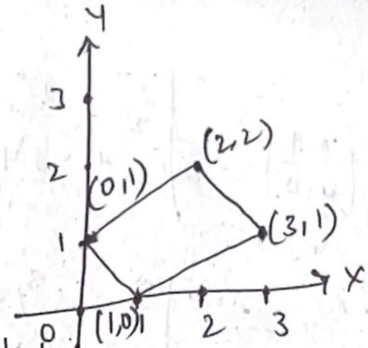
1) $\iint_R dx\,dy$. $(1,0),(3,1),(2,2)$ and $(0,1)$

$u = x+y,\ v = x-2y$.

$$\iint_R f(x,y)\,dx\,dy.$$



$$\iint_R f\big(f_1(u,v), f_2(u,v)\big)\, J\left(\frac{x,y}{u,v}\right) du\,dv$$

$u = x+y \qquad v = x-2y$.

$u - v = 3y \qquad x = u - y$.

$y = \frac{1}{3}(u-v) \qquad x = u - \left(\frac{1}{3}(u-v)\right)$

$= \frac{1}{3}\left[2u+v\right]$

$u = x+y$

$A(1,0) \Rightarrow 1+0=1$

$B(3,1) \Rightarrow 3+1=4$

$C(2,2) \Rightarrow 2+2=4$

$D(0,1) \Rightarrow 0+1=1$.

$v = x-2y$.

$A(1,0) = 1-2(0)$

$= 1$

$B(3,1) = 3-2(1)$

$= 3-2$

$= 1$

$C(2,2) = 2-2(2)$

$= 2-4$

$= -2$

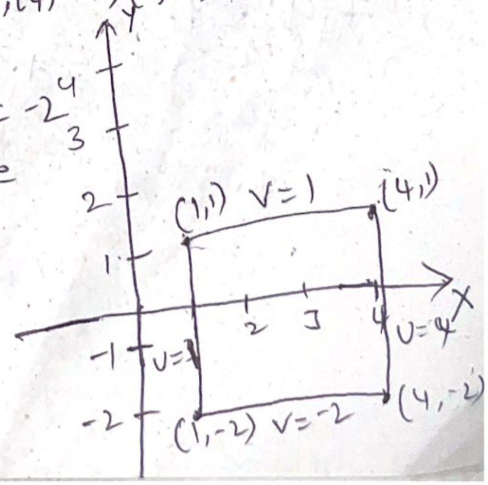
$D(0,1) = 0-2(1)$

$= -2$

Now, the points are $(1,1), (4,1), (4,-2), (1,-2)$.

The Equations $v=1,\ u=1,\ u=4,\ v=-2$

The above Equations got from the equation $y-y_1 = m(x-x_1)$



$x = \frac{1}{3}(2u+v)$, $y = \frac{1}{3}(u-v)$

$J\left(\frac{x,y}{u,v}\right)$

$$J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix}$$

$$= \frac{-2}{9} - \frac{1}{9} = \frac{-3}{9} = \left|\frac{-1}{3}\right|.$$

$$\int_{-2}^{1}\int_{1}^{4} \left(\frac{-1}{3}\right) du\, dv.$$

$$\frac{+1}{3}\int_{-2}^{1} [u]_1^4\, dv$$

$$\frac{+1}{3}\int_{-2}^{1} 3\, dv$$

$$\frac{+1}{3}(3)\,[v]_{-2}^{1}$$

$$\frac{+3}{3}[1+2] = \frac{+3}{3}[3]$$

$$= +3$$

Q) $2x+3y+4z=12$

2) $$\int_0^6 \int_0^{\frac{1}{3}(12-2x)} \int_0^{\frac{1}{4}[12-2x-3y]} dx\, dy\, dz$$

$4z = 12-2x-3y$

$\boxed{z = \frac{1}{4}[12-2x-3y]}$

$2x+3y=12$

$3y = 12-2x$

$\boxed{y = \frac{1}{3}[12-2x]}$

$2x=12$

$\boxed{x=6}$

$$\int_0^6 \int_0^{\frac{1}{3}(12-2x)} [z]_0^{\frac{1}{4}[12-2x-3y]} dx\, dy$$

$$\int_0^6 \int_0^{\frac{1}{3}(12-2x)} \frac{1}{4}[12-2x-3y]\, dx\, dy$$

$$\frac{1}{4}\int_0^6 \left[12y - 2xy - 3\frac{y^2}{2}\right]_0^{\frac{1}{3}(12-2x)} dx$$

$$\frac{1}{4}\int_0^6 \cancel{12}^{4}\left[\frac{1}{\cancel{3}}(12-2x)\right] - 2x\left[\frac{1}{3}(12-2x)\right] - \frac{3}{2}\left[\frac{1}{3}[12-2x]\right]^2 dx$$

$$\frac{1}{4}\int_0^6 4(12-2x) - \frac{2}{3}x[12-2x] - \frac{3}{2}\left[\frac{1}{9}[144+4x^2-48x]\right] dx$$

$$\frac{1}{4}\int_0^6 48-8x - \frac{\cancel{24}^{8}}{\cancel{3}}x + \frac{4x^2}{3} - \frac{\cancel{3}^{1}}{\cancel{18}6}[144+4x^2-48x] dx$$

