Question 1. Show that the following sets of numbers are fields if the usual addition and multiplication of arithmetic are used:

(1) the set of all numbers of the form $a + b\sqrt{2}$ where a and b are rational numbers.

(2) the set of all numbers of the form $a + b\sqrt{-1}$ where a and b are real numbers. What is this field?

1) $a+b\sqrt{2}$ where a and b are all real numbers.

a) Identity for Sum

$e+x=x+e=0 \quad , e=0$

$0+a+b\sqrt{2} = a+b\sqrt{2}$ ✓

b) Associativity for sum:

$(a+b)+c = a+(b+c)$

$1(a_1+b_1\sqrt{2} + a_2+b_2\sqrt{2}) + a_3+b_3\sqrt{2} =$

$a_1+b_1\sqrt{2}+a_2+b_2\sqrt{2}+a_3+b_3$

$a_1+b_1\sqrt{2}+(a_2+b_2\sqrt{2}+a_3+b_3\sqrt{3}) =$

$a_1+b_1\sqrt{2}+a_2+b_2\sqrt{2}+a_3+b_3$

therefore $(a+b)+c = a+(b+c)$ ✓

c) $a+f=f+a=0$ for any $a \in F$

F is all rational #s therefore for any a there exists a $-a$ meaning $a+f=0$ for any a holds true.

$-a-b\sqrt{2}$ exists for all rational #s

$a+b\sqrt{2}-a-b\sqrt{2}=0$ ✓

d) $a+b=b+a$

$a+b\sqrt{2} + b\sqrt{2}+a = 2a+2\sqrt{2}b$

$b\sqrt{2}+a + a+b\sqrt{2} = 2a+2\sqrt{2}b$ ✓

commutativity for sums holds.

e) $(a+b\sqrt{2}) \times 0 = 0$ ✓

f) $(a \times b) \times c = a \times (b \times c)$

$a=a_1+b_1\sqrt{2}, \quad b=a_2+b_2\sqrt{2}, \quad c=a_3+b_3\sqrt{2}$

$[(a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2})](a_3+b_3\sqrt{2}) =$

$(a_1a_2+a_2b_1\sqrt{2}+a_1b_2\sqrt{2}+2b_1b_2)(a_3+b_3\sqrt{2}) =$

$a_1a_2a_3+a_2a_3b_1\sqrt{2}+a_1a_3b_2\sqrt{2}+2a_3b_1b_2+a_1a_2b_3\sqrt{2}+2a_2b_1b_3+2a_1b_2b_3+2\sqrt{2}b_1b_2b_3$

$(a_1+b_1\sqrt{2})[(a_2+b_2\sqrt{2})(a_3+b_3\sqrt{2})] =$

$$(a_1+b_1\sqrt{2})[a_2a_3+a_2b_3\sqrt{2}+a_3b_2\sqrt{2}+2b_2b_3] =$$
$$a_1a_2a_3+a_2a_3b_1\sqrt{2}+a_1a_3b_2\sqrt{2}+2a_3b_1b_2+a_1a_2b_3\sqrt{2}+2a_2b_1b_3+2a_1b_2b_3+2\sqrt{2}b_1b_2b_3$$

therefore . . .

$$[(a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2})](a_3+b_3\sqrt{2}) = (a_1+b_1\sqrt{2})[(a_2+b_2\sqrt{2})(a_3+b_3\sqrt{2})]$$

$$(a\times b)\times c = a\times(b\times c)\ ✓$$

g) $a\times(b+c) = a\times b + a\times c$

$$(a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2}+a_3+b_3\sqrt{2}) =$$
$$a_1a_2+a_1b_2\sqrt{2}+a_1a_3+a_1b_3\sqrt{2}+a_2b_1\sqrt{2}+2b_1b_2+a_3b_1\sqrt{2}+2b_1b_3$$

$$(a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2})+(a_1+b_1\sqrt{2})(a_3+b_3\sqrt{2}) =$$
$$a_1a_2+a_1b_2\sqrt{2}+a_2b_1\sqrt{2}+2b_1b_2+a_1a_3+a_1b_3\sqrt{2}+a_3b_1\sqrt{2}+2b_1b_3$$

$$\therefore (a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2}+a_3+b_3\sqrt{2}) = (a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2})+(a_1+b_1\sqrt{2})(a_3+b_3\sqrt{2})$$

so. . .

