1 Lecture 3 3-4

1.1 Jim's Investment in DFQ and the Riskless Asset

I.L.

1. Will Jim short-sell DFQ? Jim does not want to short-sell DFQ because he prefers holding a portfolio on the positive-sloped portion of the arrowhead in the {E[R], $\sigma[R]$} space that is achievable by investing in DFQ and the riskless asset. He avoids holding portfolios on the negative-sloped portion of the arrowhead because, for any portfolio on that portion, there exists a corresponding portfolio on the positive-sloped portion with the same $\sigma[R]$ but a higher $E[R]$. Since $E[R_{\text{DFQ}}] > R_f$, short-selling DFQ would result in a portfolio on the negative-sloped portion. Consequently, Jim will not short-sell DFQ.

2. Will Jim buy DFQ on margin? Jim will buy DFQ on margin if his level of risk aversion is sufficiently low. Since $E[R_{\text{DFQ}}] > R_f$, buying DFQ using his own funds or on margin will result in a portfolio on the positive-sloped portion of the arrowhead. However, the exact portfolio Jim chooses will depend on his level of risk aversion. If his risk aversion is low, Jim may decide to buy DFQ on margin to increase his exposure to the risky asset. Otherwise, he may only invest his own funds in DFQ.

1.2 Jim's Choice Between DFQ and BML

I.M.

Jim will choose the risky asset to combine with the riskless asset based on the steepness of the capital allocation line (CAL) of the asset. The steepness of the CAL is calculated as:

$$\text{slope}[\text{CAL}_i] = \frac{|E[R_i] - R_f|}{\sigma[R_i]}$$

For DFQ:

$$\text{slope}[\text{CAL}_{\text{DFQ}}] = \frac{|11\% - 6\%|}{12.489996\%} = 0.40032$$

For BML:

$$\text{slope}[\text{CAL}_{\text{BML}}] = \frac{|10\% - 6\%|}{7.745967\%} = 0.516398$$

Since $\text{slope}[\text{CAL}_{\text{BML}}] > \text{slope}[\text{CAL}_{\text{DFQ}}]$, Jim will prefer to hold BML in combination with the riskless asset. His decision to choose BML over DFQ is independent of his level of risk aversion, as he always prefers the asset with the steeper CAL. However, the fractions Jim decides to invest in BML and the riskless asset will depend on his level of risk aversion.

Lecture 4

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1 Introduction

In this lecture, we consider the portfolio choices of two investors, Jim and John, who both have a 1-month investment horizon. Both have access to two stock portfolios and 1-month T-bills. Jim and John are only concerned with the expected 1-month return on their portfolios and the standard deviation of the returns. Jim is more risk-averse than John, so their choices differ based on their risk preferences. We analyze a portfolio with investments in value stocks and the S&P 500 index, using available data for expected returns and standard deviations, and explore the effects of correlation on portfolio diversification.

The following data is given for the 1-month returns of the portfolios:

- $E[R_{\text{Value}}] = 1.23\%$, $\sigma[R_{\text{Value}}] = 5.67\%$
- $E[R_{\text{S\&P}}] = 0.94\%$, $\sigma[R_{\text{S\&P}}] = 4.38\%$
- $\rho[R_{\text{Value}}, R_{\text{S\&P}}] = 0.73$
- The return on 1-month T-bills, $R_f = 0.39\%$

2 Part A: Portfolio Return and Risk

2.1 Expected Return on a Portfolio

We first consider a portfolio with 40% invested in the value-stock portfolio and 60% in the S&P 500 index. The expected return, $E[R_p]$, is given by:

$$E[R_p] = \omega_{\text{Value},p} \cdot E[R_{\text{Value}}] + \omega_{\text{S\&P},p} \cdot E[R_{\text{S\&P}}]$$

Substituting values:

$$E[R_p] = 0.4 \times 1.23 + 0.6 \times 0.94 = 1.056\%$$

2.2 Portfolio Standard Deviation

The standard deviation of the portfolio return, $\sigma[R_p]$, can be calculated as:

$\sigma^2[R_p] = \omega^2_{\text{Value},p} \cdot \sigma^2[R_{\text{Value}}] + \omega^2_{\text{S\&P},p} \cdot \sigma^2[R_{\text{S\&P}}] + 2 \cdot \omega_{\text{Value},p} \cdot \omega_{\text{S\&P},p} \cdot \rho[R_{\text{Value}}, R_{\text{S\&P}}] \cdot \sigma[R_{\text{Value}}] \cdot \sigma[R_{\text{S\&P}}]$

Substituting values:

$$\sigma^2[R_p] = 0.4^2 \times 5.67^2 + 0.6^2 \times 4.38^2 + 2 \times 0.4 \times 0.6 \times 0.73 \times 5.67 \times 4.38$$

$$\sigma^2[R_p] = 20.75225 \quad \Rightarrow \quad \sigma[R_p] = 4.55546\%$$

3 Part B: Effect of Varying Correlation on Portfolio Risk

Now, we examine how changes in the correlation, $\rho[R_{\text{Value}}, R_{\text{S\&P}}]$, affect the portfolio's standard deviation. The expected return remains unchanged.

3.1 Case 1: $\rho = 1$

When $\rho[R_{\text{Value}}, R_{\text{S\&P}}] = 1$, the portfolio variance becomes:

$$\sigma^2[R_p] = 23.97082 \quad \Rightarrow \quad \sigma[R_p] = 4.896\%$$

3.2 Case 2: $\rho = 0.5$

When $\rho[R_{\text{Value}}, R_{\text{S\&P}}] = 0.5$, the portfolio variance becomes:

$$\sigma^2[R_p] = 18.01051 \quad \Rightarrow \quad \sigma[R_p] = 4.24388\%$$

3.3 Case 3: $\rho = 0$

When $\rho[R_{\text{Value}}, R_{\text{S\&P}}] = 0$, the portfolio variance becomes:

$$\sigma^2[R_p] = 12.05021 \quad \Rightarrow \quad \sigma[R_p] = 3.47134\%$$

3.4 Case 4: $\rho = -1$

When $\rho[R_{\text{Value}}, R_{\text{S\&P}}] = -1$, the portfolio variance becomes:

$$\sigma^2[R_p] = 0.1296 \quad \Rightarrow \quad \sigma[R_p] = 0.36\%$$

4 Part C: Portfolio Choices of Jim and John

Both Jim and John invest in the same risky-asset portfolio, but their allocations differ due to their varying risk aversions. The portfolio they choose is known as the tangency portfolio, T, which lies on the efficient frontier. Jim, being more risk-averse, has a smaller absolute portfolio weight in T compared to John. Specifically, Jim's portfolio consists of 71.463% in value stocks and 28.537% in the S&P 500 index.

5 Part D: Capital Allocation Line

We now calculate the slopes of the Capital Allocation Line (CAL) for different portfolios.

5.1 Slope for the Value-Firm Portfolio

The slope for the value-firm portfolio is:

$$\text{slope}_{\text{CAL}} = \frac{|E[R_{\text{Value}}] - R_f|}{\sigma[R_{\text{Value}}]} = \frac{|1.23 - 0.39|}{5.67} = 0.1481$$

5.2 Slope for the S&P 500 Index

Similarly, the slope for the S&P 500 index is:

$$\text{slope}_{\text{CAL}} = \frac{|E[R_{\text{S\&P}}] - R_f|}{\sigma[R_{\text{S\&P}}]} = \frac{|0.94 - 0.39|}{4.38} = 0.1256$$

5.3 Slope for Jim's Portfolio

The expected return and standard deviation of Jim's portfolio are calculated as follows:

$$E[R_T] = 0.71463 \times 1.23 + 0.28537 \times 0.94 = 1.14724$$

$$\sigma[R_T] = 5.03737$$

Thus, the slope of the CAL for Jim's portfolio is:

$$\text{slope}_{\text{CAL}} = \frac{|E[R_T] - R_f|}{\sigma[R_T]} = \frac{|1.14724 - 0.39|}{5.03737} = 0.15033$$

5.4 Slope for John's Portfolio

The slope of the CAL for John's portfolio is the same as Jim's, 0.15033, since they hold the same risky-asset portfolio, differing only in their allocations.

