Math 4570 – Matrix Methods for Data Analysis and Machine Learning
Homework 1
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Question 1:
(1) $a+b\sqrt{2}$:
Define +: $(a+b\sqrt{2})+(c+d\sqrt{2}) = (a+b)+(c+d)\sqrt{2}$
Define ×: $(a+b\sqrt{2})\times(c+d\sqrt{2}) = (ac+2bd)+(ad+bc)\sqrt{2}$
1) Identity for Sum:

$$e+(a+b\sqrt{2}) = (a+b\sqrt{2})+e = a+b\sqrt{2} \rightarrow e = 0$$

2) Associativity for Sum:

$$\begin{aligned}((a+b\sqrt{2})+(c+d\sqrt{2}))+(f+g\sqrt{2}) &= ((a+c)+(b+d)\sqrt{2})+(f+g\sqrt{2})\\ &= (a+c+f)+(b+d+g)\sqrt{2}\\ (a+b\sqrt{2})+((c+d\sqrt{2})+(f+g\sqrt{2})) &= (a+b\sqrt{2})+((c+f)+(d+g)\sqrt{2})\\ &= (a+c+f)+(b+d+g)\sqrt{2}\end{aligned}$$

3) Inverse for Sum:

$$\begin{aligned}(a+b\sqrt{2})+(-a-b\sqrt{2}) &= (a-a)+(b-b)\sqrt{2} = 0\\ (-a-b\sqrt{2})+(a+b\sqrt{2}) &= (a-a)+(b-b)\sqrt{2} = 0\end{aligned}$$

4) Commutativity for Sum:

$$\begin{aligned}(a+b\sqrt{2})+(c+d\sqrt{2}) &= (a+c)+(b+d)\sqrt{2}\\ (c+d\sqrt{2})+(a+b\sqrt{2}) &= (a+c)+(b+d)\sqrt{2}\end{aligned}$$

5) Identity for Product:

$$e\times(a+b\sqrt{2}) = (a+b\sqrt{2})\times e = a+b\sqrt{2} \rightarrow e = 1$$

6) Associativity:

$$\begin{aligned}((a+b\sqrt{2})\times(c+d\sqrt{2}))\times(f+g\sqrt{2}) &= ((ac+2bd)+(ad+bc)\sqrt{2})\times(f+g\sqrt{2})\\ &= (acf+2bdf+2adg+2bcg)+(adf+bcf+acg+2bdg)\sqrt{2}\\ (a+b\sqrt{2})\times((c+d\sqrt{2})\times(f+g\sqrt{2})) &= (a+b\sqrt{2})\times((cf+2dg)+(cg+df)\sqrt{2})\\ &= (acf+2bdf+2adg+2bcg)+(adf+bcf+acg+2bdg)\sqrt{2}\end{aligned}$$

7) Distributivity:

$$\begin{aligned}(a+b\sqrt{2})\times((c+d\sqrt{2})+(f+g\sqrt{2})) &= (a+b\sqrt{2})+((c+f)+(d+g)\sqrt{2})\\ &= (ac+af+2bd+2bg)+(ad+ag+bc+bf)\sqrt{2}\\ (a+b\sqrt{2})\times(c+d\sqrt{2})+(a+b\sqrt{2})\times(f+g\sqrt{2}) &= ((ac+2bd)+(ad+bc)\sqrt{2})+((af+2bg)+(ag+bf)\sqrt{2})\\ &= (ac+af+2bd+2bg)+(ad+ag+bc+bf)\sqrt{2}\end{aligned}$$

8) Commutativity for Product:

$$\begin{aligned}(a+b\sqrt{2})\times(c+d\sqrt{2}) &= (ac+2bd)+(ad+bc)\sqrt{2}\\ (c+d\sqrt{2})\times(a+b\sqrt{2}) &= (ac+2bd)+(ad+bc)\sqrt{2}\end{aligned}$$

9) Product Inverse:

$$(a+b\sqrt{2})\times(c+d\sqrt{2})=1\rightarrow(ac+2bd)+(ad+bc)\sqrt{2}=1$$

$$\begin{bmatrix} a & 2b & 1 \\ b & a & 0 \end{bmatrix}\rightarrow ref=\begin{bmatrix} 1 & 0 & \frac{a}{a^2-2b^2} \\ 0 & 1 & \frac{b}{2b^2-a^2} \end{bmatrix}$$

$$(a+b\sqrt{2})^{-1}=\frac{a-b\sqrt{2}}{a^2-2b^2}$$

Since all conditions are satisfied, $(a+b\sqrt{2},+,\times)$ forms a field.

