MATH 7241 Fall 2020: Problem Set #8

Due date: Sunday November 22

Reading: relevant background material for these problems can be found on Canvas 'Notes 4: Finite Markov Chains'. Also Grinstead and Snell Chapter 11.

$\boxed{\textbf{Exercise 1}}$ Recall the Gambler's Ruin Problem: a random walk on the integers $\{0, 1 \dots, N\}$ with probability p to jump right and $q = 1 - p$ to jump left at every step, and absorbing states at 0 and N. Starting at $X_0 = k$, the probability to reach N before reaching 0 is

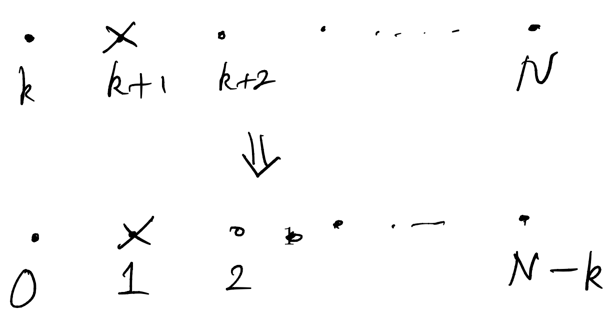
$$P_k = \frac{1 - (q/p)^k}{1 - (q/p)^N} \quad \text{for } p \neq \frac{1}{2}, \qquad\qquad P_k = \frac{k}{N} \quad \text{for } p = \frac{1}{2}.$$

Starting at $X_0 = k$, let R_k be the probability to reach state N without returning to state k. Use the Gambler's Ruin result to compute R_k for all $k = 0, \dots, N$, and for all $0 < p < 1$. [Hint: condition on the first step and use the formula given above].

Condition on step 1:

$$0$$

$$R_k = \cancel{P(\text{reach } N \text{ before } k \mid X_1 = k-1)\, q}$$
$$+ P(\text{reach } N \text{ before } k \mid X_1 = k+1)\, p$$
$$= p\, P(\text{reach } N \text{ before } k \mid X_1 = k+1)$$



$$\Rightarrow R_k = p\, P(\text{reach } N-k \text{ before } 0 \mid X_0 = 1)$$

Use formula above for P_k:

$$\Rightarrow R_k = \begin{cases} p \dfrac{1 - q/p}{1-(q/p)^{N-k}} & p \neq \frac{1}{2} \\ \frac{1}{2} \dfrac{1}{N-k} & p = \frac{1}{2} \end{cases}$$

$\boxed{\textbf{Exercise 2}}$ Consider a biased random walk X_n on the semi-infinite line $\{0, 1, 2, \dots\}$, where at each step the walker either goes left with probability q or goes right with probability p, where $p + q = 1$. The point 0 is absorbing. Recall the solution of the Gambler's Ruin problem:

$$Q_k = \frac{(q/p)^N - (q/p)^k}{(q/p)^N - 1}, \quad q \neq p, \qquad\qquad Q_k = 1 - \frac{k}{N} \quad \text{for } p = q$$

was the probability of being absorbed at 0 before being absorbed at N, starting at k. By taking appropriate limits, use this result to find the probability that the walk X_n on the semi-infinite line is absorbed at 0, after starting at k (your answer will depend on q and p – be sure to include the case $q = p$).

Let $A_{N,k} = \{\text{reach } 0 \text{ before } N \mid X_0 = k\}$

then $A_{N,k} \subset A_{N+1,k}$ for all N.

let $A_k = \bigcup_{N=1}^{\infty} A_{N,k}$

then $\mathbb{P}(\text{reach } 0 \mid X_0 = k) = \mathbb{P}(A_k)$

By continuity, since $\{A_{N,k}\}$ are increasing,

$$\mathbb{P}(A_k) = \lim_{N \to \infty} \mathbb{P}(A_{N,k})$$

$$= \begin{cases} 1 & q \geq p \\ (q/p)^k & q < p. \end{cases}$$

Exercise 3 The Markov chain $X = \{X_n\}$ is defined on the state space $S = \{0, 1, 2, \dots\}$. The chain is irreducible, aperiodic and positive persistent, with stationary distribution $\{w_k\}$ $(k = 0, 1, 2, \dots)$. Let $Y = \{Y_n\}$ be an independent copy of X, and define $Z = (X, Y)$.

a). Write down the transition matrix for Z, and compute its stationary distribution (your answer will depend on w).

b). Given that the chain Z starts at the state (k, k) (so that $X_0 = Y_0 = k$), find an expression for the expected number of steps until the first return to (k, k).

