1. Verify Rolle's theorem for $f(x) = \log\left(\frac{x^2+ab}{x(a+b)}\right)$ on $[a,b]$ $a>0$

Sol: Here,

$$f(x) = \log\left(\frac{x^2+ab}{x(a+b)}\right)$$

1) Since $a>0$, $b>0$, and log function is continuous in the positive side of x-axis so $f(x)$ is Continuous.

2) $$f(x) = \log\left(\frac{x^2+ab}{x(a+b)}\right)$$

$$= \log(x^2+ab) - \log(x(a+b)) - \log x$$

$$f'(x) = \frac{1}{x^2+ab}(2x) - \frac{1}{x(a+b)}(0) - \frac{1}{x}$$

$$f'(x) = \frac{2x}{(x^2+ab)x}$$

So, $f(x)$ is differentiable

3) $$f(a) = \log\left[\frac{x^2+ab}{x(a+b)}\right]$$

$$= \log\left[\frac{a^2+ab}{a(a+b)}\right]$$

$$= \log\left[\frac{a^2+ab}{a^2+ab}\right]$$

$$f(a) = \log 1$$

$$f(a) = 0$$

$$f(b) = \log\left[\frac{x^2+ab}{x(a+b)}\right]$$

$$f(b) = \log\left[\frac{b^2+ab}{b(a+b)}\right]$$

$$f(b) = \log\left[\frac{b^2+ab}{b^2+ab}\right]$$

$$f(b) = \log 1$$

$$f(b) = 0$$

$$f(a) = f(b)$$

Since,

$f(x)$ is satisfied all the three conditions. so,

$\exists$ at least one c $a<c<b$

$$f(x) = \log\left(\frac{x^2+ab}{x(a+b)}\right)$$

$$f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x}$$

Now,

$$f'(c) = 0$$

$$\frac{2c}{c^2+ab} - \frac{1}{c} = 0$$

$$\frac{2c}{c^2+ab} = \frac{1}{c}$$

$$2c^2 = c^2+ab$$

$$c^2 = ab$$

$$\boxed{\therefore c = \pm\sqrt{ab}}$$

$$\therefore c = \sqrt{ab}$$

Because, It should be +ve.

2. Verify Rolle's theorem for $f(x) = 2x^3+x^2+4x-2$ on $[-\sqrt{2}, \sqrt{2}]$

Sol:- Here,

$$f(x) = 2x^3+x^2+4x-2$$

1) $f(x)$ is continuous since $f(x)$ is a polynomial

2) $f(x) = 2x^3+x^2+4x-2$

$$= 6x^2+2x+4$$

So, $f(x)$ is differentiable

3) $f(-\sqrt{2}) = 2(-\sqrt{2})^3 + (-\sqrt{2})^2 + 4(-\sqrt{2}) - 2$

$$= 2(-2\sqrt{2}) - 2 - 4\sqrt{2} - 2$$

$$= -8\sqrt{2}$$

$f(\sqrt{2}) = 2(\sqrt{2})^3 + (\sqrt{2})^2 + 4(\sqrt{2}) - 2$

$$= 8\sqrt{2}$$

$f(-\sqrt{2}) \neq f(\sqrt{2})$

since, $f(x)$ is not satisfied the 3^{rd} condition of Rolle's theorem.

So, we cannot find the c

3. If $a<b$, prove that $\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$ using lagrange's theorem. Hence Deduce $\frac{5\pi+4}{20} < \tan^{-1}2 < \frac{\pi+3}{4}$

Sol: Here $f(x) = \tan^{-1}x$

1) $f(x)$ is continuous

2) $f(x)$ is differentiable.

So, we can apply lagrange's theorem $\exists$ at least one c such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} \rightarrow (1)$$

$$f(x) = \tan^{-1}x$$

$$f(b) = \tan^{-1}b$$

$$f(a) = \tan^{-1}a$$

So, $$\frac{f(b)-f(a)}{b-a} = \frac{\tan^{-1}b - \tan^{-1}a}{b-a} \rightarrow (2)$$

$$f(x) = \tan^{-1}x$$

$$f'(x) = \frac{1}{1+x^2}$$

$$\therefore f'(c) = \frac{1}{1+c^2} \rightarrow (3)$$

Now, c will lie b/w a and c

$$a<c<b \Rightarrow a^2<c^2<b^2$$

$$1+a^2<1+c^2<1+b^2$$

$$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2} \Rightarrow \frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$$

