Black-Scholes Option Pricing Model: Inputs and Effects

A. The Five Inputs into the Black-Scholes Option Pricing Model

The five inputs into the Black-Scholes option pricing model are:

- $S(0)$: The current price of the underlying asset.
- X: The exercise price.
- σ: The volatility of the continuously compounded annual return on the underlying asset.
- r': The continuously compounded annual risk-free rate (1 invested today at the risk-free rate is worth $1e^{r't}$ in t years).
- T: The time to expiration.

How Each Input Affects the Price of a European Call Option on a Non-Dividend Paying Stock

- $S(0)$: $C_{X,T}(0)$ is monotonically increasing in $S(0)$ as expected.
- X: $C_{X,T}(0)$ is monotonically decreasing in X as expected.
- σ: $C_{X,T}(0)$ is monotonically increasing in σ.
  - *Explanation*: The option feature of the call truncates the payoff at 0 when the underlying's value is less than the strike price. When σ increases, the volatility of $S(T)$ increases. The call option holder benefits from greater upside payoff potential due to the greater upside of $S(T)$, but because the option payoff is truncated at 0, does not bear greater downside payoff potential due to the greater downside of $S(T)$. Thus, the value of the call increases relative to $S(0)$.
- r': $C_{X,T}(0)$ is monotonically increasing in r'.
  - *Explanation*: The exercise price does not have to be paid until time T. When r' increases, the current value of X paid at T decreases, making the option more valuable for a given $S(0)$.
- T: $C_{X,T}(0)$ is monotonically increasing in T.

- *Explanation*: There are two effects, both acting in the same direction:
    1. The exercise price does not have to be paid until time T. When T increases, the current value of X paid at T decreases, making the option more valuable for the same reason that an increase in r' increases the call's value today.
    2. With a longer time to maturity, the volatility of $S(T)$ increases for a given σ. So the value of the call today increases for the same reason that an increase in σ increases the call's value today.

How Each Input Affects the Price of a European Put Option on a Non-Dividend Paying Stock

- $S(0)$: $P_{X,T}(0)$ is monotonically decreasing in $S(0)$ as expected.
- X: $P_{X,T}(0)$ is monotonically increasing in X as expected.
- σ: $P_{X,T}(0)$ is monotonically increasing in σ.
  - *Explanation*: The option feature of the put truncates the payoff at 0 when the underlying's value is more than the strike price. When σ increases, the volatility of $S(T)$ increases. The put option holder benefits from greater upside payoff potential due to the greater downside of $S(T)$, but because the option payoff is truncated at 0, does not bear greater downside payoff potential due to the greater upside of $S(T)$. Thus, the value of the put increases relative to $S(0)$.
- r': $P_{X,T}(0)$ is monotonically decreasing in r'.
  - *Explanation*: The exercise price is not received until time T. When r' increases, the current value of X received at T decreases, making the option less valuable for a given $S(0)$.
- T: The effect of T on $P_{X,T}(0)$ is ambiguous.
  - *Explanation*: There are two effects, each acting in a different direction:
    1. The exercise price is not received until time T. When T increases, the current value of X received at T decreases, making the option less valuable for the same reason that an increase in r' decreases the put's value today.
    2. With a longer time to maturity, the volatility of $S(T)$ increases for a given σ. So the value of the put today increases for the same reason that an increase in σ increases the call's value today.

Futures and Forward Contracts - Valuation

I. Treasury Strip Yields and Arbitrage Opportunities

A. Gold Forward Price and Arbitrage Opportunity

Given:

- Spot price of gold today: $S(0) = 300$
- Forward price of gold for delivery in 2 years: $F_2 = 370$
- 2-year U.S. Treasury yield (EAR): $r = 10\%$

Check forward-spot parity (no carrying costs):

$$F_2 = S(0) \times (1 + r)^2 = 300 \times (1.10)^2 = 300 \times 1.21 = 363$$

Since $370 > 363$, there is an arbitrage opportunity.
Arbitrage strategy:

- **Today (Time 0):**
  - Sell a forward contract delivering 1 oz of gold in 2 years: 0
  - Buy 1 oz of gold: -300
  - Sell 2-year U.S. T-bills with a face value of 370: $+\frac{370}{(1+0.10)^2} = +305.79$
- **In 2 years (Time 2):**
  - Deliver the gold and receive the forward price: $+370$
  - Sell the gold purchased today: $-S(2)$
  - Repay the T-bill face value: -370

Net cash flows:

$$\text{Time 0: } +305.79 - 300 = +5.79$$

$$\text{Time 2: } +370 - S(2) - 370 = 0$$

This strategy provides a risk-free profit of 5.79 at Time 0, confirming an arbitrage opportunity.

