Homework 1.

Q1

1) $S = \{a+b\sqrt{2} : a, b \in \mathbb{Q}\}$

1. Closed under addition

For any $a_1+b_1\sqrt{2}$, $a_2+b_2\sqrt{2} \in S$,

$(a_1+b_1\sqrt{2})+(a_2+b_2\sqrt{2}) = (a_1+a_2)+(b_1+b_2)\sqrt{2} \in S$

$[\because (a_1+a_2) \in \mathbb{Q}$ and $(b_1+b_2) \in \mathbb{Q}]$

2. Associativity of addition

For any $a_1+b_1\sqrt{2}$, $a_2+b_2\sqrt{2}$, $a_3+b_3\sqrt{2} \in S$.

$[(a_1+b_1\sqrt{2})+(a_2+b_2\sqrt{2})]+(a_3+b_3\sqrt{2})$

$=(a_1+a_2)+(b_1+b_2)\sqrt{2}+(a_3+b_3\sqrt{2})$

$=(a_1+a_2+a_3)+(b_1+b_2+b_3)\sqrt{2}$

$=(a_1+(a_2+a_3))+(b_1+(b_2+b_3))\sqrt{2}$

$=(a_1+b_1\sqrt{2})+(a_2+a_3)+(b_2+b_3)\sqrt{2}$

$=(a_1+b_1\sqrt{2})+[(a_2+b_2\sqrt{2})+(a_3+b_3\sqrt{2})]$

3. Additive Identity

$(0+0\sqrt{2})+(a+b\sqrt{2}) = a+b\sqrt{2} = (a+b\sqrt{2})+(0+0\sqrt{2})$

4. Additive Inverse

$(a+b\sqrt{2})+(-a+(-b)\sqrt{2}) = (a-a)+(b-b)\sqrt{2} = 0+0\sqrt{2}$

5. Commutativity of addition

$(a_1+b_1\sqrt{2})+(a_2+b_2\sqrt{2}) = (a_1+a_2)+(b_1+b_2)\sqrt{2}$

$=(a_2+a_1)+(b_2+b_1)\sqrt{2}$

$=(a_2+b_2\sqrt{2})+(a_1+b_1\sqrt{2})$

6. Closed under multiplication

For any $a+b\sqrt{2}$, $c+d\sqrt{2} \in S$

$(a+b\sqrt{2})(c+d\sqrt{2}) = ac+ad\sqrt{2}+bc\sqrt{2}+bd\cdot 2$

$= (ac+2bd)+(ad+bc)\sqrt{2} \in S$

$[\because ac+2bd \in Q$ and $ad+bc \in Q$

when $a, b, c, d \in Q]$

7. Associativity of Multiplication

$a_1+b_1\sqrt{2}$, $a_2+b_2\sqrt{2}$, $a_3+b_3\sqrt{2} \in S$.

