Math 4570 HW #1

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1. **Show that the following sets of numbers are fields if the usual addition and multiplication of arithmetic are used:**

a) The two operations on the set $\mathbb{F}$ are $+$ and $\times$ such that:
(a+b$\sqrt{2}$)+(c+d$\sqrt{2}$) := a+c+(b+d)$\sqrt{2}$ and
(a+b$\sqrt{2}$)$\times$(c+d$\sqrt{2}$) := ac+2bd+(ad+bc)$\sqrt{2}$

identity for sum: there exists 0 ϵ $\mathbb{F}$, such that 0 + a+b$\sqrt{2}$ = a+b$\sqrt{2}$ + 0 = a+b$\sqrt{2}$

associativity for sum: [(a+b$\sqrt{2}$)+(c+d$\sqrt{2}$)]+(e+f$\sqrt{2}$) = (a+b$\sqrt{2}$)+[(c+d$\sqrt{2}$)+(e+f$\sqrt{2}$)]

inverse for sum: for any (a+b$\sqrt{2}$) ϵ $\mathbb{F}$, there exists an element -(a+b$\sqrt{2}$) ϵ $\mathbb{F}$ such that (a+b$\sqrt{2}$) + (-a-b$\sqrt{2}$) = (-a-b$\sqrt{2}$) + (a+b$\sqrt{2}$) = 0

commutativity for sum: (a+b$\sqrt{2}$) + (c+d$\sqrt{2}$) = (c+d$\sqrt{2}$) + (a+b$\sqrt{2}$)

multiplicative identity: 1$\cdot$(a+b$\sqrt{2}$) = (a+b$\sqrt{2}$)$\cdot$1 = (a+b$\sqrt{2}$)

associativity for product: [(a+b$\sqrt{2}$)$\cdot$(c+d$\sqrt{2}$)]$\cdot$(e+f$\sqrt{2}$) = (a+b$\sqrt{2}$)$\cdot$[(c+d$\sqrt{2}$)$\cdot$(e+f$\sqrt{2}$)]

distributivity: (a+b$\sqrt{2}$)$\cdot$[(c+d$\sqrt{2}$)+(e+f$\sqrt{2}$)]=(a+b$\sqrt{2}$)$\cdot$(c+d$\sqrt{2}$)+(a+b$\sqrt{2}$)$\cdot$(e+f$\sqrt{2}$)

commutativity for product: (a+b$\sqrt{2}$)$\cdot$(c+d$\sqrt{2}$) = (c+d$\sqrt{2}$)$\cdot$(a+b$\sqrt{2}$)

inverse for product: there exists (a+b$\sqrt{2}$)$^{-1}$:= (x+y$\sqrt{2}$) ϵ $\mathbb{F}$ such that
(a+b$\sqrt{2}$)$\cdot$(x+y$\sqrt{2}$) = 1

$$\begin{bmatrix} a & 2b & 1 \\ b & a & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2b/a & 1/a \\ b/a & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2b/a & 1/a \\ 0 & (a^2-2b^2)/a^2 & -b/a^2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2b/a & 1/a \\ 0 & 1 & b/(2b^2-a^2) \end{bmatrix}$$
$$\longrightarrow \begin{bmatrix} 1 & 0 & -a/(2b^2-a^2) \\ 0 & 1 & b/(2b^2-a^2) \end{bmatrix}$$

$(a+b\sqrt{2})^{-1} := (x+y\sqrt{2})$ for $x = -a/(2b^2 - a^2)$ and $y = b/(2b^2 - a^2)$

b) The two operations on the set $\mathbb{F}$ are + and $\times$ such that:
$(a+b\sqrt{-1})+(c+d\sqrt{-1}) := a+c+(b+d)\sqrt{-1}$ and
$(a+b\sqrt{-1})\times(c+d\sqrt{-1}) := ac-bd+(ad+bc)\sqrt{-1}$

identity for sum: there exists $0 \epsilon \mathbb{F}$, such that $0 + a+b\sqrt{-1} = a+b\sqrt{-1} + 0 = a+b\sqrt{-1}$

associativity for sum: $[(a+b\sqrt{-1})+(c+d\sqrt{-1})]+(e+f\sqrt{-1}) = (a+b\sqrt{-1})+[(c+d\sqrt{-1})+(e+f\sqrt{-1})]$

inverse for sum: for any $(a+b\sqrt{-1}) \epsilon \mathbb{F}$, there exists an element $-(a+b\sqrt{-1}) \epsilon \mathbb{F}$ such that $(a+b\sqrt{-1}) + (-a-b\sqrt{-1}) = (-a-b\sqrt{-1}) + (a+b\sqrt{-1}) = 0$

