An Estimation-Control Duality and its extension to unknown distributions

Self Study Seminar

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Objective

- To **understand** the research paper:

  S. Talebi, A. Taghvaei, and M. Mesbahi, "*Data-driven Optimal Filtering for Linear Systems with Unknown Noise Covariances*". NeurIPS, 2023.

- **Regenerate** the simulation results.

Content

Introduction

- Dynamical System
- State Estimation
- Kalman Filter

Problem Formulation

- Define the Problem
- Estimation-Control Duality
- Optimization problem

Algorithm & Convergence Analysis

- Methodology
- Analysis
- Simulation Results

Conclusion

Motivating Example

Want to know my exact position?

- Can't see due to Heavy Fog
- GPS Sensor (noisy)
- How much noisy are the sensors !!

Want to **Estimate !**

Generated by Gemini Veo

Dynamical System (DS)

- Any system that evolves (changes) in time according to some rules.

Mathematical Model:

state

$$\frac{d}{dt}x = f(x, t)$$

dynamics

time

System Dynamics

Observation

$$y = g(x, t)$$

Observation Model

Dynamical System (DS)

- Any system that evolves (changes) in time according to some rules.

Mathematical Model:

$$\frac{d}{dt}x = \underbrace{f}_{\text{dynamics}}(\overbrace{x}^{\text{state}}, \underbrace{t}_{\text{time}}) + \underbrace{\xi_t}_{\text{Process Noise}}$$

System Dynamics

$$\overbrace{y}^{\text{Observation}} = g(x, t) + \underbrace{\omega_t}_{\text{Measurement Noise}}$$

Observation Model

Discrete Dynamical System

Consider the **Linear Time-invariant (LTI) dynamical system**,

State: $x_t \in \mathbb{R}^n$

System Dynamics

$$x_{t+1} = Ax_t + \xi_t$$

Observation: $y_t \in \mathbb{R}^m$

Observation Model

$$y_t = Hx_t + \omega_t$$

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- Noises are coming from unknown distribution with zero mean and given covariances.

$$\xi_t \sim (\mathbf{0}, \mathbf{Q}) \qquad \omega_t \sim (\mathbf{0}, \mathbf{R})$$

- Noises are uncorrelated with x_0 and with each other.

Discrete Dynamical System

Timestep (t)

