1. a)

① Identity for sum: $e = 0$, $a + b\sqrt{2} + 0 = a + b\sqrt{2}$

② Associativity for sum: $(a + b\sqrt{2} + c + d\sqrt{2}) + e + f\sqrt{2} = a + b\sqrt{2} + (c + d\sqrt{2} + e + f\sqrt{2})$

③ Inverse for sum: $(a + b\sqrt{2})^{-1} = -a - b\sqrt{2}$, $a + b\sqrt{2} + (-a - b\sqrt{2}) = 0 = e$

④ Commutativity for sum: $a + b\sqrt{2} = b\sqrt{2} + a$

⑤ Multiplicative identity: $e' = 1 + 0\sqrt{2}$, $(a + b\sqrt{2})(1 + 0\sqrt{2}) = a + b\sqrt{2}$

⑥ Associativity for multiplication: $\left((a + b\sqrt{2}) \cdot (c + d\sqrt{2})\right) e + f\sqrt{2} =$

$(ac + bc\sqrt{2} + ad\sqrt{2} + 2bd)(e + f\sqrt{2}) = ace + bce\sqrt{2} + ade\sqrt{2} + 2bde + acf\sqrt{2} + 2bcf + 2adf + 2\sqrt{2}\,bdf =$

$(a + b\sqrt{2})(ce + \sqrt{2}\,de + \sqrt{2}\,cf + 2df) = (a + b\sqrt{2})\left((c + d\sqrt{2})(e + f\sqrt{2})\right)$

Distributive → ⑦ $(a + b\sqrt{2})\left((c + d\sqrt{2}) + (e + f\sqrt{2})\right) = ac + ad\sqrt{2} + ae + af\sqrt{2} + bc\sqrt{2} + 2bd + be\sqrt{2} + 2bf =$

$(a + b\sqrt{2})(c + d\sqrt{2}) + (a + b\sqrt{2})(e + f\sqrt{2})$

Commutativity for multiplication

→ ⑧ $(a + b\sqrt{2})(c + d\sqrt{2}) = ac + ad\sqrt{2} + bc\sqrt{2} + 2bd = (c + d\sqrt{2})(a + b\sqrt{2})$

Inverse for product → ⑨

$(a + b\sqrt{2}) \cdot (a - b\sqrt{2}) = a^2 - 2b^2$

$$(a + b\sqrt{2})^{-1} = \frac{a - b\sqrt{2}}{a^2 - 2b^2} = \frac{a}{a^2 - 2b^2} - \frac{b}{a^2 - 2b^2}\sqrt{2}$$

b)

① Identity for sum: $e = 0$, $a + b\sqrt{-1} + 0 = a + b\sqrt{-1}$

② Associativity for sum: $(a + b\sqrt{-1} + c + d\sqrt{-1}) + e + f\sqrt{-1} = a + b\sqrt{-1} + (c + d\sqrt{-1} + e + f\sqrt{-1})$

③ Inverse for sum: $(a + b\sqrt{-1})^{-1} = -a - b\sqrt{-1}$, $a + b\sqrt{-1} + (-a - b\sqrt{-1}) = 0 = e$

④ Commutativity for sum: $a + b\sqrt{-1} = b\sqrt{-1} + a$

⑤ Multiplicative identity: $e' = 1 + 0\sqrt{-1}$, $(a + b\sqrt{-1})(1 + 0\sqrt{-1}) = a + b\sqrt{-1}$

⑥ Associativity for multiplication: $\left((a + b\sqrt{-1}) \cdot (c + d\sqrt{-1})\right)(e + f\sqrt{-1}) =$

$(ac + bc\sqrt{-1} + ad\sqrt{-1} - bd)(e + f\sqrt{-1}) = ace + bce\sqrt{-1} + ade\sqrt{-1} - bde + acf\sqrt{-1} - bcf$

