Portfolio Selection of Jim and John

Portfolio Description

Jim and John choose to hold the same portfolio of risky assets in combination with the riskless asset. This chosen risky-asset portfolio lies on the minimum-variance frontier for the N risky assets. It is the risky-asset portfolio that has the steepest sloped Capital Allocation Line (CAL) of all portfolios on the minimum-variance portfolio for the N risky assets. This portfolio is referred to as the **tangency portfolio**, denoted as T.

Characteristics of the Tangency Portfolio

The characteristics of the tangency portfolio are as follows:

- If $E[R_T] > R_f$, the tangency portfolio lies on the **efficient frontier**. Otherwise, it lies on the **inefficient frontier**.
- Both Jim and John either have a positive weight in the tangency portfolio (if $E[R_T] > R_f$) or both have a negative weight (if $E[R_T] < R_f$).
- The absolute value of Jim's portfolio weight in the tangency portfolio is smaller than the absolute value of John's portfolio weight, as Jim is more risk averse than John.

Portfolio Weights

Let w_T^J be the weight of Jim's investment in the tangency portfolio and w_T^J be the weight of John's investment in the tangency portfolio. Then:

$$|w_T^J| < |w_T^J|$$

This indicates that Jim's investment in the tangency portfolio is less than John's investment in the same portfolio due to their differing levels of risk aversion.

Relation Between the Portfolios

The portfolios chosen by Jim and John are related in that they both hold the tangency portfolio as part of their overall investment strategy. The difference in their weights reflects their individual risk tolerances, with Jim's lower weight indicating a preference for less risk compared to John.

In conclusion, while Jim and John may choose the same risky-asset portfolio, the extent of their investment in that portfolio varies based on their personal risk preferences.

Lecture 5: Capital Asset Pricing Model (CAPM)

1 Assumptions

Assume that the CAPM holds in the economy.

2 Portfolios Held by Investors

1. The market portfolio M is defined as the portfolio of all risky assets in the economy, with each asset weighted by its value relative to the total value of all assets.

2. All investors hold combinations of the riskless asset and the market portfolio M, as M serves as the tangency portfolio for the N risky assets in the economy given the riskless rate.

3. The CAPM assumes that all investors perceive the same opportunity set, hence all investors identify the same tangency portfolio T.

4. By portfolio theory, if investors only consider $E[R]$ and $\sigma[R]$, they seek to hold a combination of the riskless asset and T.

5. Since all investors hold the same risky portfolio T, the market portfolio M must correspond to the tangency portfolio T, meaning all investors hold M as their risky portfolio.

6. Thus, all individuals combine the value-weighted market portfolio M and the riskless asset, forming **efficient portfolios** with a correlation of 1 with the market portfolio.

3 Expected Return of an Arbitrary Asset

- In a CAPM economy, all assets lie on the Security Market Line (SML):

$$E[R_p] = R_f + \beta_{p,M} \left(E[R_M] - R_f \right)$$

where $\beta_{p,M} = \frac{\sigma[R_p, R_M]}{\sigma^2[R_M]}$ determines $E[R_p]$.

4 Appropriate Measure of Risk

- The relevant measure of risk for any asset or portfolio p is **Beta** with respect to the market portfolio, as defined by $\beta_{i,M}$.
- This beta risk measures how asset i impacts the market portfolio's variance $\sigma^2[R_M]$ and thus is the appropriate measure of the riskiness of asset i.
- The expected return $E[R_i]$ is therefore dependent on $\beta_{i,M}$ as per the SML.

5 Exercises

5.1 Exercise I

Assume the CAPM holds in the economy with the following data:

- Market Portfolio M: $E[R_M] = 16\%$, $\sigma[R_M] = 12\%$
- Risk-free rate $R_f = 6\%$
- Asset ABC: $\sigma[R_{ABC}] = 9\%$, $\beta_{ABC,M} = 0.75$
- Asset WXY: $\sigma[R_{WXY}] = 10\%$, $\beta_{WXY,M} = 0.75$

Questions and Answers

1. **What is $\beta_{M,M}$?**
$$\beta_{M,M} = \frac{\sigma[R_M, R_M]}{\sigma^2[R_M]} = 1$$
2. **What is $E[R_{ABC}]$?**
$$E[R_{ABC}] = R_f + \beta_{ABC,M}\left(E[R_M] - R_f\right) = 6\% + 0.75 \times (16\% - 6\%) = 13.5\%$$
3. **What is $E[R_{WXY}]$?**
$$E[R_{WXY}] = R_f + \beta_{WXY,M}\left(E[R_M] - R_f\right) = 6\% + 0.75 \times (16\% - 6\%) = 13.5\%$$
4. **Does ABC plot on the SML and CML?**
   (a) **SML:** Yes, as calculated.
   (b) **CML:**
$$R_f + \sigma[R_{ABC}]\frac{E[R_M] - R_f}{\sigma[R_M]} = 6\% + 9\% \times \frac{16\% - 6\%}{12\%} = 13.5\%$$
5. **Does WXY plot on the SML and CML?**
   (a) **SML:** Yes.

(b) **CML:**

$$R_f+\sigma[R_{WXY}]\frac{E[R_M]-R_f}{\sigma[R_M]} = 6\%+10\%\times\frac{16\%-6\%}{12\%} = 14.33\% > E[R_{WXY}]$$

6. **Could any investor be holding ABC or WXY as their entire portfolio?**
   - ABC: Yes, since it lies on the CML.
   - WXY: No, as it does not lie on the CML.

