromagnetic Ising model

The Kramers-Wannier duality on the homogenous ferromagnetic Ising model is usually described using hight-temperature and low-temprature expansion [20]. However, we propose here to follow a geometric derivation that can be found in [21] and gives better insight in how variables are mapped, and is clearer to generalise to the antiferromagnetic case. The partition function of the Ising model in two dimensions with the spin variable $s_i = \pm 1$ reads:

$$Z = \sum_{\{s\}} \prod_{\langle ij\rangle} e^{K s_i s_j} = \sum_{\{s\}} \prod_{\langle ij\rangle} \sum_{k=0}^{1} C_k(K)(s_i s_j)^k, \tag{2.2}$$

where we have introduced a new variable k associated with every link : $k_{\mu,i}$. i indicates the lattice position. μ indicates the direction (x or y). The variable k_μ is a vector field in $\mathbb{Z}_2$ above the lattice. We reorganise the sum in order to regroup the occurence of s_i for a particular site i:

$$Z = \sum_{\{s\}} \sum_{\{k\}} \prod_{\text{link}} C_{k_\mu}(K) \prod_i (s_i)^{\sum_i k_\mu} = \sum_{\{s\}} \sum_{\{k\}} \prod_{\text{link}} C_{k_\mu}(K) \prod_i 2\delta_2(\sum_i k_\mu), \tag{2.3}$$

where δ_2 stands for the Kronecker symbol modulo 2, and $\sum_i k_\mu$ stands for the sum of the four links around the site i. We now introduce a new variable to generate the k_μ. Doing so, we introduce a spin variable σ_i which lies at the dual lattice as:

$$k_{\mu,i} = \frac{1 - \sigma_i \sigma_{i-\nu}}{2}, \quad \text{with} \quad \mu \neq \nu. \tag{2.4}$$

The sum over k_μ can be expressed with the σ variable as:

$$\sum_i k_\mu = 2 - \frac{1}{2}(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_4 + \sigma_4\sigma_1), \tag{2.5}$$

which is always even and enable us to drop the δ-function in using the σ variable. With this dual spin variable, the partition function reads:

$$Z = \frac{1}{2} 2^N \sum_{\sigma} \prod_{\text{link}} C_{\frac{1-\sigma_i\sigma_j}{2}}(K). \tag{2.6}$$

Now, if we look at the factor C_k:

$$C_k(K) = \cosh K\big(1 + k(\tanh K - 1)\big) = \cosh K \exp(k \ln \tanh K) = \cosh K e^{-2k\tilde{K}}, \tag{2.7}$$

where we have defined:

$$\exp(-2\tilde{K}) = \tanh K. \tag{2.8}$$

We now express C_k using the dual spin variable σ:

$$C_k(K) = \big(\cosh K \sinh K\big)^{1/2} \exp(\tilde{K}\sigma_i\sigma_j). \tag{2.9}$$

The partition function thus reads:

$$Z = \frac{1}{2}\big(2\cosh K \sinh K\big)^N \sum_{\{\sigma\}} e^{\tilde{K}\sigma_i\sigma_j} = \frac{1}{2}\big(\sinh 2\tilde{K}\big)^{-N} \sum_{\{\sigma\}} e^{\tilde{K}\sigma_i\sigma_j}, \tag{2.10}$$

this relation, which constitute the Kramers-Wannier (KW) duality can be written in a more symetric way introducing the symetric partition function $Y\{K\}$ [22]:

$$Y\{K\} = Z\{K\} 2^{-N/2} \cosh(2K)^{-N} = Y\{\tilde{K}\}. \tag{2.11}$$

with $\tilde{K}$ defined by 2.8.

2.1.2 Duality in the Antiferomagnetic Ising system

We now consider an antiferromagnetic Ising system. The partition function reads:

$$Z = \sum_{\{s\}} \prod_{\langle ij \rangle} e^{-Ks_is_j} = \sum_{\{s\}} \prod_{\langle ij \rangle} \sum_{k=0}^{1} C_k(-K)(s_is_j)^k. \tag{2.12}$$

But now the minus sign introduce a phase term, modifing eq. 2.7:

$$C_k(-K) = \cosh K e^{-2k(\tilde{K}+i\pi/2)}, \tag{2.13}$$

where $K > 0$ and $\tilde{K}$ is defined by eq. 2.8. The C_k express with the dual variable σ reads:

$$C_k(-K) = C_k(K)e^{-ik\pi} = (\cosh K \sinh K)^{1/2} e^{(\tilde{K}+i\pi/2)\sigma_i\sigma_j - i\pi/2}. \tag{2.14}$$

The partition function thus reads:

$$Z = \frac{1}{2}\big(2\cosh K \sinh K\big)^N \sum_{\{\sigma\}} e^{(\tilde{K}+i\pi/2)\sigma_i\sigma_j - i\pi/2} = \frac{1}{2}\big(\sinh 2\tilde{K}\big)^{-N} \sum_{\{\sigma\}} e^{(\tilde{K}+i\pi/2)\sigma_i\sigma_j - i\pi/2}, \tag{2.15}$$

We now express the duality as eq.2.11:

$$\prod_{\langle ij \rangle} [\cosh 2K^*]^{-1/2} Z\{K^*\} = \prod_{\langle ij \rangle} [\cosh 2K]^{-1/2} Z\{K\}, \tag{2.16}$$

where K^* is defined as $K^* = \tilde{K} + i\pi/2$. This just changes the sign in the term $\cosh 2K^*$ and adds $-i\pi/2$ phase in the exponential. We see that with the definition of K^*, the duality is expressed following the ferromagnetic case with K^* instead of $\tilde{K}$.