State

Observation

Discrete Dynamical System

Timestep $(\boldsymbol{t})$ $\boldsymbol{t = 0}$

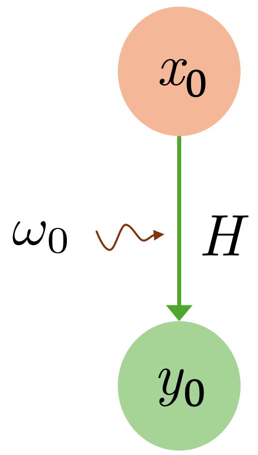
State x_0

Observation

Discrete Dynamical System

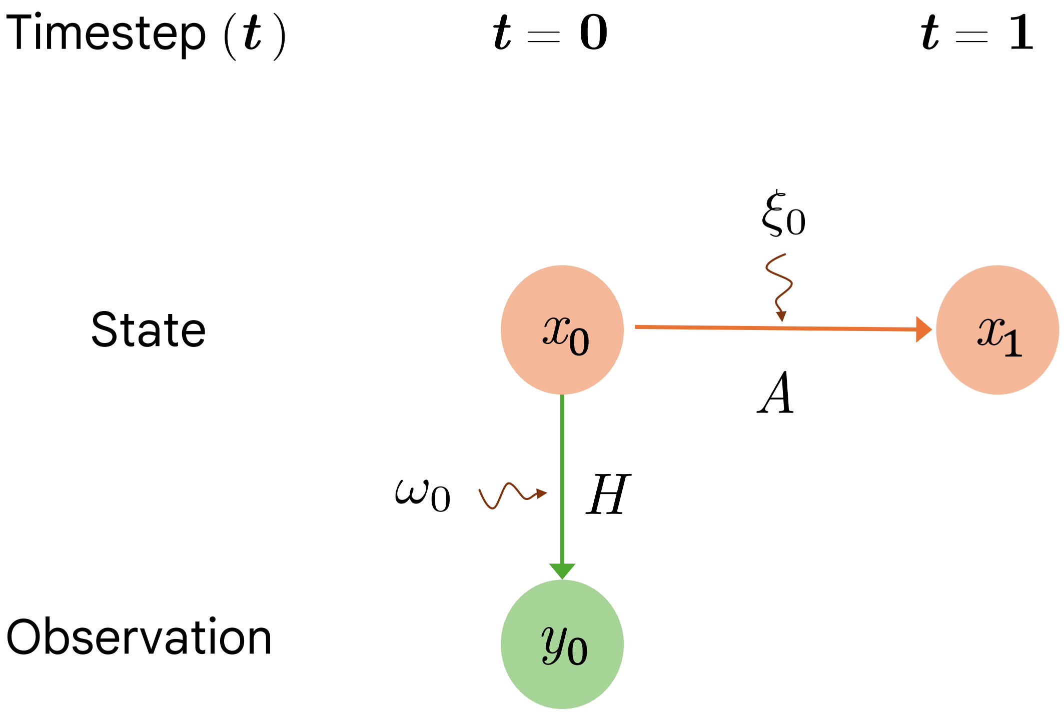
Timestep (t) $\quad t = 0$

State

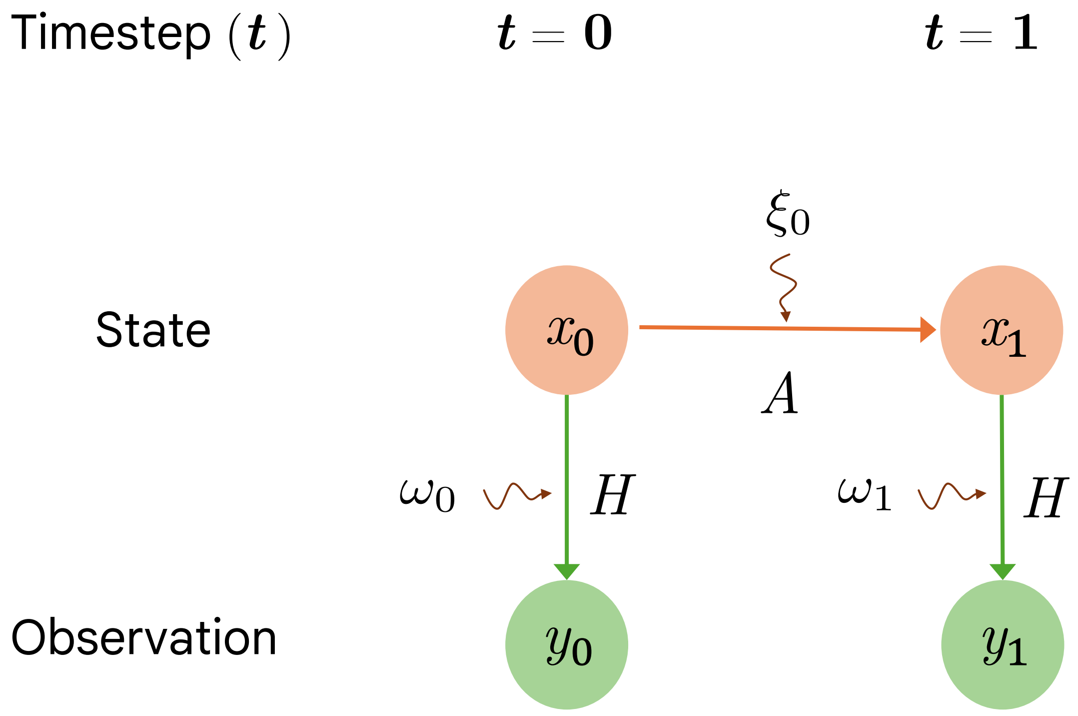


Observation

Discrete Dynamical System



Discrete Dynamical System



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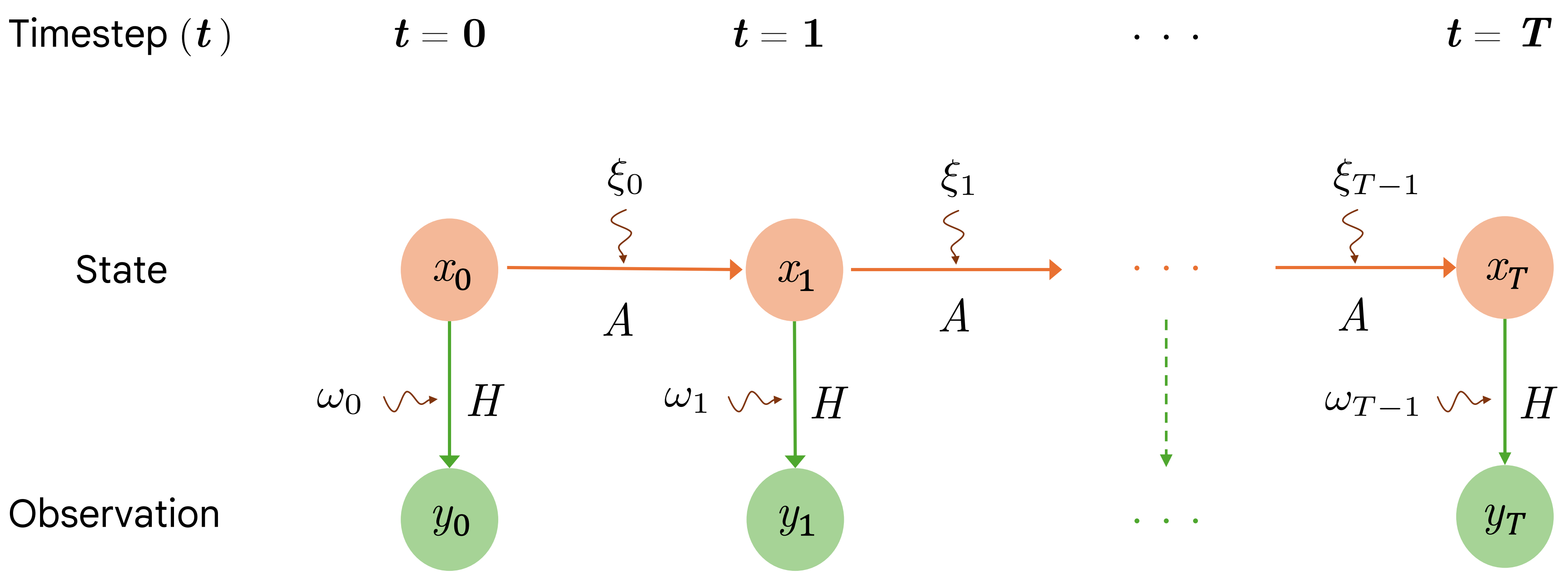

State Estimation

Consider the **Linear Time-invariant (LTI) dynamical system**,

$$\begin{aligned} x_{t+1} &= Ax_t + \xi_t \\ y_t &= Hx_t + \omega_t \end{aligned}$$

Given, History of Observations: $\mathcal{Y}_t := \{y_{0:t}\} = \{y_0, y_1, \cdots, y_{t-1}\}$

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- **Minimum Mean Squared Error (MSE) Estimator:**

$$\hat{x}_t = \underset{\hat{x} \in \mathcal{F}(\mathcal{Y}_t)}{\arg\min} \quad \mathbb{E} \left\| x_t - \hat{x} \right\|^2$$

- **Minimum MSE Linear Estimator:**

$$\hat{x}_t = \underset{\hat{x} \in \mathcal{L}(\mathcal{Y}_t)}{\arg\min} \quad \mathbb{E} \left\| x_t - \hat{x} \right\|^2$$

Here, $\mathcal{F}(\mathcal{Y}_t)$: the space of all functions of the history of the observation signal $\mathcal{Y}_t$

$\mathcal{L}(\mathcal{Y}_t)$: the space of all **linear** functions of the history of the observation signal $\mathcal{Y}_t$

Kalman Filter (KF)

KF Recursive formula by considering prior estimates only:

Kalman Gain

$$\hat{x}_{t+1} = A\hat{x}_t + L_t \underbrace{(y_t - H\hat{x}_t)}_{\text{Innovation Term}}$$

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Minimum MSE Linear Estimator:

How to get optimal L_t ?

$$\hat{x}_t = \arg\min_{\hat{x} \in \mathcal{L}(\mathcal{Y}_t)} \quad \mathbb{E}\left\|x_t - \hat{x}\right\|^2$$

R. E. Kalman, ASME. Journal of Basic Engineering, 1960

Kalman Filter (KF)

Initialization: $\hat{x}_0, P_0$

Estimation Error Covariance at time t

$$\boxed{P_t = \mathbb{E}\left[(x_t - \hat{x}_t)(x_t - \hat{x}_t)^\mathsf{T}\right]}$$

* **Chapter 2** [F. Lewis. "Optimal Estimation with an Introduction to Stochastic Control Theory", Wiley-Interscience, 1986]

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Estimation Error Covariance at time t

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*** A Priori KF Update Formula:**

Kalman Gain: $L_t := AP_tH^\intercal \left(HP_tH^\intercal + R\right)^{-1}$

State estimate (Prior): $\hat{x}_{t+1} = A\hat{x}_t + L_t\left(y_t - H\hat{x}_t\right)$

Error covariance (Prior): $P_{t+1} = AP_tA^\intercal + Q - AP_tH^\intercal \left(HP_tH^\intercal + R\right)^{-1} HP_tA^\intercal$

Depends on Q and R!

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Kalman Filter (KF)

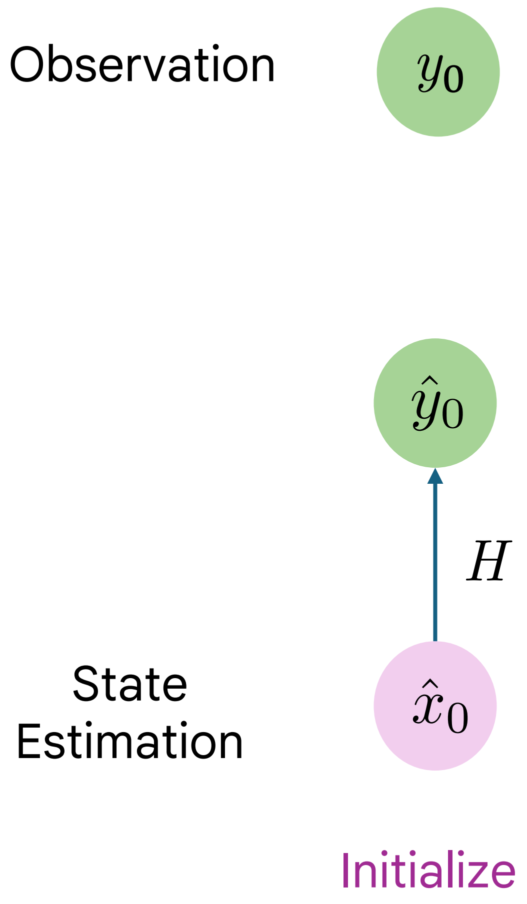
Observation y_0

State Estimation $\hat{x}_0$

Initialize

Kalman Filter (KF)

Visualization



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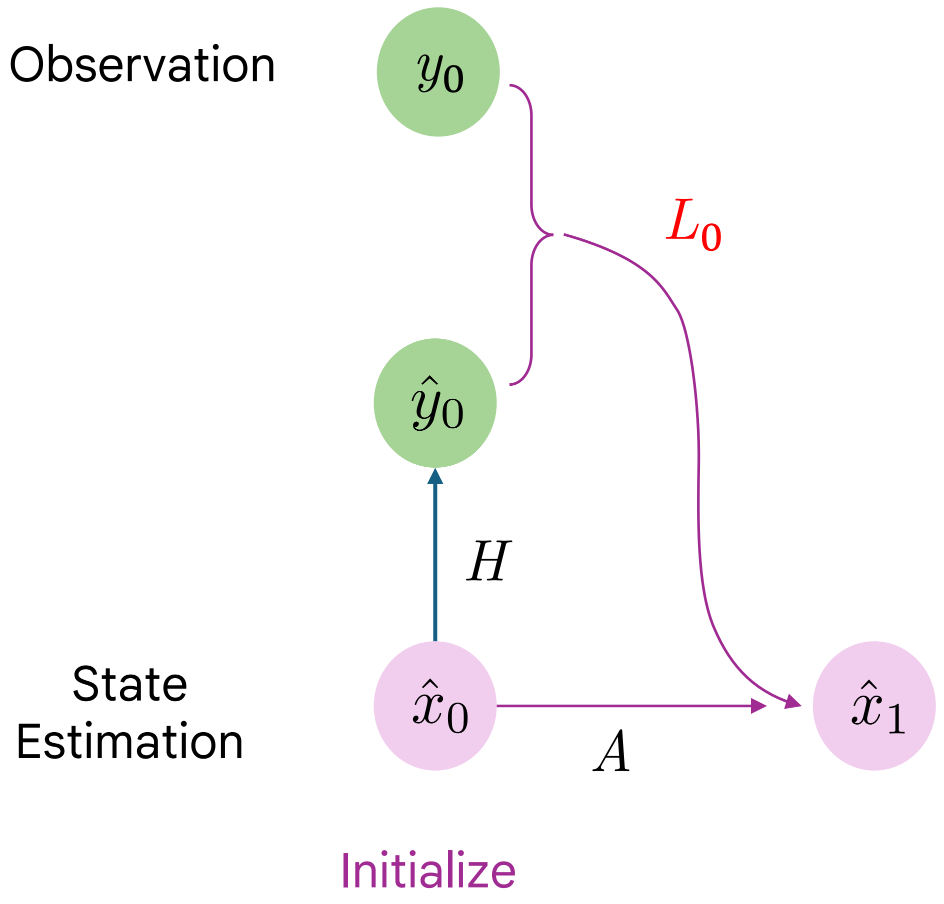

