HW1.

1 (1)

① Identity for sum

$$a+b\sqrt{2}+0=a+b\sqrt{2}$$

~~Identity~~ for

② Associativity for sum

$$(a_1+b_1\sqrt{2}+a_2+b_2\sqrt{2})+a_3+b_3\sqrt{2}=a_1+b_1\sqrt{2}+(a_2+b_2\sqrt{2}+a_3+b_3\sqrt{2})$$

$$=a_1+a_2+a_3+(b_1+b_2+b_3)\sqrt{2}$$

③ Inverse for sum

$$a+b\sqrt{2}+\cancel{(-b)}(-a-b\sqrt{2})=0$$

④ Commutativity for sum

$$a_1+b_1\sqrt{2}+a_2+b_2\sqrt{2}=a_2+b_2\sqrt{2}+a_1+b_1\sqrt{2}=(a_1+a_2)+(b_1+b_2)\sqrt{2}$$

⑤ ~~$a+b\sqrt{2}$~~ Identity for product

$$(a+b\sqrt{2})\cdot(1)=a+b\sqrt{2},$$

⑥ Associativity for product

$$[(a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2})](a_3+b_3\sqrt{2})=[a_1a_2+(a_1b_2+a_2b_1)\sqrt{2}+2b_1b_2](a_3+b_3\sqrt{2})$$

$$=a_1a_2a_3+a_1a_2b_3\sqrt{2}+(a_1a_3b_2+a_2a_3b_1)\sqrt{2}+a_1a_3b_2\sqrt{2}+2a_2b_1b_3+2a_3b_1b_2+2\sqrt{2}b_1b_2b_3$$

$$=(a_1+b_1\sqrt{2})[(a_2+b_2\sqrt{2})(a_3+b_3\sqrt{2})]$$

⑦ Distributivity

$$\overset{A}{(a_1+b_1\sqrt{2})}\left[\overset{B}{(a_2+b_2\sqrt{2})}+\overset{C}{(a_3+b_3\sqrt{2})}\right] = \overset{A}{(a_1+b_1\sqrt{2})}\overset{B}{(a_2+b_2\sqrt{2})}+\overset{A}{(a_1+b_1\sqrt{2})}\overset{C}{(a_3+b_3\sqrt{2})}$$

$$\cancel{(a_3+b_3\sqrt{2})\left[(a_1+b_1\sqrt{2})+(a_2+b_2\sqrt{2})\right] = (a_3+b_3\sqrt{2})(a_1+b_1\sqrt{2})+(a_3}$$

$$\left[\overset{B}{(a_2+b_2\sqrt{2})}+\overset{C}{(a_3+b_3\sqrt{2})}\right]\overset{A}{(a_1+b_1\sqrt{2})} = \overset{B}{(a_2+b_2\sqrt{2})}\overset{A}{(a_1+b_1\sqrt{2})}+\overset{C}{(a_3+b_3\sqrt{2})}\overset{A}{(a_1+b_1\sqrt{2})}$$

⑧ Commutativity for product

$$(a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2}) = a_1a_2+(a_1b_2+a_2b_1)\sqrt{2}+2b_1b_2$$

$$= (a_2+b_2\sqrt{2})(a_1+b_1\sqrt{2})$$

⑨ Inverse for product

$$(a+b\sqrt{2})\times(x+y\sqrt{2})=1 \Rightarrow (ax+2by)+(ay+bx)\sqrt{2}=1$$

$$\left[\begin{array}{cc|c} a & 2b & 1 \\ b & a & 0 \end{array}\right] \Rightarrow x=\frac{a}{a^2-2b^2} \quad y=\frac{b}{2b^2-a^2}$$

$$\Rightarrow \forall\ a+b\sqrt{2},\ \exists\ \frac{a}{a^2-2b^2}+\frac{b}{2b^2-a^2}\sqrt{2},\ \text{s.t}$$

$$(a+b\sqrt{2})\left(\frac{a}{a^2-2b^2}+\frac{b}{2b^2-a^2}\right)=1$$

1. (2)