commutativity for sum: $(a+b\sqrt{-1}) + (c+d\sqrt{-1}) = (c+d\sqrt{-1}) + (a+b\sqrt{-1})$

multiplicative identity: $1\cdot(a+b\sqrt{-1}) = (a+b\sqrt{-1})\cdot 1 = (a+b\sqrt{-1})$

associativity for product: $[(a+b\sqrt{-1})\cdot(c+d\sqrt{-1})]\cdot(e+f\sqrt{-1}) = (a+b\sqrt{-1})\cdot[(c+d\sqrt{-1})\cdot(e+f\sqrt{-1})]$

distributivity: $(a+b\sqrt{-1})\cdot[(c+d\sqrt{-1})+(e+f\sqrt{-1})]=(a+b\sqrt{-1})\cdot(c+d\sqrt{-1})+(a+b\sqrt{-1})\cdot(e+f\sqrt{-1})$

commutativity for product: $(a+b\sqrt{-1})\cdot(c+d\sqrt{-1}) = (c+d\sqrt{-1})\cdot(a+b\sqrt{-1})$

inverse for product: there exists $(a+b\sqrt{-1})^{-1} := (x+y\sqrt{-1}) \epsilon \mathbb{F}$ such that
$(a+b\sqrt{-1})\cdot(x+y\sqrt{-1}) = 1$

$$\begin{bmatrix} a & -b & 1 \\ b & a & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -b/a & 1/a \\ b/a & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -b/a & 1/a \\ 0 & (a^2+b^2)/a^2 & -b/a^2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -b/a & 1/a \\ 0 & 1 & -b/(b^2+a^2) \end{bmatrix}$$
$$\longrightarrow \begin{bmatrix} 1 & 0 & a/(b^2+a^2) \\ 0 & 1 & -b/(b^2+a^2) \end{bmatrix}$$

$(a+b\sqrt{-1})^{-1} := (x+y\sqrt{-1})$ for $x = a/(a^2 + b^2)$ and $y = -b/(a^2 + b^2)$
$\therefore$ This is the complex number field

2. **Show that the set of all n $\times$ n matrices $\mathbb{R}^{n\times n}$ with the usual matrix addition and multiplication is not a field if n > 1**

Proof:
Given A is an arbitrary n x n, $A^{-1} = \dfrac{1}{det(A)}adj(A)$
If $det(A) = 0 \implies A^{-1}$ does not exist
$\implies \exists$ non-zero n x n matrix whose $det(A) = 0$

So $\exists$ a n x n matrix whose inverse does not exist
$\therefore$ set of all n x n matrices is not a field if n > 1, since fields need a multiplicative inverse $\blacksquare$

3. **Write down the two operations on field $\mathbb{Z}_3$**

+	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

x	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

4. **Show that $\mathbb{C}$ is a field with the usual sum, scalar product, and product**

The two operations on the set $\mathbb{F}$ are + and $\times$ such that:
$(a+b\sqrt{-1})+(c+d\sqrt{-1}) := a+c+(b+d)\sqrt{-1}$ and
$(a+b\sqrt{-1})\times(c+d\sqrt{-1}) := ac-bd+(ad+bc)\sqrt{-1}$

identity for sum: there exists $0 \epsilon \mathbb{F}$, such that $0 + a+b\sqrt{-1} = a+b\sqrt{-1} + 0 = a+b\sqrt{-1}$

associativity for sum: $[(a+b\sqrt{-1})+(c+d\sqrt{-1})]+(e+f\sqrt{-1}) = (a+b\sqrt{-1})+[(c+d\sqrt{-1})+(e+f\sqrt{-1})]$

inverse for sum: for any $(a+b\sqrt{-1}) \epsilon \mathbb{F}$, there exists an element $-(a+b\sqrt{-1}) \epsilon \mathbb{F}$ such that $(a+b\sqrt{-1}) + (-a-b\sqrt{-1}) = (-a-b\sqrt{-1}) + (a+b\sqrt{-1}) = 0$

commutativity for sum: $(a+b\sqrt{-1}) + (c+d\sqrt{-1}) = (c+d\sqrt{-1}) + (a+b\sqrt{-1})$

multiplicative identity: $1\cdot(a+b\sqrt{-1}) = (a+b\sqrt{-1})\cdot 1 = (a+b\sqrt{-1})$

