

First Worksheet, MATH 7233

Introduction

September 8, 2021

1. In a small Graph Theory class with 9 students everyone was asked how many of their friends are also taking the class. Friendship is mutual. Which of the following sets of answers could possibly be the actual outcome? All three? Perhaps none?
 - * (a) 7,7,7,6,6,6,5,5,5?
 - * (b) 6,6,5,4,4,3,2,2,1?
 - (c) 7,7,7,6,3,3,3,2,2?
- * 2. Show that at a party there have to be two participants who know exactly the same amount of other people. (Acquaintance is mutual.)
3. Middle-Earth - after the global warming induced by the One Ring being thrown into the abyss of Mt Doom - consisted of 1000 little islands. Two airlines provided transportation between the islands: ElvenAir and AirDwarf. Between any two islands there was a single, daily non-stop flight in both directions, operated by one of the two airlines. But which islands were connected by which airlines was a mess.

Gandalf wanted join the loyalty program of one of the airlines, and then stick to that one. At the same time he wanted to be able to travel freely between all the islands. Fortunately he didn't mind multiple layovers. Can we be certain that he was able to pull it off?
4. Suppose all 30 students in the Graph Theory class have at least 5 friends in the class. Again, friendship is mutual. Some students line up in a row. We say this line is *good* if everybody is standing next to their friends. (So people in the middle stand between 2 of their friends, while people at the end next to one of their friends.) Let g denote the number of students in the longest possible good line that exists in this class.

What is the smallest possible value of g , and why?
5. In last year's Graph Theory class, it turned out that each student had exactly 3 friends there. Furthermore, when two students weren't friends, they had exactly one mutual friend. However, no 3 students were all friends with each other. How many people were there in the class?
6. Show that, given any 6 integers, there are either 3 of them that are pairwise coprime, or 3 of them that are pairwise not coprime. (Two integers are *coprime* if they have no common primes in their factorization.) Is the same true for 5 integers?
7. $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = ?$