① Identity for sum

$a+b\sqrt{-1} + 0 = a + b\sqrt{-1}$

② Associativity for sum

$(a_1+b_1\sqrt{-1} + a_2+b_2\sqrt{-1}) + a_3+b_3\sqrt{-1} = a_1+a_2+a_3+(b_1+b_2+b_3)\sqrt{-1}$

$= a_1+b_1\sqrt{-1} + (a_2+\cancel{b_2\sqrt{-1}}\,b_2\sqrt{-1} + a_3+b_3\sqrt{-1})$

③ Inverse for sum

$a+b\sqrt{-1} + (-a-b\sqrt{-1}) = 0$

④ Commutativity for sum

$a_1+b_1\sqrt{-1} + a_2+b_2\sqrt{-1} = a_2+b_2\sqrt{-1} + a_1+b_1\sqrt{-1} = (a_1+a_2) + (b_1+b_2)\sqrt{-1}$

⑤ Identity for product

$(a+b\sqrt{-1})(1) = a+b\sqrt{-1}$

⑥ Associativity for product

$[(a_1+b_1\sqrt{-1})(a_2+b_2\sqrt{-1})](a_3+b_3\sqrt{-1}) = [a_1a_2 + (a_1b_2+a_2b_1)\sqrt{-1} - b_1b_2]$
$(a_3+b_3\sqrt{-1})$

$= (a_1+b_1\sqrt{-1})[(a_2+b_2\sqrt{-1})(a_3+b_3\sqrt{-1})]$

⑦ Distributivity

$(a_1+b_1\sqrt{-1})[(a_2+b_2\sqrt{-1})+(a_3+b_3\sqrt{-1})] = (a_1+b_1\sqrt{-1})(a_2+b_2\sqrt{-1}) + (a_1+b_1\sqrt{-1})(a_3+b_3\sqrt{-1})$

$[(a_2+b_2\sqrt{-1})+(a_3+b_3\sqrt{-1})](a_1+b_1\sqrt{-1}) = (a_2+b_2\sqrt{-1})(a_1+b_1\sqrt{-1}) + (a_3+b_3\sqrt{-1})(a_1+b_1\sqrt{-1})$

⑧ Commutativity for product

$(a_1+b_1\sqrt{-1})(a_2+b_2\sqrt{-1}) = a_1a_2 + (a_1b_2+a_2b_1)\sqrt{-1} - b_1b_2$

$= (a_2+b_2\sqrt{-1})(a_1+a_2\sqrt{-1})$

(9) Inverse for product

$(a+b\sqrt{-1}) \times (x+y\sqrt{-1}) = 1 \Rightarrow ax + ay\sqrt{-1} + bx\sqrt{-1} - by = 1$

$\Rightarrow (ax - by) + (bx + ay)\sqrt{-1} = 1$

$$\left[\begin{array}{cc|c} a & -b & 1 \\ b & a & 0 \end{array}\right] \Rightarrow x = \frac{a}{a^2+b^2} \quad y = -\frac{b}{a^2+b^2}$$

$$(a+b\sqrt{-1})\left(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}\sqrt{-1}\right) = 1$$

field is $\mathbb{C}$

2. Given by the definition, A field F is a commutative ring $(F, +, \cdot)$

Counter example: $n = 2$, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 7 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 3 & 4 \end{bmatrix}$$

$\Rightarrow AB \neq BA$

$\Rightarrow$ This is not Commutative $\Leftrightarrow$ it's not a Field

3. $\mathbb{Z}_3$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

4. ① identity for sum

$$a+bi+0 = a+bi$$

② Associativity for sum

$$(a_1+b_1 i + a_2+b_2 i)+a_3+b_3 i = a_1+a_2+a_3+(b_1+b_2+b_3)i$$
$$= a_1+b_1 i + (a_2+b_2 i + a_3+b_3 i)$$