$- adf - bdf\sqrt{-1} = (a + b\sqrt{-1})(ce + de\sqrt{-1} + cf\sqrt{-1} - df) =$

$(a + b\sqrt{-1})\left((c + d\sqrt{-1})(e + f\sqrt{-1})\right)$

⑦ Distributivity: $(a+b\sqrt{-1})((c+d\sqrt{-1})+(e+f\sqrt{-1}))=$

$ac+ad\sqrt{-1}+ae+af\sqrt{-1}+bc\sqrt{-1}-bd+be\sqrt{-1}-bf=$

$(a+b\sqrt{-1})(c+d\sqrt{-1})+(a+b\sqrt{-1})(e+f\sqrt{-1})$

⑧ Commutativity for multiplication: $(a+b\sqrt{-1})(c+d\sqrt{-1})=$

$ac+ad\sqrt{-1}+bc\sqrt{-1}-bd=(c+d\sqrt{-1})(a+b\sqrt{-1})$

⑨ Multiplicative inverse: $(a+b\sqrt{-1})(a-b\sqrt{-1})=a^2+b^2$

$$(a+b\sqrt{-1})^{-1}=\frac{a-b\sqrt{-1}}{a^2+b^2}=\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}\sqrt{-1}$$

This field is $\mathbb{C}$.

2. Every element in $\mathbb{R}^{n\times n}$ must have a multiplicative inverse. For $\forall n$, $\exists A$:

$$A=\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & 0 & & 0 \end{bmatrix}.$$ Hence, every set $\mathbb{R}^{n\times n}$ has an element with determinant 0, and a matrix with determinant 0 has no multiplicative inverse, so every $\mathbb{R}^{n\times n}$ is not a field.

3.

+	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

X	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

4. same as question 1b

5. A: NO, multiple leading entries in column 3

B: YES

C: NO, row of all zeroes should be on bottom

D: YES

E: NO, multiple leading entries in column

6.

$$A+B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

7. when $\text{rref}(A) \neq I_n$

$$A = \begin{bmatrix} 6 & -1 & 1 \\ t & 0 & 1 \\ 0 & 1 & t \end{bmatrix} \begin{matrix} \div 6 \\ \updownarrow \\ \end{matrix} \rightarrow \begin{bmatrix} 1 & -1/6 & 1/6 \\ 0 & 1 & t \\ t & 0 & 1 \end{bmatrix}_{-t \cdot I} \rightarrow \begin{bmatrix} 1 & -1/6 & 1/6 \\ 0 & 1 & t \\ 0 & t/6 & 1-t/6 \end{bmatrix} \begin{matrix} +\frac{1}{6} \cdot II \\ \rightarrow \\ -t/6 \cdot II \end{matrix} \begin{bmatrix} 1 & 0 & 1/6 + 1/6t \\ 0 & 1 & t \\ 0 & 0 & 1 - t/6 - t^2/6 \end{bmatrix}$$

$\text{rref}(A) \neq I_n \rightarrow 1 - t/6 - t^2/6 = 0 \rightarrow t^2 + t - 6 = 0 \rightarrow (t+3)(t-2) = 0 \rightarrow t = 2, -3$

8.

a) $\left[\begin{array}{cc|c} 1 & h & 4 \\ 3 & 6 & 8 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & h & 4 \\ 0 & 6-3h & -4 \end{array}\right]$, consistent if $6 - 3h \neq 0$, $h \neq 2$

b) $\left[\begin{array}{cc|c} -4 & 12 & h \\ 2 & -6 & -3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -3 & -h/4 \\ 0 & 0 & -3 + h/2 \end{array}\right] \rightarrow$ consistent if $-3 + h/2 = 0$, $h = 6$

9.

(1) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

(2) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$,

$\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}$, $\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(3) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

10.

$e = 0 \rightarrow$ second row has leading entry in third or fourth column

$c = 1 \rightarrow$ fourth column of second row can't be leading entry because of other nonzero entries

$b = 0 \rightarrow$ c is leading entry

$d = 0 \rightarrow$ fifth column has leading entry below d

$a =$ anything

11.

a) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} \\ -I \\ -2\cdot I \end{matrix}$ $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -2 \\ 0 & -4 & -5 & -6 \end{bmatrix} \begin{matrix} +2\cdot II \\ \cdot -1 \\ -4\cdot II \end{matrix}$ $\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 7 & 2 \end{bmatrix} \div 7$