$[(a_1+b_1\sqrt{2}) \times (a_2+b_2\sqrt{2})] \times (a_3+b_3\sqrt{2})$

$= [(a_1a_2+2b_1b_2)+(a_1b_2+b_1a_2)\sqrt{2}] \times (a_3+b_3\sqrt{2})$

$= (a_1a_2a_3+2b_1b_2a_3+(2a_1b_2b_3+2b_1a_2b_3))$

$+ (a_1a_2b_3+2b_1b_2b_3+a_1b_2a_3+b_1a_2a_3)\sqrt{2}$

$= (a_1+b_1\sqrt{2})+[(a_2+b_2\sqrt{2}) \times (a_3+b_3\sqrt{2})]$

8. Multiplicative Identity

$(a+b\sqrt{2})(1+0\sqrt{2}) = a+b\sqrt{2} = (a+b\sqrt{2})(1+0\sqrt{2})$

Multiplicative identity : $1+0\sqrt{2}=1$

9. Multiplicative inverse

$(a+b\sqrt{2})(\frac{a-b\sqrt{2}}{a^2-2b^2}) = 1$

Hence inverse of $a+b\sqrt{2}$ is $\frac{a-b\sqrt{2}}{a^2-2b^2}$

10. Commutative under multiplication

$(a+b\sqrt{2})(c+d\sqrt{2}) = (ac+2bd)+(ad+bc)\sqrt{2}$

$= ca+2db+(da+cb)\sqrt{2}$

$= (c+d\sqrt{2})(a+b\sqrt{2})$

11. Distributive law

$$(a_1+b_1\sqrt{2})\times[(a_2+b_2\sqrt{2})+(a_3+b_3\sqrt{2})]$$
$$=(a_1+b_1\sqrt{2})\times[(a_2+a_3)+(b_2+b_3)\sqrt{2}]$$
$$=(a_1a_2+a_1a_3+2b_1b_2+2b_1b_3)+(a_2b_1+a_3b_1+b_2a_1+b_3a_1)\sqrt{2}$$
$$=(a_1a_2+2b_1b_2)+(a_2b_1+b_2a_1)\sqrt{2}+(a_1a_3+2b_1b_3)+(a_3b_1+b_3a_1)\sqrt{2}$$
$$=[(a_1+b_1\sqrt{2})\times(a_2+b_2\sqrt{2})]+[(a_1+b_1\sqrt{2})\times(a_3+b_3\sqrt{2})]$$

12. Clearly, Additive identity $\neq$ Multiplicative identity.

$\therefore$ S is a Field.

2) This field is field of complex numbers $\mathbb{C}$

1. Closed under addition

$$(a+b\sqrt{-1})(c+d\sqrt{-1})=(ac-bd)+(ad+bc)\sqrt{-1}\in\mathbb{C}$$
$$[\because ac-bd,\ ad+bc\in\mathbb{R}]$$

2. Associativity of addition

$$(a_1+b_1\sqrt{-1})+[(a_2+b_2\sqrt{-1})+(a_3+b_3\sqrt{-1})]$$
$$=(a_1+b_1\sqrt{-1}+a_2+b_2\sqrt{-1})+(a_3+b_3\sqrt{-1}).$$

3. Commutativity holds under addition.

Since, $(a+b\sqrt{-1})+(c+d\sqrt{-1})=(a+c)+(b+d)\sqrt{-1}$
$$=(c+a)+(d+b)\sqrt{-1}$$
$$=(c+d\sqrt{-1})+(a+b\sqrt{-1})$$

4. Additive identity

$$(a+b\sqrt{-1})+0=a+b\sqrt{-1}=0+(a+b\sqrt{-1})$$

5. Additive inverse

$(a+b\sqrt{-1})+(-a-b\sqrt{-1})=0$ $\quad$ $-a-b\sqrt{-1}$ is the additive inverse of $a+b\sqrt{-1}$.

6. Closed under Multiplication.

$(a+b\sqrt{-1})\times(c+d\sqrt{-1}) = (ac-bd)+(ad+bc)\sqrt{-1} \in \mathbb{C}$

$[\because ac-bd \in \mathbb{R}$ and $ad+bc \in \mathbb{R}]$

7. Associativity of Multiplication

Clearly.

8. Multiplication identity.

1 is the multiplicative identity.

$(a+b\sqrt{-1})\times 1 = a+b\sqrt{-1} = 1\times(a+b\sqrt{-1})$

9. Multiplicative inverse.

$\frac{a-b\sqrt{-1}}{a^2+b^2}$ is the inverse of $a+b\sqrt{-1}$

10. Commutative under multiplication

Clearly.

11. Distributive law.

$(a_1+b_1\sqrt{-1})\times[(a_2+b_2\sqrt{-1})+(a_3+b_3\sqrt{-1})]$

$=(a_1+b_1\sqrt{-1})\times(a_2+a_3+(b_2+b_3)\sqrt{-1})$

$= a_1a_2+a_1a_3-b_1b_2-b_1b_3+(b_1a_2+b_1a_3+a_1b_2+a_1b_3)\sqrt{-1}$

$=(a_1+b_1\sqrt{-1})\times(a_2+b_2\sqrt{2})+(a_1+b_1\sqrt{-1})\times(a_3+b_3\sqrt{-1})$

12. $0\neq 1$

Therefore $\mathbb{C}$ is a field.

Q2.

For a set to be field, it should be abelian group under addition and non-zero elements of the set form abelian group under multiplication.

But we know matrix multiplication is not commutative.

Consider $\mathbb{R}^{2\times 2}$

$$A=\begin{bmatrix}0&1\\0&0\end{bmatrix} \quad B=\begin{bmatrix}1&0\\0&0\end{bmatrix}$$

$$AB=\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$BA=\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}0&1\\0&0\end{bmatrix}$$

$AB \neq BA$ (Similarly for any $n \geq 2$ we can check)

So matrix multiplication is not commutative.