③ Inverse for sum

$$a+bi+(-a-bi) = 0$$

④ Commutativity for sum

$$a_1+b_1 i + a_2+b_2 i = a_1+a_2+(b_1+b_2)i = a_2+b_2 i + a_1+b_1 i$$

⑤ Identity for product

$$(a+bi)(1) = a+bi$$

⑥ Associativity for product

$$[(a_1+b_1 i)(a_2+b_2 i)](a_3+b_3 i) = [a_1a_2+(a_1b_2+a_2b_1)i-b_1b_2](a_3+b_3 i)$$
$$= a_1a_2a_3+a_1a_2b_3 i+(a_1a_3b_2+a_2a_3b_1)i - a_3b_1b_2 - b_1b_2b_3 i$$
$$= (a_1+b_1 i)[(a_2+b_2 i)(a_3+b_3 i)]$$

⑦ Distributivity

$$(a_1+b_1 i)[(a_2+b_2 i)+(a_3+b_3 i)] = (a_1+b_1 i)(a_2+b_2 i)+(a_1+b_1 i)(a_3+b_3 i)$$
$$[(a_2+b_2 i)+(a_3+b_3 i)](a_1+b_1 i) = (a_2+b_2 i)(a_1+b_1 i)+(a_3+b_3 i)(a_1+b_1 i)$$

⑧ Commutativity for product

$$(a_1+b_1 i)(a_2+b_2 i) = a_1a_2 + (a_1b_2 + a_2b_1) i - b_1b_2$$
$$= (a_2+b_2 i)(a_1+b_1 i)$$

⑨. Inverse for product

$$(a+bi)(x+yi) = 1 \Rightarrow ax + ayi + bxi - by = 1$$
$$\Rightarrow (ax-by) + (bx+ay) i = 1$$

$$\begin{bmatrix} a & -b & 1 \\ b & a & 0 \end{bmatrix} \Rightarrow x = \frac{a}{a^2+b^2}, \; y = \frac{-b}{a^2+b^2}$$

$$\Rightarrow (a+bi)\left(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i\right) = 1$$

5. B D are RREF

6. $A+B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

7. $A = \begin{bmatrix} 6 & -1 & 1 \\ t & 0 & 1 \\ 0 & 1 & t \end{bmatrix}$

$$\det(A) = -t \begin{bmatrix} -1 & 1 \\ 1 & t \end{bmatrix} - \begin{bmatrix} 6 & -1 \\ 0 & 1 \end{bmatrix}$$
$$= -t(-t-1) - 6$$
$$= t^2 + t - 6$$

If a matrix doesn't have an inverse, its $\det(A) = 0$

$\Rightarrow \quad t^2 + t - 6 = 0 \Rightarrow (t+3)(t-2) = 0 \Rightarrow t = -3,\ t = 2$

8.

(a) $\left[\begin{array}{cc|c} 1 & h & 4 \\ 3 & 6 & 8 \end{array}\right] \xrightarrow{3R_1} \left[\begin{array}{cc|c} 3 & 3h & 12 \\ 3 & 6 & 8 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc|c} 3 & 3h & 12 \\ 0 & 6-3h & -4 \end{array}\right]$

$\Rightarrow \quad h \neq 2$

(b) $\left[\begin{array}{cc|c} -4 & 12 & h \\ 2 & -6 & -3 \end{array}\right] \xrightarrow{2R_2} \left[\begin{array}{cc|c} -4 & 12 & h \\ 4 & -12 & -6 \end{array}\right] \xrightarrow{R_1 + R_2} \left[\begin{array}{cc|c} 0 & 0 & h-6 \\ 4 & -12 & -6 \end{array}\right]$

$\Rightarrow h = 6$

9. (1) $\begin{bmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

(2) $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$

(3) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

10 $$A=\begin{bmatrix} 1 & a & b & 3 & 0 & -2 \\ 0 & 0 & c & 1 & d & 3 \\ 0 & e & 0 & 0 & 1 & 1 \end{bmatrix}$$

a: any real number $b=0$, $c=1$ $d=0$, $e=0$

11. (1) $$A=\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_1]{R_1-R_2} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 3 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3-2R_1} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & -2 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{R_3+2R_2} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 7 & 2 \end{bmatrix} \xrightarrow[\frac{1}{7}R_3]{R_1-R_2} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix} \xrightarrow{R_2-3R_3} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & \frac{8}{7} \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix}$$

$$\xrightarrow{R_1+3R_3} \begin{bmatrix} 1 & 0 & 0 & \frac{6}{7} \\ 0 & 1 & 0 & \frac{8}{7} \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix} \Rightarrow \cancel{\vec{x}=\begin{bmatrix} -\frac{6}{7}a \\ -\frac{8}{7}a \\ -\frac{2}{7}a \\ a \end{bmatrix}} \quad \vec{x}=a\begin{bmatrix} -\frac{6}{7} \\ -\frac{8}{7} \\ -\frac{2}{7} \\ 1 \end{bmatrix}$$

