Math 4570- Applied Linear Algebra-Homework 1
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1.

A. $a + b\sqrt{2}$

$$\left(a + b\sqrt{2}\right) + \left(c + d\sqrt{2}\right) = a + b\sqrt{2} + c + d\sqrt{2}$$
$$\left(a + b\sqrt{2}\right) + \left(c + d\sqrt{2}\right) = a + c + b\sqrt{2} + d\sqrt{2}$$
$$\left(a + b\sqrt{2}\right) + \left(c + d\sqrt{2}\right) = (a + c) + (b + d)\sqrt{2}$$

$$\left(a + b\sqrt{2}\right) * \left(c + d\sqrt{2}\right) = ac + bc\sqrt{2} + ad\sqrt{2} + 2bd$$
$$\left(a + b\sqrt{2}\right) * \left(c + d\sqrt{2}\right) = ac + 2bd + ad\sqrt{2} + bc\sqrt{2}$$
$$\left(a + b\sqrt{2}\right) * \left(c + d\sqrt{2}\right) = (ac + 2bd) + (ad + bc)\sqrt{2}$$

$$0 + \left(a + b\sqrt{2}\right) = a + b\sqrt{2}$$

$$1 * \left(a + b\sqrt{2}\right) = a + b\sqrt{2}$$

$$\left(a + b\sqrt{2}\right) + \left((-a) + (-b)\sqrt{2}\right) = \left(a + b\sqrt{2}\right) + \left(-a - b\sqrt{2}\right)$$
$$\left(a + b\sqrt{2}\right) + \left((-a) + (-b)\sqrt{2}\right) = a + b\sqrt{2} - a - b\sqrt{2}$$
$$\left(a + b\sqrt{2}\right) + \left((-a) + (-b)\sqrt{2}\right) = a - a + b\sqrt{2} - b\sqrt{2}$$
$$\left(a + b\sqrt{2}\right) + \left((-a) + (-b)\sqrt{2}\right) = 0 + 0$$
$$\left(a + b\sqrt{2}\right) + \left((-a) + (-b)\sqrt{2}\right) = 0$$

$$\left(a + b\sqrt{2}\right) * \left(x + y\sqrt{2}\right) = 1$$
$$ax + bx\sqrt{2} + ay\sqrt{2} + 2by = 1$$
$$ax + 2by + bx\sqrt{2} + ay\sqrt{2} = 1$$
$$bx + ay = 0$$
$$ay = -bx$$
$$y = -\frac{bx}{a}$$
$$ax + 2by = 1$$
$$ax + 2b\left(-\frac{bx}{a}\right) = 1$$
$$ax - \frac{2b^2x}{a} = 1$$
$$a^2x - 2b^2x = a$$
$$x(a^2 - 2b^2) = a$$
$$x = \frac{a}{(a^2 - 2b^2)}$$
$$y = -\frac{b\frac{a}{(a^2 - 2b^2)}}{a}$$
$$y = -\frac{b}{(a^2 - 2b^2)}$$

$$\left(a + b\sqrt{2}\right) * \left(\left(\frac{a}{(a^2 - 2b^2)}\right) + \left(\frac{b}{(a^2 - 2b^2)}\right)\sqrt{2}\right) = 1$$

B. $a + b\sqrt{-1}$

$$\left(a + b\sqrt{-1}\right) + \left(c + d\sqrt{-1}\right) = a + b\sqrt{-1} + c + d\sqrt{-1}$$
$$\left(a + b\sqrt{-1}\right) + \left(c + d\sqrt{-1}\right) = a + c + b\sqrt{-1} + d\sqrt{-1}$$
$$\left(a + b\sqrt{-1}\right) + \left(c + d\sqrt{-1}\right) = (a + c) + (b + d)\sqrt{-1}$$

$$\left(a + b\sqrt{-1}\right) * \left(c + d\sqrt{-1}\right) = ac + bc\sqrt{-1} + ad\sqrt{-1} - bd$$
$$\left(a + b\sqrt{-1}\right) * \left(c + d\sqrt{-1}\right) = ac - bd + ad\sqrt{-1} + bc\sqrt{-1}$$
$$\left(a + b\sqrt{-1}\right) * \left(c + d\sqrt{-1}\right) = (ac - bd) + (ad + bc)\sqrt{-1}$$

$$0 + \left(a + b\sqrt{-1}\right) = a + b\sqrt{-1}$$

$$1 * \left(a + b\sqrt{-1}\right) = a + b\sqrt{-1}$$

$$\left(a + b\sqrt{-1}\right) + \left((-a) + (-b)\sqrt{-1}\right) = \left(a + b\sqrt{-1}\right) + \left(-a - b\sqrt{-1}\right)$$
$$\left(a + b\sqrt{-1}\right) + \left((-a) + (-b)\sqrt{-1}\right) = a + b\sqrt{-1} - a - b\sqrt{-1}$$
$$\left(a + b\sqrt{-1}\right) + \left((-a) + (-b)\sqrt{-1}\right) = a - a + b\sqrt{-1} - b\sqrt{-1}$$
$$\left(a + b\sqrt{-1}\right) + \left((-a) + (-b)\sqrt{-1}\right) = 0 + 0$$
$$\left(a + b\sqrt{-1}\right) + \left((-a) + (-b)\sqrt{-1}\right) = 0$$

$$\left(a + b\sqrt{-1}\right) * \left(x + y\sqrt{-1}\right) = 1$$
$$ax + bx\sqrt{-1} + ay\sqrt{-1} - by = 1$$
$$ax - by + bx\sqrt{-1} + ay\sqrt{-1} = 1$$
$$bx + ay = 0$$
$$bx = -ay$$
$$x = -\frac{ay}{b}$$
$$ax - by = 1$$
$$a\left(-\frac{ay}{b}\right) - by = 1$$
$$-\frac{a^2y}{b} - by = 1$$
$$-a^2y - b^2y = b$$
$$y(-a^2 - b^2) = b$$
$$y = \frac{b}{(-a^2 - b^2)}$$
$$x = -\frac{a\frac{b}{(-a^2 - b^2)}}{b}$$
$$x = -\frac{a}{(-a^2 - b^2)}$$

$$\left(a+b\sqrt{-1}\right)*\left(\left(-\frac{a}{(-a^2-b^2)}\right)+\left(\frac{b}{(-a^2-b^2)}\right)\sqrt{-1}\right)=1$$

2. $\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$ has $\det(0)$ and therefore has no multiplicative inverse

$\mathbb{R}^{n*n}$ is not a field

3.

+	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

*	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

4. $a+bi$

$(a+bi)+(c+di)=a+bi+c+di$
$(a+bi)+(c+di)=a+c+bi+di$
$(a+bi)+(c+di)=(a+c)+(b+d)i$

$(a+bi)*(c+di)=ac+bci+adi-bd$
$(a+bi)*(c+di)=ac-bd+adi+bci$
$(a+bi)*(c+di)=(ac-bd)+(ad+bc)i$

$0+(a+bi)=a+bi$

$1*(a+bi)=a+bi$

$(a+bi)+\left((-a)+(-b)i\right)=(a+bi)+(-a-bi)$
$(a+bi)+\left((-a)+(-b)i\right)=a+bi-a-bi$
$(a+bi)+\left((-a)+(-b)i\right)=a-a+bi-bi$
$(a+bi)+\left((-a)+(-b)i\right)=0+0$
$(a+bi)+\left((-a)+(-b)i\right)=0$

$(a+bi)*(x+yi)=1$
$ax+bxi+ayi-by=1$
$ax-by+bxi+ayi=1$
$bx+ay=0$
$bx=-ay$

$$x = -\frac{ay}{b}$$
$$ax - by = 1$$
$$a\left(-\frac{ay}{b}\right) - by = 1$$
$$-\frac{a^2y}{b} - by = 1$$
$$-a^2y - b^2y = b$$
$$y(-a^2 - b^2) = b$$
$$y = \frac{b}{(-a^2 - b^2)}$$
$$x = -\frac{a\frac{b}{(-a^2 - b^2)}}{b}$$
$$x = -\frac{a}{(-a^2 - b^2)}$$
$$(a + bi) * \left(\left(-\frac{a}{(-a^2 - b^2)}\right) + \left(\frac{b}{(-a^2 - b^2)}\right)i\right) = 1$$

5. A is not in reduced row-echelon form because the leading 1 in row three is not in a column to the right of the leading 1 in row two.

B is in reduced row-echelon form because the leading non-zero term in each row is 1 and each 1 is to the right of the 1 in the row above it, and the row of all zero elements is the final row.

C is not in reduced row-echelon form because the row of all zero elements is not the final row.

D is in reduced row-echelon form because the leading non-zero term in each row is 1 and each 1 is to the right of the 1 in the row above it, and the row of all zero elements is the final row.

E is not in reduced row-echelon form because the leading non-zero term in each row is not always 1.

6. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$
$$A + B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$
$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$
$$AB = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

7. $\det(A) \neq 0$

$$\begin{vmatrix} 6 & -1 & 1 \\ t & 0 & 1 \\ 0 & 1 & t \end{vmatrix} \neq 0$$
$$-1\begin{vmatrix} 6 & 1 \\ t & 1 \end{vmatrix} + t\begin{vmatrix} 6 & -1 \\ t & 0 \end{vmatrix} \neq 0$$

$-1(6-t)+t(0+t)\neq 0$
$-1(6-t)+t(t)\neq 0$
$t-6+t^2\neq 0$
$t^2+t-6\neq 0$
$(t-2)(t+3)\neq 0$
$t\neq -3,2$

8.

A. $\begin{bmatrix} 1 & \mathrm{h} & | & 4 \\ 3 & 6 & | & 8 \end{bmatrix}$

$-3r_1+r_2$

$\begin{bmatrix} 1 & \mathrm{h} & | & 4 \\ 0 & 6-3\mathrm{h} & | & -4 \end{bmatrix}$

$6-3\mathrm{h}\neq 0$
$3\mathrm{h}\neq 6$
$\mathrm{h}\neq 2$

B. $\begin{bmatrix} -4 & 12 & | & \mathrm{h} \\ 2 & -6 & | & -3 \end{bmatrix}$

$-\frac{1}{4}r_1$

$\begin{bmatrix} 1 & -3 & | & -\frac{\mathrm{h}}{4} \\ 2 & -6 & | & -3 \end{bmatrix}$

$-2r_1+r_2$

$\begin{bmatrix} 1 & -3 & | & -\frac{\mathrm{h}}{4} \\ 0 & 0 & | & \frac{\mathrm{h}}{2}-3 \end{bmatrix}$

$\frac{\mathrm{h}}{2}-3=0$
$\frac{\mathrm{h}}{2}=3$
$\mathrm{h}=6$

9.

A. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Three types

B. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 0 \end{bmatrix}$

Six types

C. $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

One type

10. $A = \begin{bmatrix} 1 & a & b & 3 & 0 & -2 \\ 0 & 0 & c & 1 & d & 3 \\ 0 & e & 0 & 0 & 1 & 1 \end{bmatrix}$

$a = any\,real\,number$

$b = 0$

$c = 1$

$d = 0$

$e = 0$

$A = \begin{bmatrix} 1 & * & 0 & 3 & 0 & -2 \\ 0 & 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

11.

A. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix}$

$-r_1 + r_2$

$-2r_1 + r_3$

$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -2 \\ 0 & -4 & -5 & -6 \end{bmatrix}$

$-r_2$

$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & -4 & -5 & -6 \end{bmatrix}$

$-2r_2 + r_1$

$4r_2 + r_3$

$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 7 & 2 \end{bmatrix}$

$\frac{1}{7} r_2$

$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix}$

$3r_3 + r_1$

$-3r_3 + r_2$

$\begin{bmatrix} 1 & 0 & 0 & \frac{6}{7} \\ 0 & 1 & 0 & \frac{8}{7} \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix}$

B. $A = \begin{bmatrix} [1] & [2] & [3] & [4] \\ [1] & [1] & [0] & [2] \\ [2] & [0] & [1] & [2] \end{bmatrix}$

$[6]r_1 + r_2$
$[5]r_1 + r_3$

$$A = \begin{bmatrix} [1] & [2] & [3] & [4] \\ [0] & [6] & [4] & [5] \\ [0] & [3] & [2] & [1] \end{bmatrix}$$

$[6]r_2$

$$A = \begin{bmatrix} [1] & [2] & [3] & [4] \\ [0] & [1] & [3] & [2] \\ [0] & [3] & [2] & [1] \end{bmatrix}$$

$[5]r_2 + r_1$
$[4]r_2 + r_3$

$$A = \begin{bmatrix} [1] & [0] & [4] & [0] \\ [0] & [1] & [3] & [2] \\ [0] & [0] & [0] & [2] \end{bmatrix}$$

$[4]r_3$

$$A = \begin{bmatrix} [1] & [0] & [4] & [0] \\ [0] & [1] & [3] & [2] \\ [0] & [0] & [0] & [1] \end{bmatrix}$$

$[5]r_3 + r_2$

$$A = \begin{bmatrix} [1] & [0] & [4] & [0] \\ [0] & [1] & [3] & [0] \\ [0] & [0] & [0] & [1] \end{bmatrix}$$

C.

```
GF7 = galois.GF(7)
```

```
B= GF7([[1,2,3,4],
    [1, 1, 0,2],
    [2,0, 1, 2]]);
```

```
GF7.row_reduce(B)
```

```
GF([[1, 0, 4, 0],
    [0, 1, 3, 0],
    [0, 0, 0, 1]], order=7)
```

```
GF3 = galois.GF(3)
```

```
B= GF3([[1,2,0,1],
    [1, 1, 0,2],
    [2,0, 1, 2]]);
```

```
GF7.row_reduce(B)
```

```
GF([[1, 0, 0, 0],
    [0, 1, 0, 2],
    [0, 0, 1, 2]], order=3)
```

```python
GF3 = galois.GF(2)
```

```python
B= GF3([[1,0,1,0],
    [1, 1, 0,0],
    [0,0, 1, 0]]);
```

```python
GF7.row_reduce(B)
```

```
GF([[1, 0, 0, 0],
    [0, 1, 0, 0],
    [0, 0, 1, 0]], order=2)
```

D.Yes

12.

A. $\left(A\middle|\vec{b}\right) = \begin{bmatrix} [3] & [1] & [4] & | & [1] \\ [5] & [2] & [6] & | & [5] \\ [0] & [5] & [2] & | & [1] \end{bmatrix}$

$[5]r_1$

$\begin{bmatrix} [1] & [5] & [6] & | & [5] \\ [5] & [2] & [6] & | & [5] \\ [0] & [5] & [2] & | & [1] \end{bmatrix}$

$[2]r_1 + r_2$

$\begin{bmatrix} [1] & [5] & [6] & | & [5] \\ [0] & [5] & [4] & | & [1] \\ [0] & [5] & [2] & | & [1] \end{bmatrix}$

$[3]r_2$

$\begin{bmatrix} [1] & [5] & [6] & | & [5] \\ [0] & [1] & [5] & | & [3] \\ [0] & [5] & [2] & | & [1] \end{bmatrix}$

$[2]r_2 + r_1$

$[2]r_2 + r_3$

$\begin{bmatrix} [1] & [0] & [2] & | & [4] \\ [0] & [1] & [5] & | & [3] \\ [0] & [0] & [5] & | & [0] \end{bmatrix}$

$[3]r_3$

$\begin{bmatrix} [1] & [0] & [2] & | & [4] \\ [0] & [1] & [5] & | & [3] \\ [0] & [0] & [1] & | & [0] \end{bmatrix}$

$[5]r_3 + r_1$

$[2]r_3 + r_2$

$\begin{bmatrix} [1] & [0] & [0] & | & [4] \\ [0] & [1] & [0] & | & [3] \\ [0] & [0] & [1] & | & [0] \end{bmatrix}$

B. $\vec{x} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 3 & 1 & 4 \\ 5 & 2 & 6 \\ 0 & 5 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$$

13.

```
B= Matrix([[3,11,19,-2],
    [7, 23, 39,10],
    [-4,-3,-2,6]]);
```

```
Matrix.rref(B)
```

```
(Matrix([
 [1, 0, -1, 0],
 [0, 1,  2, 0],
 [0, 0,  0, 1]]),
 (0, 1, 3))
```

No solutions

14.

```
B= Matrix([[3,6,9,5,25,53],
    [7, 14, 21,9,53,105],
    [-4,-8,-12,5,-10,11]]);
```

```
Matrix.rref(B)
```

```
(Matrix([
 [1, 2, 3, 0, 5, 6],
 [0, 0, 0, 1, 2, 7],
 [0, 0, 0, 0, 0, 0]]),
 (0, 3))
```

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6-2r-3s-5t \\ r \\ s \\ 7-2t \\ t \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 7 \\ 9 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r + \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -5 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} t$$

15.

```
B= Matrix([[2,4,3,5,6,37],
    [4,8,7,5,2,74],
    [-2,-4,3,4,-5,20],
    [1,2,2,-1,2,26],
    [5,-10,4,6,4,24]]);
```

```
Matrix.rref(B)
```

```
(Matrix([
 [1, 0, 0, 0, 0, -8221/4340],
 [0, 1, 0, 0, 0,  8591/8680],
 [0, 0, 1, 0, 0,   4695/434],
 [0, 0, 0, 1, 0,   -459/434],
 [0, 0, 0, 0, 1,    699/434]]),
 (0, 1, 2, 3, 4))
```

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\frac{8221}{4340} \\ \frac{8591}{8680} \\ \frac{4695}{434} \\ -\frac{459}{434} \\ \frac{699}{434} \end{bmatrix}$$

16.

A. $ABC = I_n$

A matrix M is invertible if $\det(M) \neq 0$

$\det(I_n) \neq 0$

$\det(ABC) \neq 0$

$\det(A)\det(B)\det(C) \neq 0$

$\det(A), \det(B), \det(C) \neq 0$

A, B, C are all invertible

B. $\det(AB) \neq 0$

$\det(A)\det(B) \neq 0$

$\det(A), \det(B) \neq 0$

A, B are both invertible

17. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$

$(AB)^2 = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 15 \\ 10 & 15 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

$B^2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$

$A^2B^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 10 & 16 \\ 10 & 16 \end{bmatrix}$

$\begin{bmatrix} 10 & 15 \\ 10 & 15 \end{bmatrix} \neq \begin{bmatrix} 10 & 16 \\ 10 & 16 \end{bmatrix}$

$(AB)^2 \neq A^2B^2$

18. $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

$A^T = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$

$A^{-1} = \frac{1}{\cos(\theta)\cos(\theta) - \sin(\theta)\,(-\sin(\theta))}\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$

$A^{-1} = \frac{1}{\cos^2(\theta) + \sin^2(\theta)}\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$
$$A^{-1} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$
$$A^T = A^{-1}$$

19.

A. Symmetric

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Skew-Symmetric

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$
$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

B. The main diagonal must be all zeros.

C. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is both symmetric and skew-symmetric

D. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$
$$A + A^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix}$$

Symmetric

$$AA^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix}$$

Symmetric

$$A^T A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{bmatrix}$$

Symmetric

$$A - A^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & b-c \\ -(b-c) & 0 \end{bmatrix}$$

Skew-symmetric

E. $\begin{bmatrix} a & b \\ b & c \end{bmatrix} + \begin{bmatrix} 0 & d \\ -d & 0 \end{bmatrix} = \begin{bmatrix} a & b+d \\ b-d & c \end{bmatrix}$

$$b + d = e$$

$b - d = f$
$\begin{bmatrix} a & e \\ f & c \end{bmatrix}$

20.

A. N/A

B. Bijective

C. Surjective

D. Injective

21. $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & \frac{15}{4} & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ 0 & \frac{4}{15} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & \frac{15}{4} & 1 & 0 \\ 0 & 0 & \frac{56}{15} & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ 0 & \frac{4}{15} & 1 & 0 \\ 0 & 0 & \frac{15}{56} & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & \frac{15}{4} & 1 & 0 \\ 0 & 0 & \frac{56}{15} & 1 \\ 0 & 0 & 0 & \frac{209}{56} \end{bmatrix}$

22. $A = \begin{bmatrix} q_1 & r_1 & 0 & 0 \\ p_1 & q_2 & r_2 & 0 \\ 0 & p_2 & q_3 & r_3 \\ 0 & 0 & p_3 & q_4 \end{bmatrix}$

$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_1 & 1 & 0 & 0 \\ 0 & l_2 & 1 & 0 \\ 0 & 0 & l_3 & 1 \end{bmatrix}$

$U = \begin{bmatrix} d_1 & u_1 & 0 & 0 \\ 0 & d_2 & u_2 & 0 \\ 0 & 0 & d_3 & u_3 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$

$A = LU$

$$\begin{bmatrix} q_1 & r_1 & 0 & 0 \\ p_1 & q_2 & r_2 & 0 \\ 0 & p_2 & q_3 & r_3 \\ 0 & 0 & p_3 & q_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_1 & 1 & 0 & 0 \\ 0 & l_2 & 1 & 0 \\ 0 & 0 & l_3 & 1 \end{bmatrix} \begin{bmatrix} d_1 & u_1 & 0 & 0 \\ 0 & d_2 & u_2 & 0 \\ 0 & 0 & d_3 & u_3 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$$

$$\begin{bmatrix} q_1 & r_1 & 0 & 0 \\ p_1 & q_2 & r_2 & 0 \\ 0 & p_2 & q_3 & r_3 \\ 0 & 0 & p_3 & q_4 \end{bmatrix} = \begin{bmatrix} d_1 & u_1 & 0 & 0 \\ l_1 d_1 & l_1 u_1 + d_2 & u_2 & 0 \\ 0 & l_2 d_2 & l_2 u_2 + d_3 & u_3 \\ 0 & 0 & l_3 d_3 & l_3 u_3 + d_4 \end{bmatrix}$$

For $1 \leq i \leq 4$ and $a_0 = 0$:

$p_i = l_i d_i$

$q_i = l_{i-1} u_{i-1} + d_i$

$r_i = u_i$

23.

24.

A. $H_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} - 2 \begin{bmatrix} \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{n}} \\ \vdots \\ \frac{1}{\sqrt{n}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \end{bmatrix}$

$$H_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} - 2 \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix}$$

$$H_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{n} & \frac{2}{n} & \cdots & \frac{2}{n} \\ \frac{2}{n} & \frac{2}{n} & \cdots & \frac{2}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{n} & \frac{2}{n} & \cdots & \frac{2}{n} \end{bmatrix}$$

$$H_n = \begin{bmatrix} 1 - \frac{2}{n} & -\frac{2}{n} & \cdots & -\frac{2}{n} \\ -\frac{2}{n} & 1 - \frac{2}{n} & \cdots & -\frac{2}{n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{2}{n} & -\frac{2}{n} & \cdots & 1 - \frac{2}{n} \end{bmatrix}$$

$$H_n^T = \begin{bmatrix} 1 - \frac{2}{n} & -\frac{2}{n} & \cdots & -\frac{2}{n} \\ -\frac{2}{n} & 1 - \frac{2}{n} & \cdots & -\frac{2}{n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{2}{n} & -\frac{2}{n} & \cdots & 1 - \frac{2}{n} \end{bmatrix}$$

$H_n = H_n^T$

H_n is a symmetric matrix

B. $H_n^T H_n = \begin{bmatrix} 1-\frac{2}{n} & -\frac{2}{n} & \cdots & -\frac{2}{n} \\ -\frac{2}{n} & 1-\frac{2}{n} & \cdots & -\frac{2}{n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{2}{n} & -\frac{2}{n} & \cdots & 1-\frac{2}{n} \end{bmatrix} \begin{bmatrix} 1-\frac{2}{n} & -\frac{2}{n} & \cdots & -\frac{2}{n} \\ -\frac{2}{n} & 1-\frac{2}{n} & \cdots & -\frac{2}{n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{2}{n} & -\frac{2}{n} & \cdots & 1-\frac{2}{n} \end{bmatrix}$

$H_n^T H_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$

$H_n^T H_n = I_n$

H_n is an orthogonal matrix

C. $H_n^2 = H_n H_n^T = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$

$H_n^2 = I_n$

D. $H_n\vec{u} = \begin{bmatrix} 1-\frac{2}{n} & -\frac{2}{n} & \cdots & -\frac{2}{n} \\ -\frac{2}{n} & 1-\frac{2}{n} & \cdots & -\frac{2}{n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{2}{n} & -\frac{2}{n} & \cdots & 1-\frac{2}{n} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{n}} \\ \vdots \\ \frac{1}{\sqrt{n}} \end{bmatrix}$

$H_n\vec{u} = \begin{bmatrix} -\sqrt{n} \\ -\sqrt{n} \\ \vdots \\ -\sqrt{n} \end{bmatrix}$

E. $H_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

$H_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$

$H_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$

$$H_3 = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$H_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - 2 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$H_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - 2 \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

$$H_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$H_4 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$