MATH 7241 Fall 2020: Problem Set #3

Due date: Sunday October 11

Reading: relevant background material for these problems can be found in the class notes, and in Rosenthal Chapter 3.

$\boxed{\textbf{Exercise 1}}$ Let $X_1, X_2, \ldots$ be a sequence of random variables (not necessarily independent), and suppose that $\mathbb{E}[X_n] = 0$ and $\mathbb{E}[(X_n)^2] = 1$ for all $n \geq 1$. Prove that

$$\mathbb{P}(X_n \geq n \text{ i.o.}) = 0$$

where i.o. means 'infinitely often'. [Hint: use the Borel-Cantelli Lemma and Markov's inequality]

Markov inequality:

$$\mathbb{P}(X_n \geq n) \leq \mathbb{P}(|X_n| \geq n) \leq \frac{\mathbb{E}[X_n^2]}{n^2} = \frac{1}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \mathbb{P}(X_n \geq n) \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

Borel–Cantelli 1 $\Rightarrow$ $\mathbb{P}(X_n \geq n \text{ i.o.}) = 0$

Exercise 2 Let $X_1, X_2, \dots$ be independent random variables and suppose that X_n is uniform on the set $\{1, 2, \dots, n\}$ for each $n \geq 1$. Compute $\mathbb{P}(X_n = 5 \text{ i.o.})$. [Hint: use the Borel-Cantelli Lemma]

$$\mathbb{P}(X_n = 5) = \frac{1}{n} \quad \text{(uniform on } \{1,..,n\})$$

$$\Rightarrow \sum_{n=1}^{\infty} \mathbb{P}(X_n = 5) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty \quad \text{(harmonic series)}$$

Borel - Cantelli 2:

- X_n independent $\Rightarrow$ events $\{X_n = 5\}$ are independent
- $\sum_{n=1}^{\infty} \mathbb{P}(X_n = 5) = \infty$

$$\Rightarrow \quad \mathbb{P}(X_n = 5 \text{ i.o.}) = 1.$$

$\boxed{\textbf{Exercise 3}}$ Define the sequence X_n inductively by setting $X_0 = 1$, and selecting X_{n+1} randomly and uniformly from the interval $[0, X_n]$. Prove that there is a number c such that for every $\epsilon > 0$, $\mathbb{P}\left(|n^{-1}\log X_n - c| \geq \epsilon\right)$ converges to zero, and evaluate the number c.

[Hint: Note that $U_n = X_n X_{n-1}^{-1}$ is a uniform random variable for each $n \geq 2$. Use the Law of Large Numbers.]

$$X_n = U_n X_{n-1} \qquad \text{where} \quad U_n \sim \text{Uniform } [0,1].$$

$$= U_n U_{n-1} X_{n-2}$$

$$= \dots = U_n U_{n-1} U_{n-2} \cdots U_1 X_0$$

where $\{U_i\}$ are IID uniform on $[0,1]$.

$$\Rightarrow \log X_n = \sum_{i=1}^{n} \log U_i + \cancel{\log X_0} \to = 0 \quad \because X_0 = 1$$

<u>LLN</u>: $\quad \frac{1}{n}\sum_{i=1}^{n} \log U_i \xrightarrow[\text{prob.}]{} \mathbb{E}[\log U] \quad \text{as } n \to \infty$

specifically $\forall \varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{1}{n}\log X_n - c\right| \geq \varepsilon\right) = 0$$

where $\quad c = \mathbb{E}[\log U] = \int_0^1 \log x \cdot 1 \, dx \qquad \log = \ln$ (base e)

$$= x(\log x - 1)\Big|_0^1$$

$$= -1.$$

$\boxed{\text{Exercise 4}}$ **Fact**: For any sequence of *pairwise disjoint* events $\{C_n\}$ we have

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} C_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(C_n) = \lim_{N\to\infty} \sum_{n=1}^{N} \mathbb{P}(C_n)$$

Definition: A sequence of events $\{A_n\}$ is *decreasing* if $A_{n+1} \subset A_n$ for all n. A sequence of events $\{B_n\}$ is *increasing* if $B_n \subset B_{n+1}$ for all n.

a) Let $\{B_n\}$ be an increasing sequence, let $C_1 = B_1$, and let $C_n = B_n \cap B_{n-1}^c$ for all $n \geq 2$. Show that the events $\{C_n\}$ are disjoint, and that for every $N \geq 1$

$$B_N = \bigcup_{n=1}^{N} B_n = \bigcup_{n=1}^{N} C_n.$$

<u>Disjoint</u>: let $m < n$, then

$$C_n \cap C_m = B_n \cap B_{n-1}^c \cap B_m \cap B_{m-1}^c$$

Now $m \leq n-1 \Rightarrow B_m \subseteq B_{n-1}$

$$\Rightarrow B_m \cap B_{n-1}^c = \phi.$$

$$\Rightarrow C_n \cap C_m = \phi \Rightarrow \text{disjoint.}$$

<u>Union</u>: use induction.