Now, put eqn ② ③ in eqn ①

$$\frac{1}{1+c^2} = \frac{\tan^{-1}b - \tan^{-1}a}{b-a} \rightarrow ⑤$$

Now eqn → ① becomes

$$\frac{1}{1+b^2} < \frac{\tan^{-1}b - \tan^{-1}a}{b-a} < \frac{1}{1+a^2}$$

multiply with $(b-a)$

$$\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$$

put $b=2, a=1$

$$\frac{2-1}{1+4} < \tan^{-1}2 - \frac{\pi}{4} < \frac{2-1}{1+1^2}$$

$$\frac{1}{5} < \tan^{-1}2 - \frac{\pi}{4} < \frac{1}{2}$$

Now, add $\frac{\pi}{4}$

$$\frac{\pi}{4} + \frac{1}{5} < \tan^{-1}2 < \frac{1}{2} + \frac{\pi}{4}$$

$$\frac{5\pi+4}{20} < \tan^{-1}2 < \frac{\pi+2}{4}$$

Hence, Proved.

4. verify mean value theorem for $f(x)=e^x$ and $g(x)=e^{-x}$ in $[a,b]$

Here $f(x)=e^x$ $g(x)=e^{-x}$

Since $f(x)$ and $g(x)$ are exponential function

So, they are continuous and differentiable

$$g(x)=e^{-x}$$
$$g'(x)=-e^{-x}\neq 0 \text{ in } [a,b]$$

then by cauchy's mean value theorem.

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)} \rightarrow ①$$

Now,

$f(x)=e^x$ $\quad$ $g(x)=e^{-x}$

$f'(x)=e^x$ $\quad$ $g'(x)=e^{-x}$

$f(a)=e^a$ $\quad$ $g(a)=e^{1/a}$

$f'(c)=e^c$ $\quad$ $g(b)=e^{1/b}$

$f(b)=e^b$ $\quad$ $g'(c)=-e^{-c}$

So, eqn $\rightarrow$ ① becomes

$$\Rightarrow \frac{e^b-e^a}{e^{1/b}-e^{1/a}}=\frac{e^c}{-e^{-c}}$$

$$\Rightarrow \frac{-(e^a-e^b)}{e^{1/b}-e^{1/a}}=-e^c e^c$$

$$\Rightarrow \frac{e^a-e^b}{e^{1/b}-e^{1/a}}=e^{2c}$$

$$\Rightarrow e^{2c}=\frac{e^a-e^b}{\frac{e^a-e^b}{e^a \cdot e^b}}$$

$$e^{2c} = e^a \cdot e^b$$

$$\Rightarrow e^{2c} = e^{a+b}$$

$$2c = a+b$$

$$\therefore \quad c = \frac{a+b}{2} \in (a,b)$$

Hence, mean value theorem is verified.

6. Using mean value theorem P.T $\frac{x}{1+x} < \log(1+x) < x$ $\forall\ x > 0$

Sol: let $f(x) = \log(1+x)$

Since, $\log(1+x)$ is continuous and differentiable for positive value of x, so by lagranges theorem $\exists$ at least one c inside the interval $[0,x]$ such that,

$$f'(c) = \frac{f(b)-f(a)}{b-a} \rightarrow ①$$

$$f(x) = \log(1+x)$$

$$f(b) = \log(1+x)$$

$$f(a) = \log(1+0)$$

$$f(a) = 0$$

So, $$\frac{f(b)-f(a)}{b-a} = \frac{\log(1+x)-0}{x-0}$$

$$= \frac{\log(1+x)}{x} \rightarrow ②$$

$$f(x) = \log(1+x)$$

$$f'(x) = \frac{1}{1+x}(1)$$

$$f'(x) = \frac{1}{1+x}$$

$$f'(c) = \frac{1}{1+c} \rightarrow ③$$

Now,

$$0 < c < x$$

$$1+0 < 1+c < 1+x$$

$$\frac{1}{1} > \frac{1}{1+c} > \frac{1}{1+x}$$

$$\frac{1}{1+x} < \frac{1}{1+c} < 1 \rightarrow (5)$$

Now eq (4) in eq (5)

$$\frac{1}{1+x} < \frac{\log(1+x)}{x} < 1$$

Multiply with x

$$\frac{x}{1+x} < \log(1+x) < x$$

Hence proved

6. verify Rolle's theorem for $f(x) = \frac{\sin x}{e^x}$ in $(0, \pi)$

Sol: Here $f(x) = \frac{\sin x}{e^x}$

1) $f(x)$ is continuous because since, e^x is never become zero, and $f(x)$ is polynomial