associativity for product: $[(a+b\sqrt{-1})\cdot(c+d\sqrt{-1})]\cdot(e+f\sqrt{-1}) = (a+b\sqrt{-1})\cdot[(c+d\sqrt{-1})\cdot(e+f\sqrt{-1})]$

distributivity: $(a+b\sqrt{-1})\cdot[(c+d\sqrt{-1})+(e+f\sqrt{-1})]=(a+b\sqrt{-1})\cdot(c+d\sqrt{-1})+(a+b\sqrt{-1})\cdot(e+f\sqrt{-1})$

commutativity for product: $(a+b\sqrt{-1})\cdot(c+d\sqrt{-1}) = (c+d\sqrt{-1})\cdot(a+b\sqrt{-1})$

inverse for product: there exists $(a+b\sqrt{-1})^{-1} := (x+y\sqrt{-1}) \epsilon \mathbb{F}$ such that
$(a+b\sqrt{-1})\cdot(x+y\sqrt{-1}) = 1$

$$\begin{bmatrix} a & -b & 1 \\ b & a & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -b/a & 1/a \\ b/a & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -b/a & 1/a \\ 0 & (a^2+b^2)/a^2 & -b/a^2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -b/a & 1/a \\ 0 & 1 & -b/(b^2+a^2) \end{bmatrix}$$
$$\longrightarrow \begin{bmatrix} 1 & 0 & a/(b^2+a^2) \\ 0 & 1 & -b/(b^2+a^2) \end{bmatrix}$$

(a+b$\sqrt{-1}$)$^{-1}$:= (x+y$\sqrt{-1}$) for x = a/(a$^2 + b^2$) and y = -b/(a$^2 + b^2$)

$\therefore$ set of all complex numbers is a field with usual sum, product, and scalar product ■

5. **Determine which of the matrices below are in reduced row-echelon form**

$$\boxed{\text{Matrices B and D are in reduced row-echelon form}}$$

6. **Compute A+B, A^2, and AB over the field $\mathbb{Z}_2$**

$$\text{A + B} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 1 & 2 & 0 \end{bmatrix},$$

$$\text{A}^2 = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

$$\text{AB} = \begin{bmatrix} 2 & 3 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

7. **For which values of t does the matrix A not have an inverse?**

$$\text{rref(A)} = \begin{bmatrix} 6 & -1 & 1 \\ t & 0 & 1 \\ 0 & 1 & t \end{bmatrix} \longrightarrow \begin{bmatrix} 6 & 0 & 1+t \\ 1 & 0 & 1/t \\ 0 & 1 & t \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & (t+t^2-6)/6t \\ 1 & 0 & 1/t \\ 0 & 1 & t \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$
$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \implies det(A) = 1$$

$$\boxed{\text{det(A) = 1 so there are no values of t for which A does not have an inverse}}$$

8. **Find all values of h that make the following matrices consistent, i.e., at least has one solution**

a) $$\text{rref(A)} = \begin{bmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & h & 4 \\ 0 & 6-3h & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 6-4h & 6-3h & 10-4h \\ 0 & 6-3h & -4 \end{bmatrix} \longrightarrow$$
$$\begin{bmatrix} 1 & 0 & \dfrac{14-4h}{6-4h} \\ 0 & 1 & \dfrac{-4}{6-3h} \end{bmatrix}$$

$$\boxed{h \neq 2 \text{ and } h \neq \frac{6}{4}}$$

b) $\text{rref}(A) = \begin{bmatrix} -4 & 12 & h \\ 2 & -6 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & \frac{-h}{4} \\ 2 & -6 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & \frac{-h}{4} \\ 0 & 0 & \frac{-6+h}{2} \end{bmatrix}$

$$\boxed{h = 6}$$

9. **We say that two m $\times$ n matrices in reduced row-echelon form are of the same type if they have the same number of leading 1's in the same position**

1) There are 4 types of 3 x 2 matrices in rref:

$$\boxed{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}$$

2) There are 7 types of 2 x 3 matrices in rref:

$$\boxed{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

3) There are 5 types of 4 x 1 matrices in rref:

$$\boxed{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}$$

10. **For which values of a, b, c, d, and e is the following matrix in reduced row-echelon form?**

$$\boxed{\text{When a} = \mathbb{R}\text{, b} = 0\text{, c} = 1\text{, d} = 0\text{, e} = 0}$$