2.1.3 Duality in a non homogenous Ising system

We now want the KW duality for an inhomogenous system *i.e.* the coupling constant varies from site to site. The partition function reads:

$$Z\{K\} = \sum_{\{s\}} \prod_{\langle ij \rangle} e^{K_{ij} s_i s_j}. \tag{2.17}$$

We just have to keep track of the sign of K_{ij}. Let's note K^* the dual bond, with the imaginary part if necessary and $\tilde{K}$ defined as $e^{-2\tilde{K}} = \tanh(|K|)$. The KW duality is summarized by:

$$K_{ij} \overset{KW}{\longmapsto} K^*_{ij} = \begin{cases} \tilde{K}_{ij} & \text{if } K_{ij} > 0 \\ \tilde{K}_{ij} + i\pi/2 & \text{if } K_{ij} < 0 \end{cases}. \tag{2.18}$$

with this convention, the duality transformation on the partition function is really simple:

$$\prod_{\langle ij \rangle} [\cosh(2K_{ij})]^{-1/2} Z\{K\} = \prod_{\langle ij \rangle} [\cosh(2K^*_{ij})]^{-1/2} Z\{K^*\} \tag{2.19}$$

2.2 Duality on the Ashkin-Teller model

We now apply the results before to the Ashkin-Teller (AT) model 2.1. The partition function reads:

$$Z_{AT} = \sum_{\{s,t\}} \prod_{\langle ij \rangle} e^{K_s s_i s_j + K_t t_i t_j + L s_i s_j t_i t_j}. \tag{2.20}$$

we want to apply the KW duality on the t spin only and express the partition function in term of the dual spin τ and the spin s.

2.2.1 The pure 4-spin interaction

First, let's study the case where $K_s = K_t = 0$ and $L > 0$. We enforce the $s_i s_j$ terme into a nonhomogenous Ising model:

$$Z = \sum_{\{s,t\}} \prod_{\langle ij \rangle} e^{L s_i s_j t_i t_j} = \sum_{\{s,t\}} \prod_{\langle ij \rangle} e^{L_{ij} t_i t_j}, \tag{2.21}$$

Where we have defined the inhomogenous coupling constant $L_{ij} = L s_i s_j$. Let's say we fixe all spin s to a special configuration denoted $\{s\}$. The partition function reads:

$$Z_{\{s\}} = \sum_{\{t\}} \prod_{\langle ij \rangle} e^{L_{ij} t_i t_j}, \tag{2.22}$$

applying the results of the previous sections:

$$Z = \sum_{\{s\}} \sum_{\{\tau\}} \prod_{\langle ij \rangle} \sqrt{\frac{\cosh 2L}{\cosh 2\tilde{L}}} \exp\left\{ -2\left(\tilde{L} + \frac{1 - s_i s_j}{2} \frac{i\pi}{2}\right)\left(\frac{1 - \tau_i \tau_j}{2}\right)\right\} \tag{2.23}$$

Thus, the dual partition function reads:

$$Z = \sum_{\{s,\tau\}} \prod_{\langle ij \rangle} \exp\left\{\tilde{L}\tau_i\tau_j - \frac{i\pi}{4}\left(1 - s_i s_j - \tau_i\tau_j + s_i s_j \tau_i \tau_j\right) + \frac{1}{2}\ln\left(\cosh 2L/\cosh 2\tilde{L}\right)\right\} \tag{2.24}$$

There is a phase which gives a minus sign for the Boltzman weight associated with some configuration. The model is then non relevant for physics. The next paragraph tries to see if allowing the Ising interaction (K_t and K_s non zero) can remove this problem.

2.2.2 The $L > K_t$ case

Let's considere the partition function of the AT model with $L > K_t$. We want to compute the duality for the t variable.

$$Z_{AT} = \sum_{\{s,t\}} \prod_{\langle ij\rangle} e^{K_s s_i s_j + K_t t_i t_j + L s_i s_j t_i t_j} = \sum_{\{s\}} \exp\Big[\sum_{\langle ij\rangle} K_s s_i s_j\Big] \sum_{\{t\}} \exp\Big[\sum_{\langle ij\rangle} K_{ij} t_i t_j\Big], \tag{2.25}$$

with $K_{ij} = K_t + L s_i s_j$. The second factor is now an inhomogenous Ising model. We can thus apply the results derived for the inhomogenous system. We define $\tilde{P}$ and $\tilde{Q}$:

$$e^{-2(\tilde{P}+\tilde{Q})} = \tanh(|K_t + L|), \tag{2.26}$$

$$e^{-2(\tilde{P}-\tilde{Q})} = \tanh(|K_t - L|). \tag{2.27}$$

And we define the dual variables:

$$\begin{cases} P^* = \tilde{P} + i\pi/4 \\ Q^* = \tilde{Q} - i\pi/4 \end{cases} \tag{2.28}$$

