Tensor Products and Exterior Products in $\mathbb{R}^n$

CS4099D Project - End Semester Report

Submitted by

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DECLARATION

"I hereby declare that this submission is my own work, and that to the best of my knowledge and belief, it contains no material previously published or written by another person nor material which has been accepted for the award of any other degree or diploma of the university or other institute of higher learning, except where due acknowledgement has been made in the text".

Place : NIT Calicut
Date : June 27, 2024

Signature :
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CERTIFICATE

This is to certify that the project report entitled : **Tensor Products and Exterior Products in** $\mathbb{R}^n$ *submitted by* **Sai Mineesh (B190305CS)** *to National Institute of Technology Calicut towards partial fulfilment of the requirements for the award of the degree of Bachelor of Technology in Computer Science and Engineering is a bonafide record of the work carried out by him/her under my supervision and guidance.*

Place : NIT Calicut
Date : June 27, 2024

Name and Signature of Guide :
(Dr. Muralikrishnan K.)

Acknowledgements

I sincerely thank Dr. Murali Krishnan K for this collaboration and guidance throughout the course of project. I admire Murali sir's approach to a subject in two axes, one that speaks the intuition behind a result and the other is the mathematical rigor to reach the result. This work is a result of his vision to develop an accessible reading material of tensor products and exterior products for computer science graduates or for any beginner in this domain. Some of the major simplifications we put forward to simplify the arguments in proof are ideas of Murali sir.

I express my deep gratitude to Dr. Michael David Spivak, an American mathematician for his pedagogical contribution [1] which forms a source inspiration for the simplifications we achieved in this work. I also thank Dr. Sanjay PK, Associate Professor, Mathematics Department for recommending this book.

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ABSTRACT

The goal of this project is to develop a simple rigorous mathematical presentation of tensor products and exterior products over real inner product spaces in a way suitable for exposition to undergraduate students aspiring to study modern mathematical tools such as tensor flow methods which use this mathematical apparatus.

We develop the theory of tensor product spaces and exterior product spaces using multi-linear maps and alternating multi-linear maps respectively which is equivalent to the abstract theory of tensor product spaces when constrained to real inner product spaces. The theory in this work is developed incrementally so that the reader is acquainted with intuition to understand the general theory.

This study is followed by the formulation of determinants and cross products in arbitrary dimensions using alternating multi-linear maps. We arrive at the notion of volume element of a real inner product space and orientation of an orthonormal basis utilizing the properties of alternating multi-linear maps. Finally, we use these notions to develop an orthogonalization algorithm that takes $\mathcal{O}(n^2)$ floating point operations.

Contents

Chapter 1

Introduction

The main objective of this thesis is to develop the theory of tensor products and exterior products in a way suitable for exposition to a typical computer science graduate. Since tensors have profound applications in machine learning, data mining, physics, differential geometry etc, the theory of tensors has been well-studied throughout the course of history (refer [26]). Tensor product spaces over arbitrary vector spaces are studied using concepts such as *universal property* from abstract algebra (see [6]). Another approach in developing the theory of tensors is to define *free algebra* on a vector space (see [2]), where for any vector space V over any field $\mathbb{F}$, for any three vectors u, v, $w \in V$, for any $\alpha \in \mathbb{F}$, we define the following,

$$u \otimes (v + w) = u \otimes v + u \otimes w$$

$$(u + v) \otimes w = u \otimes w + v \otimes w$$

$$(\alpha u) \otimes v = u \otimes (\alpha v) = \alpha u \otimes v$$

We believe that both these approaches are not suitable to introduce these topics to a typical computer science graduate or considering the applicability of this theory in computer science. The main difficulty of studying tensors using these approaches is that *abstract algebra* forms a necessary prerequisite. However, we can simplify the theory of tensors if we restrict to *finite dimensional real inner product spaces*. In chapter 2, we provide an equivalent definition of tensor product spaces using $multi - linear$ functions. The formulation of tensor products as in [1] works for both real and complex inner product spaces. But, we further constrain the formulation to real inner product spaces in order to express tensor products in terms of inner products. It is also shown that the mathematical properties defined using *free algebra* hold for the formulation in terms of multi-linear maps. We don't claim that this material is new but if the reader is comfortable in working with only orthonormal bases we present this material in a simplified manner without compromising on mathematical rigour. Moreover working on orthonormal basis is sufficient for most of the engineering applications. Next, construction of a basis for the space of all multi-linear functions is carried out and the details of the transformation of a tensor if we go from one orthonormal basis to another orthonormal basis is also worked out. This study is followed by the theory of *alternating multi-linear* functions which are used to study cross products and determinants in arbitrary dimensions in chapter 4. While developing the theory of exterior product spaces, it is easy to observe that the permutations theory is necessary to mathematically represent exterior products in higher fold spaces. Hence, some results of permutations which are used in the theory of exterior product spaces are explored in chapter 3 (see [1] and [11]). Determinants are the most familiar alternating tensors (see [2], [4]). In chapter 4, it is shown how a characterization for the determinant function can be achieved using alternating multi-linear maps. It is also shown that the inner product of a vector x_1 with the cross product of vectors $x_2, ..., x_n$ is same as the determinant of the vectors $x_1, x_2, ..., x_n$. Towards, the end of the thesis, a numerical algorithm that computes an orthonormal basis of a real inner product space is presented. This algorithm is a natural consequence of the properties of cross products and determinants formulated in chapter 4. The floating point errors of this algorithm are incompetent with the error of best known algorithm to compute an orthonormal basis. However our goal is to show the applicability of the theory we developed so far to solve a real world problem.

1.1 Motivation

The theory of tensor products has a wide span of applications ranging from machine learning to the general theory of relativity. In machine learning and optimization theory, we often encounter kth order approximation of a function (see [27] and [28]). Consider a *smooth* function $f : \mathbb{R}^n \to \mathbb{R}$. The *multivariate taylor expansion of* f at $x + h$ centered at x is,

$$\begin{aligned} f(x+h) &= f(x) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot h_i + \frac{1}{2} \cdot \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} h_i h_j + \sum_{i,j,k=1}^{n} \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k} h_i h_j h_k + ... \\ &= f(x) + \nabla f_x \cdot h + \frac{1}{2} h^* J(f)_x h + \sum_{i,j,k=1}^{n} \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k} h_i h_j h_k + ... \end{aligned}$$

where ∇f_x denotes the gradient of f at x and $J(f)_x$ denotes Jacobin of f at x. From multi variable analysis, we get that ∇f_x is a $1 \times n$ row vector and $J(f)_x$ is an $n \times n$ matrix (see [4] and [9]). In chapter 2, we see that the space of all $1 \times n$ row vectors over real field is isomorphic to the space of all $1-$tensors and the space of all $n \times n$ matrices over real field is isomorphic to the space of all $2-$tensors. In chapter 2, it is shown that an analytic form cannot be achieved for $3-$tensors. However, the third order approximation term is still a $3-$tensor. Although we don't discuss any further details about the kth order Taylor approximation and how it is used in optimization theory, we develop the theory of tensors that helps in understanding these concepts.

The most familiar alternating tensor is the determinant function. In order to characterize the determinant function, the study of alternating multi-linear functions provides a good mathematical background. In chapter 4. These notions help in defining the volume element in an $n-$dimensional space and also helps in computing the orientation of an orthonormal basis which is used in the theory of integration on manifolds (see [1]). The notion of cross product is formulated in chapter 4 where cross product of $n-1$ vectors $x_1, x_2, ..., x_{n-1}$ in an $n-$dimensional space is defined to be a vector in the *orthogonal complement* of the span of $x_1, x_2, ..., x_{n-1}$ (refer [3] or [12] or [13]). Utilizing this idea an orthogonalization algorithm is developed in chapter 5, that computes the orthonormal basis of a real inner product space.

1.2 Outline of the report

The general theory of tensor product spaces and exterior product spaces is too abstract to directly introduce it to a beginner. Hence, an incremental approach is used in discussing $1-$fold, $2-$fold and $3-$fold spaces to build the intuition that helps in gaining more insights about the general theory of $k-$fold spaces. In the next section of this chapter, we introduce some of the properties of orthonormal basis which we repeatedly use in the rest of the thesis to simplify proofs.

Chapter 2 develops the theory of tensor product spaces starting from a basic concept called dual space (which we shall see is the space of all $1-$tensors) and ends with the presentation of general theory of $k-$fold tensors. In each section, $k-$fold tensor product space is introduced as the space of all $k-$linear functions. Then, $k-$fold tensor product is defined in terms of inner products and it is shown that $k-$fold tensor product is a $k-$linear function. Defining tensor products in terms of inner products is one of the limiting points where this theory must be constrained to real inner product spaces. Following this, a basis for the space of all $k-$ linear functions is constructed in terms of tensor products and the representation of a $k-$ tensor with respect to any basis is provided. Also, the change in representation of a tensor under an orthonormal basis transformation is worked out. Finally, each section concluded by showing the invariance of computation of $k-$tensor under orthonormal basis transformations. Throughout this thesis, it shall be easy to observe that constraining the theory to orthonormal basis helped in simplifying proofs. Note that this theory works even for arbitrary bases but the proofs may be slightly more involved. Since we work on real inner product spaces we can assume with out loss of generality that there exist an orthonormal basis (refer Theorem 1.3.8).

Chapter 3 introduces the space of all alternating $k-$tensors as a subspace of $k-$fold tensor product space. We then define $k-$fold exterior product in terms of $k-$fold tensor products and show that the former is an alternating $k-$tensor. Following this, a basis for exterior product space in terms of exterior products is constructed.

In Chapter 4, the focus is on characterizing the determinant function using alternating tensors in the first part. Next, the notion of volume element and orientation

of an orthonormal basis are defined using determinant which are important concepts in higher mathematics (see [14]). Then, cross product function is defined in arbitrary dimensions and it is shown how cross product and determinant are related.
In chapter 5, an orthogonalization algorithm is developed which performs $\mathcal{O}(n^2)$ floating point operations to output an orthonormal basis given a *non-trivial vector* as input. This algorithm is completely based on the notion of cross products and determinants. We believe that this thesis is self-contained so that a student with a decent background in high school mathematics and basic linear algebra can grasp the subject.

1.3 Prerequisites

We expect the reader to be familiar with basic notions in Linear Algebra. In this section, we study some properties of orthonormal bases of finite dimensional inner product spaces which essentially help in simplifying proofs provided in the subsequent chapters.
Throughout the thesis, we consistently use $dim(V)$ to denote the dimension of vector space V and M^* to denote the transpose of a matrix $M \in \mathbb{R}^{m\times n}$.

Definition 1.3.1. Let V be a finite dimensional vector space over field $\mathbb{R}$. A function $() : V \times V \to \mathbb{R}$ is called an inner product if it satisfies the following,
Linearity : $\forall\ x, y, z \in V$, $\forall\ \alpha \in \mathbb{R}$,

$$(x + y, z) = (x, z) + (y, z)$$

$$(\alpha x, y) = \alpha(x, y)$$

Symmetry : $\forall\ x, y \in V$,

$$(x, y) = (y, x)$$

Positive Definiteness : $\forall\ x \in V$,

$$(x, x) > 0 \text{ if } x \neq 0$$

Remark :

1. $\forall\ x, y, z \in V$, $(x, y+z) = (x, y) + (x, z)$
$$(x, y+z) = (y+z, x) = (y, x) + (z, x) = (x, y) + (x, z) \tag{1.1}$$

2. $\forall\ x, y \in V$, $\forall\ \alpha \in \mathbb{R}$, $(x, \alpha y) = \alpha(x, y)$
$$(x, \alpha y) = (\alpha y, x) = \alpha(y, x) = \alpha(x, y) \tag{1.2}$$

3. $\forall\ x \in V$, $(0, x) = (x, 0) = 0$
$$(0, x) = (0+0, x) = (0, x) + (0, x) \implies (0, x) = 0 \implies (x, 0) = (0, x) = 0 \tag{1.3}$$

4. Let $x \in V$, using the above remark 3, we get
$$(x, x) = 0 \iff x = 0 \tag{1.4}$$

Let $dim(V) = n < \infty$. Let $A = \{a_1, a_2, ..., a_n\}$ be any orthonormal basis of V. Then, $\forall\ i, j \in \{1, 2, ..., n\}$,

$$\begin{aligned} (a_i, a_j) &= 1 \text{ if } i = j \\ &= 0 \text{ if } i \neq j \end{aligned} \tag{1.5}$$

Lemma 1.3.1. Let V be a finite dimensional inner product space over field $\mathbb{R}$ where $dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ be any orthonormal basis of V. Then, $\forall\ x \in V$,

$$\boxed{x = \sum_{i=1}^{n} (x, a_i) a_i}$$

Proof. $\forall\ x \in V$, since A is a basis of V, there exist unique $\alpha_i \in \mathbb{R}$ such that,

$$x = \sum_{i=1}^{n} \alpha_i a_i \tag{1.6}$$

$\forall\ j \in \{1, 2, ..., n\}$,

$$(x, a_j) = (\sum_{i=1}^{n} \alpha_i a_i, a_j)$$

Using linearity of inner product 1.3.1, we get

$$(x, a_j) = \sum_{i=1}^{n} (\alpha_i a_i, a_j) = \sum_{i=1}^{n} \alpha_i (a_i, a_j)$$

Since A is an orthonormal basis of V, using equation 1.5 we get,

$$(x, a_j) = \alpha_j \tag{1.7}$$

Combining equations 1.6 and 1.7 we get,

$$x = \sum_{i=1}^{n} (x, a_i) a_i$$

$\square$

Remark :

1. Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$ be any two orthonormal bases of V. Then, $\forall\ x \in V$, using Lemma 1.3.1 we get,

$$\boxed{x = \sum_{i=1}^{n} (x, a_i) a_i = \sum_{j=1}^{n} (x, b_j) b_j}$$

We set the convention that the coordinates of any vector $x \in V$ with respect to a basis A is a **column vector**. Formally,

Definition 1.3.2. Let V be a finite dimensional vector space over field $\mathbb{R}$ where $dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ be a basis of V, then $\forall\ x \in V$, there exist unique $\alpha_i \in \mathbb{R}$ such that

$$x = \sum_{i=1}^{n} \alpha_i a_i = \begin{bmatrix} a_1 & a_2 & . & . & a_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ . \\ . \\ \alpha_n \end{bmatrix}$$

We denote the coordinates of the vector x with respect to basis A as follows,

$$^{A}x = \begin{bmatrix} \alpha_1 & \alpha_2 & . & . & \alpha_n \end{bmatrix}^* \in \mathbb{R}^n$$

We denote the r th coordinate of vector x with respect to basis A as follows,

$$^{A}x[r] = \alpha_r \in \mathbb{R}$$

Remark :

1. If A is an orthonormal basis of V, then $\forall\ x \in V$, using Lemma 1.3.1, we get

$$x = \sum_{r=1}^{n} (x, a_i) a_i \implies {}^{A}x[i] = \alpha_i = (x, a_i)\ \forall\ i \in \{1, 2, ..., n\} \tag{1.8}$$

Lemma 1.3.2. Let V be a finite dimensional inner product space over field $\mathbb{R}$ where $dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ be any orthonormal basis of V. Then, $\forall\ x, y \in V$,

$$\boxed{(x, y) = \sum_{r=1}^{n} {}^{A}x[r] \cdot {}^{A}y[r]}$$

Proof. $\forall\ x, y \in V$, since A and B are bases of V, there exist unique α_r, $\beta_s \in \mathbb{R}$, such that

$$x = \sum_{r=1}^{n} \alpha_r a_r \qquad\qquad y = \sum_{s=1}^{n} \beta_s b_s$$

$$\begin{aligned}(x, y) &= (\sum_{r=1}^{n} \alpha_r a_r, \sum_{s=1}^{n} \beta_s a_s) \\ &= \sum_{r=1}^{n} \sum_{s=1}^{n} \alpha_r \beta_s (a_r, a_s) \\ &= \sum_{r=1}^{n} \alpha_r \beta_r \qquad \text{(since } A \text{ is an orthonormal basis of } V\text{)}\end{aligned}$$

From Remark 1.8 we get,

$$^{A}x[r] = \alpha_r \qquad\qquad ^{A}y[r] = \beta_r$$

$$\implies (x, y) = \sum_{r=1}^{n} {}^{A}x[r] \cdot {}^{A}y[r]$$

□

Definition 1.3.3. Let V be a finite dimensional inner product space over field $\mathbb{R}$ where $dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ be any orthonormal basis of V. Then, $\forall\ x, y \in V$, we define the dot product of x and y as follows,

$$\boxed{{}^{A}x \odot {}^{A}y = \sum_{r=1}^{n} {}^{A}x[r] \cdot {}^{A}y[r]}$$

Remark :

1. Lemma 1.3.2 implies that the dot product of any two vectors $x, y \in V$ has the same value irrespective of the choice of orthonormal basis. More concretely, Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$ be any 2 orthonormal bases of V. Then, $\forall\ x, y \in V$, using Lemma 1.3.1 and Lemma 1.3.2 we get

$$^{A}x \odot {}^{A}y = (x, y) = {}^{B}x \odot {}^{B}y$$

Definition 1.3.4. A matrix $M \in \mathbb{R}^{n \times n}$ is called non-singular if $\forall\ \alpha \in \mathbb{R}^n$,

$$M \cdot \alpha = 0 \implies \alpha = 0$$

Remark :

1. Several equivalent definitions for the non-singularity of a matrix M can be found in different textbooks. In the above definition we define M to be non-singular if and only if $Nullspace(M) = \{0\}$[1].

Theorem 1.3.3. Let V be a finite dimensional vectorspace over field $\mathbb{R}$ where $dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$ be any two bases of V. Then, there exists a non-singular matrix $M \in \mathbb{R}^{n\times n}$, such that,

$$\begin{bmatrix} a_1 & a_2 & . & . & a_n \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & . & . & b_n \end{bmatrix} M$$

Note that the matrix M is called the **basis transformation matrix** from A **to** B.

Proof. Since B is a basis of V, $\forall\ j \in \{1, ..., n\}$, there exist unique $M_{ij} \in \mathbb{R}$ such that,

$$a_j = \sum_{i=1}^{n} M_{ij} b_i = \begin{bmatrix} b_1 & b_2 & . & . & b_n \end{bmatrix} \begin{bmatrix} M_{1j} \\ M_{2j} \\ . \\ . \\ M_{nj} \end{bmatrix}$$

$$\implies \begin{bmatrix} a_1 & a_2 & . & . & a_n \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & . & . & b_n \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} & . & . & M_{1n} \\ M_{21} & M_{22} & . & . & M_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ M_{n1} & M_{n2} & . & . & M_{nn} \end{bmatrix}$$

Let $M = \begin{bmatrix} M_{11} & M_{12} & . & . & M_{1n} \\ M_{21} & M_{22} & . & . & M_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ M_{n1} & M_{n2} & . & . & M_{nn} \end{bmatrix} \in \mathbb{R}^{n\times n}$. Then,

$$\begin{bmatrix} a_1 & a_2 & . & . & a_n \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & . & . & b_n \end{bmatrix} M$$

[1]To explore more about Nullspace and Singularity refer [3] or [13]

To show that M is non-singular. $\forall\ \alpha \in \mathbb{R}^n$, let $\alpha = \begin{bmatrix}\alpha_1 & \alpha_2 & . & . & \alpha_n\end{bmatrix}^*$,

$$\begin{bmatrix}a_1 & a_2 & . & . & a_n\end{bmatrix} \cdot \alpha = \begin{bmatrix}b_1 & b_2 & . & . & b_n\end{bmatrix} M \cdot \alpha$$

if $M \cdot \alpha = 0$ then,

$$\begin{bmatrix}a_1 & a_2 & . & . & a_n\end{bmatrix} \cdot \alpha = \begin{bmatrix}b_1 & b_2 & . & . & b_n\end{bmatrix} M \cdot \alpha = \begin{bmatrix}b_1 & b_2 & . & . & b_n\end{bmatrix} 0 = 0$$

$$\begin{bmatrix}a_1 & a_2 & . & . & a_n\end{bmatrix} \cdot \begin{bmatrix}\alpha_1 \\ \alpha_2 \\ . \\ . \\ \alpha_n\end{bmatrix} = 0 \implies \sum_{i=1}^{n} \alpha_i a_i = 0$$

Since A is a linearly independent set we get

$$\alpha_i = 0\ \forall\ \{1, 2, ..., n\} \implies \alpha = 0$$

□

Corollary 1.3.4. Let V be a finite dimensional vectorspace over field $\mathbb{R}$ where $dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$ be any two bases of V. Let $M \in \mathbb{R}^{n\times n}$ be the basis transformation matrix from A to B i.e,

$$\begin{bmatrix}a_1 & a_2 & . & . & a_n\end{bmatrix} = \begin{bmatrix}b_1 & b_2 & . & . & b_n\end{bmatrix} M$$

Then, $\forall\ x \in V$,

$$\boxed{{}^{B}x = M \cdot {}^{A}x}$$

Proof. $\forall\ x \in V$,

$$x = \begin{bmatrix}a_1 & a_2 & . & . & a_n\end{bmatrix} \cdot {}^{A}x = \begin{bmatrix}b_1 & b_2 & . & . & b_n\end{bmatrix} \cdot {}^{B}x$$

Since M is the basis transformation from A to B we get

$$\begin{bmatrix}b_1 & b_2 & . & . & b_n\end{bmatrix} M \cdot {}^{A}x = \begin{bmatrix}b_1 & b_2 & . & . & b_n\end{bmatrix} {}^{B}x$$

$$\implies \begin{bmatrix} b_1 & b_2 & . & . & b_n \end{bmatrix} ({}^{B}x - M \cdot {}^{A}x) = 0$$

Since B is a linearly independnent set, we get

$${}^{B}x = M \cdot {}^{A}x$$

□

Remark :

1. Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$ be any two bases of V. If M is the basis transformation matrix from basis A to basis B then M^{-1} is the basis transformation matrix from B to A. Note that M^{-1} is well defined since M is non-singular. More concretely, $\forall\ x \in V$,

$$\begin{bmatrix} a_1 & a_2 & . & . & a_n \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & . & . & b_n \end{bmatrix} M \implies {}^{B}x = M \cdot {}^{A}x$$

$$M^{-1} \cdot {}^{B}x = M^{-1} \cdot M \cdot {}^{A}x \implies \boxed{{}^{A}x = M^{-1} \cdot {}^{B}x}$$

$$\implies \begin{bmatrix} b_1 & b_2 & . & . & b_n \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & . & . & a_n \end{bmatrix} M^{-1}$$

Theorem 1.3.5. Let V be a finite dimensional inner product space over field $\mathbb{R}$ where $dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$ be any two orthonormal bases of V. Let $M \in \mathbb{R}^{n \times n}$ be the basis transformation matrix from A to B i.e.,

$$\begin{bmatrix} a_1 & a_2 & . & . & a_n \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & . & . & b_n \end{bmatrix} M$$

Then, M is an orthogonal matrix. That is,

$$\boxed{M \cdot M^* = M^* \cdot M = I \implies M^{-1} = M^*}$$

Proof. Let $E = \{e_1 = \begin{bmatrix} 1 & 0 & . & . & 0 \end{bmatrix}^*, e_2 = \begin{bmatrix} 0 & 1 & . & . & 0 \end{bmatrix}^*, ..., e_n = \begin{bmatrix} 0 & 0 & . & . & 1 \end{bmatrix}^*\}$. Note that E is the standard orthonormal basis of $\mathbb{R}^n$. It is easy to notice that, $\forall$ $i \in \{1, 2, ..., n\}$,

$${}^{A}a_i = e_i$$

Since M is the basis transformation matrix from A to B we get that, $\forall\ j \in \{1, 2, ..., n\}$,

$$^{B}a_j = M \cdot {}^{A}a_j = M \cdot e_j$$

Using Lemma 1.3.2, we get that $\forall\ i, j \in \{1, 2, ..., n\}$,

$$(a_i, a_j) = ({}^{B}a_i)^* \cdot ({}^{B}a_j)$$

$$\implies (a_i, a_j) = (M \cdot e_i)^* \cdot (M \cdot e_j) = e_i^* \cdot M^* \cdot M \cdot e_j$$

Note that $\forall\ N \in \mathbb{R}^{n\times n}$,

$$e_i^* N e_j = N_{ij}$$

$$\implies (a_i, a_j) = [M^* \cdot M]_{ij}$$

Since A is an orthonormal basis of V, we get that

$$\begin{aligned}[M^* \cdot M]_{ij} &= 1 \text{ if } i = j \\ &= 0 \text{ if } i \neq j\end{aligned}$$

$$\implies M^* \cdot M = I$$

□

Remark :

1. Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$ be any two orthonormal bases of V. If M is the basis transformation matrix from A to B then, $M^{-1} = M^*$ is the basis transformation matrix from B to A.

Definition 1.3.5. Let V be a finite dimensional inner product space over $\mathbb{R}$. Then, $\forall\ u \in V$ with $\|u\| = \sqrt{(u, u)} = 1$ (i.e, u is a unit vector), we define projection operator along u, $P_u : V \to V$ as follows, $\forall\ x \in V$,

$$\boxed{P_u(x) = (x, u)u}$$

Remark :

1. Note that $P_u(x) \in Span\{u\}$

Lemma 1.3.6. Let V be a finite dimensional inner product space over $\mathbb{R}$. Then, $\forall\ u \in V$ with $\|u\| = 1$, $\forall\ x \in V$,

$$(x - P_u(x), u) = 0$$

(which means that $x - P_u(x)$ is orthogonal to u)

Proof. $\forall\ u \in V$, such that $\|u\| = 1$, $\forall\ x \in V$, from Definition 1.3.5 we get that,

$$\begin{aligned}(x - P_u(x), u) &= (x - (x, u)u, u) \\ &= (x, u) - (x, u)(u, u)\end{aligned}$$

Since $\|u\| = \sqrt{(u, u)} = 1$,

$$(x - P_u(x), u) = (x, u) - (x, u) = 0$$

$\square$

Lemma 1.3.7. If $A = \{a_1, a_2, ..., a_n\}$ is an orthonormal set of vectors in V, then A is a linearly independent set.

Proof. If $A = \{a_1, a_2, ..., a_n\}$ is an orthonormal set of vectors in V then, $\forall\ i, j \in \{1, 2, ..., n\}$,

$$\begin{aligned}(a_i, a_j) &= 1 \text{ if } i = j \\ &= 0 \text{ if } i \neq j\end{aligned}$$

Let $\alpha_i \in \mathbb{R}\ \forall\ i \in \{1, 2, ..., n\}$. Consider,

$$\sum_{i=1}^{n} \alpha_i a_i = 0$$

$\forall\ j \in \{1, 2, ..., n\}$,

$$0 = (0, a_j) = (\sum_{i=1}^{n} \alpha_i a_i, a_j, a_j) = \sum_{i=1}^{n} \alpha_i (a_i, a_j) = \alpha_j$$

$$\implies \alpha_j = 0 \ \forall \ j \in \{1, 2, ..., n\}$$

$$\implies A \text{ is a linearly independent set}$$

□

Theorem 1.3.8. (Gram Schmidt Orthogonalization) Every finite dimensional inner product space over $\mathbb{R}$ has an orthonormal basis.

Proof. Let V be a finite dimensional inner product space over $\mathbb{R}$. Let $dim(V) = n < \infty$. Since $dim(V) = n$ there exist a set $A = \{a_1, a_2, ..., a_n\}$ such that A spans V and A is a linearly independent set. Now, we use Gram-Schmidt process to define the following vectors,

$$\tilde{b}_1 = a_1 \qquad\qquad b_1 = \frac{\tilde{b}_1}{\|\tilde{b}_1\|}$$

$a_1 \neq 0$ since A is a linearly independent set $\implies \|\tilde{b}_1\| \neq 0 \implies b_1$ is well defined.

$$\tilde{b}_2 = a_2 - P_{b_1}(a_2) \qquad\qquad b_2 = \frac{\tilde{b}_2}{\|\tilde{b}_2\|} \qquad\qquad ...$$

$$\tilde{b}_i = a_i - \sum_{j=1}^{i-1} P_{b_j}(a_i) \qquad\qquad b_i = \frac{\tilde{b}_i}{\|\tilde{b}_i\|} \qquad\qquad ...$$

$$\tilde{b}_n = a_n - \sum_{j=1}^{n-1} P_{b_j}(a_n) \qquad\qquad b_n = \frac{\tilde{b}_n}{\|\tilde{b}_n\|}$$

From Remark 1.3 it follows that $Span\{b_1, b_2, ..., b_i\} = Span\{a_1, a_2, ..., a_i\} \ \forall \ i \in \{1, 2, ..., n\}$.

For $b_2, b_3, ..., b_n$ to be well-defined, we need to prove the following claim,

Claim : $\forall \ i \in \{2, ..., n\}$,

$$\tilde{b}_i = a_i - \sum_{j=1}^{i-1} P_{b_j}(a_i) \neq 0$$

If not then,

$$a_i = \sum_{j=1}^{i-1} P_{b_j}(a_i) = \sum_{j=1}^{i-1}(a_i, b_j)b_j \in Span\{b_1, b_2, ..., b_{i-1}\} = Span\{a_1, a_2, ..., a_{i-1}\}$$

This is a contradiction to the fact that A is a linearly independent set. Hence,

$$a_i - \sum_{j=1}^{i-1} P_{b_j}(a_i) \neq 0 \implies \|\tilde{b}_i\| \neq 0 \implies \text{ each } b_i \text{ is well defined}$$

From definition we get, $\forall\ i \in \{1, 2, ..., n\}$,

$$\|b_i\| = \frac{\|\tilde{b}_i\|}{\|\tilde{b}_i\|} = 1$$

Hence, it remains to show that $(b_i, b_j) = 0\ \forall\ i, j \in \{1, 2, ..., n\}$ where $i < j$.
Claim : $\forall\ i \in \{1, 2, ..., n-1\}$ if $\{b_1, b_2, ..., b_i\}$ is an orthonormal set then,

$$(b_{i+1}, b_k) = 0\ \forall\ 1 \leq k \leq i$$

$$\begin{aligned}
(\tilde{b}_{i+1}, b_k) &= (a_{i+1} - \sum_{j=1}^{i} P_{b_j}(a_{i+1}), b_k) = (a_{i+1}, b_k) - (\sum_{j=1}^{i} P_{b_j}(a_{i+1}), b_k) \\
&= (a_{i+1}, b_k) - \sum_{j=1}^{i}(P_{b_j}(a_{i+1}), b_k) = (a_{i+1}, b_k) - \sum_{j=1}^{i}((a_{i+1}, b_j)b_j, b_k) \\
&= (a_{i+1}, b_k) - \sum_{j=1}^{i}(a_{i+1}, b_j)(b_j, b_k)
\end{aligned}$$

If $\{b_1, b_2, ..., b_i\}$ is an orthonormal set, then

$$(\tilde{b}_{i+1}, b_k) = (a_{i+1}, b_k) - (a_{i+1}, b_k) = 0 \implies (b_{i+1}, b_k) = \frac{(\tilde{b}_{i+1}, b_k)}{\|\tilde{b}_{i+1}\|} = 0$$

Using Lemma 1.3.7, B is a linearly independent set. Since $dim(V) = n = |B|$ and B is a linearly independent set, B forms a basis of V. Hence, we have shown the existence of an orthonormal basis for V. □

Lemma 1.3.9. Any finite dimensional inner product space over $\mathbb{R}$ with dimension n is isomorphic to the vector space $\mathbb{R}^n$ over $\mathbb{R}$.

Proof. Let V be any finite dimensional inner product space over $\mathbb{R}$. Let $dim(V) = n < \infty$. To establish an isomorphism between V and $\mathbb{R}^n$, it is enough to provide a linear transformation that maps the basis vectors of V to basis vectors of $\mathbb{R}^n$ bijectively. Consider the standard orthonormal basis $E = \{e_1 = \begin{bmatrix} 1 & 0 & . & . & 0 \end{bmatrix}^*$, $e_2 = \begin{bmatrix} 0 & 1 & . & . & 0 \end{bmatrix}^*$, ..., $e_n = \begin{bmatrix} 0 & 0 & . & . & 1 \end{bmatrix}^*\}$ of $\mathbb{R}^n$ over $\mathbb{R}$. Since $dim(V) = n$, there exists a basis $A = \{a_1, a_2, ..., a_n\}$ of V. Now, consider the linear transformation $T : V \to \mathbb{R}^n$ defined as follows,

$$T(a_i) = e_i$$

Note that if a linear transformation is defined over each basis vector of V, then the linear transformation is well defined $\forall\ x \in V$. It remains to show that T is a bijection in order to establish the isomorphism.

Claim : T is an injective map

To prove that T is an injection, it is enough to show that $\forall\, x, y \in V$, if $T(x) = T(y)$ then $x = y$

$\forall\ x, y \in V$, since A is a basis of V, there exist unique $\alpha_i, \beta_i \in \mathbb{R}$ such that,

$$x = \sum_{i=1}^{n} \alpha_i a_i \qquad\qquad y = \sum_{i=1}^{n} \beta_i a_i$$

$$T(x) = T(y) \implies T(\sum_{i=1}^{n} \alpha_i a_i) = T(\sum_{i=1}^{n} \beta_i a_i) \implies \sum_{i=1}^{n} \alpha_i T(a_i) = \sum_{i=1}^{n} \beta_i T(a_i)$$
$$\implies \sum_{i=1}^{n} (\alpha_i - \beta_i) T(a_i) = \sum_{i=1}^{n} (\alpha_i - \beta_i) e_i = 0$$

Since E is a linearly independent set we get,

$$\alpha_i = \beta_i\ \forall\ i \in \{1, 2, ..., n\} \implies x = y$$

Claim : T is a surjective map

To prove that T is a surjection, it is enough to show that $\forall\ y \in \mathbb{R}^n$, there exist

$x \in V$ such that $T(x) = y$. $\forall\ y \in \mathbb{R}^n$, since E is a basis of $\mathbb{R}^n$, there exist unique $\alpha_i \in \mathbb{R}$ such that,

$$y = \sum_{i=1}^{n} \alpha_i e_i$$

Now, consider

$$x = \sum_{i=1}^{n} \alpha_i a_i$$

Since A spans V, $x \in V$,

$$T(x) = T(\sum_{i=1}^{n} \alpha_i a_i) = \sum_{i=1}^{n} \alpha_i T(a_i) = \sum_{i=1}^{n} \alpha_i e_i = y$$

$\square$

As a concluding remark, we can see that if we limit our framework to orthonormal basis, any finite dimensional inner product space over $\mathbb{R}$ with dimension n can be identified with $\mathbb{R}^n$ over $\mathbb{R}$ with standard dot product as the inner product.

Chapter 2

Tensor Products

In this thesis, we limit our focus to finite dimensional inner product spaces over $\mathbb{R}$. We limit the formulation to orthonormal bases in order to simplify proofs. These formulations can be extended to arbitrary bases, but the proofs of some of the theorems may be slightly more involved. As seen in the prerequisites section of chapter 1 if we limit our formulation to orthonormal basis, any finite dimensional inner product space over $\mathbb{R}$ with dimension n can be identified with $\mathbb{R}^n$ over $\mathbb{R}$ with standard dot product as the inner product. Placement of these constraints over the general theory helps in developing a more accessible material for computer science audience without compromising on mathematical rigour and practical applicability.

2.1 1-Fold Tensor Product Spaces - Dual Spaces

2.1.1 Linear Functions

Definition 2.1.1. Let V be a vector space over field $\mathbb{R}$ with inner product $() : V \times V \to \mathbb{R}$ defined on it. A function $u : V \to \mathbb{R}$ is called linear if $\forall\ x, y \in V$, $\forall\ \alpha \in \mathbb{R}$,

$$u(x + y) = u(x) + u(y) \tag{2.1}$$

$$u(\alpha \cdot x) = \alpha \cdot u(x) \tag{2.2}$$

Let set $S = \{u : V \to \mathbb{R} \mid u \text{ is linear}\}$. Define addition and multiplication on the set S as follows, $\forall\ u, v \in S$, $\forall\ x \in V$, $\forall\ \alpha \in \mathbb{R}$,

$$[u + v](x) = u(x) + v(x) \tag{2.3}$$

$$[\alpha \cdot u](x) = \alpha \cdot u(x) \tag{2.4}$$

Note that we use [.] to indicate the addition and scalar multiplication of elements of the set S.

Lemma 2.1.1. S is closed under addition and scalar multiplication

Proof. **Claim 1 :** $\forall\ u, v \in S$, $[u + v] \in S$

1. $\forall\ x, y \in V$,

$$\begin{aligned}[u + v](x + y) &= u(x + y) + v(x + y) \\ &= u(x) + u(y) + v(x) + v(y) \\ &= [u + v](x) + [u + v](y)\end{aligned}$$

2. $\forall\ x \in V$, $\forall\ \alpha \in \mathbb{R}$,

$$\begin{aligned}[u + v](\alpha x) &= u(\alpha x) + v(\alpha x) \\ &= \alpha u(x) + \alpha v(x) = \alpha[u + v](x)\end{aligned}$$

$$\implies [u + v] \text{ is linear } \implies [u + v] \in S \implies S \text{ is closed under addition}$$

Claim 2 : $\forall\ u \in V$, $\alpha \in \mathbb{R}$, $[\alpha u] \in S$,

1. $\forall\ x, y \in V$,

$$\begin{aligned}[\alpha u](x+y) &= \alpha u(x+y) \\ &= \alpha u(x) + \alpha u(y) \\ &= [\alpha u](x) + [\alpha u](y)\end{aligned}$$

2. $\forall\ x \in V$, $\beta \in \mathbb{R}$,

$$\begin{aligned}[\alpha u](\beta x) &= \alpha u(\beta x) \\ &= \beta\alpha u(x) \\ &= \beta[\alpha u](x)\end{aligned}$$

$$\implies [\alpha u] \text{ is linear } \implies [\alpha u] \in S \implies S \text{ is closed under scalar multiplication}$$

$\square$

A linear function $u \in S$ is called a **1-tensor** or a dual map on V. It is easy to verify that S is a vector space over field $\mathbb{R}$ (We already proved that S is closed under addition and scalar multiplication and rest of the axioms of vector space are easy to verify and are left to reader). The vectorspace of all 1-tensors is defined as the $1-$fold tensor product space of V denoted by $\mathcal{L}(V \to \mathbb{R})$ or V^*. In addition, $1-$fold tensor product space is also called dual space.

2.1.2 $1-$tensors using inner product

Definition 2.1.2. Let V be a finite dimensional inner product space over field $\mathbb{R}$ with $dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ be an orthonormal basis of V. Then, $\forall\ i \in \{1, 2, ..., n\}$ define $a_i^* : V \to \mathbb{R}$ as follows, $\forall\ x \in V$,

$$\boxed{a_i^*(x) = (a_i, x)}$$

Remark :

1. Note that $\forall\ i \in \{1, 2, ..., n\}$, a_i^* is linear and it follows directly from the definition of inner product 1.3.1

Illustration :
Consider $V = \mathbb{R}^2$ over $\mathbb{R}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^*, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^*\}$. We use $E = \{e_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^*, e_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^*\}$ to denote the standard orthonormal basis of $\mathbb{R}^2$. It is easy to verify that A forms an orthonormal basis of V. From the Definition 2.1.2, we get the function $a_i^* : \mathbb{R}^2 \to \mathbb{R}$ defined as follows, $\forall\ x \in V$,

$$a_i^*(x) = (a_i, x) = {}^{A}a_i \odot {}^{A}x \text{ where } 1 \leq i \leq 2$$

Note that ${}^{A}a_i = e_i$

$$\implies a_i^*(x) = e_i \odot {}^{A}x = {}^{A}x[i]$$

1. $\forall\ x, y \in V$,

$$\begin{aligned} a_i^*(x+y) &= {}^{A}(x+y)[i] \\ &= {}^{A}x[i] + {}^{A}y[i] \qquad \text{since } {}^{A}(x+y) = {}^{A}x + {}^{A}y \\ &= a_i^*(x) + a_i^*(y) \end{aligned}$$

2. $\forall\ x \in V$, $\alpha \in \mathbb{R}$,

$$\begin{aligned} a_i^*(\alpha x) &= {}^{A}(\alpha x)[i] \\ &= \alpha \cdot {}^{A}x[i] \qquad \text{since } {}^{A}(\alpha x) = \alpha \cdot {}^{A}x \\ &= \alpha a_i^*(x) \end{aligned}$$

$$\implies a_i^* \text{ is linear}$$

The linearity of coordinates of vectors can be easily verified by expressing the vectors x and y in terms of basis A. Also, observe that using this illustration we have indicated a way to prove that a_i^* is linear in terms of coordinates of vectors.

2.1.3 Basis of $1-$fold tensor product spaces

Let V be a finite dimensional inner product space over field $\mathbb{R}$ with $dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ be an orthonormal basis of V. Let $A^* = \{a_1^*, a_2^*, ..., a_n^*\}$. From Remark 2.1.2 we get that $\forall\ i \in \{1, 2, ..., n\}$, a_i^* is linear.

Theorem 2.1.2. A^* is a basis for vector space $\mathcal{L}(V \to \mathbb{R})$.