(2) $a+b\sqrt{-1}$:
Define $+$: $(a+b\sqrt{-1})+(c+d\sqrt{-1})=(a+b)+(c+d)\sqrt{-1}$
Define $\times$: $(a+b\sqrt{-1})\times(c+d\sqrt{-1})=(ac-bd)+(ad+bc)\sqrt{-1}$
1) Identity for Sum:

$$e+(a+b\sqrt{-1})=(a+b\sqrt{-1})+e=a+b\sqrt{-1}\rightarrow e=0$$

2) Associativity for Sum:

$$\begin{aligned}((a+b\sqrt{-1})+(c+d\sqrt{-1}))+(f+g\sqrt{-1})&=((a+c)+(b+d)\sqrt{-1})+(f+g\sqrt{-1})\\&=(a+c+f)+(b+d+g)\sqrt{-1}\\(a+b\sqrt{-1})+((c+d\sqrt{-1})+(f+g\sqrt{-1}))&=(a+b\sqrt{-1})+((c+f)+(d+g)\sqrt{-1})\\&=(a+c+f)+(b+d+g)\sqrt{-1}\end{aligned}$$

3) Inverse for Sum:

$$\begin{aligned}(a+b\sqrt{-1})+(-a-b\sqrt{-1})=(a-a)+(b-b)\sqrt{-1}=0\\(-a-b\sqrt{-1})+(a+b\sqrt{-1})=(a-a)+(b-b)\sqrt{-1}=0\end{aligned}$$

4) Commutativity for Sum:

$$\begin{aligned}(a+b\sqrt{-1})+(c+d\sqrt{-1})=(a+c)+(b+d)\sqrt{-1}\\(c+d\sqrt{-1})+(a+b\sqrt{-1})=(a+c)+(b+d)\sqrt{-1}\end{aligned}$$

5) Identity for Product:

$$e\times(a+b\sqrt{-1})=(a+b\sqrt{-1})\times e=a+b\sqrt{-1}\rightarrow e=1$$

6) Associativity:

$$\begin{aligned}((a+b\sqrt{-1})\times(c+d\sqrt{-1}))\times(f+g\sqrt{-1})&=((ac-bd)+(ad+bc)\sqrt{-1})\times(f+g\sqrt{-1})\\&=(acf-bdf-adg-bcg)+(adf+bcf+acg-bdg)\sqrt{-1}\\(a+b\sqrt{-1})\times((c+d\sqrt{-1})\times(f+g\sqrt{-1}))&=(a+b\sqrt{-1})\times((cf-dg)+(cg+df)\sqrt{-1})\\&=(acf-bdf-adg-bcg)+(adf+bcf+acg-bdg)\sqrt{-1}\end{aligned}$$

7) Distributivity:

$$\begin{aligned}(a+b\sqrt{-1})\times((c+d\sqrt{-1})+(f+g\sqrt{-1}))&=(a+b\sqrt{-1})+((c+f)+(d+g)\sqrt{-1})\\&=(ac+af-bd-bg)+(ad+ag+bc+bf)\sqrt{-1}\\(a+b\sqrt{-1})\times(c+d\sqrt{-1})&+(a+b\sqrt{-1})\times(f+g\sqrt{-1})\\&=((ac-bd)+(ad+bc)\sqrt{-1})+((af-bg)+(ag+bf)\sqrt{-1})\\&=(ac+af-bd-bg)+(ad+ag+bc+bf)\sqrt{-1}\end{aligned}$$

8) Commutativity for Product:

$$(a+b\sqrt{-1})\times(c+d\sqrt{-1}) = (ac-bd)+(ad+bc)\sqrt{-1}$$
$$(c+d\sqrt{-1})\times(a+b\sqrt{-1}) = (ac-bd)+(ad+bc)\sqrt{-1}$$

9) Product Inverse:

$$(a+b\sqrt{-1})\times(c+d\sqrt{-1}) = 1 \rightarrow (ac-bd)+(ad+bc)\sqrt{-1} = 1$$

$$\begin{bmatrix} a & -b & 1 \\ b & a & 0 \end{bmatrix} \rightarrow ref = \begin{bmatrix} 1 & 0 & \frac{a}{a^2+b^2} \\ 0 & 1 & \frac{-b}{a^2+b^2} \end{bmatrix}$$

$$(a+b\sqrt{-1})^{-1} = \frac{a-b\sqrt{-1}}{a^2+b^2}$$

Since all conditions are satisfied, $(a+b\sqrt{-1},+,\times)$ forms a field.

