CA - assignment - 1 (2nd half)

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1) $x = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $y = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$.

let one of the vectors in basis be $q_1 = x = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Other vector perpendicular to q_1 & part of basis

Using Gram - Schmith's process:

$q_2 = y - \text{proj}_{q_1}(y)$ $\qquad \langle q_1, y \rangle = 3+4 = 7$

$= y - \dfrac{\langle q_1, y\rangle}{\langle q_1, q_1\rangle} q_1$ $\qquad \langle q_1, q_1 \rangle = 2$

$= y - \frac{7}{2} q_1$

$= \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} - \begin{bmatrix} 7/2 \\ 7/2 \\ 0 \end{bmatrix}$

$q_2 = \begin{bmatrix} -0.5 \\ +0.5 \\ 2 \end{bmatrix}$

$\therefore$ Orthogonal Basis $= \{q_1, q_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -0.5 \\ +0.5 \\ 2 \end{bmatrix} \right\}$

2) a) Given AB ~~an~~ is invertible where A, B are square matrices.

let C be $(AB)^{-1}$ [inverse of AB].

$(AB)(AB)^{-1} = I$

$\Rightarrow (AB)C = I$

let matrix D be BC

$(AB)C = A(BC) = I$

$\boxed{A \cdot D = I}$

$\Rightarrow$ matrix A has right inverse which is D. — (i) let $E = CA$

$\because (AB)^{-1} = C$

$$EB = (CA)B = C(AB) = I$$

$$EB = I$$

$\therefore$ B has left inverse exist and its left inverse is E. (ii)

To prove - left inverse & right inverse of square matrix are same.

let A be square matrix & P, Q be its left & right inverses respectively.

$$\Rightarrow PA = AQ = I$$

Now $(PA)Q = IQ = P(AQ) = P.I$

$$IQ = PI$$

$$\boxed{Q = P} \text{ --- (iii)}$$

$\therefore$ from (i) & (iii), A has both left & right inverses exist

and $A^{-1} = D$.

$\therefore$ A is invertible.

$\therefore$ from (ii) & (iii), B has both left & right inverses exist

and $B^{-1} = E$.

$\therefore$ B is invertible.

$\therefore$ (AB) is invertible, then A & B are invertible

2) b) A is symmetric matrix. ($A^T \to$ transpose of A)

$$\Rightarrow A^T = A$$

let A^{-1} be its inverse

To prove $(A^{-1})^T = A^{-1}$

proof: $AA^{-1} = I$

Transposing both sides: [$\because I$ is symmetric]

$$(AA^{-1})^T = (I)^T = I$$

using property of transpose
$(AB)^T = B^TA^T$

$\Rightarrow (A A^{-1})^T = (A^{-1})^T (A)^T = I$

$$(A^{-1})^T A = I \qquad [\because A^T = A]$$

multity A^{-1} on Right side on Both sides.

$$((A^{-1})^T A) A^{-1} = I \cdot A^{-1} \qquad [\because I \cdot A = A]$$

$$(A^{-1})^T (A \cdot A^{-1}) = A^{-1} \qquad [\because A \cdot A^{-1} = A].$$

$$(A^{-1})^T \cdot I = A^{-1}$$

$$\boxed{(A^{-1})^T = A^{-1}}$$

Hence proved

$\Rightarrow$ Proof to $(AB)^T = B^T A^T$. Let ~~A~~ $A_{m\times n}$, and $B_{n \times m}$

$(i,j)^{th}$ element of AB is given by [~~let n be dimension~~ of ~~square matrix~~ $\cancel{A}$].

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

for $(AB)^T$, it's $(j,i)^{th}$ entry is;

$$\boxed{((AB)^T)_{j,i} = C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}}$$

Now for $B^T A^T$,

$$(B^T A^T)_{ji} = \sum_{k=1}^{n} B^T_{jk} A^T_{ki} = \sum_{k=1}^{n} B_{kj} A_{ik} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

$$\Rightarrow \boxed{(B^T A^T)_{ji} = \sum_{k=1}^{n} A_{ik} B_{kj}}$$

$\therefore (j,i)^{th}$ elements of $(AB)^T$ & $B^T A^T$ are same, both are equal

$$\boxed{(AB)^T = B^T A^T.}$$

3) for a square matrix A to be invertible it should be well defined

i.e $A^{-1} = \frac{adj(A)}{|A|}$ $\boxed{|A| = det(A)}$

$|A| \neq 0$ [$\because$ denominator cannot be '0'].

