Monday, June 29, 2020 11:42 AM

MATH 5131 - MATH MODELING

WHAT IS MATH MODELING?

WANT TO

BUILD: DESCRIBE A SITUATION IN TERMS OF FORMAL VARIABLES, RELATIONS, AND ORGANIZING PRINCIPALS.

ANALYZE: UNDERSTAND THE POSSIBLE STATES A MODEL CAN TAKE.

FIX/VALIDATE: FROM REAL WORLD DATA, FIX A SPECIFIC MODEL AND VALIDATE.

APPLIED STAT, MACHINE LEARNING

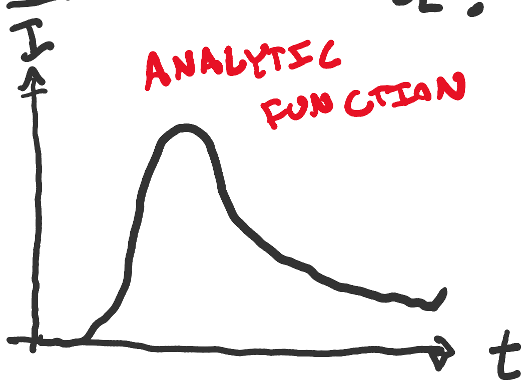
USE:

IN THIS CLASS, WE WILL FOCUS ON CONTINUOUS TIME MODELS AND DIFFERENTIAL EQUATIONS.

DIFF. EQ.: SIMPLE LOCAL ASSUMPTIONS LEAD TO COMPLICATED

LEAD TO COMPLICATED
GLOBAL BEHAVIOR.

Ex: SIR MODEL:



$$S' = -pSI$$
$$I' = pSI - rI$$
$$R' = rI$$

MODEL BUILDING:

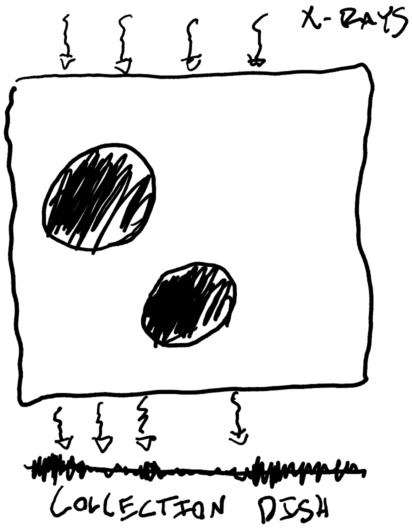
WANT TO DESCRIBE A PROBLEM IN TERMS OF VARIABLES AND RELATIONS.

PROBLEM SPACE	ABSTRACT MODEL SPACE
MEASUREABLE QUANTITIES $\rightsquigarrow$	VARIABLES IE $x \in \mathbb{N}$ OR $y \in \mathbb{R}$, OR FUNCTIONS, $y(t)$.
CONTROL VARIABLES / STATES $\rightsquigarrow$	INTERNAL VARIABLES THAT CHANGE OR CONTROL STATE OF MODEL. Ex: $t \in \mathbb{R}$ FOR TIME IN $y(t)$.
PARAMETERS $\rightsquigarrow$	CONSTANTS FIXED BY REAL WORLD

PARAMETERS $\rightsquigarrow$ CONSTANTS FIXED BY REAL WORLD OBSERVATIONS.

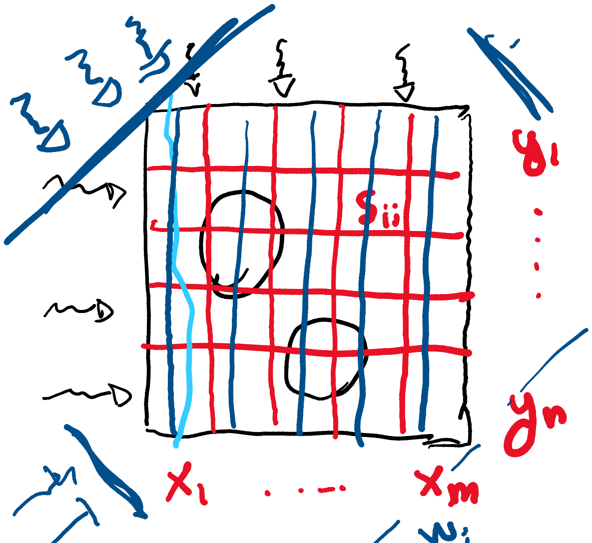
Ex: TOMOGRAPHY:

IDEA: MEDICAL IMAGING; WANT TO IMAGE OBJECTS IN $\mathbb{R}^2$, BY SHINING X-RAYS FROM BOUNDRY.



X-RAYS PASS THROUGH VOLUME BUT ARE LESS BRIGHT IF THEY PASS THROUGH A SOLID.

