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Math 4570

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Homework 1

1] 1] We define our field $F = \{a+b\sqrt{2} \mid a,b \in \mathbb{Q}\}$ with the operations sum + and product X. These are defined by the standard sum & product of polynomials. To prove this is a field, we need to prove its properties

Identity for sum & product

- Identity 0 is e for the addition operator for the field F as $a+b\sqrt{2}+0 = a+b\sqrt{2}$ and $0 \in F$ as $0 \in \mathbb{Q}$ and $a, b = 0$ then $0+0\sqrt{2}=0$
- 1 is e for the product operator for the field F as $1(a+b\sqrt{2}) = 1a+1b\sqrt{2} = a+b\sqrt{2}$, $1 \in F$ as $0, 1 \in \mathbb{Q}$ then $1+0\sqrt{2} = 1+0=1$

Therefore e exists for both the sum & product operation

Associativity

Sum For $d, f, g \in F$ as defined as $d=a_1+b_1\sqrt{2}$, $f=a_2+b_2\sqrt{2}$ and $g=a_3+b_3\sqrt{2}$. When we add $(d+f)+g = (a_1+b_1\sqrt{2}+a_2+b_2\sqrt{2})+a_3+b_3\sqrt{2} = a_1+b_1\sqrt{2}+(a_2+b_2\sqrt{2}+a_3+b_3\sqrt{2}) = d+(f+g)$. Therefore by the rules of standard sum, sum is associative.

Product For $d, f, g \in F$ where $d=a_1+b_1\sqrt{2}$, $f=a_2+b_2\sqrt{2}$, $g=a_3+b_3\sqrt{2}$. When we multiply $(d \times f) \times g = dfg = d(fg) = d \times (f \times g)$ by definition of standard multiplication

Inverse Sum As per definition of Q, for all $a, b \in Q$ there exists $-a, -b$ therefore

$$(a + b\sqrt{2}) + (-a + -b\sqrt{2}) = (a + b\sqrt{2}) - (a + b\sqrt{2}) = a + b\sqrt{2} - a - b\sqrt{2} = 0$$

Therefore inverses exist for F as Q is a group

Product By definition Q are all numbers that can be expressed as a quotient of two numbers Therefore for any $a, b \in Q$ there exists $1/a$, $1/b \in Q$ therefore for any $a + b\sqrt{2} = f$ there exists an $1/f \in Q$ Therefore $f(1/f) = 1$ thus there exists an inverse for product due to the properties of Q

Commutative Sum By properties of sum, we know that $a + b = b + a$, or that it is commutative therefore if $d, f \in F$ then $d + f = f + d$ therefore if $d = a_1 + b_1\sqrt{2}$ and $f = a_2 + b_2\sqrt{2}$ then

$$a_1 + b_1\sqrt{2} + a_2 + b_2\sqrt{2} = a_2 + b_2\sqrt{2} + a_1 + b_1\sqrt{2}$$

Therefore sum is commutative

Distributivity Product By definition of product, $f(d + h) = fd + fh$ therefore for $a, b, c \in F$ then $a(b + c) = ab + ac$. Therefore product is distributive

Commutativity for Product If $d, f \in F$ where $d = a_1 + b_1\sqrt{2}$ and $f = a_2 + b_2\sqrt{2}$

then $d \times f = (a_1 + b_1\sqrt{2}) \times (a_2 + b_2\sqrt{2}) =$

$(a_2 + b_2\sqrt{2}) \times (a_1 + b_1\sqrt{2}) = f \times d$

therefore $d \times f = f \times d$ by properties of standard product.

1] 1] <u>Inverse for Product</u> If each element in F is of the form $a+b\sqrt{2}$ then

$$(a+b\sqrt{2})^{-1} = x+y\sqrt{2}$$ which means by definition

$$(a+b\sqrt{2}) \times (x+y\sqrt{2}) = 1$$

Therefore this means that

$$ax + ay\sqrt{2} + xb\sqrt{2} + 2by = 1$$ or in other terms

$$(ax+2by) \times (ay+xb)\sqrt{2} = 1$$

We can solve this by setting $ax+2by=1$ and $(ay+xb)\sqrt{2}=0$ to solve for x & y. We can do this by using an augmented matrix

$$\begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ We want to find the inverse to solve the matrix

$$\begin{bmatrix} a & 2b \\ b & a \end{bmatrix}^{-1} = \frac{1}{a^2-2b^2} \begin{bmatrix} a & -2b \\ -b & a \end{bmatrix} =$$

$$\frac{1}{(a+\sqrt{2}b)(a-\sqrt{2}b)} \begin{bmatrix} a & -2b \\ -b & a \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

$$\frac{1}{(a+\sqrt{2}b)(a-\sqrt{2}b)} \begin{bmatrix} a \\ -b \end{bmatrix} = \begin{bmatrix} \frac{a}{a^2-2b^2} \\ \frac{-b}{a^2-2b^2} \end{bmatrix} =$$

$$x = a/a^2-2b^2 \qquad y = b/2b^2-a^2$$ where $a, b \neq 0$

Therefore x & y exist therefore the inverse exists for the product operation

1] 2] We define our field $F = \{a + b\sqrt{-1} \mid a, b \in \mathbb{R}\}$ with the sum, $+$, and product, $\times$, operators. We define these operators by the standard sum & product of polynomials:

<u>Sum</u> $(d + f\sqrt{-1}) + (h + i\sqrt{-1}) = d + h + (f + i)\sqrt{-1}$

<u>Product</u> $(d + f\sqrt{-1}) \times (h + i\sqrt{-1}) =$

$$dh - fi + (di + fh)\sqrt{-1}$$

Here is our proof step by step:

<u>Identity for sum</u> If $x \in F$ then $x = a + b\sqrt{-1}$ where $a, b \in \mathbb{R}$ for all items in $\mathbb{R}$, the identity for sum, is zero, we can see then that

$\underbrace{0 + a}_{a} + b\sqrt{-1} =$ as $a \in \mathbb{R}$ so $a + 0 = a$ therefore 0 is the identity for sum in F. Now we must see if $0 \in F$, we can see that $0 \in \mathbb{R}$ and $a, b \in \mathbb{R}$ thus $0 = a = b$ so $a + b\sqrt{-1} = 0 + 0\sqrt{-1} = 0$ thus $0 \in F$. Therefore we have proven there is an identity for sum.

<u>Associativity for sum</u> For any $x, y, z \in \mathbb{R}$ under sum, $(x + y) + z = x + (y + z)$ if we define $x, y, z \in F$ then we can say that $x = a_1 + b_1\sqrt{-1}$, $y = a_2 + b_2\sqrt{-1}$, $z = a_3 + b_3\sqrt{-1}$ then we can see that

$(x + y) + z = (a_1 + b_1\sqrt{-1} + a_2 + b_2\sqrt{-1}) + a_3 + b_3\sqrt{-1}$ by properties of sum under $\mathbb{R}$ then

$a_1 + b_1\sqrt{-1} + (a_2 + b_2\sqrt{-1} + a_3 + b_3\sqrt{-1})$ which means that $x + (y + z)$ thus

1] 2] $(x+y)+z = x+(y+z)$ therefore sum is associative for F

1) <u>Inverse for Sum</u> Under sum in $\mathbb{R}$, for any $x \in \mathbb{R}$ there exists $-x$ F such that $x+(-x)=0$ therefore for any $a, b \in \mathbb{R}$ there exists $-a, -b$ where $a+(-a)=0$ and $b+(-b)=0$ therefore if $x \in F$ where $x = a+b\sqrt{-1}$ then there exists an element $-x$ defined as $-a-b\sqrt{-1}$ thus

$$x+(-x) = a+b\sqrt{-1}+(-a-b\sqrt{-1}) =$$

$a+b\sqrt{-1}-a-b\sqrt{-1}=0$ thus sum has inverses

<u>Commutativity for Sum</u> Under sum in $\mathbb{R}$ for any $a, b \in \mathbb{R}$

$a+b = b+a$ therefore if $x, y \in F$ then if $x = a_1+b_1\sqrt{-1}$ and $y = a_2+b_2\sqrt{-1}$ then

1)

$$x+y = a_1+b_1\sqrt{-1}+a_2+b_2\sqrt{-1} =$$

$$a_1+a_2+b_1\sqrt{-1}+b_2\sqrt{-1} = a_2+b_2\sqrt{-1}+a_1+b_1\sqrt{-1}$$

$= y+x$ therefore $x+y=y+x$ thus sum is commutative in F

<u>Multiplicative Identity</u> Under product in $\mathbb{R}$, 1 is the identity and for $a \in \mathbb{R}$, $1 \times a = a$ therefore for $x \in F$ as defined by $x = a+b\sqrt{-1}$ where $a, b \in \mathbb{R}$ then

$$1x = 1(a+b\sqrt{-1}) = (a+b\sqrt{-1}) = x.$$

Next, we have to make sure $1 \in F$ since $0, 1 \in \mathbb{R}$ then $1=a$, $0=b$ then $a+b\sqrt{-1} =$

1) $1+0\sqrt{-1} = 1$ therefore $1 \in F$

Therefore F has a multiplicative identity

Associativity for Multiplication For $a, b, c \in \mathbb{R}$ under multiplication

$(a \times b) \times c = a \times (b \times c)$ therefore if $x, y, z \in F$ then $x = a_1 + b_1\sqrt{-1}$, $y = a_2 + b_2\sqrt{-1}$, $z = a_3 + b_3\sqrt{-1}$ then

$$(x \times y) \times z = ([a_1 + b_1\sqrt{-1}] \times [a_2 + b_2\sqrt{-1}]) \times [a_3 + b_3\sqrt{-1}] =$$

$$[a_1 + b_1\sqrt{-1}] \times [a_2 + b_2\sqrt{-1}] \times [a_3 + b_3\sqrt{-1}] =$$

$$[a_1 + b_1\sqrt{-1}] \times ([a_2 + b_2\sqrt{-1}] \times [a_3 + b_3\sqrt{-1}]) =$$

$x \times (y \times z)$ therefore $(x \times y) \times z = x \times (y \times z)$

by properties of product under $\mathbb{R}$ therefore product under F is associative

Distributivity Under product in $\mathbb{R}$ for any $a, b, c \in \mathbb{R}$

$a \times (b + c) = a \times b + c \times a$ Therefore if we define $x, y, z \in F$ as $x = a_1 + b_1\sqrt{-1}$, $y = a_2 + b_2\sqrt{-1}$, $z = a_3 + b_3\sqrt{-1}$ where $a_n, b_n \in \mathbb{R}$ then

$$x \times (y + z) = [a_1 + b_1\sqrt{-1}] \times (a_2 + b_2\sqrt{-1} + a_3 + b_3\sqrt{-1})$$

$$= a_2[a_1 + b_1\sqrt{-1}] + b_2\sqrt{-1}[a_1 + b_1\sqrt{-1}] +$$

$$a_3[a_1 + b_1\sqrt{-1}] + b_3\sqrt{-1}[a_1 + b_1\sqrt{-1}] =$$

$$(a_2 + b_2\sqrt{-1})(a_1 + b_1\sqrt{-1}) + (a_3 + b_3\sqrt{-1})(a_1 + b_1\sqrt{-1})$$

$= xy + xz$ Therefore product is distributive for F. We can also see this in reverse as $(y + z)x \rightarrow (a_2 + b_2\sqrt{-1} + a_3 + b_3\sqrt{-1}) \times [a_1 + b_1\sqrt{-1}] = a_2x + b_2\sqrt{-1}x + a_3x + b_3\sqrt{-1}x = x[a_2 + b_2\sqrt{-1}] + x[a_3 + b_3\sqrt{-1}] = xy + xz$. Therefore it is distributive

1] 2] <u>Commutativity for Product</u> For all $a, b \in \mathbb{R}$ under product $a \times b = b \times a$ therefore if we define $x, y \in F$ as $x = a_1 + b_1\sqrt{-1}$ and $y = a_2 + b_2\sqrt{-1}$ where $a_n, b_n \in \mathbb{R}$ then

$$x \times y = (a_1 + b_1\sqrt{-1}) \times (a_2 + b_2\sqrt{-1}) =$$

$$(a_2 + b_2\sqrt{-1}) \times (a_1 + b_1\sqrt{-1}) = y \times x$$

therefore product is commutative in F

<u>Inverse for Product</u> For all $a \in \mathbb{R}$ there exists $a \in \mathbb{R}$ where $a \times a = 1$. To find the inverse for F we define the calculation

$$(a + b\sqrt{-1}) \times (x + y\sqrt{-1}) = 1$$ then we have

$$(ax - yb^2) + (ay + bx)\sqrt{-1} = 1$$ we have

to set $ax - yb = 1$ and $(ay + bx)\sqrt{-1} = 0$

then $$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}^{-1} = \frac{1}{a^2 - b^2} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \frac{1}{(a-b)(a+b)} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

$$\frac{1}{(a-b)(a+b)} \begin{bmatrix} a \\ -b \end{bmatrix} = x = \frac{a}{a^2 - b^2}, \quad y = \frac{-b}{a^2 - b^2}$$

Therefore inverses exist under product for F

(a)

2] There is one property that $n \times n$ matrices $\mathbb{R}^{n \times n}$ have that make them not a field, that being commutativity for matrix multiplication. If we define two matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \quad AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

$$BA = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ea+fc & eb+fd \\ ag+hc & gb+hd \end{bmatrix}$$

We can see that $AB \neq BA$ therefore it is not a field as multiplication is not commutative

3]

+	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

×	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

4] This is the same question as 1.2 and I will refer back to that question for some aspects. Let's define a field $F = \{a+b\sqrt{-1} \mid a,b \in \mathbb{R}\}$ which can also be written as $F = \{a+bi \mid a,b \in \mathbb{R}\}$. We want to define two operations on this field, the standard sum & product of polynomials. For $a,b,c,d \in \mathbb{R}$

<u>Sum</u>: $(a+bi)+(c+di) := a+c+(b+d)i$

<u>Product</u>: $(a+bi)\times(c+di) := ac-bd+(ad+bc)i$

<u>Identity for Sum</u> $0 \in \mathbb{R}$ $a,b \in \mathbb{R}$ $0 = d = c$

$r,q \in F$ $r = a+bi$ $q = d+ci$

$r+q = (a+bi)+(d+ci) = (a+bi)+(d+ci)$

$= (a+bi)+(0+0i) = (a+bi)+(0+0) = (a+bi)+0 =$ $a+bi$ Therefore $0+0i$ is the identity for sum in F

<u>Identity for Product</u> $a,b \in \mathbb{R}$ then $a=1$ and $b=0$ then $1+0i \in F$

$1+0i = 1 \in F$; if we define $x \in F$ where $x = c+di$

$1x = 1(c+di) = (c+di)1 = 1c+1di = c+di$

Therefore 1 is the identity for product in F

<u>Inverse for Sum</u> If $a,b \in \mathbb{R}$ there exists $-a,-b$ such that $a+(-a)=0=b(-b)$ then we can define $c,d \in F$ such that $c=a+bi$ and $d=-a-bi$ thus $c+d = a+bi+(-a-bi) =$ $a+bi-a-bi = 0$ Therefore there exists an inverse for sum.

<u>Inverse for Product</u>: We can solve this in 2 ways, check & solve like we did in 1.2 or take the inverse by

check & guess. This time, let's try check & guess. For each $z \in F$, lets define $1/z$ as in $\mathbb{R}$ for each $a, b \in \mathbb{R}$ there is a $1/a$ and $1/b$ $\quad z \cdot \frac{1}{z} = \frac{z}{z} = 1$ Therefore for any $z \in F$ defined as $z = a + bi$, there is a $1/z$ as defined as $1/(a+bi)$ as its inverse

<u>Associativity</u>: Was shown in 1.2 that $a, b, c \in F$ $a \times (b \times c) = (a \times b) \times c$ and $a + (b + c) = (a + b) + c$

<u>Commutativity</u>: As shown in 1.2 for all $a, b \in F$ $a \times b = b \times a$ and $a + b = b + c$

<u>Distributivity</u>: As shown in 1.2 for all $a, b, c \in F$ $a \times (b + c) = a \times b + a \times c$ & $(b + c) \times a = b \times a + b \times c$ which is the same as $b \times a + a \times c$

5] B & D are in RREF

6] $A + B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 1 & 2 & 0 \end{bmatrix} =$

$A + B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} =$

$\begin{bmatrix} 1+0+0 & 1+1+1 & 1+1+0 \\ 0+0+0 & 0+1+1 & 0+1+0 \\ 0+0+0 & 0+1+0 & 0+1+0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} =$

$A^2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+1+1 & 3 & 2 \\ 0+1+1 & 2 & 1 \\ 0+1+0 & 1 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} 2 & 3 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

7] RRef of A, according to Python is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If we find the inverse of A we get

$$\begin{bmatrix} \frac{6}{-6t^2-6t+36} & \frac{t+1}{t^2+t-6} & \frac{6}{-6t^2-6t+36} \\ \frac{t^2}{-t^2-t+6} & \frac{6t}{t^2+t-6} & \frac{t-6}{t^2+t-6} \\ \frac{-t}{-t^2-t+6} & \frac{6}{-t^2-t+6} & \frac{-t}{-t^2-t+6} \end{bmatrix}$$

If we find where the denominators $= 0$ we can find what values prevent A from having an inverse

$$t^2+t-6=0 \quad (t+3)(t-2)=0$$

$t=-3, 2$; all other denominators are equivalent

$t=-3, 2$ are where A does not have an inverse

8] A] If we input the Matrix A & find its rref() we get

$$\begin{bmatrix} 1 & 0 & (8h-24)/(3h-6) \\ 0 & 1 & -4/(6-3h) \end{bmatrix}$$

$$1 = \frac{8h-24}{3h-6} \rightarrow 3h-6 = 8h-24$$

$-5h = -18$ so $h = 18/5$

$1 = \frac{-4}{6-3h} \rightarrow 6-3h = -4 \rightarrow -3h = -10$

$h = 10/3, 18/5$ are the values which make it consistent

b] If we input the Matrix into python & find its rref() we get this as a returned matrix

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{aligned} 1-3 &= 0h \\ 0+0 &= 1 \end{aligned}$$

From this, we can re-evaluate our strategy & find that

$$\left[\begin{array}{cc|c} -4 & 12 & h \\ 2 & -6 & -3 \end{array}\right]$$

-3

-3

so h must = -3 so that the matrix is consistent so

$h = -3$

9] 1] $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ There is one type of reduced row echelon form for 3x2 matrices

2] $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ There are two types

9] 3]

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

There are 4 types

10] * indicates any value

$e=0;\quad a=*;\quad d=0;\quad c=1;\quad b=0$

11] 1]

$$A=\begin{bmatrix}1&2&3&4\\1&1&0&2\\2&0&1&2\end{bmatrix}\xrightarrow{R_1-R_2}\begin{bmatrix}0&1&3&2\\1&1&0&2\\2&0&1&2\end{bmatrix}$$

$$\xrightarrow{R_3/2}\begin{bmatrix}0&1&3&2\\1&1&0&2\\1&0&1/2&1\end{bmatrix}\xrightarrow{R_2-R_3}\begin{bmatrix}0&1&3&2\\0&1&-1/2&1\\1&0&1/2&1\end{bmatrix}\xrightarrow{R_1-R_2}$$

$$\begin{bmatrix}0&0&3.5&1\\0&1&-1/2&1\\1&0&1/2&1\end{bmatrix}\xrightarrow{R_3+R_2}\begin{bmatrix}0&0&3.5&1\\0&1&-0.5&1\\1&1&0&2\end{bmatrix}\xrightarrow{R_1+R_2}$$

$$\begin{bmatrix}0&0&3.5&1\\0&1&3&2\\1&1&0&2\end{bmatrix}\xrightarrow{R_3-R_2}\begin{bmatrix}0&0&3.5&1\\0&1&3&2\\1&0&-3&0\end{bmatrix}\xrightarrow{R_3+R_1}$$

$$\begin{bmatrix}0&0&3.5&1\\0&1&3&2\\1&0&0.5&1\end{bmatrix}\xrightarrow{\times 2}\begin{bmatrix}0&0&7&2\\0&2&6&4\\1&0&1&2\end{bmatrix}\xrightarrow[R_2-6/2R_3]{R_3-1/2R_1}$$

$$\begin{bmatrix}0&0&7&2\\0&2&0&16/7\\1&0&0&6/7\end{bmatrix}\xrightarrow[\substack{1/7R_1\\1/2R_2}]{\text{Re-arrange}}\begin{bmatrix}1&0&0&6/7\\0&1&0&8/7\\0&0&1&2/7\end{bmatrix}=RREF$$

* Second part of 11.1 "Solve $A\vec{x}=\vec{0}$ & write all solutions in parametric form" is later on as I accidentally skipped that part.

1] 2] $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 0 & 1 & 3 & 2 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{6R_1} \begin{bmatrix} 0 & 6 & 4 & 5 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 + R_1}$

$\begin{bmatrix} 0 & 6 & 4 & 5 \\ 1 & 0 & 4 & 0 \\ 2 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 0 & 6 & 4 & 5 \\ 1 & 0 & 4 & 0 \\ 1 & 0 & 4 & 2 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 0 & 6 & 4 & 5 \\ 1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_3/2}$

$\begin{bmatrix} 0 & 6 & 4 & 5 \\ 1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 - 5R_3} \begin{bmatrix} 0 & 6 & 4 & 0 \\ 1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1/6} \begin{bmatrix} 0 & 1 & 3 & 0 \\ 1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{Re-arrange}}$

$\begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = RREF$

3]

```
A = Matrix([[1,2,3,4],[1,1,0,2],[2,0,1,2]])
```

To verify part 1] A.rref() verifies my answer

To verify part 2]

```
GF7 = galois.GF(7)
A = GF7([[1,2,3,4],[1,1,0,2],
[2,0,1,2]])
GF7.row_reduce(A)
```

To find the rref(A) over $\mathbb{Z}_2$ we input:

```
GF2 = galois.GF(2)
A = GF2([[1,0,1,0],[1,1,0,0],[0,0,1,0]])
GF2.row_reduce(A)
```

which returns

```
GF([1,0,0,0],
   [0,1,0,0],
   [0,0,1,0], order=2)
```

11] 3] To find the rref (A) over $\mathbb{Z}/3$ we input

```
GF3 = galois.GF(3)
A = GF3([[1,2,0,1], [1,1,0,2], [2,0,1,2]])
GF3.row_reduce(A)
```

This returns:

```
GF([[1,0,0,0],
    [0,1,0,2],
    [0,0,1,2]], order=3)
```

4] I believe that rank M would remain consistent as RREF requires reducing the matrix therefore all non-necessary elements are removed through row operations therefore the rank would remain consistent

1] "Solve $A\vec{x} = \vec{0}$ & write all solutions in parametric form"

This is to find the kernel

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 6/7 & 0 \\ 0 & 1 & 0 & 8/7 & 0 \\ 0 & 0 & 1 & 2/7 & 0 \end{array}\right] = \begin{array}{l} x_1 + 6/7 = 0 \\ x_2 + 8/7 = 0 \\ x_3 + 2/7 = 0 \end{array} \rightarrow$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -6/7 \\ -8/7 \\ -2/7 \end{bmatrix}$$

$$\left[\begin{array}{c|c} 1 & 2 \\ \hline & I \end{array}\right] \quad A = \begin{bmatrix} 3 & 1 & 4 \\ 5 & 2 & 6 \\ 0 & 5 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$$ — therefore

$$(A \mid \vec{b}) = \left[\begin{array}{ccc|c} 3 & 1 & 4 & 1 \\ 5 & 2 & 6 & 5 \\ 0 & 5 & 2 & 1 \end{array}\right]$$

For python we write:

```
GF7 = galois.GF(7)
A = GF7([[3, 1, 4, 1], [5, 2, 6, 5], [0, 5, 2, 1]])

GF7.row_reduce(A)
```

returns: GF([[1, 0, 0, 4],
[0, 1, 0, 3],
[0, 0, 1, 0]], order=7)

2] In order to find the module of the solution we must first solve for x. Therefore

$$Ax = b \rightarrow A^{-1}Ax = A^{-1} * b \rightarrow$$

$$\vec{x} = A^{-1} * b$$ then we take the modulo 7 of $\vec{x}$

I found it useful to use python for part of the calculation. If we input:

```
A = Matrix([[3, 1, 4], [5, 2, 6], [0, 5, 2]])
A_inv = A**-1

b = Matrix([[1], [5], [1]])

x = A_inv * b
print(x)
```

This returns: Matrix([[31/6], [11/6], [-49/12]])

We can rewrite this as $\begin{bmatrix} 31/6 \\ 11/6 \\ -49/17 \end{bmatrix}$ We can take the modulo 7 of this manually. We then get

$$\vec{x} = \begin{bmatrix} 31/6 \\ 11/6 \\ 35/17 \end{bmatrix}$$ which is our final answer

13] We input to Python:

```
x, y, z = symbols('x, y, z')
solution = sym.solve([3*x + 11*y +
19*z + 2, 7*x + 23*y + 39*z - 10,
-4*x - 3*y - 2*z - 6], (x, y, z))
print(solution)
```

This returns [] or an empty list since there are no solutions for this equation and cannot be written in parametric form

14] We input into Python:

```
x, y, z, v, w = symbols('x, y, z, v, w')
problem = [3*v + 6*w + 9*x + 5*y + 25*z
- 53, 7*v + 14*w + 21*x + 9*y + 57*z
- 105, -4*v - 6*w - 12*x + 5*y - 10*z
- 11]
```

24] solution = sym.solve(problem, (v, w, x, y, z))

print(solution)

This then returns

{v: -2*w - 3*x - 5*z + 6, y: 7 - 2*z}

which translates to $v = -2w - 3x - 5z + 6$
$y = 7 - 2z$

Since v, y are the only ones w/ solutions they are the only ones w/ a true parametric form.

For notational purposes

$x_1 = v$, $x_2 = w$, $x_3 = x$, $x_4 = y$, $x_5 = z$

$$\text{solution} = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_3 - 5x_5 + 6 \\ 7 - 2x_5 \end{bmatrix} =$$

$$x_2 \begin{bmatrix} -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ -2 \end{bmatrix} + \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

25] We input into Python:

x, y, z, v, w = symbols('x, y, z, v, w')

problem = [2*v + 4*w + 3*x + 5*y + 6*z - 37, 4*v + 8*w + 7*x + 5*y + 2*z - 74, -2*v - 4*w + 3*x + 4*y - 5*z - 20, v + 2*w + 2*x - y + 2*z - 26, 5*v - 10*w + 4*x + 6*y + 4*z - 24]

solution = sym.solve(problem, (v, w, x, y, z))

print(solution)

which returns {v: -8221/4340, w: 8591/8680, x: 4695/434, y: -459/434, z: 699/434}

which when written in parametric form where $v = x_1$, $w = x_2$, $x = x_3$, $y = x_4$, $z = x_5$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -8221/4340 \\ 8591/8680 \\ 4695/434 \\ -459/434 \\ 699/434 \end{bmatrix}$$

16] 1] Each of the matrices must be invertible as

$$\underbrace{(AB)}_{C^{-1}}C = I \quad \text{and} \quad A\underbrace{(BC)}_{A^{-1}} = I$$

so each of the matrices must be invertible

Their inverses are A's inverse is $A^{-1} = BC$, C's inverse is $C^{-1} = AB$, and B's inverse is B^{-1}

2] If $(AB)^{-1}$ exists then by property of inverse $A^{-1}B^{-1}$ must exist thus A & B are invertible

17] $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$A^2 B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Therefore $A^2B^2 \neq (AB)^2$

$$(AB)^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

18] $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ $A^T = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$$AA^T = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Therefore A is a non-identity matrix whose transpose is its inverse

19 1 Symmetric 2 x 2 : $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = A$ $A^T = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Thus $A = A^T$

Skew Symmetric 2 x 2 $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = A$ $A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Therefore $A^T = -A$

Symmetric 3x3

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$$

Therefore $A = A^T$ and is symmetric

Skew Symmetric 3x3

$$A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$$

Therefore $A^T = -A$ is skew symmetric

Symmetric 4x4

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 4 & 5 & 6 \\ 2 & 5 & 7 & 8 \\ 3 & 6 & 8 & 9 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 4 & 5 & 6 \\ 2 & 5 & 7 & 8 \\ 3 & 6 & 8 & 9 \end{bmatrix}$$ Therefore $A = A^T$

<u>Skew Symmetric 4x4</u> $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & 0 & -6 \\ -2 & 0 & 0 & -8 \\ -3 & 6 & 8 & 0 \end{bmatrix}$

$$A^T = \begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & 0 & 6 \\ 2 & 0 & 0 & 8 \\ 3 & -6 & -8 & 0 \end{bmatrix}$$

Therefore $A^T = -A$ and is skew symmetric

2] The main diagonal of a skew symmetric matrix must consist of only zeros as no other number would be able to satisfy the definition of skew symmetric as if $a \in \mathbb{R}$, $-1a \neq a$ unless $a = 0$

3] The only matrix that can be both symmetric & skew-symmetric is one made of only zeros, for example

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad A^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Therefore $A = A^T$ and $A^T = -A$; no other type of matrix can satisfy both conditions as for other $a \in \mathbb{R}$, $-1a \neq a$ unless $a = 0$

19] 3] Therefore a matrix consisting of only zeros is both symmetric and skew symmetric

4] <u>$A+A^T$ are symmetric</u> Let's define a matrix $B = A + A^T$ then lets take the transpose of B:

$$B^T = (A+A^T)^T = A^T + (A^T)^T$$ by properties of transpose $(A^T)^T = A$ thus

$$A^T + (A^T)^T = A^T + A$$ by the fact that matrix addition is commutative

$$A^T + A = A + A^T$$ thus

$B^T = A + A^T = B$ and $B^T = B$ and $A + A^T$ is symmetric

<u>AA^T is symmetric</u> Let's define a matrix C as $C = AA^T$ then we need to find that $C = C^T$ then

$$C^T = (AA^T)^T = (A^T)^T A^T$$ by properties of matrix transposition therefore

$$(A^T)^T A^T = AA^T$$ therefore

$C^T = AA^T = C$ thus $C^T = C$ so

AA^T is symmetric

A^TA is symmetric Let's define a matrix $D = A^TA$ then

$D^T = (A^TA)^T = A^T(A^T)^T$ by properties of matrix transposition so

$A^T(A^T)^T = A^TA = D$ therefore

$D^T = A^TA = D$ so $D^T = D$ therefore A^TA is symmetric

$A - A^T$ is skew symmetric Let's define a matrix $E = A - A^T$ then $E^T = (A - A^T)^T$ then

$(A - A^T)^T = (A + (-A^T))^T =$

$A^T + (-A^T)^T = A^T + (-A) =$ by the property of commutativity in matrix addition

$-A + A^T$ that can be written as

$-(-(A - A^T)) = -E$ thus

$E^T = -(A - A^T) = -E$ thus

$A - A^T$ is skew symmetric

5] Let's define a matrix A as a $n \times n$ matrix and let's define $B = A + A^T$, which is symmetric by the previous section, and define matrix $C = A - A^T$, which is skew symmetric by a previous proof, then we can express A as

$A = \frac{1}{2}C + \frac{1}{2}B = A = \frac{1}{2}(C + B) =$

19] 5] $A = \frac{1}{2}(C+B) = \frac{1}{2}\left([A-A^T]+[A+A^T]\right) =$

$\frac{1}{2}\left(A - A^T + A + A^T\right)$ which because matrix addition is commutative can be written as

$$\frac{1}{2}\left(A + A \underbrace{+A^T - A^T}_{0}\right) =$$

$$\frac{1}{2}\left(A + A\right) = \frac{1}{2}\cdot(2A) = A$$

Therefore A can be expressed as a sum of symmetric & skew symmetric matrices

20] a] $F(x) = x^2$ is surjective

b] $F(x) = x^3/x^2+1$ is surjective

c] $F(x) = x(x-1)(x-2)$ is surjective

d] $F(x) = e^x + 2$ is injective

21]

$$A = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{bmatrix}$$

$L_{11}U_{11} = 4$ $\quad L_{41}U_{11} = 0$

$L_{21}U_{11} = 1$ $\quad L_{11} = 4 \quad L_{41} = 0$

$L_{31}U_{11} = 0$ $\quad L_{21} = 1 \quad U_{11} = 1$

$L_{31} = 0$

$\rightarrow$

$$\begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 1 & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{bmatrix} \begin{bmatrix} 1 & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{bmatrix}$$

$4U_{12} = 1$

$U_{12} + L_{22}U_{22} = 4$

$L_{31}U_{12} + L_{32}U_{22} = 1$

$L_{41}U_{12} + L_{42}U_{22} = 0$

$L_{22}U_{22} = 3.75$

$1/4\, L_{31} + L_{32}U_{22} = 1$

$U_{12} = 1/4$

$L_{41} = 0$ $L_{22} = 4$

$L_{41} = 0$

$L_{31} = 0$

$L_{32} = 1$

$U_{22} = 1$

24] 1] The definition of symmetric is for a matrix A, $A^T = A$, therefore to prove H_n is symmetric we have show that it possesses this property

$$H_n^T = (I_n - 2\vec{u}\vec{u}^T)^T =$$

$$I_n^T - 2(\vec{u}\vec{u}^T)^T$$ by properties of transpose then $I_n^T - 2(u^T)^T \vec{u}^T =$

$$I_n^T - 2\vec{u}\vec{u}^T$$ Then we must check whether the identity matrix is symmetric

$$IA = A \rightarrow (IA)^T = A^T \longrightarrow$$

$A^T I^T = A^T$ thus $I^T = I$ so we have by that property

$$I_n - 2\vec{u}\vec{u}^T = H_n$$ thus

$H_n^T = H_n$ and H_n is symmetric

2] For this we must prove that

$H_n^T H_n = I_n$ by our previous prop we know that $H_n^T = H_n$ so

$$H_n H_n = (I_n - 2u\vec{u}^T)(I_n - 2u\vec{u}^T) =$$

$$I_n^2 - 2I_n u\vec{u}^T - 2u\vec{u}^T I_n + 4(uu^T)(uu^T)$$

by definition of identity this can be written as $I_n - 2uu^T - 2uu^T + 4(uu^T)(uu^T)$ which can be written as

$I_n - 4uu^T + 4u(u^Tu)u^T$ which by properties of unit vectors can be written as

$I_n - 4uu^T + 4uu^T = I_n$ Therefore

H_n is an orthogonal matrix

3] H^2_n is very similar to our previous proof.

$H^2_n = H_n(H_n) = H^T_n H_n = I_n$ by our last proof & by the definition of symmetric. This can be seen if fully written out a tad clearer

$H^2_n = H_n H_n = (I_n - 2u\vec{u}^T)(I_n - 2uu^T) =$

$I_n - 4uu^T + 4(uu^T)(uu^T) =$

$I_n - 4uu^T + 4uu^T = I_n$ which is the result of H^2_n

4] $H_n\vec{u} = (I - 2uu^T)\vec{u} =$

$I_n\vec{u} - 2uu^Tu =$ by properties of identity $\vec{u} - 2uu^Tu$ by properties of unit vectors $\vec{u} - 2u(u^Tu) = \vec{u} - 2u =$

$-\vec{u} = H_n\vec{u}$

5] $H_3 = I_3 - 2u\vec{u}^T =$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\left(\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)\left(\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\right)$$

24]
5]

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - 2\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1&1&1\end{bmatrix} =$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1&1&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix} =$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}2/3&2/3&2/3\\2/3&2/3&2/3\\2/3&2/3&2/3\end{bmatrix} = \begin{bmatrix}1/3&-2/3&-2/3\\-2/3&1/3&-2/3\\-2/3&-2/3&1/3\end{bmatrix} = H_3$$

$$H_4 = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} = I_n \qquad \vec{u} = \frac{1}{\sqrt{4}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$$H_4 = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} - 2\left(\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right)\left(\frac{1}{2}\begin{bmatrix}1&1&1&1\end{bmatrix}\right) =$$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\begin{bmatrix}1&1&1&1\end{bmatrix} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix} =$$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} - \begin{bmatrix}1/2&1/2&1/2&1/2\\1/2&1/2&1/2&1/2\\1/2&1/2&1/2&1/2\\1/2&1/2&1/2&1/2\end{bmatrix} = \begin{bmatrix}1/2&-1/2&-1/2&-1/2\\-1/2&1/2&-1/2&-1/2\\-1/2&-1/2&1/2&-1/2\\-1/2&-1/2&-1/2&1/2\end{bmatrix} =$$

$$H_4 = \begin{bmatrix}1/2&-1/2&-1/2&-1/2\\-1/2&1/2&-1/2&-1/2\\-1/2&-1/2&1/2&-1/2\\-1/2&-1/2&-1/2&1/2\end{bmatrix}$$