$\therefore$ for A to be invertible, $|A| \neq 0$.

a) $A = \begin{bmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ k & -8 & k-1 \end{bmatrix}$

taking determinant using first column;

$det(A) = |A| = k \begin{vmatrix} k+1 & 1 \\ -8 & k-1 \end{vmatrix} - 0 \begin{vmatrix} -k & 3 \\ -8 & k-1 \end{vmatrix} + k \begin{vmatrix} -k & 3 \\ k+1 & 1 \end{vmatrix}$

$= k(k^2 - 1 + 8) + 0 + k(-k - 3k - 3)$

$= k^3 - k + 8k - 4k^2 - 3k$

$det(A) = \cancel{k^3 - 4k^2 + 4k + 8}$

$= k^3 - 4k^2 + 4k$

$det(A) \neq 0$

$k(k^2 - 4k + 4) \neq 0$

$k(k-2)(k-2) \neq 0$

$k \neq 0$ / $k \neq 2$

$\therefore$ for any $k \in \mathbb{R} - \{0, 2\}$, matrix A is invertible.

a value for such k is 1.

b) $A = \begin{bmatrix} k & k & 0 \\ k^2 & 2 & k \\ 0 & k & k \end{bmatrix}$

det through 1st row:

$det(A) = k \begin{vmatrix} 2 & k \\ k & k \end{vmatrix} - k \begin{vmatrix} k^2 & k \\ 0 & k \end{vmatrix} + 0 \begin{vmatrix} k^2 & 2 \\ 0 & k \end{vmatrix}$

$= k(2k - k^2) - k(k^3)$

$\det(A) = 2k^2 - k^3 - k^4$

$\det(A) \neq 0$

$k^2(2 - k - k^2) \neq 0$

$(k^2)(-k+1)(k+2) \neq 0$

$(k)^2(k-1)(k+2) \neq 0$

$k \neq 0 \mid k \neq 1 \mid k \neq -2$

$\therefore$ for any $k \in \mathbb{R} - \{0, 1, -2\}$ A is invertible.

a value for such k is 5.

4) given, $\langle a, b \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$ for $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

for u_1, u_2, u_3 to be orthogonal basis they should be mutually orthogonal (i.e $\langle u_1, u_2 \rangle = \langle u_2, u_3 \rangle = \langle u_3, u_1 \rangle = 0$]

$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $u_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $u_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

$\langle u_1, u_2 \rangle = 1.1 + 1.(-1) + 1(0) = 0$

$\langle u_2, u_3 \rangle = 1.1 + (-1)(1) + 0(-2) = 0$

$\langle u_3, u_1 \rangle = 1.1 + 1.1 + 1.(-2) = 0$

$\therefore$ they are mutually orthogonal, u_1, u_2, u_3 forms basis for $\mathbb{R}^3$

for finding coordinates of $w = \begin{bmatrix} 7 \\ 9 \\ 10 \end{bmatrix}$ under basis u_1, u_2, u_3, w is needed to be expressed in the form;

$w = c_1 u_1 + c_2 u_2 + c_3 u_3$

where, c_1, c_2, c_3 are coordinates wrt each basis vector.

and $c_i = \dfrac{\langle w, u_i \rangle}{\langle u_i, u_i \rangle}$ (projection of w on respective basis vector)

$$C_1 = \frac{\langle w, u_1 \rangle}{\langle u_1, u_1 \rangle} = \frac{7\cdot 1 + 9\cdot 1 + 10\cdot 1}{1\cdot 1 + 1\cdot 1 + 1\cdot 1} = \frac{26}{3} = C_1$$

$$C_2 = \frac{\langle w, u_2 \rangle}{\langle u_2, u_2 \rangle} = \frac{7\cdot 1 + 9(-1) + 10(0)}{1\cdot 1 + (-1)(-1) + 0\cdot 0} = \frac{-2}{2} = -1 = C_2$$

$$C_3 = \frac{\langle w, u_3 \rangle}{\langle u_3, u_3 \rangle} = \frac{7\cdot 1 + 9\cdot 1 + 10(-2)}{1\cdot 1 + 1\cdot 1 + (-2)(-2)} = \frac{-4}{6} = -2/3 = C_3$$

$\therefore$ coordinates of w w.r.t $u_1, u_2, u_3 = (C_1, C_2, C_3)$

$$= \left(\frac{26}{3}, -1, -\frac{2}{3}\right).$$

$$w = \frac{26}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

~~AA~~

5)a) For a matrix A to be orthogonal, its columns and rows should form orthonormal set of vectors.

given $A = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{5} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{5} \\ -\frac{1}{3} & 0 & \frac{2}{5} \end{bmatrix}$.