$$\frac{1}{4}\int_0^6 [48-8x-\cancel{8x}+\frac{4}{3}x^2 - 24 - \frac{\cancel{4}^{2}}{\cancel{6}_{3}}x^2 + \cancel{8x}] dx$$

$$\frac{1}{4}\int_0^6 24-8x+\frac{2}{3}x^2\, dx$$

$$\frac{1}{4}\left[24x - 8\frac{x^2}{2} + \frac{2}{3}\cdot\frac{x^3}{3}\right]_0^6$$

$$\frac{1}{4}\left[24(6) - \frac{8}{2}(6^2) + \frac{2}{9}(6^3)\right]$$

$$\frac{1}{4}\left[144 - \frac{8}{\cancel{2}}\left[\cancel{36}\right] + \frac{2}{\cancel{9}}\left[\cancel{216}\right]\right]$$

$$\frac{1}{4}\left[144 - 144 + 48\right]$$

$$\frac{1}{4}\left[48\right]$$

$$= 12$$

3) $\iiint (x^2+y^2+z^2)\, dx\, dy\, dz$ , $x^2+y^2+z^2=1$

$z^2 = 1-x^2-y^2$

$\boxed{z = \sqrt{1-x^2-y^2}}$

$\iiint$

$x^2+y^2=1$

$x^2 = 1-y^2$

$\boxed{x = \sqrt{1-y^2}}$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$x^2 = 1$

$dx\,dy\,dz = r^2\sin\theta\, dr\, d\theta\, d\phi$

$\boxed{x=1}$

$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{1} r^2\, r^2 \sin\theta\, dr\, d\theta\, d\phi$

$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^4 \sin\theta\, dr\, d\theta\, d\phi.$

$\int_0^{2\pi}\int_0^{\pi} \left[\frac{r^5}{5}\right]_0^1 \sin\theta\, d\theta\, d\phi.$

$\int_0^{2\pi}\int_0^{\pi} \frac{1}{5}\sin\theta\, d\theta\, d\phi$

$\frac{1}{5}\int_0^{2\pi} [-\cos\theta]_0^{\pi}\, d\phi \Rightarrow \frac{1}{5}\int_0^{2\pi} -\cos\pi + \cos 0 \quad d\phi$

$\Rightarrow \frac{1}{5}\int_0^{2\pi} 2\, d\phi$

$\Rightarrow \frac{2}{5}[\phi]_0^{2\pi} \Rightarrow \frac{2}{5}[2\pi]$

$= \frac{4\pi}{5}$

4)
a) $$\int_0^1 \int_1^2 \int_2^3 (x+y+z)\, dx\, dy\, dz.$$

$$\int_0^1 \int_1^2 \left[xz + yz + \frac{z^2}{2}\right]_2^3 dx\, dy$$

$$\int_0^1 \int_1^2 \left[3x + 3y + \frac{9}{2}\right] - \left[2x + 2y + 2\right] dx\, dy$$

$$\int_0^1 \int_1^2 x + y + \frac{5}{2}\, dx\, dy.$$

$$\frac{9}{2} - 2$$
$$= \frac{\phantom{5}}{2}$$

$$\int_0^1 \left[xy + \frac{y^2}{2} + \frac{5}{2}y\right]_1^2 dx$$

$$\int_0^1 \left(2x + \frac{1}{2}[4] + \frac{5}{2}[2]\right) - \left[x + \frac{1}{2} + \frac{5}{2}\right] dx$$

$$\int_0^1 x + 7 - 3\, dx$$

$$\int_0^1 x + 4\, dx$$

$$= \left[\frac{x^2}{2} + 4x\right]_0^1$$

$$= \frac{1}{2} + 4 = 9/2$$

b) $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z} x+y+z \, dx\, dy\, dz$. Here
y varies from $x-z$ to $x+z$
x varies from 0 to z
z varies from -1 to 1

$\int_{-1}^{1}\int_{0}^{z}\left[xy+\frac{y^2}{2}+zy\right]_{x-z}^{x+z} dx\, dz$

$\int_{-1}^{1}\int_{0}^{z}\left[x(x+z)+\frac{1}{2}[x+z]^2+z[x+z]\right]-\left[x(x-z)+\frac{1}{2}[x-z]^2+z[x-z]\right]$

$\int_{-1}^{1}\int_{0}^{z} x^2+xz+\frac{1}{2}\left[x^2+z^2+2xz\right]+xz+z^2$
$-\left[x^2-xz+\frac{1}{2}(x^2+z^2-2xz)+xz-z^2\right]$

$\int_{-1}^{1}\int_{0}^{z} \cancel{x^2}+xz+\frac{x^2}{2}+\frac{z^2}{2}+xz+xz+z^2$
$-\cancel{x^2}+xz-\frac{x^2}{2}-\frac{z^2}{2}+\cancel{xz}-\cancel{xz}+z^2$