So set of all matrix is not a field

We can also see that in a field if $ab=0$, then either $a=0$ or $b=0$.

But here $A\neq 0$, $B\neq 0$, but $AB=0$

i.e. This set has zero-divisors, it can't be field.

Q3.

+	[0]	[1]	[2]
[0]	0	1	2
[1]	1	2	0
[2]	2	0	1

x	[0]	[1]	[2]
[0]	0	0	0
[1]	0	1	2
[2]	0	2	1

Q4.

1) Closed under addition

let $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$ be two complex numbers,

$z_1 \in \mathbb{C}$, $z_2 \in \mathbb{C}$

$\therefore z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2)$

$= (a_1 + a_2) + i(b_1 + b_2)$

$\Rightarrow z_1 + z_2 \in C$

$\therefore \mathbb{C}$ is closed under addition.

2). Associativity.

let $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$, $z_3 = a_3 + ib_3$

$(z_1 + z_2) + z_3 = ((a_1 + ib_1) + (a_2 + ib_2)) + a_3 + ib_3$

$= (a_1 + a_2) + i(b_1 + b_2) + a_3 + ib_3$

$= (a_1 + a_2 + a_3) + i(b_1 + b_2 + b_3)$

$z_1 + (z_2 + z_3) = (a + ib_1) + ((a_2 + ib_2) + (a_3 + ib_3))$

$= (a_1 + ib_1) + ((a_2 + a_3) + i(b_2 + b_3))$

$= (a_1 + a_2 + a_3) + i(b_1 + b_2 + b_3)$

$\Rightarrow (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

3) Existence of Identity.

For all $z \in \mathbb{C}$, there exists an element, $0 + 0i$ such that $(a + bi) + (0 + 0i) = a + bi = (0 + 0i) + (a + bi)$

$\Rightarrow \mathbb{C}$ has the identity element that is 0.

4) Existence of Inverse.

For each $z = a + ib \in \mathbb{C}$, there exists $(-a) + i(-b) \in \mathbb{C}$ such that $(a + ib) + ((-a) + i(-b)) = (a - a) + i(b - b)$

$= 0 + i0 = 0$

$$= (-a) + i(-b) + a + ib = (-a+a) + i(-b+b) = 0.$$

$\therefore z = -a-ib$ is inverse of $z = a+ib$

5) Commutativity : $z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2)$

$$= (a_1 + a_2) + i(b_1 + b_2)$$

$$z_2 + z_1 = (a_2 + ib_2) + (a_1 + ib_1)$$

$$= (a_2 + a_1) + i(b_2 + b_1) = (a_1 + a_2) + i(b_1 + b_2)$$

$$\Rightarrow z_1 + z_2 = z_2 + z_1$$

6) Closed under multiplication.

let $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2 \in \mathbb{C}$

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)$$

$$\text{Again, } \left.\begin{matrix} a_1, a_2 \in \mathbb{R} \\ b_1, b_2 \in \mathbb{R} \end{matrix}\right\} \Rightarrow \begin{matrix} a_1 a_2 - b_1 b_2 \in \mathbb{R} \\ a_1 b_2 + b_1 a_2 \in \mathbb{R} \end{matrix}$$

$\therefore z_1 z_2 \in \mathbb{C}$

7). Commutative $z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2)$

$$= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

$$z_2 z_1 = (a_2 + ib_2)(a_1 + ib_1)$$

$$= (a_2 a_1 - b_2 b_1) + i(b_2 a_1 + b_1 a_2) = z_1 z_2$$

8). Associative.

$$z_1(z_2 z_3) = (z_1 z_2) z_3 = (a_1 a_2 a_3 - a_1 b_2 b_3 - b_1 a_2 b_3 - b_1 a_3 b_2) + i(a_1 a_2 b_3 + a_1 a_3 b_2 + b_1 a_2 a_3 - b_1 b_2 b_3)$$

9) Existence of Multiplicative Identity

For $z = a+ib \in \mathbb{C}$ these exists $1 = 1 + 0i \in \mathbb{C}$ such that

$$(a+ib)(1+0i) = (a+ib) = (1+0i)(a+ib)$$

$z \cdot 1 = z = 1 \cdot z$ Multiplicative Identity is 1

10). Existence of Multiplicative Inverse.

