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Graph automorphic approaches to the robustness of complex networks

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ABSTRACT

Leveraging on graph automorphic properties of complex networks (CNs), this study investigates three robustness aspects of CNs including the robustness of controllability, disturbance decoupling, and fault tolerance against failure in a network element. All these aspects are investigated using a quantified notion of graph symmetry, namely the automorphism group, which has been found implications for the network controllability during the last few years. The typical size of automorphism group is very big. The study raises a computational issue related to determining the whole set of automorphism group and proposes an alternative approach which can attain the emergent symmetry characteristics from the significantly smaller groups called generators of automorphisms. Novel necessary conditions for network robust controllability following a failure in a network element are attributed to the properties of the underlying graph symmetry. Using a symmetry related concept called determining set and a geometric control property called controlled invariant, the new necessary and sufficient conditions for disturbance decoupling are proposed. In addition, the critical nodes/edges of the network are identified by determining their role in automorphism groups. We verify that nodes with more repetition in symmetry groups of the network are more critical in characterizing the network robustness. Further, the impact of elimination of critical network elements on its robustness is analyzed by calculating a new improved index of symmetry which considers the orbital impacts of automorphisms. The importance of all symmetry inspired findings of this paper is highlighted via simulation on various networks.

1. Introduction

A review on the related works is presented in this section following by motivations on the necessity for addressing CN robustness using graph symmetry. It then proceeds with the contributions and organization of the paper.

1.1. Literature review and motivations

Graph theory, rooted in discrete algebra, has formed increasing applications in CN analysis during the last decade (Abeysinghe et al., 2018; Albert, Albert, & Nakarado, 2004; Chapman & Mesbahi, 2014, 2015; Chapman, Nabi-Abdolyousefi, & Mesbahi, 2014; Dorfler, Simpson-Porco, & Bullo, 2018; Holmgren, 2006; Li, Ma, Zhang, & Sun, 2015; MacArthur, Sanchez-Garcia, & Anderson, 2018; Nasiruzzaman & Pota, 2011; Pagani & Aiello, 2013; Pecora, Sorrentino, Hagerstorm, Murphy, & Roy, 2014; Siddique, Pecora, Hart, & Sorrentino, 2018; Sorrentino & Pecora, 2016; Tran, Garratt, & Petersen, 2020; Wang, Hill, Chen and Dong, 2017; Xiao, Lao, Hou, & Bai, 2014; Yuan, Zhao, Di, Wang, & Lai, 2013). However, in spite of these extensions of graph properties in complex networks, many other graph characteristics and their potential applications in networked systems have remained rather unknown for the community. The wealth of fundamental knowledge of graph theoretic properties that have not yet been widely studied, the significant improvement in the analysis and synthesis of CNs using the abstract network modeling facilitated by CN theory (Pagani & Aiello, 2013), and the necessity to answer many open problems related to CN analysis (Dorfler et al., 2018) motivate investigating the impact of other graph properties on network behavior. Among these graph properties, graph symmetry, as described by automorphism group (MacArthur et al., 2018), is the focus of this study.

Although the implications of graph symmetry in engineering is not investigated extensively, significant results have attained in Chapman and Mesbahi (2014, 2015), Chapman et al. (2014), MacArthur et al. (2018) (in a general form of networks) and Pecora et al. (2014), Siddique et al. (2018), Sorrentino and Pecora (2016) (in power networks) explaining some emergent behaviors of complex networks by employing the underlying graph symmetry. In particular, symmetry has been employed to investigate the controllability (Chapman & Mesbahi, 2014) and robustness (MacArthur et al., 2018) of CN in general form of networked systems, and synchronization in power networks (Pecora et al.,

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Nomenclature

CN	Complex network
ECM	Exact controllability method
lcm	The lowest common multiple
LTI	Linear time invariant
ε	Identity (or trivial) permutation
c	A complex number
δ	A permutation
δ_i	Maximum algebraic multiplicity of $\lambda_{(i)}$
λ^M	Maximum algebraic multiplicity of the eigenvalue λ^M
λ_i	The ith eigenvalue
σ	Permutation
$\mathcal{A}$, A	Adjacency matrix
$\mathcal{D}$	The degree matrix
$\mathcal{G}$	Graph
$\mathcal{J}$	The incidence vector of driver nodes
$\mathcal{L}$	Laplacian matrix
$\mathcal{S}$	Determining set
$\mathcal{V}$	Controlled invariant subspace
Aut($\mathcal{G}$)	Automorphism group
dimV_{λ_i}	Dimension of eigenspace associated with $\lambda(i)$
Gen($\mathcal{G}$)	Generators of automorphism
Mov$_{v_l}^{Gen}$	The set of generators mapping v_l
$\mu(\lambda_i)$	Maximum geometric multiplicity of $\lambda(i)$
ζ	A permutation
σ	A permutation
ζ_{dis}, δ_{dis}	Disjoint generators
a_{ij}	Element of $\mathcal{A}$
B	Input matrix
E	The set of edges
$F(x_i)$	The individual node's dynamical equation
$Im\,C$	Image of C
K	The feedback gain
$ker\,L$	Kernel or null space of L
l_{ij}	The (i,j)th element of the Laplacian matrix
N_D	Number of required driver nodes
$r_{\mathcal{G}}^*$	Normalized measure of network redundancy
$r_{\mathcal{G}}$	Network redundancy
$s(t)$	The desired state
$S_{\mathcal{O}}(\mathcal{G})$	Symmetry index based on orbits
u	Control signal
V	The set of nodes
$w(t)$	Disturbance
$\rho_{\mathcal{O}}$	The orbital ratio of symmetry after node removal
ρ_{aut}	The ratio of symmetry after node removal

2014; Siddique et al., 2018; Sorrentino & Pecora, 2016). It has been verified in MacArthur et al. (2018) and Chapman and Mesbahi (2014) that symmetry is an obstruction to controllability but it reinforces the robustness. In fact, by attributing the controllability of complex networks to the number of required driver nodes (Li et al., 2015; Yuan et al., 2013), it is shown that the more symmetric network necessitates more driver nodes to be fully controllable (Chapman & Mesbahi, 2014). Conversely, a network with higher number of automorphisms has more robust characteristics (MacArthur et al., 2018). These significant impacts of symmetry on fundamental network's behavior stimulate conducting further research in this direction. For this purpose, three aspects of network robustness, i.e., (i) the robustness of controllability, (ii) disturbance decoupling, and (iii) fault tolerance against node failure are discussed below and will be addressed by graph symmetry.

Serious concerns about the robustness of complex networks have been raised during the last decade (Abeysinghe et al., 2018; Albert et al., 2004; Como, Savla, Acemoglu, Dahleh, & Frazzoli, 2013; Holmgren, 2006; Nasiruzzaman & Pota, 2011; Tononi, Sporns, & Edelman, 1999; Tootaghaj et al., 2018; Wang, Hill et al., 2017; Xiao et al., 2014) which necessitate a better understanding of influencing factors and methods for protecting the network. *Robustness* and *fault tolerance* entail wide ranges of topics in networked systems. However, here, *fault tolerance* is defined as the network's ability to tolerate failure in a network element with no significant performance degradation (Albert & Barabasi, 2002). Also, the *robustness of controllability* is defined as the network ability to preserve the controllability with a fixed number of driver nodes after a node failure.

It is important to know which network components are critical and if compromised can cause significant malfunctioning of the whole network. For example, a failure in a critical element of a power network can lead to cascading failure or large blackouts (Albert et al., 2004 and Como et al., 2013; Tootaghaj et al., 2018). *Robustness* and fault tolerance are addressed by improving the network's ability to tolerate disturbance and node elimination, respectively (Albert & Barabasi, 2002). To this end, robustness is investigated via the impact of graph symmetry on (1), robustness of controllability (2), disturbance decoupling and (3) fault tolerance against node failure. The necessity to investigate on these issues using symmetry is discussed below.

The traditional rank of controllability is not applicable, in particular, to complex networks as the accurate system parameters are difficult to acquire (Li et al., 2015; Yuan et al., 2013 and Lin, 1974; Liu, Slotine, & Barabási, 2011; Wu, Wang, Gu, & Jiang, 2020). Structural controllability is proposed as a solution in Lin (1974) from the graph perspective. Then, the structural controllability problem converted to the problem of finding the minimum number of external inputs to fully control a network (Liu et al., 2011). These external inputs are usually referred to as driver nodes (Yuan et al., 2013 and Amani, Jalili, Yu, & Stone, 2017; Hou, Li, & Chen, 2016; Jalili & Yu, 2017; Lou, Wang, & Chen, 2018; Ouyang et al., 2018; Xiang, Chen, Ren, & Chen, 2019a; Zhang, Wang, Cheng, & Jia, 2019) which act as the control nodes through which the control signals can be injected. They could be specified as the non-zero elements of the input matrix of the system. The external control vector is then applied to each node correspondent to each non-zero element of the input matrix. This reformulation of controllability in CNs via finding the set o required driver nodes stimulated many research studies in recent years (Amani et al., 2017; Hou et al., 2016; Jalili & Yu, 2017; Lou et al., 2018; Ouyang et al., 2018; Xiang et al., 2019a; Zhang et al., 2019).

Symmetry has been verified to have an important role in controllability of complex networks (Chapman & Mesbahi, 2014). However, the necessary conditions of CN controllability attained in Chapman et al. (2014) and Chapman and Mesbahi (2015) impose a serious computation burden as it relies on computing and sweeping over a very big set of automorphisms in order to find the set of driver nodes. The cardinality of automorphism group of typical networks can be 10^{17} – 10^{159} (this is verified in MacArthur et al. (2018) and also calculated in simulation section). As a result, it is not practical to compute and sweep over all automorphisms.