6 Part E: Jim's Investment in T-bills

Jim has invested 75% of his portfolio in T-bills. The fraction of his portfolio invested in the value-stock portfolio is:

$$\omega_{\text{Value},p} = 0.25 \times 0.71463 = 0.17866$$

And the fraction of his portfolio invested in the S&P 500 index is:

$$\omega_{\text{S\&P},p} = 0.25 \times 0.28537 = 0.07134$$

Since John is less risk-averse than Jim, the fraction of John's portfolio invested in T-bills is less than Jim's 75%.

Lecture 4-5

1 Equal Weighted Portfolio of N Risky Assets

1.1 Example with N = 2

An equal weighted portfolio has 50% in each asset.

1.2 Same Expected Return and Standard Deviation

Suppose all the risky assets have the same expected return, $E[R] = \bar{R}$, and the same standard deviation of return, $\sigma[R] = \sigma$, with returns that are correlated.

1.2.1 1. What is the expected return on the portfolio?

The expected return on the portfolio is the weighted average of the expected returns of the assets:

$$E[R_p] = \frac{1}{N}\sum_{i=1}^{N} E[R_i] = \bar{R}$$

1.2.2 2. What is the standard deviation of the return on the portfolio?

The standard deviation of the return on the portfolio can be written as:

$$\sigma_p = \sqrt{\frac{\sigma^2}{N} + \frac{(N-1)\sigma^2\rho}{N}}$$

where ρ is the average correlation between asset returns.

1.2.3 3. Fixing N, what happens to the standard deviation as the average correlation varies?

Fixing N, lowering the correlations between asset returns improves diversification by reducing the variance (and hence the standard deviation) of the portfolio return without affecting the portfolio's expected return. An average correlation of zero substantially reduces the portfolio variance.

1.2.4 4. Fixing the average correlation, what happens to the standard deviation as N varies?

As N increases, holding the average correlation fixed, the expected portfolio return is unaffected, while the portfolio variance converges to the average pairwise correlation between the assets.

1.2.5 5. What happens when the average correlation is 1?

When all returns are perfectly positively correlated ($\rho = 1$), the standard deviation of the portfolio is:

$$\sigma_p = \sigma$$

Thus, there are no diversification benefits as N increases.

1.2.6 6. What happens when the average correlation is 0?

When all returns are uncorrelated ($\rho = 0$), the standard deviation of the portfolio is:

$$\sigma_p = \frac{\sigma}{\sqrt{N}}$$

Hence, diversification benefits increase as N increases.

1.2.7 7. What do these results tell us about the benefits of diversification?

The benefits of diversification increase with the number of assets in the portfolio and as correlations between asset returns decrease. When asset returns are perfectly positively correlated, there are no diversification benefits, whereas when they are uncorrelated, diversification can reduce variance toward zero as N increases.

1.3 Non-zero Covariances, Differing Expected Returns, and Standard Deviations

The expected return on the portfolio is the weighted average of the expected returns:

$$E[R_p] = \frac{1}{N} \sum_{i=1}^{N} E[R_i]$$

The variance of the portfolio return is:

$$\sigma_p^2 = \frac{\sigma^2}{N} + \frac{(N-1)\text{Cov}}{N}$$

where Cov is the average covariance between the assets.

As N increases, the portfolio variance depends on the average covariance between assets, implying that diversification benefits increase as the average covariance declines relative to the average variance of the assets.

2 Jim and John: 1-Month Investment Horizon

Suppose Jim and John have a 1-month investment horizon and access to N risky assets and a riskless asset. Jim is more risk averse than John.

2.1 A. Minimum Variance Frontier and Efficient Frontier

The minimum variance frontier is constructed by finding, for each possible expected portfolio return, the smallest possible standard deviation that can be attained by combining the N assets. The efficient frontier is the positive-sloped portion of the minimum variance frontier.

2.2 B. Adding Another Risky Asset

When another non-redundant risky asset is added to the N risky assets, the minimum variance frontier shifts to the left, reducing the minimum standard deviation for each possible expected portfolio return. The efficient frontier also shifts to the left in $\{E[R], \sigma[R]\}$ space.

2.3 C. Portfolios Chosen by Jim and John

Jim and John both choose to hold the same portfolio of risky assets in combination with the riskless asset. This portfolio is the tangency portfolio T, which lies on the minimum-variance frontier and has the steepest capital allocation line. Jim's portfolio has a smaller weight in the tangency portfolio than John's because Jim is more risk averse.