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Homework #1

9/19/21

1.1) show the set of all numbers of the form $a+b\sqrt{2}$ where a and b are rational numbers.

- The sum on set F is
  $(a+b\sqrt{2})+(c+d\sqrt{2})=(a+c)+(b+d)\sqrt{2}$
- The product on set F is
  $(a+b\sqrt{2})\times(c+d\sqrt{2})=ac+2bd+(ad+bc)\sqrt{2}$

1) (Identity for sum) $=0$, since $0+a+b\sqrt{2}=a+b\sqrt{2}$ ✓

2) (Associativity for sum):

Consider $((a+b\sqrt{2})+(c+d\sqrt{2}))+(e+f\sqrt{2})$
$=(a+c+(b+d)\sqrt{2})+(e+f\sqrt{2})$
$=a+c+e+(b+d+e)\sqrt{2}$

consider $(a+b\sqrt{2})+((c+d\sqrt{2})+(e+f\sqrt{2}))$
$=a+b\sqrt{2}+(c+e+(d+f)\sqrt{2})$
$=a+c+e+(b+d+e)\sqrt{2}$ which is equivalent to $((a+b\sqrt{2})+(c+d\sqrt{2}))+(e+f\sqrt{2})$ so associativity for sum holds

3) (inverse for sum): For each element $a+b\sqrt{2}\in F$, the inverse for sum is $-a-b\sqrt{2}$ since $a+b\sqrt{2}+(-a-b\sqrt{2})=0$

4.) (communitivity for sum)
Consider $(a+b\sqrt{2})+(c+d\sqrt{2})=(c+d\sqrt{2})+(a+b\sqrt{2})$ is true so it holds

5.) (multiplicative identity) For each element $a+b\sqrt{2}\in F$, $1\times(a+b\sqrt{2})=(a+b\sqrt{2})$ so the identity for product is 1.

7) (Distributivity)
$(a+b\sqrt{2})(c+d\sqrt{2}+e+f\sqrt{2})$
$=(a+b\sqrt{2})(c+d\sqrt{2})+(a+b\sqrt{2})(e+f\sqrt{2})$ and
$(c+d\sqrt{2}+e+f\sqrt{2})(a+b\sqrt{2})$
$=(c+d\sqrt{2})(a+b\sqrt{2})+(e+f\sqrt{2})(a+b\sqrt{2})$
so it holds

1.1 cont) 8.) (communitivity for product):

$(a+b\sqrt{2})\times(c+d\sqrt{2}) = (c+d\sqrt{2})\times(a+b\sqrt{2})$ is true

6.) (associativity):

$((a+b\sqrt{2})\times(c+d\sqrt{2}))\times(e+f\sqrt{2})$

$= (a+b\sqrt{2})\times((c+d\sqrt{2})\times(e+f\sqrt{2}))$ is true

9.) (inverse for product):

$(a+b\sqrt{2})^{-1} = x+y\sqrt{2}$ of $a+b\sqrt{2}\in F$ is the calculated condition $(a+b\sqrt{2})\times(x+y\sqrt{2})=1$

Solving: $(ax+2by)+(ay+bx)\sqrt{2}=1$

$ax+2b=1$
$ay+bx=0$
$\left[\begin{array}{cc|c} a & 2b & 1 \\ b & a & 0 \end{array}\right] \xrightarrow[R_2=1/b R_2]{R_1=1/a R_1} \left[\begin{array}{cc|c} 1 & 2b/a & 1/a \\ 1 & a/b & 0 \end{array}\right]$

$\xrightarrow{R_2=R_2-R_1} \left[\begin{array}{cc|c} 1 & 2b/a & 1/a \\ 0 & a/b-2b/a & -1/a \end{array}\right]$ $y(a/b-2b/a) = -1/a$
$y = -1/a\,/\,(a/b-2b/a)$

$y = \dfrac{-1/a}{\frac{a^2-2b^2}{ab}} = \dfrac{-ab}{a(a^2-2b^2)} = \dfrac{-b}{a^2-2b^2}$

$x + \frac{2b}{a}\left(-\frac{b}{a^2-2b^2}\right) = 1/a$ $x = 1/a + \frac{2b^2}{a(a^2-2b^2)}$ $x = \frac{a}{a^2-2b^2}$