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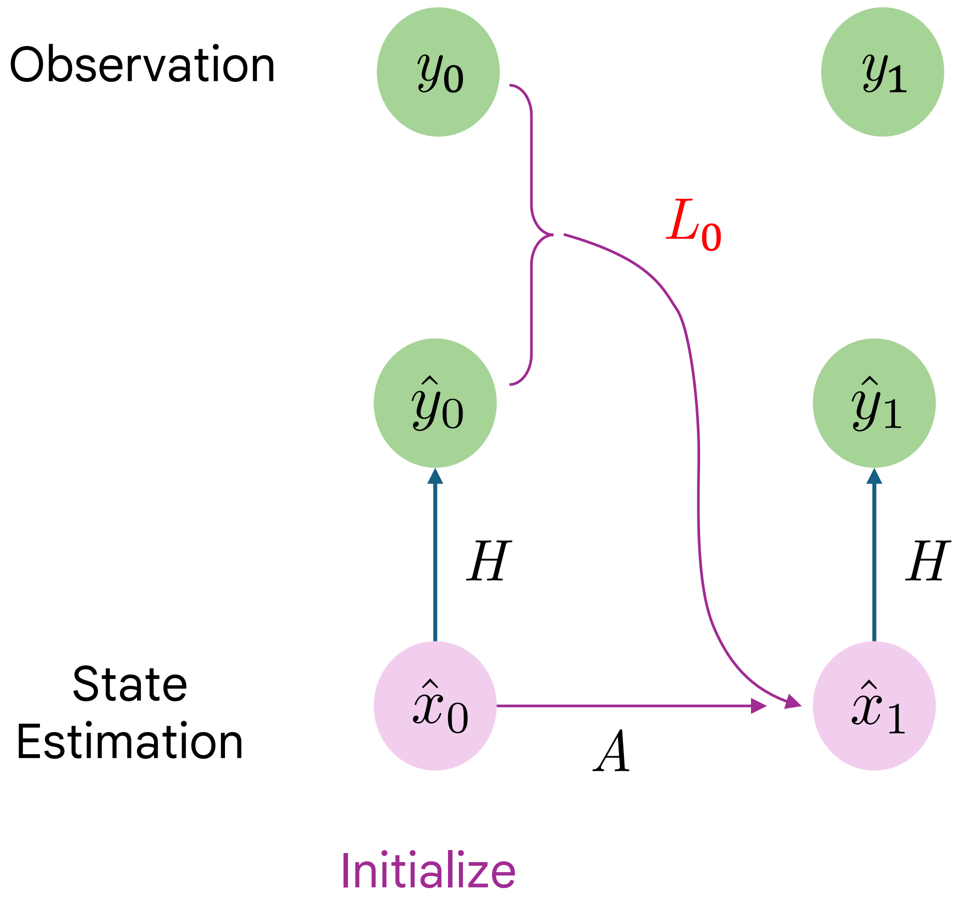

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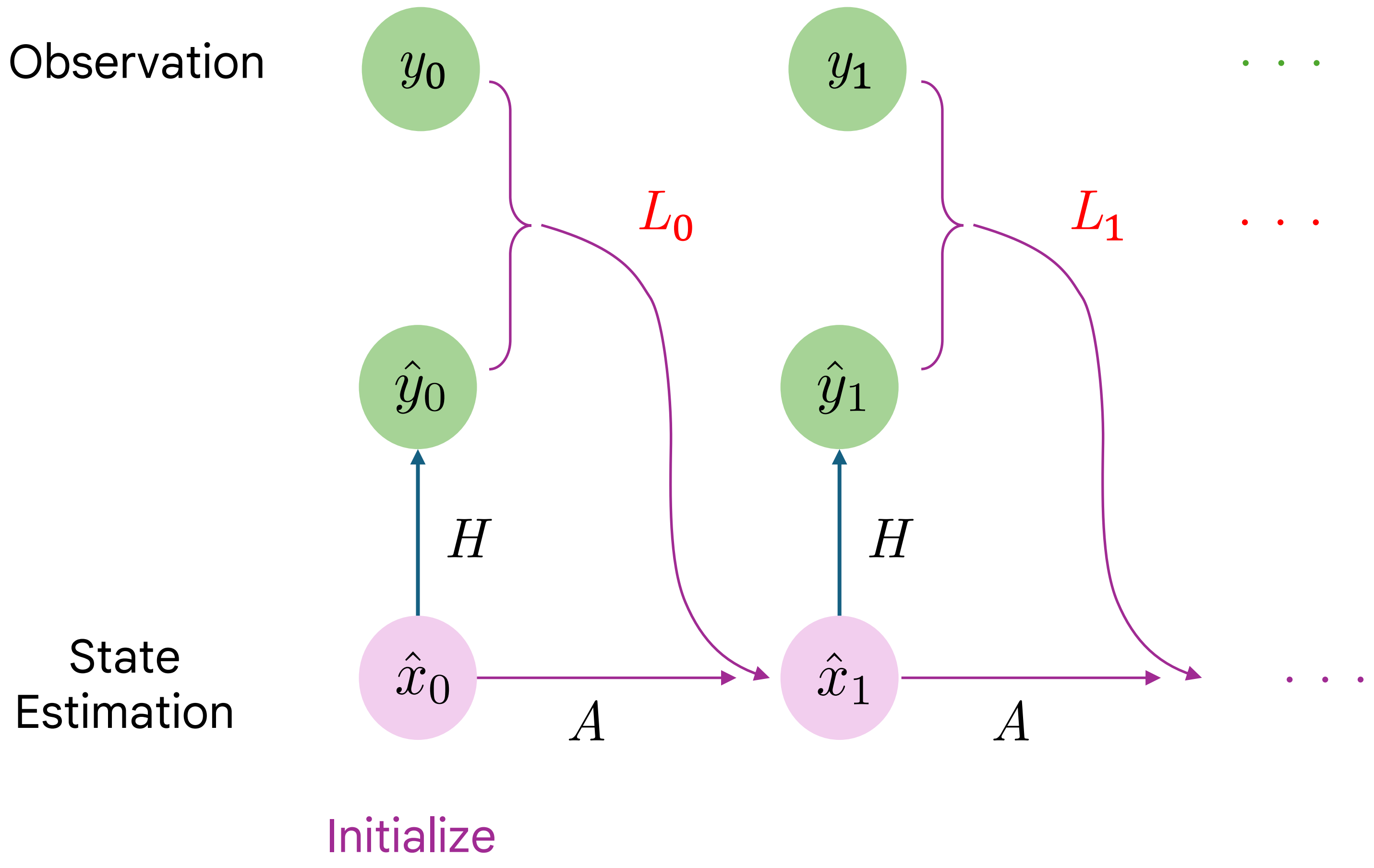

