1. 1) ① $e=0,\ e+a+b\sqrt{2} = a+b\sqrt{2}+e = a+b\sqrt{2}$

② $(a_1+b_1\sqrt{2} + a_2+b_2\sqrt{2}) + (a_3+b_3\sqrt{2}) = a_1+b_1\sqrt{2} + (a_2+b_2\sqrt{2} + a_3+b_3\sqrt{2})$

③ $a+b\sqrt{2} + (-a-b\sqrt{2}) = 0$

④ $(a_1+b_1\sqrt{2}) + (a_2+b_2\sqrt{2}) = (a_2+b_2\sqrt{2}) + (a_1+b_1\sqrt{2})$

⑤ $e'=1,\ e' \times (a+b\sqrt{2}) = (a+b\sqrt{2}) \times e' = (a+b\sqrt{2})$

⑥ $[(a_1+b_1\sqrt{2}) \times (a_2+b_2\sqrt{2})] \times [(a_3+b_3\sqrt{2})]$

$= (a_1+b_1\sqrt{2}) \times [(a_2+b_2\sqrt{2}) \times (a_3+b_3\sqrt{2})]$

⑦ $(a_1+b_1\sqrt{2}) \times (a_2+b_2\sqrt{2} + a_3+b_3\sqrt{2})$

$= (a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2}) + (a_1+b_1\sqrt{2})(a_3+b_3\sqrt{2})$

⑧ $(a_1+b_1\sqrt{2}) \times (a_2+b_2\sqrt{2}) = (a_2+b_2\sqrt{2}) \times (a_1+b_1\sqrt{2})$

⑨ $(a+b\sqrt{2})(x+y\sqrt{2}) = 1$

$(ax+2by) + (ay+bx)\sqrt{2} = 1$

$$\left[\begin{array}{cc|c} a & 2b & 1 \\ b & a & 0 \end{array}\right] \Rightarrow \begin{cases} x = \frac{a}{a^2-2b^2} \\ y = -\frac{b}{a^2-2b^2} \end{cases}$$

2) ① $e=0,\ e+a+b\sqrt{-1} = a+b\sqrt{-1}+e = a+b\sqrt{-1}$

② $(a_1+b_1\sqrt{-1}) + (a_2+b_2\sqrt{-1}+a_3+b_3\sqrt{-1}) = (a_1+b_1\sqrt{-1}+a_2+b_2\sqrt{-1}) + a_3+b_3\sqrt{-1}$

③ $a+b\sqrt{-1} + (-a-b\sqrt{-1}) = 0$

④ same as 1)

⑤ same as 1)

⑥ same as 1)

⑦ same as 1)

⑧ same as 1)

⑨ $(a+b\sqrt{-1})(x+y\sqrt{-1}) = 1$

$ax-by + (ay+bx)\sqrt{-1} = 1$

$$\left[\begin{array}{cc|c} a & -b & 1 \\ b & a & 0 \end{array}\right] \Rightarrow \begin{cases} x = \frac{a}{a^2+b^2} \\ y = \frac{-b}{a^2+b^2} \end{cases}$$

2. $AB \neq BA$ for matrix, so it's not a field

#3. $[0] = \{0, \pm 3, \pm 6 \cdots\} = [3]$

$[1] = \{1, 1\pm 3, 1\pm 6 \cdots\}$

$[2] = \{2, 2\pm 3, 2\pm 6 \cdots\}$

$\therefore$

+	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

×	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

#4, same as 1)

#5. B, D

#6. $A+B = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ in $\mathbb{Z}_2$

$A^2 = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ in $\mathbb{Z}_2$

$AB = \begin{bmatrix} 2 & 3 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ in $\mathbb{Z}_2$

#7. $\det A = 0$, $\det = 6(0-1) - (-1)(t^2) + 1(t-0)$

$= -6 + t^2 + t$

$= t^2 + t - 6 = 0$

$t = -3, 2$

8\. a) rref: $\begin{bmatrix} 1 & h & 4 \\ 0 & 6-3h & -4 \end{bmatrix}$; $6-3h \neq 0$

$h \neq 2$

$\therefore$ ~~h~~ $\neq 2$

b) rref: $\begin{bmatrix} 2 & -6 & -3 \\ 0 & 0 & -6+h \end{bmatrix}$; $-6+h = 0$

$h = 6$

10\. $a =$ any number, $b = 0$, $c = 1$, $d = 0$, $e = 0$.