We can now express K^*_{ij}:

$$K^*_{ij} = P^* + s_i s_j Q^*. \tag{2.29}$$

The Kadanoff's equation eq.2.19 gives:

$$Z\{K\} = \prod_{\langle ij\rangle} \sqrt{\frac{\cosh 2K_{ij}}{\cosh 2K^*_{ij}}} Z\{K^*\} \tag{2.30}$$

We finaly get:

$$Z = \sum_{\{\tau\}} \prod_{\langle ij\rangle} \exp\Big\{ -\frac{i\pi}{4}(1 - s_i s_j - t_k t_l + s_i s_j t_k t_l) + \tilde{P} t_k t_l + (A + \tilde{Q}) s_i s_j - \tilde{Q} s_i s_j t_k t_l + B\Big\}, \tag{2.31}$$

with:

$$\begin{cases} A = \dfrac{1}{4} \ln\left(\dfrac{\cosh(2K + 2L)\cosh(2\tilde{P} - 2\tilde{Q})}{\cosh(2K - 2L)\cosh(2\tilde{P} + 2\tilde{Q})}\right), \\ B = \dfrac{1}{4} \ln\left(\dfrac{\cosh(2K + 2L)\cosh(2K - 2L)}{\cosh(2\tilde{P} + 2\tilde{Q})\cosh(2\tilde{P} - 2\tilde{Q})}\right). \end{cases} \tag{2.32}$$

There, the AT partitionnal function reads:

$$Z_{AT} = \sum_{\{s\}} \sum_{\{\tau\}} \prod_{\langle ij\rangle} \exp\Big\{ -\frac{i\pi}{4}(1 - s_i s_j - t_k t_l + s_i s_j t_k t_l) + \tilde{P} t_k t_l + (K_s + A + \tilde{Q}) s_i s_j - \tilde{Q} s_i s_j t_k t_l + B\Big\} \tag{2.33}$$

Here, we see that the minus sign problem cannot be removed by adjusting some parameters. Therefore, this approach don't gives physical results.

3 Classical system equivalent to the Haldane Chain

It's possible to find connection between quantum models and classical models. One way is to use the Suzuki-Trotter decomposition. This enable to express a d dimension quantum model as a $d+1$ classical model. Here, we aimed at finding a statistical mechanics model equivalent to a quantum model which exihit SPT order. We are thus looking for positive Boltzman weights (in order to have physical significance). We first describe vertex model and show how the Trotter-Suzuki decomposition maps a spin chain to a vertex model. We will also describe this mapping in the other way, the integrable system [23]. We start with the spin $S = 1/2$ which is simpler and can be easily generalised to the spin $S = 1$ case. After, we discuss some of models which are known to have SPT phases.

3.1 Vertex model

We propose to describe vertex model in two dimensions as an example of tensor network. Let a spin variable S live on each bond of the lattice and take values in some range. The simplest example is $S \in \{-1, 1\}$. To each vertex, we attach a Boltzman weight $R^{\alpha\beta}_{\mu\nu}$ which depends on the value of the spin on the legs of the vertex. This Boltzman weight can be seen as a tensor whose entries are the values of the adjacents spins, see figure 3. We can thus write the partition function of a vertex model as the sum over every configuration of the product of the Boltzman weight associated with each vertex on a $N \times M$ lattice:

$$Z_V = \sum_{\text{configurations}} \prod_{\text{vertices}} R^{\alpha\beta}_{\mu\nu}, \tag{3.1}$$

$$= \operatorname{tr} T^M. \tag{3.2}$$

Where we have defined the row-to-tow tranfer matrix:

$$T^{j_1 j_2 \cdots j_N}_{i_1 i_2 \cdots i_N} = R^{k_N i_1}_{k_1 j_1} R^{k_1 i_2}_{k_2 j_2} \cdots R^{k_1 i_N}_{k_N j_N}. \tag{3.3}$$

where we have assumed the Einstein summation convention for repeated indices. Therefore, the partition function is obtained by contracting the tensors network along each bond.

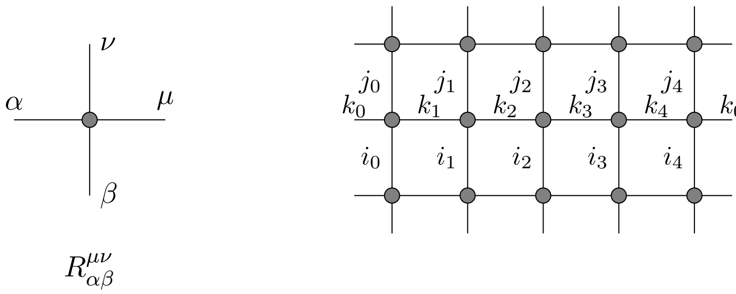


Figure 3: Vertex model. The value of the vertex depends on the four adjacent spins.