Question 2:
A field must have a product inverse. For a matrix $A_{n\times n}$ to be invertible, $det(A)\neq 0$. One example of a matrix with zero determinant is:

$$A_{2\times 2} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Furthermore, $det(A)=0$ for any matrix $A_{n\times n}$ with $a_{ij}=1$, so the set of all $n\times n$ matrices do not form a field. However, the case of $n=1$ is just a scalar and will have a product inverse for all values not equal to zero.

Question 3:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\times$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Question 4:
$a+bi$:
Where: $i\equiv\sqrt{-1}$
Define $+$: $(a+bi)+(c+di)=(a+b)+(c+d)i$
Define $\times$: $(a+bi)\times(c+di)=(ac-bd)+(ad+bc)i$
1) Identity for Sum:

$$e+(a+bi)=(a+bi)+e=a+bi \rightarrow e=0$$

2) Associativity for Sum:

$$
\begin{aligned}
((a+bi)+(c+di))+(f+gi) &= ((a+c)+(b+d)i)+(f+gi) \\
&= (a+c+f)+(b+d+g)i \\
(a+bi)+((c+di)+(f+gi)) &= (a+bi)+((c+f)+(d+g)i) \\
&= (a+c+f)+(b+d+g)i
\end{aligned}
$$

3) Inverse for Sum:

$$
\begin{aligned}
(a+bi)+(-a-bi) &= (a-a)+(b-b)i = 0 \\
(-a-bi)+(a+bi) &= (a-a)+(b-b)i = 0
\end{aligned}
$$

4) Commutativity for Sum:

$$
\begin{aligned}
(a+bi)+(c+di) &= (a+c)+(b+d)i \\
(c+di)+(a+bi) &= (a+c)+(b+d)i
\end{aligned}
$$

5) Identity for Product:

$$
e \times (a+bi) = (a+bi) \times e = a+bi \to e = 1
$$

6) Associativity:

$$
\begin{aligned}
((a+bi)\times(c+di))\times(f+gi) &= ((ac-bd)+(ad+bc)i)\times(f+gi) \\
&= (acf-bdf-adg-bcg)+(adf+bcf+acg-bdg)i \\
(a+bi)\times((c+di)\times(f+gi)) &= (a+bi)\times((cf-dg)+(cg+df)i) \\
&= (acf-bdf-adg-bcg)+(adf+bcf+acg-bdg)i
\end{aligned}
$$

7) Distributivity:

$$
\begin{gathered}
(a+bi)\times((c+di)+(f+gi)) = (a+bi)+((c+f)+(d+g)i) \\
= (ac+af-bd-bg)+(ad+ag+bc+bf)i \\
(a+bi)\times(c+di)+(a+bi)\times(f+gi) \\
= ((ac-bd)+(ad+bc)i)+((af-bg)+(ag+bf)i) \\
= (ac+af-bd-bg)+(ad+ag+bc+bf)i
\end{gathered}
$$

8) Commutativity for Product:

$$
\begin{aligned}
(a+bi)\times(c+di) &= (ac-bd)+(ad+bc)i \\
(c+di)\times(a+bi) &= (ac-bd)+(ad+bc)i
\end{aligned}
$$

9) Product Inverse:

$$
(a+bi)\times(c+di) = 1 \to (ac-bd)+(ad+bc)i = 1
$$

$$
\begin{bmatrix} a & -b & 1 \\ b & a & 0 \end{bmatrix} \to ref = \begin{bmatrix} 1 & 0 & \frac{a}{a^2+b^2} \\ 0 & 1 & \frac{-b}{a^2+b^2} \end{bmatrix}
$$

$$
(a+bi)^{-1} = \frac{a-bi}{a^2+b^2}
$$

Since all conditions are satisfied, $(a+bi,+,\times)$ forms a field.

Question 5:
Matrices B and D are in rref. Matrices A, C, and E are not.

