MATH 7241 Fall 2020: Problem Set #5

Due date: Sunday October 25

Reading: relevant background material for these problems can be found on Canvas 'Notes 4: Finite Markov Chains'. Also Grinstead and Snell Chapter 11.

Grinstead and Snell: see pages 442-443 and pages 444, 467-468 on Canvas. The text is available online (free!) at

http://www.dartmouth.edu/~chance/

Click on the link "A GNU book".

Exercise 1 Grinstead & Snell, p. 442: #3.

$$P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$

a) Absorbing if $1-a=1$ or $1-b=1 \iff ab=0$

b) if $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow P^{2n} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $P^{2n+1} = P$.

$\Rightarrow$ irreducible but not regular

So irreducible, not regular $\iff ab = 1$

c) $$P^2 = \begin{pmatrix} 1-a(2-a-b) & a(2-a-b) \\ b(2-a-b) & 1-b(2-a-b) \end{pmatrix}$$

If $0<a<1$, $b>0 \Rightarrow P^2 > 0$

Same for $0<b<1$, $a>0$

So regular $\iff 0 < ab < 1$.

Exercise 2 Grinstead & Snell, p. 442: #5.

a) $P = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$w_1 = \frac{3}{4} w_1 + \frac{1}{2} w_2$

$\Rightarrow w_1 = 2w_2$

$\Rightarrow w = \left(\frac{2}{3}, \frac{1}{3}\right)$

b) $P = \begin{pmatrix} \frac{9}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{9}{10} \end{pmatrix} \Rightarrow w = \left(\frac{1}{2} \quad \frac{1}{2}\right)$

c) $P = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$

$w_1 = \frac{3}{4} w_1 + \frac{1}{4} w_3 \Rightarrow w_1 = w_3$

$w_3 = \frac{1}{3} w_2 + \frac{1}{2} w_3 \Rightarrow w_2 = \frac{3}{2} w_3$

so $w_3 + \frac{3}{2} w_3 + w_3 = 1 \Rightarrow w_3 = \frac{2}{7}$

$w = \left(\frac{2}{7}, \frac{3}{7}, \frac{2}{7}\right)$

Exercise 3 'Finite Markov Chains – Problems' file: Exercise 1

Doubly ~~stochastic~~ $\Rightarrow \sum_{i=1}^{M} P_{ij} = 1$ all j.

Claim: $W = \left(\frac{1}{M} \quad \frac{1}{M} \quad \cdots \quad \frac{1}{M}\right)$ is stationary.

Proof: $w_j = \frac{1}{M}$

$$\sum_{i=1}^{M} w_i P_{ij} = \sum_{i=1}^{M} \frac{1}{M} P_{ij} = \frac{1}{M} \sum_{i=1}^{M} P_{ij}$$

$$= \frac{1}{M}$$

$\Rightarrow \quad w_j = \sum_{i=1}^{M} w_i P_{ij}$ for all j

$\Rightarrow \quad W$ is stationary.

Exercise 4 'Finite Markov Chains – Problems' file: Exercise 3

Claim 1: $\{Y_n\}$ is Markov.

Proof:
$$\mathbb{P}(Y_{n+1}=(j_1,j_2) \mid Y_n=(i_1,i_2),\ Y_{n-1}=(k_1,k_2),\dots)$$
$$= \mathbb{P}(X_n=j_1,\ X_{n+1}=j_2 \mid X_{n-1}=i_1,\ X_n=i_2,\ X_{n-2}=k_1,\ X_{n-1}=k_2,\ \dots)$$

Only need to check cases where this event has nonzero prob. $\Rightarrow i_1=k_2, \dots$ (all consistent)

$$= \mathbb{P}(X_{n+1}=j_2 \mid X_n=j_1,\ X_{n-1}=i_1,\ X_n=i_2,\ X_{n-2}=k_1,\dots).$$
$$\mathbb{P}(X_n=j_1 \mid X_{n-1}=i_1,\ X_n=i_2,\ X_{n-2}=k_1,\dots)$$

nonzero only if $j_1=i_2$ in which case it equals 1

$$= \mathbb{P}(X_{n+1}=j_2 \mid X_n=i_2,\ X_{n-1}=i_1,\dots)\cdot\delta_{j_1,i_2}$$
$$= P_{i_2,j_2}\cdot\delta_{j_1,i_2} \quad \text{by Markov property for } X$$
$$= \mathbb{P}(Y_{n+1}=(j_1,j_2) \mid Y_n=(i_1,i_2))$$

$\Rightarrow Y_n$ is Markov chain.

By assumption,

$$\lim_{n\to\infty} \mathbb{P}(X_n = j) = w_j \quad \text{all } j.$$

$$\Rightarrow \lim_{n\to\infty} \mathbb{P}(Y_n = (i,j))$$

$$= \lim_{n\to\infty} \mathbb{P}(X_{n-1} = i,\ X_n = j)$$

$$= \lim_{n\to\infty} \mathbb{P}(X_n = j \mid X_{n-1} = i)\, \mathbb{P}(X_{n-1} = i)$$

$$= P_{ij} \lim_{n\to\infty} \mathbb{P}(X_{n-1} = i)$$

$$= w_i P_{ij}$$

Exercise 5 GS, p. 444: #17.