$\therefore$ The mult inverse is $\dfrac{a}{a^2-2b^2} - \dfrac{b}{a^2-2b^2}\sqrt{2}$

$\therefore$ Since all properties hold, $a+b\sqrt{2}$ is a field

1.2) $a+b\sqrt{-1}$ where a and b are $\mathbb{R}$

$= a+bi$

- The sum on F is $(a+bi)+(c+di) = (a+c)+(b+d)i$
- The product on set F is $(a+bi)(c+di) = ac+adi+bci-bd$

1.) (Identity for sum): since $0+(a+bi) = a+bi$ the identity for sum is 0

2.) (Associativity for sum)

$((a+bi)+(c+di))+(e+fi) = (a+bi)+((c+di)+(e+fi))$ is true

3.) (inverse for sum): For each element $a+bi\in F$, the inverse of sum is $-a-bi$ since $(a+bi)+(-a-bi)=0$

1.2 (cont) 4.) (commutativity for sum):
$(a+bi)+(c+di)=(c+di)+(a+bi)$ is true

5.) (multiplicative identity) For each element $a+bi \in F$, $1 \times (a+bi) = a+bi$ so the identity for product is 1.

6.) (Associativity): $((a+bi)\times(c+di))\times(e+fi)$
$=(a+bi)\times((c+di)\times(e+fi))$ is true

7.) (Distributivity):
$(a+bi)\times((c+di)+(e+fi)) = (a+bi)(c+di)+(a+bi)(e+fi)$
and $((c+di)+(e+fi))\times(a+bi) = (c+di)(a+bi)+(e+fi)(a+bi)$
are true

8.) (commutativity for product)
$(a+bi)\times(c+di)=(c+di)\times(a+bi)$ is true

9.) (inverse for product)
$(a+bi)^{-1} = x+yi$ of $a+bi \in F$ is calculated the condition $(a+bi)\times(x+yi)=1$
solving: $ax+ayi+bxi-by=1$

$ax-by=1$
$ay+bx=0$

$$\left[\begin{array}{cc|c} a & -b & 1 \\ b & a & 0 \end{array}\right] \xrightarrow[R_2 = 1/b R_2]{R_1 = 1/a R_1} \left[\begin{array}{cc|c} 1 & -b/a & 1/a \\ 1 & a/b & 0 \end{array}\right] \xrightarrow{R_2 = R_2 - R_1} \left[\begin{array}{cc|c} 1 & -b/a & 1/a \\ 0 & \frac{a}{b}+\frac{b}{a} & -1/a \end{array}\right]$$

$y\left(\frac{a}{b}+b/a\right) = -1/a$

$$y = -\frac{1/a}{\frac{a}{b}+b/a} = \frac{-1/a}{\frac{a^2+b^2}{ab}} = -\frac{ab}{a(a^2+b^2)} = -\frac{b}{a^2+b^2}$$

$x - \frac{b}{a}\left(-\frac{b}{a^2+b^2}\right) = 1/a \quad x + \frac{b^2}{a(a^2+b^2)} = 1/a$

$$x = \frac{1}{a} - \frac{b^2}{a(a^2+b^2)} = \frac{a^2+b^2-b^2}{a(a^2+b^2)} = \frac{a}{a^2+b^2}$$

$\therefore$ The mult inverse is $\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$

$\therefore$ Since all properties hold $a+b\sqrt{-1}$ is a field

2.) The set of all $n \times n$ matrices $\mathbb{R}^{n \times n}$ is not a field if $n > 1$.

By def of field, any non-zero element has a multiplicative inverse, but not all $n \times n$ matrices have an inverse (ex: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ s)

Therefore, the set of all $n \times n$ matrices $\mathbb{R}^{n \times n}$ can only be considered a ring, not a field if $n > 1$.

3.) on field $\mathbb{Z}_3$:

+	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

×	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

4.) Show that $\mathbb{C}$ is a field.

Proof:

See proof for question 1.2 for proof of all properties.

identity for sum = 0

inverse for sum = $-a-bi$

identity for prod = 1

inverse for prod = $\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$

$\therefore$ Because all properties hold $\mathbb{C}$ is a field

Homework 1 cont:

5.) $A = \begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ is ~~not~~ in rref b/c all entries in col 3 below the leading entry are not zero.

