Problem Set 2 Solution: Portfolio Management and the CAPM

I. Expected Return, Return Standard Deviation, Covariance, and Portfolios

State Probabilities and Returns:

State Riskless Asset	Probability	Asset A	Asset B
Boom 7%	0.25	24%	14%
Normal Growth 7%	0.5	18%	9%
Recession 7%	0.25	2%	5%

A. Expected Return on Each Asset

$$
\begin{aligned}
E[R_A] &= 0.25 \times 24\% + 0.5 \times 18\% + 0.25 \times 2\% = 15.5\%, \\
E[R_B] &= 0.25 \times 14\% + 0.5 \times 9\% + 0.25 \times 5\% = 9.25\%, \\
E[R_f] &= 0.25 \times 7\% + 0.5 \times 7\% + 0.25 \times 7\% = 7\%.
\end{aligned}
$$

B. Standard Deviation of Return on Each Asset

First, calculate the variance for each asset.

$$
\begin{aligned}
\sigma^2[R_A] &= 0.25 \times (24 - 15.5)^2 + 0.5 \times (18 - 15.5)^2 + 0.25 \times (2 - 15.5)^2 \\
&= 18.0625 + 3.125 + 45.5625 = 66.75, \\
\sigma^2[R_B] &= 0.25 \times (14 - 9.25)^2 + 0.5 \times (9 - 9.25)^2 + 0.25 \times (5 - 9.25)^2 \\
&= 5.640625 + 0.03125 + 4.515625 = 10.1875, \\
\sigma^2[R_f] &= 0.25 \times (7 - 7)^2 + 0.5 \times (7 - 7)^2 + 0.25 \times (7 - 7)^2 = 0.
\end{aligned}
$$

Then calculate the standard deviation.

$$
\begin{aligned}
\sigma[R_A] &= \sqrt{66.75} = 8.1701\%, \\
\sigma[R_B] &= \sqrt{10.1875} = 3.1918\%, \\
\sigma[R_f] &= 0\%.
\end{aligned}
$$

C. Correlation and Covariance between Returns

1. **Covariance and Correlation between Assets A and B:**

$$
\begin{aligned}
\sigma[R_A, R_B] &= 0.25 \times (24 - 15.5)(14 - 9.25) + 0.5 \times (18 - 15.5)(9 - 9.25) \\
&\quad + 0.25 \times (2 - 15.5)(5 - 9.25) \\
&= 10.09375 + (-0.3125) + 14.34375 = 24.125, \\
\rho[R_A, R_B] &= \frac{\sigma[R_A, R_B]}{\sigma[R_A] \cdot \sigma[R_B]} = \frac{24.125}{8.1701 \cdot 3.1918} = 0.9251.
\end{aligned}
$$

2. **Covariance and Correlation between Asset A and the Riskless Asset:**

$$\begin{aligned}\sigma[R_A, R_f] &= 0.25 \times (24 - 15.5)(7 - 7) + 0.5 \times (18 - 15.5)(7 - 7) \\ &\quad + 0.25 \times (2 - 15.5)(7 - 7) = 0, \\ \rho[R_A, R_f] &= \frac{\sigma[R_A, R_f]}{\sigma[R_A] \cdot \sigma[R_f]} = \frac{0}{0} \text{ (undefined)}.\end{aligned}$$

3. **Covariance and Correlation between Asset B and the Riskless Asset:**

$$\begin{aligned}\sigma[R_B, R_f] &= 0.25 \times (14 - 9.25)(7 - 7) + 0.5 \times (9 - 9.25)(7 - 7) \\ &\quad + 0.25 \times (5 - 9.25)(7 - 7) = 0, \\ \rho[R_B, R_f] &= \frac{\sigma[R_B, R_f]}{\sigma[R_B] \cdot \sigma[R_f]} = \frac{0}{0} \text{ (undefined)}.\end{aligned}$$