Question 6:

$$A+B=\begin{bmatrix}1&0&0\\1&0&0\\1&0&0\end{bmatrix}$$

$$A^2=\begin{bmatrix}1&1&0\\0&0&1\\0&1&1\end{bmatrix}$$

$$AB=\begin{bmatrix}0&1&0\\0&0&1\\1&1&1\end{bmatrix}$$

Question 7:

$$\begin{bmatrix}6&-1&1\\t&0&1\\0&1&t\end{bmatrix}\underset{R_1+R_3}{\rightarrow}\begin{bmatrix}6&0&1+t\\t&0&1\\0&1&t\end{bmatrix}\underset{R_2-\frac{t}{6}R_1}{\rightarrow}\begin{bmatrix}6&0&1+t\\0&0&\frac{-t(1+t)}{6}\\0&1&t\end{bmatrix}\underset{R_2\leftrightarrow R_1}{\rightarrow}\begin{bmatrix}6&0&1+t\\0&1&t\\0&0&\frac{t(1+t)}{6}\end{bmatrix}\underset{\frac{1}{6}R_1}{\rightarrow}\begin{bmatrix}1&0&\frac{1+t}{6}\\0&1&t\\0&0&\frac{t(1+t)}{6}\end{bmatrix}$$

In order for the matrix to not have an inverse, $\frac{t(1+t)}{6}\neq 1$. Therefore, $t\neq 2,-3$.

Question 8:

$$\begin{bmatrix}1&h&4\\3&6&8\end{bmatrix}\underset{ref}{\rightarrow}\begin{bmatrix}1&2&\frac{8}{3}\\0&h-2&\frac{4}{3}\end{bmatrix}$$

This matrix is consistent if $h\neq 2$.

$$\begin{bmatrix}-4&12&h\\2&-6&-3\end{bmatrix}\underset{ref}{\rightarrow}\begin{bmatrix}2&-6&-3\\0&0&h-6\end{bmatrix}$$

This matrix is consistent if $h=6$.

Question 9:

There are three types of 3×2 rref matrices:

$$\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}\begin{bmatrix}1&\star\\0&0\\0&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix}$$

There are six types of 2×3 rref matrices:

$$\begin{bmatrix}1&0&\star\\0&1&\star\end{bmatrix}\begin{bmatrix}1&\star&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&\star&\star\\0&0&0\end{bmatrix}\begin{bmatrix}0&1&\star\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}$$

There is one 4×1 rref matrix:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Question 10:
In order for the matrix to be rref, $c = 1$ and $b, d, e = 0$, but a can have any value. This yields:

$$\begin{bmatrix} 1 & a & 0 & 3 & 0 & -2 \\ 0 & 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Question 11:
(1)

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \underset{R_2-R_1, R_3-2R_1}{\rightarrow} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -2 \\ 0 & -4 & -5 & -6 \end{bmatrix} \underset{R_3-4R_2, -R_2}{\rightarrow} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 7 & 2 \end{bmatrix}$$

$$\underset{R_1-2R_2, \frac{1}{7}R_3}{\rightarrow} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix} \underset{3R_3+R_1, R_2-3R_3}{\rightarrow} \begin{bmatrix} 1 & 0 & 0 & \frac{6}{7} \\ 0 & 1 & 0 & \frac{8}{7} \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix}$$

$$A\vec{x} = \vec{0} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{6}{7} \\ 0 & 1 & 0 & \frac{8}{7} \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} \frac{-6}{7} \\ \frac{-8}{7} \\ \frac{-2}{7} \\ 1 \end{bmatrix}$$

(2)

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \underset{R_2-R_1}{\rightarrow} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -6 \\ 2 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 6 & 4 & 1 \\ 2 & 0 & 1 & 2 \end{bmatrix}$$

$$\underset{R_3-2R_1}{\rightarrow} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 6 & 4 & 1 \\ 0 & -4 & -5 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 6 & 4 & 1 \\ 0 & 3 & 2 & 1 \end{bmatrix} \underset{R_2-2R_3}{\rightarrow} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & -2 \\ 0 & 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 5 \\ 0 & 3 & 2 & 1 \end{bmatrix}$$

$$\underset{3R_2, R_2 \leftrightarrow R_3}{\rightarrow} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 15 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underset{R_1-4R_3, R_2-R_3}{\rightarrow} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{5R_2}{\rightarrow} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 15 & 10 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underset{R_1-R_2}{\rightarrow} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(3) The answer to part 2 was verified using:

```
GF7 = galois.GF(7)
A = GF7([[1, 2, 3, 4], [1, 1, 0, 2], [2, 0, 1, 2]]);
GF7.row_reduce(A)
```