For each non-zero $z=a+ib \in \mathbb{C}$, there exists $\frac{1}{z} \in \mathbb{C}$ such that

$$(a+ib)\cdot\left(\frac{1}{a+ib}\right) = (a+ib)\left(\frac{a-ib}{a^2+b^2}\right)$$
$$= \frac{a^2}{a^2+b^2} - \frac{abi}{a^2+b^2} + \frac{abi}{a^2+b^2} + \frac{b^2}{a^2+b^2}$$
$$= \frac{a^2+b^2}{a^2+b^2} = 1$$

Similarly $\left(\frac{1}{a+ib}\right)(a+ib) = 1$

Multiplicative Inverse of $z = \frac{1}{z}$, $z=0$

11) Distributive law

For all $z_1, z_2, z_3 \in \mathbb{C}$, $z_1(z_2+z_3) = z_1z_2 + z_1z_3$

$$z_1(z_2+z_3) = (a_1+ib_1)((a_2+ib_2)+(a_3+ib_3))$$
$$= (a_1a_2+a_1a_3-b_1b_2-b_1b_3) + i(a_1b_2+a_1b_3+b_1a_2$$
$$+b_1a_3) = z_1z_2+z_1z_3$$

$\mathbb{C}$ satisfies all properties of Field

$\Rightarrow$ Yes, $\mathbb{C}$ is a field with usual addition, scalar product and product.

Q5.

B and D are in reduced row-echelon form.

Q6.

$$A+B=\begin{bmatrix}1&1&1\\0&1&1\\0&1&0\end{bmatrix}+\begin{bmatrix}0&1&1\\1&1&1\\1&1&0\end{bmatrix}=\begin{bmatrix}1&0&0\\1&0&0\\1&0&0\end{bmatrix}$$

$$A^2=\begin{bmatrix}1&1&1\\0&1&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&1\\0&1&0\end{bmatrix}=\begin{bmatrix}1&3&2\\0&2&1\\0&1&1\end{bmatrix}\pmod 2$$

$$=\begin{bmatrix}1&1&0\\0&0&1\\0&1&1\end{bmatrix}$$

$$AB=\begin{bmatrix}1&1&1\\0&1&1\\0&1&0\end{bmatrix}\begin{bmatrix}0&1&1\\1&1&1\\1&1&0\end{bmatrix}=\begin{bmatrix}2&3&2\\2&2&1\\1&1&1\end{bmatrix}\pmod 2$$

$$=\begin{bmatrix}0&1&0\\0&0&1\\1&1&1\end{bmatrix}$$

Q7.

$$A = \begin{bmatrix} 6 & -1 & 1 \\ t & 0 & 1 \\ 0 & 1 & t \end{bmatrix}$$

$$\begin{vmatrix} 6 & -1 & 1 \\ t & 0 & 1 \\ 0 & 1 & t \end{vmatrix} = 0 \Rightarrow 6 \begin{vmatrix} 0 & 1 \\ 1 & t \end{vmatrix} + 1 \begin{vmatrix} t & 1 \\ 0 & t \end{vmatrix} + 1 \begin{vmatrix} t & 0 \\ 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow -6 + t^2 + t = 0$$

$$t^2 + t - 6 = 0$$

$$(t+3)(t-2) = 0$$

$$t = -3, 2.$$

Q8.

a). $\left[\begin{array}{cc|c} 1 & h & 4 \\ 3 & 6 & 8 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & h & 4 \\ 0 & 3h-6 & 8 \end{array}\right].$

$-\frac{h}{2}+3$

$3h-6 \neq 0 \Rightarrow h \neq 2.$

b). $\left[\begin{array}{cc|c} -4 & 12 & h \\ 2 & -6 & -3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -3 & -h/4 \\ 2 & -6 & -3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -3 & -h/4 \\ 0 & 0 & -\frac{h}{2}+3 \end{array}\right]$

$-\frac{h}{2}+3=0.$

$\frac{h}{2}=3 \Rightarrow h=6.$

Q9.

① $\begin{bmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ rank 1 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ rank 2.

$\therefore$ 2 types.

② $\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}$ rank 1 $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}$ rank 2. $\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}$ rank 2.