$C_1 = B_1$ ✓

suppose $\bigcup_{n=1}^{N} C_n = B_N$.

$$\Rightarrow \bigcup_{n=1}^{N+1} C_n = \bigcup_{n=1}^{N} C_n \cup C_{N+1}$$

$$= B_N \cup (B_{N+1} \cap B_N^c)$$

$$= (B_N \cup B_{N+1}) \cap (B_N \cup B_N^c)$$

$$= B_{N+1} \cap S$$

$$= B_{N+1}$$ ✓

$\Rightarrow$ true for all N by induction

b) Show that

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} C_n.$$

[Hint: let x be any element of the left side, show that this implies that x belongs to the right side. Then show that the converse is also true].

Let $x \in \bigcup_{n=1}^{\infty} B_n$

$\Rightarrow x \in B_n$ for some n.

$\Rightarrow x \in \bigcup_{i=1}^{n} C_i$

$\Rightarrow x \in \bigcup_{i=1}^{\infty} C_i \Rightarrow \bigcup_{n=1}^{\infty} B_n \subseteq \bigcup_{n=1}^{\infty} C_n.$

Conversely, let $x \in \bigcup_{n=1}^{\infty} C_n$

$\Rightarrow x \in C_n$ some n

$\Rightarrow x \in B_n \cap B_{n-1}^c$

$\Rightarrow x \in B_n$

$\Rightarrow x \in \bigcup_{n=1}^{\infty} B_n \Rightarrow \bigcup_{n=1}^{\infty} C_n \subseteq \bigcup_{n=1}^{\infty} B_n$

Hence $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} C_n$

c) Using the results of (b), the Fact, and (a), show that

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{N\to\infty} \mathbb{P}(B_N)$$

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} B_n\right) = \mathbb{P}\left(\bigcup_{n=1}^{\infty} C_n\right)$$

$$= \sum_{n=1}^{\infty} \mathbb{P}(C_n) \quad \text{(disjoint)}$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} \mathbb{P}(C_n) \quad \text{(definition)}$$

$$= \lim_{N\to\infty} \mathbb{P}\left(\bigcup_{n=1}^{N} C_n\right) \quad \text{(disjoint)}$$

$$= \lim_{N\to\infty} \mathbb{P}(B_N)$$

d) Let $\{A_n\}$ be a decreasing sequence. By taking the complement in the result (c) show that

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{N\to\infty} \mathbb{P}(A_N)$$

$\{A_n\}$ decreasing

$\Rightarrow \{A_n^c\}$ increasing

$$\Rightarrow \quad \mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n^c\right) = \lim_{N\to\infty} \mathbb{P}(A_N^c)$$

$$\begin{aligned}
\Rightarrow \quad \mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n\right) &= 1 - \mathbb{P}\left(\left(\bigcap_{n=1}^{\infty} A_n\right)^c\right) \\
&= 1 - \mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n^c\right) \\
&= 1 - \lim_{N\to\infty} \mathbb{P}(A_N^c) \\
&= 1 - \lim_{N\to\infty} \left(1 - \mathbb{P}(A_N)\right) \\
&= \lim_{N\to\infty} \mathbb{P}(A_N)
\end{aligned}$$

e) Use the result of (d) to show that the cdf of any random variable X is right continuous, that is

$$F(a) = \lim_{h \to 0,\, h>0} F(a+h)$$

for all a. [Hint: Recall that the cdf of X is the function $F(a) = P(X \leq a)$. Define appropriate sets $\{A_n\}$ and use the result of part (d). Note that it is sufficient to prove convergence along any decreasing sequence of points $\{h_1, h_2, \dots\}$ which converge to 0.]

Let $\{h_1, h_2, \dots\}$ be a decreasing sequence converging to 0. Define

$$A_n = \{X \leq a + h_n\}.$$

Then if A_{n+1} is true

$$\Rightarrow X \leq a + h_{n+1}$$

$$\Rightarrow X \leq a + h_n \qquad (\text{b/c } h_{n+1} \leq h_n)$$

$$\Rightarrow A_n \text{ is true.}$$

Therefore $A_{n+1} \subseteq A_n$, so $\{A_n\}$ are decreasing.

$$\Rightarrow \quad P\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{N \to \infty} P(A_N)$$

$$= \lim_{N \to \infty} P(X \leq a + h_N) = \lim_{N\to\infty} F(a+h_N)$$

Now if $\bigcap_{n=1}^{\infty} A_n$ is true $\Leftrightarrow A_n$ is true for all n

$$\Leftrightarrow X \leq a + h_n \text{ for all } n$$

$$\Leftrightarrow X \leq a \quad (\text{b/c } h_n \to 0)$$

$$\Rightarrow \quad P\left(\bigcap_{n=1}^{\infty} A_n\right) = P(X \leq a) = F(a).$$

$$\Rightarrow \quad F(a) = \lim_{N \to \infty} F(a + h_N)$$

True for any decreasing sequence $\{h_n\}$ which converges to 0.

$$\Rightarrow \quad F(a) = \lim_{h \downarrow 0} F(a+h).$$

$\Rightarrow$ the cdf is right continuous at every point.

<u>Note</u>: for every a,

$$\lim_{h \downarrow 0} F(a-h) = \mathbb{P}(X < a).$$

so $\mathbb{P}(X=a) = F(a) - \lim_{h \downarrow 0} F(a-h)$