11. **Let A** $= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix}$

1) rref(A) $= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 3 & 2 \\ 1 & 1 & 0 & 2 \\ 0 & -2 & 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 7 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 7 & 2 \end{bmatrix}$
$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & 6/7 \\ 0 & 1 & 0 & 8/7 \\ 0 & 0 & 1 & 2/7 \end{bmatrix}$

$$\boxed{\mathrm{x} = \begin{bmatrix} 6/7 \\ 8/7 \\ 2/7 \end{bmatrix}}$$

2) rref(A) over field $\mathbb{Z}_7 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 3 & 2 \\ 1 & 1 & 0 & 2 \\ 0 & 6 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 3 & 2 \\ 1 & 0 & 4 & 0 \\ 0 & 6 & 1 & 0 \end{bmatrix} \longrightarrow$
$\begin{bmatrix} 0 & 1 & 3 & 2 \\ 1 & 0 & 4 & 0 \\ 0 & 6 & 4 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 6 & 1 & 5 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$\boxed{\text{rref(A) over field } \mathbb{Z}_7 = \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

3)

$$\boxed{\begin{array}{l} \text{rref(A) over field } \mathbb{Z}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ \\ \text{rref(A) over field } \mathbb{Z}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \end{array}}$$

4)

No, it is not possible for a matrix M to have different rank over different fields $\mathbb{Z}_p$

12. **Let A =** $\begin{bmatrix} 3 & 1 & 4 \\ 5 & 2 & 6 \\ 0 & 5 & 2 \end{bmatrix}$ **and b =** $\begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$

1)

rref(A|b) over $\mathbb{Z}_7$ = $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

2)

x = $\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$

13. **Solve the linear system:** $\begin{cases} 3x_1 + 11x_2 + 19x_3 = -2 \\ 7x_1 + 23x_2 + 39x_3 = 10 \\ -4x_1 - 3x_2 - 2x_3 = 6 \end{cases}$

rref(A) = $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$0 \neq 1 \implies$ system is inconsistent, no solution

14. **Solve the linear system:** $\begin{cases} 3x_1 + 6x_2 + 9x_3 + 5x_4 + 25x_5 = 53 \\ 7x_1 + 14x_2 + 21x_3 + 9x_4 + 53x_5 = 105 \\ -4x_1 - 8x_2 - 12x_3 + 5x_4 - 10x_5 = 11 \end{cases}$

x = r $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ + s $\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ + t $\begin{bmatrix} -5 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ + $\begin{bmatrix} 6 \\ 0 \\ 0 \\ 7 \\ 0 \end{bmatrix}$

15. **Solve the linear system:** $\begin{cases} 2x_1 + 4x_2 + 3x_3 + 5x_4 + 6x_5 = 37 \\ 4x_1 + 8x_2 + 7x_3 + 5x_4 + 2x_5 = 74 \\ -2x_1 - 4x_2 + 3x_3 + 4x_4 - 5x_5 = 20 \\ x_1 + 2x_2 + 2x_3 - x_4 + 2x_5 = 26 \\ 5x_1 - 10x_2 + 4x_3 + 6x_4 + 4x_5 = 24 \end{cases}$

$$\boxed{\mathrm{x} = \begin{bmatrix} \frac{-8221}{4340} \\ \frac{8591}{8680} \\ \frac{4695}{434} \\ \frac{-459}{434} \\ \frac{699}{434} \end{bmatrix}}$$

16. **1)If A, B, and C are n $\times$ n matrices and ABC = $\mathbf{I}_n$, is each of the matrices invertible? What are their inverses?**
2) Suppose A and B are n $\times$ n matrices. If AB is invertible, are both A and B are invertible?

1)

$$\boxed{\begin{array}{l} \text{Yes, if ABC} = \mathrm{I}_n \text{ each of thee matrices is invertible.} \\ \mathrm{A}^{-1} = \mathrm{BC},\ \mathrm{C}^{-1} = \mathrm{AB},\ \mathrm{B}^{-1} = \mathrm{CA} \end{array}}$$

2)

$$\boxed{\text{Yes, if AB is invertible, A and B are invertible}}$$

17. **Provide a counter-example to the statement: For any 2 $\times$ 2 matrices A and B, $(\mathbf{AB})^2 = \mathbf{A}^2\mathbf{B}^2$**

$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} 6 & 7 \\ 11 & 11 \end{bmatrix}$, $(AB)^2 = \begin{bmatrix} 113 & 119 \\ 187 & 198 \end{bmatrix}$

$A^2B^2 = \begin{bmatrix} 123 & 94 \\ 212 & 163 \end{bmatrix}$

$$\boxed{\text{If } A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix}, (AB)^2 \neq A^2B^2}$$

18. **Find an example of a 2 x 2 non-identity matrix whose transpose is its inverse**

$$\boxed{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}$$

19. **An n x n matrix A is symmetric provided A^T = A and skew-symmetric provided A^T = -A**