$\therefore$ 3 types.

③. $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ rank 1

$\therefore$ 1 type.

Q10.

$a =$ arbitrary

$b = 0$

$c = 1$

$d = 0$

$e = 0$

Q11.

1) $A=\begin{bmatrix}1&2&3&4\\1&1&0&2\\2&0&1&2\end{bmatrix}\xrightarrow[2R_1-R_3]{R_1-R_2}\begin{bmatrix}1&2&3&4\\0&1&3&2\\0&4&5&6\end{bmatrix}$

$\xrightarrow[4R_2-R_3]{}\begin{bmatrix}1&2&3&4\\0&1&3&2\\0&0&7&2\end{bmatrix}\xrightarrow[R_3/7]{}\begin{bmatrix}1&2&3&4\\0&1&3&2\\0&0&1&2/7\end{bmatrix}\xrightarrow[R_2-3R_3]{}\begin{bmatrix}1&2&3&4\\0&1&0&8/7\\0&0&1&2/7\end{bmatrix}$

$\xrightarrow{R_1-2R_2}\begin{bmatrix}1&0&3&12/7\\0&1&0&8/7\\0&0&1&2/7\end{bmatrix}\xrightarrow{R_1-3R_3}\begin{bmatrix}1&0&0&6/7\\0&1&0&8/7\\0&0&1&2/7\end{bmatrix}$

$\therefore \mathrm{rref}(A)=\begin{bmatrix}1&0&0&6/7\\0&1&0&8/7\\0&0&1&2/7\end{bmatrix}$

$A\vec{x}=0$

$\Rightarrow \begin{bmatrix}1&0&0&6/7\\0&1&0&8/7\\0&0&1&2/7\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}\Rightarrow \begin{aligned}x_1+6/7x_4&=0\\x_2+8/7x_4&=0\\x_3+2/7x_4&=0\end{aligned}$

Let $x_4=t \Rightarrow x_1=-6/7t$, $x_2=-8/7t$, $x_3=-2/7t$

$\therefore \vec{x}=t\begin{bmatrix}-6/7\\-8/7\\-2/7\\1\end{bmatrix}$

2) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 4 & 5 & 6 \end{bmatrix}$

In $\mathbb{Z}_7 \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 6 & 4 & 5 \\ 0 & 3 & 2 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 2 & 1 \\ 0 & 6 & 4 & 5 \end{bmatrix}$

$\xrightarrow[R_3 - 2R_2]{} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \xrightarrow[R_3/3]{R_2/3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2/3 & 1/3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 5/3 & 10/3 \\ 0 & 1 & 2/3 & 1/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

rref(A) over field $\mathbb{Z}_7 = \begin{bmatrix} 1 & 0 & 5/3 & 10/3 \\ 0 & 1 & 2/3 & 1/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Q12.

1) $\left[\begin{array}{ccc|c} 3 & 1 & 4 & 1 \\ 5 & 2 & 6 & 5 \\ 0 & 5 & 2 & 1 \end{array}\right] \xrightarrow{5R_1} \left[\begin{array}{ccc|c} 15 & 5 & 20 & 5 \\ 5 & 2 & 6 & 5 \\ 0 & 5 & 2 & 1 \end{array}\right]$ $[\because 15=(7\times 2)+1$, $20=(7\times 2)+6]$

$= \left[\begin{array}{ccc|c} 1 & 5 & 6 & 5 \\ 5 & 2 & 6 & 5 \\ 0 & 5 & 2 & 1 \end{array}\right] \xrightarrow{R_2-5R_1} \left[\begin{array}{ccc|c} 1 & 5 & 6 & 5 \\ 0 & -23 & -24 & -20 \\ 0 & 5 & 2 & 1 \end{array}\right]$ $[\because -23=(7\times(-4))+5$, $-24=(7\times(-4))+4$, $-20=(7\times(-3))+1]$

$= \left[\begin{array}{ccc|c} 1 & 5 & 6 & 5 \\ 0 & 5 & 4 & 1 \\ 0 & 5 & 2 & 1 \end{array}\right] \xrightarrow{R_2-R_3} \left[\begin{array}{ccc|c} 1 & 5 & 6 & 5 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 2 & 0 \end{array}\right] \xrightarrow{R_3/2} \left[\begin{array}{ccc|c} 1 & 5 & 6 & 5 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\xrightarrow{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1-2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_2-4R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 5 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\xrightarrow{3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 15 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$ $[\because 15=7\times(2)+1]$

$\therefore$ rref$(A|\vec{b})$ over field $\mathbb{Z}_7$ is $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$

Q12/27.