D. Expected Return and Standard Deviation for Portfolio of Asset A and Riskless Asset

ω	$E[R_p]$	$\sigma[R_p]$
−0.2	5.3%	1.6340%
0.6	12.1%	4.9020%
1.2	17.2%	9.8041%

E. Expected Return and Standard Deviation for Portfolio of Asset B and Riskless Asset

ω	$E[R_p]$	$\sigma[R_p]$
−0.2	6.55%	0.6383%
0.6	8.35%	1.9151%
1.2	9.7%	3.8301%

F. Risk-Averse Investor's Preferred Combination

For a risk-averse investor, the preferred asset combination is the one with the higher slope of the Capital Allocation Line (CAL).

$$\begin{aligned}\text{Slope of CAL (A)} &= \frac{E[R_A] - R_f}{\sigma[R_A]} = \frac{15.5\% - 7\%}{8.1701\%} = 1.0404, \\ \text{Slope of CAL (B)} &= \frac{E[R_B] - R_f}{\sigma[R_B]} = \frac{9.25\% - 7\%}{3.1918\%} = 0.7049.\end{aligned}$$

Since CAL(A) has a higher slope, a risk-averse investor would prefer asset A combined with the riskless asset.

G. Expected Return and Standard Deviation for Portfolio of Asset A and Asset B

ω	$E[R_p]$	$\sigma[R_p]$
−0.2	8%	2.4%
0.8	14.25%	7.1307%
1.2	16.75%	9.2167%

II. The Two Risky Asset Case

A. Note that ! denotes the tangency portfolio T in the following graph. Since the pension fund manager wants to minimize portfolio standard deviation holding expected portfolio return fixed at 15%, she holds the combination of T and the riskless asset that produces an expected portfolio return of 15%.

1. Since her portfolio is a combination of the risk-free fund and the tangency portfolio, the capital allocation line (CAL) for her portfolio equals the CAL for the tangency portfolio T. So the slope of the CAL for her portfolio is the slope of the CAL for the tangency portfolio T. Therefore, we need to calculate the slope of the CAL for the tangency portfolio T.

2. The weights of the two risky funds in the tangency portfolio T are as follows. The weight of S in the tangency portfolio T is given by:

$$\omega_{S,T} = 0.7075$$

and the weight of B in the tangency portfolio T is given by:

$$\omega_{B,T} = 1 - \omega_{S,T} = 0.2925.$$

3. The expected return of the tangency portfolio T is:

$$E[R_T] = \omega_{S,T} E[R_S] + (1 - \omega_{S,T}) E[R_B]$$

$$= 0.7075 \times 22\% + 0.2925 \times 13\% = 19.3671\%.$$

4. The variance of R_T is calculated as:

$$\sigma^2[R_T] = \omega_{S,T}^2 \sigma^2[R_S] + \omega_{B,T}^2 \sigma^2[R_B] + 2\,\omega_{S,T}\,\omega_{B,T}\,\sigma[R_S, R_B]$$

$$= (0.7075^2) \times 1024 + (0.2925^2) \times 529 + 2 \times 0.7075 \times 0.2925 \times 110.4$$

$$= 512.57 + 45.26 + 45.69 = 603.52.$$

$$\sigma[R_T] = 24.5667\%.$$

5. The slope of the CAL for the tangency portfolio T and her portfolio is given by:

$$\text{slope-CAL}(T) = \frac{E[R_T] - R_f}{\sigma[R_T]} = \frac{19.3671\% - 9\%}{24.5667\%} = 0.4220.$$

6. (a) One Answer: Her portfolio consists of the tangency portfolio T and the riskless asset. So for her portfolio p and letting $\omega_{T,p}$ be the weight of the tangency portfolio in her portfolio:

$$
\begin{aligned}
E[R_p] = 15\% &= \omega_{T,p}E[R_T] + (1 - \omega_{T,P})R_f \\
&= R_f + \omega_{T,p}(E[R_T] - R_f). \\
&= 9\% + \omega_{T,p}(19.3671\% - 9\%).
\end{aligned}
$$

