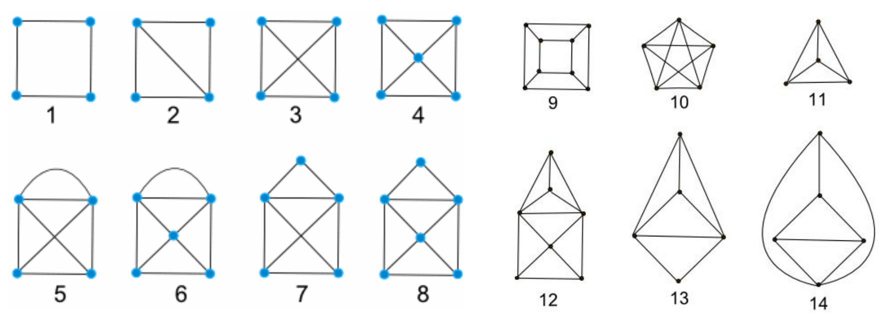
# Third Worksheet, MATH 7233

September 22, 2021

1. Show that any group of people can be arranged in two rooms such that each person has at most as many friends in their own room group than in the other room.

2. In a tennis tournament everybody played a match against everybody. Show that you can arrange the players in one long line such that everybody won their match against the person behind them.

3. Which of the following pictures can you draw without lifting your pencil? You're only allowed to draw each line once!




4. We want to move a knight around a $7 \times 7$ chessboard so that it visits every square exactly once, and then arrives back at the starting position. Can this be done? (Bonus question: what about on a regular chessboard?)

5. A *cherry* is simply a node $s$ together with a pair of its neighbors, $\{a, b\}$. (The stem is $s$, and $a, b$ are the fruits of the cherry.... ) We denote the number of cherries in $G$ by $ch(G)$. In other words, $ch(G)$ is the number of subgraphs of $G$ that are isomorphic to $P_3$.

   Let $G$ be an $n$-vertex graph with $e$ edges, and whose degree sequence is $(d_1, d_2, \ldots, d_n)$.

   (a) Prove that $ch(G) = \sum_i \binom{d_i}{2}$.

   (b) Using this, show that
   $$ch(G) \geq \frac{2e^2}{n} - e.$$
   Hint: $(x_1 + \cdots + x_n)/n \leq \sqrt{(x_1^2 + \cdots + x_n^2)/n}$ may be useful!

6. Let $G$ be a graph on 10 vertices that has no triangles. Show that $G^c$ must have a $K_4$ subgraph.