2) $f(x) = \frac{\sin x}{e^x}$

$$f'(x) = \frac{e^x \cos x - \sin x e^x}{(e^x)^2}$$

$$f'(x) = \frac{e^x(\cos x - \sin x)}{(e^x)^2}$$

So, $f(x)$ is differentiable

3) $f(0) = \frac{\sin 0}{e^0} = 0$

$$f(\pi) = \frac{\sin \pi}{e^\pi} = 0$$

$\frac{0}{e^{\pi}} = 0$

$f(0) = f(\pi)$

Since,

$f(x)$ satisfied all the three conditions so, $\exists$ at least one c' $0 < c < \pi$

$f(x) = \frac{\sin x}{e^x}$

$f'(c) = 0$

Now,

$f'(x) = \frac{e^x \cos x - e^x \sin x}{e^{2x}}$

$f'(c) = \frac{e^c \cos c - e^c \sin c}{e^{2c}}$

Now,

$f'(c) = 0$

$e^c \cos c - e^c \sin c = 0$

$\cancel{e^c} \cos c = \cancel{e^c} \sin c$

$\cos c = \sin c$

$1 = \frac{\sin c}{\cos c}$

$1 = \tan c$

$\boxed{\therefore c = \frac{\pi}{4}}$

7. Expand $\log(1+x)$ in power of x?

let $f(x) = \log(1+x)$

$f'(x) = \frac{1}{1+x}$

$f''(x) = (-1)\frac{1}{1+x^2}$

$f'''(x) = (-1)(-2)\frac{1}{1+x^2}$

$f(0) = 0$

$f(0) = 1$

$f(0) = -1$

$f(0) = 2$

$f(0) = 6$

$$f^{iv}(x) = (-1)(-2)(-3)\frac{1}{(1+x)^4}$$

we know that the maclaurin's series of $f(x)$ is

$$f(x) = f(0) + \frac{x f'(0)}{1!} + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \frac{x^4 f^{iv}(0)}{4!} + \cdots$$

$$\log(1+x) = 0 + \frac{x(1)}{1!} + \frac{x^2(-1)}{2!} + \frac{x^3(2)}{3!} + \frac{x^4(6)}{4!} + \cdots$$

$$\log(1+x) = x - \frac{x^2}{4} + \frac{2x^3}{3!} + \frac{6x^4}{4!} + \cdots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{2x^3}{6} + \frac{6x^4}{24} + \cdots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

diff w.r. to x

$$\frac{1}{1+x} = 1 - \frac{2x}{2} + \frac{3x^2}{3} - \frac{4x^3}{4} + \frac{5x^4}{5} + \cdots$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \cdots$$

$$\frac{-1}{(1+x)^2} = -1 + 2x - 3x^2 + 4x^3 + \cdots$$

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$$

8) If $x + y^2 = 4$, $y + z^2 = w$, $z + x^2 = w$ find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

9. If $x+y+z=4$, $y+z=4v$, $z=4uvw$ then evaluate

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}$$

We know that

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

let $z = uvw \rightarrow (1)$

$y+z = uv$

$y = uv - z$

$y = uv - uvw \rightarrow (2)$

$x = u - y - z$

$x = u - (uv - uvw) - uvw$

$x = u - uv \rightarrow (3)$

Now,

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}$$

$x = u - uv$

$\frac{\partial x}{\partial u} = 1 - v$

$\frac{\partial x}{\partial v} = -u$

$\frac{\partial x}{\partial w} = 0$

$y = uv - uvw$

$\frac{\partial y}{\partial u} = v - vw$

$\frac{\partial y}{\partial v} = u - uw$

$\frac{\partial y}{\partial w} = -w$

$z = uvw$

$\frac{\partial z}{\partial u} = vw$

$\frac{\partial z}{\partial v} = uw$

$\frac{\partial z}{\partial w} = uv$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -w \\ vw & uw & uv \end{vmatrix}$$

$= 1 - v[u - uw(uv) + uw(w)] + u[v - vw(uv) + vw(w)]$

$1 - v[u^2v - u^2wv + uw^2] + u[uv^2 - v^2uw + vw^2]$

$= u^2v - u^2wv + uw^2v - u^2v^2 - u^2wv^2 - uvw^2 + u^2v^2 - v^2u^2w$

$+ vwu^2$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$$

10. State that the functions $u = x+y+z$, $y = x^2+y^2+z^2-2xy$ $+2yz-2zx$ and $w = x^3+y^3+z^3-3xyz$ are functionally related.

