Rotations about the axis x,y,z are as follows:

$$R(x,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix}$$

$$R(y,\theta) = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix}$$

$$R(z,\theta) = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1 Representing Rotation

There are number of ways to represent a rotation. One way is by using a matirx like above.

Second way of representing this rotation is by using axis-angle notation.
$\hat{w} = 0\hat{i} + 1\hat{j} + 0\hat{k}$

Third way of representing a rotation is by quaternions.
$q = [cos\frac{\theta}{2}, \hat{w}sin\frac{\theta}{2}]$, where $\hat{w}$ is a vector (w_x, w_y, w_z).

2 Converting axis-angle to a quaternion

Axis-angle rotation is represented by a 1x3 matrix in the form $[x, y, z]$.
From the given angle and vector we can convert to the quaternion using the following formula.

$q = [cos\frac{\theta}{2}, \vec{r}sin(\frac{\theta}{2})]$, where $\vec{r} = [x, y, z]$

3 Converting quaternion to a rotation matrix

$q = [q_s, q_x, q_y, q_z]$

$$R = \begin{bmatrix} 1 - 2q_y^2 - 2q_z^2 & 2q_xq_y - 2q_zq_s & 2q_xq_z + 2q_yq_s \\ 2q_xq_y + 2q_zq_s & 1 - 2q_x^2 - 2q_z^2 & 2q_yq_z - 2q_xq_s \\ 2q_xq_z - 2q_yq_s & 2q_yq_z + 2q_xq_s & 1 - 2q_x^2 - 2q_y^2 \end{bmatrix}$$

4 Rotation matrix to Axis-Angle

In this excersice theta is limited to $[0, \pi]$. The general form of the equation is:
$\theta = acos(\frac{tarce(R)-1}{2})$

$\hat{r} = \frac{1}{2sin\theta}[(r_{32} - r_{23})\hat{i} + (r_{13} - r_{31})\hat{j} + (r_{21} - r_{12})\hat{k}]$
There exists singularities when $\theta = 0, \pm\pi$.

- when$\theta = 0$, there are infinite solutions (no rotation). $\hat{r}$ can be any vector
- when $\theta = \pi$

$$R(r,\theta) = \begin{bmatrix} r_x^2 v_\theta + c\theta & r_x r_y v_\theta - r_z s\theta & r_x r_z v_\theta + r_y s\theta \\ r_x r_y v_\theta + r_z s\theta & r_y^2 v_\theta + cos\theta & r_y r_z v_\theta - r_x s\theta \\ r_x r_z v_\theta - r_y s\theta & r_y r_z v_\theta + r_x s\theta & r_z^2 v_\theta + c\theta \end{bmatrix}$$

where $c\theta = cos(\theta)$, $s\theta = sin(\theta)$,$v_\theta = 1 - cos(\theta)$

substituting $\theta = pi$ in to the above equation will give the following matrix:

$$R(r,\theta) = \begin{bmatrix} 2x^2 - 1 & 2xy & 2xz \\ 2xy & 2y^2 - 1 & 2yz \\ 2xz & 2yz & 2z^2 - 1 \end{bmatrix}$$

$x = \sqrt{\frac{R_{1,1}+1}{2}}$ (1)

$y = \sqrt{\frac{R_{2,2}+1}{2}}$ (2)

$z = \sqrt{\frac{R_{3,3}+1}{2}}$ (3)

solve for [x,y,z] using the diagonals (1),(2),(3). To find the signs of x,y,z, it is better to find y and z using the first row (interdependence of variables) instead of the diagonals.

If x=0, solve for y and use second row to solve for x and y. If y=0, solve for z and use third row to solve for z and x.

For $\theta = \pi$ there are two solution with different signs.