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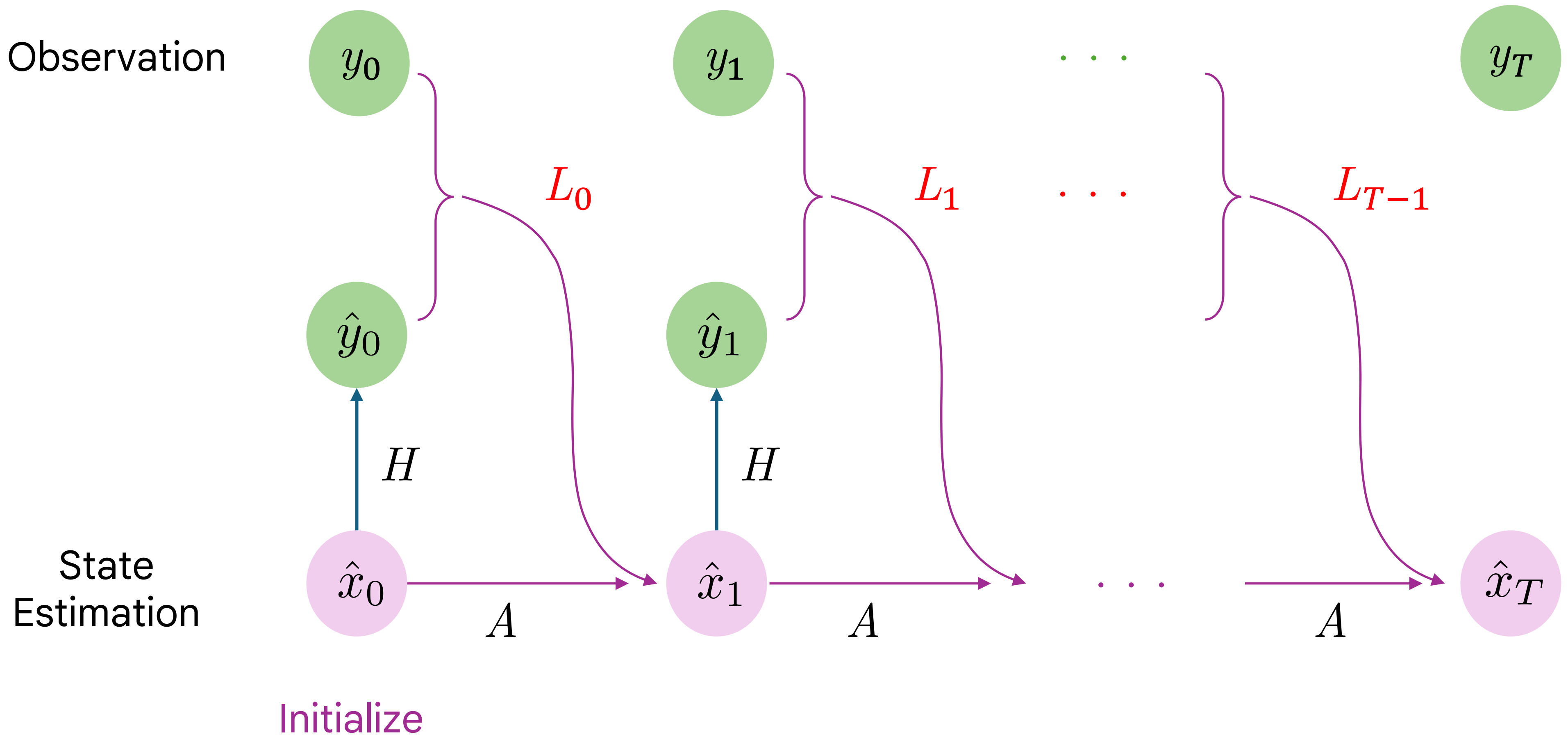

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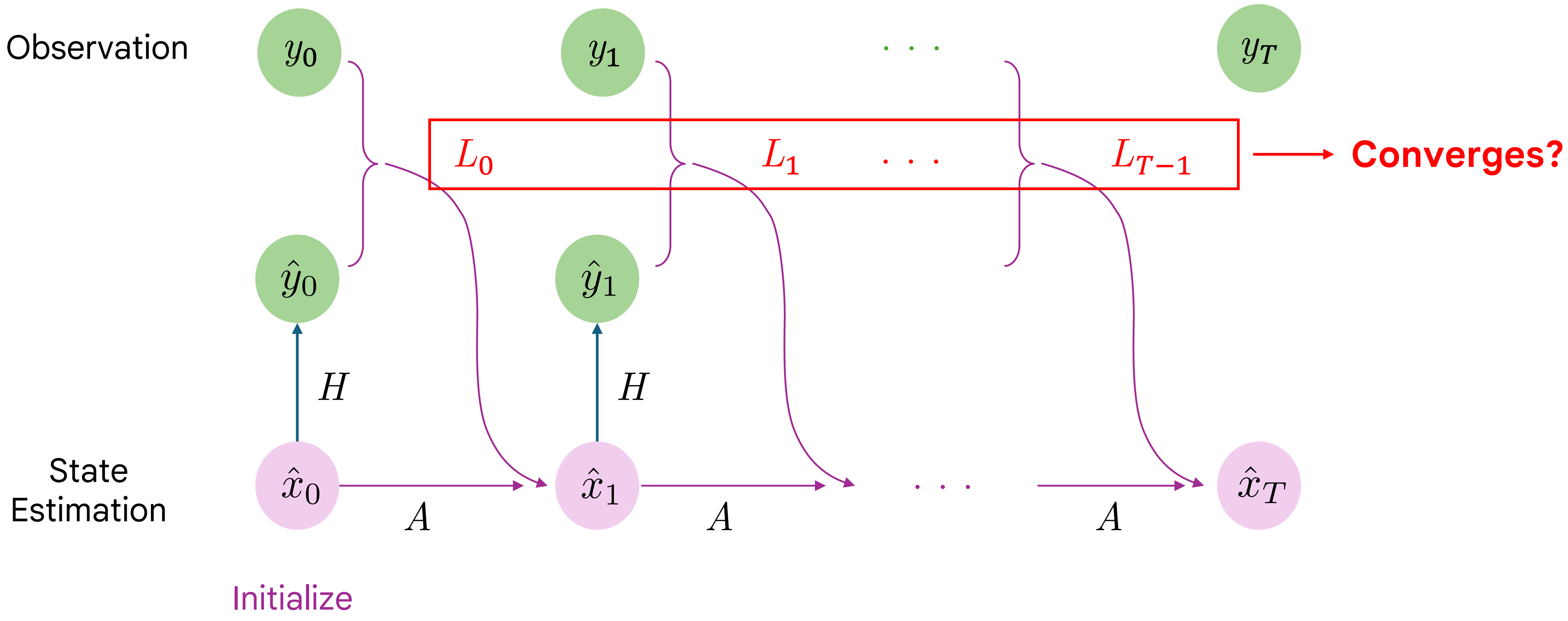

Steady-State Kalman Gain

Assumption

The pair (A, H) is **detectable**, and the pair $(A, \sqrt{Q})$ is **stabilizable**, where $\sqrt{Q}$ is the unique positive semidefinite square root of Q.

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Under this assumption *
P_t will converge to **steady-state Covariance** P_∞.

Steady-state Kalman Gain

$$\boxed{L_\infty = A P_\infty H^\top \left(H P_\infty H^\top + R\right)^{-1}}$$

* **Theorem 4.11** [H. Kwakernaak, R. Sivan. "Linear Optimal Control Systems". Wiley-interscience, 1969]

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Learning Problem *

Given

- **Known** Dynamics (A, H are **known**).
- We have the history of observations (noisy):

$$\mathcal{Y}_t = \{y_0, y_1, \cdots, y_{t-1}\}$$

Not Given

- **Unknown** noise covariances (Q, R).
- We don't have the ground-truth measurement of state (x_t).

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Learn	Steady-State Kalman Gain (L_∞)

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Learn L_∞

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- Expanding this for $t = 0$ to $T-1$, we get:

$$\hat{x}_T(L) = A_L^T\hat{x}_0 + \sum_{t=0}^{T-1} A_L^{T-t-1} L y_t$$

Get state estimate at time T using $\hat{x}_0$

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MSE: $\min_L \quad \mathbb{E}\,\|x_T - \hat{x}_T(L)\|^2$

x_t 's are not accessible !

Estimation-Control Duality

System Dynamics (Forward)

$$
\begin{aligned}
x_{t+1} &= Ax_t + \xi_t \\
y_t &= Hx_t + \omega_t
\end{aligned}
$$

* **Appendix C** [S. Talebi et al., NeurIPS '23]

Estimation-Control Duality

System Dynamics (Forward)

$$x_{t+1} = Ax_t + \xi_t$$
$$y_t = Hx_t + \omega_t$$

Dual Dynamics (Backward)

$$z_t = A^\intercal z_{t+1} - H^\intercal u_{t+1}$$

Given, $z_T = a \in \mathbb{R}^n$

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LQR Cost (Finite Horizon)

$$J_T^{\text{LQR}}\Big(a, u_{1:T+1}\Big) = z_0^\intercal P_0 z_0 + \sum_{t=1}^{T} z_t^\intercal Q z_t + u_t^\intercal R u_t$$

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Duality*

$$\mathbb{E}\left[|a^\intercal x_T - a^\intercal \hat{x}_T(L)|^2\right] = J_T^{\mathrm{LQR}}\Big(a, u_{1:T+1} \big| u_i = L^\intercal z_i\Big)$$

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Duality*

Estimation $$\underbrace{\mathbb{E}\left[|a^\intercal x_T - a^\intercal \hat{x}_T(L)|^2\right]}_{\text{Stochastic}} = \underbrace{J_T^{\mathrm{LQR}}\Big(a, u_{1:T+1} \big| u_i = L^\intercal z_i\Big)}_{\text{Deterministic}}$$ Control

* **Appendix C** [S. Talebi et al., NeurIPS '23]

Learn L_∞

- In particular, choose $\boldsymbol{a} = \boldsymbol{H_i}$ $\longleftarrow$ $\boldsymbol{H_i}^{\mathsf{T}}$ is the $\boldsymbol{i}$-th row of the matrix $\boldsymbol{H}$.

