Numerical Algorithms - Sixth lab session

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6.1) Compute the best approximation to $f(x)=x^2$ in the space generated by $\{p_1(x)=1,\ p_2(x)=x\}$. In two cases:

a) $\langle p(x), q(x)\rangle = \int_0^1 p(x)\cdot q(x)\,dx$

p^* - best approximation on $\{\alpha+\beta x\}$ $=$ $\begin{cases} p^* = \alpha+\beta x \\ f-p^* \perp 1 \\ \quad\ \ \perp x \end{cases}$

$\boxed{\langle f-p^*, 1\rangle = 0}$ $\Leftrightarrow \alpha\langle 1,1\rangle + \beta\langle x,1\rangle = \langle f,1\rangle$

$\Leftrightarrow \alpha + \frac{1}{2}\beta = \frac{1}{3}$

$\langle 1,1\rangle = \int_0^1 1\cdot 1\,dx = 1$

$\langle x,1\rangle = \int_0^1 1\cdot x\,dx = \frac{1}{2}$

$\boxed{\langle f-p^*, x\rangle = 0}$ $\Leftrightarrow \alpha\langle 1,x\rangle + \beta\langle x,x\rangle = \langle f,x\rangle$

$\Leftrightarrow \frac{1}{2}\alpha + \frac{1}{3}\beta = \frac{1}{4}$

$\langle x,x\rangle = \int_0^1 x^2\,dx = \frac{1}{3}$

$$\begin{cases} \alpha + \frac{1}{2}\beta = \frac{1}{3} \\ \frac{1}{2}\alpha + \frac{1}{3}\beta = \frac{1}{4} \end{cases} \Leftrightarrow \begin{cases} \boxed{\alpha = \frac{1}{3} - \frac{1}{2}\beta} \\ \frac{1}{2}\left(\frac{1}{3} - \frac{1}{2}\beta\right) + \frac{1}{3}\beta = \frac{1}{4} \end{cases}$$

$$\Leftrightarrow \begin{cases} \text{———} \\ \frac{1}{6} - \frac{1}{4}\beta + \frac{1}{3}\beta = \frac{1}{4} \Leftrightarrow -\frac{3\beta}{\cancel{12}} + \frac{4\beta}{\cancel{12}} = \frac{3}{\cancel{12}} - \frac{2}{\cancel{12}} \Leftrightarrow \boxed{\beta = 1} \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha = \frac{2}{6} - \frac{3}{6} = -\frac{1}{6} \\ \text{———} \end{cases}$$

b) $\langle a_0 + a_1 x + a_2 x^2, \ b_0 + b_1 x + b_2 x^2 \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$

$\langle p, q \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$

$\cancel{\langle p, q \rangle = 1 \cdot 3 + 2 \cdot 6 + 3 \cdot 7}$

$\cancel{p(x) = 1 + 2x + 3x^2}$

$\cancel{q(x) = 3 + 6x + 7x^2}$

* $\langle 1, 1 \rangle = 1 \cdot 1 + 0 \cdot 0 + 0 \cdot 0 = 1$
* $\langle x, 1 \rangle = 1 \cdot x + 0 \cdot 0 + 0 \cdot 0 = x$

p^* – best approximation on $\{\alpha + \beta x\}$ $= \begin{cases} p^* = \alpha + \beta x \\ f - p^* \perp 1 \\ \qquad\quad \perp x \end{cases}$

$\boxed{\langle f - p^*, 1 \rangle = 0}$ $\Leftrightarrow \alpha \langle 1, 1 \rangle + \beta \langle x, 1 \rangle = \langle f, 1 \rangle$ $\longrightarrow \langle x^2, 1 \rangle = 1 + x^2$

$\Leftrightarrow \alpha + \beta x = x^2 + 1$

$\boxed{\langle f - p^*, x \rangle = 0}$ $\Rightarrow \alpha \langle 1, x \rangle + \beta \langle x, x \rangle = \langle f, x \rangle$

$\Leftrightarrow \alpha x + \beta x^2 = x^2 + x$

$$\begin{cases} \alpha + \beta x = x^2 + 1 \\ \alpha x + \beta x^2 = x^2 + x \end{cases}$$