Monday, July 13, 2020 12:50 PM

TODAY: FULLY UNDERSTAND FIRST ORDER, HOMOGENEOUS, LINEAR SYSTEMS W/ CONSTANT COEFF.

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix} = A\vec{x}$$

OUR STANDARD EXAMPLE IS

$$x_1' = a_{11}x_1 + a_{12}x_2$$
$$x_2' = a_{21}x_1 + a_{22}x_2$$

HERE

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \vec{x}' = \frac{d\vec{x}}{dt} = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

UNDERSTANDING 2X2 EXAMPLE TELLS US <u>EVERYTHING</u> ABOUT HIGHER N

OF EVERYTHING ABOUT HIGHER N EXAMPLES. ALSO, THESE SYSTEMS TELL US A LOT ABOUT EQUILIBRIUM SOLUTIONS TO SYSTEMS OF NON-LINEAR EQUATIONS.

RECALL:

LET $\vec{x}' = A\vec{x}$ BE AS ABOVE. ASSUME A HAS A BASIS OF EIGENVECTORS $\vec{v}_1, \dots, \vec{v}_N$, WITH CORRESPONDING EIGEN-VALUES $\vec{\lambda}_1, \dots, \vec{\lambda}_N$, THEN THE GENERAL SOLUTION IS

$$\vec{x} = c_1 e^{\lambda_1 t} \vec{v}_1 + \dots + c_N e^{\lambda_N t} \vec{v}_N$$

RECALL:

FINDING EIGENVALUES:

AN EIGENVECTOR $\vec{v}$ OF A MATRIX A WITH EIGENVALUE λ IS ANY VECTOR SUCH THAT

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} = \lambda\vec{v}$$

TO FIND: REWRITE ABOVE AS

$$A\vec{v} - \lambda\vec{v} = (A - \lambda I)\vec{v} = \vec{0}$$

THIS MEANS $A - \lambda I$ IS DEGENERATE, SO

$$\det(A - \lambda I) = 0 \qquad (*)$$

SO NEED λ SUCH THAT $(*)$ HOLDS.
GIVEN SUCH λ, WE SOLVE

$$(A - \lambda I)\vec{v} = \vec{0}$$

FOR $\vec{v}$, USING YOUR FAVORITE TECHNIQUE.

<u>Ex</u>:

$$x_1' = -4x_1 + x_2$$

$$x_2' = 4x_1 - 4x_2$$

$x_2 = 7x_1 - 4x_2$

SOLVE:

$$A = \begin{bmatrix} -4 & 1 \\ 4 & -4 \end{bmatrix}, \quad \det(A - \lambda I) = \begin{vmatrix} -4-\lambda & 1 \\ 4 & -4-\lambda \end{vmatrix}$$

$$= (4+\lambda)^2 - 4$$

$$= \lambda^2 + 8\lambda + 12$$

$$= (\lambda + 6)(\lambda + 2)$$

TWO SOLUTIONS TO THE EIG- VALUE EQ:

$$\lambda = -2, -6.$$

FIND EIGEN VECTORS:

$\lambda_1 = -2,$

$$A - \lambda_1 I = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}$$

SOLVE VIA THE AUGMENTED SYSTEM

$$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ 4 & -2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\vec{v} = \begin{bmatrix} 1 \end{bmatrix}$

$$\Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\lambda_2 = -6$

$A - \lambda_2 I = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$, LET $\vec{v}_2 = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

WANT

$(A - \lambda_2 I)\vec{v}_2 = \vec{0} \Rightarrow \begin{array}{l} 2y_1 + y_2 = 0 \\ 4y + 2y_2 = 0 \end{array}$

$$\Rightarrow \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 4 & 2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow 2y_1 + y_2 = 0$$
$$\Rightarrow y_2 = -2y_1$$

SETTING $y_1 = 1$, GET $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

<u>Now</u>:

$$A\vec{v}_2 = \begin{bmatrix} -4 & 1 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -4-2 \\ 4+8 \end{bmatrix} = \begin{bmatrix} -6 \\ 12 \end{bmatrix}$$

$$= -6\begin{bmatrix}1\\-2\end{bmatrix}.$$

ALT: $y_1 = -2$:

$\vec{v}_2' = \begin{bmatrix}-2\\4\end{bmatrix}$ AND $A\vec{v}_2' = \begin{bmatrix}-4 & 1\\4 & -4\end{bmatrix}\begin{bmatrix}-2\\4\end{bmatrix} = \begin{bmatrix}12\\-24\end{bmatrix} = -6\begin{bmatrix}-2\\4\end{bmatrix} \checkmark$

HERE $\vec{v}_2' = -2\vec{v}_2$ SO

$$A\vec{v}_2' = A(-2)\vec{v}_2 = -2\,(A\vec{v}_2) = (-2)(-6)\vec{v}_2$$
$$= (-6)\,\vec{v}_2'.$$

<u>NOTE</u>: CHOICE IS ABSORBED INTO C LATER SO PICK EASIEST NONZERO.

<u>SO</u>: WE HAVE

$\lambda_1 = -2$, $\lambda_2 = -6$, $\vec{v}_1 = \begin{bmatrix}1\\2\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}1\\-2\end{bmatrix}$

BY

$$\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$

WE HAVE

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = c_1 e^{-2t}\begin{bmatrix}1\\2\end{bmatrix} + c_2 e^{-6t}\begin{bmatrix}1\\-2\end{bmatrix}$$

OR

$$x_1(t) = c_1 e^{-2t} + c_2 e^{-6t}$$

$$x_2(t) = 2c_1 e^{-2t} - 2c_2 e^{-6t}$$

$\square$

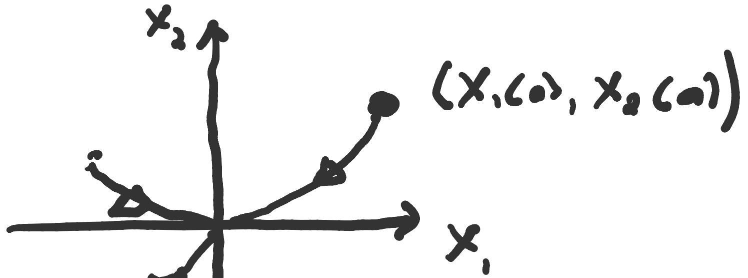
QUESTION: WHAT IS ASYMPTOTIC BEHAVIOR?

WHEN $t \to \infty$, WHERE DOES $(x_1(t), x_2(t))$ GO?

$$\lim_{t \to \infty} (x_1, x_2) = (0, 0),$$ EXPONENTIALLY.

THIS IS BECAUSE $\lambda_1, \lambda_2 < 0$.

<u>SOLUTIONS</u>:

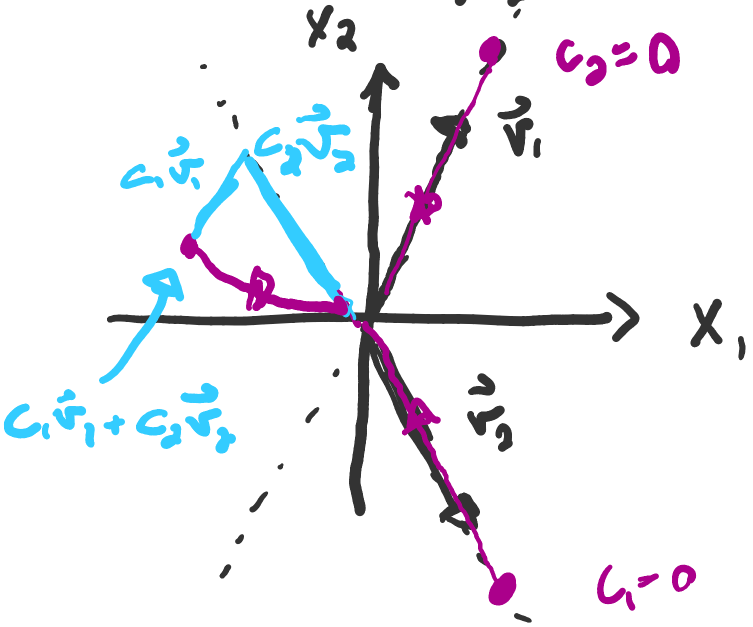


ROUGH SKETCH

MORE PRECISE: <u>RECALL</u>:

$\vec{x} = c\, e^{-2t}\begin{bmatrix} 1 \end{bmatrix} \quad c\, e^{-6t}\begin{bmatrix} 1 \end{bmatrix}$

$$\vec{x} = c_1 e^{-2t}\begin{bmatrix}1\\2\end{bmatrix} + c_2 e^{-6t}\begin{bmatrix}1\\-2\end{bmatrix}$$



SKETCH INCLUDING EIG. VECTORS.