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- By Estimation-Control Duality *,

$$\mathbb{E}\left\|y_T - H\hat{x}_T(L)\right\|^2 = \sum_{i=1}^{m} J_T^{\mathrm{LQR}}\left(H_i, u_{1:T+1} | u_i = L^\intercal z_i\right) + \operatorname{tr}\left[R\right]$$

* **Proposition 1** [S. Talebi et al., NeurIPS '23]

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Surrogate Objective (Finite time horizon)

$$J_T^{\mathrm{est}}(L) := \mathbb{E}\left\|y_T - \hat{y}_T(L)\right\|^2$$

$$\hat{y}_T(L) := H\hat{x}_T(L)$$

* **Proposition 1** [S. Talebi et al., NeurIPS ‘23]

Learn L_∞

Dual Dynamics (Backward)

$$z_t = A^\intercal z_{t+1} - H^\intercal u_{t+1} \qquad \text{Given, } \; z_T = a \in \mathbb{R}^n$$

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Substituting the dual dynamics:

$$\min_{\boldsymbol{L}} \; \boldsymbol{J}_{\boldsymbol{T}}^{\mathbf{est}}(\boldsymbol{L}) = \mathbf{tr}\left[\boldsymbol{X}_{\boldsymbol{T}}(\boldsymbol{L}) \boldsymbol{H}^\intercal \boldsymbol{H}\right] + \mathbf{tr}[\boldsymbol{R}]$$

where, $$\boldsymbol{X}_{\boldsymbol{T}}(\boldsymbol{L}) := \boldsymbol{A}_{\boldsymbol{L}}^{\boldsymbol{T}} \boldsymbol{P}_0 \left(\boldsymbol{A}_{\boldsymbol{L}}^\intercal\right)^{\boldsymbol{T}} + \sum_{\boldsymbol{t}=0}^{\boldsymbol{T}-1} \boldsymbol{A}_{\boldsymbol{L}}^{\boldsymbol{t}} \left(\boldsymbol{Q} + \boldsymbol{L}\boldsymbol{R}\boldsymbol{L}^\intercal\right) \left(\boldsymbol{A}_{\boldsymbol{L}}^\intercal\right)^{\boldsymbol{t}}$$

Learn L_∞

- **Surrogate Objective (Finite time horizon):**

$$\min_{L} \; J_T^{\text{est}}(L) = \operatorname{tr}\left[X_T(L) H^\intercal H\right] + \operatorname{tr}[R]$$

where, $$X_T(L) := A_L^T P_0 \left(A_L^\intercal\right)^T + \sum_{t=0}^{T-1} A_L^t \left(Q + L R L^\intercal\right) \left(A_L^\intercal\right)^t$$

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- When Spectral radius (Largest absolute eigen value) $\rho\,(A_L) < 1$,

  $X_T(L)$ will converge to $X_{(L)}$ as $T \to \infty$.

Learn L_∞

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- When Spectral radius (Largest absolute eigen value) $\rho(A_L) < 1$,

  $X_T(L)$ will converge to $X_{(L)}$ as $T \to \infty$.

- Set of Schur stabilizing gains, $\mathcal{S} := \{L \in \mathbb{R}^{n \times m} : \rho(A_L) < 1\}$

Learn L_∞

Optimization in Steady-State

Surrogate Objective (in Steady-state)

$$\lim_{T \to \infty} J_T^{\text{est}}(L) = \operatorname{tr}\left[X_{(L)} H^\intercal H\right] + \operatorname{tr}\left[R\right]$$

Optimization in Steady-State

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Constrained Optimization Problem in steady-state

$$\begin{aligned}
\min_{L\in\mathcal{S}} \quad & J(L) := \operatorname{tr}\left[X_{(L)} H^\intercal H\right] \\
\text{s.t.} \quad & X_{(L)} = A_L X_{(L)} A_L^\intercal + Q + LRL^\intercal \\
\text{where,} \quad & \mathcal{S} := \{L \in \mathbb{R}^{n\times m} : \rho(A_L) < 1\}
\end{aligned}$$

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$\boldsymbol{Q}, \ \boldsymbol{R}$
are unknown !

Learn L_∞

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$\boldsymbol{Q}, \ \boldsymbol{R}$

are unknown !

Standard Kalman Filter & Duality will not work

Data driven approach

Learn L_∞

Data-driven approach

Estimate the **Objective**

$J(L)$ ⟵ Steady-state Cost

Learn L_∞

Data-driven approach

Estimate the **Objective**

$J(L)$ $\longleftarrow$ Steady-state Cost

We have finite horizon data.

Learn L_∞

Data-driven approach

Estimate the **Objective**

$$J(L) \longleftarrow$$ Steady-state Cost

We have finite horizon data.

$\downarrow$

Estimate $\underbrace{\textbf{truncated objective}}$

$$J_T(L) \;=\; \mathbb{E}\,\|y_T - \hat{y}_T(L)\|^2$$

Learn L_∞

Data-driven approach

Estimate the **Objective**

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Estimate **truncated objective**

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(Squared Error)

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Estimate **gradient** of **truncated objective**

$$\nabla J_T(L)$$

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$J(L)$ ← Steady-state Cost

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Role of Duality in estimating gradient

Estimate the **gradient** of **truncated objective**

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$$= -\frac{2}{M}\sum_{i=1}^{M}\sum_{t=0}^{T-1} \underbrace{(A_L^\intercal)^{T-t-1} H^\intercal e_T^{(i)}(L)}_{z_{t+1}(L)}\, e_t^{(i)}(L)^\intercal$$

$z_{t+1}(L) \longleftarrow$ Adjoint state from Dual Dynamics

Note: $\nabla \widehat{J}_T(L)$ does not depend on $\boldsymbol{Q}$, $\boldsymbol{R}$.

[**Lemma 3,** S. Talebi et al,. NeurIPS, 2023]

Algorithm

Batch Gradient Descent

Require:

$\boldsymbol{A},\ \boldsymbol{H},\ \hat{x}_0,\ \boldsymbol{P}_0.$

Hyperparameters:

$\boldsymbol{T}$: Trajectory Length, $\boldsymbol{M}$: Batch size,
η : Step Length, $\boldsymbol{k}$: No. of iterations.

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$\boldsymbol{L} = \boldsymbol{L}_0$

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Sample **M** trajectories.

$\{\mathcal{Y}_T^{(i)}\}_{i=1}^{M}$

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Initialize $L = L_0$ → Sample **M** trajectories. $\{\mathcal{y}_T^{(i)}\}_{i=1}^{M}$ → Estimate $\hat{\boldsymbol{x}}_T(\boldsymbol{L}),\ \hat{\boldsymbol{y}}_T(\boldsymbol{L})$

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Sample **M** trajectories.

