1.(h)

(0) sum: let $a_1+b_1\sqrt{2},\ a_2+b_2\sqrt{2} \in R$

$$(a_1+b_1\sqrt{2})+(a_2+b_2\sqrt{2}) = (a_1+a_2)+(b_1+b_2)\sqrt{2} \in R$$

(2) associativity let $a_1+b_1\sqrt{2},\ a_2+b_2\sqrt{2},\ a_3+b_3\sqrt{2} \in R$

$$(a_1+b_1\sqrt{2}) + \{(a_2+b_2\sqrt{2}) + (a_3+b_3\sqrt{2})\}$$
$$=(a_1+b_1\sqrt{2}) + \{(a_2+a_3)+(b_2+b_3)\sqrt{2}\}$$
$$=\{a_1+(a_2+a_3)\} + \{b_1+(b_2+b_3)\}\sqrt{2}$$
$$=\{(a_1+a_2)+a_3\} + \{(b_1+b_2)+b_3\}\sqrt{2}$$
$$=[(a_1+a_2)+(b_1+b_2)\sqrt{2}] + (a_3+b_3\sqrt{2})$$
$$=[(a_1+b_1\sqrt{2})+(a_2+b_2\sqrt{2})] + (a_3+b_3\sqrt{2}) \iff A+(B+C)=(A+B)+C$$

(1) identity for sum is 0 $\quad (a+b\sqrt{2})+0 = a+b\sqrt{2} = 0+(a+b\sqrt{2})$

(3) inverse for sum, if $a+b\sqrt{2} \in R$, then $-a-b\sqrt{2} = (-a)+(-b)\sqrt{2} \in R$ such that $(a+b\sqrt{2})+(-a-b\sqrt{2}) = 0 \quad \forall\ a+b\sqrt{2} \in R$

(4) commutative: $(a_1+b_1\sqrt{2})+(a_2+b_2\sqrt{2})$

$$=(a_1+a_2)+(b_1+b_2)\sqrt{2}$$
$$=(a_2+a_1)+(b_2+b_1)\sqrt{2}$$
$$=(a_2+b_2\sqrt{2})+(a_1+b_1\sqrt{2})$$

product $(a_1+b_1\sqrt{2})\times(a_2+b_2\sqrt{2}) = a_1a_2 + a_1b_2\sqrt{2} + b_1a_2\sqrt{2} + 2\cdot b_1b_2$

$$=(a_1a_2+2b_1b_2)+(a_1b_2+b_1a_2)\sqrt{2} \in R$$

(5) identity for product is '1' to satisfy

$$1\times(a+b\sqrt{2})\times = (a+b\sqrt{2})\times 1 = (a+b\sqrt{2})$$

(9) multiplicative inverse let $a+b\sqrt{2} \in R$ ($a\neq 0$ or/and $b\neq 0$) be any non-zero element of R

$$(a+b\sqrt{2})^{-1} = \frac{1}{a+b\sqrt{2}}$$
$$=\frac{1}{a+b\sqrt{2}}\ \frac{a-b\sqrt{2}}{a-b\sqrt{2}}$$
$$=\frac{a-b\sqrt{2}}{a^2-(b\sqrt{2})^2}$$
$$=\frac{a-b\sqrt{2}}{a^2-2b^2} = \left(\frac{a}{a^2-2b^2}\right)+\left(\frac{-b}{a^2-2b^2}\right)\sqrt{2} \in R$$

(6) $[(a_1+b_1\sqrt{2})\times(a_2+b_2\sqrt{2})]\times(a_3+b_3\sqrt{2}) = (a_1+b_1\sqrt{2})\times[(a_2+b_2\sqrt{2})\times(a_3+b_3\sqrt{2})]$