This implies that the weight of the tangency portfolio T in portfolio p is:

$$\omega_{T,p} = \frac{6\%}{10.3671\%} = 0.5788.$$

Finally:

$$\sigma[R_p] = |\omega_{T,p}|\sigma[R_T] = 0.5788 \times 24.5667\% = 14.22\%.$$

(b) Second Answer: The equation for the CAL (T) line is given by:

$$E[R_p] = R_f + \sigma[R_P]\frac{E[R_T] - R_f}{\sigma[R_T]}.$$

Thus:

$$15\% = 9\% + \sigma[R_P]\frac{19.3671\% - 9\%}{24.5667\%},$$

which implies:

$$\sigma[R_P] = \frac{6\%}{0.4220} = 14.22\%.$$

7. We know that:

$$R_p = \omega_{T,p}R_T + (1 - \omega_{T,p})R_f,$$

where $\omega_{T,p} = 0.5788$ is the weight of the tangency portfolio T in the portfolio p and:

$$R_T = \omega_{S,T}R_S + (1 - \omega_{S,T})R_B,$$

where $\omega_{S,T} = 0.7075$ is the weight of the stock fund S in the tangency portfolio T. So:

$$R_p = \omega_{T,p}(\omega_{S,T}R_S + (1 - \omega_{S,T})R_B) + (1 - \omega_{T,p})R_f.$$

This gives:

$$\omega_{S,p} = \omega_{T,p}\omega_{S,T} = 0.5788 \times 0.7075 = 0.4095$$

as the weight of the stock fund S in portfolio p.

$$\omega_{B,p} = \omega_{T,p}(1 - \omega_{S,T}) = 0.5788 \times 0.2925 = 0.1693$$

as the weight of the bond fund B in portfolio p.

$$\omega_{f,p} = (1 - \omega_{T,p}) = 0.4212$$

as the weight of the riskless asset in portfolio p.

B.

1. Her portfolio q consists only of the stock fund and the bond fund:

$$E[R_q] = 15\% = \omega_{S,q} E[R_S] + (1 - \omega_{S,q}) E[R_B].$$

$$= E[R_B] + \omega_{S,q}(E[R_S] - E[R_B]).$$

$$= 13\% + \omega_{S,q} \cdot 9\%.$$

This implies that the weight of the stock fund S in the portfolio q is given by:

$$\omega_{S,q} = \frac{2\%}{9\%} = 0.2222,$$

and the weight of the bond fund B in q is:

$$\omega_{B,q} = 1 - \omega_{S,q} = 0.7778.$$

2. Portfolio q's standard deviation is:

$$\sigma^2[R_q] = \omega_{S,q}^2 \sigma^2[R_S] + \omega_{B,q}^2 \sigma^2[R_B] + 2\omega_{S,q}\omega_{B,q}\sigma[R_S, R_B].$$

$$= (0.2222^2) \times 1024 + (0.7778^2) \times 529 + 2 \times 0.2222 \times 0.7778 \times 110.4.$$

$$= 50.558 + 320.031 + 38.160 = 408.749.$$

$$\sigma[R_q] = 20.2175\%.$$

We can see that although portfolio p from the previous question and portfolio q both have the same expected return, portfolio p has the lower standard deviation. Thus, any risk-averse individual would prefer to hold portfolio p.

3. The slope of the CAL for portfolio q is given by:

$$\text{slope-CAL}(q) = \frac{E[R_q] - R_f}{\sigma[R_q]} = \frac{15\% - 9\%}{20.2175\%} = 0.2968.$$

Solution Set for Foundations of Finance

III. SML and the CAPM

A. In a CAPM world, all assets lie on the SML.

$$E[R_p] = R_f + \beta_{p,m}(E[R_m] - R_f)$$
$$20\% = 5\% + \beta_{p,m}(15\% - 5\%)$$
$$\beta_{p,m} = \frac{15\%}{10\%} = 1.5.$$