Sol: Given

$u = x+y+z \quad v = x^2+y^2+z^2-2xy+2yz-2xz$

$w = x^3+y^3+z^3-3xyz$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2(x-y-z) & 2(y-x-z) & 2(x-y-z) \\ 3(x^2-yz) & 3(y^2-xz) & 3(z^2-xy) \end{vmatrix}$$

$$= 6 \begin{vmatrix} 1 & 1 & 1 \\ x-y-z & y-x-z & z-y-x \\ x^2-yz & y^2-xz & z^2-xy \end{vmatrix}$$

$C_1 - C_2$

$C_2 - C_3$

$$6 \begin{vmatrix} 0 & 0 & 1 \\ 2(x-y) & 2(y-z) & z-y-x \\ (x-y)(x+y+z) & (y-z)(x+y+z) & z^2-xy \end{vmatrix}$$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = 12 \begin{vmatrix} x-y & y-z \\ x-y & y-z \\ x+y+z & x+y+z \end{vmatrix}$$

$$= 12(x-y)(y-z) \begin{vmatrix} 1 & 1 \\ x+y+z & x+y+z \end{vmatrix}$$

$$= 12(x-y)(y-z)(0)$$

$$= 0$$

Hence the functional relationship exists b/w u, v, w.

11. Find the max and min values of the function

$f(x) = x^5 - 3x^4 + 5$

Given

$f(x) = x^5 - 3x^4 + 5$

$f(x) = \frac{\partial f}{\partial x} = 5x^4 - 12x^3$

let

$f(x) = 0$

$5x^4 - 12x^3 = 0$

$x^3[5x - 12] = 0$

$x = 0, \; x = 12/5$

The stationary values are

$x=0,\ x=12/5$

$r = f_{xx} = 20x^3 - 36x^2$

at $x=0$ then $r=0$

It needs further Investigation

at $x=\frac{12}{3}$ then $r=20\left(\frac{12}{5}\right)^3-36\left(\frac{12}{5}\right)^2$

$=20(2.4)^3-36(2.4)^2$

$r=69.12>0$

∴ The $f(x)$ has min value at $x=\frac{12}{5}$ and the min values

$\Rightarrow f\left(\frac{12}{5}\right) = \left(\frac{12}{5}\right)^5 - 3\left(\frac{12}{5}\right)^4+5$

$f\left(\frac{12}{5}\right) = -14.92$

12. Find three positive numbers whose sum is 100 and whose product is maximum.

Sol: x, y, z be three +ve numbers

$x+y+z=100 \Rightarrow z=100-x-y$

$f(x,y) = xyz = xy(100-x-y)$

$f(x) = y[x(-1)+(100)(x-y)(1)]$

$= y[-x+100-x-y]$

$= y[100-2x-y]$

$= x[y(-1)+(100-x-y)(1)]$

$f_y = x(100-x-2y)$

let $f_x=0$ and $f_y=0$

$y(100-2x-y)=0$ and $x(100-x-2y)=0$

$y=0$ or $100-2x-y=0$ and $x=0$ or $100-x-2y=0$

$100-2x-y=0$ $\qquad$ $100-x-2y=0$

$2x+y=100$ $\qquad$ $4x+2y=200$

$2\left(\frac{100}{3}\right)+y=100$ $\qquad$ $x+2y=100$

$3x=100$

$y=100-\frac{200}{3}$ $\qquad$ $\boxed{x=\frac{100}{3}}$

$\boxed{y=\frac{100}{3}}$

$\left(\frac{100}{3},\frac{100}{3}\right)$ is stationary point

$f_x=y(100-2x-y)$ $\quad$ $f_y=x(100-x-2y)$

$r=f_{xx}=y(-2)=-2y$

$s=f_{xy}=y(-1)+(100-2x-y)(1)$

$s=100-2x-2y$

$t=f_{yy}=x(-2)=-2x$

at $\left(\frac{100}{3},\frac{100}{3}\right)$ $\quad r=\frac{-200}{3}$

$r=100-\frac{200}{3}-\frac{200}{3}$

$r=\frac{-100}{3}$ $\qquad t=\frac{-200}{3}$

$rt-s^2=\left(-\frac{200}{3}\right)\left(-\frac{200}{3}\right)-\left(-\frac{100}{3}\right)^2$