BOTH COEFFICIENTS WILL CONTRACT WITH e^{-2t} AND e^{-6t} RESPECTIVELY.

IF $\lambda_1, \lambda_2 < 0$, ALL SOLUTIONS ARE ASYMPT. TO $\vec{0}$, WE CALL $\vec{0}$ A STABLE NODE.

EX:

$$\begin{aligned} x_1' &= 2x_1 + 2x_2 \\ x_2' &= 5x_1 - x_2 \end{aligned}, \quad A = \begin{bmatrix}2 & 2\\5 & -1\end{bmatrix},$$

EIG. VALUES:

EIG. VALUES:

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 2 \\ 5 & -1-\lambda \end{vmatrix} = (2-\lambda)(-1-\lambda) - 10$$
$$= \lambda^2 - \lambda - 12$$
$$= (\lambda - 4)(\lambda + 3)$$

SO $\lambda_1 = -3$, $\lambda_2 = 4$.

EIG. VECTORS:

$\lambda_1 = -3$:

$$\left[\begin{array}{cc|c} 5 & 2 & 0 \\ 5 & 2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 5 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow 5y_1 + 2y_2 = 0$$
$$\Rightarrow \vec{v}_1 = \begin{bmatrix} 2 \\ -5 \end{bmatrix} \checkmark$$

$\lambda_2 = 4$:

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 5 & -5 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow -2y_1 + 2y_2 = 0$$
$$\Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \checkmark$$

So

>0

$$\vec{x} = \underbrace{c_1 e^{-3t}\begin{bmatrix}2\\-5\end{bmatrix}}_{\text{SHRINK}} + \underbrace{c_2 e^{4t}\begin{bmatrix}1\\1\end{bmatrix}}_{\text{INFLATE}} \quad \checkmark$$

<u>QUESTION</u>: WHAT HAPPENS AS WE TAKE $t \to \infty$?

$$x_1 = 2c_1 \cancelto{0}{e^{-3t}} + c_2 e^{4t} \quad \to \pm\infty \text{ AS } t \to \infty$$

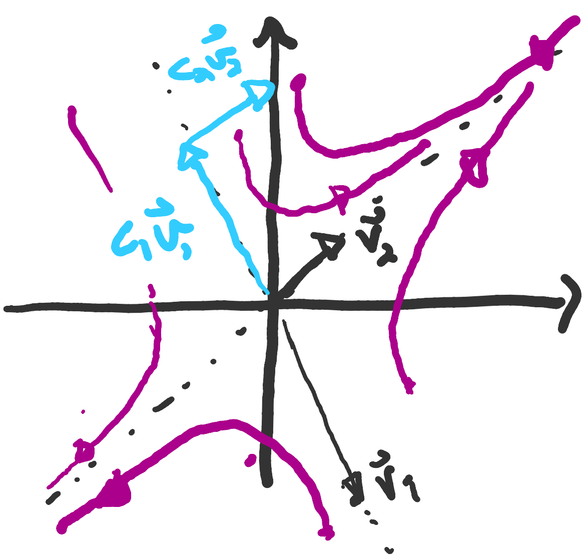
DEP ON SIGN(c_2)

$$x_2 = -5\cancelto{0}{c_1 e^{-3t}} + c_2 e^{4t} \quad \to \pm\infty \quad //$$

BUT THERES MORE!

FOR $t \gg 0$, $\vec{x}(t) \approx \left(c_2 e^{4t}, c_2 e^{4t}\right)$

SO GOES TO ∞ ALONG $\vec{v}_2$:



SO SOLUTION "GO TO ∞" BUT ALONG $\vec{v}_2$.

SADDLE OR A .

SADDLE OR A SEMI-STABLE NODE.

FOR

$x(t) = c_1 \vec{v}_1 e^{-3t} + c_2 \vec{v}_2 e^{4t}$

WHEN $\lambda_1 < 0,\ \lambda_2 > 0$ HAVE SADDLE.