Proof. **Span :**
$\forall\ x \in V$, since A is a basis of V, there exist unique $\alpha_i \in \mathbb{R}$ such that

$$x = \sum_{i=1}^{n} \alpha_i a_i$$

$\forall\ u \in \mathcal{L}(V \to \mathbb{R})$, since u is linear , we get

$$u(x) = u(\sum_{i=1}^{n} \alpha_i a_i) = \sum_{i=1}^{n} \alpha_i u(a_i) \tag{2.5}$$

$\forall\ j \in \{1, ..., n\}$, since A is an orthonormal basis of V we get

$$a_j^*(x) = (a_j, x) = (a_j, \sum_{i=1}^{n} \alpha_i a_i) = \sum_{i=1}^{n} \alpha_i (a_j, a_i) = \alpha_j \tag{2.6}$$

Combining 2.5 and 2.6 we get,

$$u(x) = \sum_{i=1}^{n} u(a_i) a_i^*(x) \implies u = \sum_{i=1}^{n} u(a_i) a_i^*$$

$$\implies A^* \text{ spans } \mathcal{L}(V \to \mathbb{R})$$

Linear Independence :
Let $\alpha_i \in \mathbb{R}\ \forall\ 1 \leq i \leq n$. Consider,

$$\sum_{i=1}^{n} \alpha_i a_i^* = 0$$

$\forall\ j \in \{1, 2, ..., n\}$, since A is an orthonormal basis of V we get,

$$0 = (0, a_j) = (\sum_{i=1}^{n} \alpha_i a_i, a_j) = \sum_{i=1}^{n} \alpha_i (a_i, a_j) = \alpha_j$$

$$\implies \alpha_j = 0\ \forall\ j \in \{1, 2, ..., n\}$$

$$\implies A^* \text{ is a linearly independent set and a basis of } \mathcal{L}(V \to \mathbb{R})$$

$\square$

Corollary 2.1.3.

$$\boxed{dim(\mathcal{L}(V \to \mathbb{R})) = dim(V^*) = dim(V) = n}$$

Illustration :
Consider $V = \mathbb{R}^2$ over $\mathbb{R}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^*, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^*\}$. Let $A^* = \{a_1^*, a_2^*\}$ where each $a_i^* : \mathbb{R}^2 \to \mathbb{R}$ is defined as follows, $\forall\ x \in V$,

$$a_i^*(x) = (x, a_i) = {}^{A}x \odot {}^{A}a_i = {}^{A}x[i] \text{ where } 1 \leq i \leq 2$$

From the above Theorem, we get that A^* is a basis of $\mathcal{L}(\mathbb{R}^2 \to \mathbb{R})$. Consider the following linear function,

$$u(\begin{bmatrix} x \\ y \end{bmatrix}) = x + y \qquad \text{where } \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$$

It is straight forward to verify that u is linear. Since A is a basis of V, $\forall \begin{bmatrix} x & y \end{bmatrix}^* \in \mathbb{R}^2$ there exist unique α, $\beta \in \mathbb{R}$, such that

$$\begin{bmatrix} x & y \end{bmatrix}^* = \alpha a_1 + \beta a_2$$

Since u is linear we get that

$$u(\begin{bmatrix} x \\ y \end{bmatrix}) = u(\alpha a_1 + \beta a_2) = \alpha u(a_1) + \beta u(a_2)$$

From the above expression, it is quite clear that computing $u(a_1)$ and $u(a_2)$ is sufficient to determine the action of u on any $\begin{bmatrix} x & y \end{bmatrix}^*$.

$$u(a_1) = u(\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}) = \sqrt{2} \qquad\qquad u(a_2) = u(\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}) = 0$$

$$\implies u(\begin{bmatrix} x \\ y \end{bmatrix}) = \sqrt{2}\alpha$$

Now, let's inspect the action of a_1^* and a_2^* on $\begin{bmatrix} x & y \end{bmatrix}^*$,

$$a_1^*(\begin{bmatrix} x \\ y \end{bmatrix}) = (\alpha a_1 + \beta a_2, a_1) = \alpha \qquad\qquad a_2^*(\begin{bmatrix} x \\ y \end{bmatrix}) = (\alpha a_1 + \beta a_2, a_2) = \beta$$

$$\implies u(\begin{bmatrix} x \\ y \end{bmatrix}) = \sqrt{2}a_1^*(\begin{bmatrix} x \\ y \end{bmatrix}) = u(a_1)a_1^*(\begin{bmatrix} x \\ y \end{bmatrix}) + u(a_2)a_2^*(\begin{bmatrix} x \\ y \end{bmatrix})$$

With this illustration, you can observe how $\{a_1^*, a_2^*\}$ works as a basis of $\mathcal{L}(\mathbb{R}^2 \to \mathbb{R})$ and also note that the values $u(a_1)$, $u(a_2)$ is sufficient to compute $u(\begin{bmatrix} x \\ y \end{bmatrix})$ for any $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$

Remark :

1. $\forall\ u \in \mathcal{L}(V \to \mathbb{R})$. From Theorem 2.1.2, we get that,

$$u = \sum_{i=1}^{n} u(a_i)a_i^*$$

From Definition 1.3.2, we get coordinates of the vector $u \in \mathcal{L}(V \to \mathbb{R})$ with respect to basis A^* as follows,

$$^{A^*}u = \begin{bmatrix} u(a_1) & u(a_2) & . & . & u(a_n) \end{bmatrix}^*$$

2.1.4 Basis Transformation

Theorem 2.1.4. Let V be a finite dimensional inner product space over field $\mathbb{R}$ with $dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$ be any two orthonormal basis of V. Let $A^* = \{a_1^*, a_2^*, ..., a_n^*\}$ and $B^* = \{b_1^*, b_2^*, ..., b_n^*\}$. From Theorem 2.1.2, we get that both A^* and B^* form bases for $\mathcal{L}(V \to \mathbb{R})$. Let $M \in \mathbb{R}^{n\times n}$ be the transformation matrix from basis A to B i.e,

$$\begin{bmatrix} a_1 & a_2 & . & . & . & a_n \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & . & . & . & b_n \end{bmatrix} M$$

Then, $\forall\ u \in \mathcal{L}(V \to \mathbb{R})$,

$$\boxed{{}^{B^*}u = M \cdot {}^{A^*}u}$$

Proof. $\forall\ u \in \mathcal{L}(V \to \mathbb{R})$, since A^* and B^* form bases for $\mathcal{L}(V \to \mathbb{R})$,

$$u = \sum_{j=1}^{n} u(a_j)a_j^* = \sum_{i=1}^{n} u(b_i)b_i^*$$

$$\implies {}^{A^*}u = \begin{bmatrix} u(a_1) & u(a_2) & . & . & . & u(a_n) \end{bmatrix}^* \qquad {}^{B^*}u = \begin{bmatrix} u(b_1) & u(b_2) & . & . & . & u(b_n) \end{bmatrix}^*$$

Since M is the transformation matrix from basis A to B, we get that M^* is the transformation matrix from B to A i.e,

$$\begin{bmatrix} b_1 & b_2 & . & . & . & b_n \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & . & . & . & a_n \end{bmatrix} M^*$$

$\forall\ i \in \{1, 2, ..., n\}$,

$$b_i = \sum_{j=1}^{n} M_{ji}^* a_j = \sum_{j=1}^{n} M_{ij} a_j$$

Since u is linear, we get

$$u(b_i) = u(\sum_{j=1}^{n} M_{ij}a_j) = \sum_{j=1}^{n} M_{ij}u(a_j)$$

$$\implies u(b_i) = \begin{bmatrix} M_{i1} & M_{i2} & . & . & M_{in} \end{bmatrix} \begin{bmatrix} u(a_1) \\ u(a_2) \\ . \\ . \\ u(a_n) \end{bmatrix}$$

$$\implies \begin{bmatrix} u(b_1) \\ u(b_2) \\ . \\ . \\ u(b_n) \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & . & . & M_{1n} \\ M_{21} & M_{22} & . & . & M_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ M_{n1} & M_{n2} & . & . & M_{nn} \end{bmatrix} \begin{bmatrix} u(a_1) \\ u(a_2) \\ . \\ . \\ u(a_n) \end{bmatrix} \implies {}^{B^*}u = M \cdot {}^{A^*}u$$

$\square$

Remark :

1. Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$ be any two orthonormal basis of V. Then, $A^* = \{a_1^*, a_2^*, ..., a_n^*\}$ and $B^* = \{b_1^*, b_2^*, ..., b_n^*\}$ form bases of $\mathcal{L}(V \to \mathbb{R})$. Let $M \in \mathbb{R}^{n \times n}$ be the transformation matrix from basis A to B. Then, M is the basis transformation matrix from basis A^* to basis B^* and $M^{-1} = M^*$ is the basis transformation matrix from B to A. More concretely, $\forall\ u \in \mathcal{L}(V \to \mathbb{R})$,

$$\begin{bmatrix} a_1^* & a_2^* & . & . & a_n^* \end{bmatrix} = \begin{bmatrix} b_1^* & b_2^* & . & . & b_n^* \end{bmatrix} M \implies {}^{B^*}u = M \cdot {}^{A*}u$$

$$\implies \boxed{{}^{A^*}u = M^* \cdot {}^{B}u} \implies \begin{bmatrix} b_1^* & b_2^* & . & . & b_n^* \end{bmatrix} = \begin{bmatrix} a_1^* & a_2^* & . & . & a_n^* \end{bmatrix} M^*$$

Illustration :

Consider $V = \mathbb{R}^2$ over $\mathbb{R}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^*, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^*\}$. Let $A^* = \{a_1^*, a_2^*\}$ where each $a_i^* : \mathbb{R}^2 \to \mathbb{R}$ is defined as follows, $\forall\ x \in V$,

$$a_i^*(x) = (a_i, x) = {}^{A}a_i \odot {}^{A}x = {}^{A}x[i] \text{ where } 1 \leq i \leq 2$$

Let $B = \{b_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^*, b_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^*\}$. Let $B^* = \{b_1^*, b_2^*\}$ where each $b_i^* : \mathbb{R}^2 \to \mathbb{R}$

is defined as follows, $\forall\ x \in V$,

$$b_i^*(x) = (b_i, x) = {}^{B}b_i \odot {}^{B}x = {}^{B}x[i] \text{ where } 1 \leq i \leq 2$$

It is easy to see that both A and B are orthonormal bases of $\mathbb{R}^2$. From Theorem 2.1.2, we get that A^* and B^* are bases of $\mathcal{L}(V \to \mathbb{R})$.

Computing the basis transformation matrix from A^* to B^* :

$$a_1 = \frac{1}{\sqrt{2}}b_1 - \frac{1}{\sqrt{2}}b_2 \qquad a_2 = \frac{1}{\sqrt{2}}b_1 + \frac{1}{\sqrt{2}}b_2$$

$$\begin{bmatrix} a_1 & a_2 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \implies M = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

We get that M is the transformation matrix from basis A to B of $\mathbb{R}^2$ which implies that M^* is the transformation matrix from basis B to A i.e,

$$\begin{bmatrix} b_1 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\implies b_1 = \frac{1}{\sqrt{2}}a_1 + \frac{1}{\sqrt{2}}a_2 \qquad\qquad b_2 = -\frac{1}{\sqrt{2}}a_1 + \frac{1}{\sqrt{2}}a_2$$

$\forall\ u \in \mathcal{L}(\mathbb{R}^2 \to \mathbb{R})$, since A^* and B^* form bases for $\mathcal{L}(\mathbb{R}^2 \to \mathbb{R})$ we get,

$$u = u(a_1)a_1^* + u(a_2)a_2^* = u(b_1)b_1^* + u(b_2)b_2^*$$

Since u is linear we get that,

$$u(b_1) = \frac{1}{\sqrt{2}}u(a_1) + \frac{1}{\sqrt{2}}u(a_2) \qquad u(b_2) = -\frac{1}{\sqrt{2}}u(a_1) + \frac{1}{\sqrt{2}}u(a_2)$$

$$\implies \begin{bmatrix} u(b_1) \\ u(b_2) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} u(a_1) \\ u(a_2) \end{bmatrix} \implies {}^{B^*}u = M \cdot {}^{A^*}u$$

2.1.5 Invariance of computation of $1-$tensors under any orthonormal basis transformations

Theorem 2.1.5. Let V be a finite dimensional inner product space over field $\mathbb{R}$ where $dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ be any orthonormal basis of V. Let $A^* = \{a_1^*, a_2^*, ..., a_n^*\}$. From Theorem 2.1.2, we get that A^* forms a basis of $\mathcal{L}(V \to \mathbb{R})$. Then, $\forall\ u \in \mathcal{L}(V \to \mathbb{R})$, $\forall\ x \in V$,

$$\boxed{u(x) = \sum_{r=1}^{n} {}^{A^*}u[r] \cdot {}^{A}x[r] = {}^{A^*}u \odot {}^{A}x}$$

Proof. $\forall\ x \in V$, since A is a basis of V, there exist unique $\alpha_r \in \mathbb{R}$, such that

$$x = \sum_{r=1}^{n} \alpha_r a_r$$

$\forall\ u \in \mathcal{L}(V \to \mathbb{R})$ since u is linear,

$$u(x) = u(\sum_{r=1}^{n} \alpha_r a_r) = \sum_{r=1}^{n} \alpha_r u(a_r)$$

Since A^* is a basis of $\mathcal{L}(V \to \mathbb{R})$,

$$u(a_r) = {}^{A^*}u[r]$$

Since A is an orthonormal basis of V,

$$\alpha_r = {}^{A}x[r]$$

$$\implies u(x) = \sum_{r=1}^{n} {}^{A^*}u[r] \cdot {}^{A}x[r] = {}^{A^*}u \odot {}^{A}x$$

□

Remark :

1. Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$ be any two orthonormal bases of V. Let $A^* = \{a_1^*, a_2^*, ..., a_n^*\}$ and $B^* = \{b_1^*, b_2^*, ..., b_n^*\}$. From Theorem

2.1.2, we get that both A^* and B^* form bases for $\mathcal{L}(V \to \mathbb{R})$. Then, $\forall$ $u \in \mathcal{L}(V \to \mathbb{R})$, $\forall$ $x \in V$,

$$u(x) = {}^{A^*}u \odot {}^{A}x = {}^{B^*}u \odot {}^{B}x$$

2. It is easy to observe that $\forall$ $x \in V$, $u(x)$ can be determined by the action of $({}^{A^*}u)^*$ on ${}^{A}x$. Hence, if we fix computations with respect to an orthonormal basis A we can identify u with $({}^{A^*}u)^*$

3. Let $dim(V) = n$ then, $\forall$ $u \in \mathcal{L}(V \to \mathbb{R})$, $({}^{A^*}u)^* = \begin{bmatrix} u(a_1) & u(a_2) & . & . & u(a_n) \end{bmatrix}^*$ is a $1 \times n$ row vector in $\mathbb{R}^n$. Hence, it is easy to show that the $1-$fold tensor product space, $\mathcal{L}(V \to \mathbb{R})$ is isomorphic to $\mathbb{R}^{1\times n} \cong \mathbb{R}^n$. (It is straightforward to verify and is left to the reader. For the proof technique you may refer Lemma 1.3.9)

Illustration :

Consider $V = \mathbb{R}^2$ over $\mathbb{R}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^*, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^*\}$. Let $A^* = \{a_1^*, a_2^*\}$ where each $a_i^* : \mathbb{R}^2 \to \mathbb{R}$ is defined as follows, $\forall$ $x \in V$,

$$a_i^*(x) = (a_i, x) = {}^{A}a_i \odot {}^{A}x = {}^{A}x[i] \text{ where } 1 \leq i \leq 2$$

Let $B = \{b_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^*, b_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^*\}$. Let $B^* = \{b_1^*, b_2^*\}$ where each $b_i^* : \mathbb{R}^2 \to \mathbb{R}$ is defined as follows, $\forall$ $x \in V$,

$$b_i^*(x) = (b_i, x) = {}^{B}b_i \odot {}^{B}x = {}^{B}x[i] \text{ where } 1 \leq i \leq 2$$

It is easy to see that both A and B form orthonormal bases of $\mathbb{R}^2$. From Theorem 2.1.2, we get A^* and B^* form bases of $\mathcal{L}(V \to \mathbb{R})$. In the previous illustration, we have already shown that the basis transformation matrix M from A to B is,

$$M = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

We have also seen that, $\forall\ x \in \mathbb{R}^2$,

$$^{B}x = M \cdot {}^{A}x$$

$\forall\ u \in \mathcal{L}(\mathbb{R}^2 \to \mathbb{R})$,

$$^{B^*}u = M \cdot {}^{A^*}u$$

Since B is an orthonormal bases of $\mathbb{R}^2$, we get

$$x = {}^{B}x[1]b_1 + {}^{B}x[2]b_2$$

Since u is linear, we get

$$\begin{aligned} u(x) &= u({}^{B}x[1]b_1 + {}^{B}x[2]b_2) = {}^{B}x[1]u(b_1) + {}^{B}x[2]u(b_2) \\ &= \begin{bmatrix} u(b_1) & u(b_2) \end{bmatrix} \begin{bmatrix} {}^{B}x[1] \\ {}^{B}x[2] \end{bmatrix} = {}^{B^*}u \odot {}^{B}x \end{aligned}$$

$$\implies u(x) = (M \cdot {}^{A^*}u)^* \cdot (M \cdot {}^{A}x) = ({}^{A^*}u)^* \cdot M^* \cdot M \cdot {}^{A}x = ({}^{A^*}u)^* \cdot {}^{A}x$$

$$\implies u(x) = {}^{B^*}u \odot {}^{B}x = {}^{A^*}u \odot {}^{A}x$$

2.2 2-Fold Tensor Product Spaces

2.2.1 Bi-linear Functions

Definition 2.2.1. Let V, W be any two vector spaces over field $\mathbb{R}$ with inner products $()_1 : V \times V \to \mathbb{R}$ and $()_2 : W \times W \to \mathbb{R}$ defined on V and W respectively. Note that subscripts $()_1$ and $()_2$ will be dropped if the context is clear. A function $u : V \times W \to \mathbb{R}$ is called bi-linear if the following holds,

1. $\forall\ x, y \in V$, $\forall\ z \in W$,

$$u(x + y, z) = u(x, z) + u(y, z)$$

2. $\forall\ x \in V$, $\forall\ y, z \in W$,

$$u(x, y + z) = u(x, y) + u(x, z)$$

3. $\forall\ x \in V$, $\forall\ y \in W$, $\forall\ \alpha \in \mathbb{R}$,

$$u(\alpha x, y) = \alpha u(x, y) = u(x, \alpha y)$$

Let set $S = \{u : V \times W \to \mathbb{R} \mid u \text{ is bi-linear}\}$. Define addition and multiplication on the set S as follows, $\forall\ u, v \in S$, $\forall\ x \in V$, $\forall\ y \in W$, $\forall\ \alpha \in \mathbb{R}$,

$$[u + v](x, y) = u(x, y) + v(x, y)$$
$$[\alpha u](x, y) = \alpha u(x, y)$$

Recall that we use [.] to indicate the addition and scalar multiplication of elements of the set S.

Lemma 2.2.1. *S is closed under addition and scalar multiplication*

Proof. **Claim 1 :** $\forall\ u, v \in S$, $[u + v] \in S$,

1. $\forall\ x, y \in V$, $\forall\ z \in W$,

$$[u + v](x + y, z) = u(x + y, z) + v(x + y, z)$$

$$= u(x,z) + u(y,z) + v(x,z) + v(y,z)$$
$$= [u+v](x,z) + [u+v](y,z)$$

2. $\forall\ x \in V$, $\forall\ y, z \in W$,

$$\begin{aligned} [u+v](x, y+z) &= u(x, y+z) + v(x, y+z) \\ &= u(x,y) + u(x,z) + v(x,y) + v(x,z) \\ &= [u+v](x,y) + [u+v](x,z) \end{aligned}$$

3. $\forall\ x \in V$, $\forall\ y \in W$, $\forall\ \alpha \in \mathbb{R}$,

$$\begin{aligned} [u+v](\alpha x, y) &= u(\alpha x, y) + v(\alpha x, y) = \alpha u(x,y) + \alpha v(x,y) = \alpha[u+v](x,y) \\ &= u(x, \alpha y) + v(x, \alpha y) = [u+v](x, \alpha y) \end{aligned}$$

$$[u+v] \text{ is bi-linear } \implies [u+v] \in S \implies S \text{ is closed under addition}$$

Claim 2 : $\forall\ u \in S$, $\alpha \in \mathbb{R}$, $[\alpha u] \in S$,

1. $\forall\ x, y \in V$, $\forall\ z \in W$,

$$[\alpha u](x+y, z) = \alpha u(x+y, z) = \alpha u(x,z) + \alpha u(y,z) = [\alpha u](x,z) + [\alpha u](y,z)$$

2. $\forall\ x \in V$, $\forall\ y, z \in W$,

$$[\alpha u](x, y+z) = \alpha u(x, y+z) = \alpha u(x,y) + \alpha u(x,z) = [\alpha u](x,y) + [\alpha u](x,z)$$

3. $\forall\ x \in V$, $\forall\ y \in W$, $\forall\ \beta \in \mathbb{R}$,

$$\begin{aligned} [\alpha u](\beta x, y) &= \alpha u(\beta x, y) = \alpha\beta u(x,y) = \beta[\alpha u](x,y) \\ &= \alpha u(x, \beta y) = [\alpha u](x, \beta y) \end{aligned}$$

$$[\alpha u] \text{ is bi-linear } \implies [\alpha u] \in S \implies S \text{ is closed under scalar multiplication}$$

$\square$

A bi-linear function $u \in S$ is called a **2-tensor** or a bi-linear map on $V \times W$. It is easy to verify that S is a vector space over field $\mathbb{R}$ (We already proved that S is closed under addition and scalar multiplication and rest of the axioms of vector space are easy to verify and are left to the reader). The vector space of all 2-tensors is defined as the tensor product space of V and W denoted by $\mathcal{L}(V \times W \to \mathbb{R})$.

2.2.2 Tensor Products on vector spaces V and W

Definition 2.2.2. Let V, W be any two finite dimensional inner product spaces over field $\mathbb{R}$ where $dim(V) = n$ and $dim(W) = m$. Then, $\forall\ u \in V$, $\forall\ v \in W$, we define the tensor product of u and v as a function $[u \otimes v] : V \times W \to \mathbb{R}$ as follows, $\forall\ x \in V$, $\forall\ y \in W$,

$$\boxed{[u \otimes v](x, y) = (u, x)(v, y)}$$

Remark :

1. $\forall\ u \in V$, $\forall\ v \in W$, notice that $u \otimes v \neq v \otimes u$ in general.

Lemma 2.2.2. Let V, W be any two finite dimensional inner product spaces over field $\mathbb{R}$. Then, $\forall\ u \in V$, $\forall\ v \in W$, $[u \otimes v]$ is bi-linear.

Proof. 1. $\forall\ x, y \in V$, $\forall\ z \in W$,

$$\begin{aligned}[u \otimes v](x + y, z) &= (u, x + y)(v, z) = (u, x)(v, z) + (u, y)(v, z) \\ &= [u \otimes v](x, z) + [u \otimes v](y, z)\end{aligned}$$

2. $\forall\ x \in V$, $\forall\ y, z \in W$,

$$\begin{aligned}[u \otimes v](x, y + z) &= (u, x)(v, y + z) = (u, x)(v, y) + (u, x)(v, z) \\ &= [u \otimes v](x, y) + [u \otimes v](x, z)\end{aligned}$$

3. $\forall\ x \in V$, $\forall\ y \in W$, $\forall\ \alpha \in \mathbb{R}$,

$$\begin{aligned}[u \otimes v](\alpha x, y) &= (u, \alpha x)(v, y) = (u, x)(v, \alpha y) = [u \otimes v](x, \alpha y) \\ &= \alpha(u, x)(v, y) = \alpha[u \otimes v](x, y)\end{aligned}$$

$$[u \otimes v] \text{ is bi-linear} \implies [u \otimes v] \in \mathcal{L}(V \times W \to \mathbb{R})$$

$\square$

Lemma 2.2.3. Let V, W be any two vector spaces over field $\mathbb{R}$. Then,

1. $\forall\ u, v \in V$, $\forall\ w \in W$,

$$[u + v] \otimes w = u \otimes w + v \otimes w$$

2. $\forall\ u \in V$, $\forall\ v, w \in W$,

$$u \otimes [v + w] = u \otimes v + u \otimes w$$

3. $\forall\ u \in V$, $\forall\ v \in W$, $\forall\ \alpha \in \mathbb{R}$,

$$[\alpha u] \otimes v = u \otimes [\alpha v] = \alpha[u \otimes v]$$

Proof. $\forall\ x \in V$, $\forall\ y \in W$,

1. $\forall\ u, v \in V$, $\forall\ w \in W$,

$$\begin{aligned}[][u + v] \otimes w](x, y) &= (u + v, x)(w, y) = (u, x)(w, y) + (v, x)(w, y) \\ &= [u \otimes w](x, y) + [v \otimes w](x, y)\end{aligned}$$

2. $\forall\ u \in V$, $\forall\ v, w \in W$,

$$\begin{aligned}[u \otimes [v + w]](x, y) &= (u, x)(v + w, y) = (u, x)(v, y) + (u, x)(w, y) \\ &= [u \otimes v](x, y) + [u \otimes w](x, y)\end{aligned}$$

3. $\forall\ u \in V$, $\forall\ v \in W$, $\forall\ \alpha \in \mathbb{R}$,

$$\begin{aligned}[][\alpha u] \otimes v](x, y) &= (\alpha u, x)(v, y) = \alpha(u, x)(v, y) = \alpha[u \otimes v](x, y) \\ &= (u, x)(\alpha v, y) = [u \otimes [\alpha v]](x, y)\end{aligned}$$

$\square$

Illustration :
Consider $V = \mathbb{R}^2$ over $\mathbb{R}$ and $W = \mathbb{R}^3$ over $\mathbb{R}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^*, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^*\}$ and $B = \{b_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}^*, b_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}^*, b_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}^*\}$. It is easy to see that A and B form orthonormal basis of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively.

Let $u = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

$\forall\ x \in \mathbb{R}^2$, $\forall\ y \in \mathbb{R}^3$,

$$[u \otimes v](x, y) = (u, x)(v, y) = ({}^{A}u \odot {}^{A}x) \cdot ({}^{B}v \odot {}^{B}y) = ({}^{A}x \odot {}^{A}u) \cdot ({}^{B}v \odot {}^{B}y)$$

$$\implies [u \otimes v](x, y) = \begin{bmatrix} {}^{A}x[1] & {}^{A}x[2] \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} {}^{B}y[1] \\ {}^{B}y[2] \\ {}^{B}y[3] \end{bmatrix}$$

$$\implies [u \otimes v](x, y) = \begin{bmatrix} {}^{A}x[1] & {}^{A}x[2] \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 6 & 4 & 2 \end{bmatrix} \begin{bmatrix} {}^{B}y[1] \\ {}^{B}y[2] \\ {}^{B}y[3] \end{bmatrix}$$

From the linearity of inner product (dot product) in both the first and second arguments, we can easily conclude that $[u \otimes v]$ is bi-linear and all the properties in Lemma 2.2.3 hold. Note that computing ${}^{A}u \cdot ({}^{B}v)^*$ is sufficient to determine the action of $[u \otimes v]$ on any $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^3$.

2.2.3 Basis of 2−fold tensor product spaces

Let V, W be any two inner product spaces over field $\mathbb{R}$ where $dim(V) = n$ and $dim(W) = m$. Let $A = \{a_1, a_2, ..., a_n\}$ be an orthonormal basis of V and $B = \{b_1, b_2, ..., b_m\}$ be an orthonormal basis of W. Define $A \otimes B = \{a_i \otimes b_j \mid 1 \le i \le n, 1 \le j \le m\}$. From Lemma 2.2.2 we get that, $\forall\ i \in \{1, 2, ..., n\}$, $\forall\ j \in \{1, 2, ..., m\}$, $a_i \otimes b_j$ is bi-linear.

Theorem 2.2.4. *$A \otimes B$ is a basis for vectorspace $\mathcal{L}(V \times W \to \mathbb{R})$*

Proof. **Span :**
$\forall\ x \in V$, since A is a basis of V, there exist unique $\alpha_i \in \mathbb{R}$ such that

$$x = \sum_{i=1}^{n} \alpha_i a_i$$

$\forall\ y \in W$, since B is a basis of W, there exist unique $\beta_j \in \mathbb{R}$ such that

$$y = \sum_{j=1}^{m} \beta_j b_j$$

$\forall\ u \in \mathcal{L}(V \times W \to \mathbb{R})$, since u is bi-linear, we get

$$u(x, y) = u(\sum_{i=1}^{n} \alpha_i a_i, \sum_{j=1}^{m} \beta_j b_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j u(a_i, b_j)$$

$\forall\ i \in \{1, 2, ..., n\}$, $\forall\ j \in \{1, 2, ..., m\}$, since A and B are orthonormal bases, we get

$$[a_i \otimes b_j](x, y) = (a_i, x)(b_j, y) = {}^{A}x[i] \cdot {}^{B}y[j] = \alpha_i \beta_j$$

$$\implies u(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{m} u(a_i, b_j)[a_i \otimes b_j](x, y)$$

$$\implies u = \sum_{i=1}^{n} \sum_{j=1}^{m} u(a_i, b_j)[a_i \otimes b_j]$$

$$\implies A \otimes B \text{ spans } \mathcal{L}(V \times W \to \mathbb{R})$$

Linear Independence :
Let $\alpha_{ij} \in \mathbb{R}$, $\forall\ i \in \{1, 2, ..., n\}$, $\forall\ j \in \{1, 2, ..., m\}$. Consider,

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{ij} [a_i \otimes b_j] = 0$$

$\forall\ p \in \{1, 2, ..., n\}$, $\forall\ q \in \{1, 2, ..., m\}$,

$$\sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_{ij}[a_i \otimes b_j](a_p, b_q) = 0$$

$$\implies \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_{ij}(a_i, a_p)(b_j, b_q) = 0$$

Since A and B are orthonormal bases, we get

$$\sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_{ij}(a_i, a_p)(b_j, b_q) = \alpha_{pq} = 0$$

$$\implies A \otimes B \text{ is a linearly independent set and a basis of } \mathcal{L}(V \times W \to \mathbb{R})$$

$\square$

Corollary 2.2.5.

$$\boxed{dim(\mathcal{L}(V \times W \to \mathbb{R})) = n \cdot m = dim(V) \cdot dim(W)}$$

Illustration :
Consider $V = \mathbb{R}^2$ over $\mathbb{R}$ and $W = \mathbb{R}^3$ over $\mathbb{R}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^*, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^*\}$ and $B = \{b_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}^*, b_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}^*, b_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}^*\}$. It is easy to see that A and B form orthonormal basis of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. $\forall\ i \in \{1, 2\}$, $\forall\ j \in \{1, 2, 3\}$, $\forall\ x \in \mathbb{R}^2$, $\forall\ y \in \mathbb{R}^3$,

$$\begin{aligned} [a_i \otimes b_j](x, y) &= (a_i, x)(b_j, y) \\ &= ({}^{A}a_i \odot {}^{A}x) \cdot ({}^{B}b_j \odot {}^{B}y) \\ &= {}^{A}x[i] \cdot {}^{B}y[j] \end{aligned}$$

Let $A \otimes B = \{a_i \otimes b_j \mid 1 \leq i \leq 2, 1 \leq j \leq 3\}$. From the above Theorem, we get that $A \otimes B$ is a basis of $\mathcal{L}(\mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R})$. Consider the following linear function,

$$u(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}) = (x_1 + 2x_2) \cdot (3y_1 + y_3) \qquad \text{where } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{R}^3$$

It is straight forward to verify that u is bi-linear. Since A and B form bases of V and W respectively we get that,

$\forall \begin{bmatrix} x_1 & x_2 \end{bmatrix}^* \in \mathbb{R}^2$ there exist unique α_1, $\alpha_2 \in \mathbb{R}$, such that

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix}^* = \alpha_1 a_1 + \alpha_2 a_2$$

$\forall \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^* \in \mathbb{R}^3$ there exist unique β_1, β_2, $\beta_3 \in \mathbb{R}$, such that

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^* = \beta_1 b_1 + \beta_2 b_2 + \beta_3 b_3$$

Since u is bi-linear we get that

$$u(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}) = u(\sum_{i=1}^{2} \alpha_i a_i, \sum_{j=1}^{3} \beta_j b_j) = \sum_{i=1}^{2} \sum_{j=1}^{3} \alpha_i \beta_j u(a_i, b_j)$$

From the above expression, it is quite clear that computing $u(a_i, b_j)$ is sufficient to determine the action of u on any $\begin{bmatrix} x_1 & x_2 \end{bmatrix}^*, \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^*$.

$$u(a_1, b_1) = 2\sqrt{6} \qquad u(a_1, b_2) = \frac{9}{2} \qquad u(a_1, b_3) = \frac{\sqrt{3}}{2}$$

$$u(a_2, b_1) = -\frac{2\sqrt{2}}{3} \qquad u(a_2, b_2) = -\frac{3}{2} \qquad u(a_2, b_3) = -\frac{1}{2\sqrt{3}}$$

$$\implies u(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}) = \sum_{i=1}^{2} \sum_{j=1}^{3} u(a_i, b_j)[a_i \otimes b_j](x, y)$$

With this illustration, you can observe how $A \otimes B$ works as a basis of $\mathcal{L}(\mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R})$ and also note that the values of $u(a_i, b_j)$ is sufficient to compute $u(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix})$ for any $\begin{bmatrix} x_1 & x_2 \end{bmatrix}^* \in \mathbb{R}^2$ and $\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^* \in \mathbb{R}^3$.

2.2.4 Basis Transformation

Definition 2.2.3. Let V, W be any two inner product spaces over field $\mathbb{R}$ where $dim(V) = n$ and $dim(W) = m$. Let $A = \{a_1, a_2, ..., a_n\}$ be an orthonormal basis of V and $B = \{b_1, b_2, ..., b_m\}$ be an orthonormal basis of W. Let $A \otimes B = \{a_i \otimes b_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$. From Theorem 2.2.4, we get that $A \otimes B$ forms a basis of $\mathcal{L}(V \times W \to \mathbb{R})$. Hence, $\forall\ u \in \mathcal{L}(V \times W \to \mathbb{R})$, we get that

$$u = \sum_{i=1}^{n} \sum_{j=1}^{m} u(a_i, b_j)[a_i \otimes b_j]$$

We define coordinates of the bi-linear function u as follows,

$$_{A \otimes B}u = \begin{bmatrix} u(a_1, b_1) & u(a_1, b_2) & . & . & . & u(a_1, b_m) \\ u(a_2, b_1) & u(a_2, b_2) & . & . & . & u(a_2, b_m) \\ . & . & . & . & . & . \\ u(a_n, b_1) & u(a_n, b_2) & . & . & . & u(a_n, b_m) \end{bmatrix}$$

Note :

1. Note that we used general column vector representation for $1-$tensors, but we use matrix representation for $2-$tensors.

Theorem 2.2.6. Let V, W be two inner product spaces over field $\mathbb{R}$ where $dim(V) = n$ and $dim(W) = m$. Let $A = \{a_1, ..., a_n\}$ and $C = \{c_1, ..., c_n\}$ be any two orthonormal basis of V. Let $B = \{b_1, ..., b_m\}$ and $D = \{d_1, ..., d_m\}$ be any two orthonormal basis of W. Let $A \otimes B = \{a_i \otimes b_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ and $C \otimes D = \{c_i \otimes d_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$. From Theorem 2.2.4, we get that $A \otimes B$ and $C \otimes D$ form bases of $\mathcal{L}(V \times W \to \mathbb{R})$. Let $M \in \mathbb{R}^{n \times n}$ be the transformation matrix from A to C i.e,

$$\begin{bmatrix} a_1 & a_2 & . & . & . & a_n \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & . & . & . & c_n \end{bmatrix} M$$

Let $N \in \mathbb{R}^{m\times m}$ be the transformation matrix from B to D i.e,

$$\begin{bmatrix} b_1 & b_2 & . & . & . & b_m \end{bmatrix} = \begin{bmatrix} d_1 & d_2 & . & . & . & d_m \end{bmatrix} N$$

Then, $\forall\ u \in \mathcal{L}(V \times W \to \mathbb{R})$,

$$\boxed{{}^{C\otimes D}u = M \cdot {}^{A\otimes B}u \cdot N^*}$$

Proof. $\forall\ u \in \mathcal{L}(V \times W \to \mathbb{R})$, since $A \otimes B$, $C \otimes D$ form bases of $\mathcal{L}(V \times W \to \mathbb{R})$ we get,

$$u = \sum_{i=1}^{n}\sum_{j=1}^{m} u(a_i, b_j)[a_i \otimes b_j] = \sum_{p=1}^{n}\sum_{q=1}^{m} u(c_p, d_q)[c_p \otimes d_q]$$

$$\implies {}^{A\otimes B}u = \begin{bmatrix} u(a_1,b_1) & . & . & u(a_1,b_m) \\ . & . & . & . \\ u(a_n,b_1) & . & . & u(a_n,b_m) \end{bmatrix} \qquad {}^{C\otimes D}u = \begin{bmatrix} u(c_1,d_1) & . & . & u(c_1,d_m) \\ . & . & . & . \\ u(c_n,d_1) & . & . & u(c_n,d_m) \end{bmatrix}$$

Since M is the transformation matrix from A to C, we get that M^* is the transformation matrix from C to A i.e,

$$\begin{bmatrix} c_1 & c_2 & . & . & . & c_n \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & . & . & . & a_n \end{bmatrix} M^*$$

$\forall\ p \in \{1, 2, ..., n\}$,

$$c_p = \sum_{i=1}^{n} M^*_{ip} a_i = \sum_{i=1}^{n} M_{pi} a_i$$

Since N is the transformation matrix from B to D, we get that N^* is the transformation matrix from D to B i.e,

$$\begin{bmatrix} d_1 & d_2 & . & . & . & d_m \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & . & . & . & b_m \end{bmatrix} N^*$$

$\forall\ q \in \{1, 2, ..., m\}$,

$$d_q = \sum_{j=1}^{m} N^*_{jq} b_j = \sum_{j=1}^{m} N_{qj} b_j$$

Since u is bi-linear, we get

$$u(c_p, d_q) = u(\sum_{i=1}^{n} M_{pi} a_i, \sum_{j=1}^{m} N_{qj} b_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} M_{pi} N_{qj} u(a_i, b_j)$$

$$\implies u(c_p, d_q) = \begin{bmatrix} M_{p1} & . & . & M_{pn} \end{bmatrix} \begin{bmatrix} u(a_1, b_1) & . & . & u(a_1, b_m) \\ . & . & . & . \\ u(a_n, b_1) & . & . & u(a_n, b_m) \end{bmatrix} \begin{bmatrix} N_{q1} \\ . \\ . \\ N_{qm} \end{bmatrix}$$

$$\begin{bmatrix} u(c_1, d_1) & .. & u(c_1, d_m) \\ . & .. & . \\ u(c_n, d_1) & .. & u(c_n, d_m) \end{bmatrix} = \begin{bmatrix} M_{11} & .. & M_{1n} \\ . & .. & . \\ M_{n1} & .. & M_{nn} \end{bmatrix} \begin{bmatrix} u(a_1, b_1) & .. & u(a_1, b_m) \\ . & .. & . \\ u(a_n, b_1) & .. & u(a_n, b_m) \end{bmatrix} \begin{bmatrix} N_{11} & .. & N_{m1} \\ . & .. & . \\ N_{1m} & .. & N_{mm} \end{bmatrix}$$

$$\implies {}^{C \otimes D}u = M \cdot {}^{A \otimes B}u \cdot N^*$$

□

Illustration :

Consider $V = \mathbb{R}^2$ over $\mathbb{R}$ and $W = \mathbb{R}^3$ over $\mathbb{R}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^*, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^*\}$ and $B = \{b_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}^*, b_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}^*, b_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}^*\}$. It is easy to see that A and B form orthonormal basis of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. Let $C = \{c_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^*, c_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^*\}$, $D = \{d_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^*, d_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^*, d_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^*\}$. It is easy to see that C and D are standard orthonormal bases of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. $\forall\ i \in \{1, 2\}$, $\forall\ j \in \{1, 2, 3\}$, $\forall\ x \in \mathbb{R}^2$, $\forall\ y \in \mathbb{R}^3$,

$$[a_i \otimes b_j](x, y) = {}^{A}x[i] \cdot {}^{B}y[j]$$

Let $A \otimes B = \{a_i \otimes b_j \mid 1 \leq i \leq 2, 1 \leq j \leq 3\}$. Similarly, $\forall\ p \in \{1, 2\}$, $\forall$

$q \in \{1, 2, 3\}$, $\forall\, x \in \mathbb{R}^2$, $\forall\, y \in \mathbb{R}^3$,

$$[c_p \otimes d_q](x, y) = {}^{C}x[i] \cdot {}^{D}y[j]$$

Let $C \otimes D = \{c_p \otimes d_q \mid 1 \le p \le 2, 1 \le q \le 3\}$. From Theorem 2.2.4, we get that $A \otimes B$ and $C \otimes D$ form bases of $\mathcal{L}(\mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R})$.

Computing the basis transformation matrix from basis A to C :

$$a_1 = \frac{1}{\sqrt{2}}c_1 + \frac{1}{\sqrt{2}}c_2 \qquad a_2 = \frac{1}{\sqrt{2}}c_1 - \frac{1}{\sqrt{2}}c_2$$

$$\begin{bmatrix} a_1 & a_2 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \implies M = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

We get that M is the transformation matrix from basis A to B of $\mathbb{R}^2$ which implies that M^* is the transformation matrix from basis B to A i.e,

$$\begin{bmatrix} c_1 & c_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\implies c_1 = \frac{1}{\sqrt{2}}a_1 + \frac{1}{\sqrt{2}}a_2 \qquad\qquad c_2 = -\frac{1}{\sqrt{2}}a_1 + \frac{1}{\sqrt{2}}a_2$$

Computing the basis transformation matrix from basis B to D :

$$b_1 = \frac{1}{\sqrt{3}}d_1 + \frac{1}{\sqrt{3}}d_2 + \frac{1}{\sqrt{3}}d_3 \qquad b_2 = \frac{1}{\sqrt{2}}d_1 - \frac{1}{\sqrt{2}}d_2 \qquad b_1 = \frac{1}{\sqrt{6}}d_1 + \frac{1}{\sqrt{6}}d_2 - \frac{2}{\sqrt{6}}d_3$$

$$\begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} d_1 & d_2 & d_3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix} \implies N = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}$$

We get that N is the transformation matrix from basis C to D of $\mathbb{R}^3$ which implies that N^* is the transformation matrix from basis D to C i.e,

$$\begin{bmatrix} d_1 & d_2 & d_3 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}$$

$$\implies d_1 = \frac{1}{\sqrt{3}}b_1 + \frac{1}{\sqrt{2}}b_2 + \frac{1}{\sqrt{6}}b_3 \qquad d_2 = \frac{1}{\sqrt{3}}b_1 + -\frac{1}{\sqrt{2}}b_2 + \frac{1}{\sqrt{6}}b_3 \qquad d_3 = \frac{1}{\sqrt{3}}b_1 - \frac{2}{\sqrt{6}}b_3$$

$\forall\ u \in \mathcal{L}(\mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R})$, since u is bi-linear, $\forall\ p \in \{1,2\}$, $\forall\ q \in \{1,2,3\}$ we get,

$$\begin{aligned} u(c_p, d_q) &= u(\sum_{i=1}^{2} M^*_{ip} a_i, \sum_{j=1}^{3} N^*_{jq} b_i) \\ &= \sum_{i=1}^{2}\sum_{j=1}^{3} M_{pi} N_{qj} u(a_i, b_j) \\ &= \begin{bmatrix} M_{p1} & M_{p2} \end{bmatrix} \begin{bmatrix} u(a_1,b_1) & u(a_1,b_2) & u(a_1,b_3) \\ u(a_2,b_1) & u(a_2,b_2) & u(a_2,b_3) \end{bmatrix} \begin{bmatrix} N_{q1} \\ N_{q2} \\ N_{q3} \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} u(c_1,d_1) & u(c_1,d_2) & u(c_1,d_3) \\ u(c_2,d_1) & u(c_2,d_2) & u(c_2,d_3) \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} u(a_1,b_1) & u(a_1,b_2) & u(a_1,b_3) \\ u(a_2,b_1) & u(a_2,b_2) & u(a_2,b_3) \end{bmatrix} \begin{bmatrix} N_{11} & N_{21} & N_{31} \\ N_{12} & N_{22} & N_{32} \\ N_{13} & N_{23} & N_{33} \end{bmatrix}$$

$$\implies {}^{C\otimes D}u = M \cdot {}^{A\otimes B}u \cdot N^*$$

2.2.5 Invariance of computation of $2-$tensor under any orthonormal basis transformations

Theorem 2.2.7. Let V, W be any two inner product spaces over field $\mathbb{R}$ where $dim(V) = n$ and $dim(W) = m$. Let $A = \{a_1, ..., a_n\}$ be any orthonormal basis of V. Let $B = \{b_1, ..., b_m\}$ be any orthonormal basis of W. Let $A \otimes B = \{a_i \otimes b_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$. From Theorem 2.2.4, we get that $A \otimes B$ form basis of $\mathcal{L}(V \times W \to \mathbb{R})$. Then, $\forall\ u \in \mathcal{L}(V \times W \to \mathbb{R})$, $\forall\ x \in V$, $\forall\ y \in W$,

$$\boxed{u(x,y) = \sum_{r=1}^{n}\sum_{s=1}^{m} {}^{A}x[r] \cdot {}^{A\otimes B}u[r,s] \cdot {}^{B}y[s] = \left({}^{A}x\right)^* \cdot {}^{A\otimes B}u \cdot {}^{B}y}$$

Note that we use ${}^{A\otimes B}u[r,s]$ to denote $u(a_r, b_s)$.