Disturbance decoupling is another characteristics of a reliable network (Alshehri & Khalid, 2019; Djilali, Sanchez, & Belkheiri, 2019; Selvaraj, Kwon, Lee, & Sakthivel, 2014; Wang, Li, Li, & Sun, 2014) that is investigated in this paper under the impact of symmetry. To decouple the disturbance, a state feedback must be designed in such a way that the disturbance could not affect the output. Although, various aspects of disturbance rejection in complex networks are addressed in literature, the majority of the proposed approaches rely on designing a specific controller which cannot be used when the disturbance or operating conditions are changed.

All the proposed approaches in literature present a method for controller design to address disturbance rejection for specific disturbance and system dynamics. The lack of a more comprehensive approach to deal with disturbance without depending to system's dynamics motivates exploring the common topological properties of the underlying networks. Symmetric structures, as verified in MacArthur et al. (2018) and also in our simulation results, is present in all networks and can be used to find the set of driver nodes (Chapman et al., 2014). This study verifies that, under a controlled invariant subspace, the disturbance rejection is dependent to the selection of the set of driver nodes attained from symmetry analysis. A significant advantage of the proposed approach is that, instead of proceeding with complicated algorithms for controller design, the disturbance rejection is accomplished via a systematic selection of set of driver nodes under a controlled invariant subspace.

Ideally, we try to protect all network elements. However, this is not feasible in practice as it imposes a high cost, in particular, in physical systems such as power grids. Therefore, it is crucial to identify and protect the most important (critical) nodes of the network. Previous studies on critical node/edge identification have mostly relied on an assessment of graph centrality such as closeness, betweenness, and node degree distribution (Koc, Warnier, Kooij, & Brazier, 2013; Lawyer, 2015; Nasiruzzaman & Pota, 2011). The idea is to find the set of hub nodes that have significant role in determining the network performance. Node degree is verified as a key to characterize various properties of electricity distribution networks (Abeysinghe et al., 2018). The node degree distribution then have a critical role in the network vulnerability as nodes with higher degree are considered more critical (Lawyer, 2015; Nasiruzzaman & Pota, 2011). Therefore, a common approach to enhance the network tolerance against a node failure is to manipulate the node degree distribution. This can entail adding/eliminating lots of nodes/edges over the network (Pang & Hao, 2017; Wang, Chen, Wang and Lai, 2017; Xiang, Chen, Ren, & Chen, 2019b) or changing the directions of some edges (Pang & Hao, 2017; Xiao et al., 2014). As a result, the proposed modification algorithms are not cost effective.

1.2. Contributions

In this paper, computation issue related to calculating the whole set of automorphism groups is resolved by proposing an alternative approach based on computing and sweeping over generators of automorphisms which are significantly smaller groups than automorphisms. The necessary conditions for full controllability can then be computed effectively by the proposed approach. Furthermore, the necessary conditions for robustness of controllability is attained in this paper. It is verified that the network can function after a change in its topology if some conditions satisfy on its symmetry group determining the set of driver nodes.

An inherent network property, the multiplicity of a node in the network symmetry factors, called the generator set, is leveraged to find the critical components of complex networks. This study verifies that even a node with low centrality can have significant impact on network performance if it highly contributes to the construction of the generator set. It will be shown that, using the symmetry analysis, it is enough to manipulate only one or two nodes to modify the symmetry strength of the network. A significant advantage of using symmetry is that the idea relies on an inherent characteristic of the networks, and no network synthesizing or severe structural manipulation are required. Moreover, the paper will show that symmetry, in the context of automorphisms, is present in all networks independent of network size, order, and degree distribution.

Finally, some interesting results are observed in simulation section which further emphasizes on the importance of symmetry in determining the network behavior. For example, it is observed that the network symmetry and in turn, its robustness, can be independent to nodes degree distribution. This is not aligned with the majority of related literature where only the importance of the most central nodes or hub nodes have been highlighted. According to our symmetry analysis, even a node with low degree distribution can have significant impact on the network robustness. Also, we noticed that there is an overlap between the set of driver nodes attained from symmetry analysis and the set of driver nodes attained from another established method called exact controllability method (ECM). Such observations motivate conducting more studies in this direction.

The rest of this article is organized as follows. Section 2 reviews the mathematical preliminaries and definitions. The main results on network robustness using the symmetry characteristics of the underlying graph of the network are presented in Section 3. The simulation is carried out on various networked systems in Section 4. Finally, the conclusion is presented in Section 5.

2. Preliminaries

Some mathematical preliminaries and definitions are given in this section. These include some notions from linear algebra, graph theory and, in particular, graph symmetry.

Definition 2.1. The column space (also called image) of a matrix A is the set of all linear combinations of its column vectors. If A is an $m \times n$ matrix where its column vectors are $\mathbf{v}_1, \dots, \mathbf{v}_n$ and a linear combination of these vectors form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n,$$

where $c_1, c_2, \dots, c_n$ are scalars, the image of A is the column space of all possible compositions $A\mathbf{x}$ for $\mathbf{x} \in \mathbf{c}^n$.

Definition 2.2. The kernel (also called null space) of a linear map $L : V \to W$ between two vector spaces V and W is the set of all elements $\mathbf{v}$ of V for which $L(\mathbf{v}) = 0$, where $\mathbf{0}$ denotes the zero vector in W, i.e., $\ker L = \{\mathbf{v} \in V \mid L(\mathbf{v}) = \mathbf{0}\}$.

2.1. Graph theory

An undirected graph $\mathcal{G}$ is composed of a set of un-ordered pairs of nodes V and edges E and is denoted by $\mathcal{G}(V, E)$ or simply $\mathcal{G}$. Two nodes v_1 and v_2 are said to be adjacent if there is an edge between them. The size and order of $\mathcal{G}$ are denoted by $|V|$ and $|E|$, respectively. An *adjacency matrix*, denoted by $\mathcal{A}$, is a square $|V| \times |V|$ matrix whose element, a_{ij}, indicates whether pairs of nodes i and j are connected together or not. For a simple graph, if there is an edge from node i to node j then a_{ij} is one, otherwise it is zero. The degree matrix $\mathcal{D}$ is a diagonal matrix where d_{ii} is the number of edges attached to node i. The Laplacian matrix is a form of representing $\mathcal{G}$ and is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$ where $\mathcal{D}$ is the degree matrix,.

Definition 2.3. A subgraph of $\mathcal{G}$ is a graph $\mathcal{H}$ such that $V_{\mathcal{H}} \subset V_{\mathcal{G}}$ and $E_{\mathcal{H}} \subset E_{\mathcal{G}}$. In general, any isomorphic graph to a subgraph of $\mathcal{G}$ is also said to be a subgraph of $\mathcal{G}$.

Definition 2.4. The composition or product of two functions ζ and δ, denoted by $\zeta \circ \delta$ is the pointwise action of ζ to the result of δ which generates a third function. The notation $\zeta \circ \delta$ is read as "ζ composed with δ" and $(\zeta \circ \delta)\big|_{(i)}$ denotes the pointwise composition of ζ and δ" on point i. Intuitively, by composition of two functions, the pointwise output of the inner function becomes the input of the outer function.

The composition of two permutations is calculated in the simple example below.

Example 2.1. Let ζ and δ be given by

$\zeta = (1 \quad 2 \quad 3 \quad 4)$ *and* $\delta = (1 \quad 3)$.

To compute the composition of ζ and δ, $\zeta \circ \delta$, first we have to check the commutation (represented by the symbol $\mapsto$) of element by δ and then its commutation by ζ. In this example

$$1 \mapsto^{\delta} 3 \mapsto^{\zeta} 4 \quad \Rightarrow \quad (\zeta \circ \delta)\Big|_{(1)} = 4$$

$$4 \mapsto^{\delta} 4 \mapsto^{\zeta} 1 \quad \Rightarrow \quad (\zeta \circ \delta)\Big|_{(4)} = 1$$

$$3 \mapsto^{\delta} 1 \mapsto^{\zeta} 2 \quad \Rightarrow \quad (\zeta \circ \delta)\Big|_{(3)} = 2$$

$$2 \mapsto^{\delta} 2 \mapsto^{\zeta} 3 \quad \Rightarrow \quad (\zeta \circ \delta)\Big|_{(2)} = 3.$$

Thus the composition of ζ and δ is $\zeta \circ \delta = (1 \quad 4)(3 \quad 2)$.

A *permutation* σ is defined as the act of rearranging a subset of nodes of $\mathcal{G}$. The *order of permutation*, denoted by order(σ), is the smallest positive integer m such that $\sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_m = \sigma^m = \varepsilon$ where ε is the identity (trivial) permutation.

Two permutations ζ and δ are *disjoint* if each node moved by ζ is fixed by δ, or equivalently, every node moved by δ is fixed by ζ, otherwise they are called *joint* permutations. The set of nodes that are not mapped by a permutation are *fixed* nodes, and the set of nodes that are mapped is called *moved* nodes. Also the set of permutations that map a node v_l is denoted by Mov_{v_l}.

Definition 2.5. Two permutations ζ and δ *commute* if $\zeta \circ \delta = \delta \circ \zeta$.

Lemma 2.1. *Disjoint cycles commute, i.e., if $\zeta = (u_1 \ldots u_r)$ and $\delta = (v_1 \ldots v_r)$ are disjoint cycles then $\zeta \circ \delta = \delta \circ \zeta$.*

Proposition 2.1. *The composition of disjoint permutations does not move an already fixed node by all of these permutations. Equivalently, the composition of disjoint permutations does not fix an already moved node by one of the permutations.*

2.2. Graph symmetry

Graph symmetry rooted in discrete mathematics can be revealed by automorphism groups. Automorphism is a form of symmetry in which the graph is mapped onto itself while preserving the graph structure, meaning the adjacency or Laplacian matrix of the underlying graph remains unchanged under mapping by an automorphism. This is illustrated by a simple example below.