B. Additional Calculations

1. All assets plot on the SML, so

$$E[R_p] = R_f + \beta_{p,m}(E[R_m] - R_f)$$
$$14\% = 4\% + 1.25 \cdot (E[R_m] - 4\%)$$
$$E[R_m] = 4\% + \frac{(14\% - 4\%)}{1.25} = 12\%.$$

2. In a CAPM world, an asset with $\beta_{p,m} = 0$ has an expected return of:

$$E[R_p] = R_f + \beta_{p,m}(E[R_m] - R_f) = 4\% + 0 \cdot (12\% - 4\%) = 4\%.$$

3. **Expected Return on Stock with Beta of -0.5:**

$$E[R] = R_f + \beta(E[R_m] - R_f) = 4\% + (-0.5)(12\% - 4\%) = 0\%.$$

The intrinsic value of the stock is given by:

$$V_0 = \frac{E[P_1 + D_1]}{1 + E[R]} = \frac{41 + 3}{1.00} = 44,$$

which is greater than its current price, indicating it is underpriced today.

IV. SML vs. CML in the CAPM

Assume that the CAPM holds in the economy. The following data is available about the market portfolio, the riskless rate, and two assets, A and B.

Asset i	$E[R_i]$	$\sigma[R_i]$	$\beta_{i,m}$
m (market)	0.15	0.08	1
A	0.096	1.2	-
B	0.07	0.6	-

Given $R_f = 0.10$.

1. **Beta of Market Portfolio**:

$$\beta_{m,m} = \frac{\sigma[R_m, R_m]}{\sigma[R_m]^2} = 1.$$

2. **Expected Return on Asset A:**

$$E[R_A] = R_f + \beta_{A,M}(E[R_M] - R_f) = 0.10 + 1.2 \cdot (0.15 - 0.10) = 0.16.$$

3. **Expected Return on Asset B:**

$$E[R_B] = R_f + \beta_{B,M}(E[R_M] - R_f) = 0.10 + 0.6 \cdot (0.15 - 0.10) = 0.13.$$

4. **Does Asset A plot on the:**
   (a) **SML (Security Market Line)?** Yes.
   (b) **CML (Capital Market Line)?**

$$E[R_A] = R_f + \sigma[R_A]\frac{E[R_M] - R_f}{\sigma[R_M]} = 0.10 + 0.096\frac{0.15 - 0.10}{0.08} = 0.16.$$

   Thus, $E[R_A] = 0.16$, confirming that A lies on the CML.

5. **Does Asset B plot on the:**
   (a) **SML?** Yes.
   (b) **CML?**

$$E[R_B] = R_f + \sigma[R_B]\frac{E[R_M] - R_f}{\sigma[R_M]} = 0.10 + 0.07\frac{0.15 - 0.10}{0.08} = 0.14375 > 0.13.$$

   So, asset B does not lie on the CML.

6. **Could any investor be holding asset A as their entire portfolio?** Yes, since it lies on the CML.

7. **Could any investor be holding asset B as their entire portfolio?** No, as it does not lie on the CML.

8. **Correlation of Asset A with the Market Portfolio:**

$$\rho[R_A, R_M] = \frac{\beta_{A,M}\sigma[R_M]}{\sigma[R_A]} = \frac{1.2 \cdot 0.08}{0.096} = 1.$$

9. **Correlation of Asset B with the Market Portfolio:**

$$\rho[R_B, R_M] = \frac{\beta_{B,M}\sigma[R_M]}{\sigma[R_B]} = \frac{0.6 \cdot 0.08}{0.07} = 0.6857.$$

10. **Composition of Asset A:** Since A lies on the CML, it must be a combination of the market portfolio and the riskless asset.

11. **Composition of Asset B:** Since B does not lie on the CML, it cannot be a combination of the market portfolio and the riskless asset. Nothing further can be inferred.