(8) commutivity for product let $a_1+b_1\sqrt{2},\ a_2+b_2\sqrt{2} \in R$

$$(a_1+b_1\sqrt{2})\times(a_2+b_2\sqrt{2}) = (a_1a_2+2b_1b_2)+(a_1b_2+b_1a_2)\sqrt{2}$$
$$=(a_2a_1+2b_2b_1)+(b_2a_1+a_2b_1)\sqrt{2}$$
$$=(a_2+b_2\sqrt{2})\times(a_1+b_1\sqrt{2})$$

(7) distributivity is considered and could be proven right

$$(a_1+b_1\sqrt{2})\times[(a_2+b_2\sqrt{2})+(a_3+b_3\sqrt{3})] = (a_1+b_1\sqrt{2})\times(a_2+b_2\sqrt{2}) + (a_1+b_1\sqrt{2})\times(a_3+b_3\sqrt{3})$$
$$[(a_2+b_2\sqrt{2})+(a_3+b_3\sqrt{2})]\times(a_1+b_1\sqrt{2}) = (a_2+b_2\sqrt{2})\times(a_1+b_1\sqrt{2}) + (a_3+b_3\sqrt{2})\times(a_2+b_2\sqrt{2})$$

(2)

any element in F is of the form $a+b\sqrt{-1}$,

Sum: $(a_1+b_1\sqrt{-1})+(a_2+b_2\sqrt{-1}) = (a_1+a_2)+(b_1+b_2)\sqrt{-1}$

product: $(a_1+b_1\sqrt{-1})\times(a_2+b_2\sqrt{-1}) = a_1a_2 + a_1b_2\sqrt{-1}, a_2b_1\sqrt{-1} + b_1b_2(\sqrt{-1})^2$
$= (a_1a_2-b_1b_2)+(a_1b_2+a_2b_1)\sqrt{-1}$

(5) identity for product is 1 to satisfy $1\times(a+b\sqrt{-1}) = a+b\sqrt{-1} = (a+b\sqrt{-1})\times 1$

(1) identity for sum is 0 since $0+(a+b\sqrt{-1}) = a+b\sqrt{-1} = (a+b\sqrt{-1})+0$

(3) inverse for sum: for each element $a+b\sqrt{-1}\in F$, the inverse for sum is $-a-b\sqrt{-1}$ because $a+b\sqrt{-1}+(-a-b\sqrt{-1})=0$

(9) multiplicative inverse $(a+b\sqrt{-1})^{-1} = \frac{1}{a+b\sqrt{-1}}$

$\forall\, a+b\sqrt{-1}\in C$,

$\exists (a+b\sqrt{-1})^{-1}\in C$

$$= \frac{1}{a+b\sqrt{-1}}\cdot\frac{a-b\sqrt{-1}}{a-b\sqrt{-1}}$$
$$= \frac{a-b\sqrt{-1}}{a^2+b^2}$$
$$= \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}\sqrt{-1} \in C$$

follow the same logic and proof in 1(c)

(2) associativity for sum: $(a_1+b_1\sqrt{-1})+[(a_2+b_2\sqrt{-1})+(a_3+b_3\sqrt{-1})] = [(a_1+b_1\sqrt{-1})+(a_2+b_2\sqrt{-1})] + (a_3+b_3\sqrt{-1})$

(4) commutativity: $(a_1+b_1\sqrt{-1})+(a_2+b_2\sqrt{-1}) = (a_2+b_2\sqrt{-1})+(a_1+b_1\sqrt{-1})$

(7) distributivity: could be proven right

$(a_1+b_1\sqrt{-1})\times[(a_2+b_2\sqrt{-1})+(a_3+b_3\sqrt{-1})] = (a_1+b_1\sqrt{-1})\times(a_2+b_2\sqrt{-1}) + (a_1+b_1\sqrt{-1})\times(a_3+b_3\sqrt{-1})$

$[(a_2+b_2\sqrt{-1})+(a_3+b_3\sqrt{-1})]\times(a_1+b_1\sqrt{-1}) = (a_2+b_2\sqrt{-1})\times(a_1+b_1\sqrt{-1}) + (a_3+b_3\sqrt{-1})\times(a_1+b_1\sqrt{-1})$