Example 2.2. Consider the simple triangle graph of Fig. 1. The adjacency matrix of this graph is

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

An automorphism of this graph is $(1\,2\,3)$ which permute nodes 1, 2, and 3 to nodes 2, 3, and 1, respectively. However, the adjacency matrix of the graph remain the same as the above matrix under the act of this automorphism.

Definition 2.6. An automorphism of $\mathcal{G}$ is a permutation σ for which $\{i, j\} \in E(\mathcal{G})$ (where $E(\mathcal{G})$ is the set of edges of $\mathcal{G}$) if and only if $\big(\sigma(i), \sigma(j)\big) \in E(\mathcal{G})$. The automorphism group of $\mathcal{G}$ and its size are denoted by $Aut(\mathcal{G})$ and $|Aut(\mathcal{G})|$, respectively.

Each automorphism can be attained by the multiplication of some elementary automorphisms (also called generator of automorphisms). Throughout this paper, automorphisms are divided into two categories: (i) generators of automorphisms, and (ii) ordinary automorphisms. Each ordinary automorphisms can be attained from the compositions of generators or other ordinary automorphisms. This is illustrated by the simple example below.

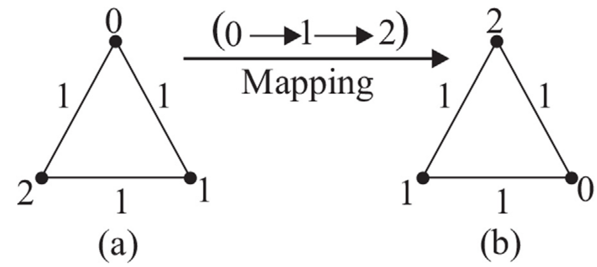

Fig. 1. The graph of Example 2.2.

Example 2.3. For the simple graph of Fig. 1.(a), the set of generators and automorphisms are computed in Sage as

$$\mathrm{Gen}(\mathcal{G}) = \{(1, 2), (0, 1)\}$$

$$\mathrm{Aut}(\mathcal{G}) = \{(I), (1, 2), (0, 1), (0, 2, 1), (0, 1, 2), (0, 2)\}$$

In the above set of automorphism, (I) is identity or trivial automorphism. The second and third automorphisms are also generators of automorphisms. The last three automorphisms can be attained from the compositions of other automorphisms as:

$$(1, 2) \circ (0, 1) = (0, 2, 1)$$

$$(0, 1) \circ (0, 2, 1) = (0, 2)$$

$$(1, 2) \circ (0, 2) = (0, 1, 2).$$

No non-repetitive automorphism can be generated by any composition of the automorphisms.

A permutation matrix, denoted by $\mathcal{J}$, is a square binary matrix attained by permuting the rows of an identity matrix where every permutation matrix associates to a unique permutation (here the permutation is either an automorphism or a generator).

Definition 2.7. The incidence vector or indicator vector or characteristic vector of a subset T of a set S is the vector $x_T := (x_s)_{s \in S}$ such that $x_s = \mathbf{1}$ if $s \in T$ and $x_s = 0$ if $s \notin T$.

Lemma 2.2. *Let two nodes $a, b \in \mathcal{G}$ and σ be a permutation on $\mathcal{G}$. By defining $a \sim b$ if and only $b = \sigma^n(a)$ for $n \in \mathbb{Z}$, then $\sim$ is an equivalence relation on $\mathcal{G}$. This equivalence classes are called orbits of $\mathcal{G}$.*

2.3. The exact controllability method (ECM)

In this study, the exact controllability method (Yuan et al., 2013) is used to compute the number of required driver nodes satisfying the full CN controllability. The approach is based on maximum multiplicity of eigenvalues to attain the number of driver nodes N_d after each modification. It has been verified in Yuan et al. (2013) that the minimum number of driver nodes can be computed by the maximum geometric multiplicity $\mu(\lambda_i)$ of the eigenvalue λ_i of the coupling matrix $\mathcal{A}$:

$$N_d = \max_i \{\mu(\lambda_i)\} \tag{1}$$

where

$$\mu(\lambda_i) = \dim V_{\lambda_i} = N - \mathrm{rank}(\lambda_i I_N - A) \tag{2}$$

and $\lambda_i (i = 1, \ldots, l)$ represents the eigenvalues of A. For undirected weighted networks, the maximum algebraic multiplicity $\delta(\lambda_i)$ determine the number of driver nodes as

$$N_d = \max_i \{\delta(\lambda_i)\} \tag{3}$$

where $\delta(\lambda_i)$ is also the eigenvalue degeneracy of matrix A (Yuan et al., 2013). Using the elementary column transformation on the matrix $(\lambda^M I_N - A)$ a reduced row echelon form of the adjacency matrix can be attained. Subsequently, the set of linearly dependent rows corresponds to the driver nodes which is equivalent to the maximum geometric multiplicity. Consider the general form of linear time invariant system as

$$\dot{x} = A\boldsymbol{x}(t) + B\boldsymbol{u}(t) \tag{4}$$

where A is the coupling or (adjacency) matrix, B is the input matrix, $\boldsymbol{x}(t)$ is the state vector, and $\boldsymbol{u}(t)$ is the control vector. According to the exact controllability method, system (4) is controllable if and only if

$$\mathrm{rank}(cI_N - A, B) = N \tag{5}$$

is satisfied for any complex number c, where I_N is the identity matrix of dimension N.

3. Symmetry impact on complex network robustness

This section focuses on the concept of graph symmetry to address a few issues related to the robustness of complex networks. Three aspects of network robustness are addressed. First, the network's ability to function after failure in a critical element is examined. This is attributed to the *robustness of controllability* referring to network controllability after failure. This parameter is assessed by the number of required driver nodes for full controllability before and after a node failure. The Second aspect of robustness points to the robustness of the grid against disturbance (disturbance decoupling) which is analyzed under the impact of symmetry. The third aspect of robustness deals with the identification of the most critical network elements in terms of their contributions in symmetry group and, in turn, grid robustness. *Fault tolerance* points to the network ability to preserve controllability (or tolerate failure or loss of a node) with a fixed set off driver nodes. The set of "critical" nodes/edges should be defined: A node/edge that has the most impact on the cardinality of automorphism group is considered as a *critical* element since, as will be verified, the elimination of nodes/edges with more multiplicity in the symmetry group can significantly change the number of required driver nodes. Subsequently, *controllability* means that the network can be driven from any initial state to any final state in finite time using a set of driver nodes. All findings of this paper are attained by leveraging on the concept of symmetry.

In controlled consensus problem, the external control signal $u(t) \in \mathbb{R}^q$ should be applied to node i through input matrix $\overline{B}_i \in \mathbb{R}^q$. The individual node's dynamics can be written as

$$\dot{x}_i(t) = -\sum_{\{i,j\}\in E} (x_i - x_j) + \overline{B}_i^T u(t) \tag{6}$$

where $x_i(t) \in \mathbb{R}$ is the state of node $i \in V$ at time t. The network's dynamics at time t is observed by a $y(t) \in \mathbb{R}^p$ via an output matrix $C \in \mathbb{R}^{p\times n}$. The full network dynamics is then given by

$$\begin{aligned} \dot{x}(t) &= -\mathcal{L}(\mathcal{G})x(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{7}$$

where $B = [\overline{B}_1, \overline{B}_1, \dots, \overline{B}_n]^T \in \mathbb{R}^{n\times q}$. For a set of input nodes or driver nodes in the graph as $S = [i_1, i_2, \dots, i_q]$ for $i_1 < i_2 < \cdots < i_q$ the associated input matrix is given by $B(S) = [e_{i_1}, e_{i_2}, \dots, e_{i_q}] \in \mathbb{R}^{n\times q}$.

The network dynamics given in (7) is said to be controllable if the pair $(-\mathcal{L}(\mathcal{G}), B)$ is controllable. This criterion is restated in proposition below according to the network symmetric characteristics.

Proposition 3.1 (*Chapman & Mesbahi, 2014*). *A graph $\mathcal{G}$ is uncontrollable from any pair $(-\mathcal{L}(\mathcal{G}), B(S))$, if there exist a nontrivial automorphism of $\mathcal{G}$ which fixes all inputs in the set S.*

In Chapman and Mesbahi (2014), the choice of driver nodes is attributed to the determining set via Corollary 3.1 where the state matrix is replaced by the adjacency matrix.

Corollary 3.1 (*Chapman & Mesbahi, 2014*). *Let $\mathcal{A}(\mathcal{G})$ represents the adjacency matrix of the graph $\mathcal{H}$. A necessary condition for controllability of the pair $(A(\mathcal{G}), B(S))$ is that S is a determining set.*

3.1. CN controllability and robustness of controllability based on symmetry groups

The necessary conditions for controllability (or sufficient conditions for uncontrollability) have been attained in Chapman and Mesbahi (2014) as proposition below.

Proposition 3.2 (*Chapman & Mesbahi, 2014*). *The system of Eqs. (4) is uncontrollable if there exists a nontrivial automorphism of $\mathcal{G}$ which fixes all inputs in the set of driver nodes S (or equivalently matrix B).*

This proposition can be restated using the concept of determining set defined as below.

Definition 3.1. A set of vertices S is a determining set for a graph $\mathcal{G}$ if every automorphism of $\mathcal{G}$ is uniquely determined by its action on S. Equivalently, a subset S of the vertices of a graph G is called a determining set if whenever $g, h \in Aut(\mathcal{G})$ so that $g(s) = h(s)$ for all $s \in S$, then $g = h$.