Let $X_n = S_n \bmod 7$ (i.e. X_n is remainder when S_n is divided by 7, so $S_n = 7 \cdot k + X_n$ some k).

$\text{Ran}(X_n) = \{0,1,2,3,4,5,6\}$

X_n is a Markov chain with ~~transition~~ matrix

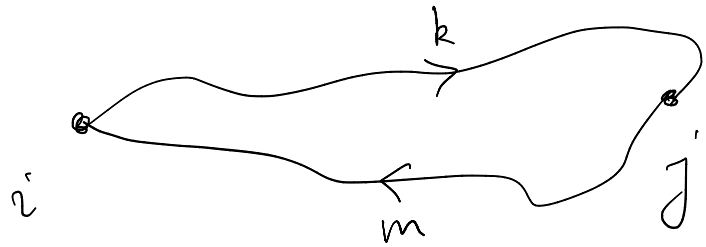
$$P = \begin{array}{c} \begin{pmatrix} 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \end{pmatrix} \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

P is doubly stochastic $\Rightarrow$ stationary distribution $w_i = \frac{1}{7}$

$P^2 > 0 \Rightarrow P$ is regular $\Rightarrow P(X_n = 0) \to \frac{1}{7}$

$\Rightarrow$ proportion of values of S_n divisible by 7 converges to $\frac{1}{7}$

Exercise 6 For a Markov chain, suppose that state i is transient, and that state i is accessible from state j (meaning that there is some integer m such that $p_{ji}(m) > 0$). Show that $p_{ij}(n) \to 0$ as $n \to \infty$. [Hint: find an inequality relating $\sum_n p_{ii}(n)$ and $\sum_n p_{ij}(n)$]



Let $h = P_{ji}(m) > 0$.

For all k, $P_{ii}(k+m) = \mathbb{P}(X_{k+m} = i \mid X_0 = i)$

$$\geq \mathbb{P}(X_{k+m} = i \mid X_k = j, X_0 = i).$$
$$\mathbb{P}(X_k = j \mid X_0 = i)$$

Markov property $\Rightarrow$ $= \mathbb{P}(X_{k+m} = i \mid X_k = j).$
$$\mathbb{P}(X_k = j \mid X_0 = i)$$

$$= P_{jk}(m)\, P_{ij}(k)$$

$$= h\, P_{ij}(k)$$

$$\Rightarrow \quad P_{ij}(k) \leq \frac{1}{h} P_{ii}(k+m) \qquad (\text{all } k \geq 1)$$

Now i is transient $\Rightarrow \sum_{n=1}^{\infty} P_{ii}(n) < \infty$

$\Rightarrow P_{ii}(n) \to 0$ as $n \to \infty$

$$\Rightarrow \quad P_{ii}(k+m) \to 0 \text{ as } k \to \infty$$

Hence from the inequality,

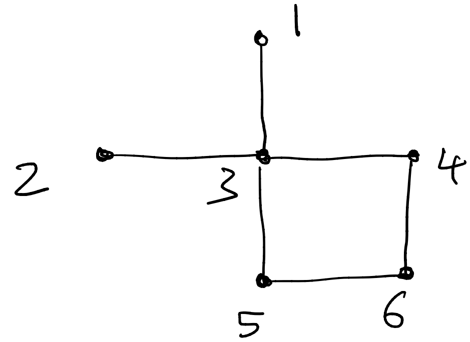
$$\lim_{k\to\infty} P_{ij}(k) \leq \frac{1}{h} \lim_{k\to\infty} P_{ii}(k+m)$$

$$= 0$$

Exercise 7 GS, p. 467: #7

[Note: ergodic means irreducible, and fixed vector means stationary distribution. For part (c), guess that the chain is reversible and solve the reversibility equations. For part (d), set up a recursion formula by conditioning on the first step].

a)



$$P = \left(\begin{array}{cccccc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right) \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

b) Can reach every room from any other room $\Rightarrow$ it is irreducible.

But can return to a room only after an even number of steps.

$\Rightarrow$ $P_{ii}(2n+1) = 0$ all n

$\Rightarrow$ P is not regular (period $= 2$)

c) $w = \left(\frac{1}{12} \quad \frac{1}{12} \quad \frac{1}{3} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \right)$

d) $m_{ab} = \mathbb{E}[\#$ steps $a \to b$ for first time$]$

$$m_{15} = 1 + m_{35}$$

$$m_{35} = 1 + \frac{1}{4}(m_{15} + m_{25} + m_{45})$$

$$m_{25} = 1 + m_{35}$$

$$m_{45} = 1 + \frac{1}{2}(m_{35} + m_{65})$$

$$m_{65} = 1 + \frac{1}{2} m_{45}$$

$$\Rightarrow \quad m_{15} = 7.$$