Homework 1

1) a) $a+b\sqrt{2}$ where a and b are rational numbers.

1) Identity for sum : $(a+b\sqrt{2})+0 = (a+b\sqrt{2})$; Identity for mult. : $(a+b\sqrt{2})\cdot 1 = a+b\sqrt{2}$

2) Associativity for sum : $[(a+b\sqrt{2})+(c+d\sqrt{2})]+(e+f\sqrt{2}) = (a+b\sqrt{2})+[(c+d\sqrt{2})+(e+f\sqrt{2})]$ ✓

3) Inverse for sum : $(a+b\sqrt{2})+(-a-b\sqrt{2}) = 0$

$= -(a+\sqrt{2}\cdot b)+(a+b\sqrt{2}) = 0$

4) Commutativity for sum :

$(a+b\sqrt{2})+(c+d\sqrt{2}) = (c+d\sqrt{2})+(a+b\sqrt{2})$ ✓

5) Inverse for sum : $-(a+b\sqrt{2}) = -a-b\sqrt{2}$

$(a+b\sqrt{2})-a-b\sqrt{2} = 0$

6) Inverse for multiplicative :

$(a+b\sqrt{2})\cdot(x+y\sqrt{2}) = 1$

$x = \dfrac{1}{(a+b\sqrt{2})} - y\cdot\sqrt{2}$

$y = \dfrac{1}{\sqrt{2}\cdot(a+b\sqrt{2})} - \dfrac{x}{\sqrt{2}}$

7) Associativity for x : $(a\cdot b)\cdot c = a\cdot(b\cdot c)$

$(1)\rightarrow (a+b\sqrt{2})\cdot(c+d\sqrt{2})\cdot(e+f\sqrt{2}) = (1)$

$=(ac+ad\sqrt{2}+bc\cdot\sqrt{2}+bd\sqrt{2}\cdot\sqrt{2})\cdot(e+f\sqrt{2})$

$(*)\rightarrow\ = ace+acf\cdot\sqrt{2}+ade\sqrt{2}+adf\cdot 2\sqrt{2}+bce\cdot\sqrt{2}+bcf\cdot\sqrt{2}+bde+dbf\cdot\sqrt{2}^3$

$(2)\rightarrow (c+d\sqrt{2})\cdot(e+f\sqrt{2})\cdot(a+b\sqrt{2}) = (2)$

$=(ce+\sqrt{2}\cdot fc+\sqrt{2}\cdot de+2fd)\cdot(a+b\sqrt{2})$

$(+)\rightarrow\ = ace+bce\cdot\sqrt{2}+acf\cdot\sqrt{2}+bcf\cdot 2+aed\cdot\sqrt{2}+bed\cdot 2+2\cdot fda+fbd\cdot\sqrt{2}^3$

$(*) = (+) \Rightarrow (1) = (2).$

8) Commutativity for product:

$(a+b\sqrt{2})(c+d\sqrt{2}) = ac+ad\sqrt{2}+bc\sqrt{2}+2bd$

$(c+d\sqrt{2})(a+b\sqrt{2}) = ac+cb\sqrt{2}+ad\sqrt{2}+2bd$] = ✓

9) Distributivity : $a(b+c) = ab+ac$

$(b+c)(a) = ba+ca.$

$(a+b\sqrt{2})(c+d\sqrt{2}+e+f\sqrt{2}) = ac+ad\sqrt{2}+ae+\sqrt{2}\,af+bc\sqrt{2}+2bd+be\sqrt{2}+2bf$

$= ac+ae+2bd+2bf+\sqrt{2}(ad+af+bc+be)$ ✓

$(c+d\sqrt{2}+e+f\sqrt{2})(a+b\sqrt{2}) = ac+cb\sqrt{2}+ad\sqrt{2}+2bd+ea+eb\sqrt{2}+af\sqrt{2}+2bf$

$= ac+ae+2bd+2bf+\sqrt{2}(cb+ad+eb+af)$ ✓

1) b) This is field $\mathbb{Q}(i)$

$a + b\sqrt{-1}$, where a and b are real numbers.

1) Identity for sum $= 0$: $(a + b\sqrt{-1}) + 0 = a + b\sqrt{-1}$

Identity for mult $= 1$: $(a + b\sqrt{-1}) \cdot 1 = a + b\sqrt{-1}$

2) Associativity for sum : $((a + b\sqrt{-1}) + (c + d\sqrt{-1})) + e + f\sqrt{-1} = (a + b\sqrt{-1}) + [(c + d\sqrt{-1}) + (e + f\sqrt{-1})]$

3) Inverse for sum : $(a + b\sqrt{-1}) + (-a - b\sqrt{-1}) = 0$

$= -(a + \sqrt{-1} \cdot b) + (a + b\sqrt{-1}) = 0$

4) Commutativity for sum

$(a + b\sqrt{-1}) + (c + d\sqrt{-1}) = a + c + b\sqrt{-1} + d\sqrt{-1}$

$(c + d\sqrt{-1}) + (a + b\sqrt{-1}) = a + c + b\sqrt{-1} + d\sqrt{-1}$

5) Inverse for sum : $-(a + b\sqrt{-1}) = -a - b\sqrt{-1}$

$(a + b\sqrt{-1}) - a - b\sqrt{-1} = 0$

6) Inverse for multiplicative:

$(a + b\sqrt{-1})\ (x + y\sqrt{-1}) = 1$

$x = \dfrac{1}{a + b\sqrt{-1}} - y\sqrt{-1}$

$y = \dfrac{1}{\sqrt{-1} \cdot (a + b\sqrt{-1})} - \dfrac{x}{\sqrt{-1}}$

$y = \dfrac{1}{-b + a \cdot \sqrt{-1}} - x\,(-1)^{-1/2}$

7) Associativity for mult : $(a \cdot b) c = a \cdot (b \cdot c)$

(1) $(a + b\sqrt{-1})(c + d\sqrt{-1})(e + f\sqrt{-1})$

$= (ac + ad\sqrt{-1} + bc\sqrt{-1} - bd)\ (e + f\sqrt{-1})$

$= ace + acf\sqrt{-1} + ade\sqrt{-1} - adf + bce\sqrt{-1} - bcf - bde - bdf\sqrt{-1}$ $(*)$

(2) $(c + d\sqrt{-1})(e + f\sqrt{-1})\ (a + b\sqrt{-1})$

$= (ce + cf\sqrt{-1} + de\sqrt{-1} - df)\ (a + b\sqrt{-1})$

$= (ace + ceb\sqrt{-1} + acf\sqrt{-1} - bcf + ead\sqrt{-1} - bed - daf - dfb\sqrt{-1}$ $(+)$

Since $(*) = (+) \Rightarrow (1) = (2)$

8) Commutativity for product

$(a + b\sqrt{-1})(c + d\sqrt{-1}) = ac + ad\sqrt{-1} + bc\sqrt{-1} - bd$

$(c + d\sqrt{-1})(a + b\sqrt{-1}) = ac + ab\sqrt{-1} + ad\sqrt{-1} - bd$

$\Big\} = \checkmark$

9) Distributivity: $a(b+c) = ab + ac$; $(b+c)a = ba + ca$

$\checkmark$ $(a + b\sqrt{-1})(c + d\sqrt{-1} + e + f\sqrt{-1}) = ac + ad\sqrt{-1} + ae + \sqrt{-1}\,af + \sqrt{-1}\,bc - bd + be\sqrt{-1} - bf$

$=$

$\checkmark$ $(c + d\sqrt{-1} + e + f\sqrt{-1})(a + b\sqrt{-1}) = ac + cb\sqrt{-1} + ad\sqrt{-1} - bd + ea + eb\sqrt{-1} + af\sqrt{-1} - bf$

Homework 1 (1, 4, 8a, 11.2-4, 12-15, 16-17

2) Set of all $n \times n$ matrices $\mathbb{R}^{n \times n}$ is not a field b/c it is not commutative.
$a \cdot b \neq b \cdot a$.

Eg:
$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 5 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}.$$

3) Operations on $\mathbb{Z}_3$

+	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

x	0	[1]	[2]
0	[0]	[0]	[0]
1	[0]	[1]	[2]
2	[0].	[2]	[1]

5) D.

6) $A+B$ in $\mathbb{Z}_2$ $= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

A^2 in $\mathbb{Z}_2$ $= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$

7) $A = \begin{bmatrix} 6 & -1 & 1 \\ t & 0 & 1 \\ 0 & 1 & t \end{bmatrix}.$

$$\det(A) = -\begin{vmatrix} 6 & 1 \\ t & 1 \end{vmatrix} + t\begin{vmatrix} 6 & -1 \\ t & 0 \end{vmatrix} = -(6-t) + t(t)$$
$$= -6 + t + 2t = 3t - 6$$

$3t - 6 \neq 0$
$3t \neq 6$
$t \neq 2$

$\rightarrow$ If $t = 2$, A doesn't have an inverse

8) a) $\left[\begin{array}{cc|c} 1 & h & 4 \\ 3 & 6 & 8 \end{array}\right] = \left[\begin{array}{cc|c} 1 & h & 4 \\ 0 & 6-3h & -4 \end{array}\right]$

$\cancel{6} \neq \cancel{3h} \neq -4$
$\cancel{6} + 4 \neq \cancel{3h}$
$10/3 \neq h.$

$= \left[\begin{array}{cc|c} 3 & 3h & 12 \\ 3 & 6 & 8 \end{array}\right] \rightarrow h \neq 2.$

$6 - 3h \neq 0$
$6 \neq 3h$
$2 \neq h$

$x_1 + hx_2 = 4$
$3x_1 + 6x_2 = 8$
$(6-3h)x_2 = -4$
$(6-3h)x_2 + 8 = x_1 + hx_2$
$6x_2 - 3hx_2 + 8 = x_1 + hx_2$
$6x_2 - 4hx_2 = x_1 - 8$
$(6-4h)x_2 = x_1 - 8.$

$\left[\begin{array}{cc|c} 1 & h & 4 \\ 0 & 1 & \frac{-4}{6-3h} \end{array}\right]$

b) $\left[\begin{array}{cc|c} -4 & 12 & h \\ 2 & -6 & -3 \end{array}\right]$

$h - 6 = 0$
$h = 6$

$= \left[\begin{array}{cc|c} 0 & 0 & h-6 \\ 2 & -6 & -3 \end{array}\right]$

9) $\begin{bmatrix} x & x \\ x & x \\ x & x \end{bmatrix}$

$\begin{matrix} x & x & x \\ x & x & x \end{matrix}$

$[x \quad x \quad x \quad x]$

a) 3
b) 5
c) $1 + 4 + 4 + 6 = 15.$

a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$

b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$

c) ... $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}; \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}; \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$

10). $a = 0$
$b = 0$
$c \geq 0$
$d = 0$
$e = 1$

1) 12) Using Python:

$$\left[\begin{array}{ccc|c} 3 & 1 & 4 & 1 \\ 5 & 2 & 6 & 5 \\ 0 & 5 & 2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right] \text{ over } \mathbb{Z}_7$$

$\Rightarrow$ $x_1 = 4$

$x_2 = 3$ in $\mathbb{Z}_7$.

$x_3 = 0$

13) $\left[\begin{array}{ccc|c} 3 & 11 & 19 & -2 \\ 7 & 23 & 39 & 10 \\ -4 & -3 & -2 & 6 \end{array}\right] \rightarrow$ no solution

14) $\left[\begin{array}{ccccc|c} 3 & 6 & 9 & 5 & 25 & 53 \\ 7 & 14 & 21 & 9 & 53 & 105 \\ -4 & -8 & -12 & 5 & -10 & 11 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 0 & 5 & 6 \\ 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = 6 - 2x_2 - 3x_3 - 5x_5$

$x_2 = x_2$

$x_3 = x_3$

$x_4 = 7 - 2x_5$

$x_5 = x_5$

$\rightarrow$ Ans $\vec{x} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 7 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

15) $\left[\begin{array}{ccccc|c} 2 & 4 & 3 & 5 & 6 & 37 \\ 4 & 8 & 7 & 5 & 2 & 74 \\ -2 & -4 & 3 & 4 & -5 & 20 \\ 1 & 2 & 2 & -1 & 2 & 26 \\ 5 & -10 & 4 & 6 & 4 & 24 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -8221/4340 \\ 0 & 1 & 0 & 0 & 0 & 8571/8680 \\ 0 & 0 & 1 & 0 & 0 & 4695/434 \\ 0 & 0 & 0 & 1 & 0 & -459/434 \\ 0 & 0 & 0 & 0 & 1 & 699/434 \end{array}\right]$

$\vec{x} = \begin{bmatrix} -8221/4340 \\ 8591/8680 \\ 4695/434 \\ -459/434 \\ 699/434 \end{bmatrix}$

16) b) $\det(AB) = \det(A) \cdot \det(B)$

if AB is invertible, $\det(AB) \neq 0$

$\rightarrow \det(A) \cdot \det(B) \neq 0 \rightarrow$ A and B are invertible

$\rightarrow$ a) If $ABC = I_n$; $\det(ABC) \neq 0$; hence ABC is invertible

from part b), we can conclude that if $\det(ABC) \neq 0 \Rightarrow \det(AB) \neq 0$; $\det(C) \neq 0$

$\Rightarrow \det(A) \neq 0$; $\det(B) \neq 0 \Rightarrow A, B, C$ are invertible

If $A, B = n \times n = I_n$; $AB = BA$; $A = B^{-1}$; $A^{-1} = B$

$\Rightarrow I_n = (AB)C = A(BC) \Rightarrow C^{-1} = AB$; $A^{-1} = BC$

$\Rightarrow I_n = C(AB) = CA(B) \Rightarrow B^{-1} = CA$

17) Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$(AB)^2 = \begin{bmatrix} 3 & 4 \\ 11 & 16 \end{bmatrix}^2 = \begin{bmatrix} 53 & 76 \\ 209 & 300 \end{bmatrix} \ (*)$$

$$A^2 B^2 = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}^2 \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^2 = \begin{bmatrix} 2 & 3 \\ 6 & 11 \end{bmatrix} \cdot \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} = \begin{bmatrix} 59 & 86 \\ 207 & 302 \end{bmatrix} \ (+)$$

$(*) \neq (+)$

$\therefore (AB)^2 \neq A^2 B^2$

18) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

19) a) <u>Symmetric matrix:</u>

$2\times2 \rightarrow \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}$ $\quad 3\times3 \rightarrow \begin{bmatrix} 0 & 2 & 3 \\ 4 & 0 & 4 \\ 3 & 2 & 0 \end{bmatrix}$ $\quad 4\times4 \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 2 & 2 \\ 2 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

<u>Skew - symmetric</u> :

$2\times2 \rightarrow \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ $\quad 3\times3 \rightarrow \begin{bmatrix} 0 & 1 & -2 \\ 3 & 0 & 3 \\ 2 & -1 & 0 \end{bmatrix}$ $\quad 4\times4 \rightarrow \begin{bmatrix} 0 & 2 & -2 & 0 \\ -1 & 0 & 0 & 3 \\ -3 & 0 & 0 & 1 \\ 0 & 2 & -2 & 0 \end{bmatrix}$

b) it stays the same. ; made no of 0s

c) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

d) We know $(A^T)^T = A$ and $(AB)^T = B^T \cdot A^T$

$\therefore (A^T A)^T = A^T (A^T)^T = A^T A$ $\rightarrow A \cdot A^T$ is symmetric

$(A A^T)^T = (A^T)^T \cdot A^T = A \cdot A^T$ $\rightarrow A^T \cdot A$ is symmetric

We know $(A+B)^T = A^T + B^T$

$\therefore (A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T \rightarrow A + A^T$ is symmetric

$(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T) \rightarrow A - A^T$ is skew symmetric

e) Let $B = A + A^T$

$B^T = (A + A^T)^T$

$B^T = A^T + A$ (proof from part d))

$B^T = B$

$\therefore B = A + A^T$ is symmetric

Let $C = A - A^T$

$C^T = (A - A^T)^T$

$C^T = -(A - A^T)$ (proof from part d)

$C^T = -C$

$\therefore C = -A + A^T$ is skew-symmetric

20. a) bijective
b) bijective
c) surjective
d) injective.

21)

$$A = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \qquad [L] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 0 & 4/15 & 1 & 0 \\ 0 & 0 & 15/56 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 15/4 & 1 & 0 \\ 0 & 0 & 56/15 & 1 \\ 0 & 0 & 0 & 209/56 \end{bmatrix} = [U]$$

22)

$$LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_1 & 1 & 0 & 0 \\ 0 & l_2 & 1 & 0 \\ 0 & 0 & l_3 & 1 \end{bmatrix} \begin{bmatrix} d_1 & u_1 & 0 & 0 \\ 0 & d_2 & u_2 & 0 \\ 0 & 0 & d_3 & u_3 \\ 0 & 0 & 0 & d_4 \end{bmatrix} = \begin{bmatrix} d_1 & u_1 & 0 & 0 \\ l_1 d_1 & l_1 u_1 + d_2 & u_2 & 0 \\ 0 & l_2 d_2 & l_2 u_2 + d_3 & u_3 \\ 0 & 0 & l_3 d_3 & l_3 d_3 + d_4 \end{bmatrix}$$

$$A = \begin{bmatrix} q_1 & v_1 & 0 & 0 \\ p_1 & q_2 & v_2 & 0 \\ 0 & p_2 & q_3 & v_3 \\ 0 & 0 & p_3 & q_4 \end{bmatrix}$$

$\rightarrow v_1 = u_1$; $v_2 = u_2$; $v_3 = u_3$ $\Rightarrow v_i = u_i$

$p_i = l_i d_i$

$q_i = d_i + l_{i-1} \cdot u_{i-1}$

23)

$$[U] = \begin{bmatrix} d_1 & u_1 & \cdots & \\ & d_2 & u_2 & \vdots \\ & & \ddots & u_n \\ & & & d_n \end{bmatrix} = \begin{array}{c} d_1 \\ \shortparallel \\ \begin{bmatrix} 4 & 1 & 0 & 0 & \cdots \\ 0 & (4 - 1/d_1) & 1 & \cdots & \\ 0 & 0 & (4 - 1/d_2) & 1 & \\ \vdots & & & \ddots & 1 \\ \vdots & \cdots & & & 4 - 1/d_{n-1} \end{bmatrix} \end{array}$$

$$[L] = \begin{bmatrix} 1 & \cdots & & & 0 \\ 1/4 & 1 & & & \vdots \\ & 1/(4-l_1) & 1 & & \vdots \\ \vdots & & 1/(4-l_2) & 1 & \vdots \\ 0 & \cdots & & 1/(4-l_{n-1}) & 1 \end{bmatrix}$$

24) 1) H_n is symmetric if $H_n = H_n^T$

$H_n = I_n - 2\vec{u}\cdot\vec{u}^T$

$\rightarrow H_n^T = (I_n - 2\vec{u}\cdot\vec{u}^T)^T$

$= I_n^T - 2(\vec{u})^T\cdot(\vec{u}^T)^T$

$= I_n - 2\vec{u}^T\cdot\vec{u}$

$= H_n$

$\Rightarrow H_n^T = H_n$

$\Rightarrow H_n$ is symmetrical

2) Since $H_n^T = H_n$; H_n is orthogonal if $H_n \cdot H_n = I_n$.

$H_n H_n = (I_n - 2\vec{u}\vec{u}^T)(I_n - 2\vec{u}\vec{u}^T)$

$= I_n^2 - 2\vec{u}\cdot\vec{u}^T\cdot I_n - 2\vec{u}\cdot\vec{u}^T\cdot I_n + 4\vec{u}\cdot\vec{u}^T\cdot\vec{u}\cdot\vec{u}^T$

$= I_n - 4(\vec{u}\cdot\vec{u}^T) + 4(\vec{u}\cdot\vec{u}^T)(\vec{u}\cdot\vec{u}^T)$

$= I_n - 4(1) + 4(1)$

$= I_n$

$= H_n H_n^T$

* Since $\vec{u}$ is an unit vector, $\vec{u}\cdot\vec{u}^T = \|u\|^2 = 1$

Hence, H_n is orthogonal.

3) From (2), we can conclude that $(H_n)^2 = I_n$

4) $H_n\vec{u} = (I_n - 2\vec{u}\cdot\vec{u}^T)\cdot\vec{u}$

$= \vec{u} - 2(1)\cdot\vec{u}$

$= \vec{u} - 2\vec{u}$

$= -(\vec{u})$

5) $H_3 = I_3 - 2\cdot\vec{u}\cdot\vec{u}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\cdot\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}\cdot\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$

$H_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \frac{2}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} +1 & -1 & -1 & -1 \\ -1 & +1 & -1 & -1 \\ -1 & -1 & +1 & -1 \\ -1 & -1 & -1 & +1 \end{bmatrix}$