The above lemma implies that to find the whole set of driver nodes, one have to compute and sweep over the whole set of automorphism group $Aut(\mathcal{G})$. The typical cardinality of automorphism group for complex networks can be as big as 10^{17} (this is verified in MacArthur et al. (2018) and also computed in simulation section for several networks). Obviously, computing and sweeping over this set is not computationally effective. Lemma below presents an alternative approach based on generators of automorphisms.

Lemma 3.1. *Let σ be an automorphism attained from the composition of some generators of automorphisms $\sigma_1, \sigma_2, \dots, \sigma_r$ where their orders are $\epsilon_1, \epsilon_2, \dots, \epsilon_r$, respectively. The necessary conditions for controllability of $\mathcal{G}^*$ are*

- *(i) if order(σ) $=$ lcm($\epsilon_1, \epsilon_2, \dots, \epsilon_r$) where lcm stands for the lowest common multiple, then there is at least one node n^* in the set of nodes $V^* \in \{\sigma_1 \cap \sigma_2 \cap \dots \sigma_r\}$ for which $n^* \in S^*$,*
- *(ii) if order(σ) $\neq$ lcm($\epsilon_1, \epsilon_2, \dots, \epsilon_r$) then for the joint node/s n_j where $n_j \in V^*$, $n_j^* \notin S^*$,*
- *(iii) if all nodes in V^* are joint nodes then $V^* \in S^*$, and*
- *(iv) if a node $n_a \notin Gen(\mathcal{G})$ then n_a can be excluded from S^*.*

Proof. Any automorphism σ is a composition of either joint or disjoint generators and/or mediator automorphisms. If $\sigma_1, \sigma_2, \dots, \sigma_r$ are disjoint, according to commuting property of permutations, one can write

$$(\sigma_1 \circ \sigma_2 \circ \dots \sigma_r)^z = \sigma_1^z \circ \sigma_2^z \circ \dots \circ \sigma_r^z$$

for all positive integers z. Let m be the lowest common multiple of the integers $\epsilon_1, \epsilon_2, \dots, \epsilon_r$. Then

$$(\sigma_1 \circ \sigma_2 \circ \dots \sigma_r)^m = \sigma_1^m \circ \sigma_2^m \circ \dots \sigma_r^m$$

and clearly $(\sigma_1 \circ \sigma_2 \circ \dots \sigma_r)^m = \varepsilon$. On the other hand, if $(\sigma_1 \circ \sigma_2 \circ \dots \sigma_r)^m = \varepsilon$ it follows (from the permutations being disjoint) that the order of each σ_i is m or $\sigma_i^m = \varepsilon$. Therefore m is divisible by each ϵ_i which leads to

$$\mathrm{order}(\sigma_1 \circ \sigma_2 \circ \dots \sigma_r) = m = \mathrm{lcm}(\epsilon_1, \epsilon_2, \dots, \epsilon_r).$$

Then for S^* to be a determining set there should be at least one node n^* in S^* that also belongs to set of nodes in $\sigma_1 \cap \sigma_2 \cap \dots \sigma_r$. Similarly, if $(\sigma_1 \circ \sigma_2 \circ \dots \sigma_r)$ contain two or more joint permutations then order(σ) $\neq$ lcm($\epsilon_1, \epsilon_2, \dots, \epsilon_r$). Consequently, joint nodes have to be excluded from the determining set, i.e., $n_j \notin S^*$. Finally, if all nodes in the composition $(\sigma_1 \circ \sigma_2 \circ \dots \sigma_r)$ are joint nodes (which is very rare) then all nodes V^* are in S^*. Finally, condition (iv) is straightforward considering that if a node is fixed by all generators then the composition of generators or/and mediator automorphisms will not move it. □

The above lemma relates the necessary conditions of CN controllability to the selection of a set of driver nodes attained from generators of automorphisms $Gen(\mathcal{G})$. The cardinality of $Gen(\mathcal{G})$ is very smaller than $Aut(\mathcal{G})$. It is thus computationally more effective to attain the set of driver nodes from $Gen(\mathcal{G})$ instead of $Aut(\mathcal{G})$.

The network controllability (or the number of required drive nodes) can be affected once the network topology is changed following a failure in a network element. Robustness of controllability can be guaranteed if the selection of nodes in determining set are constrained to some conditions on generators of automorphisms. These conditions are formalized in the theorem below.

Theorem 3.1. *Assume that the adjacency matrix of $\mathcal{G}$ is diagonalizable and symmetry preserving and b is the indicator vector associated to the set S of N_d driver nodes. Let the network topology after attacks be denoted by $\mathcal{G}^*$. If the robustness of controllability of $\mathcal{G}$ in case of a successful attack manipulating the network topology to $\mathcal{G}^*$ is guaranteed then*

- *(i) Given g_k^* as a generator of automorphism of $\mathcal{G}^*$, if all $j \in V_{g_k^*}$ are joint nodes then all nodes of g_k^* are in S, otherwise $j \notin S$ where j is the joint node/s of the pairwise joint generators, and*
- *(ii) there is no non-trivial generator g^* of $\mathcal{G}$ that fixes S, and*
- *(iii) if all eigenvalues of $M = J - I - 2\mathcal{A}$, where I and J are the identity and unit matrices, be simple then $S \nsubseteq \Phi_g$ where Φ_g is the set of fixed nodes by all generators of automorphisms.*

Proof. Consider the matrix W defined as

$$W = [\mathcal{J} \quad \mathcal{A}\mathcal{J} \quad \dots \quad \mathcal{A}^{n-1}\mathcal{J}] \tag{8}$$

where $\mathcal{J}$ is the incidence vector associated with the set of driver nodes. Then W is invertible if the network is robustly controllable. Considering $\mathcal{M}_p$ as the permutation matrix associated with $Aut(\mathcal{G})$, then the set of generators that fix the driver nodes as a set need to satisfy $\mathcal{M}_p\mathcal{J} = \mathcal{J}$. Subsequently, it can be written

$$\begin{aligned}\mathcal{M}_pW &= [\mathcal{M}_p\mathcal{J} \quad \mathcal{M}_p\mathcal{A}\mathcal{J} \quad \dots \quad \mathcal{M}_p\mathcal{A}^{n-1}\mathcal{J}] \\ &= [\mathcal{M}_p\mathcal{J} \quad \mathcal{A}\mathcal{M}_p\mathcal{J} \quad \dots \quad \mathcal{A}^{n-1}\mathcal{M}_p\mathcal{J}]\end{aligned} \tag{9}$$

where along with $\mathcal{M}_p\mathcal{J} = \mathcal{J}$ one can conclude $\mathcal{M}_pW = W$. Since W is invertible then $\mathcal{M}_p$ needs to be identity matrix which, in turn, implies that the only generator of $\mathcal{G}$ that fixes the driver nodes is the identity or trivial permutation. Contrastingly, the joint nodes of generators have to be excluded from the driver nodes since the composition of joint generators may fix the joint node. Consequently, there will be an automorphism that fixes the joint node which is also a driver node. This contradicts the required conditions for controllability mentioned in Corollary 3.1. In the case where all nodes of a generator are joint nodes all nodes need to be included in driver nodes so that there will not be an automorphism that fixes all nodes in S except for identity. From the properties of compositions of permutations one can write

$$Mov\{\zeta_{dis}^r \circ \delta_{dis}^s\} \subset Mov\{Gen_{dis}\}$$

where ζ_{dis} and δ_{dis} are disjoint generators and r and s are the orders of generators in composition. This simply means that the compositions of disjoint generators does not move an already fixed node by all of those generators. Then the proof of part (iii) is a straightforward result of Theorem 5.8 in Farrugia and Sciriha (2014). □

The above theorem relates the selection of driver nodes to some permutation properties of generator set. The use of this theorem is computationally effective as it relies on sweeping over a limited number of permutations.

3.2. Robustness against disturbance by employing the concepts of determining set and controlled invariant subspace

In this section, a model of network with disturbance is considered as

$$\dot{x}(t) = f(x(t), u(t), w(t)) \tag{10}$$

where $w(t) \in \mathbb{R}^n$ is the disturbance which can possibly represent external disturbances, modeling uncertainty, parameter uncertainty, and unmodeled nonlinear dynamics, etc. The disturbance decoupling problem is thus to find a state feedback so that the outputs are not disturbed by w.

Definition 3.2. In the state space model of the systems, if there is a subspace that contains the initial states then it is possible to preserve the states in that subspace at all times. This subspace is called a *controlled invariant subspace*. For the LTI system of (4), the subspace $\mathcal{V}$ is an (A, B)-invariant or controlled invariant subspace if

$$(A + Bk)\mathcal{V} \subseteq \mathcal{V} \tag{11}$$

where k is the feedback gain.

In configuring the choice of determining set of automorphism group (or generator set) according to the concept of controlled invariant subspace, a new lemma is proposed below which presents the necessary and sufficient conditions for disturbance decoupling. The resulted determining set also preserves the necessary condition of controllability as verified earlier in Theorem 3.1.

The general form of LTI systems considering the disturbance can be written as

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + Ew(t) \\ y &= Cx(t)\end{aligned} \tag{12}$$

where $w(t) \in \mathbb{R}^n$ is a vector of unmeasured signals which lumps the impacts of all disturbances. In a network with controlled consensus protocol of (7), a node's dynamics with disturbance is given by

$$\begin{aligned}\dot{x}(t) &= -\mathcal{L}x(t) + Bu(t) + Ew(t) \\ y &= Cx(t),\end{aligned} \tag{13}$$

where $B = [\overline{B}_1, \overline{B}_2, \dots, \overline{B}_n]^T \in \mathbb{R}^{n\times q}$.