$\int_{-1}^{1}\int_{0}^{z} 4xz+2z^2$

$\int_{-1}^{1}\left[4\frac{x^2}{2}z+2xz^2\right]_{0}^{z}$

$\int_{-1}^{1} 4\frac{z}{2}[z^2]+2z^3$

$\int_{-1}^{1} 4z^3 \Rightarrow 4\left[\frac{z^4}{4}\right]_{-1}^{1}$

$\Rightarrow \frac{4}{4}-4\left[\frac{(-1)^4}{4}\right]$

$=\frac{4}{4}-\frac{4}{4}=0$

5) a) $\bar{F} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$

$\nabla \times \bar{F} = 0.$

$$\text{Curl } \bar{F} = \nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right)$$
$$-j\left(\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right)$$
$$+k\left(\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right]$$

$$= i(c+1) - j(4-a) + k(b-2]$$

$= c+1=0,\ -4+a=0,\ b-2=0$

$$\boxed{c=-1,\ a=4,\ b=2}$$

$\therefore \bar{F} = (x+2y+4z)\hat{i} + (2x-3y-z)\hat{j} + (4x-y+2z)\hat{k} \rightarrow (1)$

Curl $\bar{F} = 0.$

Curl grad $\phi = 0$

$\bar{F} = \text{grad}\,\phi$

$= i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z} \rightarrow (2)$

Comparing Equations (1) and (2)

$\frac{\partial \phi}{\partial x} = \cancel{x}$

$$\frac{\partial \phi}{\partial x} = x + 2y + 4z$$

$$\frac{\partial \phi}{\partial y} = 2x - 3y - z$$

$$\frac{\partial \phi}{\partial z} = 4x - y + 2z$$

$$\int x + 2y + 4z \, dx \quad \frac{x^2}{2} + 2xy + 4xz + f(y, z)$$

$$\int 2x - 3y - z \, dy = 2xy - \frac{3}{2} y^2 - yz + f(x, z)$$

$$\int 4x - y + 2z \, dz = 4xz - yz + \cancel{2} \frac{z^2}{\cancel{2}} + f(x, y)$$

$$= 4xz - yz + z^2 + f(x, y)$$

Now

$$\phi(x, y, z) = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz + c.$$

b) $r^n \bar{r}$ is solenoidal if $n=-3$.

$\operatorname{div}(r^{-3}\bar{r})$

$$\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)(r^{-3}\bar{r})$$

$\bar{r}=xi+yj+zk$

$r=\sqrt{x^2+y^2+z^2}$

$r^2=x^2+y^2+z^2$

$2r\frac{\partial r}{\partial x}=2x$

$\frac{\partial r}{\partial x}=\frac{x}{r},\ \frac{\partial r}{\partial y}=\frac{y}{r},\ \frac{\partial r}{\partial z}=\frac{z}{r}$

$\left[\because \sum i\frac{\partial}{\partial x}r^n x = \sum\left[\left(n r^{n-1}\cdot\frac{\partial r}{\partial x}\cdot x\right)+r^n\right]\right]$

$\sum i\cdot\frac{\partial r}{\partial x}\cdot r^{-3}\cdot x$

$$\sum\left(-3\cdot r^{-4}\frac{\partial r}{\partial x}\cdot x + r^{-3}\right)$$

$$\sum\left(-3\cdot r^{-4}\cdot\frac{x^2}{r}+r^{-3}\right)$$

$$\sum\left(-3r^{-5}x^2+r^{-3}\right)$$

$-3r^{-5}x^2-3r^{-5}y^2-3r^{-5}z^2+3r^{-3}$

$3r^{-3}-3r^{-5}\left[x^2+y^2+z^2\right]$

$3r^{-3}-3r^{-5}\cdot r^2$

$3r^{-3}-3r^{-3}=0$.

6) $\bar{F} = (2xyz^2)\bar{i} + (x^2z^2 + z\cos yz)\hat{j} + (2x^2yz + y\cos yz)$

$$\operatorname{Curl} \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^2 & x^2z^2 + x\cos yz & 2x^2yz + y\cos yz \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}(2x^2yz + y\cos yz) - \frac{\partial}{\partial z}(x^2z^2 + x\cos yz)\right)$$

$$-j\left(\frac{\partial}{\partial x}(2x^2yz + y\cos yz) - \frac{\partial}{\partial z}(2xyz^2)\right)$$

$$+k\left(\frac{\partial}{\partial x}(x^2z^2 + x\cos yz) - \frac{\partial}{\partial y}(2xyz^2)\right)$$

$$= i\left(\cancel{2x^2z} - y\sin(yz)(z) + \cos yz - \cancel{2x^2z} + z\cos yz(y) - \cos yz\right]$$

$$+\bar{j}\left(4xy\cancel{z} - 4xy\cancel{z}\right] + \bar{k}\left[\cancel{2xz^2} + y\cos yz \;\; 2xz^2\right] = 0$$

$= 0$.

∴ The above function is Conservative.

$\bar{f} = \operatorname{grad} \phi$.

$= i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$

$\frac{\partial \phi}{\partial x} = 2xyz^2$

$\frac{\partial \phi}{\partial y} = x^2z^2 + z\cos yz$

$\frac{\partial \phi}{\partial z} = 2x^2yz + y\cos yz$

$$\int 2xy^2z\,dx \Rightarrow \cancel{2}\frac{x^2}{\cancel{2}}y^2z^2 \Rightarrow x^2yz^2 + f(y,z)$$

$$\int x^2z^2 + z\cos yz\,dy \Rightarrow x^2z^2y + \frac{z(\sin yz)}{\cancel{z}} + f(x,z)$$

$$\int 2x^2yz + y\cos yz\,dz \Rightarrow \cancel{2}x^2y\frac{z^2}{\cancel{2}} + \frac{y(\sin yz)}{\cancel{y}} + f(x,y)$$

$$\boxed{\phi = x^2yz^2 + \sin yz + c}$$

(b)

7) $\phi = 4xy^2 + 2x^2yz$ $\quad P = (1,2,3)$, $Q(5,0,4)$.