Column vectors : $u_1 = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$ $u_2 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$, $u_3 = \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \\ \frac{2}{5} \end{bmatrix}$

using standard inner product, $\langle a, b \rangle = a_1b_1 + a_2b_2 + a_3b_3$ for $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$\|u_1\| = \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)$

$= \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{3}{9} = \frac{1}{3} \neq 1.$

$\because$

$\|u_2\| = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \neq 1$

$\|u_2\| \neq 1$

$\|u_3\| = \left(\frac{1}{5}\right)\left(\frac{1}{5}\right) + \left(\frac{1}{5}\right)\left(\frac{1}{5}\right) + \left(\frac{2}{5}\right)\left(\frac{2}{5}\right)$

$= \frac{1}{25} + \frac{1}{25} + \frac{4}{25} = \frac{6}{25} \neq 1$

$\because$ $\|u_1\| \neq 1$ $\|u_2\| \neq 1$, $\|u_3\| \neq 1$

non of vectors are normalized.

$\therefore$ A is not an Orthogonal matrix.

5) b) ~~let~~ Q be 2×2 orthogonal matrix

let $Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

for a matrix to be orthogonal its vectors formed from columns and rows should be orthonormal.

let $Q = [\bar{q}_1, \bar{q}_2] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\left(q_1 = \begin{bmatrix} a \\ c \end{bmatrix}, \; q_2 = \begin{bmatrix} b \\ d \end{bmatrix} \right)$

using column Ortho normality

$\langle q_1 \cdot q_2 \rangle = 0$

using standard inner product for $\mathbb{R}^2$ for $a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$\langle a, b \rangle = a_1 b_1 + a_2 b_2$

$\langle q_1, q_2 \rangle = ab + cd = 0$ — (i)

$\langle q_1, q_1 \rangle = 1$

$a^2 + c^2 = 1$ — (ii) $\quad$ a,c are coordinates of unit circle

$\Rightarrow a = \cos\theta, \; c = \sin\theta$ for some θ.

$\langle q_2, q_2 \rangle = 1$

$b^2 + d^2 = 1$ — (iii)

b,d are coordinates of unit circle

$\Rightarrow b = \cos\alpha, \; d = \sin(\alpha)$

for $ab + cd = 0$;

$\cos\theta \cos\alpha + \sin\theta \sin\alpha = 0$

$\frac{1}{2}[\cancel{\cos(\theta+\alpha)} + \cos(\theta-\alpha)] + \frac{1}{2}[\cos(\theta-\alpha) - \cancel{\cos(\theta+\alpha)}] = 0$

$\frac{1}{2}[\cos(\theta-\alpha) + \cos(\theta-\alpha)] = 0$

$\cos(\theta-\alpha) = 0$

$\theta - \alpha = \frac{\pi}{2} + n\pi$ $\quad$ for $n \in \mathbb{Z}$

$\boxed{\theta = \left(n + \frac{1}{2}\right)\pi + \alpha}$

if $n \% 2 = 0$

~~$\cos\alpha$~~

~~$\cos$~~ $b = \cos\alpha$

$d = \sin\alpha$

$a = \cos\theta = \cos\left(\left(n+\frac{1}{2}\right)\pi + \alpha\right)$

$a = -\sin\alpha$

$c = \sin\theta = \sin\left(\left(n+\frac{1}{2}\right)\pi + \alpha\right)$

$c = \cos\alpha$

$\Rightarrow a = -d \Rightarrow d = -a$

$b = c$

$\Rightarrow Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$

if $n \% 2 = 1$

$b = \cos\alpha$

$d = \sin\alpha$

$a = \cos\theta = \cos\left(\left(n+\frac{1}{2}\right)\pi + \alpha\right)$

$= \sin\alpha$

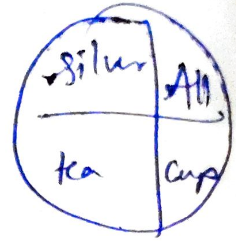
$c = \sin\theta = \sin\left(\left(n+\frac{1}{2}\right)\pi + \alpha\right)$

$= -\cos\alpha$

$\Rightarrow a = d$

$\Rightarrow b = -c \Rightarrow c = -b$

$\Rightarrow Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$



$\therefore$ Q can be of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ (or) $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$

for Q to be orthogonal.