Proof. $\forall\ x \in V$, since A is a basis of V, there exist unique $\alpha_r \in \mathbb{R}$, such that

$$x = \sum_{r=1}^{n} \alpha_r a_r$$

$\forall\ y \in W$, since B is a basis of W, there exist unique $\beta_s \in \mathbb{R}$, such that

$$y = \sum_{s=1}^{m} \beta_s b_s$$

$\forall\ u \in \mathcal{L}(V \times W \to \mathbb{R})$, we get

$$u(x,y) = u(\sum_{r=1}^{n} \alpha_r a_r, \sum_{s=1}^{m} \beta_s b_s) = \sum_{r=1}^{n} \sum_{s=1}^{m} \alpha_r \beta_s u(a_r, b_s)$$

since A and B are orthonormal bases of V and W, we get

$$\alpha_r = {}^{A}x[r] \qquad\qquad \beta_s = {}^{B}y[s]$$

since $A \otimes B$ forms basis of $\mathcal{L}(V \times W \to \mathbb{R})$, we get

$${}^{A\otimes B}u = \begin{bmatrix} u(a_1,b_1) & u(a_1,b_2) & . & . & . & u(a_1,b_m) \\ u(a_2,b_1) & u(a_2,b_2) & . & . & . & u(a_2,b_m) \\ . & . & . & . & . & . \\ u(a_n,b_1) & u(a_n,b_2) & . & . & . & u(a_n,b_m) \end{bmatrix}$$

$$\implies u(x,y) = \sum_{r=1}^{n} \sum_{s=1}^{m} {}^{A}x[r] \cdot {}^{A\otimes B}u[r,s] \cdot {}^{B}y[s]$$

□

Remark :

1. Let $A = \{a_1, ..., a_n\}$ and $C = \{c_1, ..., c_n\}$ be any two orthonormal basis of V. Let $B = \{b_1, ..., b_m\}$ and $D = \{d_1, ..., d_m\}$ be any two orthonormal basis of W. Let $A \otimes B = \{a_i \otimes b_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ and $C \otimes D = \{c_i \otimes d_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$. From Theorem 2.2.4, we get that $A \otimes B$ and $C \otimes D$ form bases of $\mathcal{L}(V \times W \to \mathbb{R})$. Then, $\forall\ u \in \mathcal{L}(V \times W \to \mathbb{R})$, $\forall\ x \in V$, $\forall$

$y \in W$, we get

$$u(x,y) = \left({}^{A}x\right)^{*} \cdot {}^{A\otimes B}u \cdot {}^{B}y = \left({}^{C}x\right)^{*} \cdot {}^{C\otimes D}u \cdot {}^{D}y$$

2. It is easy to observe that $\forall\ (x,y) \in V \times W$, $u(x,y)$ can be determined by the action of ${}^{A\otimes B}u$ on ${}^{A}x$ and ${}^{B}y$. Hence, if we fix the computations with respect to orthonormal bases A and B of V and W respectively we can identify u with ${}^{A\otimes B}u$.

3. Let $dim(V) = n$, $dim(W) = m$ then, $\forall\ u \in \mathcal{L}(V \times W \to \mathbb{R})$,

$$ {}^{A\otimes B}u = \begin{bmatrix} u(a_1,b_1) & u(a_1,b_2) & . & . & . & u(a_1,b_m) \\ u(a_2,b_1) & u(a_2,b_2) & . & . & . & u(a_2,b_m) \\ . & . & . & . & . & . \\ u(a_n,b_1) & u(a_n,b_2) & . & . & . & u(a_n,b_m) \end{bmatrix}$$

   Hence, it is easy to show that the $2-$fold tensor product space, $\mathcal{L}(V \times W \to \mathbb{R})$ is isomorphic to $\mathbb{R}^{n\times m}$. (It is straight-forward to verify and is left to the reader. For the proof technique you may refer Lemma 1.3.9)

Illustration :

Consider $V = \mathbb{R}^2$ over $\mathbb{R}$ and $W = \mathbb{R}^3$ over $\mathbb{R}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^*, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^*\}$ and $B = \{b_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}^*, b_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}^*, b_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}^*\}$. It is easy to see that A and B form orthonormal basis of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. Let $C = \{c_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^*, c_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^*\}$, $D = \{d_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^*, d_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^*, d_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^*\}$. It is easy to see that C and D are standard orthonormal bases of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. $\forall\ i \in \{1,2\}$, $\forall\ j \in \{1,2,3\}$, $\forall\ x \in \mathbb{R}^2$, $\forall\ y \in \mathbb{R}^3$,

$$[a_i \otimes b_j](x,y) = {}^{A}x[i] \cdot {}^{B}y[j]$$

Let $A \otimes B = \{a_i \otimes b_j \mid 1 \leq i \leq 2, 1 \leq j \leq 3\}$. Similarly, $\forall\ p \in \{1,2\}$, $\forall\ q \in \{1,2,3\}$, $\forall\ x \in \mathbb{R}^2$, $\forall\ y \in \mathbb{R}^3$,

$$[c_p \otimes d_q](x,y) = {}^{C}x[i] \cdot {}^{D}y[j]$$

Let $C \otimes D = \{c_p \otimes d_q \mid 1 \leq p \leq 2, 1 \leq q \leq 3\}$. From Theorem 2.2.4, we get that $A \otimes B$ and $C \otimes D$ form bases of $\mathcal{L}(\mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R})$. In the previous illustration, we have already shown that the basis transformation matrix M from A to C is,

$$M = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

We have also seen that $\forall\ x \in \mathbb{R}^2$,

$$^{C}x = M \cdot {}^{A}x$$

We also that the basis transformation matrix N from B to D is,

$$N = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}$$

We have also seen that $\forall\ y \in \mathbb{R}^3$,

$$^{D}y = N \cdot {}^{B}y$$

Since C and D are orthonomal bases, we get that

$$x = {}^{C}x[1]c_1 + {}^{C}x[2]c_2 \qquad y = {}^{D}y[1]d_1 + {}^{D}y[2]d_2 + {}^{D}y[3]d_3$$

$\forall\ u \in \mathcal{L}(\mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R})$, we have already shown in previous illustration that,

$$^{C\otimes D}u = M \cdot {}^{A\otimes B}u \cdot N^*$$

Since u is bi-linear, we get

$$u(x,y) = u(\sum_{i=1}^{2} {}^{C}x[i]c_i, \sum_{j=1}^{3} {}^{D}y[j]d_j) = \sum_{i=1}^{2}\sum_{j=1}^{3} {}^{C}x[i]\,{}^{D}y[j]u(c_i, d_j)$$

$$\implies u(x,y) = \begin{bmatrix} {}^{C}x[1] & {}^{C}x[2] \end{bmatrix} \begin{bmatrix} u(c_1,d_1) & u(c_1,d_2) & u(c_1,d_3) \\ u(c_2,d_1) & u(c_2,d_2) & u(c_2,d_3) \end{bmatrix} \begin{bmatrix} {}^{B}y[1] \\ {}^{B}y[2] \\ {}^{B}y[3] \end{bmatrix}$$

$$\begin{aligned} u(x,y) &= \left({}^{C}x\right)^* \cdot {}^{C\otimes D}u \cdot {}^{D}y = \left(M \cdot {}^{A}x\right)^* \cdot M \cdot {}^{A\otimes B}u \cdot N^* \cdot \left(N \cdot {}^{B}y\right) \\ &= \left({}^{A}x\right)^* \cdot M^* \cdot M \cdot {}^{A\otimes B}u \cdot N^* \cdot N \cdot {}^{B}y \end{aligned}$$

$$\implies u(x,y) = \left({}^{A}x\right)^* \cdot {}^{A\otimes B}u \cdot {}^{B}y = \left({}^{C}x\right)^* \cdot {}^{C\otimes D}u \cdot {}^{D}y \text{ (use } M^*M = N^*N = I\text{)}$$

2.3 3−Fold Tensor Product Spaces

2.3.1 Tri-linear Functions

Definition 2.3.1. Let U, V and W be three vector spaces over field $\mathbb{R}$ with inner products $()_1 : U \times U \to \mathbb{R}$, $()_2 : V \times V \to \mathbb{R}$ and $()_3 : W \times W \to \mathbb{R}$ defined on U, V and W respectively. Note that subscripts $()_1$, $()_2$ and $()_3$ will be dropped if the context is clear. A function $u : U \times V \times W \to \mathbb{R}$ is called tri-linear if the following holds,

1. $\forall\ x, y \in U$, $\forall\ z \in V$, $\forall\ w \in W$,

$$u(x + y, z, w) = u(x, z, w) + u(y, z, w)$$

2. $\forall\ x \in U$, $\forall\ y, z \in V$, $\forall\ w \in W$,

$$u(x, y + z, w) = u(x, y, w) + u(x, z, w)$$

3. $\forall\ x \in U$, $\forall\ y \in V$, $\forall\ z, w \in W$,

$$u(x, y, z + w) = u(x, y, z) + u(x, y, w)$$

4. $\forall\ x \in U$, $\forall\ y \in V$, $\forall\ z \in W$, $\forall\ \alpha \in \mathbb{F}$,

$$u(\alpha x, y, z) = u(x, \alpha y, z) = u(x, y, \alpha z) = \alpha u(x, y, z)$$

Let set $S = \{u : U \times V \times W \to \mathbb{R} \mid u \text{ is tri-linear}\}$. Define addition and multiplication on the set S as follows, $\forall\ u, v \in S$, $\forall\ x \in U$, $\forall\ y \in V$, $\forall\ z \in W$, $\forall$ $\alpha \in \mathbb{R}$,

$$[u + v](x, y, z) = u(x, y, z) + v(x, y, z)$$

$$[\alpha u](x, y, z) = \alpha u(x, y, z)$$

Recall that we use [.] to indicate the addition and scalar multiplication of elements of the set S.

Lemma 2.3.1. S is closed under addition and scalar multiplication

Proof. **Claim 1 :** $\forall\ u, v \in S$, $[u+v] \in S$,

1. $\forall\ x, y \in U$, $\forall\ z \in V$, $\forall\ w \in W$,

$$\begin{aligned}[u+v](x+y,z,w) &= u(x+y,z,w) + v(x+y,z,w) \\ &= u(x,z,w) + u(y,z,w) + v(x,z,w) + v(y,z,w) \\ &= [u+v](x,z,w) + [u+v](y,z,w)\end{aligned}$$

2. $\forall\ x \in U$, $\forall\ y, z \in V$, $\forall\ w \in W$,

$$\begin{aligned}[u+v](x,y+z,w) &= u(x,y+z,w) + v(x,y+z,w) \\ &= u(x,y,w) + u(x,z,w) + v(x,y,w) + v(x,z,w) \\ &= [u+v](x,y,w) + [u+v](x,z,w)\end{aligned}$$

3. $\forall\ x \in U$, $\forall\ y \in V$, $\forall\ z, w \in W$,

$$\begin{aligned}[u+v](x,y,z+w) &= u(x,y,z+w) + v(x,y,z+w) \\ &= u(x,y,z) + u(x,y,w) + v(x,y,z) + v(x,y,w) \\ &= [u+v](x,y,z) + [u+v](x,y,w)\end{aligned}$$

4. $\forall\ x \in U$, $\forall\ y \in V$, $\forall\ z \in W$, $\forall\ \alpha \in \mathbb{R}$,

$$\begin{aligned}[u+v](\alpha x,y,z) &= u(\alpha x,y,z) + v(\alpha x,y,z) \\ &= u(x,\alpha y,z) + v(x,\alpha y,z) = [u+v](x,\alpha y,z) \\ &= u(x,y,\alpha z) + v(x,y.\alpha z) = [u+v](x,y,\alpha z) \\ &= \alpha u(x,y,z) + \alpha v(x,y,z) = \alpha[u+v](x,y,z)\end{aligned}$$

$$[u+v] \text{ is tri-linear } \implies [u+v] \in S \implies S \text{ is closed under addition}$$

Claim 2 : $\forall\ u \in S\ \forall\ \alpha \in \mathbb{R}$, $[\alpha u] \in S$,

1. $\forall\ x, y \in U$, $\forall\ z \in V$, $\forall\ w \in W$,

$$[\alpha u](x+y,z,w) = \alpha u(x+y,z,w) = \alpha u(x,z,w) + \alpha u(y,z,w)$$

$$= [\alpha u](x, z, w) + [\alpha u](y, z, w)$$

2. $\forall\ x \in U$, $\forall\ y, z \in V$, $\forall\ w \in W$,

$$\begin{aligned}[\alpha u](x, y+z, w) &= \alpha u(x, y+z, w) = \alpha u(x, y, w) + \alpha u(x, z, w) \\ &= [\alpha u](x, y, w) + [\alpha u](x, z, w)\end{aligned}$$

3. $\forall\ x \in U$, $\forall\ y \in V$, $\forall\ z, w \in W$,

$$\begin{aligned}[\alpha u](x, y, z+w) &= \alpha u(x, y, z+w) = \alpha u(x, y, z) + \alpha u(x, y, w) \\ &= [\alpha u](x, y, z) + [\alpha u](x, y, w)\end{aligned}$$

4. $\forall\ x \in U$, $\forall\ y \in V$, $\forall\ z \in W$, $\forall\ \beta \in \mathbb{R}$,

$$\begin{aligned}[\alpha u](\beta x, y, z) &= \alpha u(\beta x, y, z) = \alpha u(x, \beta y, z) = [\alpha u](x, \beta y, z) \\ &= \alpha u(x, y, \beta z) = [\alpha u](x, y, \beta z) \\ &= \alpha\beta u(x, y, z) = \beta[\alpha u](x, y, z)\end{aligned}$$

$[\alpha u]$ is tri-linear $\implies [\alpha u] \in S \implies S$ is closed under scalar multiplication

$\square$

A tri-linear function $u \in S$ is called a $3-$**tensor** or a tri-linear map on $U \times V \times W$. It is easy to verify that S is a vector space over field $\mathbb{R}$ (We already proved that S is closed under addition and scalar multiplication and rest of the axioms of vector space are easy to verify and are left to the reader). The vector space of all $3-$tensors is defined as the tensor product space of U, V and W denoted by $\mathcal{L}(U \times V \times W \to \mathbb{R})$.

2.3.2 Tensor Products on vector spaces U, V and W

Definition 2.3.2. Let U, V, W be any three finite dimensional inner product spaces over field $\mathbb{R}$. Then, $\forall\ u \in U$, $\forall\ v \in V$, $\forall\ w \in W$, we define the tensor product of u, v and w as a function $[u \otimes v \otimes w] : U \times V \times W \to \mathbb{R}$ as follows, $\forall$

$x \in U$, $\forall\ y \in V$, $\forall\ z \in W$,

$$\boxed{[u \otimes v \otimes w](x, y, z) = (u, x)(v, y)(w, z)}$$

Remark :

1. $\forall\ u \in U$, $\forall\ v \in V$, $\forall\ w \in W$ notice that the tensor products $u \otimes v \otimes w$, $u \otimes w \otimes v$, $v \otimes u \otimes w$, $v \otimes w \otimes u$, $w \otimes u \otimes v$, $w \otimes v \otimes u$ are not equal (pairwise) in general.

Lemma 2.3.2. Let U, V, W be any three finite dimensional inner product spaces over field $\mathbb{R}$. Then, $\forall\ u \in U$, $v \in V$, $w \in W$, $[u \otimes v \otimes w]$ is tri-linear

Proof. 1. $\forall\ x, y \in U$, $\forall\ z \in V$, $\forall\ t \in W$,

$$\begin{aligned}[u \otimes v \otimes w](x + y, z, t) &= (u, x + y)(v, z)(w, t) = (u, x)(v, z)(w, t) + (u, y)(v, z)(w, t) \\ &= [u \otimes v \otimes w](x, z, t) + [u \otimes v \otimes w](y, z, t)\end{aligned}$$

2. $\forall\ x \in U$, $\forall\ y, z \in V$, $\forall\ t \in W$,

$$\begin{aligned}[u \otimes v \otimes w](x, y + z, t) &= (u, x)(v, y + z)(w, t) = (u, x)(v, y)(w, t) + (u, x)(v, z)(w, t) \\ &= [u \otimes v \otimes w](x, y, t) + [u \otimes v \otimes w](x, z, t)\end{aligned}$$

3. $\forall\ x \in U$, $\forall\ y \in V$, $\forall\ z, t \in W$,

$$\begin{aligned}[u \otimes v \otimes w](x, y, z + t) &= (u, x)(v, y)(w, z + t) = (u, x)(v, y)(w, z) + (u, x)(v, y)(w, t) \\ &= [u \otimes v \otimes w](x, y, z) + [u \otimes v \otimes w](x, y, t)\end{aligned}$$

4. $\forall\ x \in U$, $\forall\ y \in V$, $\forall\ z \in W$, $\forall\ \alpha \in \mathbb{R}$,

$$\begin{aligned}[u \otimes v \otimes w](\alpha x, y, z) &= (u, \alpha x)(v, y)(w, z) = (u, x)(v, \alpha y)(w, z) = [u \otimes v \otimes w](x, \alpha y, z) \\ &= (u, x)(v, y)(w, \alpha z) = [u \otimes v \otimes w](x, y, \alpha z) \\ &= \alpha(u, x)(v, y)(w, z) = \alpha[u \otimes v \otimes w](x, y, z)\end{aligned}$$

$$\implies [u \otimes v] \otimes w] \text{ is tri-linear}$$

$\square$

Lemma 2.3.3. Let U, V, W be any three finite dimensional inner product spaces over field $\mathbb{R}$. Then,

1. $\forall\ u, v \in U$, $\forall\ w \in V$, $t \in W$,

$$[u+v] \otimes w \otimes t = u \otimes w \otimes t + v \otimes w \otimes t$$

2. $\forall\ u \in U$, $\forall\ v, w \in V$, $t \in W$,

$$u \otimes [v+w] \otimes t = u \otimes v \otimes t + u \otimes w \otimes t$$

3. $\forall\ u \in U$, $\forall\ v \in V$, $w, t \in W$,

$$u \otimes v \otimes [w+t] = u \otimes v \otimes w + u \otimes v \otimes t$$

4. $\forall\ u \in U$, $\forall\ v \in V$, $\forall\ w \in W$, $\forall\ \alpha \in \mathbb{R}$,

$$[\alpha u] \otimes v \otimes w = u \otimes [\alpha v] \otimes w = u \otimes v \otimes [\alpha w] = \alpha[u \otimes v \otimes w]$$

Proof. 1. $\forall\ x \in U$, $\forall\ y \in V$, $\forall\ z \in W$,

$$\begin{aligned}[][u+v] \otimes w \otimes t](x,y,z) &= (u+v,x)(w,y)(t,z) = (u,x)(w,y)(t,z) + (v,x)(w,y)(t,z) \\ &= [u \otimes w \otimes t](x,y,z) + [v \otimes w \otimes t](x,y,z)\end{aligned}$$

2. $\forall\ x \in U$, $\forall\ y \in V$, $\forall\ z \in W$,

$$\begin{aligned}[u \otimes [v+w] \otimes t](x,y,z) &= (u,x)(v+w,y)(t,z) = (u,x)(v,y)(t,z) + (u,x)(w,y)(t,z) \\ &= [u \otimes v \otimes t](x,y,z) + (u \otimes w \otimes t)(x,y,z)\end{aligned}$$

3. $\forall\ x \in U$, $\forall\ y \in V$, $\forall\ z \in W$,

$$\begin{aligned}[u \otimes v \otimes [w+t]](x,y,z) &= (u,x)(v,y)(w+t,z) = (u,x)(v,y)(w,z) + (u,x)(v,y)(t,z) \\ &= [u \otimes v \otimes w](x,y,z) + [u \otimes v \otimes t](x,y,z)\end{aligned}$$

4. $\forall\, x \in U$, $\forall\, y \in V$, $\forall\, z \in W$,

$$\begin{aligned}
[[\alpha u] \otimes v \otimes w](x,y,z) &= (\alpha u, x)(v,y)(w,z) \\
&= (u,x)(\alpha v, y)(w,z) = [u \otimes [\alpha v] \otimes w](x,y,z) \\
&= (u,x)(v,y)(\alpha w, z) = [u \otimes v \otimes [\alpha w]](x,y,z) \\
&= \alpha (u,x)(v,y)(w,z) = \alpha [u \otimes v \otimes w](x,y,z)
\end{aligned}$$

$\square$

Illustration :

Consider $U = \mathbb{R}^2$ over $\mathbb{R}$, $V = \mathbb{R}^3$ over $\mathbb{R}$ and $W = \mathbb{R}^4$ over $\mathbb{R}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^*, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^*\}$, $B = \{b_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}^*, b_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}^*, b_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}^*\}$ and $C = \{c_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}^*, c_2 = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}^*, c_3 = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}^*, c_4 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}^*\}$. It is easy to verify that A, B and C form orthonormal basis of $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^4$ respectively. Let $u = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$,

w=$\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 4 \\ 1 \end{bmatrix}$.

$\forall\, x \in \mathbb{R}^2$, $\forall\, y \in \mathbb{R}^3$, $\forall\, z \in \mathbb{R}^4$,

$$[u \otimes v \otimes w](x,y,z) = (u,x)(v,y)(w,z) = ({}^A u \odot {}^A x) \cdot ({}^B v \odot {}^B y) \cdot ({}^C w \odot {}^C z)$$

$$= \begin{bmatrix} {}^A x[1] & {}^A x[2] \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} {}^B y[1] & {}^B y[2] & {}^B y[3] \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} {}^C z[1] & {}^C z[2] & {}^C z[3] & {}^C z[4] \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

$$= ({}^A x[1] + 2\,{}^A x[2]) \cdot (3\,{}^B y[1] + 2\,{}^B y[2] + {}^B y[3]) \cdot ({}^C z[1] + 4\,{}^C z[3] + {}^C z[4])$$

From the linearity of inner product (dot product) in both the first and second arguments, we can easily conclude that $[u \otimes v \otimes w]$ is tri-linear and all the properties in Lemma 2.3.3 hold. Recall that we had an analytic expression to identify $1-$tensor $\left({}^{A}u\right)^{*}$ and $2-$ tensor ${}^{A}u \cdot \left({}^{B}v\right)^{*}$ once a basis is fixed. It is quite clear that such an expression is impossible to get here. However, computationally one can always identify a $k-$tensor as a $k-$dimensional array where $k \leq 3$ which becomes more clear once we construct the basis of $\mathcal{L}(U \times V \times W \to \mathbb{R})$.

2.3.3 Basis of $3-$fold tensor product spaces

Let U, V, W be any three inner product spaces over field $\mathbb{R}$ where $dim(U) = n$, $dim(V) = m$ and $dim(W) = l$. Let $A = \{a_1, a_2, ..., a_n\}$ be an orthonormal basis of U, $B = \{b_1, b_2, ..., b_m\}$ be an orthonormal basis of V and $C = \{c_1, c_2, ..., c_l\}$ be an orthonormal basis of W. Define $A \otimes B \otimes C = \{a_i \otimes b_j \otimes c_k \mid 1 \leq i \leq n, 1 \leq j \leq m, 1 \leq k \leq l\}$. From Lemma 2.3.2 we get that, $\forall\ i \in \{1, 2, ..., n\}$, $\forall\ j \in \{1, 2, ..., m\}$, $\forall\ k \in \{1, 2, ..., l\}$, $a_i \otimes b_j \otimes c_k$ is tri-linear.

Theorem 2.3.4. *$A \otimes B \otimes C$ is a basis for vectorspace $\mathcal{L}(U \times V \times W \to \mathbb{R})$*

Proof. **Span :**
$\forall\ x \in U$, since A is a basis of U, there exist unique $\alpha_i \in \mathbb{R}$ such that

$$x = \sum_{i=1}^{n} \alpha_i a_i$$

$\forall\ y \in V$, since B is a basis of V, there exist unique $\beta_j \in \mathbb{R}$ such that

$$y = \sum_{j=1}^{m} \beta_j b_j$$

$\forall\ z \in W$, since C is a basis of W, there exist unique $\gamma_k \in \mathbb{R}$ such that

$$z = \sum_{k=1}^{l} \gamma_k c_k$$

$\forall\ u \in \mathcal{L}(U \times V \times W \to \mathbb{R})$, since u is tri-linear, we get

$$\begin{aligned} u(x,y,z) &= u(\sum_{i=1}^{n} \alpha_i a_i, \sum_{j=1}^{m} \beta_j b_j, \sum_{k=1}^{l} \gamma_k c_k) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} \alpha_i \beta_j \gamma_k u(a_i, b_j, c_k) \end{aligned}$$

$\forall\ i \in \{1,...,n\}$, $\forall\ j \in \{1,...,m\}$, $\forall\ k \in \{1,...,l\}$, since A, B and C are orthonormal bases we get,

$$\begin{aligned} [a_i \otimes b_j \otimes c_k](x,y,z) &= (a_i, x)(b_j, y)(c_k, z) \\ &= {}^{A}x[i] \cdot {}^{B}y[j] \cdot {}^{C}z[k] \\ &= \alpha_i \beta_j \gamma_k \end{aligned}$$

$$\begin{aligned} \implies u(x,y,z) &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} u(a_i, b_j, c_k)[a_i \otimes b_j \otimes c_k](x,y,z) \\ \implies u &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} u(a_i, b_j, c_k)[a_i \otimes b_j \otimes c_k] \\ \implies & A \otimes B \otimes C \text{ spans } \mathcal{L}(U \times V \times W \to \mathbb{R}) \end{aligned}$$

Linear Independence :
Let $\alpha_{ijk} \in \mathbb{R}$, $\forall\ i \in \{1,2,...,n\}$, $\forall\ j \in \{1,2,...,m\}$, $\forall\ k \in \{1,2,...,l\}$. Consider,

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} \alpha_{ijk}[a_i \otimes b_j \otimes c_k] = 0$$

$\forall\ p \in \{1,2,...,n\}$, $\forall\ q \in \{1,2,...,m\}$, $\forall\ r \in \{1,2,...,l\}$,

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} \alpha_{ijk}[a_i \otimes b_j \otimes c_k](a_p, b_q, c_r) = 0$$

Since A, B and C are orthonormal bases we get,

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l}\alpha_{ijk}(a_i,a_p)(b_j,b_q)(c_k,c_r)=\alpha_{pqr}=0$$

$\implies A\otimes B\otimes C$ is a linearly independent set and a basis of $\mathcal{L}(U\times V\times W\to\mathbb{R})$

$\square$

Corollary 2.3.5.

$$\boxed{dim(\mathcal{L}(U\times V\times W\to\mathbb{R}))=n\cdot m\cdot l=dim(U)\cdot dim(V)\cdot dim(W)}$$

Illustration :

Consider $U=\mathbb{R}^2$ over $\mathbb{R}$, $V=\mathbb{R}^3$ over $\mathbb{R}$ and $W=\mathbb{R}^4$ over $\mathbb{R}$ with standard dot product as inner product. Let $A=\{a_1=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}^*, a_2=\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}^*\}$, $B=\{b_1=\begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{bmatrix}^*, b_2=\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{bmatrix}^*, b_3=\begin{bmatrix}\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}}\end{bmatrix}^*\}$ and $C=\{c_1=\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{bmatrix}^*, c_2=\begin{bmatrix}-\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{bmatrix}^*, c_3=\begin{bmatrix}-\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\end{bmatrix}^*, c_4=\begin{bmatrix}\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\end{bmatrix}^*\}$. It is easy to see that A, B and C form orthonormal basis of $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^4$ respectively. $\forall\ i\in\{1,2\}$, $\forall\ j\in\{1,2,3\}$, $\forall\ k\in\{1,2,3,4\}$ $\forall\ x\in\mathbb{R}^2$, $\forall\ y\in\mathbb{R}^3$, $\forall\ z\in\mathbb{R}^4$,

$$\begin{aligned}[a_i\otimes b_j\otimes c_k](x,y,z)&=(a_i,x)(b_j,y)(c_k,z)\\&=({}^{A}a_i\odot{}^{A}x)\cdot({}^{B}b_j\odot{}^{B}y)\cdot({}^{C}c_k\odot{}^{C}z)\\&={}^{A}x[i]\cdot{}^{B}y[j]\cdot{}^{C}z[k]\end{aligned}\tag{2.7}$$

Let $A\otimes B\otimes C=\{a_i\otimes b_j\otimes c_k\mid 1\le i\le 2, 1\le j\le 3, 1\le k\le 4\}$.
From the above Theorem, we get that $A\otimes B$ is a basis of $\mathcal{L}(\mathbb{R}^2\times\mathbb{R}^3\to\mathbb{R})$. Consider the following linear function,

$$u(\begin{bmatrix}x_1\\x_2\end{bmatrix},\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix},\begin{bmatrix}z_1\\z_2\\z_3\\z_4\end{bmatrix})=(x_1+2x_2)\cdot(3y_1+y_3)\cdot(z_1+4z_3)\quad\text{where}\quad\begin{bmatrix}x_1\\x_2\end{bmatrix}\in\mathbb{R}^2,\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}\in\mathbb{R}^3,\begin{bmatrix}z_1\\z_2\\z_3\\z_4\end{bmatrix}\in\mathbb{R}^4$$

It is straight forward to verify that u is tri-linear. Since A, B and C form bases of U, V and W respectively we get that,

$\forall \begin{bmatrix} x_1 & x_2 \end{bmatrix}^* \in \mathbb{R}^2$ there exist unique α_1, $\alpha_2 \in \mathbb{R}$, such that

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix}^* = \alpha_1 a_1 + \alpha_2 a_2$$

$\forall \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^* \in \mathbb{R}^3$ there exist unique β_1, β_2, $\beta_3 \in \mathbb{R}$, such that

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^* = \beta_1 b_1 + \beta_2 b_2 + \beta_3 b_3$$

$\forall \begin{bmatrix} z_1 & z_2 & z_3 & z_4 \end{bmatrix}^* \in \mathbb{R}^4$ there exist unique γ_1, γ_2, γ_3, $\gamma_4 \in \mathbb{R}$ such that

$$\begin{bmatrix} z_1 & z_2 & z_3 & z_4 \end{bmatrix}^* = \gamma_1 c_1 + \gamma_2 c_2 + \gamma_3 c_3 + \gamma_4 c_4$$

Since u is tri-linear we get that

$$u(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}) = u(\sum_{i=1}^{2} \alpha_i a_i, \sum_{j=1}^{3} \beta_j b_j, \sum_{k=1}^{4} \gamma_k c_k) = \sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{k=1}^{4} \alpha_i \beta_j \gamma_k u(a_i, b_j, c_k) \tag{2.8}$$

From the above expression, it is quite clear that computing $u(a_i, b_j, c_k)$ is sufficient to determine the action of u on any $\begin{bmatrix} x_1 & x_2 \end{bmatrix}^*$, $\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^*$, $\begin{bmatrix} z_1 & z_2 & z_3 & z_4 \end{bmatrix}^*$. We leave it for the reader to compute $u(a_i, b_j, c_k)$. Combining equations 2.7 and 2.8 we get,

$$u(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}) = \sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{k=1}^{4} u(a_i, b_j, c_k)[a_i \otimes b_j \otimes c_k](\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix})$$

With this illustration, you can observe how $A \otimes B \otimes C$ works as a basis of $\mathcal{L}(\mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}^4 \to \mathbb{R})$ and also note that the values of $u(a_i, b_j, c_k)$ is sufficient to compute $u(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix})$ for any $\begin{bmatrix} x_1 & x_2 \end{bmatrix}^* \in \mathbb{R}^2$, $\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^* \in \mathbb{R}^3$ and $\begin{bmatrix} z_1 & z_2 & z_3 & z_4 \end{bmatrix}^* \in \mathbb{R}^4$.

2.3.4 Basis Transformation

Definition 2.3.3. Let U, V, W be any three inner product spaces over field $\mathbb{R}$ where $dim(U) = n$, $dim()V = m$ and $dim(W) = l$. Let $A = \{a_1, a_2, ..., a_n\}$ be an orthonormal basis of U, $B = \{b_1, b_2, ..., b_m\}$ be an orthonormal basis of V and $C = \{c_1, c_2, ..., c_l\}$ be an orthonormal basis of W. Let $A \otimes B \otimes C = \{a_i \otimes b_j \otimes c_k \mid 1 \leq i \leq n, 1 \leq j \leq m, 1 \leq k \leq l\}$. From Theorem 2.2.4, we get that $A \otimes B \otimes C$ forms a basis of $\mathcal{L}(U \times V \times W \to \mathbb{R})$. Hence, $\forall\ u \in \mathcal{L}(U \times V \times W \to \mathbb{R})$, we get that

$$u = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l} u(a_i, b_j, c_k)[a_i \otimes b_j \otimes c_k]$$

We define coordinates of the tri-linear function u as follows,

$$^{A\otimes B\otimes C}u[i, j, k] = u(a_i, b_j, c_k)$$

where $1 \leq i \leq n, 1 \leq j \leq m, 1 \leq k \leq l$.

Note :

1. Note that column vector representation is used for $1-$tensors and matrix representation is used for $2-$tensors. Recall from illustration 2.3.2 that for $3-$tensors such an analytic representation is not possible. But, $3-$tensors can be represented as a $3-$dimensional array computationally.

Theorem 2.3.6. Let U, V, W be any three inner product spaces over field $\mathbb{R}$ where $dim(U) = n$, $dim(V) = m$ and $dim(W) = l$. Let $A = \{a_1, ..., a_n\}$ and $D = \{d_1, ..., d_n\}$ be any two orthonormal basis of U. Let $B = \{b_1, ..., b_m\}$ and $E = \{e_1, ..., e_m\}$ be any two orthonormal basis of V. Let $C = \{c_1, ..., c_l\}$ and $F = \{f_1, ..., f_l\}$ be any two orthonormal basis of W. Let $A \otimes B \otimes C = \{a_i \otimes b_j \otimes c_k \mid$

$1 \le i \le n, 1 \le j \le m, 1 \le k \le l\}$ and $D \otimes E \otimes F = \{d_p \otimes e_q \otimes f_r \mid 1 \le p \le n, 1 \le q \le m, 1 \le r \le l\}$. From Theorem 2.3.4, we get that $A \otimes B \otimes C$ and $D \otimes E \otimes F$ form bases of $\mathcal{L}(U \times V \times W \to \mathbb{R})$. Let $M \in \mathbb{R}^{n \times n}$ be the transformation matrix from A to D i.e,

$$\begin{bmatrix} a_1 & a_2 & . & . & . & a_n \end{bmatrix} = \begin{bmatrix} d_1 & d_2 & . & . & . & d_n \end{bmatrix} M$$

Let $N \in \mathbb{R}^{m \times m}$ be the transformation matrix from B to E i.e,

$$\begin{bmatrix} b_1 & b_2 & . & . & . & b_m \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & . & . & . & e_m \end{bmatrix} N$$

Let $H \in \mathbb{R}^{l \times l}$ be the transformation matrix from C to F i.e,

$$\begin{bmatrix} c_1 & c_2 & . & . & . & c_l \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & . & . & . & f_l \end{bmatrix} H$$

Then, $\forall\ u \in \mathcal{L}(U \times V \times W \to \mathbb{R})$,

$$\boxed{{}^{D\otimes E\otimes F}u[p,q,r] = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} M_{pi}N_{qj}H_{rk}{}^{A\otimes B\otimes C}u[i,j,k]}$$

where $1 \le p \le n, 1 \le q \le m, 1 \le r \le l$.

Proof. $\forall\ u \in \mathcal{L}(U \times V \times W \to \mathbb{R})$, since $A \otimes B \otimes C$, $D \otimes E \otimes F$ form bases of $\mathcal{L}(U \times V \times W \to \mathbb{R})$,

$$\begin{aligned} u &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} u(a_i, b_j, c_k)[a_i \otimes b_j \otimes c_k] \\ &= \sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{r=1}^{l} u(d_p, e_q, f_r)[d_p \otimes e_q \otimes f_r] \end{aligned}$$

Since M is the transformation matrix from A to C we get that M^* is the transformation matrix from C to A i.e,

$$\begin{bmatrix} d_1 & d_2 & . & . & . & d_n \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & . & . & . & a_n \end{bmatrix} M^*$$

$\forall\ p \in \{1, 2, ..., n\}$,

$$d_p = \sum_{i=1}^{n} M^*_{ip} a_i = \sum_{i=1}^{n} M_{pi} a_i$$

Since N is the transformation matrix from B to D we get that N^* is the transformation matrix from D to B i.e,

$$\begin{bmatrix} e_1 & e_2 & . & . & . & e_m \end{bmatrix} \begin{bmatrix} b_1 & b_2 & . & . & . & b_m \end{bmatrix} N^*$$

$\forall\ q \in \{1, 2, ..., m\}$,

$$e_q = \sum_{j=1}^{m} N^*_{jq} b_j = \sum_{j=1}^{m} N_{qj} b_j$$

Since H is the transformation matrix from C to F we get that H^* is the transformation matrix from F to C i.e,

$$\begin{bmatrix} f_1 & f_2 & . & . & . & f_l \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & . & . & . & c_l \end{bmatrix} H^*$$

$\forall\ r \in \{1, 2, ..., l\}$,

$$f_r = \sum_{k=1}^{l} H^*_{kr} c_k = \sum_{k=1}^{l} H_{rk} c_k$$

Since u is tri-linear, we get

$$\begin{aligned} u(d_p, e_q, f_r) &= u(\sum_{i=1}^{n} M_{pi} a_i, \sum_{j=1}^{m} N_{qj} b_j, \sum_{k=1}^{l} H_{rk} c_k) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l} M_{pi} N_{qj} H_{rk} u(a_i, b_j, c_k) \end{aligned}$$

$$\implies {}^{D\otimes E\otimes F}u[p, q, r] = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l} M_{pi} N_{qj} H_{rk}\, {}^{A\otimes B\otimes C}u[i, j, k]$$

$\square$

Illustration :
Consider $U = \mathbb{R}^2$ over $\mathbb{R}$, $V = \mathbb{R}^3$ over $\mathbb{R}$ and $W = \mathbb{R}^4$ over $\mathbb{R}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^*, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^*\}$, $B = \{b_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}^*, b_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}^*, b_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}^*\}$ and $C = \{c_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}^*, c_2 = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}^*, c_3 = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}^*, c_4 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}^*\}$. It is easy to verify that A, B and C form orthonormal basis of $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^4$ respectively. Let $D = \{d_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^*, d_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^*\}$, $E = \{e_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^*, e_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^*, e_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^*\}$ and $F = \{f_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^*, f_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^*, f_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^*, f_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^*\}$. It is easy to see that D, E and F are standard orthonormal bases of $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^4$ respectively. $\forall$ $i \in \{1, 2\}$, $\forall$ $j \in \{1, 2, 3\}$, $\forall$ $k \in \{1, 2, 3, 4\}$, $\forall$ $x \in \mathbb{R}^2$, $\forall$ $y \in \mathbb{R}^3$, $\forall$ $z \in \mathbb{R}^4$,

$$[a_i \otimes b_j \otimes c_k](x, y, z) = {}^{A}x[i] \cdot {}^{B}y[j] \cdot {}^{C}z[k]$$

Let $A \otimes B \otimes C = \{a_i \otimes b_j \otimes c_k \mid 1 \leq i \leq 2, 1 \leq j \leq 3, 1 \leq k \leq 4\}$. Similarly, $\forall$ $p \in \{1, 2\}$, $\forall$ $q \in \{1, 2, 3\}$, $\forall$ $r \in \{1, 2, 3, 4\}$, $\forall$ $x \in \mathbb{R}^2$, $\forall$ $y \in \mathbb{R}^3$, $\forall$ $z \in \mathbb{R}^4$,

$$[d_p \otimes e_q \otimes f_r](x, y, z) = {}^{D}x[p] \cdot {}^{E}y[q] \cdot {}^{F}z[r]$$

Let $D \otimes E \otimes F = \{d_p \otimes e_q \otimes f_r \mid 1 \leq p \leq 2, 1 \leq q \leq 3, 1 \leq r \leq 4\}$. From Theorem 2.3.4, we get that $A \otimes B \otimes C$ and $D \otimes E \otimes F$ form bases of $\mathcal{L}(\mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}^4 \to \mathbb{R})$.

Computing the basis transformation matrix from basis A to D :

$$a_1 = \frac{1}{\sqrt{2}} d_1 + \frac{1}{\sqrt{2}} d_2 \qquad a_2 = \frac{1}{\sqrt{2}} d_1 - \frac{1}{\sqrt{2}} d_2$$

$$\begin{bmatrix} a_1 & a_2 \end{bmatrix} = \begin{bmatrix} d_1 & d_2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \implies M = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

We get that M is the transformation matrix from basis A to D of $\mathbb{R}^2$ which implies that M^* is the transformation matrix from basis D to A i.e,

$$\begin{bmatrix} d_1 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\implies d_1 = \frac{1}{\sqrt{2}} a_1 + \frac{1}{\sqrt{2}} a_2 \qquad d_2 = -\frac{1}{\sqrt{2}} a_1 + \frac{1}{\sqrt{2}} a_2$$

Computing the basis transformation matrix from basis B to E :

$$b_1 = \frac{1}{\sqrt{3}} e_1 + \frac{1}{\sqrt{3}} e_2 + \frac{1}{\sqrt{3}} e_3 \qquad b_2 = \frac{1}{\sqrt{2}} e_1 - \frac{1}{\sqrt{2}} e_2 \qquad b_3 = \frac{1}{\sqrt{6}} e_1 + \frac{1}{\sqrt{6}} e_2 - \frac{2}{\sqrt{6}} e_3$$

$$\begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix} \implies N = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}$$

We get that N is the transformation matrix from basis B to E of $\mathbb{R}^3$ which implies that N^* is the transformation matrix from basis E to B i.e,

$$\begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}$$

$$\implies e_1 = \frac{1}{\sqrt{3}} b_1 + \frac{1}{\sqrt{2}} b_2 + \frac{1}{\sqrt{6}} b_3 \qquad e_2 = \frac{1}{\sqrt{3}} b_1 + -\frac{1}{\sqrt{2}} b_2 + \frac{1}{\sqrt{6}} b_3 \qquad e_3 = \frac{1}{\sqrt{3}} b_1 - \frac{2}{\sqrt{6}} b_3$$

Computing the basis transformation matrix from basis C to F :

$$c_1 = \frac{1}{2} f_1 + \frac{1}{2} f_2 + \frac{1}{2} f_3 + \frac{1}{2} f_4 \qquad c_2 = -\frac{1}{2} f_1 - \frac{1}{2} f_2 + \frac{1}{2} f_3 + \frac{1}{2} f_4$$

$$c_3 = -\frac{1}{2} f_1 + \frac{1}{2} f_2 - \frac{1}{2} f_3 + \frac{1}{2} f_4 \qquad c_4 = \frac{1}{2} f_1 - \frac{1}{2} f_2 - \frac{1}{2} f_3 + \frac{1}{2} f_4$$

$$\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \implies H = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

We get that H is the transformation matrix from basis C to F of $\mathbb{R}^4$ which implies that H^* is the transformation matrix from basis F to C i.e,

$$\begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\implies f_1 = \frac{1}{2}c_1 - \frac{1}{2}c_2 - \frac{1}{2}c_3 + \frac{1}{2}c_4 \qquad f_2 = \frac{1}{2}c_1 - \frac{1}{2}c_2 + \frac{1}{2}c_3 - \frac{1}{2}c_4$$

$$\implies f_3 = \frac{1}{2}c_1 + \frac{1}{2}c_2 - \frac{1}{2}c_3 - \frac{1}{2}c_4 \qquad f_4 = \frac{1}{2}c_1 + \frac{1}{2}c_2 + \frac{1}{2}c_3 + \frac{1}{2}c_4$$

$\forall\ u \in \mathcal{L}(\mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}^4 \to \mathbb{R})$, since u is linear, $\forall\ p \in \{1,2\}$, $\forall\ q \in \{1,2,3\}$, $\forall$ $r \in \{1,2,3,4\}$ we get

$$\begin{aligned} u(d_p, e_q, f_r) &= u(\sum_{i=1}^{2} M^*_{ip} a_i, \sum_{j=1}^{3} N^*_{jq} b_j, \sum_{k=1}^{4} H^*_{kr} c_k) \\ &= \sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{k=1}^{4} M_{pi} N_{qj} H_{rk} u(a_i, b_j, c_k) \end{aligned}$$

$$\implies {}^{D\otimes E\otimes F}u[p,q,r] = \sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{k=1}^{4} M_{pi} N_{qj} H_{rk} {}^{A\otimes B\otimes C}u[i,j,k]$$

2.3.5 Invariance of computation of $3-$tensor under any orthonormal basis transformations

Theorem 2.3.7. Let U, V, W be any three inner product spaces over field $\mathbb{R}$ where $dim(U) = n$, $dim(V) = m$ and $dim(W) = l$. Let $A = \{a_1, ..., a_n\}$ be any orthonormal basis of U. Let $B = \{b_1, ..., b_m\}$ be any orthonormal basis of V. Let $C = \{c_1, c_2, ..., c_l\}$ be any orthonormal basis of W. Let $A \otimes B \otimes C = \{a_i \otimes b_j \otimes c_k \mid 1 \le i \le n, 1 \le j \le m, 1 \le k \le l\}$. From Theorem 2.3.4, we get that $A \otimes B \otimes C$ form basis of $\mathcal{L}(U \times V \times W \to \mathbb{R})$. Then, $\forall\ u \in \mathcal{L}(U \times V \times W \to \mathbb{R})$, $\forall\ x \in U$, $\forall$

$y \in V$, $\forall\ z \in W$,

$$\boxed{u(x, y, z) = \sum_{r=1}^{n} \sum_{s=1}^{m} \sum_{t=1}^{l} {}^{A\otimes B\otimes C}u[r, s, t]\, {}^{A}x[r]\, {}^{B}y[s]\, {}^{C}z[t]}$$

Proof. $\forall\ x \in U$, since A is a basis of U, there exist unique $\alpha_r \in \mathbb{R}$, such that

$$x = \sum_{r=1}^{n} \alpha_r a_r$$

$\forall\ y \in V$, since B is a basis of V, there exist unique $\beta_s \in \mathbb{R}$, such that

$$y = \sum_{s=1}^{m} \beta_s b_s$$

$\forall\ z \in W$, since C is a basis of W, there exist unique $\gamma_t \in \mathbb{R}$, such that

$$z = \sum_{t=1}^{l} \gamma_t c_t$$

$\forall\ u \in \mathcal{L}(U \times V \times W \to \mathbb{R})$ we get

$$u(x, y, z) = u(\sum_{r=1}^{n} \alpha_r a_r, \sum_{s=1}^{m} \beta_s b_s, \sum_{t=1}^{l} \gamma_t c_t) = \sum_{r=1}^{n} \sum_{s=1}^{m} \sum_{t=1}^{l} \alpha_r \beta_s \gamma_t u(a_r, b_s, c_t)$$

Since A, B and C are orthonormal bases of U, V and W, we get

$$\alpha_r = {}^{A}x[r] \qquad\qquad \beta_s = {}^{B}y[s] \qquad\qquad \gamma_t = {}^{C}z[t]$$

Since $A \otimes B \otimes C$ forms a basis of $\mathcal{L}(U \times V \times W \to \mathbb{R})$, we get

$$u(a_r, b_s, c_t) = {}^{A\otimes B\otimes C}u[r, s, t]$$

$$\implies u(x, y, z) = \sum_{r=1}^{n} \sum_{s=1}^{m} \sum_{t=1}^{l} {}^{A\otimes B\otimes C}u[r, s, t]\, {}^{A}x[r]\, {}^{B}y[s]\, {}^{C}z[t]$$

$\square$

Remark :

1. Let $A = \{a_1, ..., a_n\}$ and $D = \{d_1, ..., d_n\}$ be any two orthonormal basis of U. Let $B = \{b_1, ..., b_m\}$ and $E = \{e_1, ..., e_m\}$ be any two orthonormal basis of V. Let $C = \{c_1, ..., c_l\}$ and $F = \{f_1, ..., f_l\}$ be any two orthonormal basis of W. Let $A \otimes B \otimes C = \{a_i \otimes b_j \otimes c_k \mid 1 \leq i \leq n, 1 \leq j \leq m, 1 \leq k \leq l\}$ and $D \otimes E \otimes F = \{d_p \otimes e_q \otimes f_r \mid 1 \leq p \leq n, 1 \leq q \leq m, 1 \leq r \leq l\}$. From Theorem 2.3.4, we get that $A \otimes B \otimes C$ and $D \otimes E \otimes F$ form bases of $\mathcal{L}(U \times V \times W \to \mathbb{R})$. Then, $\forall\ u \in \mathcal{L}(U \times V \times W \to \mathbb{R})$, $\forall\ x \in U$, $\forall\ y \in V$, $\forall\ z \in W$, we get

$$\begin{aligned} u(x, y, z) &= \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l} {}^{A\otimes B\otimes C}u[i, j, k]\, {}^{A}x[i]\, {}^{B}y[j]\, {}^{C}z[k] \\ &= \sum_{p=1}^{n} \sum_{q=1}^{m} \sum_{r=1}^{l} {}^{D\otimes E\otimes F}u[p, q, r]\, {}^{D}x[p]\, {}^{E}y[q]\, {}^{F}z[r] \end{aligned}$$

2. It is easy to observe that $\forall\ (x, y, z) \in U \times V \times W$, $u(x, y, z)$ can be completely determined by the coordinates of u i.e, ${}^{A\otimes B\otimes C}u[i, j, k]$. Hence, if we fix the computations with respect to orthonormal bases A, B and C of U, V and W respectively, we can identify u with its coordinates ${}^{A\otimes B\otimes C}u[i, j, k]$.