$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$\nabla\phi = (4y^2 + 4xyz)\hat{i} + (8xy + 2x^2z)\hat{j} + 2x^2y\bar{k}$

$\nabla\phi_{(1,2,3)} = [4(2^2) + 4(1)(2)(3)]\hat{i}$

$+ (8(1)(2) + 2(1^2)(3))\hat{j}$

$+ 2(1^2)(2)\bar{k}$

$= (16 + 24)\hat{i} + (16 + 6)\hat{j} + 4k$

$\begin{array}{r} 24 \\ 16 \\ \hline 40 \end{array}$

$= 40\hat{i} + 22j + 4k$

$\begin{array}{r} 16 \\ 6 \end{array}$

$\overline{OQ} - \overline{OP} = (5i + 4k) - (i + 2j + 3k)$

$= 4i - 2j + \bar{k}$

$\nabla\phi = \frac{\bar{a}}{|a|}$ $\qquad \frac{4i - 2j + \bar{k}}{\sqrt{4^2 + (-2)^2 + (1)^2}}$

$= 40\hat{i} + 22\hat{j} + 4k.$

$= \frac{160 - 44 + 4}{\sqrt{21}}$

$= \frac{120}{\sqrt{21}}$

8) $\phi_1 = x^2+y^2+z^2-9$. $\phi_2 = x^2+y^2-z-3$

$\nabla\phi_1 = 2xi+2yj+2zk$ $\nabla\phi_2 = 2xi+2yj-k$.

$\bar{n}_2 = \nabla\phi_2(2,-1,2)$

$\bar{n}_1 = \nabla\phi_1(2,-1,2)$

$= 2(2)i+2(-1)j-k$

$= 2(2)i+2(-1)j+2(2)k$

$= 4i-2j-k$.

$= 4\bar{i}-2\bar{j}+4\bar{k}$

$|\bar{n}_2| = \sqrt{4^2+(-2)^2+(-1)^2}$

$|\bar{n}_1| = \sqrt{4^2+(-2)^2+4^2}$

$= \sqrt{16+16+4}$

$= \sqrt{36} = 6$.

$= \sqrt{16+4+1}$

$= \sqrt{21}$

$\cos\theta = \dfrac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1| \cdot |\bar{n}_2|}$

$= \dfrac{(4i-2j+4k).(4i-2j-k)}{6\sqrt{21}}$

$= \dfrac{16+4-4}{6\sqrt{21}}$

$= \dfrac{\cancel{16}\,8}{\cancel{6}_3\sqrt{21}}$

$= \dfrac{8}{3\sqrt{21}}$

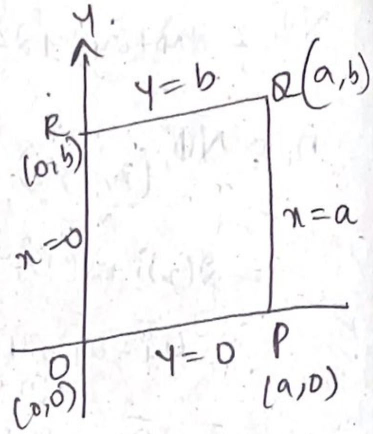
$\theta = \cos^{-1}\left(\dfrac{8}{3\sqrt{21}}\right)$.

16
16
4

9) $\bar{f} = (x^2+y^2)\hat{i} - 2xy\hat{j}$    $x=0, x=a, y=0, y=b$

$$\oint_C \bar{F}\cdot d\bar{r} = \oint_C f_1\, dx + f_2\, dy$$

$$= \oint_C (x^2+y^2)\, dx - 2xy\, dy$$




$$\int_C \bar{F}\cdot d\bar{r} = \int_{OP} + \int_{PQ} + \int_{QR} + \int_{RO}$$

Along OP.

$y=0,\ dy=0$.

$x$ varies from 0 to $a$.

$$\int_0^a x^2\, dx = \left[\frac{x^3}{3}\right]_0^a$$

$$= \frac{1}{3}[a^3] = \frac{a^3}{3}$$

Along PQ.

$x=a,\ dx=0$

$y$ varies from 0 to $b$.

$$\int_0^b (-2ay)\, dy = -ab^2$$

Along QR.

$y=b\ dy=0$

$x$ varies from $a$ to 0

$$\int_a^0 x^2 + b^2 \, dx$$

$$\left[\frac{x^3}{3}\right]_a^0 + b^2[x]_a^0$$

$$= \frac{1}{3}\left[0 - a^3\right] + b^2\left[0 - a\right]$$

$$= -\frac{a^3}{3} - ab^2$$

Along RO.

$x = 0$, $dx = 0$.