$$(a\times b)+c = a\times b + a\times c\ ✓$$

h) $a\times b = b\times a$

$$(a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2}) = a_1a_2+a_1b_2\sqrt{2}+a_2b_1\sqrt{2}+2b_1b_2$$
$$(a_2+b_2\sqrt{2})(a_1+b_1\sqrt{2}) = a_1a_2+a_1b_2\sqrt{2}+a_2b_1\sqrt{2}+2b_1b_2$$

$$\therefore (a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2}) = (a_2+b_2\sqrt{2})(a_1+b_1\sqrt{2})$$

so. . . $a\times b = b\times a$ ✓

i) Inverse multiplication property

$$(a+b\sqrt{2})(x+y\sqrt{2}) = 1$$
$$ax+ay\sqrt{2}+b\sqrt{2}+2by = 1$$
$$(ax+2by)+(ay+bx)\sqrt{2} = 1$$

$$\begin{vmatrix} a & 2b \\ b & a \end{vmatrix}\begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$$

$$A\vec{x} = \vec{b}$$
$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$
$$\vec{x} = A^{-1}\vec{b}$$

$$A^{-1} = \frac{1}{a^2-2b^2}\begin{vmatrix} a & -2b \\ -b & a \end{vmatrix}\begin{vmatrix} 1 \\ 0 \end{vmatrix} = \frac{1}{a^2-2b^2}\begin{vmatrix} a \\ -b \end{vmatrix}$$

$$= \boxed{\begin{vmatrix} \frac{a}{a^2-2b^2} \\ \frac{-b}{a^2-2b^2} \end{vmatrix}} \checkmark$$

2) Identity Sum

$$e+x = x+e = x \quad , \quad e=0$$
$$0+(a+b\sqrt{-1}) = a+b\sqrt{-1}$$
$$(a+b\sqrt{-1})+0 = a+b\sqrt{-1}$$
$$a+b\sqrt{-1} = a+b\sqrt{-1} \checkmark$$

Product identity

$$a \times e' = e' \times a = 1 \quad , \quad e'=1$$
$$(a+b\sqrt{-1})(1) = a+b\sqrt{-1}$$
$$(1)(a+b\sqrt{-1}) = a+b\sqrt{-1}$$
$$a+b\sqrt{-1} = a+b\sqrt{-1}$$

so...

$$\boxed{(a+b\sqrt{-1})(1) = (1)(a+b\sqrt{-1})}$$

Inverse multiplication

$$(a+b\sqrt{-1})(x+y\sqrt{-1}) = 1$$
$$ax+ay\sqrt{-1}+bx\sqrt{-1}-by = 1$$
$$(ax-by)+(ay-bx)(\sqrt{-1}) = 1$$
$$\begin{vmatrix} a & -b \\ b & a \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$$
$$\vec{A}x = \vec{b}$$
$$A^{-1}Ax = A^{-1}b$$
$$\vec{X} = A^{-1}\vec{b}$$

$$\boxed{\frac{1}{a^2+b^2} \begin{vmatrix} a \\ -b \end{vmatrix} = \begin{vmatrix} \frac{a}{a^2+b^2} \\ \frac{-b}{a^2+b^2} \end{vmatrix}} \checkmark$$

Question 2. Show that the set of all $n \times n$ matrices $\mathbb{R}^{n \times n}$ with the usual matrix addition and multiplication is not a field if $n > 1$.

A is an $n \times n$ matrix.

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}[A] \qquad \text{if } n > 1$$

Question 3. Write down the two operations on field $\mathbb{Z}_3$.

