Math 4310 HW 1.

#1. 1). $e=0$,

$e+a+b\sqrt{2} = a+b\sqrt{2}+e = a+b\sqrt{2}$

2). $(a_1+b_1\sqrt{2}+a_2+b_2\sqrt{2})+(a_3+b_3\sqrt{2}) = (a_1+a_2+a_3)+b_1\sqrt{2}+b_2\sqrt{2}+b_3\sqrt{2}$

$= a_1+b_1\sqrt{2}+(a_2+b_2\sqrt{2}+a_3+b_3\sqrt{2})$

3). $a+b\sqrt{2}+(-a-b\sqrt{2})=0$, $-(a+b\sqrt{2}) = -a-b\sqrt{2}$

4). $(a_1+b_1\sqrt{2})+(a_2+b_2\sqrt{2}) = (a_2+b_2\sqrt{2})+(a_1+b_1\sqrt{2})$

5). when $e'=1$,

$e'\times(a+b\sqrt{2}) = (a+b\sqrt{2})\times e' = a+b\sqrt{2}$

6). $[(a_1+b_1\sqrt{2})\times(a_2+b_2\sqrt{2})]\times[(a_3+b_3\sqrt{2})]$

$= (a_1+b_1\sqrt{2})\times[(a_2+b_2\sqrt{2})\times(a_3+b_3\sqrt{2})]$

7). $(a_1+b_1\sqrt{2})\times(a_2+b_2\sqrt{2}+a_3+b_3\sqrt{2})$

$= (a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2})+(a_1+b_1\sqrt{2})(a_3+b_3\sqrt{2})$

$(a_2+b_2\sqrt{2}+a_3+b_3\sqrt{2})(a_1+b_1\sqrt{2})$

$= (a_2+b_2\sqrt{2})(a_1+b_1\sqrt{2})+(a_3+b_3\sqrt{2})(a_1+b_1\sqrt{2})$

8) $(a_1+b_1\sqrt{2})\times(a_2+b_2\sqrt{2}) = a_1a_2+a_1b_2\sqrt{2}+b_1a_2\sqrt{2}+2b_1b_2$

$= a_1a_2+2b_1b_2+(a_1b_2+b_1a_2)\sqrt{2}$

9). $\forall\, a+b\sqrt{2}\in F$, $\exists\, x+y\sqrt{2}\in F$,

s.t. $(a+b\sqrt{2})(x+y\sqrt{2})=1$

$(ax+2by)+(ay+bx)\sqrt{2}=1$

$ax+2by=1$

$ay+bx=0$

$ay=-bx$

$a=-\frac{bx}{y}$

$-\frac{bx}{y}+2by=1$

$-bx+2by^2-y=0$

$$\begin{cases} x = \frac{a}{a^2-2b^2} \\ y = -\frac{b}{a^2-2b^2} \end{cases}$$

(2). 1) $e=0$,

$e+a+b\sqrt{-1} = a+b\sqrt{-1}+e = a+b\sqrt{-1}$

2) $(a_1+b_1\sqrt{-1}+a_2+b_2\sqrt{-1})+(a_3+b_3\sqrt{-1}) = a_1+b_1\sqrt{-1}+(a_2+b_2\sqrt{-1}+a_3+b_3\sqrt{-1})$

3). $a+b\sqrt{-1}+(-a-b\sqrt{-1})=0, \ -a-b\sqrt{-1} = -(a+b\sqrt{-1})$

4). $(a_1+b_1\sqrt{-1})+(a_2+b_2\sqrt{-1}) = (a_2+b_2\sqrt{-1})+(a_1+b_1\sqrt{-1})$

5). when $e'=1$,

$e'\times(a+b\sqrt{-1}) = (a+b\sqrt{-1})\times e' = a+b\sqrt{-1}$

6). $[(a_1+b_1\sqrt{-1})\times(a_2+b_2\sqrt{-1})]\times[(a_3+b_3\sqrt{-1})]$

$=(a_1+b_1\sqrt{-1})\times[(a_2+b_2\sqrt{-1})\times(a_3+b_3\sqrt{-1})]$

7). $(a_1+b_1\sqrt{-1})\times(a_2+b_2\sqrt{-1}+a_3+b_3\sqrt{-1})$

$=(a_2+b_2\sqrt{-1})\cdot(a_1+b_1\sqrt{-1})+(a_3+b_3\sqrt{-1})(a_1+b_1\sqrt{-2})$

8). $(a_1+b_1\sqrt{-1})\times(a_2+b_2\sqrt{-1}) = a_1a_2+a_1b_2\sqrt{-1}+a_2b_1\sqrt{-1}-b_1b_2$

