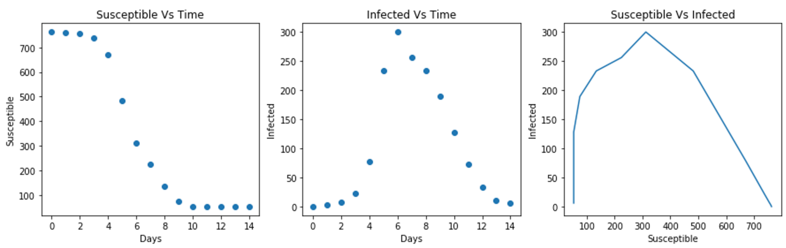
Monday, August 3, 2020 12:53 PM

PARAMETERS AND TIME DEPENDANT HEAT EQUATION



3 WAYS TO FIT PARAMETERS:

1. GO OUT AND MEASURE IT.

2. TRY TO UNDERSTAND THEM IN TERMS OF QUANTITIES MEASURABLE FROM DATA. ← WE WILL FOCUS ON THIS.

3. FIT PARAMETERS USING STATISTICS. (GRAD. DESCENT / MAXIMUM LIKELIHOOD).

SIR EXAMPLE:

DATA ABOVE COMES FROM AN INFLUENZA OUTBREAK IN A BOARDING SCHOOL.

- INFECTED - # OF STUDENTS IN HOSPITOL WARD
- SUSCEPTIBLE - ESTIMATED FROM INFECTED AND AVG. DISEASE LENGTH.

SO $S(t)$ DATA IS BEING ESTIMATED.
WANT TO US SIR MODEL:

$$S' = -\beta S I$$

$$I' = \beta S I - \gamma I$$

KNOW:
- 763 STUDENTS IN SCHOOL.
- LENGTH OF DISEASE ON AVG. IS $2\frac{1}{4}$ DAYS. SO

$$\gamma \approx \frac{1}{2\frac{1}{4}} \approx .44$$

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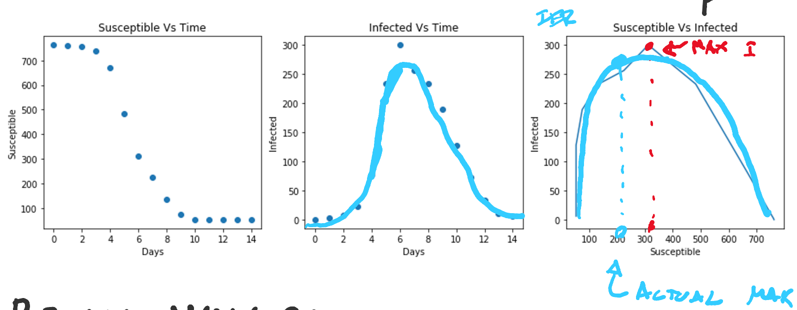
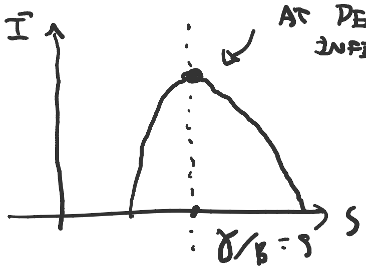
WE ONLY NEED TO DETERMINE β.



RECALL NULLCLINES:

$$0 = I' = \beta I S - \gamma I \quad \text{OR} \quad I = 0, \; S = \frac{\gamma}{\beta}.$$

NOW: WE KNOW SIR CURVES LOOK LIKE



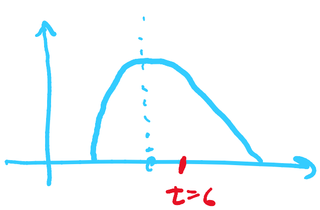
SO WHEN WE'RE AT MAX I, WHAT IS S?

AT $I_{MAX} = 300$, $S = 312$ AND SO

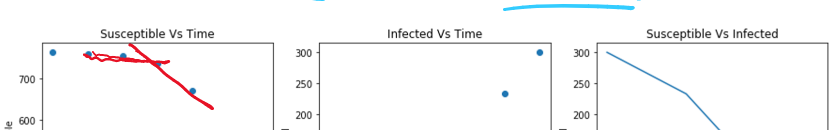
$$\gamma/\beta = S \quad \text{OR} \quad \frac{.44}{312} = \beta \approx 0.0014.$$

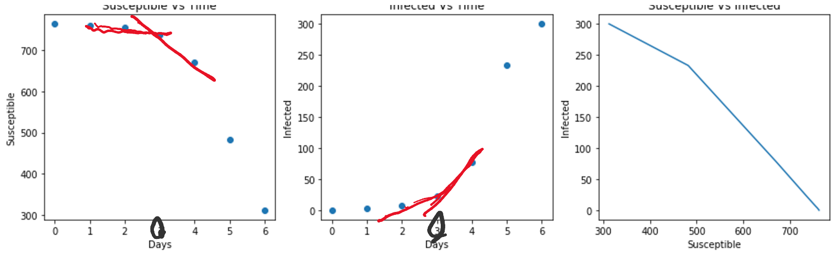
Q) DO YOU THINK THIS WILL BE AN OVER ESTIMATE OR AN UNDER ESTIMATE?

I WOULD ARGUE S SHOULD BE SMALLER. EXPECT SIR CURVE TO SMOOTH TO



I EXPECT $\underline{S}$ TO BE $\underline{\text{SMALLER}}$.





WHAT ABOUT PARTIAL DATA?

HERE, CAN'T GET A GOOD EST. FOR $\frac{\gamma}{\beta}$.

LOOKING AT

$$S' = -\beta S I$$
$$I' = \beta S I - \gamma I$$

BEFORE WE FIT TO A CRITICAL POINT.

NOW?

PICK $t=3$, PLUG IN:

$$S'(3) = -\beta(737)(22)$$
$$I'(3) = \beta(737)(22) - (.44)(22)$$

CAN ESTIMATE S' AND I', TAKE AVE. OF SLOPES OF LINES:

$$S'(3) \underset{\text{LEFT}}{\approx} \frac{737-755}{3-2} = -18,$$
$$\underset{\text{RIGHT}}{\approx} \frac{671-737}{4-3} = -66, \Big\} \text{ AVG: } -\frac{84}{2} = -42$$

SO $S'(3)$ ESTIMATES β AT

$$\frac{-42}{-(737)(22)} = \beta \approx .002590$$

NOTE: TO IMPROVE, SHOULD REPEAT AT EACH TIME t, AND FOR $I'(t)$. AVERAGING ALL TOGETHER GIVES A MUCH STRONGER EST. FOR β.

EVEN FARTHER: IF WE HAVE N PARAMETERS, THIS ALLOWS US TO GENERATE MANY EQUATIONS RELATING THEM.

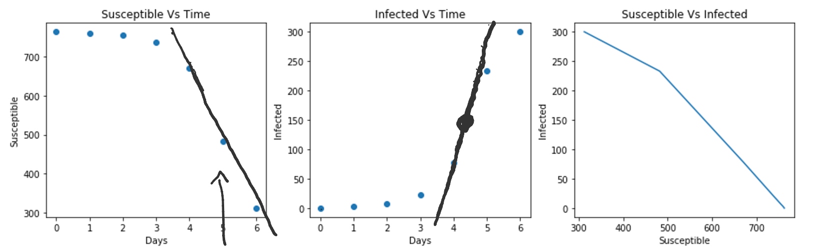
THESE ARE LOCAL ESTIMATES AT A

THESE ARE LOCAL ESTIMATES AT A POINT.

ANOTHER IDEA:

NULLCLINES CORRESPOND TO CRITICAL POINTS IN SOLUTION CURVE, AND ARE THE EASIEST GEOMETRY TO FIND.

IF NULLCLINES ARE NOT IN DOMAIN WE MAY BE ABLE TO USE INFLECTION POINTS:



INFLECTION? ROUGH INFLECTION

SLOPE OF TAN. LINE ISN'T CHANGING.

INFLECTION POINT IN S OCCURS WHEN $S''=0$:

$$0=\frac{d^2S}{dt^2}=\frac{d}{dt}\left(-\beta SI\right)=-\beta S'I-\beta I'S$$

$$=-\beta\left(-\beta SI\cdot I+(\beta SI-\gamma I)S\right)$$

OR, FACTORING OUT $-\beta SI$:

$$0=-\beta I+\beta S-\gamma$$

OR

$$\beta=\frac{\gamma}{S-I}$$ AT THE INFLECTION POINT IN S.

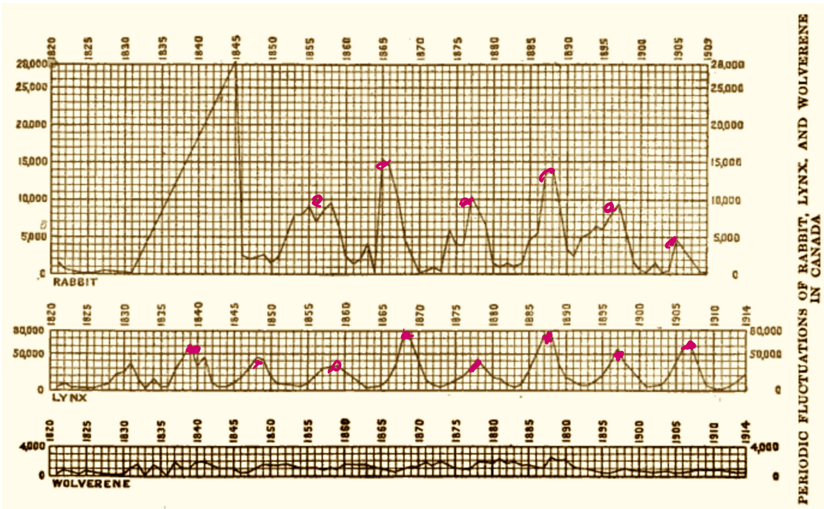
AT $t=5$:

$$\beta=\frac{.44}{482-233}$$

482-233

ANOTHER (IN?) FAMOUS EXAMPLE:

HARE AND LYNX:



TRAPPING RECORDS FROM CANADA FOR HARE, LYNX AND WOLVERENES. ONE OF THE LONGEST ECOLOGICAL POP. STUDIES ON RECORD, OFTEN SEEN AS A DIRECT VALIDATION OF THE LOTKA-VOLTERRA EQUATIONS:

$$X' = bX - c_1 XY$$
$$Y' = c_2 XY - dY$$

BECAUSE OF CYCLIC NATURE.

NOTE: CONTRAVERSIAL SINCE IN SOME PLACES SINCE LYNX POP. STARTS INCREASING BEFORE HARE POPULATION.

To FIT:

(1) FIT NULLCLINES, IE FIT CRITICAL POINTS OF X AND Y:

X SHOULD HAVE CRIT. AT

$$Y = \frac{b}{c_1}$$

Y SHOULD HAVE CRIT. AT

$$X = \frac{d}{-}$$

Y SHOULD HAVE CRIT. AT

$X = \frac{d}{c_2}$

(2) EVALUATE X', Y' AT SOME POINTS
BUT HERE DATA POINTS ARE
NOISY...

(3) USE THE PERIOD OF POPULATION
PEAK, ~9 YEARS AND
VERY STABLE.

RECALL FORMULA ROUGHLY GIVEN
BY PERIOD OF CRITICAL
POINT:

EIGS:

$\lambda = \pm i\sqrt{bd}$

SO

$\vec{X} \approx c_1 \cos \sqrt{bd}\, t + c_2 \sin \sqrt{bd}\, t$

AND PERIOD IS

$\frac{2\pi}{\sqrt{bd}}$.

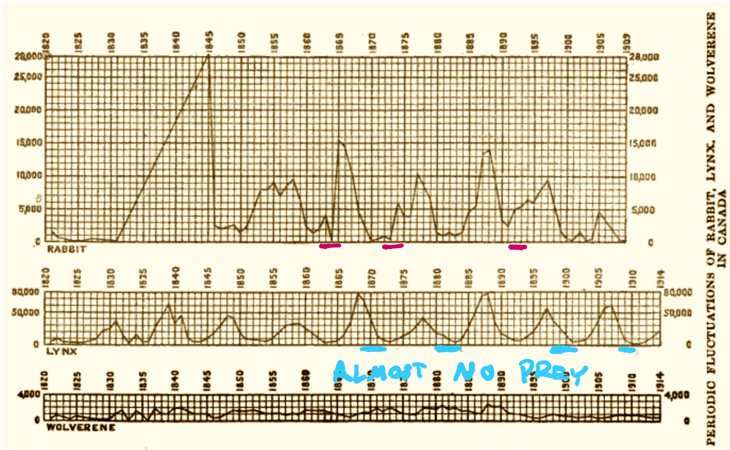
(4) LAST IDEA:

USE SPECIAL VALUES / LIMITS.
HERE, WHEN $Y \ll 1$,

$X' \approx bX$

WHEN $X \ll 1$,

$Y' \approx -dY$

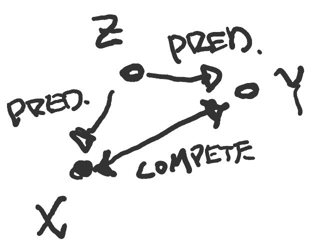


WE ESTIMATE THIS FIT IN THE NOTEBOOK.

LARGE SYSTEMS OF EQUATIONS

(NETWORKS OF POPULATIONS)

LOTKA - VOLTERA WITH MORE THAN TWO POPULATIONS, SOME COMPATITION, SOME PREDITOR – PREY. THESE FORM AN ECO WEB:



$$\Rightarrow \quad \begin{aligned} X &= \overbrace{X(b_1 - d_1 X)}^{\text{LOG. GROWTH}} - \overbrace{c_1 XY}^{\text{COMP.}} - \overbrace{e_1 XZ}^{\text{PRED.}} \\ Y &= Y(b_2 - d_2 Y) - c_2 XY - e_2 YZ \\ Z &= r_1 ZX + r_2 ZY - \delta Z \end{aligned}$$

Q) IS THERE AN EQUILIBRIUM WITH X, Y, Z NON-ZERO?

Q) WHAT KIND OF BEHAVIOR DO WE EXPECT?

Q) CAN WE GENERALIZE?

NOTE: INTERESTING EXAMPLE
YOU CAN SHOW INTRODUCING A PREDITOR CAN STABALIZE A COMPETITION SYSTEM.

PREDATOR CAN STABILIZE A COMPETITION SYSTEM.

EXC: FIND THESE VALUES.

Q) HOW MANY EQUILIBRIUM SOLUTIONS ARE THERE TO THIS SYSTEM? (X, Y, Z NOT NESS. ≥ 0) AT MOST 2^N SOLUTIONS.

SUB QUESTION: HOW MANY EQ. SOL. TO

$$X' = b_1 X - r_1 X Y - d_1 X^2$$
$$Y' = r_2 X Y - d_1 Y$$

3 SOLUTIONS

$(0,0)$, $(\alpha, 0)$, (γ, δ), NO STAB EQ. W/ $X = 0$ $Y \neq 0$.

HOW ABOUT TO

$$X' = X(b_1 - d_1 X) - c_1 X Y$$
$$Y' = Y(b_2 - d_1 Y) - c_2 X Y$$

? 4 SOLUTION

BOTH HAVE SOLUTIONS OF FORM:

$(0,0)$, $(0, \alpha)$, $(\beta, 0)$, (γ, δ)

GOING TO FIND AT MOST

ONE SOLUTION: $X, Y, Z \neq 0$

THREE SOL: ONE OF $X, Y, Z = 0$

THREE SOL: TWO OF $X, Y, Z = 0$

ONE SOL: $X = Y = Z = 0$.

WRITE EQUATIONS AS:

$$X' = X(b_1 - d_1 X - c_1 Y - e_1 Z)$$
$$Y' = Y(b_2 - d_2 Y - c_2 X - e_2 Z)$$
$$Z' = Z(\ldots)$$

$$Y' = Y\,(b_2 - d_2 Y - c_2 X - e_2 Z)$$
$$Z' = Z\,(r_1 X + r_2 Y - S)$$

WRITE AS MATRIX MULT:

$$\vec{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \quad D_{\vec{X}} = \begin{bmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Z \end{bmatrix}, \quad A = \begin{bmatrix} -d_1 & -c_1 & -e_1 \\ -c_2 & -d_2 & -e_2 \\ r_1 & r_2 & 0 \end{bmatrix}$$

COMPETITION MATRIX

$$\vec{R} = \begin{bmatrix} b_1 \\ b_2 \\ -S \end{bmatrix}$$

NATURAL GROWTH RATE VECTOR

SO

$$\vec{X}' = D_{\vec{X}} \cdot A \cdot \vec{X} + D_{\vec{X}} \cdot \vec{R}$$
$$= D_{\vec{X}} \left(A \vec{X} + \vec{R} \right)$$

<u>Now</u>: CAN VERIFY THAT THERE IS AT MOST <u>ONE</u> EQUILIBRIUM SOLUTION WITH $X_1, \dots, X_N > 0$:

EQ. SOLUTION:

$$0 = \vec{X}' = D_{\vec{X}} \left(A\vec{X} + \vec{R} \right)$$

IF $X_1, \dots, X_N \neq 0$, THEN

$$A\vec{X} + \vec{R} = 0$$

OR

$$\vec{X}_* = -A^{-1}\vec{R}$$

AT MOST <u>ONE</u> SOLUTION. NOTE,

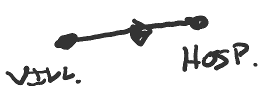
$\vec{X}$

AT MOST ONE SOLUTION. NOTE, $\vec{X}_*$ IS ONLY IN MODEL DOMAIN IF ALL ENTRIES ARE POSITIVE.

EXAMPLE: SIR AND NETWORKS OF POPULATIONS

VILLAGE + HOSPITOL SYSTEM:

WANT TO MODEL BOTH POPULATIONS USING SIR, HERE INFECTED PEOPLE ARE TAKEN TO HOSPITOL:



Q) HOSPITOL ONLY HAS 1000 BEDS, WHAT INFECTIVITY RATE MUST VILLAGE AND HOSPITOL HAVE TO STAY UNDER THIS NUMBER?

PROPOSE EQUATIONS:

$$S_v' = -\beta_v S_v I_v$$
$$I_v' = \beta_v S_v I_v - \gamma_v I_v - \delta I_v$$

$$S_H' = -\beta_H S_H I_H$$
$$I_H' = \beta_H S_H I_H - \gamma_H I_H + \delta I_v$$

HERE, δ IS PERCENT OF INFECTED VILLIGERS TAKEN TO HOSPITOL EACH DAY.

ASSUME

- $\gamma_v = \gamma_H$
- $\beta_v = a \beta_H$

- $\beta_\omega = a\beta_H$

FIND γ IN LIT., THIS IS EXAMPLE OF A NONTRIVIAL EXTENSION OF SIR.