3. Let $dim(U) = n\ dim(V) = m$, $dim(W) = l$ then, $\forall\ u \in \mathcal{L}(U \times V \times W \to \mathbb{R})$,

$$ {}^{A\otimes B\otimes C}u[i, j, k] = u(a_i, b_j, c_k)$$

   Hence, it is easy to show that the 3−fold tensor product space, $\mathcal{L}(U \times V \times W \to \mathbb{R})$ is isomorphic to $\mathbb{R}^{n\times m\times l}$. (It is straight-forward to verify and is left to the reader. For the proof technique you may refer Lemma 1.3.9)

Illustration :

Consider $U = \mathbb{R}^2$ over $\mathbb{R}$, $V = \mathbb{R}^3$ over $\mathbb{R}$ and $W = \mathbb{R}^4$ over $\mathbb{R}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^*, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^*\}$, $B = \{b_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}^*, b_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}^*, b_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}^*\}$ and $C = \{c_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}^*, c_2 = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}^*, c_3 = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}^*, c_4 =$

$\begin{bmatrix}\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\end{bmatrix}^*\}$. It is easy to verify that A, B and C form orthonormal basis of $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^4$ respectively. Let $D = \{d_1 = \begin{bmatrix}1 & 0\end{bmatrix}^*, d_2 = \begin{bmatrix}0 & 1\end{bmatrix}^*\}$, $E = \{e_1 = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^*, e_2 = \begin{bmatrix}0 & 1 & 0\end{bmatrix}^*, e_3 = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^*\}$ and $F = \{f_1 = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^*, f_2 = \begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}^*, f_3 = \begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix}^*, f_4 = \begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^*\}$. It is easy to see that D, E and F are standard orthonormal bases of $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^4$ respectively. $\forall$ $i \in \{1, 2\}$, $\forall$ $j \in \{1, 2, 3\}$, $\forall$ $k \in \{1, 2, 3, 4\}$ $\forall$ $x \in \mathbb{R}^2$, $\forall$ $y \in \mathbb{R}^3$, $\forall$ $z \in \mathbb{R}^4$,

$$[a_i \otimes b_j \otimes c_k](x, y, z) = {}^{A}x[i] \cdot {}^{B}y[j] \cdot {}^{C}z[k]$$

Let $A \otimes B \otimes C = \{a_i \otimes b_j \otimes c_k \mid 1 \leq i \leq 2, 1 \leq j \leq 3, 1 \leq k \leq 4\}$. Similarly, $\forall$ $p \in \{1, 2\}$, $\forall$ $q \in \{1, 2, 3\}$, $\forall$ $r \in \{1, 2, 3, 4\}$, $\forall$ $x \in \mathbb{R}^2$, $\forall$ $y \in \mathbb{R}^3$, $\forall$ $z \in \mathbb{R}^4$,

$$[d_p \otimes e_q \otimes f_r](x, y, z) = {}^{D}x[p] \cdot {}^{E}y[q] \cdot {}^{F}z[r]$$

Let $D \otimes E \otimes F = \{d_p \otimes e_q \otimes f_r \mid 1 \leq p \leq 2, 1 \leq q \leq 3, 1 \leq r \leq 4\}$. From Theorem 2.3.4, we get that $A \otimes B \otimes C$ and $D \otimes E \otimes F$ form bases of $\mathcal{L}(\mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}^4 \to \mathbb{R})$. In the previous illustration, we have already shown that the basis transformation matrix M from A to D is,

$$M = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$$

$\forall$ $x \in \mathbb{R}^2$,

$$^{D}x = M \cdot {}^{A}x$$

The basis transformation matrix N from B to E is,

$$N = \begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}}\end{bmatrix}$$

$\forall$ $y \in \mathbb{R}^3$,

$$^{E}y = N \cdot {}^{B}y$$

The basis transformation matrix from C to F is,

$$H = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$\forall\ z \in \mathbb{R}^4$,

$$^{F}z = H \cdot {}^{C}z$$

In this illustration, we will show that $\forall\ u \in \mathbb{R}^2$, $v \in \mathbb{R}^3$, $w \in \mathbb{R}^4$, $[u \otimes v \otimes w]$ has the same value irrespective of the choice of orthonormal basis and this is sufficient to claim that any $t \in \mathcal{L}(\mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}^4 \to \mathbb{R})$ has the same value irrespective of the choice of orthonormal basis. Since any $t \in \mathcal{L}(\mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}^4 \to \mathbb{R})$ can be written as a linear combination of tensor products in $A \otimes B \otimes C$.

$$\begin{aligned}
[u \otimes v \otimes w](x, y, z) &= (u, x)(v, y)(w, z) = ({}^{D}u \odot {}^{D}x) \cdot ({}^{E}v \odot {}^{E}y) \cdot ({}^{F}w \odot {}^{F}z) \\
&= (({}^{D}u)^{*} \cdot {}^{D}x) \cdot (({}^{E}v)^{*} \cdot {}^{E}y) \cdot (({}^{F}w)^{*} \cdot {}^{F}z) \\
&= ((M \cdot {}^{A}u)^{*} \cdot M \cdot {}^{A}x) \cdot ((N \cdot {}^{B}v)^{*} \cdot N \cdot {}^{B}y) \cdot ((H \cdot {}^{C}w)^{*} \cdot H \cdot {}^{C}z) \\
&= (({}^{A}u)^{*} \cdot M^{*} \cdot M \cdot {}^{A}x) \cdot (({}^{B}v)^{*} \cdot N^{*} \cdot N \cdot {}^{B}y) \cdot (({}^{C}w)^{*} \cdot H^{*} \cdot H \cdot {}^{C}z) \\
&= (({}^{A}u)^{*} \cdot {}^{A}x) \cdot (({}^{B}v)^{*} \cdot {}^{B}y) \cdot (({}^{C}w)^{*} \cdot {}^{C}z) \text{ use } M^{*}M = N^{*}N = H^{*}H = I \\
&= ({}^{A}u \odot {}^{A}x) \cdot ({}^{B}v \odot {}^{B}y) \cdot ({}^{C}w \odot {}^{C}z)
\end{aligned}$$

Tensor product of any three vectors $u, v, w \in V$ is defined as a function in terms of inner products $[u \otimes v \otimes w](x, y, z) = (u, x)(v, y)(w, z)$ which is tri-linear. Subsequently, a basis is constructed for $\mathcal{L}(U \times V \times W \to \mathbb{R})$ in terms of tensor products of orthonormal bases of vector spaces U, V and W. The formulation of tensor product of vectors in terms of multiplication of inner products forms the basic intuition with which we generalize this theory to higher fold tensor product spaces.

2.4 k-Fold Tensor Product Spaces

2.4.1 Multi-linear Functions

Definition 2.4.1. Let V_1, V_2, ..., V_k be vector spaces over field $\mathbb{R}$ with inner products $()_1 : V_1 \times V_1 \to \mathbb{R}$, $()_2 : V_2 \times V_2 \to \mathbb{R}$, ..., $()_k : V_k \times V_k \to \mathbb{R}$ defined on V_1, V_2, ..., V_k respectively. Note that subscripts $()_1$, $()_2$, ..., $()_k$ will be dropped if the context is clear. A function $u : V_1 \times V_2 ... \times V_k \to \mathbb{R}$ is called multi-linear if the following holds,

1. $\forall\ x_1 \in V_1$, $\forall\ x_2 \in V_2$, ..., $\forall\ x_i, \tilde{x}_i \in V_i$, ..., $\forall\ x_k \in V_k$, where $1 \leq i \leq k$,

$$u(x_1, x_2, ..., x_i + \tilde{x}_i, ..., x_k) = u(x_1, x_2, ..., x_i, ..., x_k) + u(x_1, x_2, ..., \tilde{x}_i, ..., x_k)$$

2. $\forall\ x_1 \in V_1$, $\forall\ x_2 \in V_2$, ..., $\forall\ x_i \in V_i$, ..., $\forall\ x_k \in V_k$, where $1 \leq i \leq k$, $\forall\ \alpha \in \mathbb{R}$,

$$u(x_1, x_2, ..., \alpha x_i, ..., x_k) = \alpha u(x_1, x_2, ..., x_k)$$

Let set $S = \{u : V_1 \times V_2 \times ... \times V_k \to \mathbb{R} \mid u \text{ is multi-linear}\}$. Define addition and multiplication on the set S as follows, $\forall\ u, v \in S$, $\forall\ x_1 \in V_1$, $\forall\ x_2 \in V_2$, ..., $\forall\ x_k \in V_k$, $\forall\ \alpha \in \mathbb{R}$,

$$[u + v](x_1, x_2, ..., x_k) = u(x_1, x_2, ..., x_k) + v(x_1, x_2, ..., x_k)$$

$$[\alpha u](x_1, x_2, ..., x_k) = \alpha u(x_1, x_2, ..., x_k)$$

Recall that we use [.] to indicate the addition and scalar multiplication of elements of the set S.

Lemma 2.4.1. S is closed under addition and scalar multiplication.

Proof. **Claim 1 :** $\forall\ u, v \in S$, $[u + v] \in S$,

1. $\forall\ x_1 \in V_1$, $\forall\ x_2 \in V_2$, ..., $\forall\ x_i, \tilde{x}_i \in V_i$, ..., $\forall\ x_k \in V_k$, where $1 \leq i \leq k$,

$$[u + v](x_1, .., x_i + \tilde{x}_i, .., x_k) = u(x_1, .., x_i + \tilde{x}_i, .., x_k) + v(x_1, .., x_i + \tilde{x}_i, .., x_k)$$

$$= u(x_1, .., x_i, .., x_k) + u(x_1, .., \tilde{x}_i, .., x_k) + v(x_1, .., x_i, .., x_k) + v(x_1, .., \tilde{x}_i, .., x_k)$$
$$= [u+v](x_1, .., x_i, .., x_k) + [u+v](x_1, .., \tilde{x}_i, .., x_k)$$

2. $\forall\ x_1 \in V_1$, $\forall\ x_2 \in V_2$, ..., $\forall\ x_i \in V_i$, ..., $\forall\ x_k \in V_k$, where $1 \leq i \leq k$, $\forall\ \alpha \in \mathbb{R}$,

$$\begin{aligned} [u+v](x_1, .., \alpha x_i, .., x_k) &= u(x_1, .., \alpha x_i, .., x_k) + v(x_1, .., \alpha x_i, .., x_k) \\ &= \alpha u(x_1, .., x_i, .., x_k) + \alpha v(x_1, .., x_i, .., x_k) \\ &= \alpha [u+v](x_1, .., x_i, .., x_k) \end{aligned}$$

$$[u+v] \text{ is multi-linear } \implies [u+v] \in S \implies S \text{ is closed under addition}$$

Claim 2 : $\forall\ u \in S$, $\forall\ \alpha \in \mathbb{R}$, $[\alpha u] \in S$,

1. $\forall\ x_1 \in V_1$, $\forall\ x_2 \in V_2$, ..., $\forall\ x_i, \tilde{x}_i \in V_i$, ..., $\forall\ x_k \in V_k$, where $1 \leq i \leq k$,

$$\begin{aligned} [\alpha u](x_1, .., x_i + \tilde{x}_i, .., x_k) &= \alpha u(x_1, .., x_i + \tilde{x}_i, .., x_k) \\ &= \alpha u(x_1, .., x_i, .., x_k) + \alpha u(x_1, .., \tilde{x}_i, .., x_k) \\ &= [\alpha u](x_1, .., x_i, ..., x_k) + [\alpha u](x_1, .., x_i, ..., x_k) \end{aligned}$$

2. $\forall\ x_1 \in V_1$, $\forall\ x_2 \in V_2$, ..., $\forall\ x_i \in V_i$, ..., $\forall\ x_k \in V_k$, where $1 \leq i \leq k$, $\forall\ \beta \in \mathbb{R}$,

$$\begin{aligned} [\alpha u](x_1, .., \beta x_i, .., x_k) = \alpha u(x_1, .., \beta x_i, .., x_k) &= \alpha\beta u(x_1, .., x_i, .., x_k) \\ &= \beta[\alpha u](x_1, .., x_i, .., x_k) \end{aligned}$$

$$[\alpha u] \text{ is multi-linear } \implies [\alpha u] \in S \implies S \text{ is closed under scalar multiplication}$$

$\square$

A multi-linear function $u \in S$ is called a $k-$**tensor** or a multi-linear map on $V_1 \times V_2 \times ... \times V_k$. It is easy to verify that S is a vector space over field $\mathbb{R}$ (We already proved that S is closed under addition and scalar multiplication and rest of the axioms of vector space are easy to verify and are left to the reader). The vector space of all $k-$tensors is defined as the tensor product space of V_1, V_2, ..., V_k denoted by $\mathcal{L}(V_1 \times V_2 \times ... \times V_k \to \mathbb{R})$.

2.4.2 Tensor Products on vector spaces V_1, V_2, ..., V_k

Definition 2.4.2. Let V_1, V_2, ..., V_k be finite dimensional inner product spaces over field $\mathbb{R}$ where $dim(V_i) = n_i \ \forall \ i \in \{1, 2, ..., k\}$. Then, $\forall \ u_1 \in V_1$, $\forall \ u_2 \in V_2$, ..., $\forall \ u_k \in V_k$, we define the tensor product of u_1, u_2, ..., u_k as a function $[u_1 \otimes u_2 \otimes ... \otimes u_k] : V_1 \times V_2 ... \times V_k \to \mathbb{R}$ as follows, $\forall \ x_1 \in V_1$, $\forall \ x_2 \in V_2$, ..., $\forall \ x_k \in V_k$,

$$\boxed{[u_1 \otimes u_2 \otimes ... \otimes u_k](x_1, x_2, ..., x_k) = (u_1, x_1)(u_2, x_2)...(u_k, x_k) = \prod_{i=1}^{k} (u_i, x_i)}$$

Lemma 2.4.2. Let V_1, V_2, ..., V_k be finite dimensional inner product spaces over field $\mathbb{R}$. Then, $\forall \ u_1 \in V_1, \forall \ u_2 \in V_2, ..., \forall \ u_k \in V_k$, $[u_1 \otimes u_2 \otimes ... \otimes u_k]$ is multi-linear.

Proof. 1. $\forall \ x_1 \in V_1, \forall \ x_2 \in V_2, ..., \forall \ x_i, \tilde{x}_i \in V_i, ..., \forall \ x_k \in V_k$, where $1 \leq i \leq k$,

$$[u_1 \otimes ... \otimes u_k](x_1, ..., x_i + \tilde{x}_i, ..., x_k) = (u_1, x_1)...(u_i, x_i + \tilde{x}_i)...(u_k, x_k)$$

$$\begin{aligned} &= (u_1, x_1)...(u_i, x_i)...(u_k, x_k) + (u_1, x_1)...(u_i, \tilde{x}_i)...(u_k, x_k) \\ &= [u_1 \otimes ... \otimes u_k](x_1, ..., x_i, ..., x_k) + [u_1 \otimes ... \otimes u_k](x_1, ..., \tilde{x}_i, ..., x_k) \end{aligned}$$

2. $\forall \ x_1 \in V_1, \forall \ x_2 \in V_2, ..., \forall \ x_i, \tilde{x}_i \in V_i, ..., \forall \ x_k \in V_k$, where $1 \leq i \leq k$, $\forall \ \alpha \in \mathbb{R}$,

$$\begin{aligned} [u_1 \otimes ... \otimes u_k](x_1, ..., \alpha x_i, ..., x_k) &= (u_1, x_1)...(u_i, \alpha_i x_i)...(u_k, x_k) \\ &= \alpha(u_1, x_1)...(u_i, x_i)...(u_k, x_k) \\ &= \alpha[u_1 \otimes ... \otimes u_k](x_1, x_2, ..., x_k) \end{aligned}$$

$$\implies [u_1 \otimes ... \otimes u_k] \text{ is multi-linear}$$

□

Lemma 2.4.3. Let V_1, V_2, ..., V_k be finite dimensional inner product spaces over field $\mathbb{R}$. Then,

1. $\forall \ u_1 \in V_1, ..., \forall \ u_i, \tilde{u}_i \in V_i, ..., \forall \ u_k \in V_k$, where $1 \leq i \leq k$,

$$[u_1 \otimes ... \otimes [u_i + \tilde{u}_i] \otimes ... \otimes u_k] = [u_1 \otimes ... \otimes u_i \otimes ... \otimes u_k] + [u_1 \otimes ... \otimes \tilde{u}_i \otimes ... \otimes u_k]$$

2. $\forall\, u_1 \in V_1,\, ...,\, \forall\, u_i \in V_i,\, ...,\, \forall\, u_k \in V_k$, where $1 \leq i \leq k$, $\forall\, \alpha \in \mathbb{R}$,

$$[u_1 \otimes ... \otimes [\alpha u_i] \otimes ... \otimes u_k] = \alpha[u_1 \otimes ... \otimes u_i \otimes ... \otimes u_k]$$

Proof. 1. $\forall\, x_1 \in V_1,\, ...,\, \forall\, x_k \in V_k$,

$$[u_1 \otimes ... \otimes [u_i + \tilde{u}_i] \otimes ... \otimes u_k](x_1, ..., x_k) = (u_1, x_1, u_1)...(u_i + \tilde{u}_i, x_i)...(u_k, x_k)$$

$$= (u_1, x_1)...(u_i, x_i)...(u_k, x_k) + (u_1, x_1)...(\tilde{u}_i, x_i)...(u_k, x_k)$$
$$= [u_1 \otimes ...u_i... \otimes u_k](x_1, ..., x_k) + [u_1 \otimes ...\tilde{u}_i... \otimes u_k](x_1, ..., x_k)$$

2. $\forall\, x_1 \in V_1,\, ...,\, \forall\, x_k \in V_k,\, \forall\, \alpha \in \mathbb{R}$,

$$\begin{aligned}[u_1 \otimes ... \otimes [\alpha u_i] \otimes ... \otimes u_k](x_1, ..., x_k) &= (u_1, x_1)...(\alpha u_i, x_i)...(u_k, x_k) \\ &= \alpha(u_1, x_1)...(u_i, x_i)...(u_k, x_k) \\ &= \alpha[u_1 \otimes ... \otimes u_i \otimes ... \otimes u_k](x_1, ..., x_k)\end{aligned}$$

□

Illustration :
Consider $V_i = \mathbb{R}^2$ over $\mathbb{R}$, $\forall\, i \in \{1, 2, ..., k\}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^*, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^*\}$. It is easy to see that A form orthonormal basis of each $V_i = \mathbb{R}^2$. Let $u_i = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\forall\, i \in \{1, 2, ..., k\}$. $\forall\, x_1, x_2, ..., x_k \in \mathbb{R}^2$,

$$(u_1 \otimes ... \otimes u_k)(x_1, ..., x_k) = \prod_{i=1}^{k} (u_i, x_i) = \prod_{i=1}^{k} ({}^{A}u_i \odot {}^{A}x_i) = \prod_{i=1}^{k} \begin{bmatrix} {}^{A}x[1] & {}^{A}x[2] \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

From the linearity of inner product (dot product) in both the first and second arguments, we can easily conclude that $[u_1 \otimes u_2 \otimes ... \otimes u_k]$ is multi-linear and all the properties in Lemma 2.4.3 hold. Recall that we had an analytic expression to identify $1-$tensor $(({}^{A}u)^*)$ and $2-$ tensor $({}^{A}u \cdot ({}^{B}v)^*)$ once a basis is fixed. It is quite clear that such an expression is impossible to get if $k \geq 3$. However, a $k-$tensor can be identified as a $k-$dimensional array computationally $\forall\, k \in \mathbb{N}$.

2.4.3 Basis of $k-$fold tensor product spaces

Let V_1, V_2, ..., V_k be finite dimensional inner product spaces over field $\mathbb{R}$, where $dim(V_i) = n_i \ \forall \ i \in \{1, 2, ..., k\}$. Let $A_1 = \{a_{11}, a_{12}, ..., a_{1n_1}\}$ be an orthonormal basis of V_1, $A_2 = \{a_{21}, a_{22}, ..., a_{2n_2}\}$ be an orthonormal basis of V_2, ..., $A_k = \{a_{k1}, a_{k2}, ..., a_{kn_k}\}$ be an orthonormal basis of V_k. Define $A_1 \otimes A_2 ... \otimes A_k = \{a_{1i_1} \otimes a_{2i_2} ... \otimes a_{ki_k} \mid 1 \leq i_1 \leq n_1, 1 \leq i_2 \leq n_2, ..., 1 \leq i_k \leq n_k\}$. Lemma 2.4.2 implies that $\forall \ i_1 \in \{1, 2, ..., n_1\}$, $\forall \ i_2 \in \{1, 2, ..., n_2\}$, ..., $\forall \ i_k \in \{1, 2, ..., n_k\}$, $a_{1i_1} \otimes a_{2i_2} ... \otimes a_{ki_k}$ is multi-linear.

Theorem 2.4.4. *$A_1 \otimes A_2 ... \otimes A_k$ is a basis for vectorspace $\mathcal{L}(V_1 \times V_2 ... \times V_k \to \mathbb{R})$*

Proof. **Span :**
$\forall \ x_1 \in V_1$, since A_1 is a basis of V_1, there exist unique $\alpha_{1i_1} \in \mathbb{R}$ such that,

$$x_1 = \sum_{i_1=1}^{n_1} \alpha_{1i_1} a_{1i_1}$$

$\forall \ x_2 \in V_2$, since A_2 is a basis of V_2, there exist unique $\alpha_{2i_2} \in \mathbb{R}$ such that,

$$x_2 = \sum_{i_2=1}^{n_2} \alpha_{2i_2} a_{2i_2}$$

$$...,$$

$\forall \ x_k \in V_k$, since A_k is a basis of V_k, there exist unique $\alpha_{ki_k} \in \mathbb{R}$ such that,

$$x_k = \sum_{i_1=1}^{n_k} \alpha_{ki_k} a_{ki_k}$$

$\forall \ u \in \mathcal{L}(V_1 \times V_2 \times ... \times V_k \to \mathbb{R})$, since u is multi-linear we get

$$\begin{aligned} u(x_1, x_2, ..., x_k) &= u(\sum_{i_1=1}^{n_1} \alpha_{1i_1} a_{1i_1}, \sum_{i_2=1}^{n_2} \alpha_{2i_2} a_{2i_2}, ..., \sum_{i_k=1}^{n_3} \alpha_{ki_k} a_{ki_k}) \\ &= \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} \alpha_{1i_1} \alpha_{2i_2} ... \alpha_{ki_k} u(a_{1i_1}, a_{2i_2}, ..., a_{ki_k}) \end{aligned}$$

$\forall \ i_1 \in \{1, ..., n_1\}$, $i_2 \in \{1, ..., n_2\}$, ..., $i_k \in \{1, ..., n_k\}$, since A_1, A_2, ..., A_k are orthonormal bases we get,

$$[a_{1i_1} \otimes a_{2i_2} \otimes ... \otimes a_{ki_k}](x_1, x_2, ..., x_k) = (a_{1i_1}, x_1)(a_{2i_2}, x_2)...(a_{ki_k}, x_k) = \prod_{j=1}^{k}{}^{A_j} x_i[j] = \prod_{j=1}^{k} \alpha_{ji_j}$$

$$\implies u(x_1, x_2, ..., x_k) = \sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}...\sum_{i_k=1}^{n_k} u(a_{1i_1}, a_{2i_2}, ..., a_{ki_k})[a_{1i_1} \otimes a_{2i_2} \otimes ... \otimes a_{ki_k}](x_1, x_2, ..., x_k)$$

$$\implies u = \sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}...\sum_{i_k=1}^{n_k} u(a_{1i_1}, a_{2i_2}, ..., a_{ki_k})[a_{1i_1} \otimes a_{2i_2} \otimes ... \otimes a_{ki_k}]$$

$$\implies A_1 \otimes A_2 \otimes ... \otimes A_k \text{ spans } \mathcal{L}(V_1 \times V_2 ... \times V_k \to \mathbb{R})$$

Linear Independence :

Let $\alpha_{i_1,i_2,...,i_k} \in \mathbb{R}$, $\forall\ i_1 \in \{1, ..., n_1\}$, $i_2 \in \{1, ..., n_2\}$, ..., $i_k \in \{1, ..., n_k\}$. Consider,

$$\sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}...\sum_{i_k=1}^{n_k} \alpha_{i_1,i_2,...,i_k}[a_{1i_1} \otimes a_{2i_2} ... \otimes a_{ki_k}] = 0$$

$\forall\ j_1 \in \{1, ..., n_1\}$, $\forall\ j_2 \in \{1, ..., n_2\}$, ..., $\forall\ j_k \in \{1, ..., n_k\}$,

$$\sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}...\sum_{i_k=1}^{n_k} \alpha_{i_1,i_2,...,i_k}[a_{1i_1} \otimes a_{2i_2} ... \otimes a_{ki_k}](a_{1j_1}, a_{2j_2}, ..., a_{kj_k}) = 0$$

Since A_1, A_2, ..., A_k are orthonormal bases we get,

$$\sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}...\sum_{i_k=1}^{n_k} \alpha_{i_1,i_2,...,i_k}(a_{1i_1}, a_{1j_1})(a_{2i_2}, a_{2j_2})...(a_{ki_k}, a_{kj_k}) = \alpha_{j_1,j_2,...,j_k} = 0$$

$$\implies A_1 \otimes A_2 \otimes ... \otimes A_k \text{ is a linearly independent set and a basis of } \mathcal{L}(V_1 \times V_2 ... \times V_k \to \mathbb{R})$$

□

Corollary 2.4.5.

$$\boxed{dim(\mathcal{L}(V_1 \times V_2 \times ... \times V_k \to \mathbb{R})) = \prod_{i=1}^{k} n_i = \prod_{i=1}^{k} dim(V_i)}$$

Illustration :

Consider $V_i = \mathbb{R}^2$ over $\mathbb{R}$, $\forall\ i \in \{1, 2, ..., k\}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^*, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^*\}$. It is easy to see that A form orthonormal basis of each $V_i = \mathbb{R}^2$.

$\forall\ i_1 \in \{1, 2\}$, $\forall\ i_2 \in \{1, 2\}$, ..., $\forall\ i_k \in \{1, 2\}$, $\forall\ x_1, x_2, ..., x_k \in \mathbb{R}^2$,

$$\begin{aligned}(a_{i_1} \otimes a_{i_2} \otimes ... \otimes a_{i_k})(x_1, x_2, ..., x_k) &= (a_{i_1}, x_1)(a_{i_2}, x_2)...(a_{i_k}, x_k) \\ &= ({}^{A}a_{i_1} \odot {}^{A}x_1) \cdot ({}^{A}a_{i_2} \odot {}^{A}x_2) \cdot ... \cdot ({}^{A}a_{i_k} \odot {}^{A}x_k) \\ &= {}^{A}x_1[i_1] \cdot {}^{A}x_2[i_2] \cdot ... \cdot {}^{A}x_k[i_k]\end{aligned}$$

Let $A \otimes A \otimes ... \otimes A(k \text{ times}) = \otimes_{i=1}^{k} A = \{a_{i_1} \otimes a_{i_2} ... \otimes a_{i_k} \mid 1 \leq i_1 \leq 2, 1 \leq i_2 \leq 2, ..., 1 \leq i_k \leq 2\}$.

From the above Theorem, we get that that $\otimes_{i=1}^{k} A$ is a basis of $\mathcal{L}((\mathbb{R}^2)^k \to \mathbb{R})$,

$$u(\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}, \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix}, ..., \begin{bmatrix} x_{k1} \\ x_{k2} \end{bmatrix}) = x_{11} \cdot x_{21} ... \cdot x_{k1} = \prod_{i=1}^{k} x_{i1} \text{ where } \begin{bmatrix} x_{j1} \\ x_{j2} \end{bmatrix} \in \mathbb{R}^2\ \forall\ j \in \{1, 2, ..., k\}$$

It is straight-forward to verify that u is multi-linear. Since $\otimes_{i=1}^{k} A$ is a basis of $\mathcal{L}((\mathbb{R}^2)^k \to \mathbb{R})$, $\forall\ \begin{bmatrix} x_{11} & x_{12} \end{bmatrix}^* \in \mathbb{R}^2$ there exist unique α_{11}, $\alpha_{12} \in \mathbb{R}$, such that

$$\begin{bmatrix} x_{11} & x_{12} \end{bmatrix}^* = \alpha_{11} a_1 + \alpha_{12} a_2$$

$\forall\ \begin{bmatrix} x_{21} & x_{22} \end{bmatrix}^* \in \mathbb{R}^2$ there exist unique α_{21}, $\alpha_{22} \in \mathbb{R}$, such that

$$\begin{bmatrix} x_{21} & x_{22} \end{bmatrix}^* = \alpha_{21} a_1 + \alpha_{22} a_2$$

...,

$\forall\ \begin{bmatrix} x_{k1} & x_{k2} \end{bmatrix}^* \in \mathbb{R}^2$ there exist unique α_{k1}, $\alpha_{k2} \in \mathbb{R}$, such that

$$\begin{bmatrix} x_{k1} & x_{k2} \end{bmatrix}^* = \alpha_{k1} a_1 + \alpha_{k2} a_2$$

Since u is multi-linear we get that

$$u(\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}, \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix}, ..., \begin{bmatrix} x_{k1} \\ x_{k2} \end{bmatrix}) = u(\sum_{i_1=1}^{2} \alpha_{1i_1} a_{i_1}, \sum_{i_2=1}^{2} \alpha_{2i_2} a_{i_2}, ..., \sum_{i_k=1}^{2} \alpha_{ki_k} a_{i_k})$$
$$= \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} ... \sum_{i_k=1}^{2} \alpha_{1i_1} \alpha_{2i_2} ... \alpha_{ki_k} u(a_{i_1}, a_{i_2}, ..., a_{i_k})$$

From the above expression, it is quite clear that computing $u(a_{i_1}, a_{i_2}, ..., a_{i_k})$ is sufficient to determine the action of u on any $\begin{bmatrix} x_{11} & x_{12} \end{bmatrix}^*, \begin{bmatrix} x_{21} & x_{22} \end{bmatrix}^*, ..., \begin{bmatrix} x_{k1} & x_{k2} \end{bmatrix}^*$. Note that $u(a_{i_1}, a_{i_2}, ..., a_{i_k})$ can be easily computed given k.

$$\implies u(\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}, ..., \begin{bmatrix} x_{k1} \\ x_{k2} \end{bmatrix}) = \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} ... \sum_{i_k=1}^{2} u(a_{i_1}, ..., a_{i_k})[a_{1i_1} \otimes ... \otimes a_{ki_k}](\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}, ..., \begin{bmatrix} x_{k1} \\ x_{k2} \end{bmatrix})$$

With this illustration, you can observe how $\otimes_{i=1}^{k} A$ works as a basis of $\mathcal{L}((\mathbb{R}^2)^k \to \mathbb{R})$ and also note that the values of $u(a_{i_1}, a_{i_2}, ..., a_{i_k})$ is sufficient to compute u on any $\begin{bmatrix} x_{11} & x_{12} \end{bmatrix}^*, \begin{bmatrix} x_{21} & x_{22} \end{bmatrix}^*, ..., \begin{bmatrix} x_{k1} & x_{k2} \end{bmatrix}^* \in \mathbb{R}^2$.

2.4.4 Basis Transformation

Definition 2.4.3. Let V_1, V_2, ..., V_k be finite dimensional inner product spaces over field $\mathbb{R}$ where $dim(V_i) = n_i \ \forall \ i \in \{1, 2, ..., k\}$. Let $A_1 = \{a_{11}, a_{12}, ..., a_{1n_1}\}$ be an orthonormal basis of V_1, $A_2 = \{a_{21}, a_{22}, ..., a_{2n_2}\}$ be an orthonormal basis of V_2, ..., $A_k = \{a_{k1}, a_{k2}, ..., a_{kn_k}\}$ be an orthonormal basis of A_k. Let $A_1 \otimes A_2 \otimes ... \otimes A_k = \{a_{1i_1} \otimes a_{2i_2} \otimes ... \otimes a_{ki_k} \mid 1 \leq i_1 \leq n_1, 1 \leq i_2 \leq n_2, ..., 1 \leq i_k \leq n_k\}$. From Theorem 2.4.4, we get that $A_1 \otimes A_2 ... \otimes A_k$ forms a basis of $\mathcal{L}(V_1 \times V_2 ... \times V_k \to \mathbb{R})$. Hence, $\forall \ u \in \mathcal{L}(V_1 \times V_2 \tilde{...} \times V_k \to \mathbb{R})$ we get that,

$$u = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} u(a_{1i_1}, a_{2i_2}, ..., a_{ki_k})[a_{1i_1} \otimes a_{2i_2} ... \otimes a_{ki_k}]$$

We define coordinates of the multi-linear function u as follows,

$$^{A_1 \otimes A_2 \otimes ... \otimes A_k} u[i_1, i_2, ..., i_k] = u(a_{1i_1}, a_{2i_2}, ..., a_{ki_k})$$

where $1 \leq i_1 \leq n_1, 1 \leq i_2 \leq n_2, ..., 1 \leq i_k \leq n_k$.

Note :

1. Note that column vector representation is used for $1-$tensors and matrix representation is used for $2-$tensors. Recall from the illustration 2.4.2 that for $k-$tensors such an analytic representation is not possible when $k \geq 3$. But, $k-$tensors can be represented as $k-$dimensional array computationally.

Theorem 2.4.6. Let V_1, V_2, ..., V_k be finite dimensional inner product spaces over field $\mathbb{R}$ where $dim(V_i) = n_i \ \forall \ i \in \{1, 2, ..., k\}$. Let $A_1 = \{a_{11}, ..., a_{1n_1}\}$ and $B_1 = \{b_{11}, ..., b_{1n_1}\}$ be any two orthonormal basis of V_1, let $A_2 = \{a_{21}, ..., a_{2n_2}\}$ and $B_2 = \{b_{21}, ..., b_{2n_2}\}$ be any two orthonormal basis of V_2, ..., let $A_k = \{a_{k1}, ..., a_{kn_k}\}$ and $B_k = \{b_{k1}, ..., b_{kn_k}\}$ be any two orthonormal basis of V_k. Let $A_1 \otimes A_2 \otimes ... \otimes A_k = \{a_{1i_1} \otimes a_{2i_2} \otimes ... \otimes a_{ki_k} \mid 1 \leq i_1 \leq n_1, 1 \leq i_2 \leq n_2, ..., 1 \leq i_k \leq n_k\}$ and $B_1 \otimes B_2 \otimes ... \otimes B_k = \{b_{1j_1} \otimes b_{2j_2} \otimes ... \otimes b_{kj_k} \mid 1 \leq j_1 \leq n_1, 1 \leq j_2 \leq n_2, ..., 1 \leq j_k \leq n_k\}$. From Theorem 2.3.4, we get that $A_1 \otimes A_2 \otimes ... \otimes A_k$ and $B_1 \otimes B_2 \otimes ... \otimes B_k$ form bases of $\mathcal{L}(V_1 \times V_2 \times ... \times V_k \to \mathbb{R})$. Let ${}^{i}M \in \mathbb{R}^{n_i \times n_i}$ be the transformation matrix from A_i to B_i i.e,

$$\begin{bmatrix} a_{i1} & a_{i2} & . & . & . & a_{in_i} \end{bmatrix} = \begin{bmatrix} b_{i1} & b_{i2} & . & . & . & b_{in_i} \end{bmatrix} {}^{i}M$$

where $1 \leq i \leq k$.Then, $\forall \ u \in \mathcal{L}(V_1 \times V_2 \times ... \times V_k \to \mathbb{R})$,

$$\boxed{{}^{B_1 \otimes B_2 \otimes ... \otimes B_k}u[j_1, j_2, ..., j_k] = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} {}^{1}M_{j_1 i_1} {}^{2}M_{j_2 i_2} ... {}^{k}M_{j_k i_k} {}^{A_1 \otimes A_2 \otimes ... \otimes A_k}u[i_1, i_2, ..., i_k]}$$

where $1 \leq j_1 \leq n_1, 1 \leq j_2 \leq n_2, ..., 1 \leq j_k \leq n_k$.

Proof. $\forall \ u \in \mathcal{L}(V_1 \times V_2 ... \times V_k \to \mathbb{R})$, since $A_1 \otimes A_2 \otimes ... \otimes A_k$, $B_1 \otimes B_2 \otimes ... \otimes B_k$ form bases of $\mathcal{L}(V_1 \times V_2 \times ... \times V_k \to \mathbb{R})$,

$$\begin{aligned} u &= \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} u(a_{1i_1}, a_{2i_2}, ..., a_{ki_k})[a_{1i_1} \otimes a_{2i_2} ... \otimes a_{ki_k}] \\ &= \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} ... \sum_{j_k=1}^{n_k} u(b_{1j_1}, b_{2j_2}, ..., b_{kj_k})[b_{1j_1} \otimes b_{2j_2} ... \otimes b_{kj_k}] \end{aligned}$$

Since ${}^{1}M$ is the transformation matrix from A_1 to B_1, we get that ${}^{1}M^*$ is the transformation from B_1 to A_1 i.e,

$$\begin{bmatrix} b_{11} & b_{12} & . & . & . & b_{1n_1} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1n_1} \end{bmatrix} {}^{1}M^*$$

$\forall\ j_1 \in \{1, 2, ..., n_1\}$,

$$b_{1j_1} = \sum_{i_1=1}^{n_1} {}^{1}M^*_{i_1 j_1} a_{1i_1} = \sum_{i_1=1}^{n_1} {}^{1}M_{j_1 i_1} a_{1i_1}$$

Since ${}^{2}M$ is the transformation matrix from A_2 to B_2, we get that ${}^{2}M^*$ is the transformation from B_2 to A_2 i.e,

$$\begin{bmatrix} b_{21} & b_{22} & . & . & . & b_{2n_2} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & . & . & . & a_{2n_2} \end{bmatrix} {}^{2}M^*$$

$\forall\ j_2 \in \{1, 2, ..., n_2\}$,

$$b_{2j_2} = \sum_{i_2=1}^{n_2} {}^{2}M^*_{i_2 j_2} a_{2i_2} = \sum_{i_2=1}^{n_2} {}^{2}M_{j_2 i_2} a_{2i_2}$$

$$...,$$

Since ${}^{k}M$ is the transformation matrix from A_k to B_k, we get that ${}^{k}M^*$ is the transformation from B_k to A_ki.e,

$$\begin{bmatrix} b_{k1} & b_{k2} & . & . & . & b_{kn_k} \end{bmatrix} = \begin{bmatrix} a_{k1} & a_{k2} & . & . & . & a_{kn_k} \end{bmatrix} {}^{k}M^*$$

$\forall\ j_k \in \{1, 2, ..., n_k\}$,

$$b_{kj_k} = \sum_{i_k=1}^{n_k} {}^{k}M^*_{i_k j_k} a_{ki_k} = \sum_{i_k=1}^{n_k} {}^{k}M_{j_k i_k} a_{ki_k}$$

Since u is multi-linear, we get

$$\begin{aligned} u(b_{1j_1}, b_{2j_2}, ..., b_{kj_k}) &= u(\sum_{i_1=1}^{n_1} {}^1M_{j_1i_1}a_{1i_1}, \sum_{i_2=1}^{n_2} {}^2M_{j_2i_2}a_{2i_2}, ..., \sum_{i_k=1}^{n_k} {}^kM_{j_ki_k}a_{ki_k}) \\ &= \sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} M_{j_1i_1}M_{j_2i_2}...M_{j_ki_k}u(a_{1i_1}, a_{2i_2}, ..., a_{ki_k}) \end{aligned}$$

$$\implies {}^{B_1\otimes B_2\otimes ...\otimes B_k}u[j_1, j_2, ..., j_k] = \sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} M_{j_1i_1}M_{j_2i_2}...M_{j_ki_k}{}^{A_1\otimes A_2\otimes ...\otimes A_k}u[i_1, i_2, ..., i_k]$$

$\square$

Illustration :

Consider $V_i = \mathbb{R}^2$ over $\mathbb{R}$, $\forall\ i \in \{1, 2, ..., k\}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^*, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^*\}$. It is easy to see that A form orthonormal basis of each $V_i = \mathbb{R}^2$. Let $B = \{b_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^*, b_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^*\}$. It is easy to see that B is the standard orthonormal bases of $\mathbb{R}^2$.
$\forall\ i_1 \in \{1, 2\}$, $\forall\ i_2 \in \{1, 2\}$, ..., $\forall\ i_k \in \{1, 2\}$, $\forall\ x_1, x_2, ..., x_k \in \mathbb{R}^2$,

$$\begin{aligned} [a_{i_1} \otimes a_{i_2} ... \otimes a_{i_k}](x_1, x_2, ..., x_k) &= (a_{i_1}, x_1)(a_{i_2}, x_2)...(a_{i_k}, x_k) \\ &= ({}^Aa_{i_1} \odot {}^Ax_1) \cdot ({}^Aa_{i_2} \odot {}^Ax_2) \cdot ... \cdot ({}^Aa_{i_k} \odot {}^Ax_k) \\ &= {}^Ax_1[i_1] \cdot {}^Ax_2[i_2] \cdot ... \cdot {}^Ax_k[i_k] \end{aligned}$$

Let $A \otimes A \otimes ... \otimes A(k \text{ times}) = \otimes_{i=1}^k A = \{a_{i_1} \otimes a_{i_2} ... \otimes a_{i_k} \mid 1 \le i_1 \le 2, 1 \le i_2 \le 2, ..., 1 \le i_k \le 2\}$.
$\forall\ j_1 \in \{1, 2\}$, $\forall\ j_2 \in \{1, 2\}$, ..., $\forall\ j_k \in \{1, 2\}$, $\forall\ x_1, x_2, ..., x_k \in \mathbb{R}^2$,

$$\begin{aligned} [b_{j_1} \otimes b_{j_2} ... \otimes b_{j_k}](x_1, x_2, ..., x_k) &= (b_{j_1}, x_1)(b_{j_2}, x_2)...(b_{j_k}, x_k) \\ &= ({}^Bb_{j_1} \odot {}^Bx_1) \cdot ({}^Bb_{j_2} \odot {}^Bx_2) \cdot ... \cdot ({}^Bb_{j_k} \odot {}^Bx_k) \\ &= {}^Bx_1[j_1] \cdot {}^Bx_2[j_2] \cdot ... \cdot {}^Bx_k[j_k] \end{aligned}$$

Let $B \otimes B \otimes ... \otimes B(k \text{ times}) = \otimes_{i=1}^k B = \{b_{j_1} \otimes b_{j_2} ... \otimes b_{j_k} \mid 1 \le j_1 \le 2, 1 \le j_2 \le 2, ..., 1 \le j_k \le 2\}$. From Theorem 2.4.4, we get that $\otimes_{i=1}^k A$ and $\otimes_{j=1}^k B$ form bases of $\mathcal{L}((\mathbb{R}^2)^k \to \mathbb{R})$.