$\{\mathcal{Y}_T^{(i)}\}_{i=1}^{M}$

→

Estimate

$\hat{\boldsymbol{x}}_T(\boldsymbol{L}),\ \hat{\boldsymbol{y}}_T(\boldsymbol{L})$

→

Calculate Estimated Cost

$\hat{\mathbf{J}}_T(\boldsymbol{L})$

& Estimated Gradient

$\nabla\, \hat{\mathbf{J}}_T(\boldsymbol{L})$

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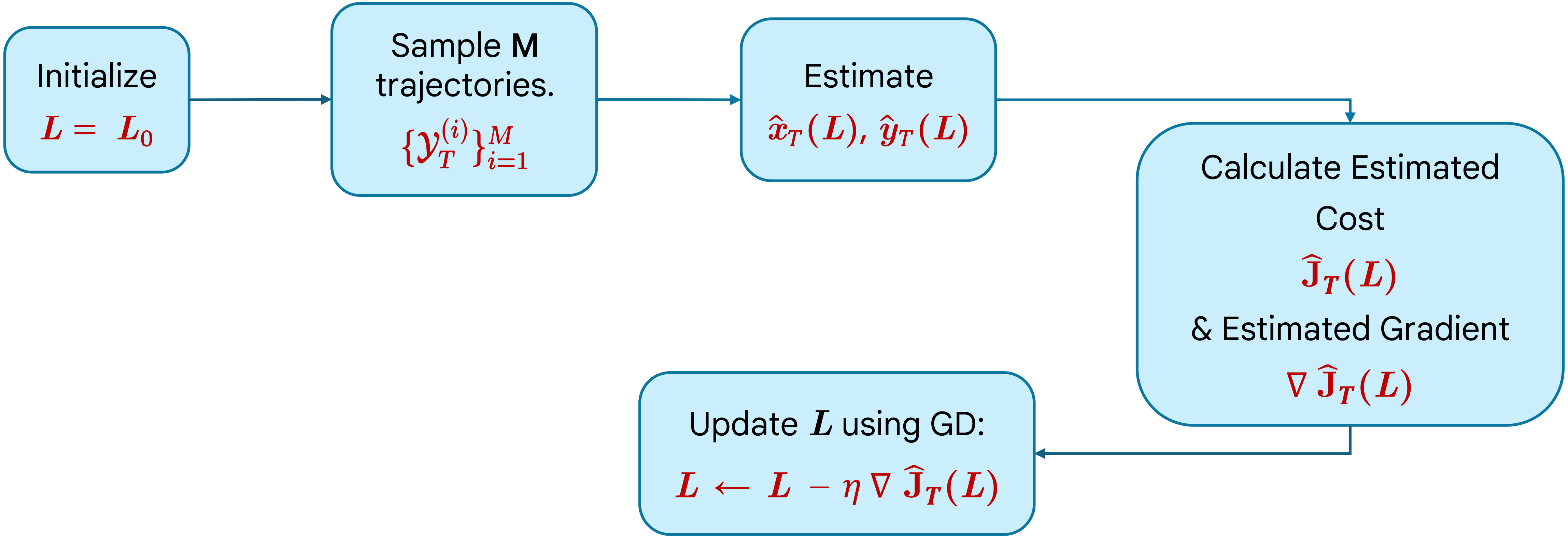

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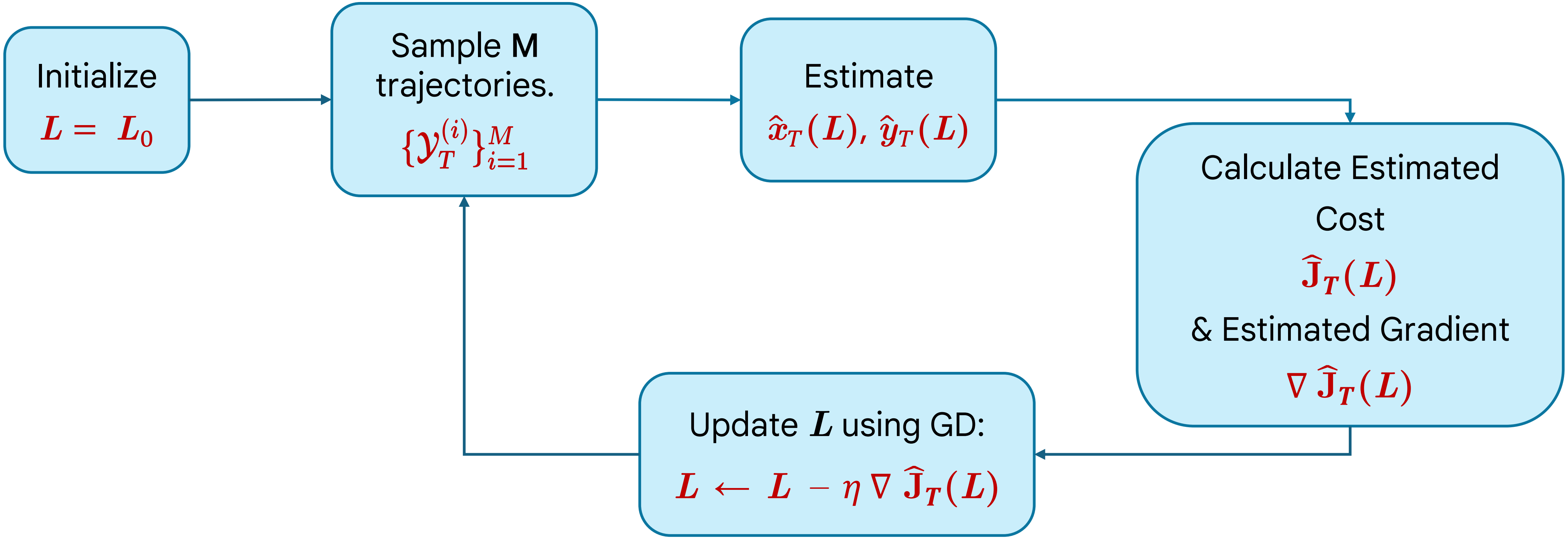

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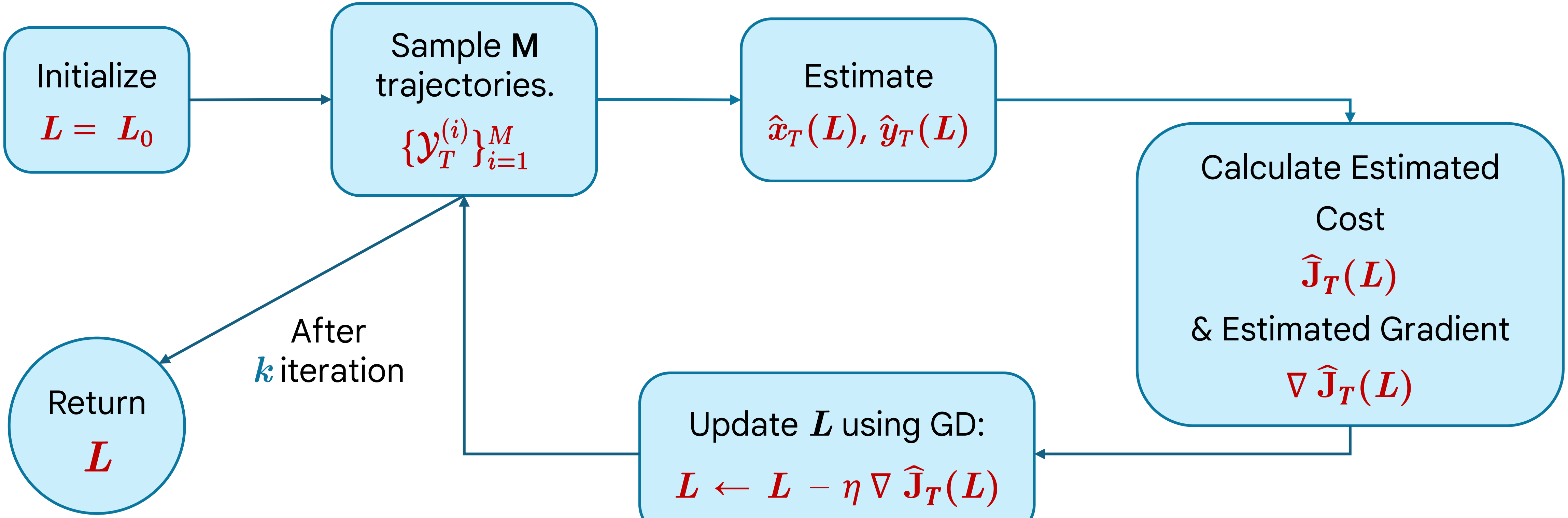

Research Questions

- How to initialize $\boldsymbol{L_0}$?
- How to choose $\boldsymbol{T}$ (Trajectory length), $\boldsymbol{M}$ (Batch-size) ?
- How the choices of $\boldsymbol{L_0}$, $\boldsymbol{T}$, $\boldsymbol{M}$ will affect the convergence rate?