Theorem 3.2. *The set of driver nodes attained from the determining set characterizes the network model of (13) as*

$$\begin{aligned}\dot{x}(t) &= -\mathcal{L}x(t) + Su(t) + Ew(t) \\ y &= Cx(t)\end{aligned} \tag{14}$$

where $S \in \mathbb{R}^q$ is the characteristic matrix associated to the determining set attained from the $Aut(\mathcal{G})$ or $Gen(\mathcal{G})$ such that $S_{i,1} = 1$ if $v_i \in S$ otherwise $S_{i,1} = 0$. Assume that the adjacency matrix of the network $\mathcal{G}$ is diagonalizable and symmetry preserving. With fixed control feedback k, the disturbance E can be decoupled if S^* satisfies

$$Im\, E \subseteq \langle -\mathcal{L} + S^*k \mid Im\, E\rangle \subseteq ker\, C. \tag{15}$$

Proof. The Eq. (13) can be written as

$$\dot{y}(t) = C(-\mathcal{L} + Sk)x(t) + ECw(t)$$

To decouple the disturbance, the second term in the above equation must be zero, i.e., $CE = 0$. Subsequently, by assuming $C(\mathcal{L} + Sk)^{i-2}E = 0$ where $i \geq 2$, one can write

$$y^{(i)} = C(-\mathcal{L} + Sk)^i x + C(-\mathcal{L} + Sk)^{i-1}Ew.$$

Then the following conditions must be satisfied

$$\begin{aligned}&CE = 0 \\ &C(-\mathcal{L} + Sk)^{i-1}E = 0\end{aligned} \tag{16}$$

where $i = \{0, 1, \dots, n\}$. The above equation can be restated as

$$\begin{aligned}&\text{Im } E \subseteq \ker C \\ &(-\mathcal{L} + Sk)^{i-1}\text{Im } E \subseteq \ker C.\end{aligned} \tag{17}$$

Equivalently,

$$\langle -\mathcal{L} + S^*k \mid \text{Im } E\rangle \subseteq \ker C \tag{18}$$

i.e., $\langle \mathcal{L} + S^*k \mid \mathrm{Im}\ E\rangle \subseteq \ker C$ features $(\mathcal{L}, B)$-invariant or controlled invariant property. If $E \subseteq \langle \mathcal{L} + S^*k \mid \mathrm{Im}\ E\rangle$ then

$$\langle -\mathcal{L} + S^*k \mid \mathrm{Im}\ E\rangle \subseteq \langle -\mathcal{L} + S^*k\rangle \subseteq \ker C$$

and it is necessary that

$$\langle -\mathcal{L} + Sk \mid \mathrm{Im}\ E\rangle \subseteq \ker C.$$

Since $\langle -\mathcal{L} + Sk \mid \mathrm{Im}\ E\rangle$ is an $(-\mathcal{L}, S)$ invariant subspace of $\ker C$, then $\langle -\mathcal{L} + Sk \mid \mathrm{Im}\ E\rangle \subseteq \langle -\mathcal{L} + S^*k \mid \mathrm{Im}\ E\rangle$. Subsequently $\mathrm{Im}\ E \subseteq \langle -\mathcal{L} + Sk \mid \mathrm{Im}\ E\rangle$ results in $\mathrm{Im}\ E \subseteq \langle -\mathcal{L} + S^*k \mid \mathrm{Im}\ E\rangle$. □

Theorem 3.2 presents the necessary and sufficient conditions for disturbance decoupling in the networked system of (14). It relates the disturbance decoupling to the appropriate selection of determining set (attained from symmetry groups, i.e., $Aut(\mathcal{G})$ and $Gen(\mathcal{G})$) and control feedback gain.

3.3. Critical nodes identification based on symmetry

As verified in MacArthur et al. (2018), symmetry can improve the network's robustness by inducing redundancy which, in turn, provides structural backups against attacks (Tononi et al., 1999). The more often any given node is repeated in $Aut(\mathcal{G})$, the more effective it is in the structural robustness of the underlying network. However, since $|Aut(\mathcal{G})|$ for a typical network is usually a very big number (MacArthur et al., 2018) it is not cost effective to find the whole set of automorphisms (this is also verified in simulation section in Table 1). Instead, we propose another criteria to quantify the role of each node in symmetry and, in turn, in network robustness. To this end, the elementary factors of automorphisms or generators of automorphisms ($Gen(\mathcal{G})$) are used. Lemma 3.2 formalizes this adaption from automorphism group to the set of generators.

Lemma 3.2. *For a given graph $\mathcal{G}$, if the multiplicity of a node v_l in $Gen(\mathcal{G})$ is p, then the multiplicity of v_l in $Aut(\mathcal{G})$ is greater or equal to $p(1+q)$ where q is the cardinality (size) of the set of generators that fix v_l.*

Proof. Let the set of generators that map v_l be denoted by

$$\mathrm{Mov}_{v_l}^{Gen} := \{\delta_1, \delta_2, \dots, \delta_p\}$$

and the set of generators that fix v_l be denoted by $\{\sigma_1, \sigma_2, \dots, \sigma_q\}$. Given the pointwise product of $\{\sigma_1, \sigma_2, \dots, \sigma_q\}$ and $\{\delta_1, \delta_2, \dots, \delta_p\}$ be denoted by $\Delta\circ\Sigma$, we can write

$$|\mathrm{Mov}_{v_l}^{Aut}| \geq |\mathrm{Mov}_{v_l}^{Gen}| + |\mathrm{Mov}_{v_l}^{\Delta\circ\Sigma}|$$

where $|.|$ stands for the cardinality of the set. Clearly, the above inequality can be written as

$$|\mathrm{Mov}_{v_l}^{Aut}| \geq (p + p.q) = p(1+q). \quad \square$$

Lemma 3.2 guarantees a lower bound on the multiplicity of a node v_l in $Aut(\mathcal{G})$ as long as v_l is permuted by at least one generator. Implied by the above lemma, one can determine the most repeated nodes in $Aut(\mathcal{G})$ by only assessing $Gen(\mathcal{G})$ without imposing a large computation burden. The node multiplicity can be used as a measure of each node/edge's impact on symmetry strength of the network. In fact, it is enough to compute the size of automorphism group $|Aut(\mathcal{G})|$ after deleting each node with maximum multiplicity. The higher increase in $|Aut(\mathcal{G})|$ after each modification means that the modified node/edge is more effective in the structural robustness of the network and, in turn, is considered as a more critical node. This is also verified by simulation on several networks and discussed in the simulation section.

3.4. Fault tolerance against node elimination based on a novel symmetry index

We can compute the size of automorphism group before and after node elimination. Then the consistency between the number of required driver nodes with the network's symmetry level can be investigated. We attain the ratio of the symmetry after node removal to the original network's symmetry using equation below:

$$\rho_{aut} = \% \frac{\overline{|\mathrm{Aut}(\mathcal{G})|}}{|\mathrm{Aut}^*(\mathcal{G})|} \times 100 \tag{19}$$

where $|\mathrm{Aut}^*(\mathcal{G})|$ and $\overline{|\mathrm{Aut}(\mathcal{G})|}$ are the size of automorphism group of original network and the average size of automorphism group of the network after removing a node. The size of the automorphism group, as the traditional measure of symmetry, provides a fair approximation of symmetry. However, to create a metric that realizes the complex structure of symmetry, a novel notion of symmetry, network redundancy, has been proposed in MacArthur et al. (2018) as

$$r_{\mathcal{G}} = \frac{|\mathcal{O}| - 1}{V} \tag{20}$$

where $r_{\mathcal{G}}$ is a measure of network redundancy and $|\mathcal{O}|$ is the number of orbits. A normalized measure of redundancy is

$$r_{\mathcal{G}}^* = 1 - \frac{|\mathcal{O}| - 1}{V - 1} \tag{21}$$

presented in Ball and Geyer-Schulz (2018) that captures the asymmetric case as well. In (20) and (21), the presence of symmetric structures in a graph is quantified employing the orbits of automorphisms. In fact, all nodes in the same orbit are structurally equivalent. Consequently, non-trivial orbits have been associated with structural redundancy which according to MacArthur et al. (2018) and Tononi et al. (1999) reinforces the robustness of the network against attacks on the nodes by providing structural backups. However, the difference between the induced symmetry by two orbits of different sizes cannot be projected by (20) and (21).

Inspired by Shannon's entropy formula that provides a measure of disorder, uniformity, or randomness in networks (Bonchev, 1983), the orbits' impact on the network symmetry could be better described as

$$S_{\mathcal{O}}(\mathcal{G}) = \frac{\sum_{i=1}^{l} |\mathcal{O}_l| \ln |\mathcal{O}_l|}{n} \tag{22}$$

where $S_{\mathcal{O}}(\mathcal{G})$ is the impact of orbits of automorphisms on symmetry level featuring the orbits' structures, $|\mathcal{O}_l|$ is the number of elements of the lth orbit, and n is the number of nodes. The bigger the orbit sizes, the more symmetric the underlying graph is. Computing (22) before and after node elimination determines how robust the network is against node elimination. We define the ratio of symmetry after a node removal to the original network's symmetry as

$$\rho_{\mathcal{O}} = \% \frac{\overline{S_{\mathcal{O}}(\mathcal{G})}}{S_{\mathcal{O}}^*(\mathcal{G})} \times 100 \tag{23}$$

where $S_{\mathcal{O}}^*(\mathcal{G})$ and $\overline{S_{\mathcal{O}}(\mathcal{G})}$ are the symmetry index of the original network and the average symmetry index of the network after removing a node.

4. Simulation

The symmetric characteristics of several networks are initially investigated to verify that the quantified form of symmetry studied in this paper is present in all networks independent of network size and order. This is carried out on Sage with Python programming for several real and synthetic networks of various sizes and orders. The symmetry specifications of a few of these networks are presented in Table 1. It is observed that all networks of various sizes entail a certain level of symmetry.