(6) association $[(a_1+b_1\sqrt{-1})\times(a_2+b_2\sqrt{-1})]\times(a_3+b_3\sqrt{-1})$
$= (a_1+b_1\sqrt{-1})\times[(a_2+b_2\sqrt{-1})\times(a_3+b_3\sqrt{-1})]$

(8) $(a_1+b_1\sqrt{-1})\times(a_2+b_2\sqrt{-1}) = (a_2+b_2\sqrt{-1})\times(a_1+b_1\sqrt{-1})$

the field is $\mathbb{C}$

2. for example $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $AB \neq BA \Rightarrow$ not communitive

so it is not a field

Q3 $\mathbb{Z}_3 = \{[0], [1], [2]\}$

$+$	$[0]$	$[1]$	$[2]$
$[0]$	$[0]$	$[1]$	$[2]$
$[1]$	$[1]$	$[2]$	$[0]$
$[2]$	$[2]$	$[0]$	$[1]$

$\times$	$[0]$	$[1]$	$[2]$
$[0]$	$[0]$	$[0]$	$[0]$
$[1]$	$[0]$	$[1]$	$[2]$
$[2]$	$[0]$	$[2]$	$[1]$

4.

sum: let a_1+b_1i, $a_2+b_2i \in \mathbb{C}$, such that

$(a_1+b_1i)+(a_2+b_2i)$

$=(a_1+a_2)+(b_1+b_2)i \quad \in \mathbb{C}$

product $(a_1+b_1i)(a_2+b_2i) = (a_1a_2-b_1b_2)+(a_1b_2+a_2b_1)i$

① identity for sum: $e=0$ $(a+bi)+e = a+bi = e+(a+bi)$

⑤ identity for product is 1 to satisfy $1\times(a+bi) = a+bi$

③ inverse of sum: for each element $a+bi$ in the field, the inverse of sum is $-a-bi$, since $a+bi+(-a-bi)=0$

② $[(a_1+b_1i)+(a_2+b_2i)]+(a_3+b_3i) = a_1+b_1i+[(a_2+b_2i)+(a_3+b_3i)]$

④ $(a_1+b_1i)+(a_2+b_2i)=(a_2+b_2i)+(a_1+b_1i)$

⑤ $e'=1$, $e'\times(a+bi)= a+bi=(a+bi)\times e'$

⑥ $[(a_1+b_1i)\times(a_2+b_2i)]\times(a_3+b_3i) = (a_1+b_1i)\times[(a_2+b_2i)\times(a_3+b_3i)]$

⑦ distributive: same logic

⑧ $(a_1+b_1i)\times(a_2+b_2i) = (a_2+b_2i)\times(a_1+b_1i)$

⑨ $\forall\, a+bi \in \mathbb{C}$, $\exists\, (a+bi)^{-1} \in \mathbb{C}$

$(a+bi)^{-1} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$ same logic

$\vdash(2)$

5 A is not because not all entries above leading 1 are 0's

B is rref

C is not since all zero rows do not lie beneath all non-zero rows

D is rref

E is not since

6. $A+B=\begin{bmatrix}1&2&2\\1&2&2\\1&2&0\end{bmatrix}$ $=\begin{bmatrix}[1]&[0]&[0]\\ [1]&[0]&[0]\\ [1]&[0]&[0]\end{bmatrix}$

$A^2=\begin{bmatrix}1&3&2\\0&2&1\\0&1&1\end{bmatrix}$ $=\begin{bmatrix}[1]&[1]&[0]\\ [0]&[0]&[1]\\ [0]&[1]&[1]\end{bmatrix}$

$AB=\begin{bmatrix}2&3&2\\2&2&1\\1&1&1\end{bmatrix}=\begin{bmatrix}[0]&[1]&[0]\\ [0]&[0]&[1]\\ [1]&[1]&[1]\end{bmatrix}$