Convergence Analysis

We are getting the sequence of gain: $\boldsymbol{L_0},\ \boldsymbol{L_1},\ \ldots\ ,\boldsymbol{L_k}$

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  Define Sublevel Set for some $\boldsymbol{\alpha} > 0,\ \boldsymbol{\mathcal{S}}_\alpha := \{\boldsymbol{L} \in \mathbb{R}^{\boldsymbol{n \times m}} : \boldsymbol{J}(\boldsymbol{L}) \leq \alpha\}$

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- Ensure **Cost Decay** in each step of GD **?**

Convergence Analysis

$$\|\nabla \widehat{J}_T(L) - \nabla J(L)\|$$

Convergence Analysis

Error Bounds

$$\|\nabla \widehat{J}_T(L) - \nabla J(L)\| \leq \|\nabla \widehat{J}_T(L) - \nabla J_T(L)\| + \|\nabla J_T(L) - \nabla J(L)\|$$

Convergence Analysis

Error Bounds

$$\|\nabla \widehat{J}_T(L) - \nabla J(L)\| \leq \underbrace{\|\nabla \widehat{J}_T(L) - \nabla J_T(L)\|}_{\text{Concentration Error}} + \underbrace{\|\nabla J_T(L) - \nabla J(L)\|}_{\text{Truncation Error}}$$

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- **Concentration Error Bound** [1]

$$\mathbb{P}\Big[\, \Big\|\nabla \widehat{J}_T(L) - \nabla J_T(L)\Big\| \ge s \Big] \;\le\; 2n \exp\Big[- M\, c_1(s, L)\Big]$$

⟵ Depends on **M, L**

[1] **Proposition 4,** [2] **Proposition 5** [S. Talebi et al,. NeurIPS '23]

Convergence Analysis

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- **Truncation Error Bound** [2]

$$\|\nabla J(L) - \nabla J_T(L)\| \le \bar{\gamma}_L \sqrt{\rho(A_L)}^{\,T+1}$$

Depends on **T, L**

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Convergence Analysis

$$\|\nabla \widehat{J}_T(L) - \nabla J(L)\| \leq \underbrace{\|\nabla \widehat{J}_T(L) - \nabla J_T(L)\|}_{\text{Concentration Error}} + \underbrace{\|\nabla J_T(L) - \nabla J(L)\|}_{\text{Truncation Error}}$$

Decays as **M** grows. Decays as **T** grows.

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Convergence Analysis

Given **T** length **M** trajectories with **M** $> l(\delta)$, $\delta > 0$

Convergence Analysis

Biased Gradient

Given **T** length **M** trajectories with **M** $> l(\delta)$, $\delta > 0$

Suppose we have access to a biased estimate of the gradient $\nabla \widehat{J}(L)$

The following holds with probability $> 1 - \delta$

Theorem 2 [S. Talebi et al,. NeurIPS '23]

Convergence Analysis

Given **T** length **M** trajectories with **M** $> l(\delta)$, $\delta > 0$

Suppose we have access to a biased estimate of the gradient $\nabla \widehat{J}(L)$

The following holds with probability $> 1 - \delta$

There exists constants $s, s_0 > 0$ implying

$$\|\nabla \widehat{J}(L) - \nabla J(L)\|_F \leq s \|\nabla J(L)\|_F + s_0 \quad \text{for all } L \in S_\alpha \setminus \mathcal{C}_\tau$$

for some $\alpha > 0$.

τ-neighborhood of
L^* (Optimal Gain)

Theorem 2 [S. Talebi et al,. NeurIPS '23]

Convergence Analysis

Linear rate of Convergence

Now, we have biased Gradient

Convergence Analysis

Linear rate of Convergence

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GD algorithm starting from any $L_0 \in \mathcal{S}_\alpha \setminus \mathcal{C}_\tau$ with fixed step-size $\eta(\alpha, \tau)$

Theorem 1 [S. Talebi et al,. NeurIPS '23]

Convergence Analysis

Linear rate of Convergence

Now, we have biased Gradient

GD algorithm starting from any $L_0 \in \mathcal{S}_\alpha \setminus \mathcal{C}_\tau$ with fixed step-size $\eta(\alpha, \tau)$

- Then it generates a sequence of policies $\{\boldsymbol{L_k}\}$ that are stable (i.e. each $L_k \in S_\alpha$)

Satisfying the Constraint ✓

Convergence Analysis

Linear rate of Convergence

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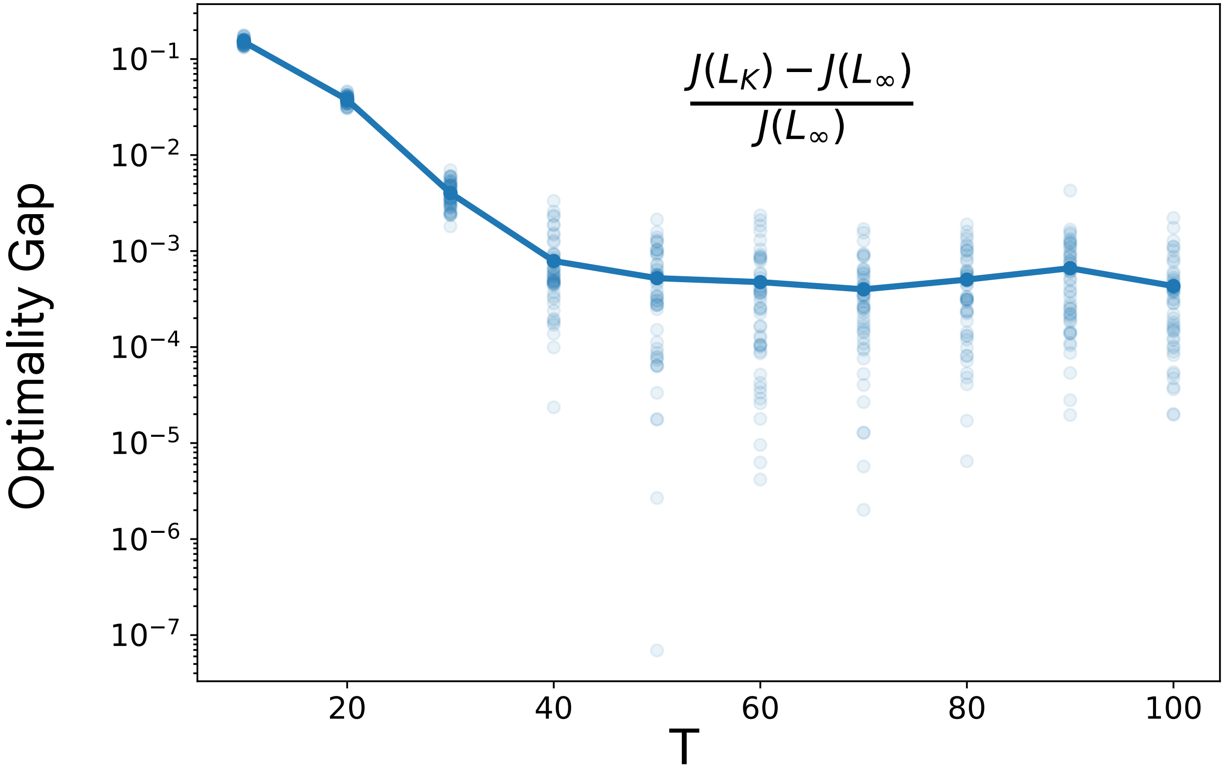
- Then it generates a sequence of policies $\{\boldsymbol{L_k}\}$ that are stable (i.e. each $L_k \in S_\alpha$) **Satisfying the Constraint** ✓

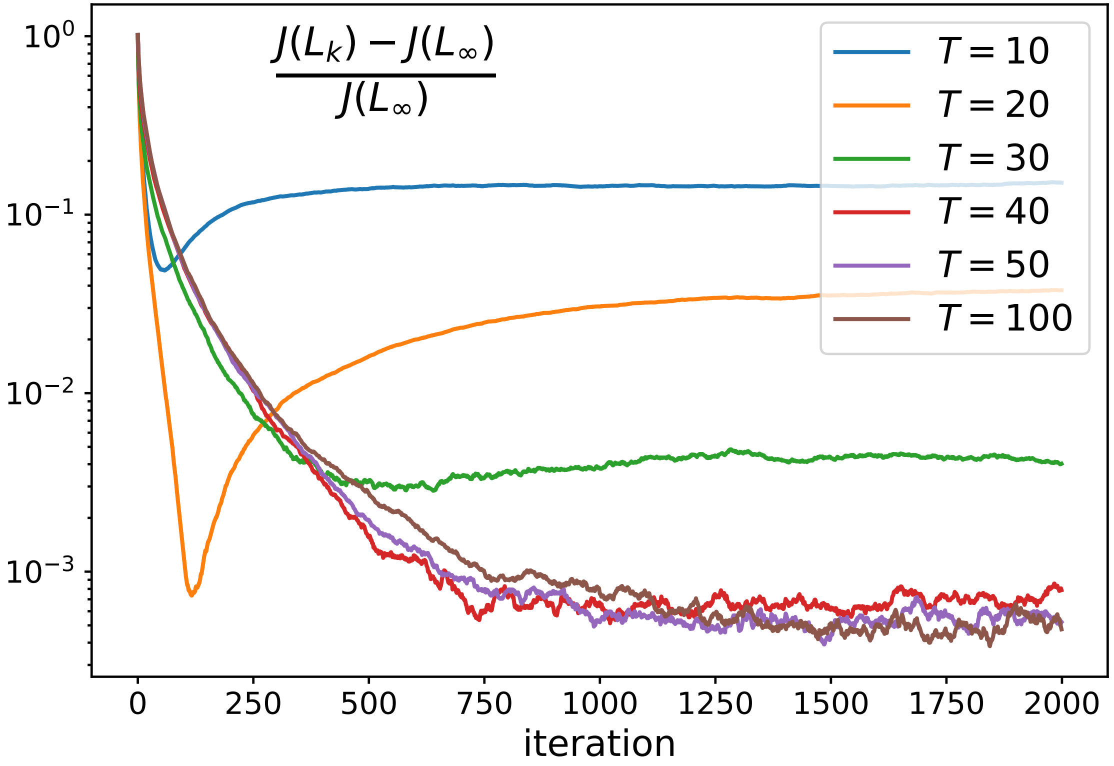
- **Decay** in cost value with **Linear convergence rate** before entering $\mathcal{C}_\tau$

$$J(L_{k+1}) - J(L^*) \leq c_1(\alpha, \tau, \eta)\left[J(L_k) - J(L^*)\right]$$

Simulation Result

Trajectory Length (T)