$B = \begin{bmatrix} 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is in rref.

$C = \begin{bmatrix} 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 5 \end{bmatrix}$ is not in rref b/c row 2 doesn't contain a leading entry further to the left of row 3's.

$D = \begin{bmatrix} 0 & 1 & 0 & 2 & 4 \end{bmatrix}$ is in rref

$E = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}$ is not in rref b/c all entries in a column below a leading entry are not zero.

6.) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ over field $\mathbb{Z}_2$

$$A+B = \begin{bmatrix} 1+0 & 1+1 & 1+1 \\ 0+1 & 1+1 & 1+1 \\ 0+1 & 1+1 & 0+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1\times1+1\times0+1\times0 & 1\times1+1\times1+1\times1 & 1\times1+1\times1+1\times0 \\ 0\times1+1\times0+1\times0 & 1\times0+1\times1+1\times1 & 0\times1+1\times1+1\times0 \\ 0\times1+1\times0+0\times0 & 0\times1+1\times1+0\times1 & 0\times1+1\times1+0\times0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 1+1+1 & 1+1+0 \\ 0+0+0 & 0+1+1 & 0+1+0 \\ 0+0+0 & 0+1+0 & 0+1+0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times0+1\times1+1\times1 & 1\times1+1\times1+1\times1 & 1\times1+1\times1+1\times0 \\ 0\times0+1\times1+1\times1 & 0\times1+1\times1+1\times1 & 0\times1+1\times1+1\times0 \\ 0\times0+1\times1+0\times1 & 0\times1+1\times1+0\times1 & 0\times1+1\times1+0\times0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1+1 & 1+1+1 & 1+1+0 \\ 0+1+1 & 0+1+1 & 0+1+0 \\ 0+1+0 & 0+1+0 & 0+1+0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

7) $A = \begin{bmatrix} 6 & -1 & 1 \\ t & 0 & 1 \\ 0 & 1 & t \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 6 & -1 & 1 \\ 0 & 1 & t \\ t & 0 & 1 \end{bmatrix} \xrightarrow{R_3 = tR_1 - 6R_3}$

$\rightarrow \begin{bmatrix} 6 & -1 & 1 \\ 0 & 1 & t \\ 0 & -t & t-6 \end{bmatrix} \xrightarrow{R_3 = tR_2 + R_3} \begin{bmatrix} 6 & -1 & 1 \\ 0 & 1 & t \\ 0 & 0 & t^2+t-6 \end{bmatrix}$

no inv when $t^2+t-6=0$

$(t+3)(t-2)=0$

$t=-3, 2$

$\therefore$ There is no inverse when $t=-3$ or $t=2$

Homework 6 Cont.

8.a.) $\left[\begin{array}{cc|c} 1 & h & 4 \\ 3 & 6 & 8 \end{array}\right] \xrightarrow{R_2=-3R_1+R_2} \left[\begin{array}{cc|c} 1 & h & 4 \\ 0 & -3h+6 & -4 \end{array}\right]$

$-3h+6=0$ ∴ the matrix is consistent when $h \neq 2$.

$-3h=-6$

$h=2$

8b.) $\left[\begin{array}{cc|c} -4 & 12 & h \\ 2 & -6 & -3 \end{array}\right] \xrightarrow{R_2=R_1+2R_2} \left[\begin{array}{cc|c} -4 & 12 & h \\ 0 & 0 & h-6 \end{array}\right]$

$h-6=0$ ∴ the matrix is consistent when $h=6$

$h=6$

9.) (1) 3x2 matrices in rref

rank=2 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & \star \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, 2 types — rank = 1

(2) 2x3 matrices in rref

$\begin{bmatrix} 1 & 0 & \star \\ 0 & 1 & \star \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ → 2 types rank = 2

(3) 4x1 matrices

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, → 1 type rank = 1

10.) $\begin{bmatrix} 1 & a & b & 3 & 0 & -2 \\ 0 & 0 & c & 1 & d & 3 \\ 0 & e & 0 & 0 & 1 & 1 \end{bmatrix}$ $a=\star$, $b=0$, $c=1$, $d=0$, $e=0$

resulting matrix: $\begin{bmatrix} 1 & \star & 0 & 3 & 0 & -2 \\ 0 & 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

11.) 1.) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[R_3=2R_1-R_3]{R_2=R_1-R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 4 & 5 & 6 \end{bmatrix} \xrightarrow{R_3=4R_2-R_3}$

$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 7 & 2 \end{bmatrix} \xrightarrow{R_3=1/7R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 2/7 \end{bmatrix} \xrightarrow[R_2=-3R_3+R_2]{R_1=-3R_3+R_1}$

$\rightarrow \begin{bmatrix} 1 & 2 & 0 & 22/7 \\ 0 & 1 & 0 & 8/7 \\ 0 & 0 & 1 & 2/7 \end{bmatrix} \xrightarrow{R_1=-2R_2+R_1} \begin{bmatrix} 1 & 0 & 0 & 6/7 \\ 0 & 1 & 0 & 8/7 \\ 0 & 0 & 1 & 2/7 \end{bmatrix} = rref(A)$

$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 6/7 & 0 \\ 0 & 1 & 0 & 8/7 & 0 \\ 0 & 0 & 1 & 2/7 & 0 \end{array}\right]$ $\begin{array}{ll} x_1 + 6/7 x_4 = 0 & x_1 = -6/7 x_4 \\ x_2 + 8/7 x_4 = 0 & x_2 = -8/7 x_4 \\ x_3 + 2/7 x_4 = 0 & x_3 = -2/7 x_4 \end{array}$

(free

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -6/7 x_4 \\ -8/7 x_4 \\ -2/7 x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -6/7 \\ -8/7 \\ -2/7 \\ 1 \end{bmatrix}$

2) calculate rref(A) over field $\mathbb{Z}_7$

	add inv	mult inv
$[0] = \{0, \pm 7, \pm 14, \dots\}$		
$[1] = \{1, 1\pm 7, 1\pm 14, \dots\}$	[6]	[1]
$[2] = \{2, 2\pm 7, 2\pm 14, \dots\}$	[5]	[4]
$[3] = \{3, 3\pm 7, 3\pm 14, \dots\}$	[4]	[5]
$[4] = \{4, 4\pm 7, 4\pm 14, \dots\}$	[3]	[2]
$[5] = \{5, 5\pm 7, 5\pm 14, \dots\}$	[2]	[3]
$[6] = \{6, 6\pm 7, 6\pm 14, \dots\}$	[1]	[6]

$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[R_3=R_3-2R_1\ (R_3+[-2]R_1 \to R_3+[5]R_1)]{R_2=R_2-R_1\ (R_2+[-1]R_1 \to R_2+[6]R_1)}$

$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 6 & 4 & 5 \\ 0 & 3 & 2 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 2 & 1 \\ 0 & 6 & 4 & 5 \end{bmatrix} \rightarrow$

Homework 1 cont:

11.2(cont) $R_3 = R_3 - 2R_2$ $(R_3 + [-2]R_2 = R_3 + 5R_2)$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \quad \begin{array}{l} R_2 = \frac{1}{3}R_2 \left(\frac{1}{[3]}R_2 = 5R_2\right) \\ R_3 = \frac{1}{3}R_3 \left(\frac{1}{[3]}R_3 = 5R_2\right) \end{array}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} R_1 = R_1 - 4R_3 \ (R_1 + [-4]R_3 = R_1 + 3R_3) \\ R_2 = R_2 - 5R_3 \ (R_2 + [-5]R_3 = R_2 + 2R_3) \end{array}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_1 = R_1 - 2R_2 \ (R_1 + [-2]R_2 = R_1 + 5R_2)$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \text{ref}(A)$$

11.3.) (by python)

rref(A) over $\mathbb{Z}_7$ = $\begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

rref(A) over $\mathbb{Z}_2$ $= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

rref(A) over $\mathbb{Z}_3$ $= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix}$

11.4) Yes, it is possible that a matrix M have a different rank over different fields of $\mathbb{Z}_p$.