$= a_1a_2-b_1b_2+(a_1b_2+a_2b_1)\sqrt{-1}$

9). $\forall a+b\sqrt{-1}\in\mathbb{C}, \ \exists x+y\sqrt{-1}\in\mathbb{C}$,

s.t. $(a+b\sqrt{-1})(x+y\sqrt{-1})=1$

$ax+ay\sqrt{-1}+xb\sqrt{-1}-by=1$

$ax-by+(ay+xb)\sqrt{-1}=1$

$\Rightarrow \begin{cases} ax-by=1 \\ ay+xb=0 \end{cases}$

$\begin{cases} x=\dfrac{a^3}{a^2(a^2+b^2)} = \dfrac{a}{a^2+b^2} \\ y=-\dfrac{ab}{a(a^2+b^2)} = -\dfrac{b}{a^2+b^2} \end{cases}$

$$\left[\begin{array}{cc|c} a & -b & 1 \\ b & a & 0 \end{array}\right] \xrightarrow[\frac{1}{b}R_2]{\frac{1}{a}R_1} \left[\begin{array}{cc|c} 1 & -\frac{b}{a} & \frac{1}{a} \\ 1 & \frac{a}{b} & 0 \end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{cc|c} 1 & -\frac{b}{a} & \frac{1}{a} \\ 0 & \frac{a^2+b^2}{ab} & -\frac{1}{a} \end{array}\right] \xrightarrow{\frac{ab}{a^2+b^2}R_2}$$

$$\left[\begin{array}{cc|c} 1 & -\frac{b}{a} & \frac{1}{a} \\ 0 & 1 & -\frac{ab}{a(a^2+b^2)} \end{array}\right] \xrightarrow{R_1+\frac{b}{a}R_2} \left[\begin{array}{cc|c} 1 & 0 & \frac{a^3}{a^2(a^2+b^2)} \\ 0 & 1 & -\frac{ab}{a(a^2+b^2)} \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & \frac{a^{\cancel{2}}}{\cancel{a}a^2+b^2} \\ 0 & 1 & -\frac{b}{a^2+b^2} \end{array}\right]$$

in field $\mathbb{C}$

#2. Because $AB \neq BA$ for matrices, thus the set of all $n\times n$ matrices $\mathbb{R}^{n\times n}$ is not a field.

$[6] = [3] = [0] = \{0, \pm 3, \pm 6, \dots\}$

$[1] = \{1, 1\pm 3, 1\pm 6, \dots\}$

$[2] = \{2, 2\pm 3, 2\pm 6, \dots\}$

#3.

+	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

x	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

#4. ① $e + a + b\sqrt{-1} = a + b\sqrt{-1} + e = a + b\sqrt{-1}$

②. $(a_1 + b_1\sqrt{-1}) + a_2 + b_2\sqrt{-1} + (a_3 + b_3\sqrt{-1}) = a_1 + b_1\sqrt{-1} + (a_2 + b_2\sqrt{-1} + a_3 + b_3\sqrt{-1})$

③. $a + b\sqrt{-1} + (-a - b\sqrt{-1}) = 0$, $(-a - b\sqrt{-1}) = -(a + b\sqrt{-1})$

④. $(a_1 + b_1\sqrt{-1}) + (a_2 + b_2\sqrt{-1}) = (a_2 + b_2\sqrt{-1}) + (a_1 + b_1\sqrt{-1})$

⑤. multiplicative identity:

⑥. $\forall\, a + b\sqrt{-1} \in \mathbb{C}, \exists\, x + y\sqrt{-1} \in \mathbb{C},$

$$\text{s.t. } (a + b\sqrt{-1})(x + y\sqrt{-1}) = 1$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{a}{a^2+b^2} \\ 0 & 1 & -\frac{b}{a^2+b^2} \end{array}\right]$$

#5. B D $\quad \mathbb{Z}_2$: $[0] = [2] = \{0, \pm 2, \pm 4, \dots\}$ $\quad [1] = \{1, 1\pm 2, 1\pm 4, \dots\}$

#6. $A + B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} [1] & [0] & [0] \\ [1] & [0] & [0] \\ [1] & [0] & [0] \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} [1] & [1] & [0] \\ [0] & [0] & [1] \\ [0] & [1] & [1] \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} [0] & [1] & [0] \\ [0] & [0] & [1] \\ [1] & [1] & [1] \end{bmatrix}$