7 the ref of A is $\begin{bmatrix}t&0&1\\0&-1&\frac{t-6}{t}\\0&0&\frac{t^2+t-6}{t}\end{bmatrix}$ $\Rightarrow\begin{cases}t\neq0\\t^2+t-6\neq0\end{cases}\Rightarrow\begin{cases}t\neq0\\t\neq2\\t\neq-3\end{cases}$

8. (a) the rref is $\begin{pmatrix}1&0&\frac{8h-24}{3(a-2)}\\0&1&\frac{4}{3(h-2)}\end{pmatrix}$, h can be any number other than 2

(b) the rref is $\begin{pmatrix}1&-3&-\frac{h}{4}\\0&0&\frac{-6+h}{2}\end{pmatrix}$, $-6+h=0$ $h=6$, h can be any number other than 6

9 (1) 4 types $\begin{bmatrix}0&0\\0&0\\0&0\end{bmatrix}$, $\begin{bmatrix}1&x\\0&0\\0&0\end{bmatrix}$ $\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$ $\begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix}$

(2) 7 types $\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}$, $\begin{bmatrix}1&x&x\\0&0&0\end{bmatrix}$, $\begin{bmatrix}1&0&x\\0&1&x\end{bmatrix}$, $\begin{bmatrix}1&x&0\\0&0&1\end{bmatrix}$, $\begin{bmatrix}0&1&x\\0&0&0\end{bmatrix}$ $\begin{bmatrix}0&1&x\\0&0&1\end{bmatrix}$ $\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}$

(3) $\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$ $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$

10. a = any b=0, c=1 d=0, e=0

11. (1) $\begin{bmatrix}1&2&3&4\\1&1&0&2\\2&0&1&2\end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix}1&2&3&4\\0&-1&-3&-2\\2&0&1&2\end{bmatrix} \xrightarrow{R_3-2R_1} \begin{bmatrix}1&2&3&4\\0&-1&-3&-2\\0&-4&-5&-6\end{bmatrix} \xrightarrow{(-1)\cdot R_2} \begin{bmatrix}1&2&3&4\\0&1&3&2\\0&-4&-5&-6\end{bmatrix}$

$\xrightarrow{R_1-2R_2} \begin{bmatrix}1&0&-3&0\\0&1&3&2\\0&-4&-5&-6\end{bmatrix} \xrightarrow{R_3+4R_2} \begin{bmatrix}1&0&-3&0\\0&1&3&2\\0&0&7&2\end{bmatrix} \xrightarrow{\frac{1}{7}\cdot R_3} \begin{bmatrix}1&0&-3&0\\0&1&3&2\\0&0&1&\frac{2}{7}\end{bmatrix} \xrightarrow{R_1+3R_3} \begin{bmatrix}1&0&0&\frac{6}{7}\\0&1&3&2\\0&0&1&\frac{2}{7}\end{bmatrix}$

$\xrightarrow{R_2-3R_3} \begin{bmatrix}1&0&0&\frac{6}{7}\\0&1&0&\frac{8}{7}\\0&0&1&\frac{2}{7}\end{bmatrix}$ solution $\vec{x} = x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\\0\end{bmatrix} + x_3\begin{bmatrix}0\\0\\1\end{bmatrix} + x_4\begin{bmatrix}\frac{6}{7}\\\frac{8}{7}\\\frac{2}{7}\end{bmatrix}$

(2) $\begin{bmatrix}1&2&3&4\\1&1&0&2\\2&0&1&2\end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix}0&1&3&2\\1&1&0&2\\2&0&1&2\end{bmatrix} \xleftrightarrow{R_1\leftrightarrow R_2} \begin{bmatrix}1&1&0&2\\0&1&3&2\\2&0&1&2\end{bmatrix} \xrightarrow{R_3-2R_1} \begin{bmatrix}1&1&0&2\\0&1&3&2\\0&-2&1&-2\end{bmatrix}$

$\xrightarrow{R_3+2R_2} \begin{bmatrix}1&1&0&2\\0&1&3&2\\0&0&0&2\end{bmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{bmatrix}1&1&0&2\\0&1&3&2\\0&0&0&1\end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix}1&0&4&0\\0&1&3&2\\0&0&0&1\end{bmatrix} \xrightarrow{R_2-2R_3} \begin{bmatrix}1&0&4&0\\0&1&3&0\\0&0&0&1\end{bmatrix}$.