The rref was calculated over $\mathbb{Z}_2$ using:

```
GF2 = galois.GF(2)
A = GF2([[1, 0, 1, 0], [1, 1, 0, 0], [0, 0, 1, 0]]);
GF2.row_reduce(A)
```

$$rref(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The rref was calculated over $\mathbb{Z}_3$ using:

```
GF3 = galois.GF(3)
A = GF3([[1, 2, 0, 1], [1, 1, 0, 2], [2, 0, 1, 2]]);
GF3.row_reduce(A)
```

$$rref(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(4) Based on the results above, the rank of a matrix is the same over different fields $\mathbb{Z}_2$.

Question 12:
(1) $rref(A|\vec{b})$ was calculated using:

```
GF7 = galois.GF(7)
A = GF7([[3, 1, 4, 1], [5, 2, 6, 5], [0, 5, 2, 1]]);
GF7.row_reduce(A)
```

$$rref(A|\vec{b}) = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(2) From this result, the equation $A\vec{x} = \vec{b}$ was solved for:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$$

Question 13:
The system of equations was written as an augmented matrix:

$$\begin{bmatrix} 3 & 11 & 19 & -2 \\ 7 & 23 & 39 & 10 \\ -4 & -3 & -2 & 6 \end{bmatrix}$$

and solved in Python using:

```
M = Matrix([[3, 11, 19, -2], [7, 23, 39, 10], [-4, -3, -2, 6]])
M.rref()
```

yielding the result:

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This system is inconsistent based on the last row of the matrix and has no solutions.

Question 14:
The system of equations was written as an augmented matrix:

$$\begin{bmatrix} 3 & 6 & 9 & 5 & 25 & 53 \\ 7 & 14 & 21 & 9 & 53 & 105 \\ -4 & -8 & -12 & 5 & -10 & 11 \end{bmatrix}$$

and solved in Python using:

```
M = Matrix([[3, 6, 9, 5, 25, 53], [7, 14, 21, 9, 53, 105], [-4, -8, -12, 5, -10, 11]])
M.rref()
```

yielding the result:

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 5 & 6 \\ 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Written in parametric form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \\ 0 \\ 7 \\ 0 \end{bmatrix}$$

Question 15:
The system of equations was written as an augmented matrix:

$$\begin{bmatrix} 2 & 4 & 3 & 5 & 6 & 37 \\ 4 & 8 & 7 & 5 & 2 & 74 \\ -2 & -4 & 4 & 4 & -5 & 20 \\ 1 & 2 & 2 & -1 & 2 & 26 \\ 5 & -10 & 4 & 6 & 4 & 24 \end{bmatrix}$$

and solved in Python using:

```
M = Matrix([[2, 4, 3, 5, 6, 37], [4, 8, 7, 5, 2, 74], [-2, -4, 3, 4, -5, 20], [1, 2, 2, -1, 2, 26], [5
M.rref()
```

yielding the result:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \frac{-8221}{4340} \\ 0 & 1 & 0 & 0 & 0 & \frac{8591}{8680} \\ 0 & 0 & 1 & 0 & 0 & \frac{4695}{434} \\ 0 & 0 & 0 & 1 & 0 & \frac{-459}{434} \\ 0 & 0 & 0 & 0 & 1 & \frac{699}{434} \end{bmatrix}$$

Written in parametric form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \frac{-8221}{4340} \\ \frac{8591}{8680} \\ \frac{4695}{434} \\ \frac{-459}{434} \\ \frac{699}{434} \end{bmatrix}$$

Question 16:
(1)

$$A^{-1} = (BC), \text{ so } A \text{ is invertible.}$$

$$C^{-1} = (AB), \text{ so } C \text{ is invertible.}$$

$$A^{-1}ABCC^{-1} = A^{-1}I_nC^{-1} \to B = A^{-1}C^{-1} = (CA)^{-1}, \text{ so } B \text{ is invertible.}$$

(2)

$$(AB)^{-1}(AB) = I_n \to ((AB)^{-1}A)B = B^{-1}B = I_n$$

$$(AB)(AB)^{-1} = I_n \to A(B(AB)^{-1}) = AA^{-1} = I_n$$

Both matrices are invertible.

