Fourth Worksheet, MATH 7233

September 29, 2021

Definition (Chromatic number)**.** Let $G(V,E)$ be a graph, and S be a finite set. A function $c : V \to S$ is a *proper coloring* if for all edges $xy \in E$ one has $c(x) \neq c(y)$. The set S is usually called the set of colors.

The chromatic number of G, denoted $\chi(G)$, is the smallest integer k such that there is a proper coloring $c : V \to S$ with $|S| = k$.

* 1. Compute $\chi(K_n), \chi(C_n), \chi(S_n)$. What can we say about $\chi(T)$ is T is a tree?

2. Show that in the graph connectedness game, if the answering person plays according to the "Evil Elf" strategy, the game will not end in fewer than $n(n-1)/2$ questions.

3. Let G be a graph without isolated nodes. Show that G has an Euler tour if and only if it is connected and all its degrees are even.

4. Let $G(V,E)$ be a graph. We define it's Laplace matrix $L = L_G := D_G - A_G$, where D_G is the diagonal matrix containing the degrees. Let $f : V \to \mathbb{R}$ be a function, also considered as a vector in $\mathbb{R}^V$.

   (a) Show that
   $$\sum_{i\in V} \deg(i) f(i)^2 - \sum_i \sum_{j\sim v} f(i)f(j) = f^T L f = \sum_{(ij)\in E} (f(i) - f(j))^2$$

   (b) Let us say that f is *nice* if
   $$f(v) = \frac{1}{\deg(v)} \sum_{w\sim v} f(w),$$
   that is, if the value of the function at any node is equal to the average of the values of its neighbors. Describe all nice functions on a graph!

5. Let G be a tournament (aka complete graph with all edges oriented). We say a node $x \in V(G)$ is a *pseudo champion* if for any $y \neq x$ there is an oriented path of length 1 or 2 from x to y. (In tournament language x beat y or x beat some z who beat y.)

   Prove that any tournament has a pseudo champion!

6. Let $k \geq 3$. Clearly, if a graph has K_k as a subgraph, then $\chi(G) \geq k$. Show an example of a graph **for each** k that does not contain K_k as a subgraph, yet $\chi(G) = k$. (Hint: start with $k = 3$.)

7. Show that if a simple graph has no triangles, then $e \leq n^2/4$. (Hint: count cherries!)

8. Let G be a graph on 9 vertices. Show that either G contains a triangle, or G^c contains a K_4 subgraph.