1) Give examples of symmetric and skew-symmetric 2×2, 3×3, and 4×4 matrices

Symmetric matrices: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Skew-symmetric matrices: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{bmatrix}$

2) What can you say about the main diagonal of a skew-symmetric matrix?

$$\boxed{\text{The diagonal of a skew-symmetric matrix consists of only zero elements}}$$

3) Give an example of a matrix that is both symmetric and skew-symmetric

$$\boxed{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ matrix is both symmetric and skew-symmetric}}$$

4) Prove that for any n x n matrix A, the matrices A+A^T, AA^T, and A^TA are symmetric and A-A^T is skew-symmetric

$(A+A^T)^T = (A^T)^T + A^T = A + A^T \implies A + A^T$ is symmetric
$(AA^T)^T = (A^T)^T A^T = AA^T \implies AA^T$ is symmetric
$(A^T A)^T = A^T(A^T)^T = A^T A \implies A^T A$ is symmetric
$(A-A^T)^T = A^T - A = -(A - A^T) \implies A - A^T$ is skew-symmetric

5) Prove that any n x n can be written as the sum of a symmetric and skew-symmetric matrix

Proof:
Suppose A + A^T is symmetric and A - A^T is skew-symmetric
$\frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = \frac{1}{2}A + \frac{1}{2}A + \frac{1}{2}A^T - \frac{1}{2}A^T = A$
$\therefore$ any n x n matrix A can be written as the sum of a symmetric and skew-symmetric matrix $\blacksquare$

20. **Mark each of the following functions F : R $\rightarrow$ R injective, surjective, or bijective**

a) bijective

b) bijective

c) surjective

d) surjective

21. **Find LU-factorization of matrix A**

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 0 & 4/15 & 1 & 0 \\ 0 & 0 & 15/56 & 1 \end{bmatrix},\ U = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 15/4 & 1 & 0 \\ 0 & 0 & 56/15 & 1 \\ 0 & 0 & 0 & 209/56 \end{bmatrix}$$

24. **Let $\mathbf{I}_n$ be the n x n identity matrix. Let u be a unit vector in $\mathbb{R}^n$. Define $\mathbf{H}_n = I_n - 2uu^T$**

1) Is H_n a symmetric matrix?

$\mathrm{H}_n^2 = I_n^2 - 4uu^T + 4uuu^Tu^T$
Since $\mathrm{uu}^T = 1$, $\mathrm{H}_n^2 = I_n^2 - 4 + 4 = I_n$
$\implies \mathrm{H} = \mathrm{H}^T$

$$\boxed{\text{Yes, } \mathrm{H}_n \text{ is symmetric because } \mathrm{H} = \mathrm{H}^T}$$

2) Is H_n an orthogonal matrix?

$$\boxed{\text{Yes, } \mathrm{H}_n \text{ is orthogonal because } \mathrm{H} = \mathrm{H}^T \text{ and } \mathrm{H}{\cdot}\mathrm{H}^T = \mathrm{I}_n}$$

3) What is H_n^2?

$$\boxed{\mathrm{H}_n^2 = I_n^2 - 4uu^T + 4uuu^Tu^T = I_n}$$

4) What is $\mathrm{H}_n\mathrm{u}$?
$\mathrm{H}_n u = (I_n - 2uu^T)u = u - 2uu^Tu = u - 2u = -u$

$$\boxed{\mathrm{H}_n = -u}$$

5) Suppose $\mathrm{u} = \dfrac{1}{\sqrt{n}}\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}$, write H_3 and H_4?

$$\mathrm{H}_3 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - 2\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1&1&1\end{bmatrix}\cdot\frac{1}{\sqrt{3}} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix} = \begin{bmatrix}1/3&-2/3&-2/3\\-2/3&1/3&-2/3\\-2/3&-2/3&1/3\end{bmatrix}$$

$$\mathrm{H}_4 = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} - 2\cdot\frac{1}{\sqrt{4}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1&1&1&1\end{bmatrix}\cdot\frac{1}{\sqrt{4}} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix} =$$
$$\begin{bmatrix}1/2&-1/2&-1/2&-1/2\\-1/2&1/2&-1/2&-1/2\\-1/2&-1/2&1/2&-1/2\\-1/2&-1/2&-1/2&1/2\end{bmatrix}$$

$$\mathrm{H}_3 = \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$\mathrm{H}_4 = \begin{bmatrix} 1/2 & -1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & -1/2 \\ -1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$$