Computing the basis transformation matrix from basis A to B :

$$a_1 = \frac{1}{\sqrt{2}}b_1 - \frac{1}{\sqrt{2}}b_2 \qquad a_2 = \frac{1}{\sqrt{2}}b_1 + \frac{1}{\sqrt{2}}b_2$$

$$\begin{bmatrix} a_1 & a_2 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \implies M = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

We get that M is the transformation matrix from basis A to B of $\mathbb{R}^2$ which implies that M^* is the transformation matrix from basis B to A i.e,

$$\begin{bmatrix} b_1 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\implies b_1 = \frac{1}{\sqrt{2}}a_1 + \frac{1}{\sqrt{2}}a_2 \qquad\qquad b_2 = -\frac{1}{\sqrt{2}}a_1 + \frac{1}{\sqrt{2}}a_2$$

It is easy to notice that each ${}^{i}M = M$ since each $V_i = \mathbb{R}^2$.
$\forall\ u \in \mathcal{L}((\mathbb{R}^2)^k \to \mathbb{R})$, since u is multi-linear, $\forall\ j_1 \in \{1,2\}$, $\forall\ j_2 \in \{1,2\}$, ..., $\forall$ $j_k \in \{1,2\}$ we get

$$\begin{aligned} u(b_{1j_1}, b_{2j_2}, ..., b_{kj_k}) &= u(\sum_{i=1}^{2} M^*_{i_1j_1} a_{i_1}, \sum_{i_2=1}^{2} M^*_{i_2j_2} a_{i_2}, ..., \sum_{i_k=1}^{2} M^*_{i_kj_k} a_{i_k}) \\ &= \sum_{i_1=1}^{2}\sum_{i_2=1}^{2} ... \sum_{i_k=1}^{2} M_{j_1i_1} M_{j_2i_2} ... M_{j_ki_k} u(a_{i_1}, a_{i_2}, ..., a_{i_k}) \end{aligned}$$

$$\implies {}^{B_1\otimes B_2\otimes ...\otimes B_k}u[j_1, j_2, ..., j_k] = \sum_{i_1=1}^{2}\sum_{i_2=1}^{2} ... \sum_{i_k=1}^{2} M_{j_1i_1} M_{j_2i_2} ... M_{j_ki_k} {}^{A_1\otimes A_2\otimes ...\otimes A_k}u[i_1, i_2, ..., i_k]$$

2.4.5 Invariance of computation of $k-$tensor under any orthonormal basis transformations

Theorem 2.4.7. Let V_1, V_2, ..., V_k be finite dimensional inner product spaces over field $\mathbb{R}$ where $dim(V_i) = n_i$ $\forall\ i \in \{1,2,...,k\}$. Let $A_1 = \{a_{11}, ..., a_{1n_1}\}$ be any orthonormal basis of V_1, let $A_2 = \{a_{21}, ..., a_{2n_2}\}$ be any orthonormal basis of V_2, ..., let $A_k = \{a_{k1}, ..., a_{kn_k}\}$ be any orthonormal basis of V_k. Let $A_1 \otimes A_2 \otimes ... \otimes A_k = \{a_{1i_1} \otimes a_{2i_2} \otimes ... \otimes a_{ki_k} \mid 1 \le i_1 \le n_1, 1 \le i_2 \le n_2, ..., 1 \le i_k \le n_k\}$. From Theorem

2.4.4, we get that $A_1 \otimes A_2 \otimes ... \otimes A_k$ form basis of $\mathcal{L}(V_1 \times V_2 \times ... \times V_k \to \mathbb{R})$, $\forall$ $x_1 \in V_1$, $\forall$ $x_2 \in V_2$, ..., $\forall$ $x_k \in V_k$,

$$\boxed{u(x_1, x_2, ..., x_k) = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} {}^{A_1 \otimes A_2 \otimes ... \otimes A_k} u[i_1, i_2, ..., i_k]^{A_1} x_1[i_1]^{A_2} x_2[i_2] ...^{A_k} x_k[i_k]}$$

Proof. $\forall$ $x_1 \in V_1$, since A_1 is a basis of V_1, there exist unique $\alpha_{1i_1} \in \mathbb{R}$, such that

$$x_1 = \sum_{i_1=1}^{n_1} \alpha_{1i_1} a_{1i_1}$$

$\forall$ $x_2 \in V_2$, since A_2 is a basis of V_2, there exist unique $\alpha_{2i_2} \in \mathbb{R}$, such that

$$x_2 = \sum_{i_2=1}^{n_2} \alpha_{2i_2} a_{2i_2}$$

$$...,$$

$\forall$ $x_k \in V_k$, since A_k is a basis of V_k, there exist unique $\alpha_{ki_k} \in \mathbb{R}$, such that

$$x_k = \sum_{i_k=1}^{n_k} \alpha_{ki_k} a_{ki_k}$$

$\forall$ $u \in \mathcal{L}(V_1 \times V_2 \times ... \times V_k \to \mathbb{R})$ we get,

$$\begin{aligned} u(x_1, x_2, ..., x_k) &= u(\sum_{i_1=1}^{n_1} \alpha_{1i_1} a_{1i_1}, \sum_{i_2=1}^{n_2} \alpha_{2i_2} a_{2i_2}, ..., \sum_{i_k=1}^{n_k} \alpha_{ki_k} a_{ki_k}) \\ &= \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} \alpha_{1i_1} \alpha_{2i_2} ... \alpha_{ki_k} u(a_{1i_1}, a_{2i_2}, ..., a_{ki_k}) \end{aligned}$$

Since A_1, A_2, ..., A_k are orthonormal bases of V_1, V_2, ..., V_k respectively, we get

$$\alpha_{1i_1} = {}^{A_1} x_1[i_1] \qquad \alpha_{2i_2} = {}^{A_2} x_2[i_2] \qquad ... \qquad \alpha_{ki_k} = {}^{A_k} x_k[i_k]$$

Since $A_1 \otimes A_2 \otimes ... \otimes A_k$ forms basis of $\mathcal{L}(V_1 \times V_2 \times ... \times V_k \to \mathbb{R})$, we get

$$u(a_{1i_1}, a_{2i_2}, ..., a_{ki_k}) = {}^{A_1 \otimes A_2 ... \otimes A_k} u[i_1][i_2]...[i_k]$$

$$\implies u(x_1, x_2, ..., x_k) = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} {}^{A_1 \otimes A_2 ... \otimes A_k} u[i_1, i_2, ..., i_k] {}^{A_1} x_1[i_1] {}^{A_2} x_2[i_2] ... {}^{A_k} x_k[i_k]$$

$\square$

Remark :

1. Let V_1, V_2, ..., V_k be finite dimensional inner product spaces over field $\mathbb{R}$ where $dim(V_i) = n_i \ \forall \ i \in \{1, 2, ..., k\}$. Let $A_1 = \{a_{11}, ..., a_{1n_1}\}$ and $B_1 = \{b_{11}, ..., b_{1n_1}\}$ be any two orthonormal basis of V_1, let $A_2 = \{a_{21}, ..., a_{2n_2}\}$ and $B_2 = \{b_{21}, ..., b_{2n_2}\}$ be any two orthonormal basis of V_2, ..., let $A_k = \{a_{k1}, ..., a_{kn_k}\}$ and $B_k = \{b_{k1}, ..., b_{kn_k}\}$ be any two orthonormal basis of V_k. Let $A_1 \otimes A_2 \otimes ... \otimes A_k = \{a_{1i_1} \otimes a_{2i_2} \otimes ... \otimes a_{ki_k} \mid 1 \leq i_1 \leq n_1, 1 \leq i_2 \leq n_2, ..., 1 \leq i_k \leq n_k\}$ and $B_1 \otimes B_2 \otimes ... \otimes B_k = \{b_{1j_1} \otimes b_{2j_2} \otimes ... \otimes b_{kj_k} \mid 1 \leq j_1 \leq n_1, 1 \leq j_2 \leq n_2, ..., 1 \leq j_k \leq n_k\}$. From Theorem 2.3.4, we get that $A_1 \otimes A_2 \otimes ... \otimes A_k$ and $B_1 \otimes B_2 \otimes ... \otimes B_k$ form bases of $\mathcal{L}(V_1 \times V_2 \times ... \times V_k \to \mathbb{R})$, $\forall \ x_1 \in V_1$, $\forall \ x_2 \in V_2$, ..., $\forall \ x_k \in V_k$, we get

$$\begin{aligned} u(x_1, x_2, ..., x_k) &= \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} {}^{A_1 \otimes A_2 \otimes ... \otimes A_k} u[i_1, i_2, ..., i_k] {}^{A_1} x_1[i_1] {}^{A_2} x_2[i_2] ... {}^{A_k} x_k[i_k] \\ &= \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} ... \sum_{j_k=1}^{n_k} {}^{B_1 \otimes B_2 \otimes ... \otimes B_k} u[j_1, j_2, ..., j_k] {}^{B_1} x_1[j_1] {}^{B_2} x_2[j_2] ... {}^{B_k} x_k[j_k] \end{aligned}$$

2. It is easy to observe that $\forall \ (x_1, x_2, ..., x_k) \in V_1 \times V_2 \times ... \times V_k$, $u(x_1, x_2, ..., x_k)$ can be completely determined by the coordinates of u i.e, ${}^{A_1 \otimes A_2 \otimes ... \otimes A_k} u[i_1, i_2, ..., i_k]$. Hence, if we fix the computations with respect to orthonormal bases A_1, A_2, ..., A_k of V_1, V_2, ..., V_k respectively, we can identify u with its coordinates ${}^{A_1 \otimes A_2 \otimes .... A_k} u[i_1, i_2, ..., i_k]$.

3. Let $dim(V_i) = n_i$ then, $\forall \ u \in \mathcal{L}(V_1 \times V_2 \times ... \times V_k \to \mathbb{R})$,

$$ {}^{A_1 \otimes A_2 \otimes ... \otimes A_k} u[i_1, i_2, ..., i_k] = u(a_{1i_1}, a_{2i_2}, ..., a_{ki_k})$$

   Hence, it is easy to show that the $k-$fold tensor product space, $\mathcal{L}(V_1 \times V_2 \times ... \times V_k \to \mathbb{R})$ is isomorphic to $\mathbb{R}^{n_1 \times n_2 \times ... \times n_k}$. (It is straight-forward to verify and is left to the reader. For the proof technique you may refer Lemma 1.3.9)

Illustration :

Consider $V_i = \mathbb{R}^2$ over $\mathbb{R}$, $\forall\ i \in \{1, 2, ..., k\}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^*, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^*\}$. It is easy to see that A form orthonormal basis of each $V_i = \mathbb{R}^2$. Let $B = \{b_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^*, b_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^*\}$. It is easy to see that B is the standard orthonormal bases of $\mathbb{R}^2$.

$\forall\ i_1 \in \{1, 2\}$, $\forall\ i_2 \in \{1, 2\}$, ..., $\forall\ i_k \in \{1, 2\}$, $\forall\ x_1, x_2, ..., x_k \in \mathbb{R}^2$,

$$\begin{aligned}
[a_{i_1} \otimes a_{i_2} \otimes ... \otimes a_{i_k}](x_1, x_2, ..., x_k) &= (a_{i_1}, x_1)(a_{i_2}, x_2)...(a_{i_k}, x_k) \\
&= ({}^{A}a_{i_1} \odot {}^{A}x_1) \cdot ({}^{A}a_{i_2} \odot {}^{A}x_2) \cdot ... \cdot ({}^{A}a_{i_k} \odot {}^{A}x_k) \\
&= {}^{A}x_1[i_1] \cdot {}^{A}x_2[i_2] \cdot ... \cdot {}^{A}x_k[i_k]
\end{aligned}$$

Let $A \otimes A \otimes ... \otimes A (k \text{ times}) = \otimes_{i=1}^{k} A = \{a_{i_1} \otimes a_{i_2} \otimes ... \otimes a_{i_k} \mid 1 \leq i_1 \leq 2, 1 \leq i_2 \leq 2, ..., 1 \leq i_k \leq 2\}$. $\forall\ j_1 \in \{1, 2\}$, $\forall\ j_2 \in \{1, 2\}$, ..., $\forall\ j_k \in \{1, 2\}$, $\forall\ x_1, x_2, ..., x_k \in \mathbb{R}^2$,

$$\begin{aligned}
[b_{j_1} \otimes b_{j_2} \otimes ... \otimes b_{j_k}](x_1, x_2, ..., x_k) &= (b_{j_1}, x_1)(b_{j_2}, x_2)...(b_{j_k}, x_k) \\
&= ({}^{B}b_{j_1} \odot {}^{B}x_1) \cdot ({}^{B}b_{j_2} \odot {}^{B}x_2) \cdot ... \cdot ({}^{B}b_{j_k} \odot {}^{B}x_k) \\
&= {}^{B}x_1[j_1] \cdot {}^{B}x_2[j_2] \cdot ... \cdot {}^{B}x_k[j_k]
\end{aligned}$$

Let $B \otimes B \otimes ... \otimes B (k \text{ times}) = \otimes_{i=1}^{k} B = \{b_{j_1} \otimes b_{j_2} \otimes ... \otimes b_{j_k} \mid 1 \leq j_1 \leq 2, 1 \leq j_2 \leq 2, ..., 1 \leq j_k \leq 2\}$. From Theorem 2.4.4, we get that $\otimes_{i=1}^{k} A$ and $\otimes_{j=1}^{k} B$ form bases of $\mathcal{L}((\mathbb{R}^2)^k \to \mathbb{R})$. In the previous illustration, we have already shown that the basis transformation matrix M from A to D is,

$$M = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$\forall\ x \in \mathbb{R}^2$,

$$^{B}x = M \cdot {}^{A}x$$

In this illustration, we will show that $\forall\ u_1, u_2, ..., u_k \in \mathbb{R}^2$, $[u_1 \otimes u_2 ... \otimes u_k]$ has the same value irrespective of the choice of orthonormal basis and this is sufficient to claim that any $v \in \mathcal{L}((\mathbb{R}^2)^k \to \mathbb{R})$ has the same value irrespective of the choice of orthonormal basis. Since any $t \in \mathcal{L}((\mathbb{R}^2)^k \to \mathbb{R})$ can be written as a linear combination of tensor products in $\otimes_{i=1}^k A$.

$$
\begin{aligned}
[u_1 \otimes u_2 ... \otimes u_k](x_1, x_2, ..., x_k) &= (u_1, x_1)(u_2, x_2)...(u_k, x_k) \\
&= ({}^B u_1 \odot {}^B x_1) \cdot ({}^B u_2 \odot {}^B x_2) \cdot ... \cdot ({}^B u_k \odot {}^B x_k) \\
&= \prod_{i=1}^k ({}^B u_i \odot {}^B x_i) = \prod_{i=1}^k ((M \cdot {}^A u_i)^* \cdot M \cdot {}^A x_i) \\
&= \prod_{i=1}^k (({}^A u_i)^* \cdot {}^A x_i) = \prod_{i=1}^k ({}^A u_i \odot {}^A x_i)
\end{aligned}
$$

Chapter 3

Exterior Product Spaces

3.1 Prerequisites - Permutations

Definition : Let $S = \{1, 2, ..., n\}$ be the set of first n natural numbers. A function $\sigma : S \to S$ is called a permutation if and only if σ is a bijection.

Note :
Note that if $S = \{a_1, a_2, ..., a_n\}$ is an arbitrary set of n integers then we can identify each a_i with i. Hence, the following theory can be extended to arbitrary set of integers.

Lemma 3.1.1. Let $S = \{1, 2, ..., n\}$,

1. If $\sigma : S \to S$ and $\rho : S \to S$ are permutations, then $\sigma \circ \rho$ is also a permutation where $\circ$ denotes the composition of functions σ and ρ.
2. If $\sigma : S \to S$ is a permutation then inverse of σ exists and is also a permutation.

Proof. 1. To prove $\sigma \circ \rho$ is an injection, we need to show that $\forall\ x, y \in S$, if $\sigma \circ \rho(x) = \sigma \circ \rho(y)$ then $x = y$. $\forall\ x, y \in S$, if

$$\sigma \circ \rho(x) = \sigma \circ \rho(y) \implies \rho(x) = \rho(y) \text{ since } \sigma \text{ is an injection}$$

$$\implies x = y \text{ since } \rho \text{ is an injection}$$

To prove $\sigma \circ \rho$ is a surjection, we need to show that $\forall\ y \in S$, there exist $x \in S$ such that $\sigma \circ \rho(x) = y$. $\forall\ y \in S$, since σ is a surjection, there exists $x \in S$ such that,

$$\sigma(x) = y$$

Since ρ is a surjection, there exists $z \in S$ such that,

$$\rho(z) = x \implies \sigma \circ \rho(z) = y$$

2. Since σ is a bijection, the inverse of σ exists, let it be σ^{-1}. Note that $\forall$ $x \in S$, $\sigma^{-1} \circ \sigma(x) = \sigma \circ \sigma^{-1}(x) = x$. $\forall\ x, y \in S$,

$$\sigma^{-1}(x) = \sigma^{-1}(y) \implies \sigma \circ \sigma^{-1}(x) = \sigma \circ \sigma^{-1}(y) \implies x = y$$

$\forall\ y \in S$, let $\sigma(y) = x$ then,

$$\sigma^{-1} \circ \sigma(y) = \sigma^{-1}(x) \implies \sigma^{-1}(x) = y$$

$$\sigma^{-1} : S \to S \text{ is a bijection.}$$

$\square$

Definition 3.1.1. Let $S = \{1, 2, ..., n\}$. Then, we define P_n to be the set of all permutations from $S \to S$ i.e, the set of all bijective functions from S to S.

Remark :

1. Recall from basic combinatorics that the number of ways to place n distinct balls into n distinct bins in such a way that each bin gets exactly one ball is $n!$ where $n \geq 1$. Here we can use the same argument to conclude that $|P_n| = n!$

2. We use $id : S \to S$ to denote the identity permutation i.e, $\forall\ x \in S$, $id(x) = x$. Verify that $\forall\ \sigma \in P_n$, $id \circ \sigma = \sigma \circ id = \sigma$

3. From above Lemma, it is obvious that if $\sigma \in S$ then, $\sigma^{-1} \in S$. Moreover, $\sigma \circ \sigma^{-1} = \sigma^{-1} \circ \sigma = id$

4. We use the following matrix notation to denote a permutation $\sigma \in P_n$. Let $\sigma(1) = y_1$, $\sigma(2) = y_2$, ..., $\sigma(n) = y_n$. We represent σ as follows (just for convenience),
$$\sigma = \begin{bmatrix} 1 & 2 & . & . & n \\ y_1 & y_2 & . & . & y_n \end{bmatrix}$$

5. In general $\forall\ \sigma, \rho \in S$, $\sigma \circ \rho \neq \rho \circ \sigma$. For instance, let $n = 3$ and
$$\sigma = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} \qquad \rho = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$
Then,
$$\sigma \circ \rho = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} \qquad \rho \circ \sigma = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

Definition 3.1.2. Let $\sigma \in P_n$, we define σ to be a $k-$cycle if there exists $x_1, x_2, ..., x_k \in S$ such that

$$\sigma(x_1) = x_2 \quad \sigma(x_2) = x_3 \quad ... \quad \sigma(x_{k-1}) = x_k \quad \sigma(x_k) = x_1$$

$\forall\ y \notin \{x_1, x_2, ..., x_k\}$, $\sigma(y) = y$

Example : Let $n = 6$. If

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 3 & 5 & 6 & 2 \end{bmatrix}$$

It is easy to notice that σ is a $4-$cycle.

$$\sigma(2) = 4 \qquad \sigma(4) = 5 \qquad \sigma(5) = 6 \qquad \sigma(6) = 2$$

Definition 3.1.3. Let σ and ρ be an m and n cycle respectively. σ and ρ are called disjoint if $\forall\ x \in \{1, 2, ..., n\}$,

$$\sigma(x) \neq x \implies \rho(x) = x \tag{3.1}$$

Example : Let $n = 6$. If

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 3 & 5 & 6 & 2 \end{bmatrix} \qquad \rho = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 4 & 5 & 6 \end{bmatrix}$$

σ is a $4-$cycle and ρ is a $2-$cycle. Notice that whenever $\sigma(x) \neq x$, $\rho(x) = x$ which implies that σ and ρ are disjoint.

Remark :

1. $\forall\ \sigma, \rho \in P_n$ if σ and ρ are disjoint cycles then, $\forall\ x \in \{1, 2, ..., n\}$,

$$\rho \circ \sigma(x) = \sigma \circ \rho(x)$$

$$\text{if } \sigma(x) = y \neq x \implies \sigma(y) = z \neq y \text{ since } \sigma \text{ is an injection}$$

$$\implies \rho(x) = x \text{ and } \rho(y) = y \implies \sigma \circ \rho(x) = y = \rho \circ \sigma(x)$$

If $\sigma(x) = x$ then there are two cases, $\rho(x) = x$ which is straight-forward to verify. Let's look at what happens if $\rho(x) = y \neq x$

$$\implies \rho(y) = z \neq y \text{ since } \rho \text{ is also an injection}$$

$$\implies \sigma(y) = y \implies \sigma \circ \rho(x) = y = \rho \circ \sigma(x)$$

Theorem 3.1.2. $\forall\ \sigma \in P_n$, σ can be expressed as a composition of disjoint cycles. Moreover, this composition is unique except for the order in which we compose cycles. (Note : since cycles are disjoint order of composition doesn't matter).

Proof. Proof is by induction on n.

Base Case $(n = 1)$ **:** For $n = 1$, $|P_n| = 1$ i.e, $P_1 = \{\sigma = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$ and notice that σ can be expressed uniquely as a composition of $1-$ disjoint cycle.

Induction Hypothesis : Let all permutations of P_i be expressed uniquely as a composition of disjoint cycles where $1 \leq i \leq n-1$. It is enough to show that the result holds for any $\sigma \in P_n$. Now, let's define the following recursive formula,

$$a_{i+1} = \sigma(a_i)$$

where the initialization case is $a_1 = n$. Let $A = \{a_i \mid i \in \mathbb{N}\}$. Since $S = \{1, 2, ..., n\}$ is finite, we must have some repetitions such that $a_i = a_j$, without loss of generality assume $i < j$. Now, pick the smallest such pair let it be (i_0, j_0) (samllest in the sense of dictionary order). Then,
Claim : $i_0 = 1$
If not then $i_0 > 1$,

$$a_{i_0} = a_{j_0} \implies \sigma(a_{j_0-1}) = a_{j_0} = \sigma(a_{i_0-1}) \implies a_{j_0-1} = a_{i_0-1}$$

This contradicts our choice that (i_0, j_0) is the smallest pair such that $a_{i_0} = a_{j_0}$. Hence, $i_0 = 1$.
Let ρ be the k cycle obtained using the above recursive formula,

$$\rho(a_1 = n) = a_2 \qquad \rho(a_2) = a_3 \qquad ... \qquad \rho(a_k) = a_1 = n$$

Notice that $\rho^{-1} \circ \sigma$ fixes all the elements in A to their respective positions i.e, $\rho^{-1} \circ \sigma(a_i) = a_i$ and if we limit the permutation $\rho^{-1} \circ \sigma$ to the first $\{1, 2, ..., n-1\}$ elements. Using induction hypothesis we get that $\rho^{-1} \circ \sigma$ can be written uniquely as a composition of disjoint cycles where each cycle(τ)'s definition shall be extended at n as $\tau(n) = n$ i.e,

$$\rho^{-1} \circ \sigma = \tau_1 \circ \tau_2 \circ ... \circ \tau_k$$

where τ_i's are disjoint cycles and $\tau_i(n) = n \ \forall \ i$.

$$\implies \sigma = \rho \circ \tau_1 \circ \tau_2 \circ ... \circ \tau_k$$

Recall that $\rho^{-1} \circ \sigma$ fixes all elements of set A. Hence, $\tau_i(a_j) = a_j \ \forall \ i \in \{1, 2, ..., k\}$, $\forall \ a_j \in A$. This implies that ρ and each τ_i are disjoint.
Let's now try to prove uniqueness. Let $\sigma = \rho_1 \circ \rho_2 \circ ... \circ \rho_m = \tau_1 \circ \tau_2 \circ ... \circ \tau_n$ be any two compositions of σ. Without loss of generality, assume $\sigma(n) = x \neq n$, then there exist unique cycles ρ_i and τ_j such that $\rho_i(n) = \tau_j(n) = x$. From the disjointness of cycles, we must have

$$\rho_i = \tau_j$$

Note that $\rho_i^{-1} \circ \sigma$ and $\tau_j^{-1} \circ \sigma$ fixes the n'th element. From the induction hypothesis, we get that $\rho_i^{-1} \circ \sigma = \tau_j \circ \sigma$ can be expressed uniquely as a composition of disjoint cycles which concludes the proof. □

Definition 3.1.4. A $2-$cycle $\sigma \in P_n$ is defined as a transposition i.e, if there exist $x, y \in \{1, 2, ..., n\}$ such that $x < y$ and

$$\sigma(x) = y \qquad\qquad \sigma(y) = x$$

Moreover, a transposition is called an adjacent transposition if $y = x + 1$. We use the following notation to define a transposition more concisely, $\sigma = (x_1, x_2)$ where σ is parameterized with elements x_1, x_2 it exchanges.

Lemma 3.1.3. $\forall\, \sigma \in P_n$, if σ is a cycle then σ can be expressed as a composition of transpositions.

Proof. $\forall\, \sigma \in P_n$, since σ is a cycle, there exist $x_1, x_2, ..., x_k \in S$ such that,

$$\sigma(x_1) = x_2 \qquad \sigma(x_2) = x_3 \qquad ... \qquad \sigma(x_k) = x_1$$

Now, consider following transpositions,

$$\tau_1 = (x_1, x_2) \qquad \tau_2 = (x_2, x_3) \qquad ... \qquad \tau_{k-1}(x_{k-1}, x_k)$$

It is easy to verify that,

$$\sigma = \tau_{k-1} \circ \tau_{k-2} \circ ... \circ \tau_1$$

□

Remark :

1. Since each cycle can be written as a composition of transpositions. From Theorem 3.1.2 and Lemma 3.1.3 we get that, $\forall\, \sigma \in P_n$, σ can be expressed as a composition of transpositions.

2. Notice that $\tau^{-1} = \tau$, since
$$\tau \circ \tau = id$$

Definition 3.1.5. Let $\sigma \in P_n$. There is an inversion of σ between i and j if $i < j$ and $\sigma(i) > \sigma(j)$. Let $N(\sigma)$ denote the set of all inversions of σ. Note that given σ, $N(\sigma)$ can be determined uniquely. $|N(\sigma)|$ is used to denote the inversion number.

Remark :

1. $\forall\ \sigma \in P_n$,
$$0 \leq |N(\sigma)| \leq \frac{n(n-1)}{2}$$
Note that identity permutation (id) has minimum number of inversions and the descending order permutation (i.e, $\sigma(1) = n, \sigma(2) = n-2, ..., \sigma(n-1) = 2, \sigma(n) = 1$) has maximum number of inversions.

Lemma 3.1.4. Every transposition τ can be written as a composition of odd number of adjacent transpositions only. (as a composition of even number of adjacent transpositions is impossible)

Proof. Let τ be any transposition. Then, there exist $x, y \in \{1, 2, ..., n\}$ where $x < y$ and
$$\tau = (x, y)$$
Now, consider the following adjacent transpositions,
$$\tau_1 = (x, x+1) \qquad \tau_2 = (x+1, x+2) \qquad ... \qquad \tau_k = (y-1, y)$$
$$\rho_1 = (y-1, y-2) \qquad \rho_2 = (y-2, y-3) \qquad ... \qquad \rho_{k-1} = (x+1, x)$$
Now it can be easily verified that,
$$\sigma = \rho_{k-1} \circ \rho_{k-2} \circ ... \circ \rho_1 \circ \tau_k \circ \tau_{k-1} \circ ... \circ \tau_1$$
Note that $k + k - 1$ is odd. To prove that a transposition must be a composition of odd number of adjacent transpositions. Notice that any transposition has odd number of inversions $\implies$ $(-1)^{|N(\sigma)|} = -1$. Also, observe that for any permutation π and an adjacent transposition τ, $\pi \circ \tau$ has either one inversion more than π or one inversion less than π. Let
$$\tau = \rho_1 \circ \rho_2 \circ ... \circ \rho_m$$

where each ρ_i is an adjacent transposition. Then from above observations we get,

$$(-1)^m = (-1)^{|N(\sigma)|} = -1$$

which implies that m must be odd. $\square$

Example :
To make the above Lemma concrete consider the transposition $\tau = (3, 7)$. Then, the composition of adjacent transpositions that gives τ is

$$(3, 7) = (4, 3) \circ (5, 4) \circ (6, 5) \circ (6, 7) \circ (5, 6) \circ (4, 5) \circ (3, 4)$$

Note that the composition of functions on an argument is evaluated from right to left.

Theorem 3.1.5. $\forall\ \sigma \in P_n$. Let $\sigma = \tau_1 \circ \tau_2 \circ ... \circ \tau_m$ be any composition of transpositions. Then, parity of m and $|N(\sigma)|$ is same i.e,

$$m \text{ is odd} \iff |N(\sigma)| \text{ is odd}$$

Proof. Given that $\sigma = \tau_1 \circ \tau_2 \circ ... \circ \tau_m$ be a composition of transpositions τ_i. From Lemma 3.1.4, we get that each transposition τ_i can be written as a composition of odd number of adjacent transpositions ρ_j's. Hence, there exist $\rho_1, \rho_2, ..., \rho_l$ such that each ρ_j is an adjacent transposition and

$$\sigma = \rho_1 \circ \rho_2 \circ ... \circ \rho_l \tag{3.2}$$

where each ρ_i is an adjacent transposition. Recall that for any permutation π and an adjacent transposition τ, $\pi \circ \tau$ has either one inversion more than π or one inversion less than π. This implies that for each adjacent transposition, parity of count of inversions must change and from equation 3.2 we get that the parity of l is same as the parity of $|N(\sigma)|$. Since $|N(\sigma)|$ is a fixed number we get that any such composition leads to the fact that the parity of l should be same as $|N(\sigma)|$. Also, notice that each τ_i must be expressed as a composition of odd number of adjacent transpositions ρ_j's which implies that if m is odd then, l is odd and parity of $|N(\sigma)|$ must be odd. If m is even then, l is even and parity of $|N(\sigma)|$ has to be even. $\square$

Definition 3.1.6. Let $\sigma \in P_n$ and $\sigma = \tau_1 \circ \tau_2 \circ ... \circ \tau_m$ be any composition of transpositions. Then, we define

$$sgn(\sigma) = (-1)^m$$

Note that $sgn(\sigma)$ is well-defined because from Theorem 3.1.5 we get that $(-1)^m = (-1)^{|N(\sigma)|}$ where $|N(\sigma)|$ is a fixed number.

Lemma 3.1.6. $\forall\ \sigma, \rho \in P_n$, $sgn(\sigma \circ \rho) = sgn(\sigma) \cdot sgn(\rho)$

Proof. Let σ be a composition of $k-$transpositions. Let $\sigma = \alpha_1 \circ \alpha_2 \circ ... \circ \alpha_k$ where each α_i is a transposition. Let ρ be a composition of $l-$transpositions. Let $\sigma = \beta_1 \circ \beta_2 \circ ... \circ \beta_k$ where each β_j is a transposition. Then,

$$\sigma \circ \rho = \alpha_1 \circ ... \circ \alpha_k \circ \beta_1 \circ ... \circ \beta_l$$

$$\implies sgn(\sigma \circ \rho) = (-1)^{k+l} = (-1)^k \cdot (-1)^l = sgn(\sigma) \cdot sgn(\rho)$$

$\square$

Remark :

1. We already observed that $id \circ id = id$,

$$sgn(id \circ id) = sgn(id) \cdot sgn(id) = sgn(id) \implies sgn(id) = 1$$

2. $\forall\ \sigma \in P_n$, $\sigma \circ \sigma^{-1} = id$

$$sgn(\sigma \circ \sigma^{-1}) = sgn(\sigma) \cdot sgn(\sigma^{-1}) = sgn(id) = 1 \implies sgn(\sigma) = sgn(\sigma^{-1})$$

3. For any transposition $\tau \in P_n$, $sgn(\tau) = -1$

Definition 3.1.7. Let $\tau = (x, y) \in P_n$ be any transposition. Then, we define

$$(x, y) \circ P_n = \{(x, y) \circ \sigma \mid \sigma \in P_n\} \text{ and } P_n \circ (x, y) = \{\sigma \circ (x, y) \mid \sigma \in P_n\}$$

Lemma 3.1.7.

$$(x, y) \circ P_n = P_n$$

Proof. Let $f : P_n \to (x, y) \circ P_n$ be defined as follows,

$$f(\sigma) = (x, y) \circ \sigma$$

It is enough to show that f is a bijection.

1. (One-One Condition) Let σ and $\rho \in P_n$ such that

$$f(\sigma) = f(\rho) \implies (x, y) \circ \sigma(z) = (x, y) \circ \rho(z) \ \forall \ z \in \{1, 2, ..., n\}$$

   Recall that $(y, x) \circ (x, y) = id$,

$$\implies \sigma(z) = \rho(z)$$

2. (Onto Condition) Straight-forward from the definition.

$\square$

Remark :

1. $\forall \ \sigma \in P_n$, there exist $\tau_1, \tau_2, ..., \tau_m$ such that,

$$\sigma = \tau_1 \circ ... \circ \tau_m \implies \sigma \circ P_n = P_n$$

2. Note that a similar proof can be given for $P_n \circ (x, y) = P_n$

$$\implies P_n \circ \sigma = P_n \ \forall \ \sigma \in P_n$$

3.2 $2-$Fold Exterior Product Spaces

3.2.1 Alternating $2-$Tensors

Definition 3.2.1. Let V be a finite dimensional inner product space over field $\mathbb{R}$. A tensor $u \in \mathcal{L}(V \times V \cong V^2 \to \mathbb{R})$ is called an alternating $2-$tensor if $\forall\ x, y \in V$

$$u(x, y) = -u(y, x)$$

Let $S = \{u \in \mathcal{L}(V^2 \to \mathbb{R}) \mid u \text{ is an alternating } 2-\text{tensor}\}$

Lemma 3.2.1. *S forms a subspace of $\mathcal{L}(V^2 \to \mathbb{R})$.*

Proof. We have already seen that $\mathcal{L}(V^2 \to \mathbb{R})$ is a vector space over R. To see that S is a subspace of $\mathcal{L}(V^2 \to \mathbb{R})$, it is enough to show the following,

1. $\forall\ u, v \in S$, $\forall\ x, y \in V$,

$$\begin{aligned}[u+v](x,y) &= u(x,y) + v(x,y) = -u(y,x) - v(y,x) \\ &= -[u+v](y,x) \implies [u+v] \in S\end{aligned}$$

2. $\forall\ u \in S$, $\forall\ \alpha \in \mathbb{R}$, $\forall\ x, y \in V$,

$$[\alpha u](x,y) = \alpha u(x,y) = -\alpha u(y,x) = -[\alpha u](y,x) \implies [\alpha u] \in S$$

3. It is easy to see that $0 \in S$

$\square$

A function $u \in S$ is called an **alternating** $2-$**tensor**. The subspace of all alternating $2-$tensors of V is denoted by $\mathcal{L}_{ALT}(V^2 \to \mathbb{R})$.

Remark :

1. Recall that P_2 is used to denote the set of all permutations of $\{1, 2\}$. Also, $\forall\ \sigma \in P_2$, $sgn(\sigma)$ is used to denote the sign of the permutation. It is easy to see that $P_2 = \{(1, 2), (2, 1)\}$.

2. $\forall\ u \in \mathcal{L}_{ALT}(V^2 \to \mathbb{R})\ \forall\ x, y \in V$ if $x = y$ then,

$$u(x,y) = -u(x,y) \implies u(x,y) = 0$$

Lemma 3.2.2. $\forall\ \sigma \in P_2$, $\forall\ u \in \mathcal{L}_{ALT}(V^2 \to \mathbb{R})$, $\forall\ x_1, x_2 \in V$,

$$u(x_{\sigma(1)}, x_{\sigma(2)}) = sgn(\sigma)u(x_1, x_2)$$

Proof. Since $P_2 = \{(1,2), (2,1)\}$,

If $\sigma = (1,2) = id$, then $sgn(\sigma) = +1$,

$$\implies u(x_{\sigma(1)}, x_{\sigma(2)}) = u(x_1, x_2) = sgn(\sigma)u(x_1, x_2)$$

If $\sigma = (2,1)$, then $sgn(\sigma) = -1$,

$$\implies u(x_{\sigma(1)}, x_{\sigma(2)}) = u(x_2, x_1) = -u(x_1, x_2) = sgn(\sigma)u(x_1, x_2)$$

□

3.2.2 2−fold Exterior Product on V

Definition 3.2.2. Let V be a finite dimensional inner product space over $\mathbb{R}$ where $dim(V) = n$. Then, $\forall\ u, v \in V$, we define the exterior product (also called wedge product) of u, v as a function $\wedge : V^2 \to \mathbb{R}$ as follows,

$$\boxed{u \wedge v = u \otimes v - v \otimes u}$$

Remark :

1. $\forall\ x, y \in V$,

$$[u \wedge v](x,y) = [u \otimes v](x,y) - [v \otimes u](x,y) = (u,x)(v,y) - (v,x)(u,y)$$

Lemma 3.2.3. Let V be a finite dimensional inner product space over $\mathbb{R}$. Then, $\forall\ u, v \in V$, $u \wedge v \in \mathcal{L}_{ALT}(V^2 \to \mathbb{R})$

Proof. $\forall\ x, y \in V$,

$$\begin{aligned}[u \wedge v](x,y) &= [u \otimes v](x,y) - [v \otimes u](x,y) = (u,x)(v,y) - (v,x)(u,y) \\ &= -((u,y)(v,x) - (v,y)(u,x)) = -([u \otimes v](y,x) - [v \otimes u](y,x)) \\ &= -[u \wedge v](y,x)\end{aligned}$$

$$\implies u \wedge v \in \mathcal{L}_{ALT}(V^2 \to \mathbb{R})$$

□

Lemma 3.2.4. Let V be a finite dimensional inner product space over $\mathbb{R}$. Then, $\forall\ u, v, w \in V$, $\forall\ \alpha \in \mathbb{R}$,

1. $$[u+v] \wedge w = u \wedge w + v \wedge w$$

2. $$u \wedge [v+w] = u \wedge v + u \wedge w$$

3. $$u \wedge [\alpha v] = [\alpha u] \wedge v = \alpha[u \wedge v]$$

Proof. 1.

$$\begin{aligned}[u+v] \wedge w &= [u+v] \otimes w - w \otimes [u+v] \\ &= u \otimes w + v \otimes w - w \otimes u - w \otimes v = u \wedge w + v \wedge w\end{aligned}$$

2.

$$\begin{aligned}u \wedge [v+w] &= u \otimes [v+w] - [v+w] \otimes u \\ &= u \otimes v + u \otimes w - v \otimes u - w \otimes u = u \wedge v + u \wedge w\end{aligned}$$

3.

$$[\alpha u] \wedge v = [\alpha u] \otimes v - v \otimes [\alpha u] = u \otimes [\alpha v] - [\alpha v] \otimes u = u \wedge [\alpha u]$$
$$= \alpha(u \otimes v - v \otimes u) = \alpha[u \wedge v]$$

□

Lemma 3.2.5. Let V be a finite dimensional inner product space over $\mathbb{R}$. Then, $\forall\ u, v \in V$,

$$u \wedge v = -v \wedge u$$

Proof.

$$u \wedge v = u \otimes v - v \otimes u = -(v \otimes u - u \otimes v) = -v \wedge u$$

□

Remark :

1. Let $v_1, v_2 \in V$, if $v_1 = v_2$ then,

$$v_1 \wedge v_2 = 0$$

Illustration :

Consider $V = \mathbb{R}^2$ over $\mathbb{R}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^*, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^*\}$. Recall that A is an orthonormal basis of $\mathbb{R}^2$.

Let $u = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

$\forall\ x, y \in \mathbb{R}^2$,

$$[u \wedge v](x, y) = [u \otimes v](x, y) - [v \otimes u](x, y) = (u, x)(v, y) - (v, x)(u, y)$$
$$= ({}^{A}x \odot {}^{A}u) \cdot ({}^{A}v \odot {}^{A}y) - ({}^{A}x \odot {}^{A}v) \cdot ({}^{A}u \odot {}^{A}y)$$
$$= ({}^{A}x)^* \cdot ({}^{A}u \cdot ({}^{A}v)^* - {}^{A}v \cdot ({}^{A}u)^*) \cdot {}^{B}y$$
$$= ({}^{A}x)^* \cdot (\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}) \cdot {}^{B}y = ({}^{A}x)^* \cdot \begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix} \cdot {}^{B}y$$

All the previously proved Lemmas can be concretely worked out using this illustration. Note that computing ${}^{A}u \cdot ({}^{A}v)^{*} - {}^{A}v \cdot ({}^{A}u)^{*}$ is sufficient to determine the action of $u \wedge v$ on any $(x, y) \in (\mathbb{R}^2)^2$.