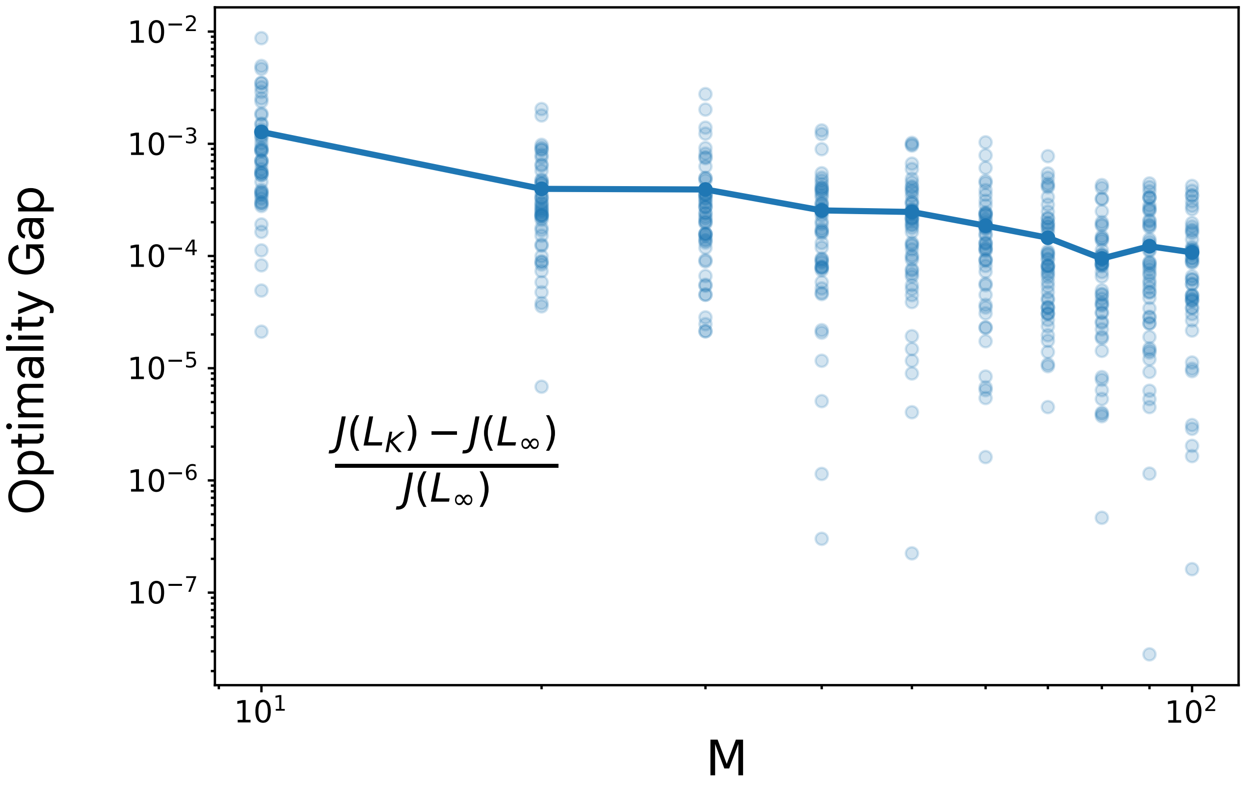


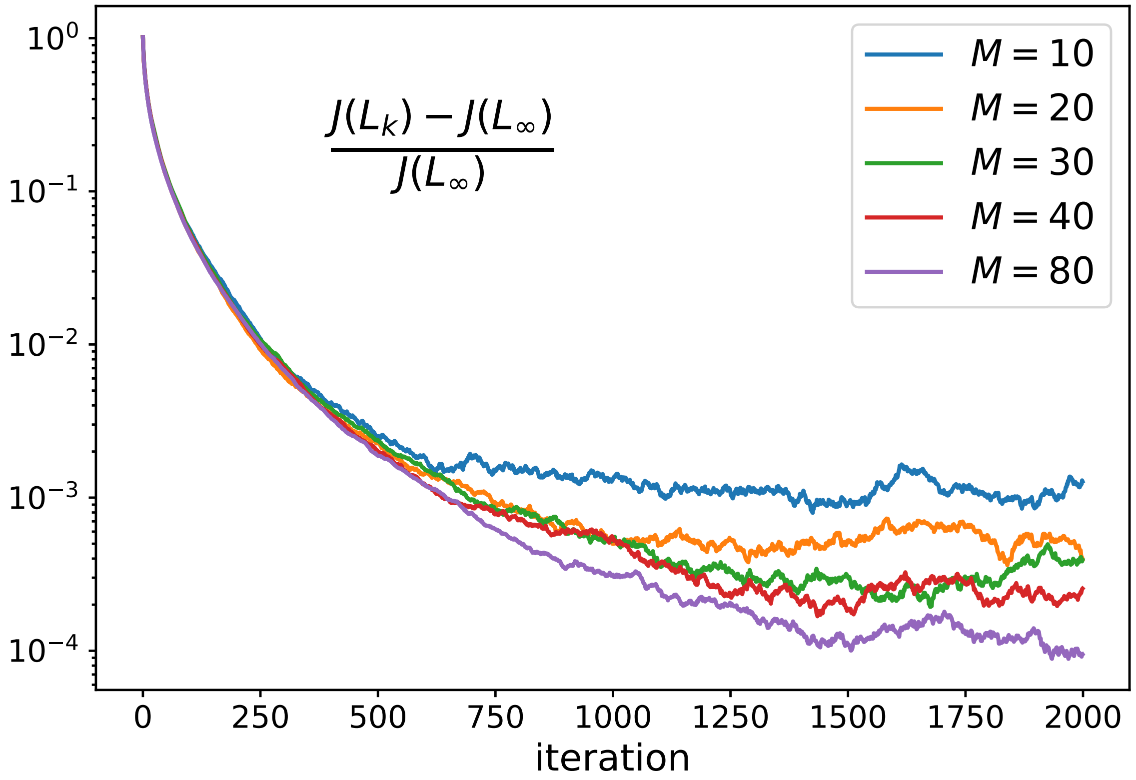
It shows **linear decay of optimality gap** w.r.t. **T**.

* Regenerated using code provided by authors.

Simulation Result

Batch-size (M)





It shows **linear decay of optimality gap** when the amount of data increases.

* Regenerated using code provided by authors.

Convergence Analysis

Consider **observable** $(\boldsymbol{A}, \boldsymbol{H})$, **bounded noise** $\boldsymbol{\xi_t}, \boldsymbol{\omega_t}$,

with (Stability Constraint) $\boldsymbol{L_0} \in \boldsymbol{\mathcal{S}}$ and step-size $\bar{\eta} := \dfrac{2}{9\,\ell(J(L_0))}$

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For all $\epsilon > 0$, if

$$T \geq O(\ln(\frac{1}{\epsilon})), \quad M \geq O\Big(\frac{1}{\epsilon}\ln(\frac{1}{\delta})\ln(\ln(\frac{1}{\epsilon}))\Big) \quad \text{and} \quad k \geq O(\ln(\frac{1}{\epsilon}))$$

Trajectory Length | Batch-size | Iteration no.

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Trajectory Length — Batch-size — Iteration no.

Then Batch GD converges to ϵ-optimal gain i.e. $J(L_k) - J(L^*) \leq \epsilon$

with higher probability $(> 1 - \delta)$

Theorem 3 [S. Talebi et al,. NeurIPS '23]

Thank You