Suppose $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a solution of the system $A\vec{x} = \vec{b}$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$$

$$\Rightarrow \vec{x} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$$

Q13.

$$\mathrm{rref}=(A)=\begin{bmatrix}1&0&-1&0\\0&1&2&0\\0&0&0&1\end{bmatrix}$$

Q14.

$$\mathrm{rref}(A)=\begin{bmatrix}1&2&3&0&5&6\\0&0&0&1&2&7\\0&0&0&0&0&0\end{bmatrix}$$

$$\Rightarrow \vec{x}_1\begin{bmatrix}1\\0\\0\end{bmatrix}+\vec{x}_2\begin{bmatrix}2\\0\\0\end{bmatrix}+\vec{x}_3\begin{bmatrix}3\\0\\0\end{bmatrix}+\vec{x}_4\begin{bmatrix}0\\1\\0\end{bmatrix}+\vec{x}_5\begin{bmatrix}5\\2\\0\end{bmatrix}$$

$$=\begin{bmatrix}6\\7\\0\end{bmatrix}$$

Q15.

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}-1.89423963\\0.98974654\\10.81797235\\-1.05760369\\1.61059908\end{bmatrix}$$

```
import numpy as np
A=np.mat('2,4,3,5,6;
4,8,7,5,2;-2,-4,3,4,-5;
1,2,2,-1,2;5,-10,4,6,4')
b=np.mat('37,74,20,26,24').T
r=np.linalg.solve(A,b)
print(r)
```

Q16.

1). Yes.

$$ABC = I_n$$

$$\Rightarrow A^{-1}ABC = A^{-1}I_n$$

$$I_n BC = A^{-1}I_n$$

$$A^{-1} = BC$$

Similarly, $B^{-1} = CA$, $C^{-1} = AB$

2). Yes.

AB is invertible $\Leftrightarrow \det(AB) \neq 0$

$\det(AB) = \det(A) \cdot \det(B) \neq 0$

That is A is invertible and B is invertible.

Q17.

$$A = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

$$AB = \begin{pmatrix} 12 & 8 \\ 9 & 5 \end{pmatrix} \qquad (AB)^2 = \begin{pmatrix} 216 & 136 \\ 153 & 97 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 22 & 10 \\ 15 & 7 \end{pmatrix} \qquad B^2 = \begin{pmatrix} 9 & 5 \\ 0 & 4 \end{pmatrix}$$

$$A^2B^2 = \begin{pmatrix} 22 & 10 \\ 15 & 7 \end{pmatrix}\begin{pmatrix} 9 & 5 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 198 & 150 \\ 135 & 103 \end{pmatrix}$$

$\therefore (AB)^2 \neq A^2B^2$

Q18.

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \Rightarrow A^{-1} = \frac{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}{\cos^2\theta + \sin^2\theta} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = A^{-1}$$

Q19.

1). Examples for symmetric matrices $(A=A^T)$

$A_1=\begin{bmatrix}1&2\\2&1\end{bmatrix}$ $A_1^T=\begin{bmatrix}1&2\\2&1\end{bmatrix} \Rightarrow A_1^T=A_1$

$B_1=\begin{bmatrix}1&2&3\\2&5&4\\3&4&6\end{bmatrix}$ $B_1^T=\begin{bmatrix}1&2&3\\2&5&4\\3&4&6\end{bmatrix} \Rightarrow B_1^T=B_1$

$C_1=\begin{bmatrix}1&2&3&4\\2&8&9&5\\3&9&10&6\\4&5&6&7\end{bmatrix}=C_1^T$

Examples for skew-symmetric matrices : $(A^T=-A)$

$A_2=\begin{bmatrix}0&-2\\2&0\end{bmatrix}$ $A_2^T=\begin{bmatrix}0&2\\-2&0\end{bmatrix}=-\begin{bmatrix}0&-2\\2&0\end{bmatrix}=-A_2$

$B_2=\begin{bmatrix}0&1&2\\-1&0&3\\-2&-3&0\end{bmatrix}$ $B_2^T=-B_2$

$C_2=\begin{bmatrix}0&1&2&3\\-1&0&3&4\\-2&-3&0&5\\-3&-4&-5&0\end{bmatrix}$ $C_2^T=-C_2$

Q19/2). Let $A=[a_{ij}]_{n\times n}$ $1\le i,j\le n$

$A^T=[a_{ji}]_{n\times n}$

Now if A is skew-symmetric, then $A^T=-A$

$a_{ji}=-a_{ij}$.