(2) $$\begin{bmatrix} 1 & 0 & 0 & \frac{6}{7} \\ 0 & 1 & 0 & \frac{8}{7} \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix} \xrightarrow{7R_1,\,7R_2,\,7R_3} \begin{bmatrix} 7 & 0 & 0 & 6 \\ 0 & 7 & 0 & 8 \\ 0 & 0 & 7 & 2 \end{bmatrix} \xrightarrow{\mathbb{Z}_7} \begin{bmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

(see code in python file)

12 (1) Use Python get rref(A) in $\mathbb{Z}_7$ $$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cancel{\Rightarrow}$$

(2)

$$\Rightarrow \quad \vec{x}=\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$$

13 $\quad \text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad$ ~~do not~~ doesn't has solution

14. $\quad \text{rref}(A) = \begin{bmatrix} 1 & 2 & 3 & 0 & 5 & 6 \\ 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$\Rightarrow \vec{x_1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \vec{x_2}\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \vec{x_3}\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + \vec{x_4}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \vec{x_5}\begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 0 \end{bmatrix}$$

15. $\quad \text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -\frac{8221}{4340} \\ 0 & 1 & 0 & 0 & 0 & \frac{8591}{8680} \\ 0 & 0 & 1 & 0 & 0 & \frac{4695}{434} \\ 0 & 0 & 0 & 1 & 0 & -\frac{459}{434} \\ 0 & 0 & 0 & 0 & 1 & \frac{699}{434} \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \frac{-8221}{4340} \\ \frac{8591}{8680} \\ \frac{4695}{434} \\ \frac{-459}{434} \\ \frac{699}{434} \end{bmatrix}$$

16. (1)

$$ABC = I_n \Leftrightarrow A^{-1}ABC = A^{-1}I_n$$
$$\Rightarrow I_n BC = A^{-1} I_n$$
$$\Rightarrow A^{-1} = BC$$

$$ABC = I_n \Leftrightarrow ABCC^{-1} = I_n C^{-1}$$
$$\Rightarrow AB I_n = I_n C^{-1}$$
$$\Rightarrow C^{-1} = AB$$

$$ABC = I_n \Leftrightarrow A^{-1}ABCC^{-1} = A^{-1}I_n C^{-1}$$
$$\Rightarrow I_n B I_n = BC I_n AB \Rightarrow B = BCAB \Rightarrow I_n = CAB$$
$$\Rightarrow B^{-1} = CA$$

(2) If AB is invertible, then $\det(AB) \neq 0$

$\Rightarrow \det(AB) = \det(A) \cdot \det(B) \neq 0 \Rightarrow \det(A) \neq 0, \det(B) \neq 0$

So, A, B are invertible

17 $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

$$(AB)^2 = (AB)(AB) = \begin{bmatrix} 8 & 5 \\ 20 & 13 \end{bmatrix}\begin{bmatrix} 8 & 5 \\ 20 & 13 \end{bmatrix} = \begin{bmatrix} 164 & 105 \\ 420 & 269 \end{bmatrix}$$

$$A^2B^2 = \begin{bmatrix} 7 & 10 \\ 15 & \cancel{20}\,22 \end{bmatrix}\begin{bmatrix} 22 & 15 \\ 10 & 7 \end{bmatrix} = \begin{bmatrix} 254 & 175 \\ 550 & 379 \end{bmatrix}$$

$$\Rightarrow A^2B^2 \neq (AB)^2$$

18 $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $A^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow A^{-1} = A^T$$