Q) IF 3 FUNCTION SYSTEM

$\dot{\vec{x}} = A\vec{x}$, WHERE

$\lambda_1 = -3,\ \lambda_2 = 2,\ \lambda_3 = 7$ WITH EIG. VECTORS $\vec{v}_1, \vec{v}_2, \vec{v}_3$, WHAT WILL ASYMPT. BEHAVIOR OF SOLUTIONS BE?

SO

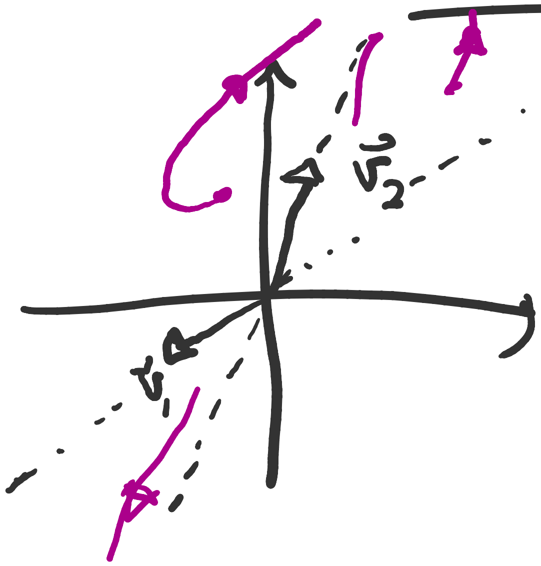
$$\vec{x} = \cancel{c_1 e^{-3t}}\vec{v}_1 + \underbrace{c_2 e^{2t}\vec{v}_2}_{\text{GOES TO INFINITY SLOWER}} + \underbrace{c_3 e^{7t}\vec{v}_3}_{\text{GOES TO } \infty \text{ FASTEST.}}$$

AND FOR $t \gg 0$,

$\vec{x} \approx c_3 e^{7t}\vec{v}_3$

$$\vec{x} \approx c_2 e^{7t} \vec{v}_2$$

FINALLY: FOR $0 < \lambda_1 < \lambda_2$, THEN HAVE AN UNSTABLE NODE.



ALL SOLUTIONS GO TO ∞ ALONG EIG. VECTOR CORR. TO LARGEST EIG. VALUE.

Ex:

CONSIDER

$x' = y$
$y' = -x$

OR $\vec{x}' = Ax$ FOR $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

LETS GO!

EIG. VALUES:

$\det(A - \lambda I) = 0$ OR $\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$ SO $\lambda^2 + 1 = 0$

$\det(A-\lambda I)=0$ OR $\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$ SO $\lambda^2+1=0$

OR $\lambda = \pm\sqrt{-1}$.

CAN WE MAKE SENSE OF THIS? YES, AND WE GET NEW BEHAVIOR.

COMPLEX NUMBERS $\mathbb{C}$

RECALL:

PICK A BRANCH.

$i^2=-1$ OR $i=\sqrt{-1}$

$z\in\mathbb{C}$, WRITTEN $z=a+ib$, $a,b\in\mathbb{R}$.

ADDITION:

$$(a+ib)+(c+id)=\underbrace{a+c}_{\text{RE}}+i\underbrace{(b+d)}_{\text{IM}}.$$

MULT:

$$(a+ib)(c+id)= ac+iad+ibc+i^2bd$$
$$=\underbrace{ac-bd}_{\text{Re}}+i\underbrace{(ad+bc)}_{\text{IM}}.$$

COMPLEX CONJUGATE:

IF $z=a+ib$, $\bar{z}=a-ib$

DEF. OF CONJUGATE:

IF $z = a + ib$, $\bar{z} = a - ib$.

NOTICE:

$$z \cdot \bar{z} = a^2 + b^2 + i(ab - ba) = a^2 + b^2 \in \mathbb{R}.$$

SO z^{-1} OR $\frac{1}{z}$ IS $\frac{\bar{z}}{z \cdot \bar{z}}$.

NOTE:

A COMPLEX NUMBER z IS REAL IF AND ONLY IF $\bar{z} = z$.

ALSO $z + \bar{z} \in \mathbb{R}$.