Ex: $M = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

Homework 1 continued:

12.) 1) rref$(A|\vec{b})$ over field $\mathbb{Z}_7$

$$A=\begin{bmatrix}3&1&4\\5&2&6\\0&5&2\end{bmatrix} \quad \vec{b}=\begin{bmatrix}1\\5\\1\end{bmatrix} \quad A|\vec{b}=\left[\begin{array}{ccc|c}3&1&4&1\\5&2&6&5\\0&5&2&1\end{array}\right]$$

$$\left[\begin{array}{ccc|c}3&1&4&1\\5&2&6&5\\0&5&2&1\end{array}\right] \xrightarrow{R_2=5R_1-3R_2\ (5R_1+[-3]R_2=5R_1+4R_2)}$$

$$\left[\begin{array}{ccc|c}3&1&4&1\\0&6&2&4\\0&5&2&1\end{array}\right] \xrightarrow{R_3=5R_2-6R_3\ (5R_2+[-6]R_3=5R_2+1R_3)}$$

$$\left[\begin{array}{ccc|c}3&1&4&1\\0&6&2&4\\0&0&5&0\end{array}\right] \xrightarrow[R_3=1/5R_3\ (1/[5]R_3=3R_3)]{R_1=1/3R_1\ (\frac{1}{[3]}R_1=5R_1),\ R_2=1/6R_2\ (1/[6]R_2=6R_2)}$$

$$\left[\begin{array}{ccc|c}1&5&6&5\\0&1&5&3\\0&0&1&0\end{array}\right] \xrightarrow[R_2=R_2-5R_3\ (R_2+[-5]R_3=R_2+2R_3)]{R_1=R_1-6R_3\ (R_1+[-6]R_3=R_1+R_3)}$$

$$\left[\begin{array}{ccc|c}1&5&0&5\\0&1&0&3\\0&0&1&0\end{array}\right] \xrightarrow{R_1=R_1-5R_2\ (R_1+[-5]R_2=R_1+2R_2)}$$

$$\left[\begin{array}{ccc|c}1&0&0&4\\0&1&0&3\\0&0&1&0\end{array}\right] = \text{rref}(A|\vec{b})$$

2.) $A\vec{x}=\vec{b} \bmod 7$

$x_1=4,\ x_2=3,\ x_3=0$ on $\mathbb{Z}_7$

$$\begin{bmatrix}4\\3\\0\end{bmatrix} = A\vec{x}=\vec{b} \bmod 7.$$

13.) $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 50.5x_4 \\ -67.5x_4 \\ 31x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 50.5 \\ -67.5 \\ 31 \\ 1 \end{bmatrix}$$ (from python)

14.) $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6-2x_2-3x_3-5x_5 \\ x_2 \\ x_3 \\ 7-2x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 7 \\ 0 \end{bmatrix} + 2x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

(from python)

15.) (from python)

$$A=\left[\begin{array}{ccccc|c} 2 & 4 & 3 & 5 & 6 & 37 \\ 4 & 8 & 7 & 5 & 2 & 74 \\ -2 & -4 & 3 & 4 & -5 & 20 \\ 1 & 2 & 2 & -1 & 2 & 26 \\ 5 & -10 & 4 & 6 & 4 & 24 \end{array}\right]$$ is input

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -8221/4340 \\ 8591/8680 \\ 4695/434 \\ -495/434 \\ 644/434 \end{bmatrix}$$

16.1) If $ABC = I_n$ then we know ABC is invertible
so $\det(ABC) \neq 0$
$\det(ABC) = \det(A)\det(B)\det(C)$
since $\det(ABC) \neq 0$, then $\det(A) \neq 0$, $\det(B) \neq 0$, $\det(C) \neq 0$
$\Rightarrow$ each matrix A, B, and C are all invertible
since their determinant does not equal 0.
since $(AB)C = I_n = A(BC)$ we can see
$A^{-1} = BC$, $C^{-1} = AB$, $B^{-1} = CA$ (since $C(AB) = I$

16.2) Yes, by similar logic to q. 16.1 if $\det(AB) \neq 0$,
since $\det(AB) = \det(A)\det(B)$ we know
$\det(A) \neq 0$ and $\det(B) \neq 0$ so A and B are
invertible.