$=\frac{40000}{9}-\frac{10000}{9}$

$=\frac{30000}{9}>0$

$r=\frac{-200}{3}<0$ $\qquad \therefore f$ has max value $\left(\frac{100}{3},\frac{100}{3}\right)$

$$z = 100 - x - y = 100 - \frac{100}{3} - \frac{100}{3}$$

$$z = \frac{100}{3}$$

$\therefore$ The three positive numbers are

$\frac{100}{3}$, $\frac{100}{3}$ and $\frac{100}{3}$.

13. locate the stationary points and examine their nature of the following functions $u = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ $(x > 0, y > 0)$

14. If $U=\frac{yz}{x}$, $V=\frac{xz}{y}$ $W=\frac{xy}{z}$ find $\frac{\partial(U,V,W)}{\partial(x,y,z)}$

Sol: $U=\frac{yz}{x}$, $V=\frac{xz}{y}$ $W=\frac{xy}{z}$

$$\frac{\partial(U,V,W)}{(x,y,z)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \\ \frac{\partial W}{\partial x} & \frac{\partial W}{\partial y} & \frac{\partial W}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} yz(-1/x^2) & z/x(1) & y/x \\ \frac{z}{y} & xz(-1/y^2) & x/y \\ \frac{y}{z} & x/z & xy(-1/z^2) \end{vmatrix}$$

$$= \frac{-yz}{x^2}\left(\frac{-xz}{y^2}\left(\frac{-xy}{z^2}\right)-\frac{x^2}{yz}\right)-\frac{z}{x}\left(\frac{z}{y}\left(\frac{-xy}{z^2}\right)-\frac{y}{z}\left(\frac{x}{y}\right)\right)$$

$$+\frac{y}{x}\left(\frac{z}{y}\left(\frac{x}{z}\right)-(y/z)\left(\frac{-xz}{yz}\right)\right)$$

$$= \frac{-yz}{x^2}\left(\frac{x^2}{yz}-\frac{x^2}{yz}\right)-\frac{z}{x}\left(\frac{-x}{z}-\frac{x}{z}\right)+\frac{y}{x}\left(\frac{x}{y}+\frac{x}{y}\right)$$

$$= -\frac{z}{x}\left(\frac{-2x}{z}\right)+\frac{y}{x}\left(\frac{2x}{y}\right)$$

$$= 2+2$$

$$\frac{\partial(U,V,W)}{\partial(x,y,z)} = 4$$

15. verify cauchy's mean value theorem for $f(x)=\log x$ $g(x)=1/x$ on $[1,e]$

let $f(x)=\log x$

$g(x)=1/x$

$f(x), g(x)$ are continuous and differentiable since, interval $(1, e)$

$$g(x) = 1/x$$

$$g'(x) = 1/x^2 \neq 0 \quad x \in (1, e)$$

then by cauchy's theorem

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Now,

$f(x) = \log x$ $\qquad$ $g(x) = 1/x$

$f'(x) = 1/x$ $\qquad$ $g'(x) = -1/x^2$

$f(e) = \log e$ $\qquad$ $g(e) = 1/e$

$f(1) = 0$ $\qquad$ $g(1) = 1$

$f'(c) = 1/c$ $\qquad$ $g'(c) = -1/c^2$

$$\frac{1-0}{\frac{1}{e} - 1} = \frac{\frac{1}{e}}{-1/e^2}$$

$$\frac{1}{\frac{1-e}{e}} = -c$$

$$\frac{e}{1-e} = -c \qquad c = \frac{e}{e-1}$$

$$c = \frac{2.71}{2.71 - 1}$$

$$\therefore \boxed{e = 2.71}$$

$$c = \frac{2.71}{1.71}$$

$$\boxed{c = 1.584}$$