$$Z = (X, Y)$$

a) $$\mathbb{P}(Z_1 = (k,l) \mid Z_0 = (i,j))$$
$$= \mathbb{P}(X_1 = k \mid X_0 = i)\,\mathbb{P}(Y_1 = l \mid Y_0 = j)$$
$$= P_{ik} P_{jl}$$

Stationary: $$w_{(k,l)} = w_k w_l.$$

Check: $$\sum_{i,j} w_{(i,j)} P_{(i,j),(k,l)} = \sum_{i,j} w_i w_j P_{ik} P_{jl}$$
$$= \Big(\sum_i w_i P_{ik}\Big)\Big(\sum_j w_j P_{jl}\Big)$$
$$= w_k w_l$$
$$= w_{(k,l)}$$

b) Mean return time to (k,k) is

$$\frac{1}{w_{(k,k)}} = \frac{1}{w_k^2}$$

Exercise 4 Let $X_1, X_2, \ldots$ be IID random variables, where the moment generating function is

$$\mathbb{E}[e^{tX}] = \frac{4}{(2-t)^2}, \qquad t < 2$$

a). Find the mean $\mathbb{E}[X]$.
b). Let $Y_n = (1/n)\sum_{i=1}^n X_i$. For $x > \mathbb{E}[X]$ use Cramer's Theorem to compute

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(Y_n > x)$$

a) $\frac{d}{dt} \mathbb{E}[e^{tX}]\Big|_{t=0} = \mathbb{E}[Xe^{tX}]\Big|_{t=0} = \mathbb{E}[X]$

$\frac{d}{dt} \frac{4}{(2-t)^2} = 4(-2)(2-t)^{-3}(-1) = \frac{8}{(2-t)^3}$

$\Rightarrow \mathbb{E}[X] = \frac{8}{(2-0)^3} = 1.$

b) $Y_n = \frac{1}{n}\sum_{i=1}^n X_i$.

$\Lambda(t) = \ln \mathbb{E}[e^{tX}] = \ln 4 - 2\ln(2-t)$

$\Lambda^*(x) = \sup_t (xt - \Lambda(t))$

$\Rightarrow x = \Lambda'(t) = \frac{2}{2-t} \Rightarrow t^* = 2 - \frac{2}{x}$

$\Rightarrow \Lambda^*(x) = x\left(2 - \frac{2}{x}\right) - \ln 4 + 2\ln\left(\frac{2}{x}\right)$

$= 2x - 2 \cancel{- \ln 4} + \cancel{2\ln 2} - 2\ln x.$

Cramer's Theorem: for $x > \mathbb{E}[X] = 1$,

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{P}(Y_n > x) = - \Lambda^*(x)$$

$$= -2(x-1) + 2\ln x$$

$\boxed{\textbf{Exercise 5}}$ A fair coin is tossed n times, coming up Heads N_H times and Tails $N_T = n - N_H$ times. Let $S_n = N_H - N_T$. Use Cramer's Theorem to show that for $0 < a < 1$,

$$\lim_{n\to\infty} P(S_n > an)^{1/n} = \left[(1+a)^{1+a}(1-a)^{1-a}\right]^{-1/2}$$

Let $X_k = \begin{cases} 1 & \text{if } k^{\text{th}} \text{ Toss Heads} \\ -1 & \text{—''—— Tails} \end{cases}$

$$\Rightarrow S_N = N_H - N_T = \sum_{k=1}^{N} X_k$$

$$\Rightarrow \mathbb{P}(S_n > an) = \mathbb{P}\left(\frac{1}{n}\sum_{k=1}^{n} X_k > a\right)$$

Cramer's Theorem:

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{P}\left(\frac{1}{n}\sum_{k=1}^{n} X_k > a\right) = -\Lambda^*(a)$$

Compute Λ^*:

$$M_X(t) = \mathbb{E}\left[e^{Xt}\right] = \frac{1}{2}\left(e^{t}+e^{-t}\right)$$

$$\Lambda_X(t) = \ln\left(e^{t}+e^{-t}\right) - \ln 2$$

$$\Lambda_X'(t) = \frac{e^{t}-e^{-t}}{e^{t}+e^{-t}}$$

Solve $\Lambda_X'(t^*) = x \Rightarrow t^* = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

$$\Rightarrow \quad \Lambda^*(x) = x t^* - \Lambda(t^*)$$

$$= \frac{x}{2} \ln\left(\frac{1+x}{1-x}\right) - \ln\left(\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} + \left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}\right) + \ln 2$$

$$= \frac{1}{2}(1+x)\ln(1+x) + \frac{1}{2}(1-x)\ln(1-x)$$

$$\text{so} \quad \lim_{n\to\infty} \mathbb{P}(S_n > na)^{\frac{1}{n}} = e^{-\Lambda^*(a)}$$