Vertex model are known for a long time and the six-vertex model has been solved exactly by Lieb [24] and the eight-vertex model has been solved by Baxter [25] in the zero-field case. The six-vertex is a good way to compare new methods to exact results.

3.2 The Trotter-Suzuki decomposition

The Trotter-Suzuki (TS) decomposition is a common tool of Monte Carlo simulation [26]. It's a mapping of a d dimension quantum system to a $d+1$ classical statistical mechanics model. We will restrain here to nearest-neighbour interaction Hamiltonian $\mathcal{H} = \sum \mathcal{H}_{i,i+1}$. The main idea of this technic is to use the relation:

$$\left(e^{-\frac{\beta A}{n}} e^{-\frac{\beta B}{n}}\right)^n \underset{n\to+\infty}{\longrightarrow} e^{-\beta\mathcal{H}} \tag{3.4}$$

with $\mathcal{H} = A + B$. We can choose wisely A and B to simplify the calculs. A common decomposition is the so-called checkerboard decomposition, define A to be the odd-even interaction part of the Hamiltonian and B to be the even-odd interaction part of the Hamiltonian.

$$A = \sum_{i=0}^{N/2-1} \mathcal{H}_{2i,2i+1}, \quad \text{and} \quad B = \sum_{i=0}^{N/2-1} \mathcal{H}_{2i+1,2i+2}. \tag{3.5}$$

Using the TS decomposition 3.4, the partition function of the spin chain in the large n limit reads:

$$Z_Q = \mathrm{tr}(e^{-\beta\mathcal{H}}) \simeq Z_n = \mathrm{tr}(T^n), \tag{3.6}$$

where $T = e^{-\frac{\beta A}{n}} e^{-\frac{\beta B}{n}}$ is the transfer matrix. We insert between each transfer matrix a complete set $\sum_{\{S\}_l} |\{S\}_l\rangle\langle\{S\}_l|$:

$$\mathrm{tr}(T^n) = \sum_{\{S\}_0} \cdots \sum_{\{S\}_{2n-1}} \prod_{k=0}^{n-1} \langle\{S\}_{2k} \,|\, e^{-\frac{\beta A}{n}} \,|\, \{S\}_{2k+1}\rangle\langle\{S\}_{2k+1} \,|\, e^{-\frac{\beta B}{n}} \,|\, \{S\}_{2k+2}\rangle \tag{3.7}$$

where $\{S\}_k$ denotes a row of spin. We impose periodic boundary condition: $\{S\}_{2n} = \{S\}_0$. We have taken the basis $S_i^\theta|S_{i,k}\rangle = S_{ik}|S_{ik}\rangle$ where θ is an arbitrary direction (z for instance). We can further express:

$$\langle\{S\}_{2k} \,|\, e^{-\frac{\beta A}{n}} \,|\, \{S\}_{2k+1}\rangle = \prod_{i=0}^{N/2-1} w(2i, 2k), \tag{3.8}$$

$$\langle\{S\}_{2k+1} \,|\, e^{-\frac{\beta B}{n}} \,|\, \{S\}_{2k+2}\rangle = \prod_{i=0}^{N/2-1} w(2i+1, 2k+1) \tag{3.9}$$

with

$$w(i,k) = \langle S_{i,k}, S_{i+1,k} \,|\, e^{\beta\mathcal{H}_{i,i+1}/n} \,|\, S_{i,k+1}, S_{i+1,k+1}\rangle \tag{3.10}$$

In the following, we will impose the translation invariance. We can define a vertex model associated with this quantum chain. The tensor of each vertex in the large n limit reads:

$$R^{\mu\nu}_{\alpha\beta} = \langle\alpha\beta \,|\, e^{\mathcal{H}_{i,i+1}/n} \,|\, \mu\nu\rangle \tag{3.11}$$

The Trotter-Suziki decomposition enables us to approximate a spin chain by stat-mech model with S states in two dimensions. Assuming that nearest neigbour interaction and translation symmetry, the obtained model is simply a vertex model described in section 3.1.

3.3 The six-vertex model and the spin 1/2 Heisenberg chain

We will now see the problem from the other and, let's take a vertex model, or a q-state model. In order to simplify the discussion, we will only consider the case where the spin S on each bond can only take two values : ± 1. Generalisation to higher spin is immediate. For the 2-state case, there is 2^4 configurations possible for one vertex. We now enforce several symetries with physical meaning in order to reduce the number of freedom on each vertex:

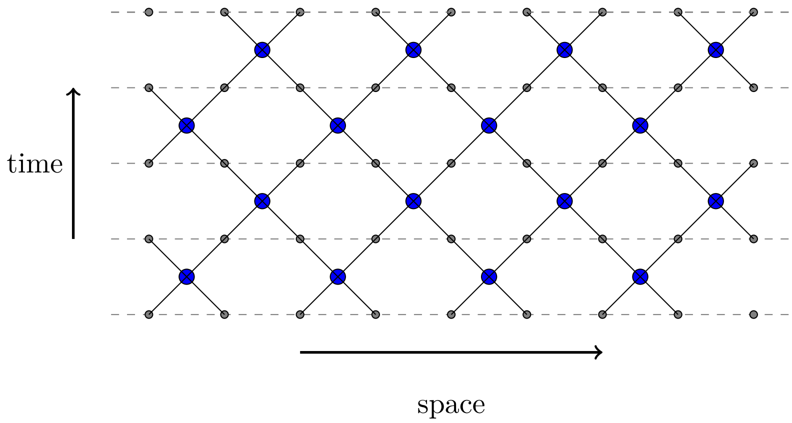


Figure 4: Trotter Suzuki decomposition, the small dots indicate the spin variable, the dashed lines indicate the spatial chain. The blue dots indicate the vertex lattice.

- the *ice rule* (which gives us the so-called *six-vertex model*):

$$R^{\alpha\beta}_{\mu\nu} = 0 \quad \text{unless} \quad \alpha + \beta = \mu + \nu. \tag{3.12}$$

- the up-down spin symetry:

$$R^{\alpha\beta}_{\mu\nu} = R^{-\alpha-\beta}_{-\mu-\nu}. \tag{3.13}$$

- invariance under reflection (for one vertex):

$$R^{\alpha\beta}_{\mu\nu} = R^{\beta\alpha}_{\nu\mu} = R^{\nu\mu}_{\beta\alpha}. \tag{3.14}$$

We can write the unfolded tensors R as a matrix in the basis : $(++, +-, -+, --)$:

$$R = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \tag{3.15}$$

We can rewrite it using the Kronecker product of the Pauli's matrices:

$$R = \frac{a}{2}(\mathbb{I} + S^z \otimes S^z) + \frac{c}{2}(\mathbb{I} - S^z \otimes S^z) + b(S^x \otimes S^x + S^y \otimes S^y). \tag{3.16}$$

We can now make a link with a spin 1/2 chain. Let's put: $a = 1 + \delta a$, $b = \delta b$ and $c = 1 + \delta c$. We can thus write the R-matrix for the ith vertex to the $(i+1)$th vertex on a same row:

$$R = \mathbb{I} + \delta b(S^x_i S^x_{i+1} + S^y_i S^y_{i+1}) + \frac{\delta a - \delta c}{2} S^z_i S^z_{i+1} + \frac{\delta a + \delta c}{2}\mathbb{I}, \tag{3.17}$$

In the limit of $\delta a, \delta b, \delta c$ small, the vertex tranfer matrix reads :

$$T \simeq \mathbb{I} + \delta\tau(\mathcal{H}_{XXZ} + u\mathbb{I}) \cdots, \tag{3.18}$$

with $\delta\tau \ll 1$ and where we have defined the Hamiltonien of the XXZ spin $S = 1$ chain:

$$\mathcal{H}_{XXZ} = \sum_i (S^x_i S^x_{i+1} + S^y_i S^y_{i+1}) + \Delta S^z_i S^z_{i+1}. \tag{3.19}$$

we identify by $\delta\tau = \delta b$ and $\Delta = (\delta a - \delta c)/2\delta b$. Hence, there is a strong relation between the quantum hamiltonien of a quantum chain with nearest neigbor interaction and the R-matrix of a vertex model. As the thermodynamical behavior of both systems is related to the eigenvalues of the transfert matrices, both systems are equivalents.

3.4 Symmetries relations

As we are dealing with symmetries in the spin model, it's interesting to see the relation between symmetries on the chain and symmetries of the vertex model.

Translational invariance: If the interaction is the same for every bond $\mathcal{H}_{i,i+}$, the A and B matrices are the same and the vertex model is homogenous. We will always consider this case.

The ice-rule symmetry: The most constrain symetry is the ice-rule as it's constrain considerably the number of non zero boltzman weights. It corresponds to the commutation of the Hamiltonian with the total spin operator:

$$[\mathcal{H}\,, S^z_{\text{tot}}] = 0, \tag{3.20}$$

if we have a nearest-neighbour interaction and $\mathcal{H} = \sum_i \mathcal{H}_{i,i+}$, this relation reads:

$$\begin{aligned} 0 &= \langle \alpha\beta \,|\, [\mathcal{H}_{i,i+1}\,, S^z \otimes \mathbb{I} + \mathbb{I} \otimes S^z] \,|\, \mu\nu \rangle, \\ &= \big[(\alpha+\beta) - (\mu+\nu)\big] \langle \alpha\beta \,|\, \mathcal{H}_{i,i+1} \,|\, \mu\nu \rangle. \end{aligned}$$

Therefore, we find the ice rule condition:

$$\langle \alpha\beta \,|\, \mathcal{H}_{i,i+1} \,|\, \mu\nu \rangle = R^{\alpha\beta}_{\mu\nu} = 0 \quad \text{unless} \quad \alpha+\beta = \mu+\nu. \tag{3.21}$$

Spin up-spin down symmetry: On the vertex model, the spin up-spin down symmetry is the exchange of the spin S to $-S$. Using eq.3.11, we can see that this symmetry means that the Hamiltonian $\mathcal{H}_{i,i+1}$ commutes with the "inversion" operator tensorised with itself:

$$[\mathcal{H}_{i,i+1}\,, A \otimes A] = 0, \tag{3.22}$$

where A is the $2S+1$ matrix with an antidiagonal of 1.