B. Forward Price of Cotton (with carrying costs)

Given:

- Spot price of cotton: $S(0) = 85$
- Cost of storing cotton for 2 years (payable at end of Year 1): $C = 20$
- 2-year U.S. Treasury yield (EAR): $r = 10\%$

Forward-spot parity with carrying costs:

$$F_2 = \left[S(0) + \frac{C}{(1+r)^1}\right] \times (1+r)^2$$

$$F_2 = \left[85 + \frac{20}{1.10}\right] \times (1.10)^2$$

$$F_2 = [85 + 18.18] \times 1.21 = 103.18 \times 1.21 = 124.85$$

The forward price for cotton in 2 years is $F_2 = 124.85$.

C. Forward Price of the S&P 500 Index (with dividends)

Given:

- Spot price of the index: $S(0) = 1000$
- Dividend yield: $q = 6\%$
- Risk-free rate: $r = 10\%$

Forward-spot parity with carrying costs:

$$F_1 = S(0) \times (1+r)^1 - S(0) \times q$$

$$F_1 = 1000 \times (1.10)^1 - 1000 \times 0.06$$

$$F_1 = 1100 - 60 = 1040$$

The forward price of the S&P 500 index in 1 year is $F_1 = 1040$.

D. Forward Price of a British Pound (Covered Interest Parity)

Given:

- Spot price of 1 British pound: $S_{\$/£}(0) = 1.55$
- Yield on a 1-year U.K. bond: $r_£ = 8\%$
- Yield on a 1-year U.S. bond: $r_\$ = 10\%$

Covered interest parity:

$$F_{\$/£}(0) = S_{\$/£}(0) \times \frac{(1 + r_\$)}{(1 + r_£)}$$

$$F_{\$/£}(0) = 1.55 \times \frac{(1 + 0.10)}{(1 + 0.08)}$$

$$F_{\$/£}(0) = 1.55 \times \frac{1.10}{1.08} = 1.55 \times 1.0185 = 1.5069$$

The forward price for 1 British pound in 1 year is $F_{\$/£}(0) = 1.5069$.

Lecture 11: Market Efficiency

I. Findings on Small Firms and High Book-to-Market Ratios

A. Consistency with Semi-Strong Form Market Efficiency under CAPM

No. The CAPM states that all assets lie on the SML. Being able to identify stocks that will plot above the SML in the future using publicly available information like market capitalization or book-to-market ratio is inconsistent with semi-strong form market efficiency.

B. Consistency with Semi-Strong Form Market Efficiency under the Fama-French 3-Factor Model

Yes. The finding is that small firms and firms with high book-to-market ratios earn subsequent returns that approximately satisfy the expected returns implied by the Fama-French 3-factor model.

C. Implications for Market Efficiency Tests

Any test of market efficiency is a joint test of market efficiency and the asset pricing model assumed to be correct. It may be possible to use information to identify mispriced securities relative to one asset pricing model, but these securities could be correctly priced relative to another model.

II. Forecastability of Stock Returns Using Yield Spread

A. Implications for Semi-Strong Form Efficiency

This finding does not necessarily imply semi-strong form inefficiency. Forecastable returns using the yield spread could reflect forecastable expected returns with the surprise component being mean-zero and unpredictable.

B. Assumption for Semi-Strong Form Inefficiency

If expected returns on U.S. stocks are constant over time, forecastable returns using the yield spread would imply forecastable return surprises, contradicting semi-strong form efficiency.

III. Forecastability Using Past Returns

A. Implications for Weak Form Efficiency

The finding does not necessarily imply weak form inefficiency. Forecastable next month's returns using the past 11 months' returns could reflect forecastable expected returns, with surprises being mean-zero and unpredictable.

B. Assumption for Weak Form Inefficiency

If expected returns are constant over time, forecastable next month's returns using the past 11 months' returns would imply forecastable return surprises, contradicting weak form efficiency.

IV. Portfolio Performance and Mispricing

Mr. X's 25% return over 6 months, compared to the market's 5%, could be due to luck or investment in risky stocks with high loadings on factors like SMB and HML. To prove skill, Mr. X must demonstrate a positive generalized alpha relative to the Fama-French model over a long period.

V. Positive Alpha Before a Dividend Announcement

A. Strong Form Efficiency

The finding does not imply strong form inefficiency since the positive alpha is earned before the announcement and cannot be used to identify mispriced stocks.

B. Semi-Strong Form Efficiency

The finding does not imply semi-strong form inefficiency since the positive alpha is earned before the announcement.

C. Weak Form Efficiency

The finding does not imply weak form inefficiency since dividend announcements are not past price data.

VI. Positive Alpha After a Dividend Announcement

A. Strong Form Efficiency

The finding implies strong form inefficiency since it is possible to use the announcement to identify mispriced stocks.

B. Semi-Strong Form Efficiency

The finding implies semi-strong form inefficiency since it is possible to use the announcement to identify mispriced stocks.

C. Weak Form Efficiency

The finding does not imply weak form inefficiency since dividend announcements are not past price data.