+	[0]	[1]	[2]
[0]	[0]	[1]	[0]
[1]	[1]	[2]	[0]
[2]	[0]	[0]	[1]

×	[0]	[1]	[2]
[0]	[0]	[1]	[0]
[1]	[1]	[1]	[2]
[2]	[0]	[2]	[1]

Question 4. Some basic knowledge of complex numbers.

- Just as $\mathbb{R}$ denotes the set of real numbers, we will use $\mathbb{C}$ to denote the set of complex numbers $z = a+ib$. Here $i = \sqrt{-1}$, and a and b are real numbers called/denoted

$$a = Re(z) = \textbf{real part of } z$$
$$b = Im(z) = \textbf{imaginary part of } z$$

- The **complex conjugate** of $z = a + bi \in \mathbb{C}$ is $\bar{z} := a - bi$
- The **absolute value** of z is $|z| = \sqrt{a^2 + b^2}$.
- $z\bar{z} = |z|^2$

Show that $\mathbb{C}$ is a **field** with the usual sum, scalar product and product.

refer to question 1 part 2.
This is the field of complex numbers

Question 5. Determine which of the matrices below are in reduced row-echelon form.

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}; B = \begin{bmatrix} 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; C = \begin{bmatrix} 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 5 \end{bmatrix}; D = \begin{bmatrix} 0 & 1 & 0 & 2 & 4 \end{bmatrix}; E = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}$$

B is in ref

$\boxed{\text{D is in rref}}$

Question 6. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ be two matrices over the field $\mathbb{Z}_2$. Compute $A + B$, A^2 and AB over the field $\mathbb{Z}_2$.

$A + B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}$$

A^2

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}}$$

AB

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$$

Question 7. For which values of t does the matrix $A = \begin{bmatrix} 6 & -1 & 1 \\ t & 0 & 1 \\ 0 & 1 & t \end{bmatrix}$ NOT have an inverse?

No inverse if determinant = 0

$$6(0t - 1) - (-1)(t^2 - 0) + (1)(t - 0) = 0$$

$$-6 + t^2 + t = 0$$

$$(t+3)(t-2) = 0$$

$$\boxed{t = -3 \text{ or } 2}$$

Question 8. Find all values of h that make the following matrices **consistent**, i.e., at least has one solution.

a) $\begin{bmatrix} 1 & h & | & 4 \\ 3 & 6 & | & 8 \end{bmatrix}$

b) $\begin{bmatrix} -4 & 12 & | & h \\ 2 & -6 & | & -3 \end{bmatrix}$

a) $1x_1 + hx_2 = 4$

$3x_1 + 6x_2 = 8$

$x_1 = \frac{8}{3} - 2x_2$

$\frac{8}{3} - 2x_2 + hx_2 =$

$(-2+h)x_2 = \frac{4}{3}$

$x_2 = \frac{4}{3(-2+h)}$

$h \neq 2$

$\boxed{h = (-\infty, 2) \cup (2, \infty)}$

b) $-4x_1 + 12x_2 = h$

$2x_1 - 6x_2 = -3$

$x_1 = -\frac{3}{2} + 3x_2$

$-4(-\frac{3}{2} + 3x_2) + 12x_2 = h$

$6 \cancel{-12x_2 + 12x_2} = h$

$\boxed{h = 6}$

Question 9. We says that two $m \times n$ matrices in reduced row-echelon form are of the same type if they have the same number of leading 1's in the same position.

(1) How many types of 3×2 matrices in reduced row-echelon form.

(2) How many types of 2×3 matrices in reduced row-echelon form.

(3) Find all 4×1 matrices in reduced row-echelon form.