3.5 The nineteen vertex model

We now move on the three-states vertex model. Each spin can take the values $\{-1, 0, 1\}$. We enforce again the *ice-rule*, and the symmetries listed for the $S = 1/2$ case. We derive the general R-matrix. The ice-rule reduces the 3^4 coefficient to a nineteen-vertex and with the other symetries one gets the following R-matrix:

$$R = \begin{pmatrix} a & & & & & & & & \\ & p & & e & & & & & \\ & & b & & g & & c & & \\ & e & & p & & & & & \\ & & g & & o & & g & & \\ & & & & & p & & e & \\ & & c & & g & & b & & \\ & & & & & e & & p & \\ & & & & & & & & a \end{pmatrix} \tag{3.23}$$

We now want to find a quantum spin chain which exihibits SPT phase which gives us positive Boltzman weights in the R-matrix. We hope that will gives us a new behavior for the vertex model. In the rest of this section, we will study a anisotropic Haldane chain and the bilinera quartic chain. Both are constructed around the Haldane phase of the Heisenberg antiferromagnet.

3.6 Example of spin 1 chain

We will consider two simple Hamiltonian develop around the Heisenberg antiferromagnet. The first one is just a generalisation to integrate the AKLT model in the Hamiltonian. It is the bilinear quartic Hamiltonian:

$$\mathcal{H}_1 = \sum_i \cos\theta \ \mathbf{S}_i\mathbf{S}_{i+1} + \sin\theta(\mathbf{S}_i\mathbf{S}_{i+1})^2, \tag{3.24}$$

The phase diagram of this model is well known, see [27] for a complete phase diagram. Several points are very well known such as the point $\tan\theta = 1/3$ which is the AKLT Hamiltonian. This model preserves the symmetries of the Heisenberg antiferromagnet. The second model is another generalisation of the Heisenberg antiferromagnet which breaking of some symmetries explicitely. We add two kinds of anisotropies along the z-axis: an Ising-like interaction parametrized by λ and a single-ion terms parametrized by D:

$$\mathcal{H}_2 = \sum_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + \lambda S_i^z S_{i+1}^z + D(S_i^z)^2. \tag{3.25}$$

This model was studied a lot and the phase diagram has been obtained using several ways [28–31]. The interesting phase for us is Haldane phase which is known to be SPT phase. The SPT phase here is the same, it is the Haldane phase of the antiferromagnet Heisenberg chain. It the same as in the Hamiltonian eq.1.3. The symmetries discussed above are all present. The tensor associated with one of thoses chain reads:

$$R^{\mu\nu}_{\alpha\beta} = \langle \alpha\beta \,|\, e^{-\beta\mathcal{H}_{i,i+1}} \,|\, \mu\nu\rangle \tag{3.26}$$

A sign problem appears here, for instance: $R^{1,0}_{1,0} < 0$ for positive temperature for $\mathcal{H}_2$. The same problem occurs with $\mathcal{H}_1$. This problem is long known and come from the Heisenberg part of the Hamiltonian. It is called the sign problem [26]. A way to avoid this problem is to divide the lattice in two sub-lattice A et B and perform a rotation on the B lattice:

$$S_i^{x,y} \mapsto -\tilde{S}_i^{x,y}, \tag{3.27}$$

$$S_i^z \mapsto \tilde{S}_i^z \quad \text{for} \quad i \in B. \tag{3.28}$$

which is a local unitary transformation. The Hamiltonian $\mathcal{H}_2$ (we choose $D = 0$ for simplicity) now reads:

$$\mathcal{H} = \sum_{i\in A, j\in B} \lambda S_i^z \tilde{S}_j^z - (S_i^x \tilde{S}_j^x + S_i^y \tilde{S}_j^y) \tag{3.29}$$

Transformation does not event staggered the Hamiltonian, therefore the Hamiltonian is invariant by translation and the vertex model associated is homogenous. One can check, with Mathematica for instance, that the associated vertex model has positives Boltzman weights for both model $\mathcal{H}_1$ and $\mathcal{H}_2$.

3.7 Edges states

One of the physical features of SPT is edges states. We have seen this in the AKLT case with the two spins half at the end of the chain. We now want to see how thoses edges states appears in the vertex model associated with the spin chain. First of all, the vertex lattice is deformed by the edge as in fig.5. According to some numericals simulation [32], magnetisation appears at the edge of the chain for the Heisenberg antiferromagnetic chain, *ie* a magnetisation appears at the edge of the 2D stat-mech model. It seems that the edge states are also present in this stat-mech model. However, a more detailed study is necessary to understand the physics of this vertex model. In the next section, we propose to expose a new technic to carry out such study.