$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 2/7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 6/7 \\ 0 & 1 & 0 & 8/7 \\ 0 & 0 & 1 & 2/7 \end{bmatrix}$ sol $= x_4 \begin{bmatrix} -6/7 \\ -8/7 \\ -2/7 \\ 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} \rightarrow \\ -I \\ -2 \cdot I \end{matrix} \begin{bmatrix} [1] & [2] & [3] & [4] \\ [0] & [6] & [4] & [5] \\ [0] & [3] & [2] & [1] \end{bmatrix} \begin{matrix} +2 \cdot II \\ \cdot 6 \\ +3 \cdot II \end{matrix} \begin{bmatrix} [1] & [0] & [4] & [0] \\ [0] & [1] & [3] & [2] \\ [0] & [0] & [0] & [2] \end{bmatrix} \rightarrow$

$\begin{bmatrix} [1] & [0] & [4] & [0] \\ [0] & [1] & [3] & [0] \\ [0] & [0] & [0] & [1] \end{bmatrix}$

c) See Python file.

$Z_2 \Rightarrow \begin{bmatrix} [1] & [0] & [0] & [0] \\ [0] & [1] & [0] & [0] \\ [0] & [0] & [1] & [0] \end{bmatrix}$

$Z_3 \rightarrow \begin{bmatrix} [1] & [0] & [0] & [0] \\ [0] & [1] & [0] & [0] \\ [0] & [0] & [1] & [0] \end{bmatrix}$

d) Yes

12. From Python

a) rref = $\begin{bmatrix} [1] & [0] & [0] & [4] \\ [0] & [1] & [0] & [3] \\ [0] & [0] & [1] & [0] \end{bmatrix}$

b) $\vec{x} = \begin{bmatrix} [4] \\ [3] \\ [0] \end{bmatrix}$

13.

$$\text{rref} = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \quad \text{NO SOLUTION}$$

14.

$$\text{rref} = \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 0 & 5 & 6 \\ 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad \vec{x} = \begin{bmatrix} 6 - 2x_2 - 3x_3 - 5x_5 \\ x_2 \\ x_3 \\ 7 - 2x_5 \\ x_5 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 0 \\ 0 \\ 7 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

15.

$$\text{rref} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -8221/4340 \\ 0 & 1 & 0 & 0 & 0 & -8591/8690 \\ 0 & 0 & 1 & 0 & 0 & 4695/434 \\ 0 & 0 & 0 & 1 & 0 & -459/434 \\ 0 & 0 & 0 & 0 & 1 & 699/434 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} -8221/4340 \\ 8591/8690 \\ 4695/434 \\ -454/434 \\ 699/434 \end{bmatrix}$$

16.

a) A is invertible. It's inverse is BC

C is invertible. It's inverse is AB

B is invertible. It's inverse is CA.

$ABC = I_n \to BC = A^{-1} \to B = A^{-1}C^{-1} \to B^{-1} = (A^{-1}C^{-1})^{-1} = CA$

b) Yes. If AB is invertible, then $\det(AB) \neq 0$. If $\det(AB) \neq 0$, then $\det(A) \neq 0$ and $\det(B) \neq 0$, so A and B are both invertible.

17.

$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$(AB)^2 = \left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right)^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \quad A^2B^2 = \left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right)\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

18.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix}$$

$a^2+b^2=1$ $\quad c^2+d^2=1$

$ac+bd=0$

$b, c = 1, \quad a, d = 0$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

19.

a)

	symmetric	skew-symmetric
2×2	$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix}$
3×3	$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 9 \end{bmatrix}$	$\begin{bmatrix} 0 & 2 & 4 \\ -2 & 0 & -6 \\ -4 & 6 & 0 \end{bmatrix}$
4×4	$\begin{bmatrix} 3 & 4 & 1 & 0 \\ 4 & 6 & 0 & -3 \\ 1 & 0 & 2 & 0 \\ 0 & -3 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$

b) The main diagonal of a skew-symmetric matrix must be all 0s, because $a_{ii} = -a_{ii}$ for all i.

c) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

d)

$A + A^T$ is symmetric: $\quad = [b_{ij}]$

Let $A = [a_{ij}]$. Let $B = A + A^T$. $\quad b_{ij} = A_{ij} + A^T_{ij} = a_{ij} + a_{ji}$,