At the main diagonal, $i=j$

$\Rightarrow a_{ii}=-a_{ii}$

$2a_{ii}=0$

$a_{ii}=0 \Rightarrow$ Main diagonal entries are 0.

3) Since matrix A is symmetric as well as skew-symmetric,

$\Rightarrow A=A^T$ and $A^T=-A$

$\Rightarrow A=A^T=-A$

$2A=0$

$A=0$

Null matrix is symmetric and skew-symmetric.

4) • $(A+A)^T=A^T+(A^T)^T=A^T+A=A+A^T$

$\Rightarrow (A+A^T)^T=A+A^T$

$\Rightarrow A+A^T$ is symmetric

• $(AA^T)^T=(A^T)^T\cdot A^T=A\cdot A^T$ $\because (AB)^T=B^T\cdot A^T$

$\Rightarrow (AA^T)^T=(AA^T)$

$\Rightarrow AA^T$ is symmetric

$(A^TA)^T=A^T(A^T)^T=A^T\cdot A \Rightarrow A^TA$ is symmetric.

• $(A-A^T)^T = A^T-(A^T)^T = A^T - A$

$= -(A-A^T)$

$\Rightarrow (A-A^T)$ is skew-symmetric matrix.

Q19/5)

Let the matrix A can be represented as the sum of symmetric and skew-symmetric matrix

$\therefore A = P+Q$ ① where P: symmetric, Q: skew-symmetric.

$\therefore P^T = P$ and $Q^T = -Q$

$\Rightarrow A^T = (P+Q)^T$

$= P^T + Q^T$ $\quad (\because P^T = P,\ Q^T = Q)$

$A^T = P - Q$ —— ②

①+② : $A + A^T = 2P$

$P = \frac{1}{2}(A+A^T)$

①−② : $A - A^T = 2Q$

$Q = \frac{1}{2}(A-A^T)$

$\therefore A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$

Q20

a). Not injective, not surjective and not bijective.

b). injective, surjective and bijective.

c). not injective, surjective and not bijective

d). injective, not surjective and not bijective.

Q21.

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 0 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{41} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

$u_{11} = 4 \quad u_{12} = 1 \quad u_{13} = 0 \quad u_{14} = 0.$

$l_{21} = \frac{1}{4} \quad l_{31} = 0 \quad l_{41} = 0$

$u_{22} = \frac{15}{4} \quad u_{23} = 1 \quad u_{24} = 0$

$l_{32} = \frac{4}{15} \quad l_{42} = 0$

$u_{33} = \frac{56}{15} \quad u_{34} = 1.$

$l_{43} = \frac{15}{56}$

$u_{44} = \frac{209}{56}$

$$\therefore A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ 0 & \frac{4}{15} & 1 & 0 \\ 0 & 0 & \frac{15}{36} & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & \frac{15}{4} & 1 & 0 \\ 0 & 0 & \frac{56}{15} & 1 \\ 0 & 0 & 0 & \frac{209}{56} \end{bmatrix}$$

Q22

$A = LU$.

$$\Rightarrow \begin{bmatrix} q_1 & r_1 & 0 & 0 \\ p_1 & q_2 & r_2 & 0 \\ 0 & p_2 & q_3 & r \\ 0 & 0 & p_3 & q_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_1 & 1 & 0 & 0 \\ 0 & l_2 & 1 & 0 \\ 0 & 0 & l_3 & 1 \end{bmatrix} \begin{bmatrix} d_1 & u_1 & 0 & 0 \\ 0 & d_2 & u_2 & 0 \\ 0 & 0 & d_3 & u_3 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$$

$$= \begin{bmatrix} d_1 & u_1 & 0 & 0 \\ l_1 d_1 & l_1 u_1 + d_2 & u_2 & 0 \\ 0 & l_2 d_2 & l_2 u_2 + d_3 & u_3 \\ 0 & 0 & l_3 d_3 & l_3 u_3 + d_4 \end{bmatrix}$$