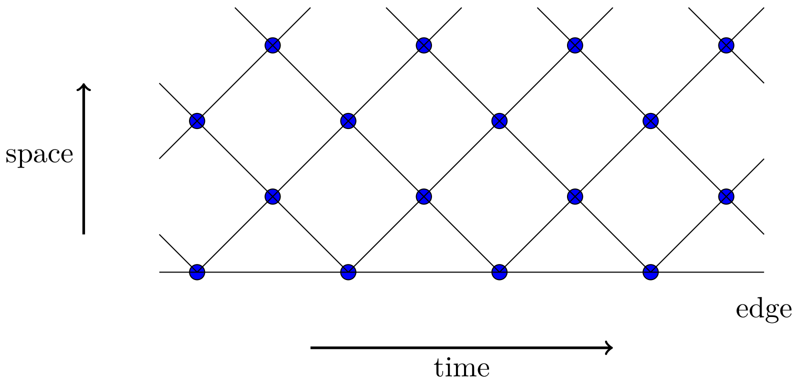


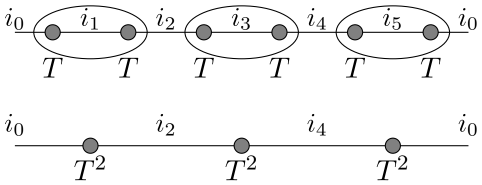
Figure 5: Trotter Suzuki decomposition with one edge. As the spin on the edge do not evolve, every other imaginary time step, there is a different configuration on the edge.

4 Tensors Network Renormalisation

The idea of this section is to explain how to apply the idea of the Renormalisation Group (RG) to system described by Tensors Network. First, we trait the easy case of chains where every tensor is just a matrix. So the computation is straighforward. Then, we explain how it can be applied to a 2D square lattice vertex model.

4.1 Chains and the example of the Ising model

Let's consider a chain of tensors *i.e.* a chain of matrices as described on figure 6. The coarse-graining is just the usual matrices product between adjacent matrices.



$$Z = T_{i_0 i_1} T_{i_1 i_2} \cdots T_{i_5 i_0}$$

$$Z = T^2_{i_0 i_2} T^2_{i_2 i_4} T^2_{i_4 i_0}$$

Figure 6: RG idea on matrix chain

The coarse RG step reads:

$$T \overset{RG}{\to} T^2 \tag{4.1}$$

$$T_{ij} \overset{RG}{\to} T_{ij} T_{jk}. \tag{4.2}$$

Thus, the fixed points are given by the limit $\{E_\infty\} = \{\lim_{n \mapsto \infty} E^n\}$ where E is the biggest eigenvalue of T.
Let's study the Ising chain. The transfer matrix is $T_{S_i S_{i+1}} = \exp(\beta S_i S_{i+1})$ or,

$$T = \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix} \tag{4.3}$$

In this case, the TRG is only the tranfer matrix method usually used to solve the Ising spin chain. It's worth to note that the results obtained by this methods are exact as the partition function is exactly $Z = \mathrm{tr}(T^N)$.

4.2 Renormalisation on a square lattice

For higher dimension, the coarse graining can become tricky as we have to use tensor and no matrices. Naively applied as in the 1D case, the coarse-graining will increase the rank of the tensor at each step exponentially and the rapidly the computation wil be intractable. Levin and Nave[33] proposed a way to effectively compute the RG flow of Tensor Network on a honeycomb lattice and is easily generalized to others lattices. Here we describe the algorithm proposed by [34]. On each site, there is a rank 4-tensor $T_{\alpha\beta\mu\nu}$. The RG idea is to coarse-grain four adjacent tensors into one new tensor and then, truncate the tensors to go back to the initial dimension D or to some manageable dimension D_{cute} see figure 7. In details, the algorithm is:

1. *Unfolding* : The first step consists in unfolding the tensors $T_{\alpha\beta\mu\nu}$, *i.e.* find some representation of this tensor in a matrix. The way to do this is to considere $T_{\alpha\beta\mu\nu}$ as a $D^2 \times D^2$ matrix by regrouping it's indicies. We need two unfolding in the algorithm:

$$M^A_{\alpha\beta;\mu\nu} = M^B_{\nu\alpha;\beta\mu} = T_{\alpha\beta\mu\nu} \tag{4.4}$$

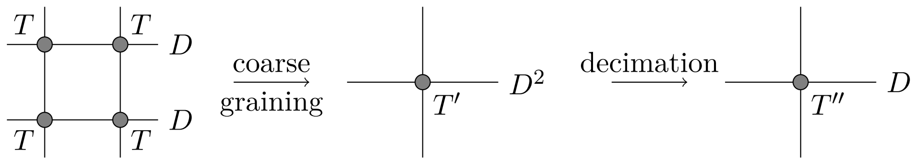


Figure 7: Scheme of the Tensors Renormalisation Group. First merge four adjacent tensors and then decimate to return to the dimension required.