$b_{ji} = A_{ji} + A^T_{ji} = a_{ji} + a_{ij}$, so $b_{ij} = b_{ji}$ $\forall i, j$ and $A + A^T$ is symmetric

AA^T is symmetric:

Let $A = [a_{ij}]$. Let $C = AA^T = [c_{ij}]$. $\quad c_{ij} = \sum_{k=0}^{n-1} A_{ik} \cdot A^T_{kj} = \sum_{k=0}^{n-1} a_{ik} \cdot a_{jk}$,

$c_{ji} = \sum_{k=0}^{n-1} A_{jk} \cdot A^T_{ki} = \sum_{k=0}^{n-1} a_{jk} \cdot a_{ik}$, so $c_{ij} = c_{ji}$ $\forall i, j$ and AA^T is symmetric

A^TA is symmetric:

Let $A = [a_{ij}]$. Let $D = A^TA = [d_{ij}]$. $d_{ij} = \sum_{k=0}^{n-1} A^T_{ik} \cdot A_{kj} = \sum_{k=0}^{n-1} a_{ki} \cdot a_{kj}$,

$d_{ji} = \sum_{k=0}^{n-1} A^T_{jk} \cdot A_{ki} = \sum_{k=0}^{n-1} a_{kj} \cdot a_{ki}$, so $d_{ij} = d_{ji}$ $\forall i,j$ and A^TA is symmetric

$A - A^T$ is skew-symmetric:

Let $A = [a_{ij}]$. Let $E = A - A^T = [e_{ij}]$. $e_{ij} = A_{ij} - A_{ij}^T = a_{ij} - a_{ji}$, $e_{ji} = A_{ji} - A_{ji}^T =$

$a_{ji} - a_{ij}$, $e_{ij} = -(e_{ji})$ $\forall i,j$ so $A - A^T$ is skew-symmetric

5 We know $(A + A^T)$ is symmetric and $(A - A^T)$ is skew-symmetric.

Let $B = 0.5 \cdot (A + A^T)$, so B is symmetric, let $C = 0.5 \cdot (A - A^T)$, so C is skew-symmetric.

$B + C = 0.5(A + A^T) + 0.5(A - A^T) = A$, so A is sum of symmetric and skew-symmetric matrices.

20.

a) none $\rightarrow$ $F(x) = F(-x)$, so not one-to-one, $F(x) \geq 0$ so not onto

b) bijective

c) surjective $\rightarrow$ $F(0) = F(1) = F(2)$, so not one-to-one

d) injective $\rightarrow$ $F(x) \geq 2$

21.

$$\begin{bmatrix}4&1&0&0\\1&4&1&0\\0&1&4&1\\0&0&1&4\end{bmatrix} \xrightarrow{-\frac{1}{4}\cdot I} \begin{bmatrix}4&1&0&0\\0&15/4&1&0\\0&1&4&1\\0&0&1&4\end{bmatrix} \xrightarrow{-\frac{4}{15}\cdot II} \begin{bmatrix}4&1&0&0\\0&15/4&1&0\\0&0&56/15&1\\0&0&1&4\end{bmatrix} \xrightarrow{-III\cdot\frac{15}{56}} \begin{bmatrix}4&1&0&0\\0&15/4&1&0\\0&0&56/15&1\\0&0&0&209/56\end{bmatrix} = U$$

$$\text{transformations} \rightarrow \begin{bmatrix}1&0&0&0\\1/4&1&0&0\\0&4/15&1&0\\0&0&15/56&1\end{bmatrix} = L$$

22.

$$\begin{bmatrix}q_1&r_1&0&0\\p_1&q_2&r_2&0\\0&p_2&q_3&r_3\\0&0&p_3&q_4\end{bmatrix} \xrightarrow{-\frac{p_1}{q_1}\cdot I} \begin{bmatrix}q_1&r_1&0&0\\0&d_2&r_2&0\\0&p_2&q_3&r_3\\0&0&p_3&q_4\end{bmatrix} \xrightarrow{-\frac{p_2}{d_2}\cdot II} \begin{bmatrix}q_1&r_1&0&0\\0&d_2&r_2&0\\0&0&d_3&r_3\\0&0&p_3&q_4\end{bmatrix} \xrightarrow{-\frac{p_3}{d_3}\cdot III}$$