3.2.3 Basis of 2-fold exterior product spaces

Let V be a finite dimensional inner product space over $\mathbb{R}$ where $dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ be an orthonormal basis of V. Then, $A \otimes A = \{a_i \otimes a_j \mid 1 \leq i, j \leq n\}$ is a basis of $\mathcal{L}(V^2 \to \mathbb{R})$. Let $A \wedge A = \{a_i \wedge a_j \mid 1 \leq i < j \leq n\}$. $\forall$ $i, j \in \{1, 2, ..., n\}$ such that $i < j$, from Lemma 3.2.3 we get that, $a_i \wedge a_j$ is an alternating $2-$tensor.

Theorem 3.2.6. $A \wedge A$ is a basis of $\mathcal{L}_{ALT}(V^2 \to \mathbb{R})$.

Proof. **Span:**
$\forall$ $u \in \mathcal{L}_{ALT}(V^2 \to \mathbb{R}) \implies u \in \mathcal{L}(V^2 \to \mathbb{R})$, since $A \otimes A$ is a basis of $\mathcal{L}(V^2 \to \mathbb{R})$ we get,

$$u = \sum_{i=1}^{n} \sum_{j=1}^{n} u(a_i, a_j)[a_i \otimes a_j]$$

If $a_i = a_j$ then $u(a_i, a_j) = 0$,

$$\implies u = \sum_{i=1}^{n} \sum_{j=i+1}^{n} u(a_i, a_j)[a_i \otimes a_j] + u(a_j, a_i)[a_j \otimes a_i]$$

$\forall$ $i, j \in \{1, ..., n\}$ where $i < j$, we have $u(a_i, a_j) = -u(a_j, a_i)$.

$$\implies u = \sum_{i=1}^{n} \sum_{j=i+1}^{n} u(a_i, a_j)[a_i \otimes a_j - a_j \otimes a_i] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} u(a_i, a_j)[a_i \wedge a_j]$$

$$\implies A \wedge A \text{ spans } \mathcal{L}_{ALT}(V^2 \to \mathbb{R})$$

Linear Independence :
Let $\alpha_{ij} \in \mathbb{R}$ $\forall$ $i, j \in \{1, 2, ..., n\}$ where $i < j$. Consider,

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \alpha_{ij}[a_i \wedge a_j] = 0$$

$$\implies \sum_{i=1}^{n}\sum_{j=i+1}^{n} \alpha_{ij}[a_i \otimes a_j - a_j \otimes a_i] = 0$$

$\forall\ p, q \in \{1, 2, ..., n\}$, where $p < q$,

$$\sum_{i=1}^{n}\sum_{j=i+1}^{n} \alpha_{ij}([a_i \otimes a_j](a_p, a_q) - [a_j \otimes a_i](a_p, a_q)) = 0$$

Since A is an orthonormal basis of V we get,

$$\sum_{i=1}^{n}\sum_{j=i+1}^{n} \alpha_{ij}((a_i, a_p)(a_j, a_q) - (a_j, a_p)(a_i, a_q)) = \alpha_{pq}(a_p, a_p)(a_q, a_q) - (a_q, a_p)(a_p, a_q) = 0$$

$$\implies \alpha_{pq} = 0$$

$$\implies A \wedge A \text{ is a linearly independent set and a basis of } \mathcal{L}_{ALT}(V^2 \to \mathbb{R})$$

□

Corollary 3.2.7.

$$\boxed{dim(\mathcal{L}_{ALT}(V^2 \to \mathbb{R})) = \binom{n}{2}}$$

Illustration :

Consider $V = \mathbb{R}^2$ over $\mathbb{R}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^*, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^*\}$. Recall that A is an orthonormal basis of $\mathbb{R}^2$. $\forall\ i, j \in \{1, 2\}$, $\forall\ x, y \in \mathbb{R}^2$,

$$\begin{aligned} [a_i \wedge a_j](x, y) &= [a_i \otimes a_j](x, y) - [a_j \otimes a_i](x, y) = (a_i, x)(a_j, y) - (a_j, x)(a_i, y) \\ &= ({}^{A}a_i \odot {}^{A}x) \cdot ({}^{A}a_j \odot {}^{A}y) - ({}^{A}a_j \odot {}^{A}x) \cdot ({}^{A}a_i \odot {}^{A}y) \\ &= {}^{A}x[i] \cdot {}^{B}y[j] - {}^{A}x[j]{}^{A}y[i] \end{aligned}$$

Let $A \wedge A = \{a_i \wedge a_j \mid 1 \leq i < j \leq 2\}$. From the above Theorem, we get that $A \wedge A$ is a basis of $\mathcal{L}_{ALT}((\mathbb{R}^2)^2 \to \mathbb{R})$. Consider the following $2-$tensor,

$$u(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}) = x_1 y_2 - x_2 y_1 \qquad \text{where } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$$

In chapter 4, it will be shown that u is a $2-$dimensional notion of determinant. It is straightforward to verify that u is an alternating $2-$tensor. Since A is a basis of $\mathbb{R}^2$, $\forall \begin{bmatrix} x_1 & x_2 \end{bmatrix}^*, \begin{bmatrix} y_1 & y_2 \end{bmatrix}^* \in \mathbb{R}^2$, there exist unique $\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2 \in \mathbb{R}$, such that

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix}^* = \alpha_1 a_1 + \alpha_2 a_2$$

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix}^* = \beta_1 a_1 + \beta_2 a_2$$

Since u is an alternating $2-$tensor we get that

$$u(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}) = u(\sum_{i=1}^{2} \alpha_i a_i, \sum_{j=1}^{2} \beta_j a_j) = (\alpha_1\beta_2 - \alpha_2\beta_1)u(a_1, a_2)$$

From the above expression, it is quite clear that computing $u(a_1, a_2)$ is sufficient to determine the action of u on any $\begin{bmatrix} x_1 & x_2 \end{bmatrix}^*, \begin{bmatrix} y_1 & y_2 \end{bmatrix}^*$.

$$u(a_1, a_2) = -1$$

$$\implies u(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}) = -(\alpha_1\beta_2 - \alpha_2\beta_1) = -({}^{A}x[1]^{A}y[2] - {}^{A}x[2]^{A}y[1])$$
$$= u(a_1, a_2)[a_1 \wedge a_2](x, y)$$

With this illustration, you can observe how $A \wedge A$ works as a basis of $\mathcal{L}_{ALT}(\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R})$ and also note that the value of $u(a_1, a_2)$ is sufficient to compute u on any $\begin{bmatrix} x_1 & x_2 \end{bmatrix}^*, \begin{bmatrix} y_1 & y_2 \end{bmatrix}^* \in \mathbb{R}^2$.

3.2.4 Basis Transformation and Invariance of computation of alternating $2-$ tensor under orthonormal basis transformations

Definition 3.2.3. $\forall\ u \in \mathcal{L}_{ALT}(V^2 \to \mathbb{R})$, from Theorem 3.2.6 we get,

$$u = \sum_{i=1}^{n_1} \sum_{j=i+1}^{n_2} u(a_i, a_j)[a_i \wedge a_j]$$

$$\text{Define },\ {}^{A\wedge A}u = \begin{bmatrix} 0 & u(a_1,a_2) & . \ . \ . & u(a_1,a_n) \\ -u(a_1,a_2) & 0 & . \ . \ . & u(a_2,a_n) \\ . & . & . \ . \ . \ . & . \\ -u(a_1,a_n) & -u(a_2,a_n) & . \ . \ . & 0 \end{bmatrix}$$

Corollary 3.2.8. Let $A = \{a_1, ..., a_n\}$ and $B = \{b_1, ..., b_n\}$ be any two orthonormal basis of V. From Theorem 3.2.6 we get, $A \wedge A$ and $B \wedge B$ form bases for $\mathcal{L}_{ALT}(V^2 \to \mathbb{R})$. Let $M \in \mathbb{R}^{n \times n}$ be the transformation matrix from basis A to B. Then, $\forall\ u \in \mathcal{L}_{ALT}(V^2 \to \mathbb{R})$,

$$\boxed{{}^{B\wedge B}u = M \cdot {}^{A\wedge A}u \cdot M^*}$$

Corollary 3.2.9. Let $A = \{a_1, ..., a_n\}$ be any orthonormal basis of V. From Theorem 3.2.6 we get that, $A \wedge A$ forms basis for $\mathcal{L}_{ALT}(V^2 \to \mathbb{R})$. Then, $\forall$ $u \in \mathcal{L}_{ALT}(V^2 \to \mathbb{R})$, $\forall\ x, y \in V$,

$$\boxed{u(x,y) = \sum_{r=1}^{n} \sum_{s=r+1}^{n} {}^{A\wedge A}u[r,s] \cdot ({}^{A}x[r] \cdot {}^{A}y[s] - {}^{A}x[s] \cdot {}^{A}y[r])}$$

Proof. Proofs of above corollaries are straight forward since $\mathcal{L}_{ALT}(V^2 \to \mathbb{R})$ is a subspace of $\mathcal{L}(V^2 \to \mathbb{R})$. We recommend the reader to go through section 2.2 first in order to get finer understanding here. $\square$

3.3 $3-$Fold Exterior Product Spaces

3.3.1 Alternating $3-$Tensors

Definition 3.3.1. Let V be a finite dimensional inner product space over field $\mathbb{R}$. A tensor $u \in \mathcal{L}(V \times V \times V \cong V^3 \to \mathbb{R})$ is called alternating $3-$tensor if $\forall$ $x_1, x_2, x_3 \in V$, for any transposition $\tau \in P_3$,

$$u(x_{\tau(1)}, x_{\tau(2)}, x_{\tau(3)}) = -u(x_1, x_2, x_3)$$

Let $S = \{u \in \mathcal{L}(V^3 \to \mathbb{R}) \mid u \text{ is an alternating } 3-\text{tensor}\}$

Lemma 3.3.1. *S forms a subspace of $\mathcal{L}(V^3 \to \mathbb{R})$.*

Proof. We have already seen that $\mathcal{L}(V^3 \to \mathbb{R})$ is a vector space over $\mathbb{R}$. To see that S is a subspace of $\mathcal{L}(V^3 \to \mathbb{R})$, it is enough to show the following,

1. $\forall$ $u, v \in S$, $\forall$ $x_1, x_2, x_3 \in V$, for any transposition $\tau \in P_3$,

$$\begin{aligned}[u+v](x_{\tau(1)}, x_{\tau(2)}, x_{\tau(3)}) &= u(x_{\tau(1)}, x_{\tau(2)}, x_{\tau(3)}) + v(x_{\tau(1)}, x_{\tau(2)}, x_{\tau(3)}) \\ &= -u(x_1, x_2, x_3) - v(x_1, x_2, x_3) \\ &= -[u+v](x_1, x_2, x_3)\end{aligned}$$

$$\implies [u+v] \in S$$

2. $\forall$ $u \in S$, $\forall$ $\alpha \in \mathbb{R}$, $\forall$ $x_1, x_2, x_3 \in V$, for any transposition $\tau \in P_3$,

$$\begin{aligned}[\alpha u](x_{\tau(1)}, x_{\tau(2)}, x_{\tau(3)}) &= \alpha u(x_{\tau(1)}, x_{\tau(2)}, x_{\tau(3)}) = -\alpha u(x_1, x_2, x_3) \\ &= -[\alpha u](x_1, x_2, x_3)\end{aligned}$$

$$\implies [\alpha u] \in S$$

3. It is easy to see that $0 \in S$

$\square$

A function $u \in S$ is called an **alternating $3-$tensor**. The subspace of all alternating $3-$tensors of V is denoted by $\mathcal{L}_{ALT}(V^3 \to \mathbb{R})$.

Remark :

1. $\forall\ u \in \mathcal{L}_{ALT}(V^3 \to \mathbb{R})$, $\forall\ x_1, x_2, x_3 \in V$, if $x_i = x_j$ for any $i, j \in \{1, 2, 3\}$ where $i < j$ then, consider the transposition $\tau = (i, j)$,

$$u(x_1, x_2, x_3) = u(x_{\tau(1)}, x_{\tau(2)}, x_{\tau(3)}) = -u(x_1, x_2, x_3) \implies u(x_1, x_2, x_3) = 0$$

Lemma 3.3.2. $\forall\ \sigma \in P_3$, $\forall\ u \in \mathcal{L}_{ALT}(V^3 \to \mathbb{R})$, $\forall\ x_1, x_2, x_3 \in V$,

$$u(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = sgn(\sigma) \cdot u(x_1, x_2, x_3)$$

Proof. $\forall\ \sigma \in P_3$, there exist $\tau_1, \tau_2, ..., \tau_m$ where each τ_i is a transposition such that

$$\sigma = \tau_1 \circ \tau_2 \circ ... \circ \tau_m$$

From the definition of u we get that for any transposition τ,

$$u(x_{\tau(1)}, x_{\tau(2)}, x_{\tau(3)}) = -u(x_1, x_2, x_3)$$

$$\begin{aligned} \implies u(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) &= u(x_{\tau_1 \circ ... \circ \tau_m(1)}, x_{\tau_1 \circ ... \circ \tau_m(2)}, x_{\tau_1 \circ ... \circ \tau_m(3)}) \\ &= (-1)^m \cdot u(x_1, x_2, x_3) \end{aligned}$$

It is already known that $sgn(\sigma) = (-1)^m$,

$$\implies u(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = sgn(\sigma) \cdot u(x_1, x_2, x_3)$$

$\square$

Remark : Notice that alternating $3-$tensor u can be defined in two equivalent ways, $\forall\ x_1, x_2, x_3 \in V$, for any transposition $\tau \in P_3$, for any $\sigma \in P_3$,

1. $$u(x_{\tau(1)}, x_{\tau(2)}, x_{\tau(3)}) = -u(x_1, x_2, x_3)$$

2. $$u(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = sgn(\sigma) \cdot u(x_1, x_2, x_3)$$

From the above Lemma it is clear that $1 \implies 2$ holds and $2 \implies 1$ is trivial.

3.3.2 3−fold Exterior Product on V

Definition 3.3.2. Let V be a finite dimensional inner product space over $\mathbb{R}$ where $dim(V) = n$. Then, $\forall\ v_1, v_2, v_3 \in V$, we define the exterior product (also called wedge product) of v_1, v_2, v_3 as a function $\wedge : V^3 \to \mathbb{R}$ as follows,

$$v_1 \wedge v_2 \wedge v_3 = v_1 \otimes v_2 \otimes v_3 - v_1 \otimes v_3 \otimes v_2 - v_2 \otimes v_1 \otimes v_3 + v_2 \otimes v_3 \otimes v_1 - v_3 \otimes v_2 \otimes v_1 + v_3 \otimes v_1 \otimes v_2$$

$$\boxed{v_1 \wedge v_2 \wedge v_3 = \sum_{\sigma \in P_3} sgn(\sigma) \cdot v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)}}$$

Remark :

1. $\forall\ x, y, z \in V$,

$$\begin{aligned}[v_1 \wedge v_2 \wedge v_3](x, y, z) &= \sum_{\sigma \in P_3} sgn(\sigma) \cdot [v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)}](x_1, x_2, x_3) \\ &= \sum_{\sigma \in P_3} sgn(\sigma) \cdot \prod_{i=1}^{3} (v_{\sigma(i)}, x_i)\end{aligned}$$

Lemma 3.3.3. Let V be a finite dimensional inner product space over $\mathbb{R}$. Then, $\forall\ v_1, v_2, v_3 \in V$, $\forall\ \sigma \in P_3$,

$$v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge v_{\sigma(3)} = sgn(\sigma) \cdot v_1 \wedge v_2 \wedge v_3$$

Proof.

$$v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge v_{\sigma(3)} = \sum_{\tilde{\sigma} \in P_3} sgn(\tilde{\sigma}) \cdot v_{\tilde{\sigma} \circ \sigma(1)} \otimes v_{\tilde{\sigma} \circ \sigma(2)} \otimes v_{\tilde{\sigma} \circ \sigma(3)}$$

$\forall\ \sigma \in P_3$, let $\sigma^{'} = \tilde{\sigma} \circ \sigma$. We already know that

$$sgn(\sigma^{'}) = sgn(\tilde{\sigma} \circ \sigma) = sgn(\sigma^{'}) \cdot sgn(\sigma) \text{ and } P_3 \circ \sigma = P_3$$

$$v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge v_{\sigma(3)} = \sum_{\sigma^{'} \in P_3} sgn(\sigma) \cdot sgn(\sigma^{'}) \cdot v_{\sigma^{'}(1)} \otimes v_{\sigma^{'}(2)} \otimes v_{\sigma^{'}(3)}$$

$$\implies v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge v_{\sigma(3)} = sgn(\sigma) \cdot v_1 \wedge v_2 \wedge v_3$$

$\square$

Remark :

1. Let $v_1, v_2, v_3 \in V$, if $v_i = v_j$ for any $i, j \in \{1, 2, 3\}$ where $i < j$ then,

$$v_1 \wedge v_2 \wedge v_3 = 0$$

Lemma 3.3.4. Let V be a finite dimensional inner product space over $\mathbb{R}$. Then, $\forall\ v_1, v_2, v_3 \in V$, $v_1 \wedge v_2 \wedge v_3 \in \mathcal{L}_{ALT}(V^3 \to \mathbb{R})$

Proof. $\forall\ x_1, x_2, x_3 \in V$, $\forall\ \sigma \in P_3$,

$$[v_1 \wedge v_2 \wedge v_3](x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = \sum_{\tilde{\sigma} \in P_3} sgn(\tilde{\sigma}) \cdot (v_{\tilde{\sigma}(1)}, x_{\sigma(1)})(v_{\tilde{\sigma}(2)}, x_{\sigma(2)})(v_{\tilde{\sigma}(3)}, x_{\sigma(3)})$$

Note that for any $\tilde{\sigma} \in P_3$,

$$(v_{\tilde{\sigma}(1)}, x_{\sigma(1)})(v_{\tilde{\sigma}(2)}, x_{\sigma(2)})(v_{\tilde{\sigma}(3)}, x_{\sigma(3)}) = (v_{\tilde{\sigma}\circ\sigma^{-1}(1)}, x_1)(v_{\tilde{\sigma}\circ\sigma^{-1}(2)}, x_2)(v_{\tilde{\sigma}\circ\sigma^{-1}(3)}, x_3)$$

This is to easy to see because if $\sigma(x) = y$ then, $\tilde{\sigma} \circ \sigma^{-1}(y) = \tilde{\sigma}(x)$.

$\forall\ \sigma \in P_3$, let $\sigma^{'} = \tilde{\sigma} \circ \sigma^{-1}$. It is already know that

$$sgn(\sigma^{'}) = sgn(\tilde{\sigma} \circ \sigma^{-1}) = sgn(\tilde{\sigma}) \cdot sgn(\sigma) \text{ and } P_3 \circ \sigma^{-1} = P_3$$

$$\implies [v_1 \wedge v_2 \wedge v_3](x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = \sum_{\sigma' \in P_3} sgn(\sigma^{'}) \cdot sgn(\sigma) \cdot (v_{\sigma'(1)}, x_1)(v_{\sigma'(2)}, x_2)(v_{\sigma'(3)}, x_3)$$

$$\implies [v_1 \wedge v_2 \wedge v_3](x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = sgn(\sigma) \cdot [v_1 \wedge v_2 \wedge v_3](x_1, x_2, x_3)$$

$$\implies [v_1 \wedge v_2 \wedge v_3] \in \mathcal{L}_{ALT}(V^3 \to \mathbb{R})$$

$\square$

Lemma 3.3.5. Let V be a finite dimensional inner product space over $\mathbb{R}$. Then,

1. $\forall\ v_1, v_1^{'}, v_2, v_3 \in V$,

$$[v_1 + v_1^{'}] \wedge v_2 \wedge v_3 = v_1 \wedge v_2 \wedge v_3 + v_1^{'} \wedge v_2 \wedge v_3$$

2. $\forall\ v_1, v_2, v_2^{'}, v_3 \in V$,

$$v_1 \wedge [v_2 + v_2^{'}] \wedge v_3 = v_1 \wedge v_2 \wedge v_3 + v_1 \wedge v_2^{'} \wedge v_3$$

3. $\forall\ v_1, v_2, v_3, v_3^{'} \in V$,

$$v_1 \wedge v_2 \wedge [v_3 + v_3^{'}] = v_1 \wedge v_2 \wedge v_3 + v_1 \wedge v_2 \wedge v_3^{'}$$

4. $\forall\ v_1, v_2, v_3 \in V$, $\forall\ \alpha \in \mathbb{R}$,

$$[\alpha v_1] \wedge v_2 \wedge v_3 = v_1 \wedge [\alpha v_2] \wedge v_3 = v_1 \wedge v_2 \wedge [\alpha v_3] = \alpha[v_1 \wedge v_2 \wedge v_3]$$

Proof. 1. Let $w_1 = v_1 + v_1^{'}$, $w_2 = v_2$, $w_3 = v_3$,

$$[v_1 + v_1^{'}] \wedge v_2 \wedge v_3 = w_1 \wedge w_2 \wedge w_3 = \sum_{\sigma \in P_3} sgn(\sigma) \cdot w_{\sigma(1)} \otimes w_{\sigma(2)} \otimes w_{\sigma(3)}$$

$$\begin{aligned}
&= \sum_{\sigma \in P_3 : \sigma(1)=1} sgn(\sigma) \cdot w_1 \otimes w_{\sigma(2)} \otimes w_{\sigma(3)} + \sum_{\sigma \in P_3 : \sigma(2)=1} sgn(\sigma) \cdot w_{\sigma(1)} \otimes w_1 \otimes w_{\sigma(3)} \\
&+ \sum_{\sigma \in P_3 : \sigma(3)=1} sgn(\sigma) \cdot w_{\sigma(1)} \otimes w_{\sigma(2)} \otimes w_1 \\
&= \sum_{\sigma \in P_3 : \sigma(1)=1} sgn(\sigma) \cdot v_1 \otimes w_{\sigma(2)} \otimes w_{\sigma(3)} + \sum_{\sigma \in P_3 : \sigma(1)=1} sgn(\sigma) \cdot v_1^{'} \otimes w_{\sigma(2)} \otimes w_{\sigma(3)} \\
&+ \sum_{\sigma \in P_3 : \sigma(2)=1} sgn(\sigma) \cdot w_{\sigma(1)} \otimes v_1 \otimes w_{\sigma(3)} + \sum_{\sigma \in P_3 : \sigma(2)=1} sgn(\sigma) \cdot w_{\sigma(1)} \otimes v_1^{'} \otimes w_{\sigma(3)} \\
&+ \sum_{\sigma \in P_3 : \sigma(3)=1} sgn(\sigma) \cdot w_{\sigma(1)} \otimes w_{\sigma(2)} \otimes v_1 + \sum_{\sigma \in P_3 : \sigma(3)=1} sgn(\sigma) \cdot w_{\sigma(1)} \otimes w_{\sigma(2)} \otimes v_1^{'} \\
&= v_1 \wedge w_2 \wedge w_3 + v_1^{'} \wedge w_2 \wedge w_3
\end{aligned}$$

2. Repeat the same steps as in the above proof using linearity of $3-$tensor in the position of argument occupied by w_2 we get,

$$v_1 \wedge [v_2 + v_2^{'}] \wedge v_3 = v_1 \wedge v_2 \wedge v_3 + v_1 \wedge v_2^{'} \wedge v_3$$

3. Repeat the same steps as in the above proof using linearity of $3-$tensor in

the position of argument occupied by w_3 we get,

$$v_1 \wedge v_2 \wedge [v_3 + v_3^{'}] = v_1 \wedge v_2 \wedge v_3 + v_1 \wedge v_2 \wedge v_3^{'}$$

4. Let $w_1 = \alpha v_1$, $w_2 = v_2$, $w_3 = v_3$,

$$[\alpha v_1] \wedge v_2 \wedge v_3 = w_1 \wedge w_2 \wedge w_3 = \sum_{\sigma \in P_3} sgn(\sigma) \cdot w_{\sigma(1)} \otimes w_{\sigma(2)} \otimes w_{\sigma(3)}$$

$$= \sum_{\sigma \in P_3 : \sigma(1)=1} sgn(\sigma) w_1 \otimes w_{\sigma(2)} \otimes w_{\sigma(3)} + \sum_{\sigma \in P_3 : \sigma(2)=1} sgn(\sigma) w_{\sigma(1)} \otimes w_1 \otimes w_{\sigma(3)}$$
$$+ \sum_{\sigma \in P_3 : \sigma(3)=1} sgn(\sigma) w_{\sigma(1)} \otimes w_{\sigma(2)} \otimes w_1$$
$$= \alpha \sum_{\sigma \in P_3 : \sigma(1)=1} sgn(\sigma) \cdot v_1 \otimes w_{\sigma(2)} \otimes w_{\sigma(3)} + \alpha \sum_{\sigma \in P_3 : \sigma(2)=1} sgn(\sigma) \cdot w_{\sigma(1)} \otimes v_1 \otimes w_{\sigma(3)}$$
$$+ \alpha \sum_{\sigma \in P_3 : \sigma(3)=1} sgn(\sigma) \cdot w_{\sigma(1)} \otimes w_{\sigma(2)} \otimes v_1$$
$$= \alpha [v_1 \wedge w_2 \wedge w_3]$$

Repeat the same steps as in the above proof using linearity of $3-$tensor in the position of argument occupied by w_2 and w_3 respectively we get,

$$v_1 \wedge [\alpha v_2] \wedge v_3 = v_1 \wedge v_2 \wedge [\alpha v_3] = \alpha [v_1 \wedge v_2 \wedge v_3]$$

□

Illustration :

Consider $V = \mathbb{R}^2$ over $\mathbb{R}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^*, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^*\}$. Recall that A is an orthonormal basis of $\mathbb{R}^2$.

Let $v_1 = v_3 = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\forall\ x, y, z \in \mathbb{R}^2$,

$$[v_1 \wedge v_2 \wedge v_3](x, y, z) = \sum_{\sigma \in P_3} sgn(\sigma) \cdot [v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)}](x, y, z)$$

$$
\begin{aligned}
&= ({}^A v_1 \odot {}^A x)({}^A v_2 \odot {}^A y)({}^A v_3 \odot {}^A z) - ({}^A v_1 \odot {}^A x)({}^A v_3 \odot {}^A y)({}^A v_2 \odot {}^A z)\\
&- ({}^A v_2 \odot {}^A x)({}^A v_1 \odot {}^A y)({}^A v_3 \odot {}^A z) + ({}^A v_2 \odot {}^A x)({}^A v_3 \odot {}^A y)({}^A v_1 \odot {}^A z)\\
&- ({}^A v_3 \odot {}^A x)({}^A v_2 \odot {}^A y)({}^A v_1 \odot {}^A z) + ({}^A v_3 \odot {}^A x)({}^A v_1 \odot {}^A y)({}^A v_2 \odot {}^A z)\\
&= {}^A x[1]{}^A y[2]{}^A z[1] - {}^A x[1]{}^A y[1]{}^A z[2] - {}^A x[2]{}^A y[1]{}^A z[1]\\
&+ {}^A x[2]{}^A y[1]{}^A z[1] - {}^A x[1]{}^A y[2]{}^A z[1] + {}^A x[1]{}^A y[1]{}^A z[2]
\end{aligned}
$$

Note that $[v_1 \wedge v_2 \wedge v_3] = 0$ here. All the previously proved Lemmas can be concretely worked out with this illustration. It is clear that given v_1, v_2, v_3, $[v_1 \wedge v_2 \wedge v_3](x, y, z)$ can be concretely calculated. From the above remark, it is easy to understand that any $3-$alternating tensor u in a $k-$dimensional space where $k \leq 2$ must to be a 0-tensor.

3.3.3 Basis of 3-fold exterior product spaces

Let V be a finite dimensional inner product space over $\mathbb{R}$ where $dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ be an orthonormal basis of V. Then, $A \otimes A \otimes A = \{a_{i_1} \otimes a_{i_2} \otimes a_{i_3} \mid 1 \leq i_1, i_2, i_3 \leq n\}$ is a basis of $\mathcal{L}(V^3 \to \mathbb{R})$. Let $A \wedge A \wedge A = \{a_{i_1} \wedge a_{i_2} \wedge a_{i_3} \mid 1 \leq i_1 < i_2 < i_3 \leq n\}$. $\forall\ i_1, i_2, i_3 \in \{1, 2, ..., n\}$ such that $i_1 < i_2 < i_3$, from Lemma 3.3.4 we get that $a_{i_1} \wedge a_{i_2} \wedge a_{i_3}$ is an alternating $3-$tensor.

Theorem 3.3.6. $A \wedge A \wedge A$ is a basis of $\mathcal{L}_{ALT}(V^3 \to \mathbb{R})$.

Proof. **Span:**
$\forall\ u \in \mathcal{L}_{ALT}(V^3 \to \mathbb{R}) \implies u \in \mathcal{L}(V^3 \to \mathbb{R})$, since $A \otimes A \otimes A$ is a basis of $\mathcal{L}(V^3 \to \mathbb{R})$,

$$u = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \sum_{i_3=1}^{n} u(a_{i_1}, a_{i_2}, a_{i_3})[a_{i_1} \otimes a_{i_2} \otimes a_{i_3}]$$

Recall that if $a_{i_1} = a_{i_2}$ or $a_{i_2} = a_{i_3}$ or $a_{i_3} = a_{i_1}$ then $u(a_{i_1}, a_{i_2}, a_{i_3}) = 0$. In addition, we have that $\forall\ 1 \leq i_1 < i_2 < i_3 \leq n$, $\forall\ \sigma \in P_3$,

$$u(a_{i_{\sigma(1)}}, a_{i_{\sigma(2)}}, a_{i_{\sigma(3)}}) = sgn(\sigma) \cdot u(a_1, a_2, a_3)$$

$$a_{i_{\sigma(1)}} \wedge a_{i_{\sigma(2)}} \wedge a_{i_{\sigma(3)}} = sgn(\sigma) \cdot a_{i_1} \wedge a_{i_2} \wedge a_{i_3}$$

$$\implies u = \sum_{(i_1,i_2,i_3)\in C(n,3)} \left(\sum_{\sigma\in P_3} sgn(\sigma) a_{i_{\sigma(1)}} \otimes a_{i_{\sigma(2)}} \otimes a_{i_{\sigma(3)}} \right) u(a_{i_1}, a_{i_2}, a_{i_3})$$
$$= \sum_{(i_1,i_2,i_3)\in C(n,3)} u(a_{i_1}, a_{i_2}, a_{i_3})[a_{i_1} \wedge a_{i_2} \wedge a_{i_3}]$$
$$\implies A \wedge A \wedge A \text{ spans } \mathcal{L}_{ALT}(V^3 \to \mathbb{R})$$

Linear Independence :
Let $\alpha_{i_1 i_2 i_3} \in \mathbb{R}\ \forall\ i_1, i_2, i_3 \in \{1, 2, .., , n\}$ where $i_1 < i_2 < i_3$. Consider,

$$\sum_{(i_1,i_2,i_3)\in C(n,3)} \alpha_{i_1 i_2 i_3}[a_{i_1} \wedge a_{i_2} \wedge a_{i_3}] = 0$$

$$\implies \sum_{(i_1,i_2,i_3)\in C(n,3)} \alpha_{i_1 i_2 i_3} \left(\sum_{\sigma\in P_3} sgn(\sigma)[a_{i_{\sigma(1)}} \otimes a_{i_{\sigma(2)}} \otimes a_{i_{\sigma(3)}} \right) = 0$$

$\forall\ j_1, j_2, j_3 \in \{1, 2, .., , n\}$ where $j_1 < j_2 < j_3$,

$$\sum_{(i_1,i_2,i_3)\in C(n,3)} \left(\sum_{\sigma\in P_3} sgn(\sigma)[a_{i_{\sigma(1)}} \otimes a_{i_{\sigma(2)}} \otimes a_{i_{\sigma(3)}}](a_{j_1}, a_{j_2}, a_{j_3}) \right) = 0$$

Since A is an orthonormal basis of V we get,

$$\alpha_{j_1 j_2 j_3} \sum_{\sigma\in P_3} sgn(\sigma)[a_{j_{\sigma(1)}} \otimes a_{j_{\sigma(2)}} \otimes a_{j_{\sigma(3)}}](a_{j_1}, a_{j_2}, a_{j_3}) = \alpha_{j_1 j_2 j_3} = 0$$

$$\implies A \wedge A \wedge A \text{ is a linearly independent set and a basis of } \mathcal{L}_{ALT}(V^3 \to \mathbb{R})$$

$\square$

Corollary 3.3.7.

$$\boxed{dim(\mathcal{L}_{ALT}(V^3 \to \mathbb{R})) = \binom{n}{3}}$$

Illustration :

Consider $V = \mathbb{R}^3$ over $\mathbb{R}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}^*, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}^*, a_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}^*\}$. Recall that A is an orthonormal basis of $\mathbb{R}^3$. $\forall\ i, j, k \in \{1, 2, 3\}$, $\forall\ x, y, z \in \mathbb{R}^2$,

$$\begin{aligned}
[a_i \wedge a_j \wedge a_k](x, y, z) &= [a_i \otimes a_j \otimes a_k](x, y, z) - [a_i \otimes a_k \otimes a_j](x, y, z) - [a_j \otimes a_i \otimes a_k](x, y, z) \\
&\quad + [a_j \otimes a_k \otimes a_i](x, y, z) - [a_k \otimes a_j \otimes a_i](x, y, z) + [a_k \otimes a_i \otimes a_j](x, y, z) \\
&= {}^A x[i] \cdot {}^A y[j] \cdot {}^A z[k] - {}^A x[i] \cdot {}^A y[k] \cdot {}^A z[j] - {}^A x[j] \cdot {}^A y[i] \cdot {}^A z[k] \\
&\quad + {}^A x[j] \cdot {}^A y[k] \cdot {}^A z[i] - {}^A x[k] \cdot {}^A y[j] \cdot {}^A z[i] - {}^A x[k] \cdot {}^A y[i] \cdot {}^A z[j]
\end{aligned} \tag{3.3}$$

Let $A \wedge A \wedge A = \{a_i \wedge a_j \wedge a_k \mid 1 \leq i < j < k \leq 3\}$. From the above Theorem, we get that $A \wedge A \wedge A$ is a basis of $\mathcal{L}_{ALT}((\mathbb{R}^3)^3 \to \mathbb{R})$. Consider the following $3-$tensor,

$$u(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}) = x_1(y_2 z_3 - y_3 z_2) - x_2(y_1 z_3 - y_3 z_1) + x_3(y_1 z_2 - y_2 z_1)$$

In chapter 4, it will be shown that u is a $3-$dimensional notion of determinant. It is straight forward to verify that u is an alternating $3-$tensor. Since A is a basis of $\mathbb{R}^3$, $\forall \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^*, \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^*, \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^* \in \mathbb{R}^3$, there exist unique α_1, α_2, α_3, β_1, β_2, β_3, γ_1, γ_2, $\gamma_3 \in \mathbb{R}$, such that

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^* = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3$$

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^* = \beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3$$

$$\begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^* = \gamma_1 a_1 + \gamma_2 a_2 + \gamma_3 a_3$$

Since u is an alternating $3-$tensor we get that

$$u(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}) = u(\sum_{i=1}^{3} \alpha_i a_i, \sum_{j=1}^{2} \beta_j a_j, \sum_{k=1}^{3} \gamma_k a_k)$$

$$= (\alpha_1\beta_2\gamma_3 - \alpha_1\beta_3\gamma_2 - \alpha_2\beta_1\gamma_3 + \alpha_2\beta_3\gamma_1 - \alpha_3\beta_2\gamma_1 + \alpha_3\beta_1\gamma_2)u(a_1, a_2, a_3) \quad (3.4)$$

From the above expression, it is quite clear that computing $u(a_1, a_2, a_3)$ is sufficient to determine the action of u on any $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^*, \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^*, \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^*$.

$$u(a_1, a_2, a_3) = 1$$

Combining equations 3.3 and 3.4 we get,

$$\implies u(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}) = [a_1 \wedge a_2 \wedge a_3](x, y, z)$$

With this illustration, you can observe how $A \wedge A \wedge A$ works as a basis of $\mathcal{L}_{ALT}((\mathbb{R}^3)^3 \to \mathbb{R})$ and also note that the value of $u(a_1, a_2, a_3)$ is sufficient to compute u on any $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^*, \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^*, \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^* \in \mathbb{R}^3$.

3.3.4 Basis Transformation and Invariance of computation of alternating 3− tensor under orthonormal basis transformations

Definition 3.3.3. $\forall\ u \in \mathcal{L}_{ALT}(V^3 \to \mathbb{R})$, from Theorem 3.3.6 we get that,

$$u = \sum_{(i,j,k) \in C(n,3)} u(a_i, a_j, a_k)[a_i \wedge a_j \wedge a_k]$$

$\forall\ 1 \leq i < j < k \leq n$, we define

$$^{A\wedge A\wedge A}u[i, j, k] = u(a_i, a_j, a_k)$$

Remark :

1. $\forall\ \sigma \in P_3$, $\forall\ 1 \leq i_1 < i_2 < i_3 < n$, we get that

$$^{A\wedge A\wedge A}u[i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)}] = sgn(\sigma) \cdot u(a_{i_1}, a_{i_2}, a_{i_3})$$

Corollary 3.3.8. Let $A = \{a_1, ..., a_n\}$ and $B = \{b_1, ..., b_n\}$ be any two orthonormal basis of V. From Theorem 3.3.6, $A \wedge A \wedge A$ and $B \wedge B \wedge B$ form bases for $\mathcal{L}_{ALT}(V^3 \to \mathbb{R})$. Let $M \in \mathbb{R}^{n\times n}$ be the transformation matrix from basis A to B. Then, $\forall\ u \in \mathcal{L}_{ALT}(V^3 \to \mathbb{R})$, $\forall\ 1 \leq j_1 < j_2 < j_3 \leq n$

$$\boxed{{}^{B\wedge B\wedge B}u[j_1, j_2, j_3] = \sum_{(i_1,i_2,i_3)\in C(n,3)} \left(\sum_{\sigma\in P_3} sgn(\sigma) \cdot M_{j_1 i_{\sigma(1)}} \cdot M_{j_2 i_{\sigma(2)}} \cdot M_{j_3 i_{\sigma(3)}} \right) \cdot {}^{A\wedge A\wedge A}u[i_1, i_2, i_3]}$$

Corollary 3.3.9. Let $A = \{a_1, ..., a_n\}$ be any orthonormal basis of V. From Theorem 3.3.6 we get that, $A \wedge A \wedge A$ forms basis for $\mathcal{L}_{ALT}(V^3 \to \mathbb{R})$. Then, $\forall$ $u \in \mathcal{L}_{ALT}(V^3 \to \mathbb{R})$, $\forall\ x, y, z \in V$,

$$\boxed{u(x, y, z) = \sum_{(r_1,r_2,r_3)\in C(n,3)} \left(\sum_{\sigma\in P_3} sgn(\sigma) \cdot {}^{A}x[r_{\sigma(1)}] \cdot {}^{A}y[r_{\sigma(2)}] \cdot {}^{A}z[r_{\sigma(3)}] \right) \cdot {}^{A\wedge A\wedge A}u[r_1, r_2, r_3]}$$

Proof. Proofs of above corollaries are straight forward since $\mathcal{L}_{ALT}(V^3 \to \mathbb{R})$ is a subspace of $\mathcal{L}(V^3 \to \mathbb{R})$. We recommend the reader to go through 2.3 first in order to get finer clarity here. $\square$

3.4 $k-$fold exterior product spaces

3.4.1 Alternating $k-$Tensors

Definition 3.4.1. Let V be a finite dimensional inner product space over field $\mathbb{R}$. A tensor $u \in \mathcal{L}(\underbrace{V \times V \times ... \times V}_{k \text{ times}} \cong V^k \to \mathbb{R})$ is called alternating $k-$tensor if $\forall\ x_1, x_2, ..., x_k \in V$, for any transposition $\tau \in P_k$,

$$u(x_{\tau(1)}, x_{\tau(2)}, ..., x_{\tau(k)}) = -u(x_1, x_2, ..., x_k)$$

Let $S = \{u \in \mathcal{L}(V^k \to \mathbb{R}) \mid u \text{ is an alternating } k\text{−tensor}\}$

Lemma 3.4.1. *S forms a subspace of $\mathcal{L}(V^k \to \mathbb{R})$.*

Proof. We have already seen that $\mathcal{L}(V^k \to \mathbb{R})$ is a vector space over $\mathbb{R}$. To see that S is a subspace of $\mathcal{L}(V^k \to \mathbb{R})$, it is enough to show the following,

1. $\forall\ u, v \in S$, $\forall\ x_1, x_2, ..., x_k \in V$, for any transposition $\tau \in P_k$,

$$\begin{aligned} [u+v](x_{\tau(1)}, x_{\tau(2)}, ..., x_{\tau(k)}) &= u(x_{\tau(1)}, x_{\tau(2)}, ..., x_{\tau(k)}) + v(x_{\tau(1)}, x_{\tau(2)}, ..., x_{\tau(k)}) \\ &= -u(x_1, x_2, ..., x_k) - v(x_1, x_2, ..., x_k) \\ &= -[u+v](x_1, x_2, ..., x_k) \end{aligned}$$

$$\implies [u+v] \in S$$

2. $\forall\ u \in S$, $\forall\ \alpha \in \mathbb{R}$, $\forall\ x_1, x_2, ..., x_k \in V$, for any transposition $\tau \in P_k$,

$$\begin{aligned} [\alpha u](x_{\tau(1)}, x_{\tau(2)}, ..., x_{\tau(k)}) &= \alpha u(x_{\tau(1)}, x_{\tau(2)}, ..., x_{\tau(k)}) = -\alpha u(x_1, x_2, ..., x_k) \\ &= -[\alpha u](x_1, x_2, ..., x_k) \end{aligned}$$

$$\implies [\alpha u] \in S$$

3. It is easy to see that $0 \in S$

$\square$

A function $u \in S$ is called an **alternating $k-$tensor**. The subspace of all alternating $k-$tensors of V is denoted by $\mathcal{L}_{ALT}(V^k \to \mathbb{R})$.

Remark :

1. $\forall\ u \in \mathcal{L}_{ALT}(V^k \to \mathbb{R})$, $\forall\ x_1, x_2, ..., x_k \in V$, if $x_i = x_j$ for any $i, j \in \{1, 2, ..., k\}$ where $i < j$ then, consider the transposition $\tau = (i, j)$,

$$\begin{aligned} u(x_1, ..., x_i, ..., x_j, ..., x_k) &= u(x_{\tau(1)}, ..., x_{\tau(i)}, ..., x_{\tau(j)}, ..., x_{\tau(k)}) \\ &= -u(x_1, ..., x_i, ..., x_j, ..., x_k) \end{aligned}$$

$$\implies u(x_1, ..., x_i, ..., x_j, ..., x_k) = 0$$

Lemma 3.4.2. $\forall\ \sigma \in P_k$, $\forall\ u \in \mathcal{L}_{ALT}(V^k \to \mathbb{R})$, $\forall\ x_1, x_2, ..., x_k \in V$,

$$u(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(k)}) = sgn(\sigma) \cdot u(x_1, x_2, ..., x_k)$$

Proof. $\forall\ \sigma \in P_k$, there exist $\tau_1, \tau_2, ..., \tau_m$ where each τ_i is a transposition such that

$$\sigma = \tau_1 \circ \tau_2 \circ ... \circ \tau_m$$

From the definition of u we get that for any transposition τ,

$$u(x_{\tau(1)}, x_{\tau(2)}, ..., x_{\tau(k)}) = -u(x_1, x_2, ..., x_k)$$

$$\begin{aligned} \implies u(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(k)}) &= u(x_{\tau_1 \circ ... \circ \tau_m(1)}, x_{\tau_1 \circ ... \circ \tau_m(2)}, ..., x_{\tau_1 \circ ... \circ \tau_m(k)}) \\ &= (-1)^m u(x_1, x_2, ..., x_k) \end{aligned}$$

We already have that $sgn(\sigma) = (-1)^m$,

$$\implies u(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(k)}) = sgn(\sigma) \cdot u(x_1, x_2, ..., x_k)$$

□

Remark : Notice that alternating $k-$tensor u can be defined in two equivalent ways, $\forall\ x_1, x_2, ..., x_k \in V$, for any transposition $\tau \in P_k$, for any $\sigma \in P_k$,

1. $$u(x_{\tau(1)}, x_{\tau(2)}, ..., x_{\tau(k)}) = -u(x_1, x_2, ..., x_k)$$

2.

$$u(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(k)}) = sgn(\sigma) \cdot u(x_1, x_2, ..., x_k)$$

From the above Lemma it is clear that 1 $\implies$ 2 holds and 2 $\implies$ 1 is trivial.