List all of them. (Use $*$ to denote any real number. Group them by rank)

1) $\begin{bmatrix} 1 & * \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\boxed{\text{3 forms}}$

2) $\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\boxed{\text{4 forms}}$

3) $[1 * * *]$ $[0 1 * *]$ $[0 0 1 *]$

$[0 0 0 1]$ $[0 0 0 0]$

$\boxed{\text{5 forms}}$

Question 10. For which values of a, b, c, d, and e is the following matrix in reduced row-echelon form?

$$A = \begin{bmatrix} 1 & a & b & 3 & 0 & -2 \\ 0 & 0 & c & 1 & d & 3 \\ 0 & e & 0 & 0 & 1 & 1 \end{bmatrix}$$

Hints: Questions 9 and 10 are about RREF. Check lecture notes or notes for my math2331 class in section 1.2 and 1.3

$$\boxed{\begin{array}{l} a = * \\ b = * \\ c = 0 \\ d = * \end{array}}$$

Question 11. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix}$.

(1) Calculation **rref**(A) over $\mathbb{R}$ by hand. Solve $A\vec{x} = \vec{0}$ and write all solutions in parametric vector forms.

(2) Calculation **rref**(A) over field $\mathbb{Z}_7$ by hand.

(3) Using Python verify your result and calculation **rref**(A) over field $\mathbb{Z}_2$ and $\mathbb{Z}_3$. (Python suggestion i uploaded on Canvas.)

(4) Is it possible that a matrix M has different rank over different fields $\mathbb{Z}_p$? (By calculation in (3))

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix}$$

1) RREF(A):

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -2 \\ 0 & -4 & -5 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & -4 & -5 & -6 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & -4 & -5 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 7 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 2/7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \frac{6}{7} \\ 0 & 1 & 0 & \frac{8}{7} \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix}$$

$$\boxed{\vec{x} = \begin{bmatrix} 6/7 \\ 8/7 \\ 2/7 \end{bmatrix}}$$

2) RREF(A) in $\mathbb{Z}_7$

$$\frac{6}{7} \; 7^6 \bmod (7) =$$
$$\bmod 7 \left(\frac{6}{7}(7^6)\right) = 0$$

$$\frac{8}{7} \; 7^6 \bmod (7)$$
$$\bmod 7 \left(\frac{8}{7}(7^6)\right) = 0$$

$$\frac{2}{7} \; 7^6 \bmod (7) =$$
$$\bmod 7 \left(\frac{2}{7}(7^6)\right) = 0$$

$$\boxed{\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

3+4)

```
def question_11():
    M = Matrix([[1, 2, 3, 4], [1, 1, 0, 2], [2, 0, 1, 2]])
    print("Input Matrix : {} ".format(M))
    M_rref = M.rref()
    print("The RREF of Matrix : {}".format(M_rref))

    GF7 = galois.GF(7)
    A = GF7([[1, 2, 3, 4],[1, 1, 0, 2], [2, 0, 1, 2]])
    print("The RREF in Z7 is", GF7.row_reduce(A))

question_11()
```

```
Input Matrix : Matrix([[1, 2, 3, 4], [1, 1, 0, 2], [2, 0, 1, 2]])
The RREF of Matrix : (Matrix([
[1, 0, 0, 6/7],
[0, 1, 0, 8/7],
[0, 0, 1, 2/7]]), (0, 1, 2))
The RREF in Z7 is GF([[1, 0, 4, 0],
    [0, 1, 3, 0],
    [0, 0, 0, 1]], order=7)
```

12)

Question 12. (Solve a linear system over field $\mathbb{Z}_7$.) Let $A = \begin{bmatrix} 3 & 1 & 4 \\ 5 & 2 & 6 \\ 0 & 5 & 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$.

(1) Calculation **rref**$(A|\vec{b})$ over field $\mathbb{Z}_7$.