$y$ varies from $b$ to $0$.

$$\int_b^0 0 = 0.$$

$$\int_C \vec{F} \cdot d\vec{r} = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 + 0.$$

$$= -2ab^2$$

10) $\phi = x^2+y^2-16$. , $z=0$ and $z=5$ in first octant

$\nabla\phi = 2x\bar{i} + 2y\bar{j}$

$$n = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\bar{i}+2y\bar{j}}{\sqrt{4x^2+4y^2}} = \frac{2(xi+yj)}{\sqrt{4(x^2+y^2)}}$$

$$= \frac{2(xi+yj)}{2\sqrt{16}}$$

$$= \frac{xi+yj}{4}$$

In yz plane.

$$ds = \frac{dy\,dz}{|n\cdot i|} \Rightarrow ds = \frac{dy\,dz}{x/4}.$$

$$\bar{F}\cdot\bar{n} = \frac{xy}{4} + \frac{xz}{4}$$

$x^2+y^2=16$.

$x=0$

$\boxed{y = \pm 4}$

Here, we have to 0 to 4 bcoz of the first octant.

$$\int_S \bar{F}\cdot\bar{n}\,\frac{dy\,dz}{|n\cdot i|} = \iint_S \frac{xz+xy}{4}\,\frac{dy\,dz}{x/4}$$

y varies from 0 to 4

z varies from 0 to 5

$$\Rightarrow \int_0^5\int_0^4 (z+y)\,dy\,dz$$

$$= \int_0^5 \left[zy + \frac{y^2}{2}\right]_0^4 dz$$

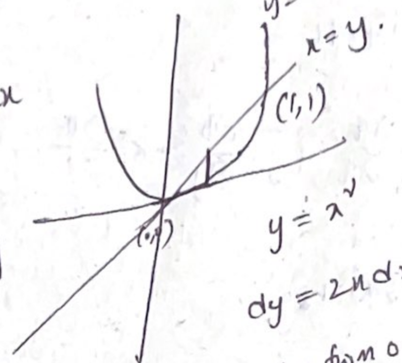
$$= \int_0^5 z(4)+8\,dz$$

$$= 4\left[\frac{z^2}{2} + 8z\right]_0^5$$

$$= 50+40 = 90.$$

11) $\int_C (xy+y^2)\,dx + x^2\,dy$.  $y = x$ and $y = x^2$

$\frac{\partial M}{\partial y} = x+2y$ . $\frac{\partial N}{\partial x} = 2x$




$y = x^2$

$dy = 2x\,dx$

$x$ varies from 0 to 1

$x = y$

$dx = dy$.

$$\iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

$$\iint_R \left(2x - (x+2y)\right) dx\,dy$$

$$\iint_R (x-2y)\,dy\,dx$$

$$\int_0^1 \int_{x^2}^{x} (x-2y)\,dy\,dx$$

$$= \int_0^1 \left[xy - \cancel{2}\frac{y^2}{\cancel{2}}\right]_{x^2}^{x} dx$$

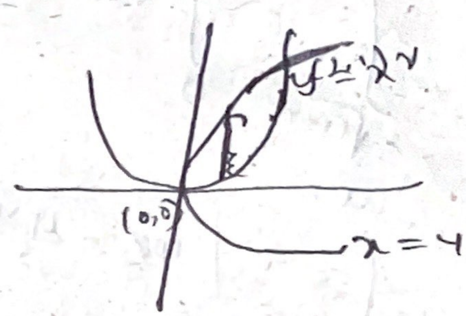
$$= \int_0^1 x\left[x - x^2\right] - \left[x^2 - x^4\right]\Big] dx$$

$$= \int_0^1 \left[\cancel{x^2} - x^3 - \cancel{x^2} + x^4\right] dx$$

$$= \left[\frac{-x^4}{4} + \frac{x^5}{5}\right]_0^1$$

$$= \left[\frac{-1}{4} + \frac{1}{5}\right] = \frac{-5+4}{20} = -1/20$$

12) $\int_C (2xy - x^2)\,dx + (x^2 + y^2)\,dy$ , $x = y^2$, $y = x^2$




$\frac{\partial M}{\partial y} = 2x$ $\frac{\partial N}{\partial x} = 2x$.

$\iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$.

$\iint_R (2x - 2x)\,dx\,dy$

$\iint_R 0\,dx\,dy = 0$.

13) Cube:-

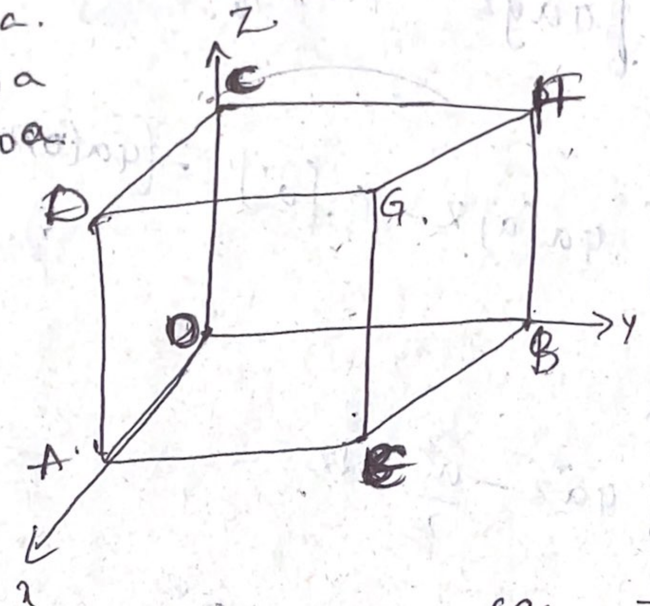
$\bar{F} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k}$

$x=0, x=a, y=0, y=a, z=0, z=a.$

x varies from 0 to a.

y varies from 0 to a

z varies from 0 to a.