$$= \left[(1+a)^{1+a}(1-a)^{1-a}\right]^{-\frac{1}{2}}$$

$\boxed{\textbf{Exercise 6}}$ Let $X_1, X_2, \ldots$ be IID r.v.'s, and $Y_n = n^{-1}\sum_{i=1}^{n} X_i$. For $x > \mathrm{E}[X]$ use Cramer's Theorem to compute

$$\lim_{n\to\infty} 1/n \log P(Y_n > x)$$

when X is a) exponential with rate λ, and b) uniform on $[0,1]$. [Note: for part (b) you will not achieve an explicit solution; instead produce a plot of the result as a function of x]

a) $X \sim$ exp. rate λ

$$M_X(t) = \mathbb{E}[e^{tX}] = \frac{\lambda}{\lambda - t}$$

$$\Rightarrow \Lambda_X(t) = \ln\left(\frac{\lambda}{\lambda - t}\right)$$

$$\Lambda_X'(t) = \frac{1}{\lambda - t}$$

Solve $\Lambda_X'(t) = x \Rightarrow t^* = \lambda - \frac{1}{x}$

$$\Rightarrow \Lambda^*(x) = \lambda x - 1 - \ln(\lambda x)$$

b) $X \sim U[0,1]$

$$M_X(t) = \frac{1}{t}(e^t - 1)$$

$$\Lambda_X(t) = \ln\left(\frac{e^t - 1}{t}\right)$$

$$\Lambda_X'(t) = \frac{e^t}{e^t - 1} - \frac{1}{t}$$

Solve $\Lambda_X'(t) = x \Rightarrow x = \frac{1}{1-e^{-t^*}} - \frac{1}{t^*}$ $(\ddagger)$

Then

$$\Lambda_X^*(x) = x t^* + \ln(1 + x t^*) - t^*$$

where t^* is solution of $(\ddagger)$

$\boxed{\textbf{Exercise 7}}$ For a branching process, calculate the probability of extinction when $p_0 = 1/6$, $p_1 = 1/2$, $p_2 = 1/3$.

$p_0 = \frac{1}{6}$, $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{3}$

let ρ = prob. of extinction.

solve $\rho = \phi(\rho)$, $\phi(t) = \mathbb{E}[t^Z]$.

$\Rightarrow \phi(t) = p_0 + p_1 t + p_2 t^2$

$= \frac{1}{6} + \frac{1}{2}t + \frac{1}{3}t^2$

solve $t = \phi(t) \Rightarrow t = \frac{1}{2}$ or $t = 1$

ρ = smallest root $= \frac{1}{2}$

Exercise 8 The number of offspring Z in a branching process has the following distribution:

$$p_0 = \mathbb{P}(Z=0) = p, \quad p_1 = \mathbb{P}(Z=1) = q, \quad p_2 = \mathbb{P}(Z=2) = 2p - \frac{1}{6}$$

where $0 \leq q \leq 1$ and $1/9 \leq p \leq 4/9$. Also $\mathbb{P}(Z > 2) = 0$.

a). Compute q as a function of p.

b). Find the mean $\mathbb{E}[Z]$ (your answer should depend on p, but not on q).

c). Find the largest value of p for which extinction is certain (your answer should be a number).

d). Let p_m be the value computed in (c). Calculate the probability of extinction for $p > p_m$ (your answer should depend on p, but not on q).

a) $p_0 + p_1 + p_2 = 1 \Rightarrow p + q + 2p - \frac{1}{6} = 1$

$$\Rightarrow q = \frac{7}{6} - 3p.$$

b)
$$\mathbb{E}[Z] = p_1 + 2p_2 = q + 2\left(2p - \frac{1}{6}\right)$$
$$= \frac{7}{6} - 3p + 4p - \frac{1}{3}$$
$$= \frac{5}{6} + p$$

c) $\mathbb{E}[Z] = 1 \Leftrightarrow p = \frac{1}{6}$

so extinction is certain for $p \leq \frac{1}{6}$. $\left(p_m = \frac{1}{6}\right)$

d) Solve
$$\rho = \phi(\rho)$$
$$t = p + qt + \left(2p - \frac{1}{6}\right)t^2$$
$$\Rightarrow \left(2p - \frac{1}{6}\right)t^2 + (q-1)t + p = 0.$$

$$(t-1)\left(\left(2p - \frac{1}{6}\right)t - p\right) = 0$$

$$\Rightarrow \rho = \frac{p}{2p-\frac{1}{6}} = \frac{6p}{12p-1} \qquad p > p_m = \frac{1}{6}$$