11\. 1) $rref(A) = \begin{bmatrix} 1 & 0 & 0 & 6/7 \\ 0 & 1 & 0 & 8/7 \\ 0 & 0 & 1 & 2/7 \end{bmatrix}$ $\begin{cases} x_1 = -\frac{6}{7}x_4 \\ x_2 = -\frac{8}{7}x_4 \\ x_3 = -2/7\, x_4 \end{cases}$ $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_4 \begin{pmatrix} -\frac{6}{7} \\ -\frac{8}{7} \\ -\frac{2}{7} \\ 1 \end{pmatrix}$

2) rref on $\mathbb{Z}_7$: $\begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

3) ~~rref on $\mathbb{Z}_2$: not defined~~

$\mathbb{Z}_2$: $[0] = \{0, \pm 2, \pm 4 \cdots\}$ $[1] = \{1, 1\pm 2, 1\pm 4, \cdots\}$

$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ rref: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

4) No

$\mathbb{Z}_3$: $[0] = \{0, \pm 3, \pm 6 \cdots\}$ $[1] = \{1, 1\pm 3, 1\pm 6 \cdots\}$ $[2] = \{2, 2\pm 3$

$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix}$, rref: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

12. 1) rref (A | $\vec{b}$) over field Z_7 = $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

2) Solution = np. linalg. solve (A, b)

= [4, 3, 0] , order = 7

13. singular matrix, rref = $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ ⇒ incosistent

14. Last 2 dimensions of the array must be square.

rref: $\begin{bmatrix} 1 & 2 & 3 & 0 & 5 & 6 \\ 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

rank = 2 < 5,

$\Rightarrow \begin{cases} x_1 = 6 - 2x_2 - 3x_3 - 5x_5 \\ \cancel{x_4 = 7 - 2x_5} \\ x_2 = x_2 \\ x_3 = x_3 \\ x_4 = 7 - 2x_5 \\ x_5 = x_5 \end{cases}$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -5 \\ 0 \\ 0 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 6 \\ 0 \\ 0 \\ 7 \\ 0 \end{pmatrix}$$

15. $$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -8221/4340 \\ 8391/8680 \\ 4695/434 \\ -439/434 \\ 699/434 \end{bmatrix}$$

16. 1) $A \cdot BC \cdot C^{-1} = I_n \cdot C^{-1}$, $\quad AB \cdot B^{-1} = C^{-1} B^{-1}$, $\quad A^{-1} \cdot ABC = A^{-1} I_n$

$C^{-1} = AB$ $\quad A = C^{-1} B^{-1}$ $\quad A^{-1} = BC$

$CA = C \cdot C^{-1} B^{-1}$

$B^{-1} = CA$

2) Yes

17. $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix}$

18. $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$, $A^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

$A^{-1} = \frac{1}{-\frac{1}{2}-\frac{1}{2}} \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = -1 \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

$\therefore A^T = A^{-1}$

19. 1) Symmetric: 2x2: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

3x3: $\begin{bmatrix} 1 & 5 & 8 \\ 5 & 1 & 0 \\ 8 & 0 & 4 \end{bmatrix}$

4x4: $\begin{bmatrix} 1 & 5 & 8 & 9 \\ 5 & 1 & 0 & 7 \\ 8 & 0 & 3 & 2 \\ 9 & 7 & 2 & 6 \end{bmatrix}$

skew-symmetric: 2x2: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

3x3: $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$

4x4: $\begin{bmatrix} 0 & 1 & 2 & -3 \\ -1 & 0 & 2 & 5 \\ -2 & -2 & 0 & -1 \\ 4 & -5 & 1 & 0 \end{bmatrix}$

2) main diagonal is 0.

3) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

4) $(A+A^T)^T = A^T + A = A + A^T$
$(AA^T)^T = A^TA$
$(A^TA)^T = AA^T$ } Symmetric
$(A-A^T)^T = A^T - A = -(A-A^T)$ — skew symmetric

5) If A is square matrix,
$$A = \underbrace{\tfrac{1}{2}(A+A^T)}_{\text{symmetric}} + \underbrace{\tfrac{1}{2}(A-A^T)}_{\text{skew symmetric}}$$

20. a) surjective
b) bijective
c) surjective
d) bijective

21. $\{q_i, p_i, r_i\}$ are multiples of $\{L_i, d_i, u_i\}$

23. $L = \begin{bmatrix} 4 & 1 & \cdots & 0 \\ 1 & \cdots & & \\ 0 & 1 & \cdots & \\ 0 & \cdots & \cdots & 4 \end{bmatrix}$ $U = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 1 & \cdots & & 0 \\ \vdots & & & \\ 0 & \cdots & & 0 \end{bmatrix}$

24. 1) $H_n^T = (I_n - 2\vec{u}\vec{u}^T)^T$

$= I_n^T - (2\vec{u}\vec{u}^T)^T$

$= I_n - 2\vec{u}\vec{u}^T$ since $(AB)^T = B^T A^T$

$= H_n$

It's symmetric matrix

2) Yes, $H_n^T H_n = (I_n - 2\vec{u}\vec{u}^T)^2$

$= I_n - 4\vec{u}\vec{u}^T + 4(\vec{u}\vec{u}^T)^2$

$=$

3) $H_n^2 = H_n \cdot H_n^T = I_n$

4) $H_n\vec{u} = (I_n - 2\vec{u}\cdot\vec{u}^T)\vec{u}$

$= I_n\vec{u} - 2\vec{u}\vec{u}^T\cdot\vec{u}$

$= I_n\vec{u} - 2\vec{u}\cdot 1$

$= \vec{u} - 2\vec{u} = -\vec{u}$

5) $H_3 = I_3 - 2\vec{u}\vec{u}^T$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\cdot\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix}$$

$H_4 = I_4 - 2\vec{u}\vec{u}^T$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & -1/2 \\ -1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$$