(3) in Z_2, we have $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}$ for rref of A

in Z_3, we have $\begin{bmatrix}1&0&0&0\\0&1&0&2\\0&0&1&2\end{bmatrix}$ for rref of A.

(4) no. they have the same rank.

Q12

(1) $\left(\begin{array}{ccc|c} 3 & 1 & 4 & 1 \\ 5 & 2 & 6 & 5 \\ 0 & 5 & 2 & 1 \end{array}\right) \xrightarrow{R_1+R_2} \left(\begin{array}{ccc|c} 1 & 3 & 3 & 6 \\ 5 & 2 & 6 & 5 \\ 0 & 5 & 2 & 1 \end{array}\right) \xrightarrow{R_2+2R_1} \left(\begin{array}{ccc|c} 1 & 3 & 3 & 6 \\ 0 & 1 & 5 & 3 \\ 0 & 5 & 2 & 1 \end{array}\right)$

$\xrightarrow{3R_3} \left(\begin{array}{ccc|c} 1 & 3 & 3 & 6 \\ 0 & 1 & 5 & 3 \\ 0 & 1 & 6 & 3 \end{array}\right) \xrightarrow{R_3-R_2} \left(\begin{array}{ccc|c} 1 & 3 & 3 & 6 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_2-5R_3} \left(\begin{array}{ccc|c} 1 & 3 & 3 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right)$

$\xrightarrow{R_1-3R_3} \left(\begin{array}{ccc|c} 1 & 3 & 0 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow{R_1-3R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right)$

(2) $x_3 = 0$, $x_2 = 3$, $x_1 = 4$, $\vec{x} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$

13. from python the rref is $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$, it is inconsistent, then we have no solution

14. from python, the rref is $\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 0 & 5 & 6 \\ 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 + 2x_2 + 3x_3 + 5x_5 = 6$

$x_4 + 2x_5 = 7$

x_2, x_3, x_5 are arbitrary

$x_1 = 6 - 2x_2 - 3x_3 - 5x_5$

$x_4 = 7 - 2x_5$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 - 2x_2 - 3x_3 - 5x_5 \\ x_2 \\ x_3 \\ 7 - 2x_5 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \\ 0 \\ 7 \\ 0 \end{bmatrix}$$

15. from python, the rref is $\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -8221/4340 \\ 0 & 1 & 0 & 0 & 0 & 8591/8680 \\ 0 & 0 & 1 & 0 & 0 & 4695/434 \\ 0 & 0 & 0 & 1 & 0 & -459/434 \\ 0 & 0 & 0 & 0 & 1 & 699/434 \end{array}\right]$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -8221/4340 \\ 8591/8680 \\ 4695/434 \\ -459/434 \\ 699/434 \end{bmatrix}$$

16. (1) since $ABC = I$, then we have $A(BC) = I$ by associative property, so BC is an inverse of A, so A is invertible.

follow the same logic C is invertible and the inverse is AB

B is invertible and the inverse is CA

(2) Let $C = B(AB)^{-1}$ and $D = (AB^{-1})A$, then

$$AC = A(B(AB)^{-1}) = (AB)(AB)^{-1} = I$$

$$DB = ((AB)^{-1}A)B = (AB)^{-1}(AB) = I$$

$\therefore$ A and B are invertible

17 $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$, from python we have $(AB)^2 = \begin{bmatrix} 6 & 8 \\ 12 & 22 \end{bmatrix}$

and $A^2B^2 = \begin{bmatrix} 6 & 0 \\ 24 & 6 \end{bmatrix}$, $(AB)^2 \neq A^2B^2$

