MATH 7241 Fall 2020: Problem Set #7

Due date: Sunday November 8

Reading: relevant background material for these problems can be found on Canvas 'Notes 4: Finite Markov Chains'. Also Grinstead and Snell Chapter 11.

Exercise 1 In each case below, determine whether or not the chain is reversible (note: the condition for reversibility is $w_i p_{ij} = w_j p_{ji}$ for all states i, j).

$$(a) \qquad P = \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{pmatrix}$$

$$(b) \qquad P = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 0 & 2/3 & 1/3 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

a) Find stationary distribution.

$W = \left(\frac{2}{3}, \frac{1}{3}\right)$,

Check reversible equation:

$W_1 P_{12} \overset{?}{=} W_2 P_{21}$

$\frac{2}{3} \frac{1}{4} \overset{?}{=} \frac{1}{3} \frac{1}{2}$ ✓ YES $\Rightarrow$ reversible

b) $P_{12} = \frac{1}{4}$, $P_{21} = 0 \Rightarrow$ impossible to satisfy $W_1 P_{12} = W_2 P_{21} \Rightarrow$ not reversible.

$\boxed{\textbf{Exercise 2}}$ A knight moves randomly on a standard 8×8 chessboard. At each step it chooses at random one of the possible legal moves available. Given that the knight starts in a corner of the chessboard, find the expected number of steps until its first return to its initial position. [Hint: model the knight's position using a Markov chain, and try to show that the chain is reversible]

$E[\#\text{steps first return to state } i] = w_i^{-1}$

Random walk on graph is reversible:

$$W_i = \frac{d_i}{\sum_j d_j} \qquad d_i = \#\text{ nearest neighbors of node } i.$$

8x8 chess board

2	3	4	4	4	4	3	2
3	4	6	6	6	6	4	3
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
3	4	6	6	6	6	4	3
2	3	4	4	4	4	3	2

← d_i for each square

$$\sum_j d_j = 336$$

$$\Rightarrow W_{\text{corner}} = \frac{2}{336} = \frac{1}{168}$$

$\Rightarrow$ mean return time $= 168$ steps

Exercise 3 Grinstead and Snell p.423, #7.

Make 0,4 into absorbing states

$$\Rightarrow P = \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

Reorder states 1, 2, 3, 0, 4.

$$P = \begin{pmatrix} Q & R \\ O & I \end{pmatrix} := \begin{pmatrix} 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} 0 & \frac{3}{4} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{3}{4} & 0 \end{pmatrix} \Rightarrow N = (I - Q)^{-1} = \begin{pmatrix} \frac{5}{2} & 3 & \frac{3}{2} \\ 2 & 4 & 2 \\ \frac{3}{2} & 3 & \frac{5}{2} \end{pmatrix}$$

$$NR = \begin{pmatrix} \frac{5}{2} & 3 & \frac{3}{2} \\ 2 & 4 & 2 \\ \frac{3}{2} & 3 & 5/2 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 0 \\ 0 & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 5/8 & 3/8 \\ 1/2 & 1/2 \\ 3/8 & 5/8 \end{pmatrix}$$

Exercise 4 Grinstead and Snell p.423, #9.

$$P = \begin{array}{c} \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

$\{1,2,3,4\}$ transient

$$\Rightarrow \quad Q = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad N = (I - Q)^{-1} = \begin{pmatrix} 1 & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \\ 0 & 1 & \frac{1}{3} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$$\text{Mean \# steps } (1 \to 5) = N_{11} + N_{12} + N_{13} + N_{14}$$

$$= 1 + \frac{1}{4} + \frac{1}{3} + \frac{1}{2} = \frac{25}{12}$$

$\boxed{\textbf{Exercise 5}}$ Grinstead and Snell p.427, #24.

Case $p=q$ done in class.