The detailed simulation is accomplished on two synthetic networks with 101 nodes and 273 nodes illustrated in Fig. 2.a-b. The symmetry

Table 1
The symmetry specifications of various networks including the sizes of automorphism group $|Aut(\mathcal{G})|$, generator set $Gen(\mathcal{G})$, orbits of automorphisms $|\mathcal{O}|$, maximum multiplicity of nodes in generator set denoted by M_{max}, cardinality of the set of nodes with maximum multiplicity $|M_{max}|$, and the symmetry index $S_{\mathcal{O}}(\mathcal{G})$ associated with the impact of orbits of automorphisms.

Network name	$\|V\|$	$\|E\|$	$\|Gen(\mathcal{G})\|$	$\mathcal{M}_{max}$	$\|\mathcal{M}_{max}\|$	$\|Aut(\mathcal{G})\|$	$\|\mathcal{O}(\mathcal{G})\|$	$S_{\mathcal{O}}(\mathcal{G})$
Net. of Fig. 2.a	101	380	53	5	42	1.27×10^{20}	7	4.22
Net. of Fig. 2.b	273	1474	23	2	43	28×10^{6}	251	0.12
Net. of Fig. 2.c	332	4252	54	3	2	2.5×10^{24}	276	0.30
Net. of Fig. 2.d	47	264	2	22	2	6	44	0.07
US power grid	4941	13 188	420	2	143	5.2×10^{152}	4466	0.15
QHPS	882	3354	82	3	52	8.5×10^{27}	336	1.08
NEPS	66	1194	35	2	52	1.7×10^{25}	7	2.25

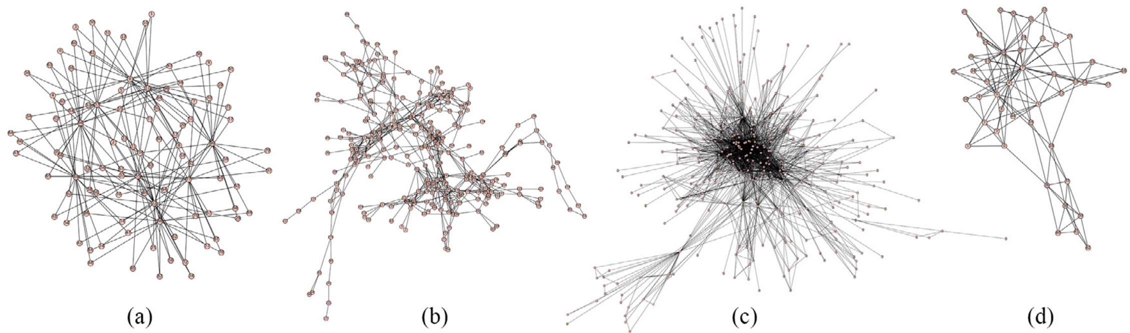

Fig. 2. Four networks with (a) 101 nodes, (b) 273 nodes, (c) 332 nodes, and (d) 47 nodes.

specifications of these networks are presented in Table 1, rows 2–3. The network of Fig. 2.a is selected as an illustrative example to examine the symmetry characteristics and the proposed adaption (of Lemma 3.1, Theorem 3.1, and Lemma 3.2) from automorphism to generators of automorphisms. As explained in Section 3, nodes with more multiplicities in symmetry group are more critical for the network robustness. The 101-node network of Fig. 2.a has 1.27×10^{20} automorphisms. According to Table 1, this can be considered as a medium cardinality for the size of automorphism group as there are networks with 10^{152} automorphisms in Table 1. Since computing and sweeping over these large sets of automorphism groups is computationally inefficient, we implement the proposed adaption of this paper, presented in Lemma 3.1, Theorem 3.1, and Lemma 3.2, in order to investigate the symmetry characteristics of the network and its impact on the network robustness.

The set of generators of automorphism of the network of Fig. 2.a are computed in Sage and, unlike the automorphism group, its size is small and is equal to 53. The full set of generators of automorphisms of this network is presented in Table 2, row 2. The number of required driver nodes can be attained by ECM explained in Section 2.3. The set of essential driver nodes for full controllability is attained in MATLAB and a big portion of nodes, 81 nodes which is approximately %81 percent of all nodes, are selected as drive nodes. This set of nodes are presented in Table 2, row 4 and also indicated in Fig. 3 with green rings.

To investigate the robustness of controllability in case of failure in a network critical element, first, we determine the set of critical nodes in terms of their impact on the cardinality of the symmetry group. Since the cardinality of Aut($\mathcal{G}$) for the network of Fig. 3 is too big, we need to implement the proposed approach of this study for finding the most critical nodes. To this end, the corresponding generator set is calculated via Python programming in Sage and 53 generators are attained (see Table 2, row 2).

The set of nodes with maximum multiplicity in the Gen($\mathcal{G}$) is computed in Sage and the results are presented in Table 2, row 5. As indicated, there are 42 nodes which are repeated 5 times (maximum multiplicity) in Gen($\mathcal{G}$). There are also 28 nodes with multiplicity 4 (Table 2, row 6), 16 nodes with multiplicity 3 (Table 2, row 7), 12 nodes with multiplicity 2 (Table 2, row 8), and 2 nodes with multiplicity 1 (Table 2, row 9). These sets of nodes are illustrated in Fig. 3 with distinct colors.

Now, we compare the characteristics of the network symmetry after failure (removing) of nodes with different multiplicity. The cardinality of automorphism group of the modified network after removing a node is computed for nodes with different multiplicities in Gen($\mathcal{G}$) and the average cardinality of Aut($\mathcal{G}$) is computed and the results are presented in Table 3, row 3. As can be seen, removing the nodes with more multiplicities is more effective in reducing the cardinality of Aut($\mathcal{G}$). The average cardinality of Aut($\mathcal{G}$) has reduced to 1.4×10^{18}, 1.8×10^{18}, 2.3×10^{18}, 1.3×10^{19}, and 1.5×10^{19} after removing a node with multiplicities 5, 4, 3, 2, and 1, respectively. This entail a significant reduction of %98.9 in |Aut($\mathcal{G}$)| when we remove a node with 5 times repetitions in the Gen($\mathcal{G}$). However, this dramatic change does not reflect on the number of required driver nodes after removing a nodes. As indicated in Table 3, row 5, the number of required driver nodes has remained unchanged (equal to 81) after removing a node with different multiplicities. This means that the cardinality of automorphism group, as a quantified measure of symmetry, is not consistent with the controllability requirement of the network. We investigate this proportional relation between symmetry level and the number of required driver nodes via the proposed index of symmetry in Eq. (22).

The set of orbits of automorphisms for the network of Fig. 3 is computed in Sage and 7 orbits are identified as listed in Table 2, row 3. Computing (22) leads to $S_{\mathcal{O}} = 4.22$ which, compared to other networks in Table 1, is considered a high value of symmetry. The number of nodes in the 7 orbits are 8, 32, 56, 2, 1, 1, and 1. The symmetry index has reduced to 3.76 for any node removal. This leads to the fixed ratio index equal to %88.89 after node removal. Comparing to the size of automorphism groups, this index indicates a significant lower reduction in the symmetry level which can explain why there is no reduction in the number of required driver nodes. This is in line with high robustness of strongly symmetric networks. The significance of index (22) could be better realized for less symmetric networks like 273-node network.

The 273-node network has the orbital symmetry index equal to $S_{\mathcal{O}} = 0.1221$ which is considered a very less symmetric network compared to 101-node network. Modifying this network by removing nodes with multiplicities 2 (maximum multiplicity) and 1 has resulted to the orbital symmetry index equal to 0.1195 and 0.1202 which means %2.13 and %1.56 reduction in the symmetry index. This reduction is in line with the reduction in the average number of required driver

Table 2
The symmetry specifications of 101-node network including the elements of generator set $Gen(\mathcal{G})$, orbits of automorphisms $|\mathcal{O}|$, the set of driver nodes N_d, set of nodes in M_i where i is the multiplicity of nodes in the generator set denoted, cardinality of the set of nodes with maximum multiplicity $|M_{max}|$, and the symmetry index $S_{\mathcal{O}}(\mathcal{G})$ associated with the impact of orbits of automorphisms.