RECOVERY ALGORITHM



$s_{ij} \in [0,1]$ $(\in \mathbb{R})$

$y_i \in \mathbb{R}$

$x_j \in \mathbb{R}$

$$X_1 = L - D \sum_{i=1}^{n} s_{i1},$$

$\vdots$

L - X-RAY INTENSITY

D - DENSITY OF

QUANTITIES:

$x_j, \underline{y_i}, w_k \ldots$

$$\vdots$$

$$x_m = L - D\sum_{i=1}^{n} S_{im}$$

$$y_1 = L - D\sum_{j=1}^{m} S_{1j}$$

$$\vdots$$

$$y_n = L - D\sum_{j=1}^{m} S_{nj}$$

D - "DENSITY OF OBJECT"

Q) WHAT IS THIS MODEL MATHEMATICALLY?

LINEAR SYSTEM

$x_j, y_i, w_k \ldots$

STATE VARIABLES: S_{ij}

PARAMETERS: L, D

WANT TO WRITE S_{ij} IN TERMS OF x_j, y_i. CAN WE? NO*

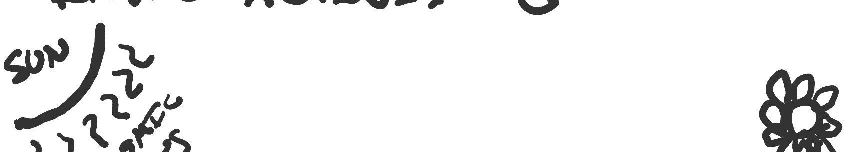
$n+m$ EQUATION IN $n \times m$ UNKNOWNS

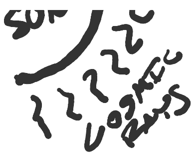
SO UNLESS SOMETHING VERY SPECIAL HAPPENS, NO.

RADIO CARBON DATING:

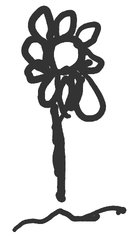
IDEA: CAVE PAINTING WERE MADE USING ORGANIC MATERIAL (IE MATERIAL FROM ANIMALS OR PLANTS).

IN 1940, DISCOVERED THAT IN ATMO. THERE ARE A SMALL NUMBER OF ISOTOPES OF CARBON THAT ARE RADIO ACTIVE: ^{14}C





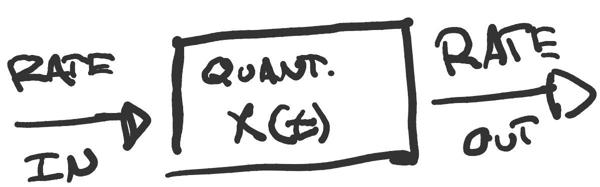
$^{14}N \rightsquigarrow {}^{14}C \rightsquigarrow CO_2$



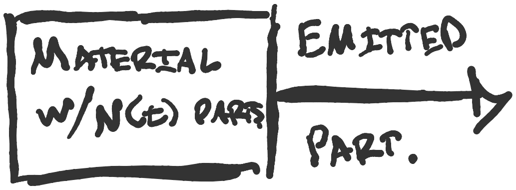
NOW: ALL CARON IN ORGANIC MATTER COMES FROM ATMO. CO_2. __SO__ PROPORTION OF ^{14}C TO STABLE ^{12}C IN LIVING ORGANIC TISSUE IS ROUGHLY THE SAME AS IN ATMO. AS SOON AS TISSUE DIES, IT STOPS REPLACING CARBON, AND ^{14}C STARTS TO DECAY. SO THE PAINT ON CAVE SHOULD HAVE STARTED AT TIME T WITH SAME ^{14}C TO ^{12}C RATIO TO ATMO, AND THEN ^{14}C DECAYS W/AGE.

- GOAL: FIND AN ACCURATE MODEL FOR RADIOACTIVE DECAY.

WE WILL US A __COMPARTMENT MODEL__



EX: RADIOACTIVE DECAY



ASSUME A MECHINISM: EACH ATOM HAS A CERTAN % CHANCE OF DECAYING OVER ANY FINITE PERIOD OF TIME Δt. ASSUME N LARGE, TIME IS CONT, AN NO OTHER MASS CHANGE.

THEN OVER ANY Δt, EXPECT

$\Delta t \cdot K \cdot N$ ATOMS TO DECAY.

$K \in \mathbb{R}^+$ IS CONST.

RATE OF CHANGE OF N OVER Δt:

$$\frac{N(t+\Delta t) - N(t)}{\Delta t}$$

$$= \frac{(\cancel{N(t)} - \Delta t \cdot K \cdot N(t)) - \cancel{N(t)}}{\Delta t}$$

$$= -KN(t).$$

TAKING LIMIT AS $\Delta t \to 0$ WE

TAKING LIMIT AS $\Delta t \to 0$, HAVE

$$\boxed{\frac{dN}{dt} = -kN(t).}$$

$$\boxed{\frac{dN}{dt} = -kN}$$ WHEN $N(t)$ IS CLEAR.