(2) Find solution of the linear system $A\vec{x} = \vec{b} \mod 7$.

```
def question_12():
    M = Matrix([[3, 1, 4, 1], [5, 2, 6, 5], [0, 5, 2, 1]])
    print("Matrix : {} ".format(M))
    M_rref = M.rref()
    print("The Row echelon form of matrix M and the pivot columns : {}".format(M_rref))p;question_12()
```

```
Matrix : Matrix([[3, 1, 4, 1], [5, 2, 6, 5], [0, 5, 2, 1]])
The Row echelon form of matrix M and the pivot columns : (Matrix([
[1, 0, 0,   31/6],
[0, 1, 0,   11/6],
[0, 0, 1, -49/12]]), (0, 1, 2))
```

RREF in $\mathbb{Z}$

$$\begin{bmatrix} 1 & 0 & 0 & 31/6 \\ 0 & 1 & 0 & 11/6 \\ 0 & 0 & 1 & -49/12 \end{bmatrix}$$

$K^{P-1} = 1 \mod (P)$

$\mathbb{Z} \to \mathbb{Z}_7$ 1) rref$(A|\vec{b})$ over $\mathbb{Z}_7$

rref$(A|\vec{b}) = \begin{bmatrix} 1 & 0 & 0 & 31/6 \\ 0 & 1 & 0 & 11/6 \\ 0 & 0 & 1 & -49/12 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 & 62/12 \\ 0 & 1 & 0 & 22/12 \\ 0 & 0 & 1 & -98/12 \end{bmatrix}$$

$12^6 = 1 \mod (7)$

$\frac{62}{12} = \frac{62}{12} 12^6 \mod (7)$

$= \mathrm{mod}\left(\left(\frac{62}{12}\right) 12^6, 7\right)$

$x_1 = \mathrm{mod}(62(12)^5, 7) = 4$

$\boxed{x_1 = 4}$

$$x_2 = \frac{22}{12}(12^6) \bmod (7)$$

$$= \bmod\left(\left(\frac{22}{12}\right)(12^6), 7\right)$$

$$\boxed{x_2 = 3}$$

$$x_3 = \frac{-98}{12} \bmod (7)$$

$$= \bmod\left(\left(-\frac{98}{12}\right)12^6, 7\right)$$

$$= \bmod\left((-98)(12^5), 7\right.$$

$$\boxed{x_3 = 0}$$

$$\boxed{\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}}$$

13) **Question 13.** (Use Python) Solve the linear system

$$\begin{cases} 3x_1 + 11x_2 + 19x_3 & = -2 \\ 7x_1 + 23x_2 + 39x_3 & = 10 \\ -4x_1 - 3x_2 - 2x_3 & = 6 \end{cases}$$

and write solutions in parametric vector forms.

```
def question_13():
    x =[[3, 11, 18], [7, 23, 39], [-4, -3, -2]]
    y = [-2, 10, 6]
    solutions = np.linalg.inv(x).dot(y)
    parametric = np.array(solutions)
    answer = parametric.reshape(3,1)
    print('x = {}'.format(answer))

question_13()
```

```
x = [[  5.23943662]
 [-14.05633803]
 [  7.6056338 ]]
```

14) **Question 14.** (Use Python) Solve the linear system

$$\begin{cases} 3x_1 + 6x_2 + 9x_3 + 5x_4 + 25x_5 & = 53 \\ 7x_1 + 14x_2 + 21x_3 + 9x_4 + 53x_5 & = 105 \\ -4x_1 - 8x_2 - 12x_3 + 5x_4 - 10x_5 & = 11 \end{cases}$$

```
def question_14():
    C = sym.Matrix([[3,6,9, 5, 25, 53], [7, 14, 21, 9, 53, 105], [-4, -8, -12, 5, -10, 11]])
    print(C.rref())

question_14()
```

```
(Matrix([
[1, 2, 3, 0, 5, 6],
[0, 0, 0, 1, 2, 7],
[0, 0, 0, 0, 0, 0]]), (0, 3))
```

15)