Param.	Elements of the symmetry parameter
Gen($\mathcal{G}$)	(89,98), (79,97), (78,87), (69,96), (68,86), (67,76), (59,95), (58,85), (57,75), (56,65), (49,94), (48,84), (47,74), (46,64), (45,54), (39,93), (38,83), (37,73), (36,63), (35,53), (34,43), (29,92), (28,82), (27,72), (26,62), (25,52), (24,42), (23,32), (19,91), (18,81), (17,71), (16,61), (15,51), (14,41), (13,31), (12,21), (9,90), (8,9) (18,19) (28,29) (38,39) (48,49) (58,59) (68,69) (78,79) (80,90) (81,91) (82,92) (83,93) (84,94) (85,95) (86,96) (87,97) (88,99), (8,80), (7,8) (17,18) (27,28) (37,38) (47,48) (57,58) (67,68) (70,80) (71,81) (72,82) (73,83) (74,84) (75,85) (76,86) (77,88) (79,89) (97,98), (7,70), (6,7) (16,17) (26,27) (36,37) (46,47) (56,57) (60,70) (61,71) (62,72) (63,73) (64,74) (65,75) (66,77) (68,78) (69,79) (86,87) (96,97), (6,60), (5,6) (15,16) (25,26) (35,36) (45,46) (50,60) (51,61) (52,62) (53,63) (54,64) (55,66) (57,67) (58,68) (59,69) (75,76) (85,86) (95,96), (5,50), (4,5) (14,15) (24,25) (34,35) (40,50) (41,51) (42,52) (43,53) (44,55) (46,56) (47,57) (48,58) (49,59) (64,65) (74,75) (84,85) (94,95), (4,40), (3,4) (13,14) (23,24) (30,40) (31,41) (32,42) (33,44) (35,45) (36,46) (37,47) (38,48) (39,49) (53,54) (63,64) (73,74) (83,84) (93,94), (3,30), (2,3) (12,13) (20,30) (21,31) (22,33) (24,34) (25,35) (26,36) (27,37) (28,38) (29,39) (42,43) (52,53) (62,63) (72,73) (82,83) (92,93), (2,20), (1,2) (10,20) (11,22) (13,23) (14,24) (15,25) (16,26) (17,27) (18,28) (19,29) (31,32) (41,42) (51,52) (61,62) (71,72) (81,82) (91,92), (1,10), (0,11) (2,12) (3,13) (4,14) (5,15) (6,16) (7,17) (8,18) (9,19) (20,21) (30,31) (40,41) (50,51) (60,61) (70,71) (80,81) (90,91)
$\mathcal{O}(\mathcal{G})$	[0, 11, 22, 33, 44, 55, 66, 77, 88, 99], [1, 2, 10, 3, 20, 12, 4, 30, 13, 21, 5, 40, 14, 31, 23, 6, 50, 15, 41, 24, 32, 7, 60, 16, 51, 25, 42, 34, 8, 70, 17, 61, 26, 52, 35, 43, 9, 80, 18, 71, 27, 62, 36, 53, 45, 90, 19, 81, 28, 72, 37, 63, 46, 54, 91, 29, 82, 38, 73, 47, 64, 56, 92, 39, 83, 48, 74, 57, 65, 93, 49, 84, 58, 75, 67, 94, 59, 85, 68, 76, 95, 69, 86, 78, 96, 79, 87, 97, 89, 98], [100]
N_d	5, 9, 10, 11, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 35, 36, 37, 38, 39, 40, 42, 43, 44, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 79, 80, 82, 83, 84, 85, 86, 87, 88, 90, 92, 93, 94, 95, 96, 97, 98, 99, 101
$v \in M_5$	13, 14, 15, 16, 17, 18, 24, 25, 26, 27, 28, 31, 35, 36, 37, 38, 41, 42, 46, 47, 48, 51, 52, 53, 57, 58, 61, 62, 63, 64, 68, 71, 72, 73, 74, 75, 81, 82, 83, 84, 85, 86
$v \in M_4$	2, 3, 4, 5, 6, 7, 8, 19, 20, 29, 30, 39, 40, 49, 50, 59, 60, 69, 70, 79, 80, 91, 92, 93, 94, 95, 96, 97
$v \in M_3$	9, 12, 21, 23, 32, 34, 43, 45, 54, 56, 65, 67, 76, 78, 87, 90
$v \in M_2$	1, 10, 11, 22, 33, 44, 55, 66, 77, 88, 89, 98
$v \in M_1$	0, 99

Table 3
The impact of removing the nodes with different multiplicities (indicated by M_i where i represents the multiplicity of the deleted nodes) on the symmetry indexes and the number of required driver nodes.

Network name	*101-node network*						*273-node network*		
Parameter	$\mathcal{G}$	M_5	M_4	M_3	M_2	M_1	$\mathcal{G}$	M_2	M_1
$\lvert\overline{\mathrm{Aut}(\mathcal{G})}\rvert$	–	$1.4e18$	$1.8e18$	$2.3e18$	$1.3e19$	$1.5e19$	–	$3,545,168$	$9,437,184$
ρ_{aut}	–	%1.10	%1.42	%1.81	%10.24	%11.81	–	%12.6	%33.70
$\overline{N_d}$	81	81	81	81	81	81	16	13.7	14.6
$\overline{S_{\mathcal{O}}}$	–	3.76	3.76	3.76	3.76	3.76	–	0.1195	0.1202
$\% \frac{S_{\mathcal{O}}^{av}}{S_{\mathcal{O}}} \times 100$	–	%88.89	%88.89	%88.89	%88.89	%88.89	–	%97.87	%98.44

nodes from 16 driver nodes for original network to 13.7 and 14.6 driver nodes after removing a node with multiplicities 2 and 1, respectively. As expected, removing a node with more multiplicity in Gen($\mathcal{G}$) is more effective in reducing the number of required driver nodes attained by ECM. This consistence between the number of required driver nodes and the network symmetry level highlights the importance of (22) in capturing the symmetry impact on the robustness of controllability.

It should be noted that although reducing the symmetry index can reduce the number of required driver nodes but the configuration of these driver nodes after node removal might totally differ from their original configuration. Thus, after network manipulation, it might not be controllable by the fixed set of initial driver nodes.

4.1. Discussing the results

Eq. (22), as the proposed measure of symmetry impact on the number of required driver nodes, captures the importance of orbits with bigger sizes. According to Table 1, the 332-node network of Fig. 2.c with 276 orbits has a small $S_{\mathcal{O}}$ equal to 0.30. This is because the sizes of majority of 276 orbits are 1 and, in practice, have no contribution in characterizing the network's symmetry strength. In contrast, the 101-node network of Fig. 2.a has only 7 orbits but it has a high symmetry index equal to 4.44 because the sizes of orbits are big (as listed in Table 2. row 3). The importance of the index (22) is more realizable when we also compare the size of automorphism groups of these two networks. According to Table 1, the size of automorphism group of 273-node network is 19,685 times bigger than 101 node network.

One interesting result that can be observed in most networks is that the majority of driver nodes are also nodes with maximum multiplicity in Aut($\mathcal{G}$) (or $Gen(\mathcal{G})$). This is the case, in particular, in networks with less symmetry than 101-node network. In fact, high symmetric networks necessitate to pin a big portion of nodes as driver node and, as such, it is sometimes unclear to realize this impact of symmetry on the selection of driver nodes. However, for networks with lower symmetry, such as the 273-node network, this is very common that the majority of driver nodes attained by ECM belong to the symmetry groups. To illustrate this, the 273-node network of Fig. 2.b with 273 nodes and 1475 edges are examined. The symmetry characteristics of this network are attained in Sage and presented in Table 1, row 3. Implementing ECM (explained in Section 2.3) has resulted in 16 driver nodes. Interestingly, 10 out of 16 driver nodes belong to the set of generator set. Considering only 46 nodes are in the generator set, this means that the nodes in symmetry group are crucial for the network controllability.

Another interesting observation is that, unlike the majority of network studies which emphasize on the impact of nodes with high centrality or high degree distribution, nodes with low degree distribution may also have significant influence on network performance. As illustrated in Fig. 3, the degree of some nodes in symmetry group is 2 while there are nodes that have higher degree but are not selected as

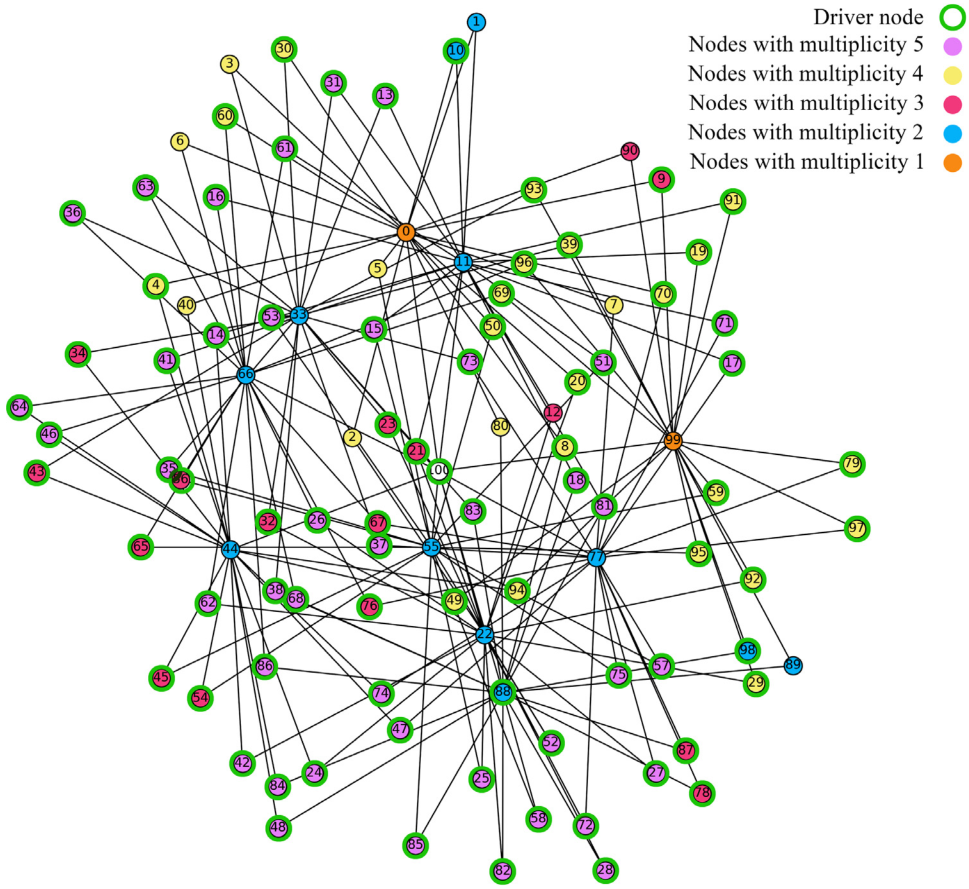

Fig. 3. 101-node system, driver nodes (green rings), nodes with maximum multiplicity $\mathcal{M}_{max} = 96$ (red circles), nodes with $\mathcal{M}_{max} = 72$ (orange circles).

driver nodes. For example, node 66 in Fig. 3 with degree 19 is neither a driver node nor within the symmetry group.