Question 17:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \ B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$(AB)^2 = \begin{bmatrix} 34 & 48 \\ 24 & 34 \end{bmatrix} \ A^2B^2 = \begin{bmatrix} 37 & 54 \\ 15 & 22 \end{bmatrix}$$

$$(AB)^2 \neq A^2B^2.$$

Question 18:

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$A^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} = A$$

$$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Question 19:
(1)
2×2 :

$$\text{Symmetric: } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{Skew-symmetric: } \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$3 \times 3:$

$$\text{Symmetric: } \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\text{Skew-symmetric: } \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

$4 \times 4:$

$$\text{Symmetric: } \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Skew-symmetric: } \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$$

(2) In a skew-symmetric matrix, $a_{ij} = -a_{ji}$. Along the main diagonal $a_{ii} = -a_{ii}$, so $a_{ii} = 0$.

(3) The zero matrix is symmetric and skew-symmetric:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(4)

$$(A + A^T)^T = A^T + (A^T)^T = A + A^T, \text{ so } (A + A^T) \text{ is symmetric.}$$

$$(AA^T)^T = (A^T)^T A^T = AA^T, \text{ so } AA^T \text{ is symmetric.}$$

$$(A^T A)^T = A^T (A^T)^T = A^T A, \text{ so } A^T A \text{ is symmetric.}$$

$$(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T), \text{ so } A - A^T \text{ is skew-symmetric.}$$

(5) $A = (A + A^T) + (A - A^T)$. The first matrix is symmetric and the second matrix is skew-symmetric.

Question 20:
(a) There are multiple solutions, so it is surjective.
(b) There is a solution for every value, so it is bijective.
(c) There are multiple solutions, so it is surjective.
(d) Some values have no solution, so it is injective.

Question 21:
Based on the results from the following two questions:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{4}{15} & 1 & 0 & 0 \\ 0 & \frac{15}{56} & 1 & 0 \\ 0 & 0 & \frac{56}{209} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & \frac{15}{4} & 1 & 0 \\ 0 & 0 & \frac{56}{15} & 1 \\ 0 & 0 & 0 & \frac{209}{56} \end{bmatrix}$$

Question 22:

$$\begin{gathered} r_i = u_i \\ p_i = d_i l_i \\ q_1 = d_1 \\ q_{i\neq 1} = d_i + u_{i-1} l_{i-1} \end{gathered}$$

Question 23:
Using the format of the L and U matrices from Question 22:

$$\begin{gathered} u_i = 1 \\ l_i = \frac{1}{d_i} \\ d_1 = 4 \\ d_i = 4 - \frac{1}{d_{i-1}} \end{gathered}$$

Question 24:

$$\vec{u}\vec{u}^T = \begin{bmatrix} u_1^2 & u_1 u_2 & \dots & u_1 u_n \\ u_1 u_2 & u_2^2 & \dots & \dots \\ \vdots & \vdots & \ddots & u_{n-1} u_n \\ u_1 u_n & \dots & u_{n-1} u_n & u_n^2 \end{bmatrix}$$

$\vec{u}\vec{u}^T$ is symmetric.

(1) $H_n^T = (I_n - 2\vec{u}\vec{u}^T)^T = I_n^T - (2\vec{u}\vec{u}^T)^T = I_n - 2\vec{u}\vec{u}^T$, so H_n is symmetric.

(2) $H_n^T H_n = (I_n - 2\vec{u}\vec{u}^T)(I_n - 2\vec{u}\vec{u}^T) = I_n^2 - 4\vec{u}\vec{u}^T + (2\vec{u}\vec{u}^T)^2 = I_n$

(3) Since H_n is symmetric and orthogonal, $H_n^2 = H_n^T H_n = I_n$.

(4) $H_n \vec{u} = (I_n - 2\vec{u}\vec{u}^T)\vec{u} = I_n \vec{u} - 2\vec{u}\vec{u}^T \vec{u} = -\vec{u}$.

(5)

$$H_3 = \frac{1}{3} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

$$H_4 = \frac{1}{2} \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$