$\underline{p \neq q}$: try

$$w_x = \frac{(q/p)^x - 1}{(q/p)^T - 1} = A (q/p)^x + B$$

$$\Rightarrow \; p w_{x+1} + q w_{x-1} = p A (q/p)^{x+1} + pB + q A (q/p)^{x-1} + qB$$

$$= q A (q/p)^x + p A (q/p)^x + A$$

$$= w_x \checkmark$$

Boundary conditions: $w_0 = 0 \Rightarrow A + B = 0$

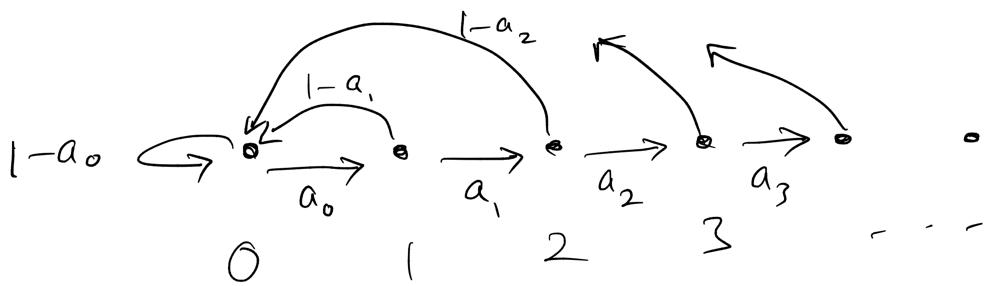
$w_T = 1 \Rightarrow A((q/p)^T - 1) = 1$.

$\boxed{\textbf{Exercise 6}}$ Consider a Markov chain on the set $S = \{0, 1, 2, \dots\}$ with transition probabilities

$$p_{i,i+1} = a_i, \quad p_{i,0} = 1 - a_i$$

for $i \geq 0$, where $\{a_i \mid i \geq 0\}$ is a sequence of constants which satisfy $0 < a_i < 1$ for all i. Let $b_0 = 1$, $b_i = a_0 a_1 \cdots a_{i-1}$ for $i \geq 1$. Show that the chain is

(a) persistent if and only if $b_i \to 0$ as $i \to \infty$ [Hint: compute $f_{00} = \sum_n f_{00}(n)$]
(b) positive persistent if $\sum_i b_i < \infty$ [Hint: compute mean return time to state 0, namely $\sum_n n f_{00}(n)$]. Compute the stationary distribution if this condition holds.



Chain is clearly irreducible b/c $0 < a_i < 1$.

a) $f_{00}(n) = \mathbb{P}(0 \to 1 \to 2 \to \cdots \to n-1 \to 0)$

$= a_0 a_1 a_2 \cdots a_{n-2}(1 - a_{n-1})$

$= b_{n-1} - b_n$

$$\Rightarrow \sum_{n=1}^{N} f_{00}(n) = b_0 - b_N = 1 - b_N$$

$$\Rightarrow \sum_{n=1}^{\infty} f_{00}(n) = \lim_{N \to \infty} (1 - b_N)$$

$$= 1 - \lim_{N\to\infty} b_N$$

$\Rightarrow$ chain is persistent $\Leftrightarrow \lim_{N \to \infty} b_N = 0.$

b) Mean return time is

$$\mu_0 = \sum_{k=1}^{\infty} k \, f_{00}(k)$$

$$= 1 - b_1 + 2(b_1 - b_2) + 3(b_2 - b_3) + \cdots$$

$$= 1 + b_1 + b_2 + b_3 + \cdots$$

$$= \sum_{k=0}^{\infty} b_k$$

So $\mu_0 < \infty \iff \sum_{k=0}^{\infty} b_k < \infty$

<u>Stationary dist.</u>

$$w_j = a_{j-1} \, w_{j-1} \qquad (j \geq 1)$$

$$\Rightarrow \quad w_j = a_{j-1} a_{j-2} \cdots a_0 w_0$$

$$= b_j w_0 \qquad , \quad w_0 = b_0 w_0$$

Normalize:

$$\sum_{j=0}^{\infty} b_j w_0 = 1$$

$$\Rightarrow \quad w_0 = \left(\sum_{j=0}^{\infty} b_j \right)^{-1}$$

$$\Rightarrow \quad w_j = \frac{b_j}{\sum_{k=0}^{\infty} b_k} .$$