Lecture 8: Bond Pricing and Forward Rates

I. Data for U.S. Treasury Notes and Strips on 8/15/10

Yields are expressed as APRs with half-year compounding.

U.S. Treasury Notes

Rate	Maturity	Price	Yield
2%	Feb 11	?	?
8%	Aug 11	?	?
10%	Feb 12	102.05	8.5153%

U.S. Treasury Strips

Type	Maturity	Price	Yield
ci	Feb 11	98	?
ci	Aug 11	?	5.1957%
ci	Feb 12	88	?

A. Price of Aug 11 Strip on 8/15/10

The price of the Aug 11 strip can be calculated using the yield expressed as an APR with semi-annual compounding.

B. Yield of Feb 12 Strip on 8/15/10

The yield of the Feb 12 strip, expressed as an APR with semi-annual compounding, is:

$$y_{1.5}(0) = 2 \times \left(\left[\frac{100}{88} \right]^{1/3} - 1 \right) = 8.7064\%.$$

C. Price of 8% Aug 11 Note on 8/15/10

$$d_{\frac{1}{2}}(0) = \frac{98}{100} = 0.98, \quad d_1(0) = \frac{95}{100} = 0.95$$

$$P_{8\%\text{Aug 11}}(0) = d_{\frac{1}{2}}(0) \times \frac{8}{2} + d_1(0) \times \left(100 + \frac{8}{2}\right)$$
$$= 0.98 \times 4 + 0.95 \times 104 = 102.72.$$

D. Arbitrage Opportunity at a Price of 102.50

The no-arbitrage price is 102.72, but the market price is 102.50. This creates an arbitrage opportunity.

Arbitrage Position

Buy 1 Aug 11 note, sell synthetic Aug 11 note.

$$a = -1.04, \quad b = -0.04$$

Arbitrage profit: $-102.5 - (-1.04 \times 95) - (-0.04 \times 98) = 0.22.$

E. Yield Difference: Feb 12 Strip vs. 10% Feb 12 Note

The yield of the Feb 12 strip (8.7064%) is higher than that of the 10% Feb 12 note (8.5153%). The difference arises because:

1. The note pays coupons in 6 months, 1 year, and 18 months, whereas the strip pays par value in 18 months.

2. The yields on shorter-term strips (Feb 11 and Aug 11) are lower than the yield on the Feb 12 strip.

F. Synthetic Aug 11 Strip

$$a \times 104 = 100 \implies a = 0.96154, \quad a \times 4 + b \times 101 = 0 \implies b = -0.03808$$

Buy 0.96154 Aug 11 notes, sell 0.03808 Feb 11 notes.

G. Holding Period Return and YTM on 10% Feb 12 Note

1. **Holding Period Return:**

$$\text{APR (semi-annual compounding)} = 7.7413\%.$$

2. **YTM on 2/15/11:** Since the holding period return is lower than the initial YTM, the YTM on 2/15/11 must be higher than 8.5153%.

H. Forward Rate for Tom

To lock in the rate for 6 months starting on 8/15/11:

$$f_{1,1.5}(0) = \frac{d_1(0)}{d_{1.5}(0)} - 1,$$

$$f_{1,1.5}(0) = 15.9091\%.$$

Lecture 8-9: Theories of the Yield Curve

I. Define the term spot rate

In the context of the yield curve, the spot rate is the yield on the shortest maturity discount bond. For our purposes, the spot rate at a point in time is the yield on a 6-month discount bond at that time.

II. Preferences of investors regarding interest rate risk under the expectations hypothesis

Under the expectations hypothesis, individuals do not care about interest rate risk. Hence, the expected return over any given future period is the same for all discount bonds and forward contracts available as investment vehicles over that period.

III. Three versions of the expectations hypothesis

The three versions are approximately equivalent, with the 1st and the 3rd being exactly equivalent. When yields, holding period returns, and forward rates are expressed as APRs with $\frac{1}{2}$-year compounding:

Version 1

Individuals with an n-year horizon: If individuals do not care about interest rate risk, the following should be equal:

1. The expected payoff at time n from buying an $(n-\frac{1}{2})$-year discount bond today and rolling the proceeds into a $\frac{1}{2}$-year discount bond for the $\frac{1}{2}$-year from $(n-\frac{1}{2})$ to n.
2. The certain payoff at time n from buying an n-year discount bond today.

So:

$$y_n(0) = \frac{1}{n}\left[\frac{n-\frac{1}{2}}{n} y_{n-\frac{1}{2}}(0) + \frac{1}{n} f_{n-\frac{1}{2},n}(0)\right].$$

Version 2

Individuals with a $\frac{1}{2}$-year horizon: If individuals do not care about interest rate risk, the following should be equal:

1. The expected return over the next $\frac{1}{2}$-year on a discount bond of any maturity.

2. The $\frac{1}{2}$-year spot rate.

So:

$$y_{\frac{1}{2}}(0) = E_{\text{at time } 0}[h_n(\frac{1}{2})],$$

where $h_n(\frac{1}{2})$ is the return on a discount bond with maturity n years in the future for the $\frac{1}{2}$-year ending at time $\frac{1}{2}$.

Version 3

Individuals who need to invest for the $\frac{1}{2}$-year starting in n years and ending in $(n+\frac{1}{2})$ years:

1. Forward rate available today for the $\frac{1}{2}$-year period starting in n years and ending in $(n+\frac{1}{2})$ years.

