Question 5: Derive the order of the error with respect to the approximations of sin and cosine.

Given: Let $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...$, and let $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...$

Approx: $sin(x) \approx x$ and $\cos(x) \approx\ 1 - \frac{x^2}{2}$.

Error Big O(): (sin series - approx) at $x - \frac{1}{N}$

(cos series - approx) at $x - \frac{1}{N}$

The equations provided are those of a Taylor series, so the error functions for both sin and cosine are the series subtracted by the approximation for that of $x - \frac{1}{N}$.

Error$_{\mathbf{sin}}$(1/N):

The results after computing the difference between the series and the approximation is as such:

$Error_{\sin}$(1/N): $-\frac{(1/N)^3}{3!} + \frac{(1/N)^5}{5!} - \frac{(1/N\ )^7}{7!} ... \rightarrow -\frac{1}{3!N^3} + \frac{1}{5!N^5} - \frac{1}{7!N^7} \cdots$

$Error_{\cos}$(1/N): $\frac{(1/N)^2}{2!} - \frac{(1/N)^4}{4!} - \frac{(1/N\ )^6}{6!} ... \rightarrow \frac{1}{4!N^4} - \frac{1}{6!N^6} + \frac{1}{8!N^8} - ...$

With this deduction in mind, and considering when N $\rightarrow \infty$, the term with the lowest power of N in the denominator is the dominant term because the higher powers with the given function x become increasingly smaller. You can observe this graphically, but also just by looking at the nature of the series given. The Ratio Test for Convergence also confirms that the series converges as N $\rightarrow \infty$

Therefore, our order of error for the sin and cosine approximation is the following: $O_{\sin}(1/N) = \frac{1}{N^3}$ and $O_{\cos}(1/N) = \frac{1}{N^4}$