NOW: WANT TO FOLLOW OUR REASONING FROM BEFORE.

HEURISTIC "SOLUTION" TO $\vec{x}' = Ax$.

WE WANT TO SOLVE THIS BY

$$\vec{X}(t) = e^{At} \cdot \vec{v}$$

HOW DO WE UNDERSTAND e^{At}?

RECALL DEF. OF e^t:

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots$$

COULD DEF

$$e^{At} = 1 + At + \frac{A^2t^2}{2} + \frac{(At)^3}{3!} + \dots$$

NOTE:

$$\frac{de^{At}}{dt} = A + A^2 t + A^3 \frac{t^2}{2!} + \dots$$
$$= A\left(e^{At}\right).$$

PROBLEM: NOT CLEAR IF A^N CAN BE EFFECTIVELY COMPUTED.

BUT WE CAN MAKE SENSE OF THIS IF $\vec{v}$ IS AN EIGENVECTOR OF A:

$$\vec{x}(t) = e^{At}\vec{v}$$

$$= \left(1 + Atv + \frac{A^2t^2v}{2} + \dots\right)$$

$$= v + v\lambda t + \frac{v\lambda^2 t^2}{2} + \cdots$$

$$= e^{\lambda t} v.$$ ← EX SOLUTION

__NOW__: WANT TO DO THIS FOR $\lambda \in \mathbb{C}$, BUT __NEED__ A REAL __SOLUTION__

__ANSWER__: USE A COMPLEX EIG. VECTOR $\vec{v} \in \mathbb{C}$ __PLUS__ COMPLEX CONJUGATE $\bar{\vec{v}} \in \mathbb{C}$. THIS WILL GIVE A __REAL__ SOLUTION.

__LAST PEICE__: COMPLEX EXP:

LET $z = a + ib \in \mathbb{C}$, THEN

$$e^z = e^{a+ib} = e^a e^{ib}$$

$$= e^a\left(1 + ib + \frac{(ib)^2}{2} + \frac{(ib)^3}{3!} + \cdots\right)$$

$$= e^a\left(1 + ib - \frac{b^2}{2!} - \frac{ib^3}{3!} + \dots\right)$$

$$= e^a\left(\underbrace{1 - \frac{b^2}{2!} + \frac{b^4}{4!} - \dots}_{\cos(b)} + \underbrace{ib - \frac{ib^3}{3!} + \dots}_{\sin(b)}\right)$$

$$= e^a(\cos b + i \sin b)$$

$$\boxed{e^{\pi i} + 1 = 0}$$ COOL!

BACK TO THE EXAMPLE:

RECALL:

$$\dot{x} = y \qquad \dot{y} = -x$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow |A - \lambda I| = 0$$

$$\Rightarrow \lambda = \pm i$$

LET FIND EIG. VECTORS:

LET FIND EIG. VECTORS:

$\lambda = i$: $\vec{v} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$(A - iI)v = 0 \Rightarrow \begin{array}{l} i\times \\ \hookrightarrow \end{array} \left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right]$

$\Rightarrow \left[\begin{array}{cc|c} -i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow -i a_1 + a_2 = 0$

$\Rightarrow a_2 = i a_1$

$\Rightarrow \vec{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}.$

CHECK:

$A\vec{v} = \begin{bmatrix} i \\ -1 \end{bmatrix} = i \cdot \vec{v}$ ✓

Now:

$\bar{\lambda}\bar{\vec{v}} = \overline{\lambda \vec{v}} = \overline{A\vec{v}} = \bar{A}\bar{\vec{v}} = A\bar{\vec{v}}$

SO IF A IS REAL AND $\lambda \in \mathbb{C} - \mathbb{R}$

SO IF A IS REAL AND $\lambda \in \mathbb{C} - \mathbb{R}$ EIG. VALUE W/ EIGENVECTOR v, THEN $\bar{\lambda}$ IS ALSO AN EIG. VALUE, W/ EIG. VECTOR $\bar{v}$.