Q: WHAT IS SOLUTION? AN EXPONENTIAL.

SOLVE: (SEPERATION OF VARIABLE)

REWRITE AS

$$g(N)\frac{dN}{dt} = f(t)$$

HERE:

$$\frac{dN}{dt} = -kN \Rightarrow \frac{1}{N}\frac{dN}{dt} = -k$$

$$\left(\Rightarrow \frac{1}{N}dN = -kdt\right)$$

THEN INTEGRATE W.R.T. t:

$$\int \frac{1}{N}\underbrace{\frac{dN}{dt}dt} = -\int kdt$$

$$\underbrace{dt}_{\text{CHANGE OF VAR IS } dN}$$

$$\Rightarrow \int \frac{1}{N}\, dN = -\int k\, dt$$

$$\Rightarrow \operatorname{Log}|N| = -kt + c$$

$$\Rightarrow N = e^{-kt+c}$$

$$\Rightarrow \boxed{N = Ae^{-kt}}, \quad A = e^{c}$$

Q: AS A MODEL: WHAT ARE

- QUANTITIES: N
- STATE VARS: t
- PARAMETERS: A, k

IN DIFF. EQ'S, $N = Ae^{-kt}$ IS

- A GENERAL SOLUTION, SINCE IT DEPENDS ON PARAMETERS THAT DON'T APPEAR IN $\frac{dN}{dt} = -kN$.

- A PARTICULAR SOLUTION HAS ALL

- A <u>PARTICULAR SOLUTION</u> HAS ALL ADDITIONAL PARAMETERS FIXED.

USUALLY, THESE ARE FIXED BY GIVING QUANTITIES AN INITIAL VALUE:

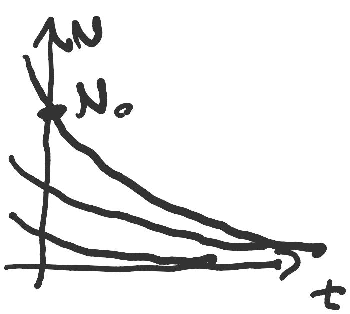
$$N(0) = N_0 ,$$

THEN THE SOLUTION TO THE I.V.P. IS

$$N(0) = N_0 = A e^{-k \cdot 0} = A$$

$$N(t) = N_0 e^{-kt} \qquad \square$$

<u>ANALYZE MODEL</u>: $\frac{dN}{dt} = -kN$, $N(0) = N_0$.



- ASYMPTOTICALLY, ALL SOLUTIONS <u>DECAY</u>.

<u>HALF LIFE</u>: FOR ALL k, $\exists \tau \in \mathbb{R}$ SUCH THAT $N(t+\tau) = \frac{1}{2} N(t)$:

$$\frac{1}{2} = \frac{N(t+\tau)}{N(t)} = \frac{\cancel{N_0} e^{\cancel{-kt} - k\tau}}{\cancel{N_0} e^{\cancel{-kt}}}$$

$2 \qquad N(t) \qquad \cancel{N_0} e^{-k\cancel{t}}$

$= e^{-k\tau}$

SO

$\log \frac{1}{2} = -k\tau \Rightarrow \boxed{\begin{array}{l} k = \frac{1}{\tau} \log 2 \\ \tau = \frac{1}{k} \log 2 \end{array}}$

BACK TO EX:

PUT SO NUMBERS IN OUR MODEL:

- IN ATMO, ^{14}C IS 1 ~ 1.5 ATOMS PER 10^{12} CARBON ATOMS.
- HALF LIFE: $\tau \approx 5568 \pm 30$
- ^{14}C DECAY RATE 13.5 ATOMS DECAYING PER MIN. PER GRAM OF ATMO CARBON.
- IN PAINTINGS, ^{14}C DECAY RATE IS 1.69 ATOMS DECAYING PER MIN. PER GRAM.

USE:

$k = \frac{\log 2}{\tau} \approx .00012$

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LET T BE TIME PAINTINGS WERE DONE, AND $t=0$ BE "NOW."

RATIO:

$$\frac{N'(T)}{N'(0)} = \frac{13.5}{1.69} = \frac{-KN(t)}{-KN(0)} = \frac{-KN_0 e^{-Kt}}{-KN_0 e^{-K0}}$$

$$= e^{-KT}$$

$$\Rightarrow e^{-T(.00012)} = \frac{13.5}{1.64}$$

$$\boxed{T = -16{,}692 \pm 90 \text{ YEARS}}$$

$\square$

Q) MATH: SOLVE

$$\frac{dy}{dt} = \frac{t^2}{\cos y}.$$

USING SEPERATION OF VARIABLES