Performance Measurement

Performance Measurement

The following information is to be used to evaluate the performance of the Bull Fund and the Boom Fund.

i	$E[R_i]$	$\sigma[R_i]$	$\sigma[R_i, R_{\text{S\&P}}]$
S&P	15%	20%	400
Bull	17%	30%	440
Boom	19%	40%	460
Risk-free	5%	0	0

A. Calculate the Sharpe ratio for

1. **The S&P 500 index fund.**

$$\text{Sharpe}_{\text{S\&P}} = \frac{E[r_{\text{S\&P}}] - R_f}{\sigma[R_{\text{S\&P}}]} = \frac{15\% - 5\%}{20\%} = 0.5$$

2. **The Bull fund.**

$$\text{Sharpe}_{\text{Bull}} = \frac{E[r_{\text{Bull}}] - R_f}{\sigma[R_{\text{Bull}}]} = \frac{17\% - 5\%}{30\%} = 0.4$$

3. **The Boom fund.**

$$\text{Sharpe}_{\text{Boom}} = \frac{E[r_{\text{Boom}}] - R_f}{\sigma[R_{\text{Boom}}]} = \frac{19\% - 5\%}{40\%} = 0.35$$

B. Calculate Jensen's alpha for

1. **The S&P 500 index fund.**

$$\beta_{\text{S\&P,S\&P}} = \frac{\text{cov}[r_{\text{S\&P}}(t), r_{\text{S\&P}}(t)]}{\text{var}[r_{\text{S\&P}}(t)]} = 1$$

$$\alpha_{\text{S\&P,S\&P}} = E[r_{\text{S\&P}}(t)] - \beta_{\text{S\&P,S\&P}} E[r_{\text{S\&P}}(t)] = (15\% - 5\%) - 1 \times (15\% - 5\%) = 0$$

2. **The Bull fund.**

$$\beta_{\text{Bull,S\&P}} = \frac{\text{cov}[r_{\text{Bull}}(t), r_{\text{S\&P}}(t)]}{\text{var}[r_{\text{S\&P}}(t)]} = \frac{440}{20^2} = 1.1$$

$$\alpha_{\text{Bull,S\&P}} = E[r_{\text{Bull}}(t)] - \beta_{\text{Bull,S\&P}} E[r_{\text{S\&P}}(t)] = (17\% - 5\%) - 1.1 \times (15\% - 5\%) = 1\%$$

3. **The Boom fund.**

$$\beta_{\text{Boom,S\&P}} = \frac{\text{cov}[r_{\text{Boom}}(t), r_{\text{S\&P}}(t)]}{\text{var}[r_{\text{S\&P}}(t)]} = \frac{460}{20^2} = 1.15$$

$$\alpha_{\text{Boom,S\&P}} = E[r_{\text{Boom}}(t)] - \beta_{\text{Boom,S\&P}} E[r_{\text{S\&P}}(t)] = (19\% - 5\%) - 1.15 \times (15\% - 5\%) = 2.5\%$$

C. An investor who only cares about the mean and standard deviation of her portfolio's return is trying to decide which of these funds to hold in combination with T-bills. Which fund should the investor choose?

The investor should choose the Bull Fund rather than the Boom Fund since $\text{Sharpe}_{\text{Bull}} > \text{Sharpe}_{\text{Boom}}$. However, the investor would prefer holding an S&P 500 index fund instead of either of these funds because $\text{Sharpe}_{\text{S\&P}}$ is higher than both $\text{Sharpe}_{\text{Bull}}$ and $\text{Sharpe}_{\text{Boom}}$.

D. An investor who only cares about the mean and standard deviation of her portfolio's return is considering combining Bull with the S&P 500 index fund (the market portfolio) and the risk-free asset. Will Bull's weight be positive, negative, or zero in the investor's portfolio?

Bull's weight will be positive since $\alpha_{\text{Bull,S\&P}} > 0$.

E. An investor who only cares about the mean and standard deviation of her portfolio's return is considering combining the Boom fund with the S&P 500 index fund (the market portfolio) and the risk-free asset. Will Boom's weight be positive, negative, or zero in the investor's portfolio?

Boom's weight will be positive since $\alpha_{\text{Boom,S\&P}} > 0$.