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(1) symmetric

2x2 $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

3x3: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4x4 $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

skew-symmetric

2x2 $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

3x3 $A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

4x4 $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{bmatrix}$

(2) Each element in the main diagonal equals to 0

(3) Null matrix is both symmetric and skew-symmetric

(4)

$$(A+A^T)^T = A^T + (A^T)^T = A^T + A \quad \text{symmetric}$$

$$(AA^T)^T = (A^T)^T A^T = AA^T \quad \text{symmetric}$$

$$(A^TA)^T = A^T (A^T)^T = A^T A \quad \text{symmetric}$$

$$(A-A^T)^T = A^T - (A^T)^T = A^T - A \quad \text{skew-symmetric}$$

(5)

let $A = B + C$

Suppose $B^T = B$, $C^T = -C$

$$A^T = (B+C)^T = B^T + C^T = B - C$$

$$\Rightarrow \begin{cases} A^T = B - C \\ A = B + C \end{cases} \Rightarrow \begin{cases} B = \dfrac{A+A^T}{2} \\ C = \dfrac{A-A^T}{2} \end{cases}$$

$\blacksquare$

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(a) surjective

(b) injective, surjective, bijective

(c) surjective

(d) injective, surjective, bijective

21. $A = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{4R_2-R_1} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 15 & 4 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{15R_3-R_2} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 15 & 4 & 0 \\ 0 & 0 & 56 & 15 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

$\xrightarrow{56R_4-R_3} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 15 & 4 & 0 \\ 0 & 0 & 56 & 15 \\ 0 & 0 & 0 & 209 \end{bmatrix} = L$

$A = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{4R_3-R_4} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 4 & 15 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{15R_2-R_3} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 15 & 56 & 0 & 0 \\ 0 & 4 & 15 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

$\xrightarrow{56R_1-R_2} \begin{bmatrix} 209 & 0 & 0 & 0 \\ 15 & 56 & 0 & 0 \\ 0 & 4 & 15 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix} = U$

$A = L \cdot U$

22 $\because A = L \cdot U$

$q_i = L_{i-1} U_{i-1} + d_i$

$u_i = r_i$

$L_i d_i = p_i$

$\{q_i \ p_i \ r_i\} = \{L_{i-1} U_{i-1} + d_i \quad L_i d_i \quad u_i\}$

23

$$U = \begin{bmatrix} 4 & 1 & \cdots & 0 & 0 \\ 0 & 15 & 4 & 0 \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & & \cdots & 4 \end{bmatrix}$$

24

(1) $H_n = I_n - 2\vec{u}\vec{u}^T$

$$(H_n)^T = (I_n)^T - (2\vec{u}\vec{u}^T)^T = I_n - 2(\vec{u}^T)^T(\vec{u})^T = I_n - 2\vec{u}\vec{u}^T = H_n$$

$\Rightarrow$ H_n is symmetric

(2)
$$H_n^T H_n = H_n H_n = H_n^2 = (I_n - 2\vec{u}\vec{u}^T)^2$$
$$= I_n^2 - 4I_n\vec{u}\vec{u}^T + 4(\vec{u}\vec{u}^T)^2$$
$$= I_n - 4\vec{u}\vec{u}^T + 4(\vec{u}\vec{u}^T)(\vec{u}\vec{u}^T)$$
$$= I_n - 4\vec{u}\vec{u}^T + 4\vec{u}(\vec{u}^T\vec{u})\vec{u}^T$$
$$= I_n - 4\vec{u}\vec{u}^T + 4\vec{u}\vec{u}^T = I_n$$

$\Rightarrow$ H_n is morthogonal matrix

(3) $H_n^2 = I_n$

(4)
$$H_n\vec{u} = (I_n - 2\vec{u}\vec{u}^T)\vec{u} = I_n\vec{u} - 2\vec{u}\vec{u}^T\vec{u}$$
$$= I_n\vec{u} - 2\vec{u} = -\vec{u}$$

(5)
$$H_3 = I_3 - 2\vec{u}\vec{u}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\cdot\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\frac{1}{\sqrt{3}}[1\ 1\ 1]$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$H_4 = I_4 - 2\vec{u}\vec{u}^T$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \cancel{2} \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$