Gauss's divergence theorem is $\iint_S \bar{F}\cdot\bar{n}\,ds = \iiint_V \operatorname{div}\bar{F}\,dv.$

$dv = dx\,dy\,dz$

$\bar{F} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k}$

$\operatorname{div}\bar{F} = \frac{\partial}{\partial x}(4xz) - \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(yz)$

$= 4z - 2y + y$

$= 4z - y$

R.H.S $= \iiint \operatorname{div}.\bar{F}\,dv$

$= \int_0^a\int_0^a\int_0^a (4z-y)\,dx\,dy\,dz$

$= \int_0^a\int_0^a \left[4xz - xy\right]_0^a dy\,dz.$

$$= \int_0^a \int_0^a 4az - ay \; dy \; dz$$

$$= \int_0^a \left[ 4ayz - a\left[\frac{y^2}{2}\right] \right]_0^a$$

$$= \int_0^a 4a[a]z - \frac{a}{2}[a^2] - \left[4a(0)z - a\left(\frac{0}{2}\right)\right] dz$$

$$= \int_0^a 4a^2 z - \frac{a^3}{2} \, dz$$

$$= \left[ 4a^2\left[\frac{z^2}{2}\right] - \frac{a^3}{2}[z] \right]_0^a$$

$$= 4a^2\left[\frac{a^2}{2}\right] - \frac{a^3}{2}[a] - 4a^2\left[\frac{0}{2}\right] - \frac{a^3}{2}[0]$$

$$= \frac{4a^4}{2} - \frac{a^4}{2}$$

$$= \frac{4a^4 - a^4}{2}$$

$$= \frac{3a^4}{2}$$

$L.H.S = \iint_S \bar{F}\cdot n\,ds = \iint_{OAEB} + \iint_{CFGD} + \iint_{GFBE} + \iint_{DCOA}$

$+ \iint_{DGAE} + \iint_{CFBO}$.

Along OAEB.

It is in xy plane.

$\hat{n} = -\hat{k}$.

$\hat{F}\cdot\hat{n} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k} \cdot (-\hat{k})$

$= -yz$

Equation $z=0$.

$\hat{F}\cdot\hat{n} = -y(0)$

$= 0$.

$\int_0^a \int_0^a 0 \, dx\, dy$

$= 0$.

Along CFGD.

It is in xy plane

$\hat{n} = \hat{k}$

$\hat{F}\cdot\hat{n} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k} \cdot (\hat{k})$

$= yz$

Equation $z = a$ $\quad \hat{F}\cdot\hat{n} = y(a)$

$$\int_0^a \int_0^a ay \, dx \, dy$$

$$\int_0^a \left[ axy \right]_0^a dy$$

$$\int_0^a a^2 y \, dy$$

$$a^2 \left[ \frac{y^2}{2} \right]_0^a$$

$$= \frac{a^2}{2} \left[ a^2 \right] = \frac{a^4}{2}$$

Along. GFBE

~~It~~ in $xz$ plane.

$\hat{n} = \hat{j}$.

$\hat{F} \cdot \hat{n} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k} \cdot (\hat{j})$

$= -y^2$

Equation $y = a$

$\hat{F} \cdot \hat{n} = -a^2$

$$\int_0^a \int_0^a -a^2 \, dx \, dz$$

$$-\int_0^a \left[ a^2 x \right]_0^a$$

$$-\int_0^a a^2[a]\,dz$$

$$\Rightarrow -\int_0^a a^3 z$$

$$\Rightarrow -a^3[\cancel{x}]_0^a$$

$$\Rightarrow -a^3[a]$$

$$\Rightarrow -a^4$$

Along . DCOA.

It is in xz plane

$\hat{n} = -\hat{j}$.

$\hat{F}.\hat{n} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k} \cdot (-\hat{j})$

$= y^2$

Equation $y=0$

$= 0$

$$\int_0^a\int_0^a 0\,dx\,dz = 0.$$

Along . DGAE.

It is in yz plane

$\hat{n} = \hat{i}$.

$\hat{F}.\hat{n} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k} \cdot (\hat{i})$

$= 4xz$

Equation $x = a$ $\hat{F}.\hat{n} = 4az$

$$\int_0^a \int_0^a 4az \, dy \, dz$$

$$\Rightarrow \int_0^a \left[4azy\right]_0^a$$

$$\Rightarrow \int_0^a 4a^2 z - 0 \, dz$$

$$\Rightarrow \left[4a^2 \frac{z^2}{2}\right]_0^a$$

$$\Rightarrow \frac{\cancel{4}^{2} a^2}{\cancel{2}}\left[a^2\right] - \frac{4a^2}{2}\left[0\right]$$

$$= 2a^4$$

Along CFBO.

