Wednesday, July 1, 2020 12:57 PM

LECTURES: 1:00 PM - 3:00 PM MW

6:00 PM - 8:00 PM MW

+ ASSIGNED READ OF CASE STUDIES.

EX: LAKE POLLUTION

BACTERIA FLOWS INTO A LAKE. BOTH INFLOW AND BACTERIAL CONCENTRATION IN THE INFLOW ARE SEASONAL:

FLOW RATE:

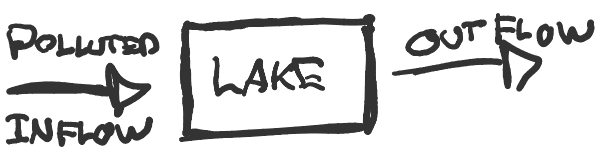
BOOK IS WRONG.

$$F(t) = 10^6\left(10 + 6\sin(2\pi t)\right)\frac{m^3}{\text{YEAR}}$$

BACTERIAL CONCENTRATION:

$$C_{IN}(t) = 10^7\left(1 + \cos(2\pi t)\right)\ \text{BAC}\cdot m^{-3}$$

MODEL SIMPLE CASE: COMPARTMENT



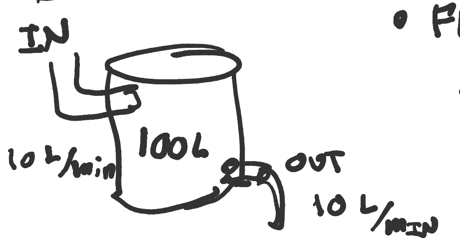
Q) VOLUME OF LAKE IS $28 \times 10^6\ m^3$, WILL THE BACTERIAL CONCENTRATION RISE TO MORE THAN $C = 4 \times 10^6\ \text{BAC}\cdot m^{-3}$.

$C = 4 \times 10^{-}$ BAC. m^{-}

SIMPLIFY: SALT TANK MODELS

LETS ASSUME INFLOW = OUTFLOW IS CONSTANT F_0, AND CONCENTRATION IN IS CONSTANT C_{IN}.

SALT TANK:



ASSUMPTIONS:

- FLOW IS CONSTANT
- SALT WATER FLOWS IN W/ CONCENTRATION C_{IN} KG/L
- SALT WATER IN TANK HAS CONST. CONCENTRATION $C(t)$.

SALT WATER $\Rightarrow$ TANK $\Rightarrow$ MIXED WATER

$\Downarrow$

SPECIFY QUANTS.

$$\xrightarrow[\frac{\text{Kg SALT}}{\text{min}}]{10 \cdot C_{IN}} \boxed{\text{SALT } S(t) \sim \text{Kg}} \xrightarrow[\frac{\text{Kg SALT}}{\text{min}}]{10 \cdot \frac{S(t)}{100}}$$

OUT: (FLOW RATE) × (CONCENTRATION IN TANK)

$$= (10) \times \left(\frac{S(t)}{V}\right)$$

$$= 10 \times \frac{S}{100} = \frac{1}{10} S$$

COMPARTMENT TO EQ'S:

RATE OF CHANGE OF S = RATE IN − RATE OUT.

HERE: $$\frac{dS}{dt} = 10 \cdot C_{IN} - \frac{1}{10} S(t).$$

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ANALYZE: SOLVE:

$$\frac{ds}{dt} = 10\, C_{IN} - \frac{1}{10} S$$

$$\Rightarrow \int \frac{1}{10 C_{IN} - \frac{1}{10} S}\, ds = \int 1\, dt$$

$$\Rightarrow -10 \operatorname{Log}\left|10 C_{IN} - \tfrac{1}{10} S\right| = t + C$$

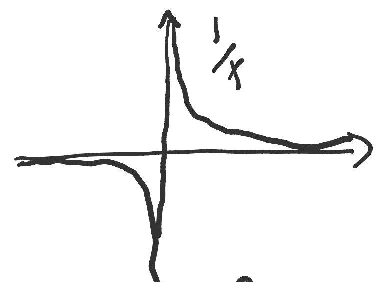
ABSORBE -10^{-1}

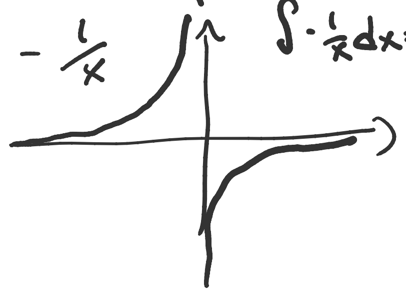
$$\Rightarrow \operatorname{Log}\left|10 C_{IN} - \tfrac{1}{10} S\right| = -\frac{1}{10} t + C$$

$$\Rightarrow 10 C_{IN} - \frac{1}{10} S = e^{-\frac{1}{10} t + C}$$

$$\Rightarrow S = 100\, C_{IN} - 10 e^{-\frac{1}{10} t + C}$$

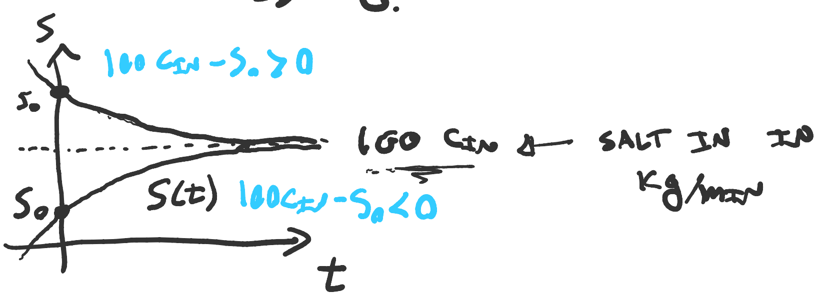
$\int \frac{1}{x} dx = \ln|x| + c$





$\int -\frac{1}{x} dx = -\ln|x| + c$

UNDETERMINED C.



SOLVE IVP $S(0) = S_0$:

$$S_0 = S(0) = 100\, C_{IN} - \underbrace{10 e^{c}}_{A}$$

$$A = 100\, C_{IN} - S_0$$

$$\boxed{S = 100\, C_{IN} - (100\, C_{IN} - S_0) e^{-\frac{1}{10} t}}$$

Q) WHAT ARE

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QUANTITIES: S - kg SALT

STATE VARS: t - min

PARAMETERS: C_{in}, S_0

GENERAL FORM:

$$\xrightarrow{F \cdot C_{in}} \boxed{S(t)} \xrightarrow{\frac{F}{V} \cdot S}$$

F - FLOW L/min

V - VOLUME L

C_{IN} - CON. kg/L

EX: $\dfrac{dS}{dt} = F C_{IN} - \dfrac{F}{V} S, \quad S(0) = S_0$

LEADS TO

$$S(t) = V C_{IN} - (V C_{IN} - S_0) e^{-t F/V}$$

$\square$

FLOW EQUATIONS IN TERMS OF CONCENTRATION:

IDEA:

$$\xrightarrow[\text{WATER}]{\text{SALT}} \boxed{\text{TANK}} \xrightarrow{\text{OUTFLOW}}$$

WANT TO WRITE IN TERMS OF CONCENTRATION OF SALT IN THE TANK

$$C(t) = S(t)/V \iff S(t) = V \cdot C(t)$$

STARTING WITH

$$\frac{dS}{dt} = F C_{IN} - \frac{F}{V} S$$

WRITE

CONSTANT

$$\frac{d(V \cdot C)}{dt} = F C_{IN} - \frac{F}{V} \cdot V C$$

$$V \frac{dC}{dt} = F C_{IN} - F C$$

$$V\frac{dc}{dt} = F\,C_{IN} - F\,C$$

$$C' = \frac{F}{V}C_{IN} - \frac{F}{V}C$$ ← SYMMETRIC IN C_{IN} AND C

Ex 3:

SHOW SOLUTION W/ $C(0) = C_0$ IS

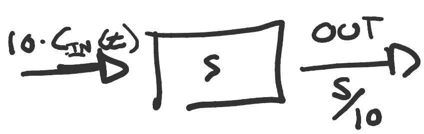
$$C(t) = C_{IN} + (C_0 - C_{IN})e^{-tF/V}.$$

NOTICE THAT THERE ARE ONLY 3 PARAMETERS, C_{IN}, C_0, $\frac{F}{V}$.

SLIGHTLY MORE COMPLEX:

WHAT IF $C_{IN} = C_{IN}(t)$ IS NOT CONSTANT?

MODEL:



START: $C_{IN}(t) = c_1 \cdot t$, $c_1 \in \mathbb{R}$, THEN

$$\frac{dS}{dt} = 10c_1 t - S/10$$

Q) CAN WE SOLVE USING SEPERATION OF VARIABLES?

$$\boxed{\frac{dy}{dx} = \frac{x^2}{\cos y}}$$ ✓

No: CANT WRITE $f(S)\,dS = g(t)\,dt$.

NEED INTEGRATING FACTORS:

II THEORY:

FOR DIFF. EQUATIONS OF THE FORM

$$\frac{dy}{dt} + P(t)y = Q(t) \qquad (*)$$

$$\frac{dy}{dt} + P(t)\,y = Q(t) \qquad (*)$$

IF 1ST ORDER, LINEAR ODE, WE CAN SOLVE $(*)$ LOCALLY USING AN INTEGRATING FACTOR:

LET $I(t) = \exp\left(\int p(t)\,dt\right)$

THEN:

$$\frac{d}{dt}\left(y I(t)\right) = \frac{d}{dt}\left(y e^{\int p(t)\,dt}\right)$$

$$= \frac{dy}{dt} e^{\int p(t)\,dt} + y \frac{d}{dt}\left(\int p(t)\,dt\right) e^{\cdots}$$

$$= \frac{dy}{dt} I(t) + p(t) I(t)$$

$$= Q(t) I(t)$$

SO GET NEW EQ:

$$\frac{d}{dt}\left(y I(t)\right) = Q(t) I(t)$$

SO

$$y I(t) = \int Q(t) I(t)\,dt, \quad y = I^{-1} \int Q I\,dt$$

Ex:

$$\frac{dS}{dt} = 10 c_1 t - \frac{S}{10}$$

REWRITE:

$$\frac{dS}{dt} + \frac{S}{10} = 10 c_1 t$$

WRITE $I(t) = \exp\left(\int \frac{1}{10}\,dt\right) = e^{\frac{1}{10}t}$.

MULT. THROUGH:

$$\frac{dS}{dt} e^{\frac{1}{10}t} + \frac{S}{10} e^{\frac{1}{10}t} = \cdots e^{\frac{1}{10}t}$$

$$\frac{dS}{dt}e^{\frac{1}{10}t} + \frac{S}{10}e^{\frac{1}{10}t} = 10c_1 t e^{\frac{1}{10}t}$$

$$\frac{d}{dt}\left(Se^{\frac{1}{10}t}\right) = \text{——}''\text{——}$$

SOLVE:

$$Se^{\frac{1}{10}t} = \int 10c_1 t e^{\frac{1}{10}t}\,dt$$

$$= 100c_1 t e^{\frac{1}{10}t} - 1000c_1 e^{\frac{1}{10}t} + A \quad \checkmark$$

NOW:

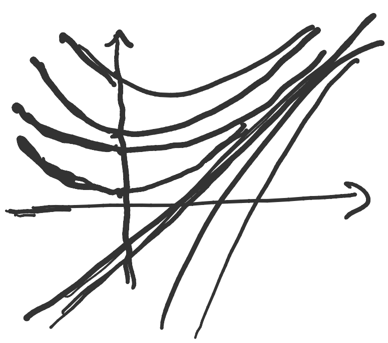
$$\underline{S = 100c_1 t - 1000c_1 + Ae^{-\frac{1}{10}t}}$$

GENERAL SOLUTION.

WHAT DOES THIS TELL US? WHAT IS $S(t)$ ASYMPTOTICALLY?

WHEN $t \to \infty$? $S \to 100c_1 t - 1000c_1$

$t \to -\infty$? $S \to \pm\infty$ FOR $\pm A > 0$



Ex: INTEGRATING FACTORS

$$\frac{dy}{dx} = -\frac{3y}{x} + \frac{e^x}{x^3}$$

REWRITE

$$\frac{dy}{dx} + \frac{3y}{x} = \frac{e^x}{x^3}$$

$$\frac{dy}{dx} + \frac{3y}{x} = \frac{e^x}{x^3}$$

IF $I(x) = \exp\left(\int \frac{3}{x}\,dx\right) = \exp(3\log|x|)$

$= x^3 \qquad (x \geq 0)$

MULTIPLYING:

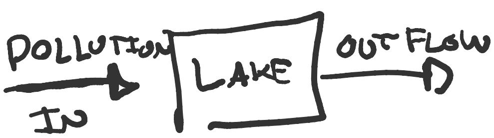
$$\frac{dy}{dx}x^3 + 3x^2y = \frac{d}{dx}(yx^3) = e^x$$

SO

$$yx^3 = e^x + C$$

OR $$y = \frac{e^x}{x^3} + \frac{C}{x^3}$$ GENERAL SOLUTION.

RETURN TO THE LAKE:



FIRST MODEL: MAKE POLLUTION IN AND FLOW CONSTANT.

SECOND MODEL: ALLOW VARIABLE FLOW

START W/ CONCENTRATION EQ'S

$$\frac{dC}{dt} = C_{IN}\frac{F(t)}{V} \quad C\frac{F(t)}{V},$$

$F(t) = 10^6(10 + 6\sin(2\pi t))$ m^3/YEAR.

$V = 28 \times 10^6$ m^2

WRITE $F(t) = F_0(1 + \varepsilon\sin(2\pi t))$.

REWRITE $\quad F_0 = 10^7, \quad \varepsilon = \frac{6}{10}.$

$$\frac{dC}{dt} = -\frac{F_0}{V}(1 + \varepsilon\sin(2\pi t))(C - C_{IN})$$

$$\frac{dC}{dt} = -\frac{F_0}{V}\left(1+\varepsilon \sin(2\pi t)\right)\left(C - C_{IN}\right).$$

$$\frac{dC}{dt} + \frac{F_0}{V}\left(1+\varepsilon \sin(2\pi t)\right) C$$
$$= \frac{F_0}{V}\left(1+\varepsilon \sin(2\pi t)\right) C_{IN}.$$

I.F. $I(t) = \exp\left(\frac{F_0}{V}\int 1+\varepsilon \sin(2\pi t)\, dt\right)$

$$= \exp\left(\frac{F_0}{V}\left(t - \frac{\varepsilon}{2\pi}\cos(2\pi t)\right)\right)$$

$$\frac{d}{dt}\left(C\, I(t)\right) = \underbrace{\frac{F_0}{V}\left(1+\varepsilon \sin(2\pi t)\right)}_{\text{TOTAL}} C_{IN} \underbrace{I(t)}_{\text{DERIV.}}.$$

SO

$$C\, I(t) = C_{IN}\, I(t) + A$$

$$C = C_{IN} + A\exp\left(-\frac{F_0}{V}\left(t + \frac{\varepsilon}{2\pi}\cos(2\pi t)\right)\right).$$

EX: FIX A GIVEN $C(0) = C_0$.

WHAT, ASYMPTOTICALLY, HAPPENS TO THE CONCENTRATION?

$$t \to \infty, \quad C(t) \to C_{IN}.$$

SO OUR MODEL ESTIMATES CONCENTRATION IN LAKE GOES TO C_{IN}.

RECALL:

$$C_{IN}(t) = 10^7\left(1+\cos(2\pi t)\right) \text{ bac}\cdot m^{-3}$$

AVG AROUND 10^7, OUR CUTOFF WAS

$$C(t) = 4\times 10^6.$$

FINAL MODEL:

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LET $C_{IN}(t) = 10^7 (1+\cos(2\pi t))$ BAC-m^3

$F(t) = 10^6 (10 + 6\sin(2\pi t))$ m^3/YEAR

$V = 28 \times 10^6$ m^3

BOX MODEL: $\longrightarrow$ $\boxed{\text{LAKE}}$ $\longrightarrow$

$$\frac{dC}{dt} = \frac{C_{IN}(t)}{V} F(t) - \frac{C(t)}{V} F(t).$$

$$= \frac{10^6}{V}\left[10 + 6\sin(2\pi t)\right].$$

$$\times \left[10^7(1+\cos(2\pi t)) + C(t)\right].$$

WHAT CAN WE DO?

- SIMULATE COMPUTATIONALLY
- REPLACE W/ SIMPLER EQ. BY FIXING SOME PARTS TO MAX/MIN VALUES.
- USE DERIVS THEMSEVES TO UNDERSTAND SOLUTIONS, USING THINGS LIKE <u>SLOPE</u> <u>PLOTS</u>.