$$\Rightarrow l_1 = \frac{p_1}{q_1} \quad l_2 = \frac{p_2 q_1}{q_2 q_1 - p_1 r_1} \quad l_3 = \frac{p_3}{\left(q_3 - \frac{p_1 q_1 r_2}{q_2 q_1 - p_1 r_1}\right)}$$

$$u_1 = r_1, \ u_2 = r_2, \ u_3 = r_3$$

$$d_1 = q_1, \ d_2 = q_2 - \frac{p_1 r_1}{q_1}, \ d_3 = q_3 - \frac{p_1 q_1 \cdot r_2}{q_2 q_1 - p_1 r_1}$$

$$d_4 = q_4 - \frac{p_3}{\left(q_3 - \frac{p_1 q_1 \cdot r_2}{q_1 q_2 - p_1 r_1}\right)}$$

Q23.

$$A = \begin{bmatrix} 4 & 1 & \cdots & 0 & 0 \\ 1 & 4 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & 4 & 1 \\ 0 & 0 & \cdots & 1 & 4 \end{bmatrix}$$

Q24. $H_n = I_n - 2\vec{u}\vec{u}^T \quad ||\vec{u}|| = 1$

1). $H_n^T = (I_n - 2\vec{u}\vec{u}^T)^T$

$= I_n^T - 2(\vec{u}\vec{u}^T)^T$

$= I_n - 2(\vec{u}^T)^T\vec{u}^T$

$= I_n - 2\vec{u}\vec{u}^T$

$= H_n$ $\quad \therefore H_n$ is symmetric matrix.

2) $H_n^T H_n = H_n H_n$

$= (I_n - 2\vec{u}\vec{u}^T)(I_n - 2\vec{u}\vec{u}^T)$

$= I_n - 2\vec{u}\vec{u}^T - 2\vec{u}\vec{u}^T + 4(\vec{u}\vec{u}^T)(\vec{u}\vec{u}^T)$

$= I_n - 4\vec{u}\vec{u}^T + 4\vec{u}(\vec{u}^T\vec{u})\vec{u}^T$

$= I_n - 4\vec{u}\vec{u}^T + 4\vec{u}\vec{u}^T \quad [\because \vec{u}^T\vec{u} = ||\vec{u}||^2 = 1]$

$= I_n$

$\Rightarrow H_n^T H_n = I_n$

$\therefore H_n$ is an orthogonal matrix.

3). $H_n^2 = (I_n - 2\vec{u}\vec{u}^T)^2 = I_n + 4\vec{u}\vec{u}^T\vec{u}\vec{u}^T - 4\vec{u}\vec{u}^T$

$= I_n + 4\vec{u}\vec{u}^T - 4\vec{u}\vec{u}^T$

$= I_n$

$\Rightarrow H_n^2 = I_n$

4). $H_n\vec{u} = (I_n - 2\vec{u}\vec{u}^T)\vec{u} = I_n\vec{u} - 2\vec{u}\vec{u}^T\vec{u}$

$= I_n - 2\vec{u} \quad [\because \vec{u}^T\vec{u} = ||\vec{u}|| = 1]$

Q24(5)

$$\vec{u} = \frac{1}{\sqrt{n}}\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}$$

$$\therefore H_3 = I_3 - 2\vec{u}\vec{u}^T = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \frac{2}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}1&1&1\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix} = \begin{bmatrix}1/3&-2/3&-2/3\\-2/3&1/3&-2/3\\-2/3&-2/3&1/3\end{bmatrix}$$

$$H_3 = \frac{1}{3}\begin{bmatrix}1&-2&-2\\-2&1&-2\\-2&-2&1\end{bmatrix}$$

$$H_4 = I_4 - 2\vec{u}\vec{u}^T = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} - 2\cdot\frac{1}{\sqrt{4}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\cdot\frac{1}{\sqrt{4}}\begin{bmatrix}1&1&1&1\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}$$

$$= \begin{bmatrix}1/2&-1/2&-1/2&-1/2\\-1/2&1/2&-1/2&-1/2\\-1/2&-1/2&1/2&-1/2\\-1/2&-1/2&-1/2&1/2\end{bmatrix}$$

$$H_4 = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$