Question 15. (Use Python) Solve the linear system

$$\begin{cases} 2x_1 + 4x_2 + 3x_3 + 5x_4 + 6x_5 &= 37 \\ 4x_1 + 8x_2 + 7x_3 + 5x_4 + 2x_5 &= 74 \\ -2x_1 - 4x_2 + 3x_3 + 4x_4 - 5x_5 &= 20 \\ x_1 + 2x_2 + 2x_3 - x_4 + 2x_5 &= 26 \\ 5x_1 - 10x_2 + 4x_3 + 6x_4 + 4x_5 &= 24 \end{cases}$$

and write solutions in parametric vector forms. (Hint: In Python, if you want precise value, use symbolic calculation A=sym(A))

```
def question_15():
    x = [[2, 4, 3, 5, 6], [4, 8, 7, 5, 2], [-2, -4, 3, 4, -5], [1, 2, 2, -1, 2], [5, -10, 4, 6, 4]]
    y = [37, 74, 20, 26, 24]
    solutions = np.linalg.inv(x).dot(y)
    parametric = np.array(solutions)
    answer = parametric.reshape(5,1)
    print('x = {}'.format(answer))

question_15()
```

```
x = [[-1.89423963]
 [ 0.98974654]
 [10.81797235]
 [-1.05760369]
 [ 1.61059908]]
```

16)

Question 16. (1) If A , B and C are $n \times n$ matrices and $ABC = I_n$, is each of the matrices invertible? What are their inverses?

(2) Suppose A and B are $n \times n$ matrices. If AB is invertible, are both A and B are invertible?

$ABC = I_n$ A, B, and C are nxn matrices

$AA^{-1}I_n = I_n$

All matrices are inversible.

say A is an nxn matrix ≠ I_n. B or C must equal either A^{-1} or I_n. or all I_n

$A = A$
$B = A^{-1}$
$C = I_n$

$A = I_n$
$B = I_n$
$C = I_n$

2) A and B are both invertible.

Question 17. Provide a counter-example to the statement: For any 2×2 matrices A and B, $(AB)^2 = A^2B^2$.

$(ij)^2 = k^2 = -1$
$i^2 j^2 = (-1)^2 = 1$

Question 18. Find an example of a 2×2 nonidentity matrix whose transpose is its inverse.

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$A^{-1} = \frac{1}{\cos^2\theta + \sin^2(\theta)} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

1

Question 19. Here are a couple of new definitions: An $n \times n$ matrix A is *symmetric* provided $A^T = A$ and *skew-symmetric* provided $A^T = -A$.

(1) Give examples of symmetric and skew-symmetric 2×2, 3×3, and 4×4 matrices.
(2) What can you say about the main diagonal of a skew-symmetric matrix?
(3) Give an example of a matrix that is both symmetric and skew-symmetric.
(4) Prove that for any $n \times n$ matrix A, the matrices $A + A^T$, AA^T, and A^TA are symmetric and $A - A^T$ is skew-symmetric.
(5) Prove that any $n \times n$ can be written as the sum of a symmetric and skew-symmetric matrices. Hint: Did you do part (4) yet?

1) $A^T = A$

1) 2×2

$A^T = A$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$A^T = -A$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

3×3

$A^T = A$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad A_T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$A^T = -A$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \qquad A_T = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

4×4

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A_T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \qquad A_T = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{bmatrix}$$

2) The main diagonal are all zeros.

3) $$\boxed{A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

zero matrix is skew-symmetric and symmetric.

4) $(A + A^T)^T = A^T + (A^T)^T = A^T + A$

$\boxed{A + A^T = (A + A^T)^T}$, therefore $A + A^T$ is symmetric

$(A^T A)^T = (A^T)^T A^T = A A^T = A^T A$

$\boxed{A^T A = (A^T A)^T}$, therefore $A^T A$ is symmetric

$$(AA^T)^T = A^T (A^T)^T = A^T A = AA^T$$

$$\boxed{\begin{array}{c} AA^T = (AA^T)^T, \text{ therefore} \\ AA^T \text{ is symmetric} \end{array}}$$

$$(A - A^T)^T$$

$$-A = A^T$$

$$(A - A^T)^T = A^T - A$$

$$-(A - A^T) = -A + A^T = A^T - A$$

$$\boxed{\begin{array}{c} (A - A^T)^T = -(A - A^T) \\ \text{therefore } A - A^T \text{ is} \\ \text{Skew symmetric.} \end{array}}$$