It is in yz plane.

$$\hat{n} = -\hat{i}$$

$$\hat{F} \cdot \hat{n} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k} \cdot (-\hat{i})$$

$$= -4xz$$

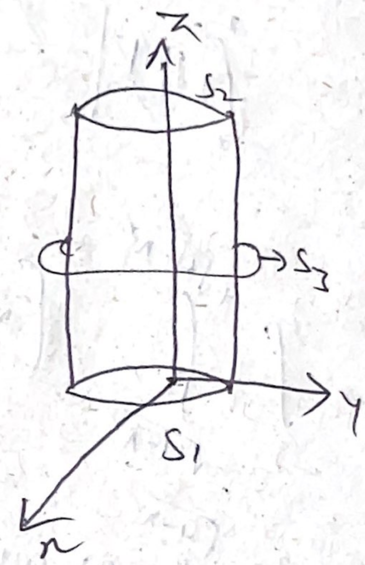
Equation $x = 0$

$$\int_0^a \int_0^a 0 \, dy \, dz = 0$$

$$\therefore \iint_S \bar{F} \cdot n \, ds = 0 + \frac{a^4}{2} + 0 - a^4 + 0 + 2a^4$$

$$= \frac{a^4}{2} + a^4$$

14) $\bar{F} = 4x\bar{i} - 2y^2\bar{j} + z^2\bar{k}$ $\quad x^2+y^2=4$, $z=0$ and $z=3$.




$\nabla \cdot \bar{F} = \frac{\partial F}{\partial x}\hat{i} + \frac{\partial F}{\partial y}\hat{j} + \frac{\partial}{\partial z}\bar{k}$

$= 4 - 4y + 2z$

$z = 0$ to $z = 3$.

$x^2 + y^2 = 4$

$y^2 = 4 - x^2$

$y = \pm\sqrt{4 - x^2}$

put $y = 0$

$x^2 = 4$

$x = \pm 2$

Now, the limits are

$z = 0$ to $3$

$y = -\sqrt{4-x^2}$ to $+\sqrt{4-x^2}$

$x = -2$ to $2$

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}}\int_{0}^{3}(4-4y+2z)\,dz\,dy\,dx.$$

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}}\left[4z-4yz+\cancel{2}\frac{z^2}{\cancel{2}}\right]_0^3 dy\,dx.$$

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}}\left[12-12y+9\right]dy\,dx.$$

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} [21-12y]\, dy\, dx.\quad \left[\because \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\right].$$

$$2\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} 21\, dy\, dx$$

$$2\int_{-2}^{2} 21\,[y]_0^{\sqrt{4-x^2}}\, dx$$

$$42\int_{-2}^{2} \sqrt{4-x^2}\, dx.$$

$$\int_{-c}$$

$$4 2 \cdot 2 \int_0^2 \sqrt{4-x^2}\, dx.$$

$$84 \left[\frac{x}{2}\sqrt{4-x^2} + \frac{2}{2}\sin^{-1}\frac{x}{2}\right]_0^2 \quad \left[\sqrt{a^2-x^2} = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a}{2}\sin^{-1}\frac{x}{a}\right]$$

$$84\left[\frac{2}{2}\sqrt{4-4} + \sin^{-1}(1)\right] - (0).$$

$$84\left[0+\pi\right]$$

$$= 84\pi. \Rightarrow R.H.S.$$

$$L.H.S. = \iint_S \bar{F}\cdot n\, ds.$$

$$= \iint_{S_1} + \iint_{S_2} + \iint_{S_3}$$

Along $S_1$

It is in $xy$ plane

$z=0,\ dz=0.$

$ds = dx\,dy \quad \hat{n} = -\hat{k}$

$$\hat{F}\cdot\hat{n} = \left(4xi - 2y^2\hat{j} + z^2\bar{k}\right)\cdot(-\hat{k})$$

$$= -z^2$$

$$= 0^2$$

$$= 0$$

Along $S_2$

It is in $xy$ plane

$z=3 \quad dz=0 \quad \hat{n} = \hat{k}$

$ds = dx\,dy \quad \hat{F}\cdot\hat{n} = \left(4xi - 2y^2\hat{j} + z^2\bar{k}\right)(\hat{k})$

$$= z^2$$

$$= 3^2 = 9.$$

$$\iint_{S_2} 9\, dx\, dy.$$

$9\,[\text{Area of the circle}]$

$= 9(\pi r^2) \quad [\because r = 2]$

$= 9[\pi \times 2^2]$

$= 36\pi.$

Along $S_3$ (curved surface).

$x^2 + y^2 = 4$

$\phi = x^2 + y^2 - 4$

$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

$= 2x\hat{i} + 2y\hat{j}$

$|\nabla\phi| = \sqrt{4x^2 + 4y^2} = \sqrt{4(x^2+y^2)}$

$= \sqrt{4(4)}$

$= \sqrt{16} = 4$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(x\hat{i} + y\hat{j})}{4}$$

$$= \frac{x\hat{i} + y\hat{j}}{2}$$

Let us take

$x = 2\cos\theta,\ y = 2\sin\theta;\quad ds = 2\,dz\,d\theta$

z varies from 0 to 3

$\theta$ varies from 0 to $2\pi$

$$\hat{F}\cdot\hat{n} = \left(4x\vec{i} - 2y^3\vec{j} + z^2\vec{k}\right)\cdot\left(\frac{x\vec{i}+y\vec{j}}{2}\right)$$

$$= 2x^2 - y^3$$

$$= 2(2\cos\theta)^2 - (2\sin\theta)^3$$

$$= 8\cos^2\theta - 8\sin^3\theta$$

$$= 8\left[\frac{1+\cos 2\theta}{2} - \left(\frac{3\sin\theta - \sin 3\theta}{4}\right)\right]$$