The impact of symmetry on the robustness of controllability (number of driver nodes) depends on the symmetry level of the network. The highly symmetric networks are very robust against node removal but, at the same time, necessitate that a high portion of nodes be driver node. It can be concluded that symmetry reinforces the robustness of controllability meaning a network with higher symmetry index of Equation (our index) is controllable with, naturally, a fixed and bigger set of driver nodes. Moreover, the highly symmetric networks are more tolerant against node failure.

Although the number of required driver nodes can be changed by the variations in the cardinality of symmetry groups, the energy cost of control is not proportionally impacted. A bigger control signal might be required when a lower number of driver nodes are needed. A future research focus can be on a trade off between the number of driver nodes and energy cost of control.

5. Conclusion

This study highlights the importance of the inherent networks' symmetry in addressing three issues related to the network robustness. The robustness of controllability is guaranteed under satisfying some conditions related to the properties of the underlying symmetry group while selecting the set of driver nodes. By incorporating the concept of controlled invariant subspace and determining set, a new necessary and sufficient condition for disturbance decoupling is presented while guaranteeing a necessary condition for CN controllability. Finally, the critical nodes and edges of the network, in terms of their impact on symmetry and in turn, robustness, are identified and the network robustness against node failure is investigated.

The proposed approach in this study is unique as it leverages on an inherent feature of complex networks, i.e., symmetry, to describe novel notions of robustness. The study shows that graph symmetry is independent of network size, order, node degree distribution or any measure of centrality. Additionally, the findings of this paper emphasize on the necessity of considering network symmetry in the pre-design and development of networks.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

Abeysinghe, S., Wu, J., Sooriyabandara, M., Abeyskera, M., Xu, T., & Wang, C. (2018). Topological properties of medium voltage electricity distribution networks. *Applied Energy*, *210*, 1101–1112.

Albert, R., Albert, I., & Nakarado, G. (2004). Structural vulnerability of the north american power grid. *Physical Review E*, *69*(2).

Albert, R., & Barabasi, A. L. (2002). Statistical mechanics of complex networks. *Reviews of Modern Physics*, *74*(1).

Alshehri, J., & Khalid, M. (2019). Power quality improvement in microgrids under critical disturbances using an intelligent decoupled control strategy based on battery energy storage system. *IEEE Access*, *7*, 147314–147326.

Amani, A. M., Jalili, M., Yu, X., & Stone, L. (2017). Finding the most influential nodes in pinning controllability of complex networks. *IEEE Transactions on Circuits and Systems. II. Experimental Briefs*, *64*(6), 685–689.

Ball, F., & Geyer-Schulz, A. (2018). How symmetric are real-world graphs? A large-scale study. *Symmetry*, *10*(1).

Bonchev, D. (1983). *Information theoretic indices for characterization of chemical structures*. UK: Research Studies Press.

Chapman, A., & Mesbahi, M. (2014). On symmetry and controllability of multi-agent systems. In *53rd IEEE Conf. Dec. Con. Los Angeles*.

Chapman, A., & Mesbahi, M. (2015). State controllability, output controllability and stabilizability of networks: A symmetry perspective. In *IEEE 54th annual conference on decision and control (CDC)*.

Chapman, A., Nabi-Abdolyousefi, M., & Mesbahi, M. (2014). Controllability and observability of networks of networks via cartesian products. *IEEE Transactions on Automatic Control*, *59*.

Como, G., Savla, K., Acemoglu, D., Dahleh, M. D., & Frazzoli, E. (2013). Robust Distributed Routing in Dynamical Networks–part II: Strong resilience, equilibrium selection and cascaded failures. *IEEE Transactions on Automatic Control*, *58*(2).

Djilali, L., Sanchez, E. N., & Belkheiri, M. (2019). Real-time Neural Input–Output Feedback Linearization control of DFIG based wind turbines in presence of grid disturbances. *Control Engineering Practice*, *83*, 151–164.

Dorfler, F., Simpson-Porco, J. W., & Bullo, F. (2018). Electrical Networks and Algebraic Graph Theory: Models, properties, and applications. *Proceedings of the IEEE*, *106*(5).

Farrugia, A., & Sciriha, I. (2014). Controllability of undirected graphs. *Linear Algebra and its Applications*, *454*.

Holmgren, A. (2006). Using graph models to analyze the vulnerability of electric power networks. *Risk Analysis*, *26*(4).

Hou, B., Li, X., & Chen, G. (2016). Structural controllability of temporally switching networks. *IEEE Transactions on Circuits and Systems. I. Regular Papers*, *63*(10), 1771–1781.

Jalili, M., & Yu, X. (2017). Enhancing pinning controllability of complex networks through link rewiring. *IEEE Transactions on Circuits and Systems. II. Experimental Briefs*, *64*(6), 690–694.

Koc, Y., Warnier, M., Kooij, R. E., & Brazier, M. T. (2013). An entropy-based metric to quantify the robustness of power grids against cascading failures. *Safety Science*, *59*.

Lawyer, G. (2015). Understanding the influence of all nodes in a network. *Nature Scientific Reports*, *5*, 8665.

Li, Y. S., Ma, D. Z., Zhang, H. G., & Sun, Q. Y. (2015). Critical nodes identification of power systems based on controllability of complex networks. *Applied Science*, *5*.

Lin, C. T. (1974). Structural controllability. *IEEE Transactions on Automatic Control*, *19*(3), 201–208.

Liu, Y.-Y., Slotine, J. J., & Barabási, A.-L. (2011). Controllability of complex networks. *Nature*, *473*(7346), 167–173.

Lou, Y., Wang, L., & Chen, G. (2018). Toward stronger robustness of network controllability: A snapback network model. *IEEE Transactions on Circuits and Systems. I. Regular Papers*, *65*(9), 2983–2991.

MacArthur, B. D., Sanchez-Garcia, R. J., & Anderson, J. W. (2018). Symmetry in Complex Neworks. *Discrete Applied Mathematics*, *156*.

Nasiruzzaman, A. B. M., & Pota, H. R. (2011). Critical node identification of smart power system using complex network framework based centrality approach. In *Annual North-American power symposium*.

Ouyang, B., Ye, Q., Patil, S., Wang, C., Lu, L., & Yan, Z. (2018). The relation between communication Range and Controllability of Networked Multi-Agent Systems. *IEEE Access*, *6*, 35901–35907.

Pagani, G. A., & Aiello, M. (2013). The Power Grid as a complex network: A survey. *Physica A. Statistical Mechanics and its Applications*, *392*, 2688–2700.

Pang, S. P., & Hao, F. (2017). Optimizing controllability of edge dynamics in complex networks by perturbing network structure. *Physica A. Statistical Mechanics and its Applications*, *470*, 217–227.

Pecora, L. M., Sorrentino, F., Hagerstorm, A. M., Murphy, T. E., & Roy, R. (2014). Cluster synchronization and isolated desynchronization in complex networks with symmetries. *Nature Communication*, *5*.

Selvaraj, P., Kwon, O. M., Lee, S. H., & Sakthivel, R. (2014). Uncertainty and disturbance rejections of complex dynamical networks via truncated predictive control. *Journal of the Franklin Institute*, *357*(8), 4901–4921.

Siddique, A. B., Pecora, L. M., Hart, J. D., & Sorrentino, F. (2018). Symmetry-and input-cluster synchronization in networks. *Physical Review*, *97*.

Sorrentino, F., & Pecora, L. M. (2016). Approximate cluster synchronization in networks with symmetries and parameter mismatches. *Chaos. An Interdisciplinary Journal of Nonlinear Science*, *26*(9).

Tononi, G., Sporns, O., & Edelman, G. M. (1999). Measures of degeneracy and redundancy in biological networks. *Proceedings of the National Academy of Sciences of the United States of America*, *96*(6).

Tootaghaj, D. Z., Bartolini, N., Khamfroush, H., He, T., & La Porta, T. (2018). Mitigation and Recovery from Cascading Failures in Interdependent Networks under uncertainty. *IEEE Transactions on Control of Network Systems*, *5*.

Tran, V. P., Garratt, M., & Petersen, I. R. (2020). Switching time-invariant formation control of a collaborative multi-agent system using negative imaginary systems theory. *Control Engineering Practice*, *95*.

Wang, L. Z., Chen, Y. Z., Wang, W. X., & Lai, Y. C. (2017). Physical controllability of complex networks. *Nature Scientific Reports*, *7*(40198).

Wang, Z., Hill, D. J., Chen, G., & Dong, Z. Y. (2017). Power system cascading risk assessment based on complex network theory. *Physica A. Statistical Mechanics and its Applications*, *482*, 532–543.

Wang, S., Li, H., Li, Y., & Sun, W. (2014). Event-triggered control for disturbance decoupling problem of mix-valued logical networks. *Journal of the Franklin Institute*, *357*(2), 796–809.

Wu, X., Wang, X., Gu, M., & Jiang, G. P. (2020). Structural Controllability for a Class of Complex Networks with Root Strongly Connected components. *IEEE Access, Early Access*.

Xiang, L., Chen, F., Ren, W., & Chen, G. (2019a). Advances in network controllability. *IEEE Circuits and Systems Magazine*, *19*(2), 8–32.

Xiang, L., Chen, F., Ren, W., & Chen, G. (2019b). Advances in network controllability. *IEEE Circuits and Systems Magazine*, Second quarter.

Xiao, Y. D., Lao, S. Y., Hou, L. L., & Bai, L. (2014). Edge orientation for optimizing controllability of complex networks. *Physical Review E*, *90*(4), Article 042804.

Yuan, Z., Zhao, C., Di, Z., Wang, W. X., & Lai, Y. C. (2013). Exact controllability of complex networks. *Nature Communication*, *4*.

Zhang, R., Wang, X., Cheng, M., & Jia, T. (2019). The evolution of network controllability in growing networks. *Physica A. Statistical Mechanics and its Applications*, *520*, 257–266.