18 $A = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ $A^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ $A^{-1} = -1\begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

19 (1) Symmetric: 2×2
$$A=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$$
3×3
$$A=\begin{bmatrix}1 & 2 & 4\\ 2 & 3 & 6\\ 4 & 6 & 5\end{bmatrix}$$
4×4
$$A=\begin{bmatrix}1 & 2 & 3 & 4\\ 2 & 5 & 6 & 7\\ 3 & 6 & 8 & 9\\ 4 & 7 & 9 & 10\end{bmatrix}$$

Skew-symmetric 2×2
$$A=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$
3×3
$$A=\begin{bmatrix}0 & 1 & 2\\ -1 & 0 & 3\\ -2 & -3 & 0\end{bmatrix}$$
4×4
$$A=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{bmatrix}$$

(2) the main diagonal entries of the skewed-symmetric matrix are zeros.

(3) $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$.

(4)

$A+A^T$ is symmetric. Let $(A+A^T)=B$
$$B^T=(A+A^T)^T=A^T+(A^T)^T$$
$$=A^T+A$$
$$\therefore B^T=B \Rightarrow A+A^T \text{ is symmetric}$$

AA^T is symmetric Let $AA^T=B$
$$B^T=(AA^T)^T=(A^T)^T\cdot A^T$$
$$=A\cdot A^T$$
$$\therefore B^T=B \Rightarrow AA^T \text{ is symmetric.}$$

19(4) continued

$\hookrightarrow$ A^TA is symmetric:

Let $B = A^TA$, $B^T = (A^TA)^T = A^T \cdot (A^T)^T$

$= A^T \cdot A$

$\therefore B^T = B \Rightarrow A^TA$ is symmetric

$A - A^T$ is skew-symmetric

Let $B = A - A^T$, $B^T = (A - A^T)^T = A^T - (A^T)^T = A^T - A$

$= -(A - A^T)$

$\therefore B^T = -B \Rightarrow A - A^T$ is skew-symmetric.

(5) Let A be the $n \times n$ matrix, we could have

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

in Q4. we have prove than $(A + A^T)$ is the symmetric matrix and $(A - A^T)$ is the skew-symmetric then we find that any $n \times n$ can be written as the sum of a symmetric and skew symmetric matrices.

20. (a) surjective (b) bijective (c) surjective (d) injective.

22

$$LU = \begin{bmatrix} d_1 & u_1 & 0 & 0 \\ l_1 d_1 & l_1 u_1 + d_2 & u_2 & 0 \\ 0 & l_2 d_2 & l_2 u_2 + d_3 & u_3 \\ 0 & 0 & l_3 d_3 & l_3 u_3 + d_4 \end{bmatrix}$$

$a_n = l_{n-1} u_{n-1} + d_n$

$r_n = u_n$

$p_n = l_{n-1} d_{n-1}$

21 from the Q22,

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ 0 & \frac{4}{15} & 1 & 0 \\ 0 & 0 & \frac{15}{56} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & \frac{15}{4} & 1 & 0 \\ 0 & 0 & \frac{56}{15} & 1 \\ 0 & 0 & 0 & \frac{209}{56} \end{bmatrix}$$

$d_1 = 4 \quad l_1 d_1 = 1, \quad l_1 = \frac{1}{4}$

$l_1 u_1 + d_2 = 4$

$\Rightarrow \frac{1}{4} \cdot 1 + d_2 = 4$

$l_2 d_2 = 1 \Rightarrow l_2 = \frac{4}{15}$

$\frac{4}{15} \cdot 1 + d_3 = 4$

$d_3 = \frac{56}{15}$

Q23 from the Q21 and Q22

$$A=\begin{bmatrix} q_1 & r_1 & 0 & 0 & \cdots & 0 \\ p_1 & q_2 & r_2 & & & 0 \\ \vdots & p_2 & q_3 & \ddots & & \vdots \\ \vdots & & p_3 & \ddots & \ddots & r_{n-1} \\ 0 & 0 & \cdots & & p_n & q_n \end{bmatrix}$$

$$LU=\begin{bmatrix} d_1 & u_1 & & & \\ l_1 d_1 & l_1 u_1 + d_2 & u_2 & & \\ & l_2 d_2 & l_2 u_2 + d_3 & & \\ & & l_3 d_3 & \ddots & \\ & & & \ddots & u_{n-1} \\ & & & l_{n-1} d_{n-1} & l_{n-1} u_{n-1} + d_n \end{bmatrix}$$