$d_2 = q_2 - \frac{p_1 r_1}{q_1}$ $\quad d_3 = q_3 - \frac{p_2 r_2}{d_2}$ $\quad d_4 = q_4 - \frac{p_3 r_3}{d_3}$

$$\begin{bmatrix}q_1&r_1&0&0\\0&d_2&r_2&0\\0&0&d_3&r_3\\0&0&0&d_4\end{bmatrix} = U, \quad L = \begin{bmatrix}1&0&0&0\\p_1/q_1&1&0&0\\0&p_2/q_2&1&0\\0&0&p_3/q_3&1\end{bmatrix}$$

22.

$$A = \begin{bmatrix}1&0&0&0\\l_1&1&0&0\\0&l_2&1&0\\0&0&l_3&1\end{bmatrix}\begin{bmatrix}d_1&\mu_1&0&0\\0&d_2&\mu_2&0\\0&0&d_3&\mu_3\\0&0&0&d_4\end{bmatrix} = \begin{bmatrix}d_1&\mu_1&0&0\\l_1d_1&l_1\mu_1+d_2&\mu_2&0\\0&l_2d_2&l_2\mu_2+d_3&\mu_3\\0&0&l_3d_3&l_3\mu_3+d_4\end{bmatrix}$$

$$q_i = \begin{cases} d_i & \text{if } i=1 \\ l_{i-1}\mu_{i-1}+d_i & \text{else}\end{cases} \qquad p_i = l_i d_i \qquad r_i = \mu_i$$

$$l_i = \frac{p_i}{d_i} \qquad d_i = \begin{cases} q_i & \text{if } i=1 \\ q_i - \frac{p_{i-1}r_{i-1}}{d_{i-1}} & \text{else}\end{cases} \qquad \mu_i = r_i$$

23.

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/d_1 & 1 & 0 & 0 \\ 0 & 1/d_2 & \ddots & 1 & 0 \\ 0 & & & 1/d_{n-1} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 4 & 1 & \cdots & 0 & 0 \\ 0 & 4-\frac{1}{d_1} & 1 & & \\ 0 & 1 & 4-\frac{1}{d_2} & \cdots & 1 \\ & & & \ddots & \\ & & & & 4-\frac{d}{d_{n-1}} \end{bmatrix}$$

24.

① H_n is symmetric.

I_n is symmetric, so H_n is symmetric $\leftrightarrow -2\vec{\mu}\mu^T$ is symmetric.

Let $\vec{\mu} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $A = \vec{\mu}\vec{\mu}^T$. Then $A_{ij} = \mu_i\mu_j = \mu_j\mu_i = A_{ji}$, so $\vec{\mu}\mu^T$ is symmetric.

$\vec{\mu}\vec{\mu}^T$ is symmetric $\rightarrow -2\vec{\mu}\vec{\mu}^T$ is symmetric $\rightarrow H_n$ is symmetric

② H_n is orthogonal.

We know H_n is symmetric, so $H_n = H_n^T$. To prove H_n orthogonal, we must prove $H_n^T H_n = H_n^2 = I_n$. $H_2 = (I_n - 2\vec{\mu}\vec{\mu}^T)^2 = I_n - 4\vec{\mu}\mu^T + 4\vec{\mu}\mu^T\mu\mu^T$.

$\vec{\mu}^T\vec{\mu} = 1$, so $I_n - 4\vec{\mu}\vec{\mu}^T + 4\vec{\mu}\vec{\mu}^T\vec{\mu}\mu^T = I_n - 4\vec{\mu}\vec{\mu}^T + 4\vec{\mu}\mu^T = I_n$, $H_n^T H_n = I_n$,

H_n orthogonal.

③ $H_n^T H_n = I_n$ and $H_n = H_n^T$ so $H_n^2 = I_n$

④ $(H_n)\vec{\mu} = \vec{\mu} - 2\vec{\mu}\vec{\mu}^T\vec{\mu} = \vec{\mu} - 2\vec{\mu} = -\vec{\mu}$

⑤ $$H_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$H_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & -1/2 \\ -1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$$