2. The expected future spot rate for the $\frac{1}{2}$-year period starting in n years and ending in $(n+\frac{1}{2})$ years.

So:

$$f_{n,n+\frac{1}{2}}(0) = E_{\text{at time } 0}[y_{\frac{1}{2}}(n)].$$

IV. Liquidity of discount bonds as a function of maturity under the liquidity preference theory

Under the liquidity preference theory, individuals require a liquidity premium to hold less liquid bonds (resulting in a price discount). Discount bonds become less liquid as maturity increases. The percentage change in the associated price discount going to the next longest maturity is positive at all maturities.

V. Preferences of investors regarding interest rate risk under the preferred habitat theory

Under the preferred habitat theory, individuals care about interest rate risk. Hence, the expected return over any future period differs across discount bonds and forward contracts depending on the interest rate offered by the various available instruments over that period. The expectations hypothesis no longer holds.

VI. Yield curve implications

A. Upward-sloping yield curve (+300 basis points yield spread)

1. **Future economic activity:** A yield spread of +300 basis points is about 2 standard deviations higher than average. Empirically, this predicts that future real economic growth will be higher than average. The yield spread is one of the best predictors of future economic activity.

2. **Under expectations hypothesis:** An upward-sloping yield curve implies that expected future spot rates are higher than the current spot rate.

3. **Under liquidity preference theory:** An upward-sloping yield curve does not provide information about expected future spot rates relative to the current spot rate.

4. **Under preferred habitat theory:** An upward-sloping yield curve does not provide information about expected future spot rates relative to the current spot rate.

B. Downward-sloping yield curve (-100 basis points yield spread)

1. **Future economic activity:** A yield spread of -100 basis points is about 2 standard deviations lower than average. Empirically, this predicts that future real economic growth will be lower than average. The yield spread is one of the best predictors of future economic activity.

2. **Under expectations hypothesis:** A downward-sloping yield curve implies that expected future spot rates are lower than the current spot rate.

3. **Under liquidity preference theory:** A downward-sloping yield curve implies that expected future spot rates are lower than the current spot rate.

4. **Under preferred habitat theory:** A downward-sloping yield curve does not provide information about expected future spot rates relative to the current spot rate.

VII. Forward rates and yield curve slopes

A. Positive slope (n to $n + \frac{1}{2}$ years)

1. **Forward rate:** The forward rate, $f_{n,n+\frac{1}{2}}(0)$, lies above the yield on the n-period discount bond.

2. **Under expectations hypothesis:** The expected spot rate in n years, $E_{\text{at time 0}}[y_{\frac{1}{2}}(n)]$, lies above the yield on the n-period discount bond.

B. Negative slope (n to $n + \frac{1}{2}$ years)

1. **Forward rate:** The forward rate, $f_{n,n+\frac{1}{2}}(0)$, lies below the yield on the n-period discount bond.

2. **Under expectations hypothesis:** The expected spot rate in n years, $E_{\text{at time 0}}[y_{\frac{1}{2}}(n)]$, lies below the yield on the n-period discount bond.

Bond Portfolio Management

I. Bond Risks

A. Reinvestment Risk

Reinvestment risk occurs when cash flows from a bond are received before the investor's time horizon. The risk is the uncertainty about yields at the time of reinvestment.

B. Liquidation Risk

Liquidation risk occurs when cash flows from a bond extend beyond the investor's time horizon. The risk is the uncertainty about the bond price at the time of liquidation.

II. Price and Macaulay Duration Calculation

Given: 6% Feb 11 Note, YTM = 4% (APR, semi-annual).

A. Price Calculation

$$P = \frac{3}{1+0.02} + \frac{103}{(1+0.02)^2} = 2.9412 + 99.0004 = 101.9416$$

B. Macaulay Duration

$$k() = \frac{2.9412}{101.9416}, \quad k(1) = \frac{99.0004}{101.9416}$$
$$D = \ \times k() + 1 \times k(1) = 0.9856 \, \text{years}.$$

III. Coupon Bonds and Strips

The price of coupon bonds and strips is always a **decreasing convex function of yield**.

IV. Firm QV's Bond Portfolio Management

A. Using 2-Year and 4-Year Strips

1. **Composition:**
   - Buy 2-Year Strips: \$5M (face value).
   - Buy 4-Year Strips: \$15M (face value).
2. **Rebalancing:** Not required with a dedication strategy.

B. Using 3-Year and 5-Year Strips

1. **Portfolio Composition**
   - **Liability Value Today:**

$$L = \frac{5M}{(1+0.04)^4} + \frac{15M}{(1+0.04)^8} = 15.2344M$$

   - **Liability Duration:**

$$D_L = 2 \times \frac{4.2740}{15.2344} + 4 \times \frac{10.9604}{15.2344} = 3.4389\,\text{years.}$$

   - **Portfolio Matching:**

$$A_3 + A_5 = 15.2344M, \quad 3.4389 = 0.78055 \times 3 + (1 - 0.78055) \times 5$$

   - **Investments:**

$$A_3 = 11.8912M, \quad A_5 = 3.3432M$$

2. **Rebalancing:** Required due to shifts in the yield curve or as time progresses.