17.) for any 2x2 matrices $(AB)^2 = A^2B^2$?

consider; $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} 6 & 7 \\ 11 & 11 \end{bmatrix}$ $(AB)^2 = \begin{bmatrix} 85 & 119 \\ 187 & 198 \end{bmatrix}$

$A^2 = \begin{bmatrix} 7 & 4 \\ 12 & 13 \end{bmatrix}$ $B^2 = \begin{bmatrix} 13 & 6 \\ 8 & 13 \end{bmatrix}$ $A^2B^2 = \begin{bmatrix} 123 & 94 \\ 250 & 241 \end{bmatrix}$

$(AB)^2 = \begin{bmatrix} 85 & 119 \\ 187 & 198 \end{bmatrix} \neq \begin{bmatrix} 123 & 94 \\ 250 & 241 \end{bmatrix}$

Therefore $(AB)^2 \neq A^2B^2$

18.) 2x2 nonidentity matrix whose transpose is its inverse

Ex: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = A$ $A^T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^{-1} = \left[\begin{array}{cc|cc} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_1 = -R_1} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right]$

$A^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = A^T = A^{-1}$

Homework 1 cont

19.) 1.) 2x2: Symmetric | skew-symmetric

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

3x3

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 5 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$$

4x4

$$\begin{bmatrix} 5 & 1 & -3 & 6 \\ 1 & 4 & -2 & -1 \\ -3 & -2 & 3 & 7 \\ 6 & -1 & 7 & 2 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & -3 & 6 \\ -1 & 0 & 2 & -1 \\ 3 & -2 & 0 & 7 \\ -6 & 1 & -7 & 0 \end{bmatrix}$$

2.) The main diagonal of a skew-symmetric matrix is all 0's.

3.) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is both symmetric and skew-symmetric

4.) for any nxn matrix A

- $A + A^T$ is symmetric
  proof: $(A + A^T)^T = (A^T + A)$ so by def $A^T = A$ for symmetric matrices so symmetric

- AA^T is symmetric
  proof: $(AA^T)^T = A^T A^{TT} = A^T A = AA^T$ by def $A^T = A$; it is symmetric.

- $A^T A$ is symmetric
  proof: $(A^T A)^T = A^{TT} A^T = AA^T = A^T A$ by def $A^T = A$, it is symmetric

• $A - A^T$ is skew-symmetric
proof: $(A-A^T)^T = A^T - A^{TT} = A^T - A = -A + A^T$
$= -(A - A^T)$ by def $A^T = -A$ it is skew-symmetric

5.) Prove that any nxn matrix can be written as the sum of symmetric and skew-symmetric matrices.
Proof: Let A be a square matrix, we can write $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$
We know $(A + A^T)$ is symmetric from pt 4 and $(A - A^T)$ is skew-symmetric from pt 4.
Thus, any square matrix can be expressed as the sum of a symmetric and skew-symmetric matrices in the above way.

20.) a.) $F(x) = x^2$: none

not injective since $f(-2) = f(2)$

not surjective since negative b values don't have an a.

b.) $F(x) = \frac{x^3}{(x^2+1)}$: injective

not surjective based on graph

c.) $F(x) = x(x-1)(x-2)$: surjective

not injective since $f(1) = f(2)$

d.) $f(x) = e^x + 2$: injective

not surjective since 0 (and other values) don't have a corresponding a value.

21.) $A = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$ based on answer to # 22 we know the format of LU (see 22) so we can say

$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 0 & 4/15 & 1 & 0 \\ 0 & 0 & 15/56 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 15/4 & 1 & 0 \\ 0 & 0 & 56/15 & 1 \\ 0 & 0 & 0 & 209/56 \end{bmatrix}$

$l_1 d_1 = 1$

$l_1(4) = 1 \quad l_1 = 1/4$

$q_2 = 4 = l_1 u_1 + d_2 = 1/4(1) + d_2 \quad d_2 = 15/4$

$l_2 = 1/d_2 = 1/15/4 = 4/15$

$q_3 = 4 = l_2 u_2 + d_3 = 4/15(1) + d_3 \quad d_3 = 56/15$

$l_3 = 1 = l_3 d_3 \quad l_3 = 1/d_3 = 1/56/15 = 15/56$

$q_4 = 4 = l_3 u_3 + d_4 \quad d_4 = 4 - l_3 u_3 = 4 - (15/56)(1) = 209/56$

22.) $LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_1 & 1 & 0 & 0 \\ 0 & l_2 & 1 & 0 \\ 0 & 0 & l_3 & 1 \end{bmatrix} \begin{bmatrix} d_1 & u_1 & 0 & 0 \\ 0 & d_2 & u_2 & 0 \\ 0 & 0 & d_3 & u_3 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$