$$\iint_S \hat{F}\cdot\hat{n} = 8\int_0^{2\pi}\int_0^3 \left(\frac{1}{2} + \frac{\cos 2\theta}{2} - \frac{3\sin\theta}{4} + \frac{\sin 3\theta}{4}\right) 2\,dz\,d\theta$$

$$= 16\int_0^{2\pi}\left[\frac{1}{2} + \frac{\cos 2\theta}{2} - \frac{3\sin\theta}{4} + \frac{\sin 3\theta}{4}\right][z]_0^3\,d\theta$$

$$= 48\int_0^{2\pi} \frac{1}{2} + \frac{\cos 2\theta}{2} - \frac{3\sin\theta}{4} + \frac{\sin 3\theta}{4}\,d\theta$$

$$= 48\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} + \frac{3\cos\theta}{4} - \frac{\cos 3\theta}{12}\right]_0^{2\pi}$$

$$= 48\left[\frac{2\pi}{2} + \frac{\sin 4\pi}{4} + \frac{3\cos 2\pi}{4} - \frac{\cos 6\pi}{12}\right]$$

$$- \left[\frac{0}{2} + \frac{\sin 0}{4} + \frac{3\cos 0}{4} - \frac{\cos 0}{12}\right]$$

$$= 48\left[\left[\pi + \frac{3}{4}\right/ - \frac{1}{12}\Big] - \left(\frac{3}{4} - \frac{1}{12}\right)\right]$$

$$= 48\pi$$

$$S_1 + S_2 + S_3 = 48\pi + 36\pi + 0\pi$$
$$= 84\pi$$

$\therefore$ L.H.S = R.H.S.

15). $\bar{F} = (2x-y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$; $x^2+y^2+z^2=1$.

It is xy-plane

$z=0$,

Strokes theorem:-

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl } \bar{F}\cdot n\, ds.$$

$$\text{curl } \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$i\left(\frac{\partial}{\partial y}(-y^2z) - \frac{\partial}{\partial z}(-yz^2)\right] - j\left(\frac{\partial}{\partial x}(-y^2z) - \frac{\partial}{\partial z}(2x-y)\right)$$
$$+k\left(\frac{\partial}{\partial x}(-yz^2) - \frac{\partial}{\partial y}(2x-y)\right)$$

$$= i(-2yz + 2yz) - j(0) + k(0+1)$$

$$= k.$$

$\hat{n} = k.$

$$ds = \frac{dx\,dy}{|n \cdot k|}.$$

$$\iint_S \text{curl}\,\bar{F} \cdot n\,ds = \iint_S k \cdot \hat{k}\,\frac{dx\,dy}{|n \cdot k|}.$$

$$= \iint_S k \cdot k\,dx\,dy \qquad [\because |n \cdot k| = 1]$$

$$= \iint_S 1\,dx\,dy. \qquad \boxed{x^2 + y^2 = 1}$$

$$= \pi \qquad [\because \text{since Area of circle is } \pi r^2,\ r = 1].$$

16) $\bar{F} = (x^2+y-4)\bar{i} + 3xy\bar{j} + (2xz+z^2)\hat{k}$.

$x^2+y^2+z^2=16$, $z=0$ [∵ It is in above xy plane].

∯ Strokes theorem:-

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot n\,ds$$

$$\text{curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y-4 & 3xy & 2xz+z^2 \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}(2xz+z^2) - \frac{\partial}{\partial z}(3xy)\right)$$
$$-j\left(\frac{\partial}{\partial x}(2xz+z^2) - \frac{\partial}{\partial z}(x^2+y-4)\right)$$
$$+k\left(\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(x^2+y-4)\right)$$

$$= i(0-0) - j(2z-0)$$
$$+k(3y-1)$$
$$= -2z + k(3y-1).$$

Since $z=0$
$$= -2(0) + k(3y-1)$$

$$\Rightarrow k(3y-1)$$

$\hat{n}=k$, $ds = \dfrac{dx\,dy}{|n\cdot k|}$.

$$\text{curl}\,\bar{F}\cdot\hat{n}\,ds = \iint_S k(3y-1)\cdot\hat{k}\cdot\frac{dx\,dy}{|\hat{k}\cdot\hat{k}|}$$
$$= \iint_S 3y-1\,dx\,dy.$$

Let us take

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$dx\,dy = r\,dr\,d\theta.$$

$\theta$ Varies from 0 to $2\pi$

$r$ Varies from 0 to 4.

$$\int_0^{2\pi}\int_0^4 \left(3(r\sin\theta) - 1\right) r\,dr\,d\theta.$$

$$\int_0^{2\pi}\int_0^4 3r^2\sin\theta - r\,dr\,d\theta.$$

$$\int_0^{2\pi}\left(\cancel{3}\frac{r^3}{\cancel{3}}\sin\theta - \frac{r^2}{2}\right]_0^4 d\theta.$$

$$\int_0^{2\pi}\left[64\sin\theta - 8\right] d\theta$$

$$\int_0^{2\pi}\left[64\left[-\cos\theta\right] - 8\theta\right]_0^{2\pi}$$

$$64\left[-\cos 2\pi - (-\cos 0)\right] - 8\left[2\pi - 0\right]$$

$$= 64\left[-1 + 1\right] - 8\left[2\pi\right]$$

$$= 0 - 16\pi$$

$$= -16\pi$$