$u_1 = r_1 \qquad l_1 = \frac{p_1}{d_1} = \frac{p_1}{q_1}$

$q_2 = l_1 u_1 + d_2 \Rightarrow d_2 = q_2 - l_1 u_1$

$= q_2 - \frac{p_1}{q_1} r_1$

$p_2 = l_2 d_2 \Rightarrow l_2 = \frac{p_2}{d_2}$

$$\begin{cases} u_i = r_i \\ d_1 = q_1 = 4 \\ l_t = \dfrac{p_t}{d_t} \quad \text{for } t = 1, 2, \cdots \\ d_t = q_{t-1} - l_{t-1} u_{t-1} \quad t = 2, 3, \cdots \end{cases}$$

in this case

$$\begin{cases} u_t = r_t = 1 \\ l_t = \dfrac{1}{d_t} \\ d_t = 4 - \dfrac{1}{d_{t-1}} \end{cases}$$

24 (1) $(H_n)^T = (I_n - 2\vec{u}\vec{u}^T)^T$

$= I_n^T - 2(\vec{u}\vec{u}^T)^T$

$= I - 2(\vec{u}^T)^T\vec{u}^T$

$= I - 2\vec{u}\vec{u}^T$

$\therefore (H_n)^T = H_n \Rightarrow H_n$ is a symmetric matrix

(2) $H_n^T H_n = H_n H_n$

$= (I_n - 2\vec{u}\vec{u}^T)(I_n - 2\vec{u}\vec{u}^T)$

$= I_n - 2\vec{u}\vec{u}^T - 2\vec{u}\vec{u}^T + 4(\vec{u}\vec{u}^T)(\vec{u}\vec{u}^T)$

$= I_n - 4\vec{u}\vec{u}^T + 4\vec{u}(\vec{u}^T\vec{u})\vec{u}^T$

$= I_n - 4\vec{u}\vec{u}^T + 4\vec{u}\vec{u}^T$ since $\vec{u}^T\vec{u} = 1$

$= I_n$

$\therefore H_n$ is an orthogonal matrix

(3)

according to (2), we figure out that $H_n^2 = H_n \cdot H_n = I_n$

(4) $H_n\vec{u} = (I_n - 2\vec{u}\vec{u}^T)\cdot\vec{u}$

$= I_n\vec{u} - 2\vec{u}\vec{u}^T\vec{u}$

$= I_n\vec{u} - 2\vec{u} = -\vec{u}$

(5) when $n=3$, $\vec{u} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$

$H_3 = I_3 - 2\vec{u}\vec{u}^T = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - 2\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix}1&1&1\end{bmatrix}$

$= \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \frac{2}{3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$

$= \begin{bmatrix}\frac{1}{3}&-\frac{2}{3}&-\frac{2}{3}\\-\frac{2}{3}&\frac{1}{3}&-\frac{2}{3}\\-\frac{2}{3}&-\frac{2}{3}&\frac{1}{3}\end{bmatrix}$

when $n=4$, $\vec{u} = \frac{1}{\sqrt{4}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

$H_4 = I_4 - 2\vec{u}\vec{u}^T = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} - 2\cdot\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\cdot\frac{1}{2}\begin{bmatrix}1&1&1&1\end{bmatrix}$

$= \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}$

$= \begin{bmatrix}\frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}\\-\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\-\frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\end{bmatrix}$