$$\begin{bmatrix} d_1 & u_1 & 0 & 0 \\ l_1 d_1 & l_1 u_1 + d_2 & u_2 & 0 \\ 0 & l_2 d_2 & l_2 u_2 + d_3 & u_3 \\ 0 & 0 & l_3 d_3 & l_3 u_3 + d_4 \end{bmatrix}$$

since $A = \begin{bmatrix} q_1 & r_1 & 0 & 0 \\ p_1 & q_2 & r_2 & 0 \\ 0 & p_2 & q_3 & r_3 \\ 0 & 0 & p_3 & q_4 \end{bmatrix}$

$r_i = u_i$, $p_i = l_i d_i$, $q_i = l_{i-1} u_{i-1} + d_i$

(identity/scale by 1) (scaling) (replacement)

23.) $A = \begin{bmatrix} 4 & 1 & \cdots & 0 & 0 \\ 1 & 4 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & 4 & 1 \\ 0 & 0 & \cdots & 1 & 4 \end{bmatrix}$ based on the answers to 21 + 22.

$A = LU = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1/4 & \cdot & 0 & \cdots & 0 \\ 0 & 4/15 & \cdots & & 0 \\ \vdots & 0 & \ddots & & 0 \\ 0 & \cdots & 0 & l_i d_i & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 & \cdots & 0 \\ 0 & 15/4 & \ddots & & \vdots \\ \vdots & & \ddots & & 0 \\ \vdots & 0 & & & 1 \\ 0 & \cdots & & 0 & l_{i-1} u_{i-1} + d_i \end{bmatrix}$

Homework #1 continued

24) I_n = nxn identity matrix

$\vec{u}$ = unit vector in $\mathbb{R}^n$

$H_n = I_n - 2\vec{u}\vec{u}^T$

1.) $H_n = I_n - 2\vec{u}\vec{u}^T$

$= (I_n - 2\vec{u}\vec{u}^T)^T$

$= I_n^T - 2(\vec{u}\vec{u}^T)^T$

$= I_n - 2(\vec{u}^T)^T\vec{u}^T$

$= I_n - 2\vec{u}\vec{u}^T = H_n$

$\therefore$ it is symmetric

2.) $H_n^T H_n = H_n H_n$ (since symmetric by pt 1).

$= (I_n - 2\vec{u}\vec{u}^T)(I_n - 2\vec{u}\vec{u}^T)$

$= I_n I_n - 2\vec{u}\vec{u}^T - 2\vec{u}\vec{u}^T + 4(\vec{u}\vec{u}^T)(\vec{u}\vec{u}^T)$

$= I_n - 4\vec{u}\vec{u}^T + 4\vec{u}(\vec{u}^T\vec{u})\vec{u}^T$

$= I_n - 4\vec{u}\vec{u}^T + 4\vec{u}\vec{u}^T$ (since $\vec{u}^T\vec{u} = ||u||^2 = 1$)

$= I_n$

$\therefore$ it is orthogonal

3.) $H_n^2 = (I_n - 2\vec{u}\vec{u}^T)(I_n - 2\vec{u}\vec{u}^T)$

$= I_n$ (by pt 2)

4.) $H_n\vec{u} = (I_n - 2\vec{u}\vec{u}^T)\vec{u}$

$= I_n\vec{u} - 2\vec{u}\vec{u}^T\vec{u}$

$= \vec{u} - 2\vec{u}$ (since $\vec{u}^T u = ||u||^2 = 1$)

$= -\vec{u}$

5.) $\vec{u} = \frac{1}{\sqrt{n}}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ $H_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} 1/\sqrt{3}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

$= I_3 - 2/3\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix}$

24.) 5 cont.) $H_4 = I_4 - \frac{2}{\sqrt{4}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$

$$H_4 = I_4 - \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & -1/2 \\ -1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$$