3.4.2 $k-$fold Exterior Product on V

Definition 3.4.2. Let V be a finite dimensional inner product space over $\mathbb{R}$ where $dim(V) = n$. Then, $\forall\ v_1, v_2, ..., v_k \in V$, we define the exterior product of $v_1, v_2, ..., v_k$ as a function $\wedge : V^k \to \mathbb{R}$ as follows,

$$\boxed{v_1 \wedge v_2 \wedge ... \wedge v_k = \sum_{\sigma \in P_k} sgn(\sigma) \cdot v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes ... \otimes v_{\sigma(k)}}$$

Remark :

1. $\forall\ x_1, x_2, ..., x_k \in V$,

$$\begin{aligned}[v_1 \wedge v_2 \wedge ... \wedge v_k](x_1, x_2, ..., x_k) &= \sum_{\sigma \in P_k} sgn(\sigma) \cdot [v_{\sigma(1)} \otimes ... \otimes v_{\sigma(k)}](x_1, ..., x_k) \\ &= \sum_{\sigma \in P_k} sgn(\sigma) \cdot \prod_{i=1}^{k} (v_{\sigma(i)}, x_i)\end{aligned}$$

Lemma 3.4.3. Let V be a finite dimensional inner product space over $\mathbb{R}$. Then, $\forall\ v_1, v_2, ..., v_k \in V$, $\forall\ \sigma \in P_k$,

$$v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge ... \wedge v_{\sigma(k)} = sgn(\sigma) \cdot v_1 \wedge v_2 \wedge ... \wedge v_k$$

Proof.

$$v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge ... \wedge v_{\sigma(k)} = \sum_{\tilde{\sigma} \in P_k} sgn(\tilde{\sigma}) \cdot v_{\tilde{\sigma}\circ\sigma(1)} \otimes v_{\tilde{\sigma}\circ\sigma(2)} \otimes ... \otimes v_{\tilde{\sigma}\circ\sigma(k)}$$

$\forall\ \sigma \in P_k$, let $\sigma^{'} = \tilde{\sigma} \circ \sigma$. We already know that

$$sgn(\sigma^{'}) = sgn(\tilde{\sigma} \circ \sigma) = sgn(\sigma^{'}) \cdot sgn(\sigma) \text{ and } P_k \circ \sigma = P_k$$

$$v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge ... \wedge v_{\sigma(k)} = \sum_{\sigma' \in P_k} sgn(\sigma) \cdot sgn(\sigma') \cdot v_{\sigma'(1)} \otimes v_{\sigma'(2)} \otimes ... \otimes v_{\sigma'(k)}$$

$$\implies v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge ... \wedge v_{\sigma(k)} = sgn(\sigma) \cdot v_1 \wedge v_2 \wedge ... \wedge v_k$$

$\square$

Remark :

1. Let $v_1, v_2, ..., v_k \in V$, if $v_i = v_j \ \forall \ i < j$ then,

$$v_1 \wedge v_2 \wedge ... \wedge v_k = 0$$

Lemma 3.4.4. Let V be a finite dimensional inner product space over $\mathbb{R}$. Then, $\forall \ v_1, v_2, ..., v_k \in V$, $v_1 \wedge v_2 \wedge ... \wedge v_k \in \mathcal{L}_{ALT}(V^k \to \mathbb{R})$

Proof. $\forall \ x_1, x_2, ..., x_k \in V$, $\forall \ \sigma \in P_k$,

$$[v_1 \wedge ... \wedge v_k](x_{\sigma(1)}, ..., x_{\sigma(k)}) = \sum_{\tilde{\sigma} \in P_k} sgn(\tilde{\sigma}) \cdot (v_{\tilde{\sigma}(1)}, x_{\sigma(1)})(v_{\tilde{\sigma}(2)}, x_{\sigma(2)})...(v_{\tilde{\sigma}(k)}, x_{\sigma(k)})$$

Note that for any $\tilde{\sigma} \in P_k$,

$$(v_{\tilde{\sigma}(1)}, x_{\sigma(1)})(v_{\tilde{\sigma}(2)}, x_{\sigma(2)})...(v_{\tilde{\sigma}(k)}, x_{\sigma(k)}) = (v_{\tilde{\sigma}\circ\sigma^{-1}(1)}, x_1)(v_{\tilde{\sigma}\circ\sigma^{-1}(2)}, x_2)...(v_{\tilde{\sigma}\circ\sigma^{-1}(k)}, x_k)$$

This is to easy to see because if $\sigma(x) = y$ then, $\tilde{\sigma} \circ \sigma^{-1}(y) = \tilde{\sigma}(x)$.

$\forall \ \sigma \in P_3$, let $\sigma' = \tilde{\sigma} \circ \sigma^{-1}$. It is already know that

$$sgn(\sigma') = sgn(\tilde{\sigma} \circ \sigma^{-1}) = sgn(\tilde{\sigma}) \cdot sgn(\sigma) \text{ and } P_k \circ \sigma^{-1} = P_k$$

$$\implies [v_1 \wedge ... \wedge v_k](x_{\sigma(1)}, ..., x_{\sigma(k)}) = \sum_{\sigma' \in P_k} sgn(\sigma') \cdot sgn(\sigma) \cdot (v_{\sigma'(1)}, x_1)(v_{\sigma'(2)}, x_2)...(v_{\sigma'(k)}, x_k)$$

$$\implies [v_1 \wedge v_2 \wedge ... \wedge v_k](x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(k)}) = sgn(\sigma) \cdot [v_1 \wedge v_2 \wedge ... \wedge v_k](x_1, x_2, ..., x_k)$$

$$\implies [v_1 \wedge v_2 \wedge ... \wedge v_k] \in \mathcal{L}_{ALT}(V^k \to \mathbb{R})$$

$\square$

Lemma 3.4.5. Let V be a finite dimensional inner product space over $\mathbb{R}$. Then,

1. $\forall\ v_1, v_2, ..., v_i, v_i^{'}, ..., v_k \in V$, where $1 \leq i \leq k$,

$$v_1 \wedge ... \wedge [v_i + v_i^{'}] \wedge ... \wedge v_k = v_1 \wedge ... \wedge v_i \wedge ... \wedge v_k + v_1 \wedge ... \wedge v_i^{'} \wedge ... \wedge v_k$$

2. $\forall\ v_1, v_2, ..., v_i, ..., v_k \in V$, where $1 \leq i \leq k$, $\forall\ \alpha \in \mathbb{R}$,

$$v_1 \wedge ... \wedge [\alpha v_i] \wedge ... \wedge v_k = \alpha[v_1 \wedge ... \wedge v_i \wedge ... \wedge v_k]$$

Proof. 1. Let $w_1 = v_1$, $w_2 = v_2$, ..., $w_i = v_i + v_i^{'}$, ..., $w_k = v_k$,

$$\begin{aligned}
v_1 \wedge ... \wedge [v_i + v_i^{'}] \wedge ... \wedge v_k &= w_1 \wedge ... \wedge w_i \wedge ... \wedge w_k \\
&= \sum_{\sigma \in P_k} sgn(\sigma) \cdot w_{\sigma(1)} \otimes ... \otimes w_{\sigma(i)} \otimes ... \otimes w_{\sigma(k)} \\
&= \sum_{\sigma \in P_k : \sigma(1) = i} sgn(\sigma) \cdot w_i \otimes w_{\sigma(2)} \otimes ... \otimes w_{\sigma(k)} \\
&+ \sum_{\sigma \in P_k : \sigma(2) = i} sgn(\sigma) \cdot w_{\sigma(1)} \otimes w_i \otimes ... \otimes w_{\sigma(k)} \\
&... \\
&+ \sum_{\sigma \in P_k : \sigma(k) = i} sgn(\sigma) \cdot w_{\sigma(1)} \otimes w_{\sigma(2)} \otimes ... \otimes w_i \\
&= \sum_{\sigma \in P_k : \sigma(1) = i} sgn(\sigma) \cdot [v_i + v_i^{'}] \otimes w_{\sigma(2)} \otimes ... \otimes w_{\sigma(k)} \\
&+ \sum_{\sigma \in P_k : \sigma(2) = i} sgn(\sigma) \cdot w_{\sigma(1)} \otimes [v_i + v_i^{'}] \otimes ... \otimes w_{\sigma(k)} \\
&... \\
&+ \sum_{\sigma \in P_k : \sigma(k) = i} sgn(\sigma) \cdot w_{\sigma(1)} \otimes w_{\sigma(2)} \otimes ... \otimes [v_i + v_i^{'}] \\
&= w_1 \wedge ... \wedge v_i \wedge ... \wedge w_k + w_1 \wedge ... \wedge v_i^{'} \wedge ... \wedge w_k
\end{aligned}$$

2. Let $w_1 = v_1$, ..., $w_i = \alpha v_i$, ..., $w_k = v_k$,

$$v_1 \wedge ... \wedge [\alpha v_i] \wedge ... \wedge v_k = w_1 \wedge ... \wedge w_i \wedge ... \wedge w_k$$

$$
\begin{aligned}
&= \sum_{\sigma \in P_k : \sigma(1)=i} sgn(\sigma) \cdot w_i \otimes w_{\sigma(2)} \otimes ... \otimes w_{\sigma(k)} \\
&+ \sum_{\sigma \in P_k : \sigma(2)=i} sgn(\sigma) \cdot w_{\sigma(1)} \otimes w_i \otimes ... \otimes w_{\sigma(k)} \\
&... \\
&+ \sum_{\sigma \in P_k : \sigma(k)=i} sgn(\sigma) \cdot w_{\sigma(1)} \otimes w_{\sigma(2)} \otimes ... \otimes w_i \\
&= \sum_{\sigma \in P_k : \sigma(1)=i} sgn(\sigma)[\alpha v_i] \otimes w_{\sigma(2)} \otimes ... \otimes w_{\sigma(k)} \\
&+ \sum_{\sigma \in P_k : \sigma(2)=i} sgn(\sigma) w_{\sigma(1)} \otimes [\alpha v_i] \otimes ... \otimes w_{\sigma(k)} \\
&... \\
&+ \sum_{\sigma \in P_k : \sigma(k)=i} sgn(\sigma) w_{\sigma(1)} \otimes w_{\sigma(2)} \otimes ... \otimes [\alpha v_i] \\
&= \alpha[w_1 \wedge ... \wedge v_i \wedge ... \wedge w_k]
\end{aligned}
$$

□

Illustration :

Consider $V = \mathbb{R}^2$ over $\mathbb{R}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^*, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^*\}$. Recall that A is an orthonormal basis of $\mathbb{R}^2$.

Let $v_i = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\forall\ i \in \{1, 2, ..., k\}$

$\forall\ x_1, x_2, ..., x_k \in \mathbb{R}^2$,

$$
\begin{aligned}
[v_1 \wedge v_2 \wedge ... \wedge v_k](x_1, x_2, ..., x_k) &= \sum_{\sigma \in P_3} sgn(\sigma) \cdot [v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes .. \otimes v_{\sigma(k)}](x_1, x_2, ..., x_k) \\
&= \sum_{\sigma \in P_k} sgn(\sigma) \prod_{i=1}^{k} ({}^{A}v_i \odot {}^{A}x_i) = \sum_{\sigma \in P_k} sgn(\sigma) \cdot \prod_{i=1}^{k} {}^{A}x_i[1]
\end{aligned}
$$

Verify that $v_1 \wedge v_2 \wedge ... \wedge v_k = 0$ here. All the previously proved Lemmas can be concretely worked out with this illustration. Note that given $v_1, v_2, ..., v_k$, $[v_1 \wedge v_2 \wedge ... \wedge v_k](x_1, x_2, ..., x_k)$ can be concretely calculated. From the above remark, it is easy to understand that any $l-$alternating tensor u in a $k-$dimensional space where $l > k$ must to be a 0-tensor.

3.4.3 Basis of $k-$fold exterior product spaces

Let V be a finite dimensional inner product space over $\mathbb{R}$ where $dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ be an orthonormal basis of V. Then, $\underbrace{A \otimes A \otimes ... \otimes A}_{\text{k times}} = \otimes_{i=1}^{k} A = \{a_{i_1} \otimes a_{i_2} \otimes ... \otimes a_{i_k} \mid 1 \leq i_1, i_2, ..., i_k \leq n\}$ is a basis of $\mathcal{L}(V^k \to \mathbb{R})$. Let $\underbrace{A \wedge A \wedge ... \wedge A}_{\text{k times}} = \wedge_{i=1}^{k} A = \{a_{i_1} \wedge a_{i_2} \wedge ... \wedge a_{i_k} \mid 1 \leq i_1 < i_2 < ... < i_k \leq n\}$. $\forall\ i_1, i_2, ..., i_k \in \{1, 2, ..., n\}$ such that $i_1 < i_2 < ... < i_k$, using Lemma 3.4.4, $a_{i_1} \wedge a_{i_2} \wedge ... \wedge a_{i_k}$ is an alternating $k-$tensor. We use $C(n, k)$ to denote the set of all k combinations of n elements arranged in increasing order. For instance, $C(4, 2) = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$

Theorem 3.4.6. $\wedge_{i=1}^{k} A$ is a basis of $\mathcal{L}_{ALT}(V^k \to \mathbb{R})$.

Proof. **Span:**
$\forall\, u \in \mathcal{L}_{ALT}(V^k \to \mathbb{R}) \implies u \in \mathcal{L}(V^k \to \mathbb{R})$, since $\otimes_{i=1}^{k} A$ is a basis of $\mathcal{L}(V^k \to \mathbb{R})$,

$$u = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} ... \sum_{i_k=1}^{n} u(a_{i_1}, a_{i_2}, ..., a_{i_k})[a_{i_1} \otimes a_{i_2} \otimes ... \otimes a_{i_k}]$$

Recall that if $a_{i_j} = a_{i_l}\ \forall\ j \neq l$, then $u(a_{i_1}, a_{i_2}, ..., a_{i_k}) = 0$, Also recall that $\forall$ $1 \leq i_1 < i_2 < ... < i_k \leq n$, $\forall\ \sigma \in P_k$,

$$u(a_{i_{\sigma(1)}}, a_{i_{\sigma(2)}}, ..., a_{i_{\sigma(k)}}) = sgn(\sigma) \cdot u(a_1, a_2, ..., a_k)$$

$$a_{i_{\sigma(1)}} \wedge a_{i_{\sigma(2)}} \wedge ... \wedge a_{i_{\sigma(k)}} = sgn(\sigma) \cdot a_{i_1} \wedge a_{i_2} \wedge ... \wedge a_{i_k}$$

$$\begin{aligned} \implies u &= \sum_{(i_1, i_2, ..., i_k) \in C(n,k)} \left(\sum_{\sigma \in P_k} sgn(\sigma) \cdot a_{i_{\sigma(1)}} \otimes a_{i_{\sigma(2)}} \otimes ... \otimes a_{i_{\sigma(k)}} \right) u(a_{i_1}, a_{i_2}, ..., a_{i_k}) \\ &= \sum_{(i_1, i_2, ..., i_k) \in C(n,k)} u(a_{i_1}, a_{i_2}, ..., a_{i_k})[a_{i_1} \wedge a_{i_2} \wedge ... \wedge a_{i_k}] \end{aligned}$$

$$\implies \wedge_{i=1}^{k} A \text{ spans } \mathcal{L}_{ALT}(V^k \to \mathbb{R})$$

Linear Independence :
Let $\alpha_{i_1 i_2 \ldots i_k} \in \mathbb{R}\ \forall\ i_1, i_2, ..., i_k \in \{1, 2, .., , n\}$ where $i_1 < i_2 < ... < i_k$. Consider,

$$\sum_{(i_1,i_2,\ldots,i_k)\in C(n,k)} \alpha_{i_1 i_2 \ldots i_k}[a_{i_1} \wedge a_{i_2} \wedge ... \wedge a_{i_k}] = 0$$

$$\implies \sum_{(i_1,i_2,\ldots,i_k)\in C(n,k)} \alpha_{i_1 i_2 \ldots i_k} \left(\sum_{\sigma \in P_k} sgn(\sigma)[a_{i_{\sigma(1)}} \otimes a_{i_{\sigma(2)}} \otimes ... \otimes a_{i_{\sigma(k)}}]\right) = 0$$

$\forall\ j_1, j_2, ..., j_k \in \{1, 2, .., , n\}$ where $j_1 < j_2 < ... < j_k$,

$$\sum_{(i_1,i_2,\ldots,i_k)\in C(n,k)} \alpha_{i_1 i_2 \ldots i_k} \left(\sum_{\sigma \in P_k} sgn(\sigma)[a_{i_{\sigma(1)}} \otimes a_{i_{\sigma(2)}} \otimes ... \otimes a_{i_{\sigma(k)}}](a_{j_1}, a_{j_2}, ..., a_{j_k})\right) = 0$$

Since A is an orthonormal basis we get that,

$$\implies \alpha_{j_1 j_2 \ldots j_k} \sum_{\sigma \in P_k} sgn(\sigma)[a_{j_{\sigma(1)}} \otimes a_{j_{\sigma(2)}} ... \otimes a_{j_{\sigma(k)}}](a_{j_1}, a_{j_2}, ..., a_{j_k}) = \alpha_{j_1 j_2 \ldots j_k} = 0$$

$$\implies \wedge_{i=1}^{k} A \text{ is a linearly independent set and a basis of } \mathcal{L}_{ALT}(V^k \to \mathbb{R})$$

□

Corollary 3.4.7.

$$\boxed{dim(\mathcal{L}_{ALT}(V^k \to \mathbb{R})) = \binom{n}{k}}$$

Illustration :
Consider $V = \mathbb{R}^k$ over $\mathbb{R}$ with standard dot product as inner product. Let $E = \{e_1 = \begin{bmatrix} 1 & 0 & . & . & 0 \end{bmatrix}^*,\ e_2 = \begin{bmatrix} 0 & 1 & . & . & 0 \end{bmatrix}^*,\ ...,\ e_k = \begin{bmatrix} 0 & 0 & . & . & 1 \end{bmatrix}^*\}$ be the standard orthonormal basis of $\mathbb{R}^k$. $\forall\ i_1, i_2, ..., i_k \in \{1, 2, ..., n\}, \forall\ x_1, x_2, ..., x_k \in \mathbb{R}^k$,

$$\begin{aligned} [e_{i_1} \wedge ... \wedge e_{i_k}](x_1, ..., x_k) &= \sum_{\sigma \in P_k} sgn(\sigma) \cdot [e_{i_{\sigma(1)}} \otimes ... \otimes e_{i_{\sigma(k)}}](x_1, ..., x_k) \\ &= \sum_{\sigma \in P_k} sgn(\sigma) \cdot {}^{E}x_1[i_{\sigma(1)}] \cdot {}^{E}x_2[i_{\sigma(2)}] \cdot ... \cdot {}^{E}x_k[i_{\sigma(k)}] \end{aligned} \tag{3.5}$$

Let $\wedge_{i=1}^{k} = \{e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k} \mid 1 \le i_1 < i_2 < ... < i_k \le k\}$. From the above Theorem, we get that $\wedge_{i=1}^{k} A$ is a basis of $\mathcal{L}_{ALT}((\mathbb{R}^k)^k \to \mathbb{R})$. Consider the following $k-$tensor,

$$u(\begin{bmatrix} x_{11} \\ x_{12} \\ . \\ . \\ x_{1k} \end{bmatrix}, \begin{bmatrix} x_{21} \\ x_{22} \\ . \\ . \\ x_{2k} \end{bmatrix}, ..., \begin{bmatrix} x_{k1} \\ x_{k2} \\ . \\ . \\ x_{kk} \end{bmatrix}) = \sum_{\sigma \in P_k} sgn(\sigma) \cdot x_{1\sigma(1)} \cdot x_{2\sigma(2)} ... x_{k\sigma(k)}$$

In chapter 4, it will be shown that u is a $k-$dimensional notion of determinant. It is straightforward to verify that u is an alternating $k-$tensor. Since E is the basis of $\mathbb{R}^k$, $\forall\ j \in \{1, 2, ..., k\}$ we have,

$$\begin{bmatrix} x_{j1} & x_{j2} & ... & x_{jk} \end{bmatrix}^* = x_{j1}e_1 + x_{j2}e_2 + ... + x_{jk}e_k$$

Since u is an alternating $k-$tensor we get that

$$u(\begin{bmatrix} x_{11} \\ x_{12} \\ . \\ . \\ x_{1k} \end{bmatrix}, ..., \begin{bmatrix} x_{k1} \\ x_{k2} \\ . \\ . \\ x_{kk} \end{bmatrix}) = \sum_{\sigma \in P_k} sgn(\sigma) \cdot x_1[\sigma(1)] \cdot x_2[\sigma(2)] \cdot ... \cdot x_k[\sigma(k)] \cdot u(e_1, e_2, ..., e_k) \tag{3.6}$$

From the above expression, it is quite clear that the value of $u(e_1, e_2, ..., e_k)$ is sufficient to determine the action of u on any $\begin{bmatrix} x_{11} & x_{12} & ... & x_{1k} \end{bmatrix}^*, ..., \begin{bmatrix} x_{k1} & x_{k2} & ... & x_{kk} \end{bmatrix}^* \in \mathbb{R}^k$.

$$u(e_1, e_2, ..., e_k) = 1$$

Combining equations 3.5 and 3.6 we get,

$$\implies u(\begin{bmatrix} x_{11} \\ x_{12} \\ . \\ . \\ x_{1k} \end{bmatrix}, \begin{bmatrix} x_{21} \\ x_{22} \\ . \\ . \\ x_{2k} \end{bmatrix}, ..., \begin{bmatrix} x_{k1} \\ x_{k2} \\ . \\ . \\ x_{kk} \end{bmatrix}) = [e_1 \wedge e_2 \wedge ... \wedge e_k](x_1, x_2, ..., x_k)$$

With this illustration, you can observe how $\wedge_{i=1}^{k} A$ works as a basis of $\mathcal{L}_{ALT}((\mathbb{R}^k)^k \to \mathbb{R})$ and also note that the value of $u(e_1, e_2, ..., e_k)$ is sufficient to compute u on any $\begin{bmatrix} x_{11} & x_{12} & ... & x_{1k} \end{bmatrix}^*, ..., \begin{bmatrix} x_{k1} & x_{k2} & ... & x_{kk} \end{bmatrix}^* \in \mathbb{R}^k$.

3.4.4 Basis Transformation and Invariance of computation of alternating $k-$tensor under orthonormal basis transformations

Definition 3.4.3. $\forall\ u \in \mathcal{L}_{ALT}(V^k \to \mathbb{R})$, from Theorem 3.3.6 we get that,

$$u = \sum_{i_1=1}^{n} \sum_{i_2=i_1+1}^{n} ... \sum_{i_k=i_{k-1}+1}^{n} u(a_{i_1}, a_{i_2}, ..., a_{i_k})[a_{i_1} \wedge a_{i_2} \wedge ... \wedge a_{i_k}]$$

$\forall\ 1 \leq i_1 < i_2 < ... < i_k \leq n$, we define

$$^{\wedge_{i=1}^{k} A}u[i_1, i_2, ..., i_k] = u(a_{i_1}, a_{i_2}, ..., a_{i_k})$$

Remark :

1. $\forall\ \sigma \in P_k$, $\forall\ 1 \leq i_1 < i_2 < ... < i_k < n$, we get that

$$^{\wedge_{i=1}^{k}}u[i_{\sigma(1)}, i_{\sigma(2)}, ..., i_{\sigma(k)}] = sgn(\sigma) \cdot u(a_{i_1}, a_{i_2}, ..., a_{i_k})$$

Corollary 3.4.8. Let $A = \{a_1, ..., a_n\}$ and $B = \{b_1, ..., b_n\}$ be any two orthonormal basis of V. From Theorem 3.4.6, $\wedge_{i=1}^{k} A$ and $\wedge_{i=1}^{k} B$ form bases for $\mathcal{L}_{ALT}(V^k \to \mathbb{R})$. Let $M \in \mathbb{R}^{n \times n}$ be the transformation matrix from basis A to B. Then, $\forall\ u \in \mathcal{L}_{ALT}(V^k \to \mathbb{R})$, $\forall\ 1 \leq j_1 < j_2 < ... < j_k \leq n$

$$\boxed{^{\wedge_{i=1}^{k} B}u[j_1, ..., j_k] = \sum_{(i_1,...,i_k) \in C(n,k)} \left(\sum_{\sigma \in P_k} sgn(\sigma) \cdot M_{j_1 i_{\sigma(1)}} \cdot ... \cdot M_{j_k i_{\sigma(k)}} \right) {}^{\wedge_{i=1}^{k}}u[i_1, ..., i_k]}$$

Corollary 3.4.9. Let $A = \{a_1, ..., a_n\}$ be any orthonormal basis of V. From Theorem 3.4.6 we get that, $\wedge_{i=1}^{k} A$ forms basis for $\mathcal{L}_{ALT}(V^k \to \mathbb{R})$. Then, $\forall$

$u \in \mathcal{L}_{ALT}(V^k \to \mathbb{R})$, $\forall\ x_1, x_2, ..., x_k \in V$,

$$\boxed{u(x_1, ..., x_k) = \sum_{(r_1,...,r_k)\in C(n,k)} \left(\sum_{\sigma \in P_k} sgn(\sigma) \cdot {}^{A}x_1[r_{\sigma(1)}] \cdot ... {}^{A}x_k[r_{\sigma(k)}] \right) \cdot {}^{\wedge_{i=1}^{k} A}u[r_1, ..., r_k]}$$

Proof. Proofs of above corollaries are straightforward since $\mathcal{L}_{ALT}(V^k \to \mathbb{R})$ is a subspace of $\mathcal{L}(V^k \to \mathbb{R})$. We recommend the reader to go through section 2.4 first in order to get finer understanding here. $\square$

Chapter 4

Cross Products and Determinants

4.1 2-dimensional determinants and cross products

Let V be a finite dimensional inner product space where $dim(V) = 2$. Let $A = \{a_1, a_2\}$ be any ordered orthonormal basis of V. From Corollary 3.2.9 we get, $\forall$ $u \in \mathcal{L}_{ALT}(V^2 \to \mathbb{R})$, $\forall$ $x, y \in V$,

$$u(x, y) = u(a_1, a_2)[a_1 \wedge a_2](x, y) = u(b_1, b_2)[b_1 \wedge b_2](x, y)$$

Lemma 4.1.1. Let V be a finite dimensional inner product space where $dim(V) = 2$. Let $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$ be any ordered orthonormal bases of V. Then, $\forall$ $u \in \mathcal{L}_{ALT}(V^2 \to \mathbb{R})$ such that $u(b_1, b_2) \neq 0$ we get,

$$u(b_1, b_2) = \pm u(a_1, a_2)$$

Proof.

$$\begin{aligned} u(b_1, b_2) &= u(a_1, a_2)[a_1 \wedge a_2](b_1, b_2) \\ &= ([a_1 \otimes a_2](b_1, b_2) - [a_2 \otimes a_1](b_1, b_2))u(a_1, a_2) \\ &= ((a_1, b_1)(a_2, b_2) - (a_2, b_1)(a_1, b_2))u(a_1, a_2) \end{aligned} \tag{4.1}$$

$$\begin{aligned} u(a_1, a_2) &= u(b_1, b_2)[b_1 \wedge b_2](a_1, a_2) \\ &= ([b_1 \otimes b_2](a_1, a_2) - [b_2 \otimes b_1](a_1, a_2))u(b_1, b_2) \\ &= ((b_1, a_1)(b_2, a_2) - (b_2, a_1)(b_1, a_2))u(b_1, b_2) \end{aligned} \tag{4.2}$$

Combining equations 4.1 and 4.2, we get

$$u(b_1, b_2) = ((a_1, b_1)(a_2, b_2) - (a_1, b_2)(a_2, b_1))^2 u(b_1, b_2)$$

$$\implies (a_1, b_1)(a_2, b_2) - (a_1, b_2)(a_2, b_1) = \pm 1$$

$$\implies u(b_1, b_2) = \pm u(a_1, a_2)$$

□

Definition 4.1.1. Let V be a finite dimensional inner product space over $\mathbb{R}$ where $dim(V) = 2$. Let $A = \{a_1, a_2\}$ be an ordered orthonormal basis of A. Then, there exist $\lambda \in \mathcal{L}_{ALT}(V^2 \to \mathbb{R})$ such that

$$\lambda(a_1, a_2) = 1$$

1. Notice that $dim(\mathcal{L}_{ALT}(V^2 \to \mathbb{R})) = \binom{n}{2} = \binom{2}{2} = 1$. Hence, any other alternating $2-$tensor λ in $\mathcal{L}_{ALT}(V^2 \to \mathbb{R})$ must be a scalar multiple of λ if $\lambda \neq 0$. Moreover, this indicates that λ is unique and it is defined as the volume element of V.

2. Let $B = \{b_1, b_2\}$ be any other ordered orthonormal basis of V then, from Lemma 4.1.1 we get, $\lambda(b_1, b_2) = \pm\lambda(a_1, a_2)$. If $\lambda(b_1, b_2) = 1$ then, we define B to be positively oriented with respect to A. Otherwise, we define basis B to be negatively oriented with respect to A.

Remark :

1. In $\mathbb{R}^2$ with standard dot product as inner product, we usually define that standard ordered orthonormal basis $E = \{e_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^*, e_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^*\}$ to be positively oriented.

2. For the purpose of the theory here, we define $B = \{b_1, b_2\}$ to be ordered positively oriented where $\lambda(b_1, b_2) = 1$. Note that once we define the orien-

tation of one ordered orthonormal basis the orientation of any other ordered orthonormal basis can be computed.

Definition 4.1.2. Let V be a finite dimensional inner product space over $\mathbb{R}$ where $dim(V) = 2$. Let $A = \{a_1, a_2\}$ be ordered positively oriented orthonormal basis of V (with respect to B 4.1). $\forall\ x, y \in V$, since A is a basis of V, there exist unique α_1, α_2, β_1, $\beta_2 \in \mathbb{R}$ such that,

$$x = \alpha_1 a_1 + \alpha_2 a_2 \qquad\qquad y = \beta_1 a_1 + \beta_2 a_2$$

Now, we define the determinant function $det : V \times V \to \mathbb{R}$ as follows,

$$det(x, y) = \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} = \alpha_1\beta_2 - \alpha_2\beta_1$$

Theorem 4.1.2. Let V be a finite dimensional inner product space over $\mathbb{R}$ where $dim(V) = 2$. Let $A = \{a_1, a_2\}$ be any ordered positively oriented orthonormal basis of V (with respect to B 4.1). Let $\lambda \in \mathcal{L}_{ALT}(V^2 \to \mathbb{R})$ be the volume element of V. Then, $\forall\ x, y \in V$,

$$\lambda(x, y) = det(x, y) \tag{4.3}$$

Note that proving this theorem also indicates that det function is well defined.

Proof. $\forall\ x, y \in V$, since A is a basis of V, there exist unique α_1, α_2, β_1, $\beta_2 \in \mathbb{R}$ such that,

$$x = \alpha_1 a_1 + \alpha_2 a_2 \qquad\qquad y = \beta_1 a_1 + \beta_2 a_2$$

$$\lambda(x, y) = \lambda(\alpha_1 a_1 + \alpha_2 a_2, \beta_1 a_1 + \beta_2 a_2) = \sum_{i=1}^{2}\sum_{j=1}^{2} \lambda(\alpha_i a_i, \beta_j a_j)$$

Since $\lambda \in \mathcal{L}_{ALT}(V^2 \to \mathbb{R})$ we get,

$$\lambda(x, y) = (\alpha_1\beta_2 - \alpha_2\beta_1)\lambda(a_1, a_2)$$

Since λ is the volume element of V, for any positively oriented ordered orthonormal basis A we have, $\lambda(a_1, a_2) = 1$,

$$\implies \lambda(x, y) = det(x, y)$$

$\square$

Remark :

1. Since $\lambda \in \mathcal{L}(V^2 \to \mathbb{R})$, $\forall\ x, y \in V$,

$$det(x, y) = \lambda(x, y)$$

   This implies determinant is also an alternating $2-$tensor.

2. $\forall\ x \in V$,

$$det(x, x) = \lambda(x, x) = 0$$

Next, we define a function $\times$ which takes a vector as input y and outputs a vector $\times(y)$ with a property that $(x, \times(y)) = det(x, y)$. Observe that the output vector $\times(x)$ is orthogonal to x.

Definition 4.1.3. Let V be a finite dimensional inner product space over $\mathbb{R}$ where $dim(V) = 2$. Let $A = \{a_1, a_2\}$ be positively oriented ordered orthonormal basis of V (with respect to B 4.1). We now define the cross product of a vector $\times : V \to V$ as follows, $\forall\ x \in V$,

$$\times(x) = a_2^*(x)a_1 - a_1^*(x)a_2$$

Theorem 4.1.3. Let V be a finite dimensional inner product space over $\mathbb{R}$ where $dim(V) = 2$. Let $A = \{a_1, a_2\}$ be positively oriented ordered orthonormal basis of V (with respect to B 4.1). Then, $\forall\ x, y \in V$,

$$(x, \times(y)) = det(x, y)$$

Proof. $\forall\ x, y \in V$,

$$\begin{aligned}(x, \times(y)) &= (x, a_2^*(y)a_1 - a_1^*(y)a_2)\\ &= (a_2, y)(x, a_1) - (a_1, y)(x, a_2)\end{aligned}$$

Since A is a basis of V, there exist unique α_1, α_2, β_1, $\beta_2 \in \mathbb{R}$ such that,

$$x = \alpha_1 a_1 + \alpha_2 a_2 \qquad\qquad y = \beta_1 a_1 + \beta_2 a_2$$

$$\implies (x, \times(y)) = \alpha_1\beta_2 - \alpha_2\beta_1 = det(x, y)$$

□

Remark :

1. With the above Theorem, we get that cross product of a vector 4.1.3 is well-defined. Consider $y \in V$. Let $\times_A(y)$ denote the cross product of y when defined using basis A. Let $\times_C(y)$ denote the cross product of y when defined using basis C. We get, $\forall\ x \in V$,

$$(x, \times_A(y) - \times_C(y)) = 0$$

   Consider $x = \times_A(y) - \times_C(y)$, from the positive definiteness of inner product we get,

$$\times_A(y) = \times_C(y)$$

2. $\forall\ x \in V$,

$$(x, \times(x)) = det(x, x) = \lambda(x, x) = 0$$

3. $\forall\ x \in V$, if $\times(x) = y$ then,

$$(y, \times(x)) = \|y\|^2 = \lambda(y, y) = det(y, y)$$

4.2 3-dimensional determinants and cross products

Let V be a finite dimensional inner product space where $dim(V) = 2$. Let $A = \{a_1, a_2, a_3\}$ be any ordered orthonormal basis of V. From Corollary 3.3.9 we get, $\forall\ u \in \mathcal{L}_{ALT}(V^3 \to \mathbb{R})$, $\forall\ x, y, z \in V$,

$$u(x, y, z) = u(a_1, a_2, a_3)[a_1 \wedge a_2 \wedge a_3](x, y, z) = u(b_1, b_2, b_3)[b_1 \wedge b_2 \wedge b_3](x, y, z)$$

Lemma 4.2.1. Let V be a finite dimensional inner product space where $dim(V) = 3$. Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ be any ordered orthonormal basis of V. Then, $\forall\ u \in \mathcal{L}_{ALT}(V^3 \to \mathbb{R})$ such that $u(b_1, b_2, b_3) \neq 0$ we get,

$$u(b_1, b_2, b_3) = \pm u(a_1, a_2, a_3)$$

Proof.

$$\begin{aligned} u(b_1, b_2, b_3) &= u_A(b_1, b_2, b_3) = u(a_1, a_2, a_3)[a_1 \wedge a_2 \wedge a_3](b_1, b_2, b_3) \\ &= (\sum_{\sigma \in P_3} sgn(\sigma)[a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes a_{\sigma(3)}](b_1, b_2, b_3)) \cdot u(a_1, a_2, a_3) \\ &= ([a_1 \otimes a_2 \otimes a_3](b_1, b_2, b_3) - [a_1 \otimes a_3 \otimes a_2](b_1, b_2, b_3) \\ &- [a_2 \otimes a_1 \otimes a_3](b_1, b_2, b_3) + [a_2 \otimes a_3 \otimes a_1](b_1, b_2, b_3) \\ &- [a_3 \otimes a_2 \otimes a_1](b_1, b_2, b_3) + [a_3 \otimes a_1 \otimes a_2](b_1, b_2, b_3))u(a_1, a_2, a_3) \end{aligned} \tag{4.4}$$

$$\begin{aligned} u(a_1, a_2, a_3) &= u_B(a_1, a_2, a_3) = u(b_1, b_2, b_3)[b_1 \wedge b_2 \wedge b_3](a_1, a_2, a_3) \\ &= (\sum_{\sigma \in P_3} sgn(\sigma)[b_{\sigma(1)} \otimes b_{\sigma(2)} \otimes b_{\sigma(3)}](a_1, a_2, a_3)) \cdot u(b_1, b_2, b_3) \\ &= ([b_1 \otimes b_2 \otimes b_3](a_1, a_2, a_3) - [b_1 \otimes b_3 \otimes b_2](a_1, a_2, a_3) \\ &- [b_2 \otimes b_1 \otimes b_3](a_1, a_2, a_3) + [b_2 \otimes b_3 \otimes b_1](a_1, a_2, a_3) \\ &- [b_3 \otimes b_2 \otimes b_1](a_1, a_2, a_3) + [b_3 \otimes b_1 \otimes b_2](a_1, a_2, a_3))u(b_1, b_2, b_3) \end{aligned} \tag{4.5}$$

Note that for any distinct $i, j, k \in \{1, 2, 3\}$,

$$[b_i \otimes b_j \otimes b_k](a_1, a_2, a_3) = [a_1 \otimes a_2 \otimes a_3](b_i, b_j, b_k)$$

Combining equations 4.4 and 4.5, we get

$$u(b_1, b_2, b_3) = \left(\sum_{\sigma \in P_3} sgn(\sigma) \cdot \prod_{i=1}^{3} (b_i, a_{\sigma(i)})\right)^2 \cdot u(b_1, b_2, b_3)$$

$$\implies \sum_{\sigma \in P_3} sgn(\sigma) \cdot \prod_{i=1}^{3} (b_i, a_{\sigma(i)}) = \pm 1$$

$$\implies u(b_1, b_2, b_3) = \pm u(a_1, a_2, a_3)$$

□

Definition 4.2.1. Let V be a finite dimensional inner product space over $\mathbb{R}$ where $dim(V) = 3$. Let $A = \{a_1, a_2, a_3\}$ be an ordered orthonormal basis of A. Then, there exist $\lambda \in \mathcal{L}_{ALT}(V^3 \to \mathbb{R})$ such that

$$\lambda(a_1, a_2, a_3) = 1$$

1. Notice that $dim(\mathcal{L}_{ALT}(V^3 \to \mathbb{R})) = \binom{n}{3} = \binom{3}{3} = 1$. Hence, any other alternating $3-$tensor λ in $\mathcal{L}_{ALT}(V^3 \to \mathbb{R})$ must be a scalar multiple of λ if $\lambda \neq 0$. Moreover, this indicates that λ is unique and it is defined as the volume element of V.

2. Let $B = \{b_1, b_2, b_3\}$ be any other ordered orthonormal basis of V then, from Lemma 4.2.1 we get, $\lambda(b_1, b_2, b_3) = \pm\lambda(a_1, a_2, a_3)$. If $\lambda(b_1, b_2, b_3) = 1$ then, we define B to be positively oriented with respect to A. Otherwise, we define basis B to be negatively oriented with respect to A.

Remark :

1. In $\mathbb{R}^3$ with standard dot product as inner product, we usually define that standard ordered orthonormal basis $E = \{e_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^*, e_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^*, e_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^*\}$ to be positively oriented.

2. For the purpose of the theory here, we define $B = \{b_1, b_2, b_3\}$ to be ordered positively oriented where $\lambda(b_1, b_2, b_3) = 1$. Note that once we define the orientation of one ordered orthonormal basis the orientation of any other ordered orthonormal basis can be computed.

Definition 4.2.2. Let V be a finite dimensional inner product space over $\mathbb{R}$ where $dim(V) = 3$. Let $A = \{a_1, a_2, a_3\}$ be ordered positively oriented orthonormal basis of V (with respect to B 4.2). $\forall\ x, y, z \in V$, since A is a basis of V, there exist unique α_1, α_2, α_3, β_1, β_2, β_3, γ_1, γ_2, $\gamma_3 \in \mathbb{R}$ such that,

$$x = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 \qquad y = \beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3 \qquad z = \gamma_1 a_1 + \gamma_2 a_2 + \gamma_3 a_3$$

Now, we define the determinant function $det : V \times V \times V \to \mathbb{R}$ as follows,

$$det(x, y, z) = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} = \alpha_1(\beta_2\gamma_3 - \beta_3\gamma_2) - \alpha_2(\beta_1\gamma_3 - \beta_3\gamma_1) + \alpha_3(\beta_1\gamma_2 - \beta_2\gamma_1)$$

Theorem 4.2.2. Let V be a finite dimensional inner product space over $\mathbb{R}$ where $dim(V) = 3$. Let $A = \{a_1, a_2, a_3\}$ be any ordered positively oriented orthonormal basis of V (with respect to B 4.2). Let $\lambda \in \mathcal{L}_{ALT}(V^3 \to \mathbb{R})$ be the volume element of V. Then, $\forall\ x, y, z \in V$,

$$\lambda(x, y, z) = det(x, y, z) \tag{4.6}$$

Note that proving this theorem also indicates that det function is well defined.

Proof. $\forall\ x, y, z \in V$, since A is a basis of V, there exist unique α_1, α_2, α_3, β_1, β_2, β_3, γ_1, γ_2, $\gamma_3 \in \mathbb{R}$ such that,

$$x = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 \qquad y = \beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3 \qquad z = \gamma_1 a_1 + \gamma_2 a_2 + \gamma_3 a_3$$

$$\lambda(x, y, z) = \lambda(\sum_{i=1}^{3} \alpha_i a_i, \sum_{j=1}^{3} \alpha_j a_j, \sum_{k=1}^{3} \alpha_k a_k) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \alpha_i \beta_j \gamma_k \lambda(a_i, a_j, a_k)$$

Since $\lambda \in \mathcal{L}_{ALT}(V^3 \to \mathbb{R})$ we get,

$$\lambda(x, y, z) = (\alpha_1(\beta_2\gamma_3 - \beta_3\gamma_2) - \beta_1(\alpha_2\gamma_3 - \alpha_3\gamma_2) + \gamma_1(\alpha_2\beta_3 - \alpha_3\beta_2))\lambda(a_1, a_2, a_3)$$

Since λ is the volume element of V, for any positively oriented ordered orthonormal basis A we have, $\lambda(a_1, a_2, a_3) = 1$,

$$\implies \lambda(x, y, z) = det(x, y, z)$$

$\square$

Remark :

1. Since $\lambda \in \mathcal{L}(V^3 \to \mathbb{R})$, $\forall\ x, y, z \in V$,

   $$det(x, y, z) = \lambda(x, y, z)$$

   This implies determinant is also an alternating $3-$tensor.

