Numerical Algorithms - Sixth lab session

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6.2) Compute the QR fatorization of $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and solve, in the sense of least squares, the system $Ax = b$, with $b = (1,1,0)^t$

Resolution:

$A = QR$ — upper triangular matrix

Q — orthogonal matrix (orthonormal columns)

$$\begin{array}{c} \\ \\ \end{array}\left[a_1 \; a_2\right] = \begin{array}{cc} a_1 & a_2 \\ \end{array}$$

$$\left[a_1 \; a_2\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \left[q_1 \; q_2\right] \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \\ 0 & 0 \end{bmatrix}$$

$\boxed{\text{* First column of QR:}}$

$\tilde{q}_1 = a_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\boxed{R_{11}} = \|\tilde{q}_1\| = \sqrt{2}$

$\boxed{q_1} = \frac{1}{R_{11}} \tilde{q}_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$

$\boxed{\text{* Second column of QR}}$

- compute $\boxed{R_{12}} = q_1^T a_2 = \frac{\sqrt{2}}{2}$

$\tilde{q}_2 = a_2 - R_{12} q_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{\sqrt{2}}{2} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$

- normalize to get

$\boxed{R_{22}} = \|\tilde{q}_2\| = \frac{\sqrt{2}}{2}$

$\boxed{q_2} = \frac{1}{R_{22}} \tilde{q}_2 = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$

Then $A = QR$

$$= \left[q_1 \; q_2\right] \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \\ 0 & 0 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}}_{Q} \times \underbrace{\begin{bmatrix} \sqrt{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} \\ \cancel{0} & \cancel{0} \end{bmatrix}}_{R}$$

Amoreno

Solve in the sense of least squares the system $Ax = b$, $b = (1, 1, 0)^t$

$$A\vec{x} = \vec{b} \quad , = \vec{b} - \vec{b}_\perp$$

$$A^T A \vec{x} = A^T \vec{b} - \cancel{A^T \vec{b}_\perp} \to 0$$

$$\boxed{A^T A \vec{x} = A^T \vec{b}}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad A^T \vec{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \boxed{\begin{aligned} x_1 &= \tfrac{1}{3} \\ x_2 &= \tfrac{1}{3} \end{aligned}}$$

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