#7. $\det A = 0$

$$\begin{vmatrix} 6 & -1 & 1 \\ t & 0 & 1 \\ 0 & 1 & t \end{vmatrix} = -t \begin{vmatrix} -1 & 1 \\ 1 & t \end{vmatrix} - \begin{vmatrix} 6 & -1 \\ 0 & 1 \end{vmatrix}$$

$$= -t(-t-1) - 6$$

$$t^2 + t - 6 = 0$$

$$(t+3)(t-2) = 0$$

$$t = -3, 2$$

#8. a). $\left[\begin{array}{cc|c} 1 & h & 4 \\ 3 & 6 & 8 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & h & 4 \\ 0 & 6-3h & -4 \end{array}\right]$ $h \neq 2$

$6-3h = 0$

$h = 2$

b). $\left[\begin{array}{cc|c} -4 & 12 & h \\ 2 & -6 & -3 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 2 & -6 & -3 \\ -4 & 12 & h \end{array}\right] \xrightarrow{2R_1 + R_2} \left[\begin{array}{cc|c} 2 & -6 & -3 \\ 0 & 0 & -6+h \end{array}\right]$

$-6 + h \neq 0$

$h \neq 6$

#9. 2). ① $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}$ ② $\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ③ $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ④ $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

⑤ $\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}$ ⑥ $\begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$ ⑦ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

7 types For 2x3 matrices

1). $\begin{bmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ 4 types for 3×2 matrices

3) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

#10. $a = 0 = b = e = d$, $c = 1$

#11. b).
$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow[R1-R3]{R2-R3} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}R3} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

c) $\mathbb{Z}_2$: $[0] = \{0, \pm2, \pm4, \dots\}$ $\quad [1] = \{1, 1\pm2, 1\pm4, \dots\}$

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\mathbb{Z}_3$: $[0] = \{0, \pm3, \pm6, \dots\}$ $\quad [1] = \{1, 1\pm3, 1\pm6, \dots\}$

$[2] = \{2, 2\pm3, 2\pm6, \dots\}$

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix}$$

d) No.

#12.
$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ (According to python) $\Rightarrow$ $x = 4$, $y = 3$, $z = 0$ $\quad \vec{x} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$

#13.
$$\begin{array}{c} \quad\quad t \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$
no solution.

#14.
$$\begin{array}{ccccc|c} x_1 & t & s & x_4 & u & \\ \hline 1 & 2 & 3 & 0 & 5 & 6 \\ 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

$x_4 + 2u = 7$

$x_4 = 7 - 2u$

$x_1 + \cancel{2t} + 2s + 5u = 6$

$x_1 = 6 - 2t - 2s - 5u$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6-2t-2s-5u \\ t \\ s \\ 7-2u \\ u \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u\begin{bmatrix} -5 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \\ 0 \\ 7 \\ 0 \end{bmatrix}$$

#15.
$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -8221/4340 \\ 0 & 1 & 0 & 0 & 0 & 8591/8680 \\ 0 & 0 & 1 & 0 & 0 & 4695/434 \\ 0 & 0 & 0 & 1 & 0 & -459/434 \\ 0 & 0 & 0 & 0 & 1 & 699/434 \end{array}\right]$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -8221/4340 \\ 8591/8680 \\ 4695/434 \\ -459/434 \\ 699/434 \end{bmatrix}$$

#16. 1). $ABC \cdot C^{-1} = I_n \cdot C^{-1}$

$$AB = C^{-1}$$

$$AB \cdot B^{-1} = C^{-1} B^{-1}$$

$$A = C^{-1} B^{-1}$$

$$CA = C \cdot C^{-1} B^{-1}$$

$$CA = B^{-1}$$

$$A^{-1} \cdot ABC = A^{-1} I_n$$

$$BC = A^{-1}$$

2). A and B are invertible

#17. $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix}$ $\quad AB = \begin{bmatrix} 6 & 7 \\ 11 & 11 \end{bmatrix}$

$$(AB)^2 = \begin{bmatrix} 6 & 7 \\ 11 & 11 \end{bmatrix}\begin{bmatrix} 6 & 7 \\ 11 & 11 \end{bmatrix} = \begin{bmatrix} 85 & 119 \\ 187 & 198 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 7 & 4 \\ 12 & 13 \end{bmatrix}, \quad B^2 = \begin{bmatrix} 13 & 6 \\ 8 & 13 \end{bmatrix}$$

$$A^2B^2 = \begin{bmatrix} 7 & 4 \\ 12 & 13 \end{bmatrix}\begin{bmatrix} 13 & 6 \\ 8 & 13 \end{bmatrix} = \begin{bmatrix} 123 & 94 \\ 250 & 241 \end{bmatrix}$$

$$\therefore (AB)^2 \neq A^2B^2$$

#18. $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$ $\quad A^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

$$A^{-1} = -1\begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A^T = A^{-1}$$