7. **Correlation of ABC with M:**

$$\rho[R_{ABC}, R_M] = \frac{\beta_{ABC,M}\sigma[R_M]}{\sigma[R_{ABC}]} = \frac{0.75\times 12\%}{9\%} = 1$$

8. **Correlation of WXY with M:**

$$\rho[R_{WXY}, R_M] = \frac{\beta_{WXY,M}\sigma[R_M]}{\sigma[R_{WXY}]} = \frac{0.75\times 12\%}{10\%} = 0.9$$

9. **Composition of ABC and WXY:**
   - ABC: Since it lies on the CML, it is a combination of the market portfolio and the riskless asset.
   - WXY: Since it does not lie on the CML, nothing specific can be said.

6 Regression Interpretation

Consider the regression $R_p = a+bR_M+e_p$. The coefficient b measures the beta, $\beta_{p,M}$, as:

$$b = \beta_{p,M} = \frac{\sigma[R_p, R_M]}{\sigma^2[R_M]}$$

Lecture 5-6: CAPM Performance Measures and Empirical Evidence

I. Regression Analysis in a CAPM World

Let R_i be the return on an arbitrary asset i, R_M be the return on the market portfolio, and R_f be the riskless rate. We examine the regression of the excess return on asset i, $r_i = R_i - R_f$, on the excess return on the market portfolio, $r_M = R_M - R_f$:

$$r_i = a_i + b_i r_M + \epsilon_{p,M}$$

A. Interpretation of b_i

The coefficient b_i measures the sensitivity of asset i's excess return to the market's excess return, and it equals the CAPM beta $\beta_{P,M}$.

B. Value of a_i in a CAPM World

Taking expectations:

$$E[r_i] = a_i + \beta_{i,M} E[r_M]$$

Since the CAPM implies that all assets lie on the SML, we have:

$$E[r_i] = 0 + \beta_{i,M} E[r_M]$$

Thus, $a_i = 0$ for any asset i under CAPM assumptions.

II. Performance Evaluation of Funds

Given data for the S&P 500 index fund, the Savvy Fund, and the Smart Fund:

Fund	$E[R_i]$	$\sigma[R_i]$	$\sigma[R_i, R_{\text{S\&P}}]$
S&P	16	25	625
Savvy	16	30	450
Smart	18.5	35	850
Risk-free	8	0	0

A. Sharpe Ratios

1. **S&P 500 Index Fund:**

$$\text{Sharpe}_{\text{S\&P}} = \frac{E[r_{\text{S\&P}}]}{\sigma[R_{\text{S\&P}}]} = \frac{16-8}{25} = 0.32$$

2. **Savvy Fund:**

$$\text{Sharpe}_{\text{Savvy}} = \frac{E[r_{\text{Savvy}}]}{\sigma[R_{\text{Savvy}}]} = \frac{16-8}{30} = 0.2667$$

3. **Smart Fund:**

$$\text{Sharpe}_{\text{Smart}} = \frac{E[r_{\text{Smart}}]}{\sigma[R_{\text{Smart}}]} = \frac{18.5-8}{35} = 0.3$$

B. Jensen's Alpha

1. **S&P 500 Index Fund:**

$$\alpha_{\text{S\&P}} = E[r_{\text{S\&P}}] - \beta_{\text{S\&P,S\&P}} E[r_{\text{S\&P}}] = (16-8) - 1 \times (16-8) = 0$$

2. **Savvy Fund:**

$$\beta_{\text{Savvy,S\&P}} = \frac{\text{Cov}[r_{\text{Savvy}}, r_{\text{S\&P}}]}{\text{Var}[r_{\text{S\&P}}]} = \frac{450}{25^2} = 0.72$$

$$\alpha_{\text{Savvy,S\&P}} = E[r_{\text{Savvy}}] - \beta_{\text{Savvy,S\&P}} E[r_{\text{S\&P}}] = (16-8) - 0.72 \times (16-8) = 2.24$$

3. **Smart Fund:**

$$\beta_{\text{Smart,S\&P}} = \frac{\text{Cov}[r_{\text{Smart}}, r_{\text{S\&P}}]}{\text{Var}[r_{\text{S\&P}}]} = \frac{850}{25^2} = 1.36$$

$$\alpha_{\text{Smart,S\&P}} = E[r_{\text{Smart}}] - \beta_{\text{Smart,S\&P}} E[r_{\text{S\&P}}] = (18.5-8) - 1.36 \times (16-8) = -0.38$$

C. Portfolio Choice for Investors Focused on Mean and Standard Deviation

An investor should choose the Smart Fund over the Savvy Fund since $\text{Sharpe}_{\text{Savvy}} < \text{Sharpe}_{\text{Smart}}$. However, the S&P 500 index fund has the highest Sharpe ratio.

D. Combining Savvy Fund with the Market Portfolio

Since $\alpha_{\text{Savvy,S\&P}} > 0$, Savvy's weight will be positive.

E. Combining Smart Fund with the Market Portfolio

Since $\alpha_{\text{Smart,S\&P}} < 0$, Smart's weight will be negative.

III. Testable Implications of CAPM

1. The market portfolio is the tangency portfolio.
2. All assets lie on the SML, so variation in expected returns is fully explained by linear variation in Beta with respect to the market.

IV. Empirical Evidence on CAPM in the U.S.

Empirical studies, such as Fama and French (1993), show inconsistencies with CAPM predictions:

- Positive and significant Jensen's alphas for high book-to-market portfolios.
- Jensen's alpha increases from large to small firms, holding book-to-market constant.

These results suggest that the CAPM does not fully hold empirically, as the market portfolio may not lie on the minimum variance frontier for individual stocks.