2. *SVD:* In order to effectively truncate, we need to selectionne the major contibution in the matrix. A way to do this is to use Singular Value Decomposition (SVD) on the unfolded matrices. For instance:
$$M^A = U^A \Lambda^A (V^A)^\dagger \tag{4.5}$$
where Λ is the diagonal matrix of singular values, and U and V are unitary matrices.

3. *Truncature:* To make the RG flow computable, we need to make an approximation and thus to truncate. we will do so in keepinp only the largest D_{cut} values λ_γ by restraining γ to D_{cut}.

4. *Decomposition:* We can now decompose the $T_{\alpha\beta\mu\nu}$ tensor by defining
$$S^{(1)}_{\mu\nu\gamma} = \sqrt{\lambda_\gamma} V^\dagger_{\gamma;\mu\nu}, \tag{4.6}$$
$$S^{(3)}_{\alpha\beta\gamma} = \sqrt{\lambda_\gamma} U_{\alpha\beta;\gamma}. \tag{4.7}$$
We do the same with the other decomposition, We do the same with M^B, which gives us $S^{(4)}_{\mu\nu\gamma} = \sqrt{\lambda_\gamma} V^\dagger_{\gamma;\mu\nu}$ and $S^{(2)}_{\alpha\beta\gamma} = \sqrt{\lambda_\gamma} U_{\alpha\beta;\gamma}$. This decomposition is described on the figure 8 and reads:
$$T_{\alpha\beta\mu\nu} \simeq \sum_{\gamma=1}^{D_{cut}} S^{(3)}_{\alpha\beta\gamma} S^{(1)}_{\mu\nu\gamma}, \tag{4.8}$$
$$\simeq \sum_{\gamma=1}^{D_{cut}} S^{(2)}_{\nu\alpha\gamma} S^{(4)}_{\beta\mu\gamma}. \tag{4.9}$$

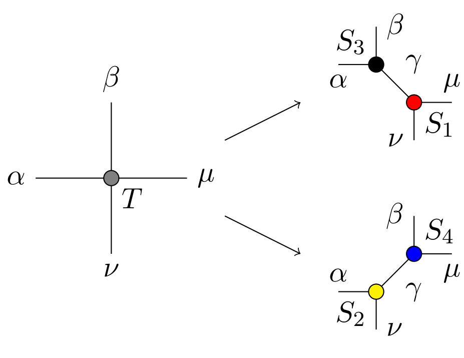


Figure 8: The tensor $T_{\alpha\beta\mu\nu}$ is decomposed two differents ways in the contraction of two 3-tensors.

5. The coarse graining is then done by summing over four adjacents matrices along the bonds as in figure 9. One get the renormalized tensor:

$$T'_{\gamma\sigma\lambda\rho} = \sum_{\alpha\beta\mu\nu} S^{(1)}_{\beta\alpha\gamma} S^{(2)}_{\mu\beta\sigma} S^{(3)}_{\nu\mu\lambda} S^{(4)}_{\alpha\nu\rho}. \tag{4.10}$$

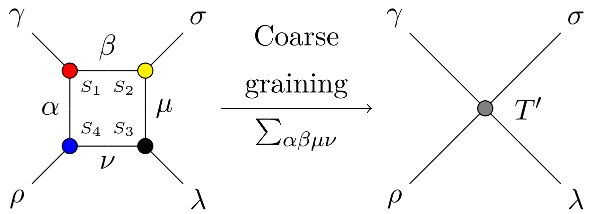


Figure 9: The coarse-graining is done by summing over the inner indices.

The range of indices for the new tensor T' is D_{cut} which can be tuned.

4.3 Perspectives: Application to Vertex model associated with SPT order

This application is under progress. Therefore, only the main idea will be developed. The TRG procedure breaks the symmetry of the model. In absence of SPT phases, this is not a problem. However, when there is SPT phase and when the symmetry that protects the topological features is broken, it becomes impossible to distinguish between the non-trivial SPT phases and the trivial phase. In a different approach to SPT phases using matrix product states, Huang, Chen, and Lin[35] developed a TRG procedure that protects the symmetry on every step of the RG scheme. We are hoping to do the same, or something equivalent, with our representation of the Haldane chain.

5 Conclusion

We have investigated on one of the most dynamic issue in Condensed Matter Theory, SPT orders. Even if the first attempt was not conclusive, we have made clear that we can use some quantum model presenting SPT order, such the Haldane chain, to have a stat-mech model with related features. We have also described a new promising tool, the Tensors renormalisation group, which is particulary adapted to the kind of (vertex) model that we have obtained with the Trotter-Suzuki decomposition. The only needed step is to prevent the RG procedure from breaking the symmetry. The direction taken with this report suggests to explore the equivalence between quantum SPT and the equivalent in stat-mech model. How behave the edges states in the stat-mech model? Can we use other caracteristics such as string parameter order?
This internship has been an opportunity to learn a lot about actual theoretical condensed matter. It was also an occasion to be autonomous during my research, deciding which direction I want to take and by thus giving me a clearer idea of what research would be.

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