2. $\forall\ x, y, z \in V$, if $x = y$ or $y = z$ or $z = x$ then,

   $$det(x, y, z) = \lambda(x, y, z) = 0$$

Next, we define a function $\times$ which takes a vector as input y, z and outputs a vector $\times(y, z)$ with a property that $(x, \times(y, z)) = det(x, y, z)$. Observe that the output vector $\times(y, z)$ is orthogonal to y and z.

Definition 4.2.3. Let V be a finite dimensional inner product space over $\mathbb{R}$ where $dim(V) = 3$. Let $A = \{a_1, a_2, a_3\}$ be positively oriented ordered orthonormal basis of V (with respect to B 4.2). We now define the cross product of a vector $\times : V^2 \to V$ as follows, $\forall\ x, y \in V$,

$$\times(x, y) = [a_2 \wedge a_3](x, y)a_1 - [a_1 \wedge a_3](x, y)a_2 + [a_1 \wedge a_2](x, y)a_3$$

Theorem 4.2.3. Let V be a finite dimensional inner product space over $\mathbb{R}$ where $dim(V) = 3$. Let $A = \{a_1, a_2, a_3\}$ be positively oriented ordered orthonormal basis of V (with respect to B 4.2). Then, $\forall\ x, y, z \in V$,

$$(x, \times(y, z)) = det(x, y, z)$$

Proof. $\forall\ x, y, z \in V$,

$$\begin{aligned}(x, \times(y, z)) &= (x, [a_2 \wedge a_3](y, z)a_1 - [a_1 \wedge a_3](y, z)a_2 + [a_1 \wedge a_2](y, z)a_3) \\ &= [a_2 \wedge a_3](y, z)(x, a_1) - [a_1 \wedge a_3](y, z)(x, a_2) + [a_1 \wedge a_2](y, z)(x, a_3)\end{aligned}$$

Since A is a basis of V, there exist unique α_1, α_2, α_3, β_1, β_2, β_3, γ_1, γ_2, $\gamma_3 \in \mathbb{R}$ such that,

$$x = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 \qquad y = \beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3 \qquad z = \gamma_1 a_1 + \gamma_2 a_2 + \gamma_3 a_3$$

$$\implies (x, \times(y, z)) = \alpha_1(\beta_2\gamma_3 - \beta_3\gamma_2) - \alpha_2(\beta_1\gamma_3 - \beta_3\gamma_1) + \alpha_3(\beta_1\gamma_2 - \beta_2\gamma_1)$$

$$\implies (x, \times(y, z)) = \lambda(x, y, z) = det(x, y, z)$$

$\square$

Remark :

1. With the above Theorem, we get that cross product of a vector 4.2.3 is well-defined. Consider $y, z \in V$. Let $\times_A(y, z)$ denote the cross product of y, z when defined using basis A. Let $\times_C(y, z)$ denote the cross product of y, z when defined using basis C. We get, $\forall\ x \in V$,

   $$(x, \times_A(y, z) - \times_C(y, z)) = 0$$

   Consider $x = \times_A(y, z) - \times_C(y, z)$, from the positive definiteness of inner product we get,

   $$\times_A(y, z) = \times_C(y, z)$$

2. $\forall\ x, y \in V$,

$$(x, \times(x, y)) = \lambda(x, x, y) = (y, \times(x, y)) = \lambda(y, x, y) = 0$$

3. $\forall\ x, y \in V$, if $\times(x, y) = z$ then,

$$(z, \times(x, y)) = \|z\|^2 = \lambda_E(z, x, y)$$

4.3 n-dimensional determinants and cross products

Let V be a finite dimensional inner product space where $dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ be any ordered orthonormal basis of V. From Corollary 3.4.9 we get, $\forall\ u \in \mathcal{L}_{ALT}(V^n \to \mathbb{R})$, $\forall\ x_1, x_2, ..., x_n \in V$,

$$\begin{aligned} u(x_1, x_2, ..., x_n) &= u(a_1, a_2, ..., a_n)[a_1 \wedge a_2 \wedge ... \wedge a_n](x_1, x_2, ..., x_n) \\ &= u(b_1, b_2, ..., b_n)[b_1 \wedge b_2 \wedge ... \wedge b_n](x_1, x_2, ..., x_n) \end{aligned}$$

Lemma 4.3.1. Let V be a finite dimensional inner product space where $dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$ be any ordered orthonormal basis of V. Then, $\forall\ u \in \mathcal{L}_{ALT}(V^n \to \mathbb{R})$ such that $u(b_1, b_2, ..., b_n) \neq 0$ we get,

$$u(b_1, b_2, ..., b_n) = \pm u(a_1, a_2, ..., a_n)$$

Proof.

$$\begin{aligned} u(b_1, b_2, ..., b_n) &= u(a_1, a_2, ..., a_n)[a_1 \wedge a_2 \wedge ... \wedge a_n](b_1, b_2, ..., b_n) \\ &= (\sum_{\sigma \in P_n} sgn(\sigma) \cdot \prod_{i=1}^{n} (b_i, a_{\sigma(i)}))u(b_1, b_2, ..., b_n) \end{aligned} \tag{4.7}$$

$$\begin{aligned} u(a_1, a_2, ..., a_n) &= u(b_1, b_2, ..., b_n)[b_1 \wedge b_2 \wedge ... \wedge b_n](a_1, a_2, ..., a_n) \\ &= (\sum_{\sigma \in P_n} sgn(\sigma) \cdot \prod_{i=1}^{n} (a_i, b_{\sigma(i)}))u(a_1, a_2, ..., a_n) \end{aligned} \tag{4.8}$$

Note that $\forall\ \sigma \in P_n$, $\forall\ x \in \{1, 2, ..., n\}$, let $\sigma(x) = y$ then $\sigma^{-1}(y) = x$

$$\prod_{i=1}^{n} (a_i, b_{\sigma(i)}) = \prod_{j=1}^{n} (a_{\sigma^{-1}(j)}, b_j)$$

Claim : Let $f : P_n \to P_n$ be defined as follows,

$$f(\sigma) = \sigma^{-1}$$

Then, f is a bijection.

1. Let σ, $\rho \in P_n$ such that $f(\sigma) = f(\rho)$ then,

$$\sigma^{-1} = \rho^{-1} \implies \sigma \circ \sigma^{-1} \circ \rho = \sigma \circ \rho^{-1} \circ \rho \implies \sigma = \rho$$

2. $\forall\ \sigma \in P_n$, $\sigma^{-1} \in P_n$ then,

$$f(\sigma^{-1}) = \left(\sigma^{-1}\right)^{-1} = \sigma$$

Also recall that $sgn(\sigma \circ \sigma^{-1}) = sgn(id) = 1$ which implies that $sgn(\sigma) = sgn(\sigma^{-1})$

$$\implies \sum_{\sigma^{-1} \in P_n} sgn(\sigma^{-1}) \cdot \prod_{i=1}^{n} (a_{\sigma^{-1}(j)}, b_j) = \sum_{\sigma \in P_n} sgn(\sigma) \cdot \prod_{i=1}^{n} (a_{\sigma(i)}, b_i)$$

Combining equations 4.7 and 4.8 we get,

$$u(b_1, b_2, ..., b_n) = \left(\sum_{\sigma \in P_n} sgn(\sigma) \cdot \prod_{i=1}^{n} (a_{\sigma(i)}, b_i) \right)^2 u(b_1, b_2, ..., b_n)$$

$$\implies \sum_{\sigma \in P_n} sgn(\sigma) \cdot \prod_{i=1}^{n} (b_i, a_{\sigma(i)}) = \pm 1$$

$$\implies u(b_1, b_2, ..., b_n) = \pm u(a_1, a_2, ..., a_n)$$

$\square$

Definition 4.3.1. Let V be a finite dimensional inner product space over $\mathbb{R}$ where $dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ be an ordered orthonormal basis of A. Then, there exist $\lambda \in \mathcal{L}_{ALT}(V^n \to \mathbb{R})$ such that

$$\lambda(a_1, a_2, ..., a_n) = 1$$

1. Notice that $dim(\mathcal{L}_{ALT}(V^n \to \mathbb{R})) = \binom{n}{n} = 1$. Hence, any other alternating $n-$tensor λ in $\mathcal{L}_{ALT}(V^n \to \mathbb{R})$ must be a scalar multiple of λ if $\lambda \neq 0$.

Moreover, this indicates that λ is unique and it is defined as the volume element of V.

2. Let $B = \{b_1, b_2, ..., b_n\}$ be any other ordered orthonormal basis of V then, from Lemma 4.2.1 we get, $\lambda(b_1, b_2, ..., b_n) = \pm\lambda(a_1, a_2, ..., a_n)$. If $\lambda(b_1, b_2, ..., b_n) = 1$ then, we define B to be positively oriented with respect to A. Otherwise, we define basis B to be negatively oriented with respect to A.

Remark :

1. In $\mathbb{R}^n$ with standard dot product as inner product, we usually define that standard ordered orthonormal basis $E = \{e_1 = \begin{bmatrix}1 & 0 & . & . & 0\end{bmatrix}^*, e_2 = \begin{bmatrix}0 & 1 & . & . & 0\end{bmatrix}^*, ..., e_n = \begin{bmatrix}0 & 0 & . & . & 1\end{bmatrix}^*\}$ to be positively oriented.
2. For the purpose of the theory here, we define $B = \{b_1, b_2, ..., b_n\}$ to be ordered positively oriented where $\lambda(b_1, b_2, ..., b_n) = 1$. Note that once we define the orientation of one ordered orthonormal basis the orientation of any other ordered orthonormal basis can be computed.

Definition 4.3.2. Let V be a finite dimensional inner product space over $\mathbb{R}$ where $dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ be ordered positively oriented orthonormal basis of V (with respect to B 4.3). $\forall\ x_1, x_2, ..., x_n \in V$, since A is a basis of V, there exist unique α_{1i_1}, α_{2i_2}, ..., $\alpha_{ni_n} \in \mathbb{R}$ where $i_1, i_2, ..., i_n \in \{1, 2, ..., n\}$ such that,

$$x_1 = \sum_{i_1=1}^{n} \alpha_{1i_1} a_{i_1} \qquad x_2 = \sum_{i_2=1}^{n} \alpha_{2i_2} a_{i_2} \qquad \dots \qquad x_n = \sum_{i_n=1}^{n} \alpha_{ni_n} a_{i_n}$$

Now, we define the determinant function $det : V^n \to \mathbb{R}$ as follows,

$$det(x_1, x_2, ..., x_n) = \begin{vmatrix} \alpha_{11} & \alpha_{12} & . & . & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & . & . & \alpha_{2n} \\ . & . & . & . & . \\ \alpha_{n1} & \alpha_{n2} & . & . & \alpha_{nn} \end{vmatrix} = \sum_{\sigma \in P_n} sgn(\sigma) \cdot \alpha_{1i_{\sigma(1)}} \cdot \alpha_{2i_{\sigma(2)}} \dots \cdot \alpha_{ni_{\sigma(n)}}$$

Theorem 4.3.2. Let V be a finite dimensional inner product space over $\mathbb{R}$ where $dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ be any ordered positively oriented orthonormal basis of V (with respect to B 4.3). Let $\lambda \in \mathcal{L}_{ALT}(V^n \to \mathbb{R})$ be the volume element of V. Then, $\forall\ x_1, x_2, ..., x_n \in V$,

$$\lambda(x_1, x_2, ..., x_n) = det(x_1, x_2, ..., x_n) \tag{4.9}$$

Note that proving this theorem also indicates that det function is well defined.

Proof. $\forall\ x_1, x_2, ..., x_n \in V$, since A is a basis of V, there exist unique α_{1i_1}, α_{2i_2}, ..., $\alpha_{ni_n} \in \mathbb{R}$ such that,

$$x_1 = \sum_{i_1=1}^{n} \alpha_{1i_1} a_{i_1} \qquad x_2 = \sum_{i_2=1}^{n} \alpha_{2i_2} a_{i_2} \qquad ... \qquad x_n = \sum_{i_n=1}^{n} \alpha_{ni_n} a_{i_n}$$

$$\begin{aligned} \lambda(x_1, x_2, ..., x_n) &= \lambda(\sum_{i_1=1}^{n} \alpha_{1i_1} a_{i_1}, \sum_{i_2=1}^{n} \alpha_{2i_2} a_{i_2}, ..., \sum_{i_n=1}^{n} \alpha_{ni_n} a_{i_n}) \\ &= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} ... \sum_{i_n=1}^{n} \alpha_{1i_1} \alpha_{2i_2} ... \alpha_{ni_n} \lambda(a_{i_1}, a_{i_2}, ..., a_{i_n}) \end{aligned}$$

Since $\lambda \in \mathcal{L}_{ALT}(V^n \to \mathbb{R})$, we get

$$\lambda(x_1, x_2, ..., x_n) = \sum_{\sigma \in P_n} sgn(\sigma) \cdot \alpha_{1\sigma(1)} \cdot \alpha_{2\sigma(2)} ... \cdot \alpha_{n\sigma(n)} \lambda(a_1, a_2, ..., a_n)$$

Since $\lambda(e_1, e_2, ..., e_n) = 1$ we get,

$$\lambda(x_1, x_2, ..., x_n) = det(x_1, x_2, ..., x_n)$$

$\square$

Remark :

1. Since $\lambda \in \mathcal{L}(V^n \to \mathbb{R})$, $\forall\ x_1, x_2, ..., x_n \in V$,

$$det(x_1, x_2, ..., x_n) = \lambda(x_1, x_2, ..., x_n)$$

   This implies determinant is also an alternating $n-$tensor.

2. $\forall\ x_1, x_2, ..., x_n \in V$, if $x_i = x_j$ for any $i, j \in \{1, 2, ..., n\}$ where $i < j$ then,

$$det(x_1, ..., x_i, ..., x_j, ..., x_n) = \lambda(x_1, ..., x_i, ..., x_j, ..., x_n) = 0$$

Next, we define a function $\times$ which takes a vector as input $x_1, x_2, ..., x_{n-1}$ and outputs a vector $\times(x_1, x_2, ..., x_{n-1})$ with a property that $(x_n, \times(x_1, x_2, ..., x_{n-1})) = det(x_n, x_1, x_2, ..., x_{n-1})$. Observe that the output vector $\times(x_1, x_2, ..., x_{n-1})$ is orthogonal to each x_i.

Definition 4.3.3. Let V be a finite dimensional inner product space over $\mathbb{R}$ where $dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ be positively oriented ordered orthonormal basis of V (with respect to B 4.3). We now define the cross product of a vector $\times : V^{n-1} \to V$ as follows, $\forall\ x_1, x_2, ..., x_{n-1} \in V$,

$$\begin{aligned}\times(x_1, ..., x_{n-1}) &= [a_2 \wedge ... \wedge a_n](x_1, x_2, ..., x_{n-1})a_1 - [a_1 \wedge a_3 \wedge ... \wedge a_n](x_1, x_2, ..., x_{n-1})a_2 \\ &+ ... + (-1)^{i-1}[a_1 \wedge ... \wedge a_{i-1} \wedge a_{i+1} \wedge ... \wedge a_n](x_1, x_2, ..., x_{n-1})a_i \\ &+ ... + (-1)^{n-1}[a_1 \wedge a_2 \wedge ... \wedge a_{n-1}](x_1, x_2, ..., x_{n-1})a_n\end{aligned}$$

We set up the notation that $\kappa_i(a_1, a_2, ..., a_n) = a_1 \wedge a_2 \wedge ... \wedge a_{j-1} \wedge a_{j+1} \wedge ... \wedge a_n$ (i.e, κ_j is the $n-1$ wedge product which doesn't contain a_j). Then,

$$\begin{aligned}\times(x_1, ..., x_{n-1}) &= \kappa_1(a_1, ..., a_n)a_1 + ... + (-1)^{i-1}\kappa_i(a_1, ..., a_n)a_i + ... + (-1)^{n-1}\kappa_n(a_1, ..., a_n)a_n \\ &= \sum_{i=1}^{n}(-1)^{i-1}\kappa_i(a_1, a_2, ..., a_n)a_i\end{aligned}$$

Lemma 4.3.3. $\forall\ j \in \{1, 2, ..., n\}$,

$$a_1 \wedge a_2 \wedge ... \wedge a_n = (-1)^{j-1}a_j \wedge \kappa_j(a_1, ..., a_n)$$

Proof. From Lemma 3.4.4 we get, $\forall\ \sigma \in P_n$,

$$a_{\sigma(1)} \wedge a_{\sigma(2)} \wedge ... \wedge a_{\sigma(n)} = sgn(\sigma) \cdot a_1 \wedge a_2 \wedge ... \wedge a_n$$

Consider the following transpositions $\tau_1 = (j-1, j), \tau_2 = (j-1, j-2), ..., \tau_{j-1} =$

$(1, 2)$ and $\sigma = \tau_{j-1} \circ \tau_{j-2} \circ ... \circ \tau_1$. It is easy to observe that,

$$a_j \wedge \kappa_j(a_1, ..., a_n) = a_{\sigma(1)} \wedge a_{\sigma(2)} \wedge ... \wedge a_{\sigma(n)}$$

$$\implies a_j \wedge \kappa_j(a_1, ..., a_n) = sgn(\sigma) \cdot a_1 \wedge a_2 \wedge ... \wedge a_n = (-1)^{j-1} a_1 \wedge a_2 \wedge ... \wedge a_n$$

$\square$

Lemma 4.3.4. $\forall\ v_2, ..., v_n \in V$, $\forall\ y_2, ..., y_n \in V$,

$$[v_2 \wedge v_3 \wedge ... \wedge v_n](y_2, ..., y_n) = \sum_{\sigma \in P_n : \sigma(1)=1} sgn(\sigma) \cdot \prod_{i=2}^{n} (v_{\sigma(i)}, y_i)$$

Proof. $\forall\ i \in \{1, 2, ..., n_1\}$ we define,

$$u_i = v_{i+1} \text{ and } z_i = y_{i+1}$$

$$\sum_{\sigma \in P_n : \sigma(1)=1} sgn(\sigma) \cdot \prod_{i=2}^{n} (v_{\sigma(i)}, y_i) = \sum_{\sigma \in P_n : \sigma(1)=1} sgn(\sigma) \cdot \prod_{i=2}^{n} (u_{\sigma(i-1)}, z_{i-1})$$

Note that for each $\sigma \in P_n$ such that $\sigma(1) = 1$, there exist unique $\tilde{\sigma} \in P_{n-1}$ where,

$$\tilde{\sigma}(k) = \sigma(k+1) - 1\ \forall\ k \in \{1, 2, ..., n-1\}$$

Also note that $sgn(\tilde{\sigma}) = sgn(\sigma)$. This is because the cycle decomposition of $\tilde{\sigma}$ and σ with $\sigma(1) = 1$ is same where each element in a cycle of σ must be subtracted by 1. For instance, let

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{bmatrix} \text{ and } \tilde{\sigma} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix}$$

Cycle in σ

$$\sigma(2) = 3\ \sigma(3) = 4\ \sigma(4) = 5\ \sigma(5) = 2$$

Cycle in $\tilde{\sigma}$ is

$$\tilde{\sigma}(1) = 2\ \tilde{\sigma}(2) = 3\ \tilde{\sigma}(3) = 4\ \tilde{\sigma}(4) = 1$$

Notice that cycle in σ is same as $\tilde{\sigma}$ except for the difference of 1. Also notice that

$|\{\sigma \in P_n \mid \sigma(1) = 1\}| = |P_{n-1}|$.

$$\begin{aligned}\implies \sum_{\sigma \in P_n : \sigma(1)=1} sgn(\sigma) \cdot \prod_{i=2}^{n} (v_{\sigma(i)}, y_i) &= \sum_{\tilde{\sigma} \in P_{n-1}} sgn(\tilde{\sigma}) \cdot \prod_{i=1}^{n-1} (u_{\tilde{\sigma}(i)}, z_i) \\ &= u_1 \wedge u_2 \wedge ... \wedge u_{n-1}(z_1, z_2, ..., z_{n-1}) \\ &= v_2 \wedge v_3 \wedge ... \wedge v_n(y_2, ..., y_n)\end{aligned}$$

□

Lemma 4.3.5. $\forall\ j \in \{1, 2, ..., n\}$, $\forall\ x_1, x_2, ..., x_{n-1} \in V$,

$$[a_1 \wedge a_2 \wedge ... \wedge a_n](a_j, x_1, ..., x_{n-1}) = (-1)^{j-1}[\kappa_j(a_1, ..., a_n)](x_1, ..., x_{n-1})$$

Proof. From Lemma 4.3.3 we get that,

$$[a_1 \wedge ... \wedge a_n](a_j, x_1, ..., x_{n-1}) = (-1)^{j-1}[a_j \wedge a_1 \wedge ... \wedge a_{j-1} \wedge a_{j+1} \wedge ... \wedge a_n](a_j, x_1, ..., x_{n-1}) \tag{4.10}$$

Let $v_1 = a_j$, $v_2 = a_1$, ..., $v_n = a_{n-1}$, $y_1 = a_j$, $y_2 = x_1$, ..., $y_n = x_{n-1}$,

$$\begin{aligned}[a_j \wedge \kappa_j(a_1, ..., a_n)](a_j, x_1, ..., x_{n-1}) &= v_1 \wedge v_2 \wedge .... \wedge v_n(y_1, y_2, ..., y_n) \\ &= \sum_{\sigma \in P_n} sgn(\sigma) \cdot \prod_{i=1}^{n} (v_{\sigma(i)}, y_i)\end{aligned}$$

Note that $\{v_1, v_2, ..., v_n\}$ is an orthonormal basis of V and $(v_j, y_1) = 0\ \forall\ j \neq 1$.

$$\begin{aligned}\implies [a_j \wedge \kappa_j(a_1, ..., a_n)](a_j, x_1, ..., x_{n-1}) &= \sum_{\sigma \in P_n : \sigma(1)=1} sgn(\sigma) \prod_{i=2}^{n} (v_{\sigma(i)}, y_i) \\ &= [v_2 \wedge v_3 \wedge ... \wedge v_n](y_2, y_3, ..., y_n) \\ &= [\kappa_j(a_1, ..., a_n)](x_1, ..., x_{n-1})\end{aligned}$$

$$\implies [a_1 \wedge ... \wedge a_n](a_j, x_1, ..., x_{n-1}) = (-1)^{j-1}[\kappa_j(a_1, ..., a_n)](x_1, ..., x_{n-1})$$

□

Theorem 4.3.6. Let V be a finite dimensional inner product space over $\mathbb{R}$ where $dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ be positively oriented ordered orthonormal basis of V (with respect to B 4.3). Then, $\forall\ x_1, x_2, ..., x_n \in V$,

$$(x_n, \times(x_1, ..., x_{n-1})) = det(x_n, x_1, x_2, ..., x_{n-1})$$

Proof.

$$\begin{aligned}
(x_n, \times(x_1, ..., x_{n-1})) &= (x_n, \sum_{i=1}^{n}(-1)^{i-1}[\kappa_i(a_1, ..., a_n)](x_1, ..., x_{n-1})) \\
&= \sum_{i=1}^{n}(x_n, [a_1 \wedge a_2 \wedge ... \wedge a_n](a_i, x_1, ..., x_{n-1})a_i) \\
&= \sum_{i=1}^{n}(x_n, a_i)[a_1 \wedge a_2 \wedge ... \wedge a_n](a_i, x_1, ..., x_{n-1}) \\
&= [a_1 \wedge a_2 \wedge ... \wedge a_n](\sum_{i=1}^{n}(x_n, a_i)a_i, x_1, ..., x_{n-1}) \\
&= [a_1 \wedge a_2 \wedge ... \wedge a_n](x_n, x_1, ..., x_{n-1}) \\
&= det(x_n, x_1, ..., x_{n-1})
\end{aligned}$$

□

Remark :

1. With the above Theorem, we get that cross product of a vector 4.3.3 is well-defined. Consider $x_1, x_2, ..., x_{n-1} \in V$. Let $\times_A(x_1, x_2, ..., x_{n-1})$ denote the cross product of $x_1, x_2, ..., x_{n-1}$ when defined using basis A. Let $\times_C(x_1, x_2, ..., x_{n-1})$ denote the cross product of $x_1, x_2, ..., x_{n-1}$ when defined using basis C. We get, $\forall\ x_n \in V$,

$$(x_n, \times_A(x_1, x_2, ..., x_{n-1}) - \times_C(x_1, x_2, ..., x_{n-1})) = 0$$

Consider $x_n = \times_A(x_1, x_2, ..., x_{n-1}) - \times_C(x_1, x_2, ..., x_{n-1})$, from the positive definiteness of inner product we get,

$$\times_A(x_1, x_2, ..., x_{n-1}) = \times_C(x_1, x_2, ..., x_{n-1})$$

2. $\forall\ x_1, x_2, ..., x_{n-1} \in V$, $\forall\ i \in \{1, 2, ..., n-1\}$

$$(x_i, \times(x_1, x_2, ..., x_{n-1})) = det(x_i, x_1, x_2, ..., x_{n-1}) = 0$$

3. $\forall\ x_1, x_2, ..., x_{n-1} \in V$, if $\times(x_1, x_2, ..., x_{n-1}) = z$ then,

$$(z, \times(x_1, x_2, ..., x_{n-1})) = \|z\|^2 = det(z, x_1, x_2, ..., x_{n-1})$$

Chapter 5

An Orthogonalization Algorithm

5.1 Introduction

In this section we describe a numerical algorithm that utilizes mathematical tools we have developed so far. Throughout this section assume that we work on a real inner product space and orthonormal basis $A = \{a_1, a_2, ..., a_n\}$ is given. You shall notice further that the proposed algorithm can't output A. So, we initially need to use Gram-Schmidt ($\mathcal{O}(n^3)$) technique to find A. However, if we reformulate the problem as "Given an orthonormal basis A of V and coordinates of a unit vector $v \in V$ with respect to basis A, then there is a unique $n-1$ dimensional hyperplane which forms the orthogonal complement of $Span\{v\}$. Our objective is to find an orthonormal basis of hyperplane". Notice that Gram-Schmidt algorithm takes $\mathcal{O}(n^3)$ time for this problem also. We propose an algorithm that solves this problem in $\mathcal{O}(n^2)$. The proposed algorithm takes a single non-trivial vector as input and outputs a set of n orthonormal vectors. We call a vector non-trivial if the first component of vector with respect to basis A is non-zero i.e, $(x, a_1) \neq 0$. Note that without loss of generality given a non-zero vector with $(x, a_1) = 0$, we can swap one of the non-zero components with the first component and re-swap those components in all of the output set of n orthonormal vectors. Major mathematical properties we use in this algorithm are,

1. $\forall\ x_1, x_2, ..., x_{n-1} \in V$, $\forall\ i \in \{1, 2, ..., n-1\}$

$$(x_i, \times(x_1, x_2, ..., x_{n-1})) = \lambda_E(x_i, x_1, x_2, ..., x_{n-1}) = \begin{vmatrix} \text{---} & x_i & \text{---} \\ \text{---} & x_1 & \text{---} \\ & ... & \\ \text{---} & x_{n-1} & \text{---} \end{vmatrix} = 0$$

2. $\forall\ x_1, x_2, ..., x_{n-1} \in V$, if $\times(x_1, x_2, ..., x_{n-1}) = z$ then,

$$(z, \times(x_1, x_2, ..., x_{n-1})) = \|z\|^2 = \lambda_E(z, x_1, x_2, ..., x_{n-1}) = \begin{vmatrix} \text{---} & z & \text{---} \\ \text{---} & x_1 & \text{---} \\ & ... & \\ \text{---} & x_{n-1} & \text{---} \end{vmatrix}$$

Since A is a basis of V, there exist unique α_{1,i_1}, α_{2,i_2}, ..., $\alpha_{n,i_n} \in \mathbb{R}$ such that,

$$x_1 = \sum_{i_1=1}^{n} \alpha_{1,i_1} a_{i_1} \qquad x_2 = \sum_{i_2=1}^{n} \alpha_{2,i_2} a_{i_2} \qquad ... \qquad x_n = \sum_{i_n=1}^{n} \alpha_{n,i_n} a_{i_n}$$

Note that we use $\times_n$ and det_n to denote $n-$dimensional cross product and determinant respectively. Also, it is easy to see that we can use following determinant notation for the cross product,

$$\times_n(x_1, x_2, ..., x_{n-1}) = \begin{vmatrix} a_1 & a_2 & . & . & a_n \\ \alpha_{1,1} & \alpha_{1,2} & . & . & \alpha_{1,n} \\ . & . & . & . & . \\ \alpha_{n-1,1} & \alpha_{n-1,2} & . & . & \alpha_{n-1,n} \end{vmatrix}$$

Note that from here we denote coordinates of a vector $x = \sum_{i=1}^{n} \alpha_i a_i$ with respect to the basis A as $x = (\alpha_1, \alpha_2, ..., \alpha_n)$. The algorithm proceeds through the following steps,

1. Project the input vector $x_1 = (\alpha_{1,1}, \alpha_{1,2}, ..., \alpha_{1,n})$ to two dimensions of first two components i.e, $\tilde{x}_1 = (\alpha_1, \alpha_2)$ and find the 2-dimensional cross product

of this vector let it be $\tilde{x}_2$ i.e,

$$x_2 = \times_2(\tilde{x}_1) = \begin{vmatrix} a_1 & a_2 \\ \alpha_{11} & \alpha_{12} \end{vmatrix} = \alpha_{12}a_1 - \alpha_{11}a_2$$

Then, we get $x_2 = (\alpha_{12}, -\alpha_{11}, 0, 0, ..., 0)$. Normalize x_2 and verify that $\{x_1, x_2\}$ is an orthonormal set.

2. $\forall\ i \in \{3, ..., n\}$, we need to compute the i'th vector which is orthonormal to any vector in $\{x_1, x_2, ..., x_{i-1}\}$. We use $i-$dimensional cross product to compute the ith vector as follows, project all $x_1, x_2, ..., x_{i-1}$ to first i dimensions and take their cross product i.e,

$$\tilde{x}_i = \times_i(\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_{i-1}) = \begin{vmatrix} a_1 & a_2 & . & . & a_i \\ \alpha_{11} & \alpha_{12} & . & . & \alpha_{1i} \\ \alpha_{21} & \alpha_{22} & 0 & . & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & . & 0 \\ . & . & . & . & . \\ \alpha_{i-11} & \alpha_{i-12} & . & \alpha_{i-1i-1} & 0 \end{vmatrix} = \sum_{j=1}^{i} \alpha_{ij} e_j$$

Now, update $x_i = (\alpha_{i1}, ..., \alpha_{ii}, 0, ..., 0)$. Normalize x_i and it is easy to verify that $\{x_1, x_2, ..., x_i\}$ is an orthonormal set. (Use Property 1 of cross products)

Before starting the algorithm we set the following notation, we denote the ith minor of the following determinant,

$$\begin{vmatrix} a_1 & a_2 & . & . & a_k \\ \alpha_{1,1} & \alpha_{1,2} & . & . & \alpha_{1,k} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & . & \alpha_{2,k} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & . & \alpha_{3,k} \\ . & . & . & . & . \\ \alpha_{k-1,1} & \alpha_{k-1,2} & . & \alpha_{k-1,k-1} & \alpha_{k-1,k} \end{vmatrix} \text{ as } \begin{vmatrix} \alpha_{2,1} & . & \alpha_{2,i-1} & \alpha_{2,i+1} & . & \alpha_{2,k-1} \\ \alpha_{3,1} & . & \alpha_{3,i-1} & \alpha_{3,i+1} & . & \alpha_{3,k-1} \\ . & . & . & . & . & . \\ \alpha_{k-1,1} & . & \alpha_{k-1,i-1} & \alpha_{k-1,i+1} & . & \alpha_{k-1,k-1} \end{vmatrix}$$

where $1 \leq i \leq k-1$.

5.2 Algorithm

A straight-forward algorithm requires computation of determinant in each of the n iterations. The time complexity of best known algorithm to compute determinant is $\mathcal{O}(n^{2.376})$ (see [15]). So, the overall time complexity of the algorithm is $\mathcal{O}(n^{3.376})$. Notice that we are padding zeros right from the first output vector to preserve orthogonality. This leads to a good optimal sub structure to solve the problem more efficiently using $1-$dimensional dynamic programming. Now, we propose the following algorithm and prove its correctness.
It is easy to see that the proposed algorithm needs $\mathcal{O}(n^2)$ floating point operations. This algorithm takes exponential time (analyze using bit complexity) if we don't normalize each output vector because the algorithm involves multiplication and we already know that multiplication of two $n-$bit numbers gives a $2n-$bit number. Next, we prove the correctness of the algorithm.

Algorithm 1: Orthogonalization Algorithm with Dynamic Programming

Input : $x_1 = [\alpha_{1,1}, \alpha_{1,2}, ..., \alpha_{1,n}]$
Let $out[1:n], dp[1:n]$ be a arrays of n elements initialized to **zero**.
$x_1 \leftarrow \frac{x_1}{\|x_1\|}$;
$out[1] \leftarrow x_1[2], out[2] \leftarrow -x_1[1], out \leftarrow \frac{out}{\|out\|}$;
$x_2 \leftarrow out$; $dp \leftarrow out$; $k \leftarrow 3$;
while $k \leq n$ **do**
    $j \leftarrow 1$; $ps \leftarrow 0$; $newout \leftarrow [0, ..., 0]$;
    while $j < k$ **do**
        $newout[j] \leftarrow (-1)^{j+k} dp[j] \cdot x_1[k]$;
        $ps \leftarrow newout[j]^2$;
        $esign \leftarrow esign \cdot -1$;
    end
    $j \leftarrow 1$;
    while $j < k$ **do**
        $newout[k] \leftarrow newout[k] + out[j]^2$;
    end
    $normxk \leftarrow \|newout\|$;
    $newout \leftarrow \frac{newout}{\|newout\|}$;
    $x_k \leftarrow newout$;
    $j \leftarrow 1$;
    while $j < k$ **do**
        $dp[j] \leftarrow dp[j] \cdot newout[k]$;
    end
    if $x_1[k] \neq 0$ **then**
        $dp[k] \leftarrow \frac{ps}{x_1[k] \cdot normxk}$
    else
        $dp[k] \leftarrow 0$;
    end
    $out \leftarrow newout$;
end
Output : $x_1, x_2, ..., x_n$

5.3 Proof of Correctness

Theorem 5.3.1. Algorithm 1 is correct

Proof. Halting of the algorithm is straight forward and it is left to the reader to verify. Next, we use the method of loop invariant (see [24]) to prove the correctness of the algorithm.

Invariant Statement : At the beginning of k'th iteration of the outer most while loop, $\{x_1, x_2, ..., x_{k-1}\}$ is an orthonormal set and in the array dp, $dp[i]$ contains the determinant of the i'th minor i.e,

$$dp[i] = \begin{vmatrix} \alpha_{2,1} & . & \alpha_{2,i-1} & \alpha_{2,i+1} & . & \alpha_{2,k-1} \\ \alpha_{3,1} & . & \alpha_{3,i-1} & \alpha_{3,i+1} & . & \alpha_{3,k-1} \\ . & . & . & . & . & . \\ \alpha_{k-1,1} & . & \alpha_{k-1,i-1} & \alpha_{k-1,i+1} & . & \alpha_{k-1,k-1} \end{vmatrix}$$

where $1 \leq i \leq k-1$. Recall that in the i'th minor of a matrix the i'th column is removed. We have written the above matrix only for clarity but $\alpha_{i,j} = 0$ $\forall$ $i \in \{2, ..., k-1\}$ and $i < j \leq k$. Note that from here we use x_i to denote $\tilde{x}_i$ also and we expect the reader to understand its meaning from the context.

Initialization : It is easy to see that $\{x_1, x_2 = \times_2(x_1)\}$ is an orthonormal set and we initialized dp to the corresponding minor that is $dp[1] = \alpha_{12}, dp[2] = -\alpha_{11}$ i.e, the minors of $\begin{vmatrix} a_1 & a_2 & a_3 \\ \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} \\ \alpha_{1,2} & -\alpha_{1,1} & 0 \end{vmatrix}$

Maintenance : Let the invariant statement be correct holds till $(k-1)$'th iteration of the algorithm and we have to show that x_k is orthonormal to each vector in $\{x_1, x_2, ..., x_{k-1}\}$ and $dp[i]$ is updated with the ith minor of $\begin{vmatrix} a_1 & a_2 & . & . & a_{k+1} \\ \alpha_{1,1} & \alpha_{1,2} & . & . & \alpha_{1,k+1} \\ \alpha_{2,1} & \alpha_{2,2} & 0 & . & 0 \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & . & 0 \\ . & . & . & . & . \\ \alpha_{k,1} & \alpha_{k,2} & . & \alpha_{k,k} & 0 \end{vmatrix}$

where $1 \leq i \leq k$. To compute x_k we use $k-$dimensional cross product,

$$x_k = \begin{vmatrix} a_1 & a_2 & . & . & a_k \\ \alpha_{1,1} & \alpha_{1,2} & . & . & \alpha_{1,n} \\ \alpha_{2,1} & \alpha_{2,2} & . & . & 0 \\ . & . & . & . & . \\ \alpha_{k-1,1} & \alpha_{k-1,2} & . & . & 0 \end{vmatrix} = \sum_{i=1}^{k} \alpha_{ki} e_i$$

It is easy to observe that $\forall\ 1 \leq i \leq k-1$,

$$\alpha_{k,i} = (-1)^{i-1} \cdot \begin{vmatrix} \alpha_{1,1} & \alpha_{1,2} & . & \alpha_{1,i-1} & \alpha_{1,i+1} & . & \alpha_{1,k} \\ \alpha_{2,1} & \alpha_{2,2} & . & 0 & 0 & . & 0 \\ \alpha_{3,1} & \alpha_{3,2} & . & 0 & 0 & . & 0 \\ . & . & . & . & . & . & 0 \\ \alpha_{k-1,1} & \alpha_{k-2,2} & . & \alpha_{k-1,i-1} & \alpha_{k-1,i+1} & . & 0 \end{vmatrix}$$

$$\implies \alpha_{k,i} = (-1)^{i+k} \alpha_{1,k} \begin{vmatrix} \alpha_{2,1} & \alpha_{2,2} & . & 0 & 0 & 0 & 0 \\ \alpha_{3,1} & \alpha_{3,2} & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & 0 \\ \alpha_{k-1,1} & \alpha_{k-2,2} & . & \alpha_{k-1,i-1} & \alpha_{k-1,i+1} & . & \alpha_{k-1,k-1} \end{vmatrix} = (-1)^{i+k} \alpha_{1,k} dp[k]$$

For $i = k$, we need to compute,

$$\alpha_{kk} = (-1)^{k-1} \begin{vmatrix} \alpha_{1,1} & \alpha_{1,2} & . & . & \alpha_{1,k-1} \\ \alpha_{2,1} & \alpha_{2,2} & . & . & 0 \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & . & 0 \\ . & . & . & . & . \\ \alpha_{k-1,1} & \alpha_{k-2,2} & . & . & \alpha_{k-1,k-1} \end{vmatrix} = (x_{k-1}, \times_{k-1}(x_1, x_2, ..., x_{k-2})) = \|x_{k-1}\|^2$$

From the algorithm it is quite clear that these steps are followed. Next, we have to see the correctness of updating dp for the $(k+1)$'th iteration. $\forall\ 1 \leq i \leq k-1$, in the $(k+1)$'th iteration we have the i'th minor as,

$$\begin{vmatrix} \alpha_{2,1} & \alpha_{2,2} & . & 0 & 0 & 0 & 0 \\ \alpha_{3,1} & \alpha_{3,2} & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & 0 \\ \alpha_{k,1} & \alpha_{k,2} & . & \alpha_{k,i-1} & \alpha_{k,i+1} & . & \alpha_{k,k} \end{vmatrix} = \alpha_{k,k} \begin{vmatrix} \alpha_{2,1} & \alpha_{2,2} & . & 0 & 0 & 0 & 0 \\ \alpha_{3,1} & \alpha_{3,2} & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & 0 \\ \alpha_{k-1,1} & \alpha_{k-2,2} & . & \alpha_{k-1,i-1} & \alpha_{k-1,i+1} & . & \alpha_{k-1,k-1} \end{vmatrix}$$
$$= \alpha_{k,k} dp[i]$$

For $i = k$, we have

$$\begin{vmatrix} \alpha_{2,1} & \alpha_{2,2} & . & 0 & 0 & 0 & 0 \\ \alpha_{3,1} & \alpha_{3,2} & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & 0 \\ \alpha_{k-1,1} & \alpha_{k-1,2} & . & . & . & . & \alpha_{k-1,k-1} \\ \alpha_{k,1} & \alpha_{k,2} & . & . & . & . & \alpha_{k,k-1} \end{vmatrix} = (-1)^{k-1} \frac{1}{\alpha_{1,k}} \begin{vmatrix} \alpha_{1,1} & \alpha_{1,2} & . & . & . & \alpha_{1,k} \\ \alpha_{2,1} & \alpha_{2,2} & . & 0 & 0 & 0 \\ . & . & . & . & . & 0 \\ \alpha_{k-1,1} & \alpha_{k-1,2} & . & . & \alpha_{k-1,k-1} & 0 \\ \alpha_{k,1} & \alpha_{k,2} & . & . & \alpha_{k,k-1} & 0 \end{vmatrix}$$

$$= \frac{1}{\alpha_{1,k}} \begin{vmatrix} \alpha_{k,1} & \alpha_{k,2} & . & . & \alpha_{k,k-1} & 0 \\ \alpha_{1,1} & \alpha_{1,2} & . & . & . & \alpha_{1,k} \\ \alpha_{2,1} & \alpha_{2,2} & . & 0 & 0 & 0 \\ . & . & . & . & . & 0 \\ \alpha_{k-1,1} & \alpha_{k-1,2} & . & . & \alpha_{k-1,k-1} & 0 \end{vmatrix}$$

$$= \frac{1}{\alpha_{1,k}} \cdot \left(\sum_{i=1}^{k-1} \alpha_{k,i}^2\right) \text{ since } (x_k, \times_k(x_1, x_2, ..., x_{k-1})) = \|x_k\|^2$$

Observe from the algorithm that the calculation of dp for the $(k+1)$'th iteration goes through the same steps. Note that for simplicity we have argued for vectors that are not normalized. We leave it to reader to see how the proof slightly changes when we consider normalized vectors. □

Say $A = [a_1, a_2, \ldots, a_n] \in \mathbb{R}^{m \times n}, B = [b_1, b_2, \ldots, b_l] \in \mathbb{R}^{n \times l}$

$$
\begin{aligned}
det(A) &= det(a_1, a_2, \ldots, a_n) \qquad \text{(definition)} && (5.1)\\
det(AB) &= det(Ab_1, Ab_2, \ldots, Ab_n) && (5.2)\\
&= det(\sum_{i_1=1}^{n} b_1(i_1)a_{i_1}, \sum_{i_2=1}^{n} b_2(i_2)a_{i_2}, \ldots, \sum_{i_n=1}^{n} b_n(i_n)a_{i_n}) && (5.3)\\
&= \sum_{i_1=1}^{n}\sum_{i_2=1}^{n} ... \sum_{i_n=1}^{n} b_1(i_1)b_2(i_2)...b_n(i_n)det(a_{i_1}, a_{i_2}, \ldots a_{i_l}) && (5.4)\\
&= \sum_{\sigma \in P_n} b_1(\sigma(1))b_2(\sigma(2)) \ldots b_n\sigma(n)det(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)}) && (5.5)\\
&= \sum_{\sigma \in P_n} sgn(\sigma)b_1(\sigma(1))b_2(\sigma(2)) \ldots b_l(\sigma(n))det(a_1, a_2, \ldots, a_n) && (5.6)\\
&= det(b_1, b_2, \ldots, b_n)det(a_1, a_2, \ldots, a_n) && (5.7)
\end{aligned}
$$

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