5) any $n \times n$ can be written as sum of symmetric and skew symmetric.

$$(A + A^T) + (A - A^T)$$

$$\frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

$$\frac{1}{2}A + \cancel{\frac{1}{2}A^T} + \frac{1}{2}A - \cancel{\frac{1}{2}A^T}$$

$= A$, which is an $n \times n$ matrix

Question 20. Mark each of the following functions $F : \mathbb{R} \to \mathbb{R}$ injective, surjective or bijective, as is most appropriate. (You may wish to draw the graph of the function in some cases.)

(a) $F(x) = x^2$;

(b) $F(x) = x^3/(x^2 + 1)$;

(c) $F(x) = x(x - l)(x - 2)$;

(d) $F(x) = e^x + 2$.

a) $F(x) = x^2$, bijective

b) $F(x) = x^3 / (x^2+1)$ injective

c) $F(x) = x(x-1)(x-2)$ bijective

d) $F(x) = e^x + 2$

$y = e^x$

$\ln(y) = \ln(e^x)$ surjective

$x = \ln y$

$f(\ln(y)) = y$

miss 3 questions!

21 - 23

Question 24. Let I_n be the $n \times n$ identity matrix. Let $\vec{u}$ be a **unit** vector in $\mathbb{R}^n$. Define $H_n = I_n - 2\vec{u}\vec{u}^T$. Here a unit vector $\vec{u}$ means that norm $||\vec{u}|| = 1$ or equivalently $\vec{u}^T\vec{u} = \vec{u} \cdot \vec{u} = 1$.

(1) Is H_n an symmetric matrix? Prove your result.

(2) Is H_n an orthogonal matrix? (i.e. is $H_n^T H_n = I_n$?)

(3) What is H_n^2?

(4) What is $H_n\vec{u}$?

(5) Suppose $\vec{u} = \frac{1}{\sqrt{n}}\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}$. Write down H_3 and H_4?

1) a matrix is symmetric if $H_n^T = H_n$

$H_n = I_n - 2\vec{u}\vec{u}^T$

$H_n^T = (I_n - 2\vec{u}\vec{u}^T)^T$

$= I_n^T - 2\vec{u}^T(\vec{u}^T)^T$

$I_n^T = I_n$, $(\vec{u}^T)^T = \vec{u}$

$H_n^T = I_n - 2\vec{u}\vec{u}^T$.

$\boxed{H_n = H_n^T\text{, therefore } H_n \text{ is symmetric}}$

2) matrix is orthogonal if $H_n^T H_n = I_n$

$H_n^T H_n = H_n H_n$

$= (I_n - 2\vec{u}\vec{u}^T)(I_n - 2\vec{u}\vec{u}^T)$

$= I_n^2 - 4\vec{u}\vec{u}^T + 4\vec{u}\vec{u}\vec{u}^T\vec{u}^T$

$\vec{u}$ is unit vector so $\vec{u}\vec{u}^T = |\vec{u}|^2 = 1$

$= I_n^2 - 4(1) + 4(1)(1)$

$= I_n^2$

I_n = identity matrix so

$I_n^2 = I_n$

$\boxed{H_n^T H_n = I_n} \checkmark$

3) $H_n^2 = H_n^T H_n$

b/c $H_n^T = H_n$

$\boxed{\therefore H_n^2 = I_n \text{ as well}}$

4) $H_n \vec{u} =$

$[I_n - 2\vec{u}\vec{u}^T]\vec{u}$

$\boxed{I_n \vec{u} - 2\vec{u}}$

5) $\vec{u} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

$$H_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$I_n - 2 \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix}}$$

$$H_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 \end{bmatrix}}$$