#19. Symmetric

2x2: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

3x3: $\begin{bmatrix} 2 & 6 & 4 \\ 6 & -1 & 0 \\ 4 & 0 & -3 \end{bmatrix}$

4x4 $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Skew-symmetric

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -2 \\ -1 & 2 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & -1 & 3 & 6 \\ 1 & 0 & 2 & -5 \\ -3 & -2 & 0 & 4 \\ -6 & 5 & -4 & 0 \end{bmatrix}$

2). all entries on the main diagonal are 0

3). $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

4).

$$\left.\begin{aligned} (A+A^T)^T &= A^T + A = A + A^T \\ (AA^T)^T &= AA^T \\ (A^TA)^T &= A^TA \end{aligned}\right\} \text{symmetric}$$

$(A-A^T)^T = A^T - A = -(A-A^T)$ — skew-symmetric

5). If A is a square matrix,

then $A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$

#20. a). surjective

b). bijective

c). surjective

d). bijective

#21. $A = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{4R_2 - R_1} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 15 & 4 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{15R_3 - R_2} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 15 & 4 & 0 \\ 0 & 0 & 56 & 15 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

$\xrightarrow{56R_4 - R_3} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 15 & 4 & 0 \\ 0 & 0 & 56 & 15 \\ 0 & 0 & 0 & 209 \end{bmatrix} = U$

$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -56 & 1 & 0 & 0 \\ 0 & -15 & 0 & 0 \\ 0 & 0 & -4 & 0 \end{bmatrix}$

#22. $A = \begin{bmatrix} q_1 & r_1 & 0 & 0 \\ p_1 & q_2 & r_2 & 0 \\ 0 & p_2 & q_3 & r_3 \\ 0 & 0 & p_3 & q_4 \end{bmatrix} \longrightarrow$

$\{q_i, p_i, r_i\}$ are multiples of $\{l_i, d_i, u_i\}$

#23. $L = \begin{bmatrix} 4 & 1 & \cdots & 0 \\ 1 & \ddots & & \\ 0 & 1 & \ddots & \\ 0 & & \cdots & 4 \end{bmatrix}$ $\quad U = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 1 & \ddots & & 0 \\ \vdots & & & \\ 0 & \cdots & & 0 \end{bmatrix}$

#24. 1) Transpose and its formulas:

$(A+B)^T = A^T + B^T$

$(AB)^T = B^T A^T$

$$\begin{aligned} H_n^T &= (I_n - 2\vec{u}\,\vec{u}^T)^T \\ &= I_n^T - (2\vec{u}\,\vec{u}^T)^T \\ &= I_n^T - 2\vec{u}\,\vec{u}^T \\ &= I_n - 2\vec{u}\,\vec{u}^T \\ &= H_n \end{aligned}$$

Thus, H_n is an symmetric matrix.

2). Yes.

Since $H_n^T = H_n$, $H_n^T H_n = (I_n - 2\vec{u}\vec{u}^T)^2 = I_n - 4\vec{u}\vec{u}^T + 4(\vec{u}\vec{u}^T)^2$

$= I_n - 4\vec{u}\vec{u}^T + 4\vec{u}(\vec{u}^T\vec{u})\vec{u}^T$

$= I_n - 4\vec{u}\vec{u}^T + 4\vec{u}\vec{u}^T = I_n$

3). $H_n^2 = H_n \cdot H_n = H_n^T H_n = I_n$

4).
$$\begin{aligned} H_n\vec{u} &= (I_n - 2\vec{u}\,\vec{u}^T)\vec{u} \\ &= I_n\vec{u} - 2\vec{u}\,\vec{u}^T\vec{u} \\ &= I_n\vec{u} - 2\vec{u}\cdot 1 \\ &= I_n\vec{u} - 2\vec{u} = -\vec{u} \end{aligned}$$

5). $H_3 = I_3 - 2\vec{u}\,\vec{u}^T$ $\quad \vec{u} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$ $\quad \vec{u}^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\cdot \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$H_4 = I_4 - 2\vec{u}\vec{u}^T \qquad \vec{u} = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\1/2\\1/2\\1/2\end{bmatrix}, \quad \vec{u}^T = \begin{bmatrix}1/2 & 1/2 & 1/2 & 1/2\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} - 2\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}\begin{bmatrix}1/2 & 1/2 & 1/2 & 1/2\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} - 2\begin{bmatrix}1/4&1/4&1/4&1/4\\1/4&1/4&1/4&1/4\\1/4&1/4&1/4&1/4\\1/4&1/4&1/4&1/4\end{bmatrix}$$

$$= \begin{bmatrix}1/2&-1/2&-1/2&-1/2\\-1/2&1/2&-1/2&-1/2\\-1/2&-1/2&1/2&-1/2\\-1/2&-1/2&-1/2&1/2\end{bmatrix}$$