SO FOR $\bar{\lambda} = -i$, $\bar{v} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

NOW:

$$\vec{x} = c_1 e^{\lambda t} v + c_2 e^{\bar{\lambda} t} \bar{v}$$

HOWEVER: LEFT HAND SIDE IS REAL SO RIGHT HAND SIDE MUST BE REAL. IN PART, NEED

$$\vec{x} = e^{At} \underbrace{\left(c v + \bar{c} \bar{v} \right)}$$

REAL IN $\mathbb{R}^2$ SINCE EACH ENTRY IS $z + \bar{z}$

$\vec{\ }$ λt

EACH ENTRY IS $z+\bar{z}$

$$\vec{x} = c e^{\lambda t} v + \bar{c} e^{\bar{\lambda} t} \bar{v},$$

FOR $c = c_1 + i c_2$.

<u>CONT</u>:

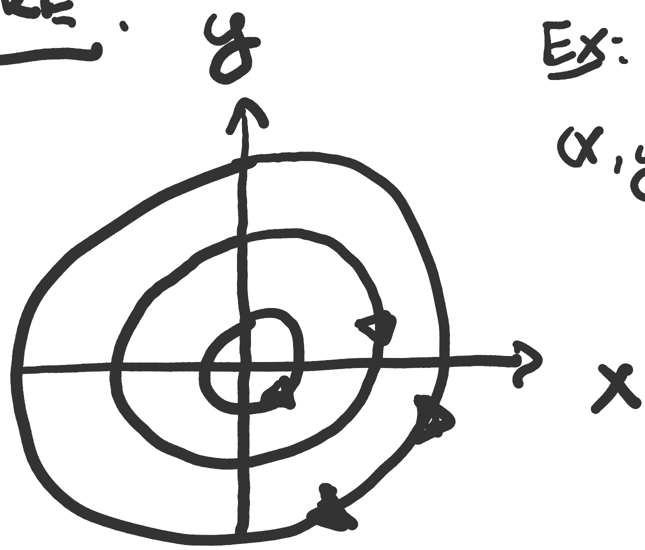
$$\vec{x} = c e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix} + \bar{c} e^{-it} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x = (c_1 + i c_2)(\cos t + i \sin t)$$
$$+ (c_1 - i c_2)(\cos t - i \sin t)$$
$$= [c_1 \cos t + \cancel{c_1 i \sin t} + \cancel{i c_2 \cos t} - c_2 \sin t]$$
$$+ [c_1 \cos t - \cancel{i c_1 \sin t} - \cancel{i c_2 \cos t} - c_2 \sin t]$$
$$= 2c_1 \cos t - 2 c_2 \sin t.$$

FOR y:

$$y = -2c_1 \sin t - 2c_2 \cos t.$$

PICTURE:



Ex: $c_1 = \frac{1}{2}$, $c_2 = 0$

$(x, y) = (\cos t, -\sin t)$

NEW BEHAVIOR: WHEN $\lambda_1 = \bar{\lambda}_2$ IS PURELY IMAGINARY, $\lambda_2, \lambda_1 \in i\mathbb{R}$, THE SOLUTIONS ARE CLOSED ORBITS, $(0,0)$ IS CALLED A CENTER.

Ex: TRULY COMPLEX EIGS:

$$x' = 2x + 4y$$

$$x' = 2x + 4y$$
$$y = -x + 2y$$

HERE:

$$\det\left(A - \lambda I\right) = (2-\lambda)(2-\lambda) + 4$$
$$= \lambda^2 \cancel{+} -4\lambda + 8$$
$$= \frac{4 \pm \sqrt{16 - 32}}{2}$$
$$= 2 \pm i2 .$$

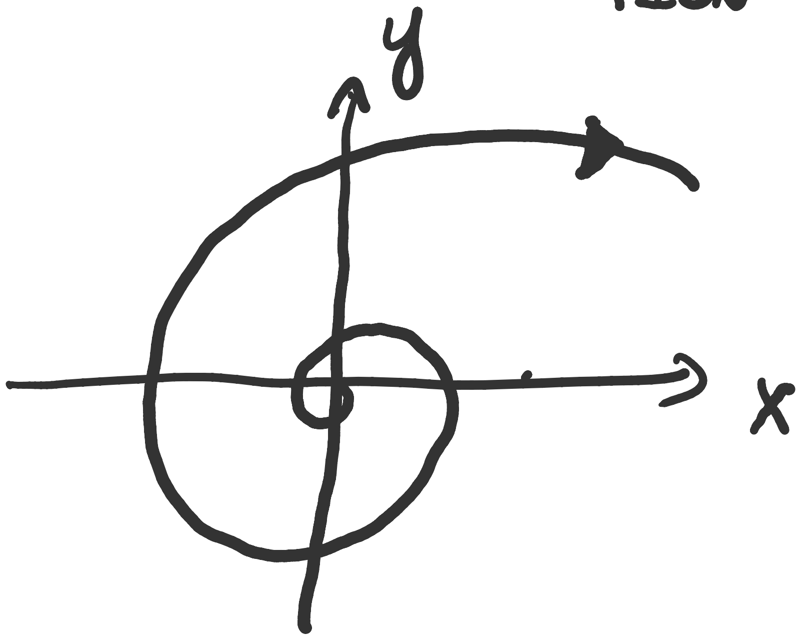
SOLUTION FORM:

$$\vec{x} = c\, e^{(2+i2)t} v + \bar{c}\, e^{(2-i2)t} \bar{v}$$

$$= e^{2t}\Big(\underbrace{c\, e^{2it} v + \bar{c}\, e^{-2it} \bar{v}}_{\text{DEPENDS ON } t \text{ ONLY THROUGH } \sin 2t \text{ AND } \cos 2t.}\Big)$$

THROUGH $\sin 2t$ AND $\cos 2t$.

SO THESE SOLUTION LOOK LIKE PRODUCT OF A CYCLIC PEICE AND A DILATION FACTOR e^{2t}.



UNSTABLE FOCUS.

SINCE $2>0$, e^{2t} EXPANDS ORBIT AS $t \to \infty$. FOR $\lambda = a + ib$, IF $a>0$, THEN $(0,0)$ IS AN UNSTABLE FOCUS.

WHAT IF $a<0$? FOR EX,

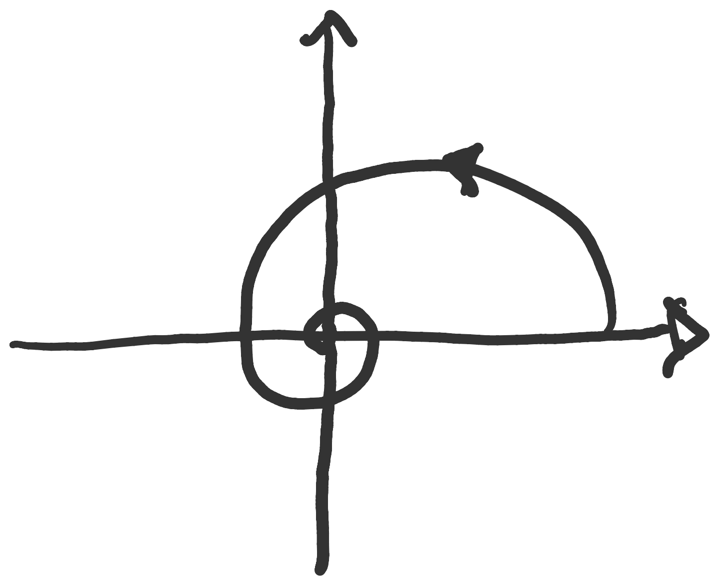
WHAT IF $\lambda = -2+i2$, $\lambda = -2-i2$?

THEN

THEN

$$\vec{x} = e^{-2t}\left(c e^{i2t} \vec{v} + \bar{c} e^{-i2t} \bar{\vec{v}} \right).$$

PICTURE: CONTRACTING SPIRAL



STABLE FOCUS.

NOTE: CLOCKWISE VS COUNTER CLOCKWISE DEPENDS ON $\vec{v}$, λ, c.

ALL TOGETHER:

BEHAVIOR CAN BE SUMMARIZED IN TRACE-DET PLANE:

IN TRACE-DET PLANE:

GIVEN $A \in \mathbb{R}^{2\times 2}$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\det(A - \lambda I) = \lambda^2 - \lambda(a+d) + (ad - cb)$$
$$= \lambda^2 - \lambda \operatorname{Tr} A + \det A$$

LET $\operatorname{Tr} A = T$, $\det A = D$, THEN

$$\lambda = \frac{1}{2}\left(T \pm \sqrt{\underbrace{T^2 - 4D}_{\Delta}}\right)$$

PHASE SPACE: