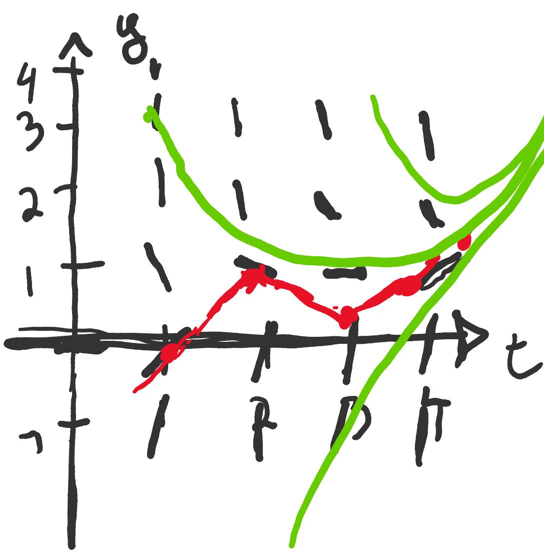
Monday, July 6, 2020 12:59 PM

# AUTONOMOUS DIFF. EQS AND POPULATION MODELS

RECALL (PROBLEM SET 3) SLOPE PLOT FOR A FIRST ORDER DIFFEQ LIKE

$$\frac{dy}{dt} = t - 3y$$

WE CAN USE EQ. TO DRAW A PLOT OF SLOPES OF SOLUTIONS:




| t \ y | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | -2 | -5 | -8 | -11 |
| 2 | -1 | -4 | -7 | -10 |
| 3 | 0 | -3 | -6 | -9 |
| 4 | 1 | -2 | -5 | -8 |

P

$\frac{dy}{dt}$ "

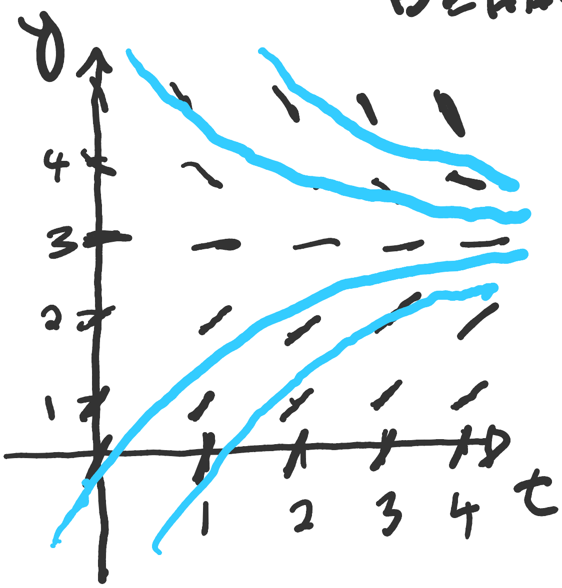
• USE PLOT TO

$\frac{dy}{dt}$

- USE PLOT TO ESTIMATE SOLUTIONS BUT DRAWING A CURVE TANGENT TO SLOPE LINE, OR PEICEWISE LINEAR. THIS IS HOW NUMERICAL SOLVES WORK IN SIMPLEST CASE.

- WANT TO GET A GLOBAL PICTURE OF POSSIBLE BEHAVIOR FOR DIFFERENT SOLUTIONS.

Ex: WHAT TO LOOK FOR IN SLOPE PLOTS? DIFFERENT FAMILIES OF BEHAVIOR / ASYMPTOTICS.




$$\frac{dy}{dt} = 3 - y, \quad y(0) = y_0$$

$$\Rightarrow y = 3 - (3 - y_0)e^{-t}$$

WE SEE THAT ALL SOLUTIONS ARE ASYMPT. TO THE CONSTANT SOLUTION

ASYMPT. TO THE CONSTANT SOLUTION $y(t) = 3$. WE CALL A CONSTANT SOLUTION AN <u>EQUILIBRIUM</u> <u>SOLUTION</u>.

<u>NOTE</u>: IN THIS CASE $\frac{dy}{dt}$ DOESN'T DEPEND ON $t$. WE CALL SUCH EQS <u>AUTONOMOUS</u>.

## <u>POPULATION GROWTH</u>:

<u>IDEA</u>: FOR A POPULATION $P$, ON SOME TIME SCALE, EACH MEMBER HAS AN $\alpha$ PERCENT CHANCE OF GIVING BIRTH.

<u>EX</u>: FOR 10 BACTERIA, EACH HAS A 50% CHANCE OF REPRODUCING EACH DAY, THEN EXPECT THAT AT

$$P(t+1) = P(t) + .5\,P(t)$$

AFTER 1/2 A DAY, EXPECT

$$P(t+1/2) = P(t) + \left(\tfrac{1}{2}\right).5\,P(t)$$

$$P(t+1/2) = P(t) + (\tfrac{1}{2})\,.5\,P(t)$$

AFTER $\Delta t$ TIME HAS PASSED, EXPECT

$$P(t+\Delta t) = P(t) + (\Delta t)(.5)\,P(t)$$

$$\frac{P(t+\Delta t) - P(t)}{\Delta t} = .5\,P(t)$$

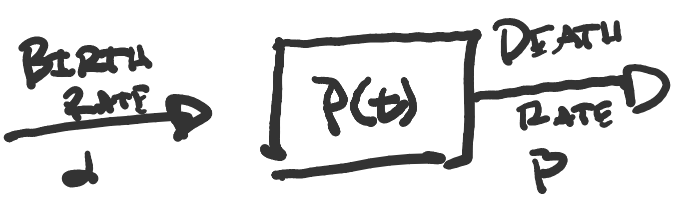
TAKE LIMIT: $\Delta t \to 0$

$$\frac{dP}{dt} = .5\,P.$$

OR IN GENERAL $\frac{dP}{dt} = \alpha P$.

## BIRTH AND DEATH:

SAY EVERY MEMBER OF POPULATION HAS AN $\alpha$ CHANCE OF REPRODUCING AND $\beta$ CHANCE OF DYING EACH TIME PERIOD.




$$\Rightarrow \frac{dP}{dt} = \alpha P - \beta P$$

SETTING $r = \alpha - \beta$. $\Rightarrow \frac{dP}{dt} = rP$

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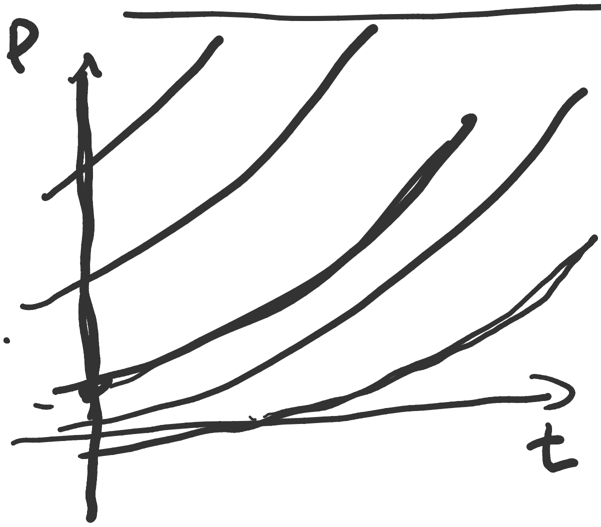
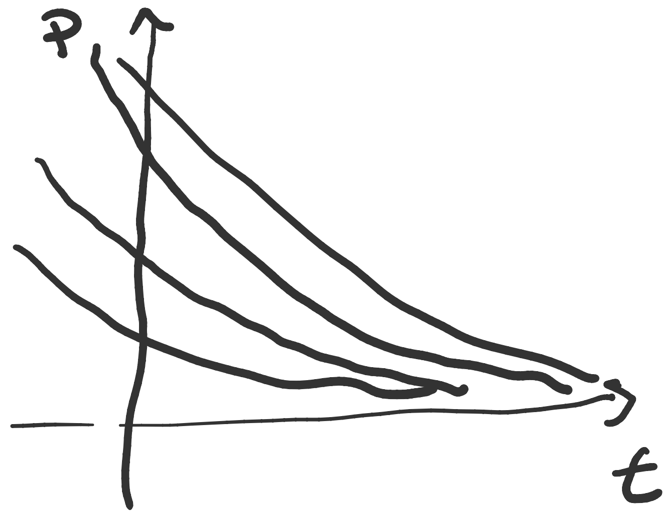
TWO PHASES FOR EQ:



GROWTH, $r > 0$



DIE OFF, $r < 0$

$$P = P_0 e^{rt}$$

# DENSITY DEPENDANT GROWTH

IDEA: LETS INCLUDE COMPITION / CROWDING WITHIN POPULATION

ONE PICTURE: ASSUME RESOURCES ARE SCARCE, AND THERE IS A $\gamma$ PERCENT CHANCE ANY TWO MEMBERS OF POP. WILL COME IN CONTACT AND <u>COMPETE</u> IN A WAY THAT KILLS ONE MEMBER OF POPULATION. THEN

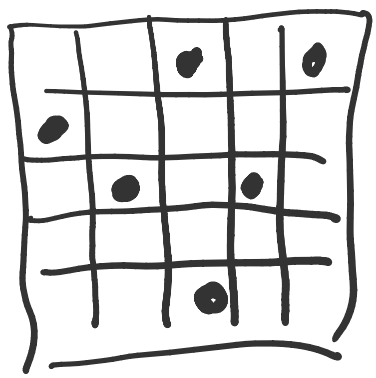
MEMBER OF POPULATION. THEN

$$\frac{dP}{dt} = \underset{\text{BIRTH RATE}}{\alpha P} - \underset{\text{OLD DEATH RATE}}{\beta P} - \underbrace{\gamma P}_{\text{DEATH RATE}} P = rP\left(1 - \frac{\gamma}{r} P\right)$$

WHERE $r = \alpha - \beta$.

<u>ALT EXP:</u>

ASSUME $N$ SITES AND $m$ HAVE FOOD



• - FOOD

☐ - SITES.

TO SURVIVE, EACH POP. MEMBER MUST GET 1 FOOD.

CHANCE OF SURVIVAL: EACH MEMBER HAS $\frac{m}{N}$ CHANCE OF RANDOMLY LANDING ON FOOD, BUT ALSO A CHANCE SOMEONE ELSE WILL ALSO LAND AND YOU WILL (BOTH) STARVE.

CHANCE IS: $\underbrace{\frac{m}{N}}_{\text{YOU FIND}} - \underbrace{\frac{m}{N}}_{\text{SOME ONE}} P$ = <u>DEATH RATE</u>

YOU FIND A FOOD SITE — SOMEONE ELSE FINDS YOUR FOOD. ← COMP.

TOTAL CHANGE:

$$\frac{dP}{dt} = \{\text{BIRTH RATE}\}P - \{\text{DEATH RATE}\}P$$

$$= \alpha P - \left(\frac{m}{N} - \frac{m}{N}P\right)P \qquad = rP\left(1 - \frac{\gamma}{r}P\right).$$

THIS IS KNOWN AS THE LOGISTIC EQUATION:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right),$$

$r$ - UNCONSTRAINED GROWTH RATE ASSUME $>0$

$K$ - CARRYING CAPACITY.

THIS IS AN AUTONOMOUS DIFF. EQ.

## GLOBAL PICTURE

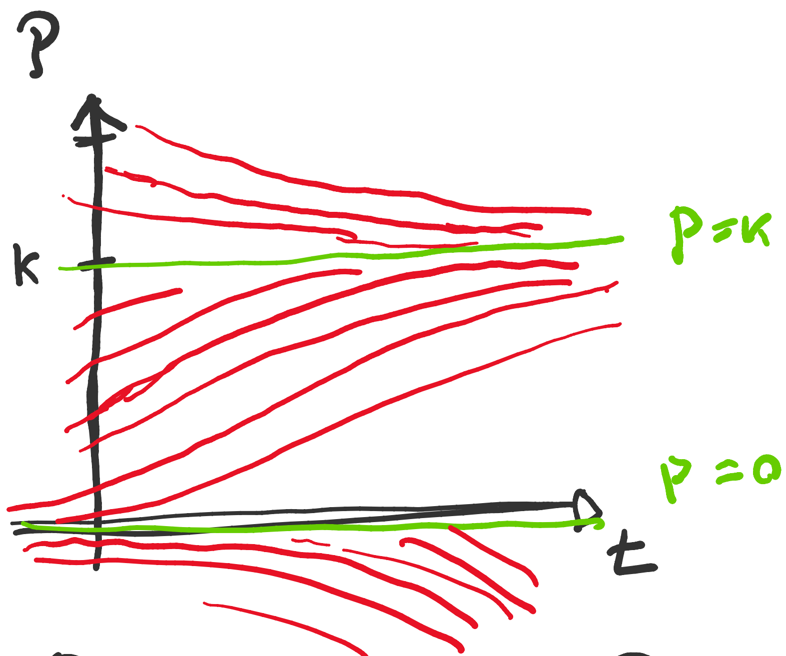
WANT TO UNDERSTAND BEHAVIOR:

1. FIND EQ. SOLUTIONS
2. CLASSIFY BEHAVIOR OF NON EQ. SOLUTIONS.

① TO FIND EQ. SOLUTIONS, IE SOLUTIONS $P(t) = P_0$, LOOK FOR SOLUTIONS WHERE $\frac{dP}{dt} = 0$.

HERE:

$$\frac{dP}{dt} = 0 = rP\left(1 - \frac{P}{K}\right) \Rightarrow \boxed{\begin{array}{l} P(t) = 0 \\ \text{OR } P(t) = K \\ \text{OR } r = 0. \end{array}}$$




- $P(t) > K$, $rP\left(1 - \frac{P}{K}\right) < 0$
  $P' < 0$, DECREASING.
- $0 < P < K$, $rP\left(1 - \frac{P}{K}\right) > 0$
  $P' > 0$, INCREASING
- $P < 0$, $rP\left(1 - \frac{P}{K}\right) < 0$

FOR AUTONOMOUS DIFF EQS, $\frac{dP}{dt}$ DOESNT DEPEND ON $t$. IN THIS CASE 3 SOLUTION FAMILIES, GIVEN BY SIGN OF $P'$, AND SEPERATED BY EQ. SOLUTIONS AT $P' = 0$.

FOR AUTONOMOUS EQS $\frac{dP}{dt} = F(P)$, WE

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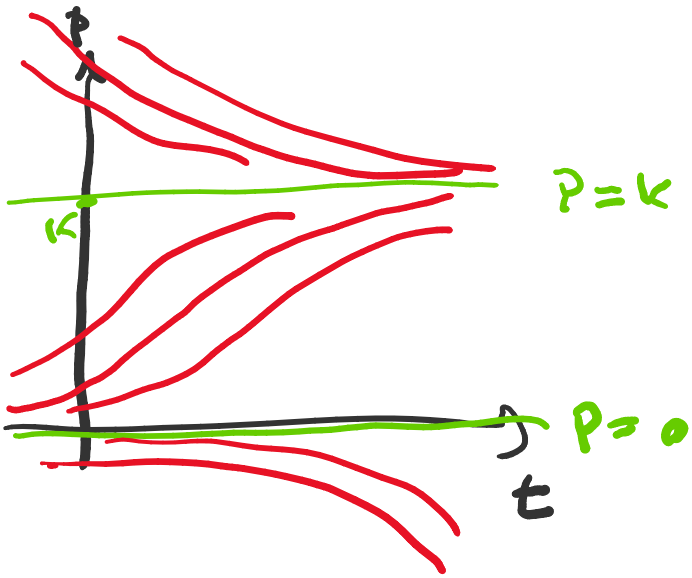
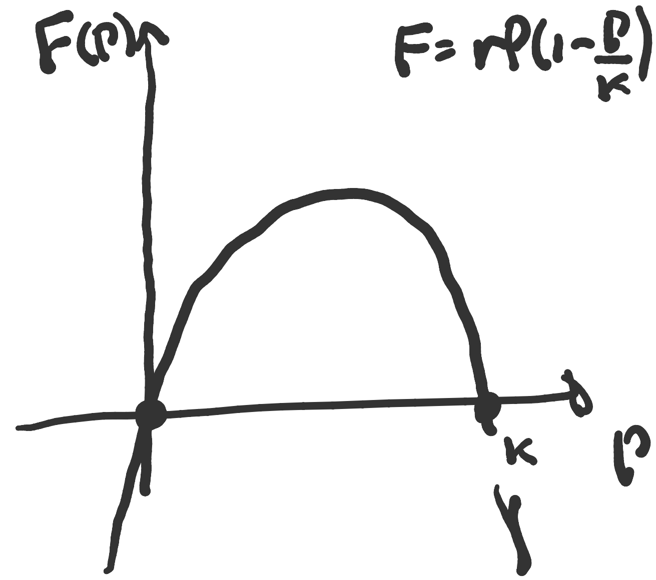
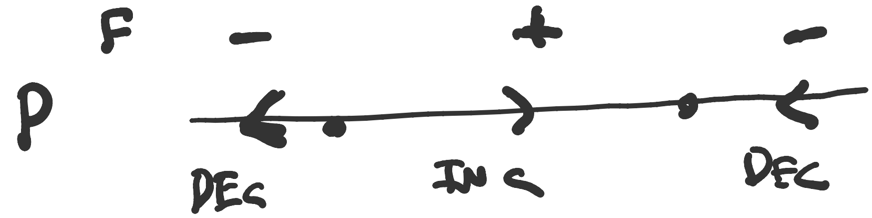



**SOLVE:**

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) \Rightarrow \int \frac{1}{P\left(1 - \frac{P}{K}\right)}\, dP = \int r\, dt$$

$$\Rightarrow \int \frac{K}{P(K-P)}\, dP$$

SOLVE W/ PARTIAL FRACTION.

**IDEA:**

WANT

$$\frac{K}{\quad} \stackrel{?}{=} \frac{A}{\quad} + \frac{B}{\quad} = \frac{A(K-P) + BP}{\quad}$$

$$\frac{K}{P(K-P)} \stackrel{?}{=} \frac{A}{P} + \frac{B}{K-P} = \frac{A(K-P) + BP}{P(K-P)}$$

EQ NUMERATORS:

$$K = AK + (B-A)P$$

SO $(B-A)P = 0 \Rightarrow A = B$

AND $AK = K \Rightarrow A = 1$

SO

$$\frac{K}{P(K-P)} = \frac{1}{P} + \frac{1}{K-P} \quad \checkmark$$

<u>Now:</u>

$$\int \frac{K}{P(K-P)}\,dP = \int \frac{1}{P} + \frac{1}{K-P}\,dP = \mathrm{Log}|P| - \mathrm{Log}|K-P|$$

$$= rt + C$$

FOR $P=0$ OR $P=K$.

SO

$$\mathrm{Log}\left|\frac{P}{K-P}\right| = rt + C$$

$$\frac{P}{K-P} = e^{rt+C}$$

BE CAREFUL ABOUT REMOVING $|\cdot|$

$$P = e^{rt+C}(K-P)$$

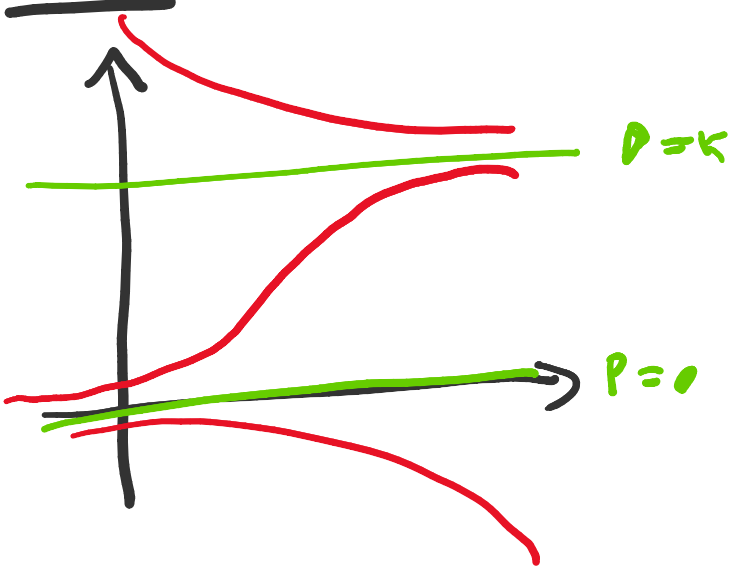
$$P = e^{rt+c}(K-P)$$

$$P(1+e^{rt+c}) = Ke^{rt+c}$$

so

$$P = \frac{Ke^{rt+c}}{1+e^{rt+c}} = \frac{KAe^{rt}}{1+Ae^{rt}}.$$

**Now:**




FIRST: FOR P>0,

$\lim_{t\to\infty} P = K$ ✓

FOR P<0, ANSWER SLIGHTLY DIFFERENT.