Sai Nikhil

NUID: 001564864

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(1) (a) $P = \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{pmatrix}$

$\pi P = \pi$

$$\left[\begin{array}{cc|c} -\frac{1}{4} & 1/2 & 0 \\ \frac{1}{4} & -\frac{1}{2} & 0 \\ 1 & 1 & 1 \end{array}\right]$$

Solving, $\pi = \begin{bmatrix} \frac{2}{3} \\ 1/3 \end{bmatrix}$

$\therefore \Pi = \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix}$

$\Pi^T * P = \begin{bmatrix} 2/3 & 2/3 \\ 1/3 & 1/3 \end{bmatrix} * \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/6 \\ 1/6 & 1/6 \end{bmatrix}$

Component-wise multiplication

This matrix is a Symmetric Matrix

$\therefore$ chain is reversible

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(b)
$$P = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 0 & 2/3 & 1/3 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -1/4 & 0 & 1/4 & 0 \\ 1/4 & -1/3 & 1/4 & 0 \\ 0 & 1/3 & -1/2 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right) \Rightarrow \pi = \begin{bmatrix} 2/7 \\ 3/7 \\ 2/7 \end{bmatrix}$$

$$\Pi = \begin{bmatrix} 2/7 & 3/7 & 2/7 \\ 2/7 & 3/7 & 2/7 \\ 2/7 & 3/7 & 2/7 \end{bmatrix}$$

$$\Pi^T * P = \begin{bmatrix} 2/7 & 2/7 & 2/7 \\ 3/7 & 3/7 & 3/7 \\ 2/7 & 2/7 & 2/7 \end{bmatrix} * \begin{bmatrix} 3/4 & 1/4 & 0 \\ 0 & 2/3 & 1/3 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

Component-wise multiplication

$$= \begin{bmatrix} 3/14 & 1/14 & - \\ 0 & - & - \\ - & - & - \end{bmatrix}$$

This matrix is not a symmetric matrix
∴ Chain is not reversible

(2) For a random-walk on an n-vertex unweighted graph,

$$\text{Expected time to reach vertex} = \frac{2N}{K}$$

where,

N = total number of edges

K = number of edges from a vertex

In case of Knight's random walk,

N = 168 , K = 2 (only 2 valid moves from corner)

$$\therefore \left.\begin{array}{l}\text{Expected time to}\\ \text{return to Corner}\end{array}\right\} = \frac{2 \times 168}{2}$$

$$= \boxed{168}$$

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(3)

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Converting it to have 0 and 4 as absorbing states, it becomes

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Converting P to its Canonical form,

$$P = \begin{pmatrix} 0 & 3/4 & 0 & 1/4 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 3/4 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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$$Q = \begin{pmatrix} 0 & 3/4 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 3/4 & 0 \end{pmatrix}$$

$$I - Q = \begin{pmatrix} 1 & -3/4 & 0 \\ -1/2 & 1 & -1/2 \\ 0 & -3/4 & 1 \end{pmatrix}$$

$$\therefore \; N = (I - Q)^{-1} = \begin{pmatrix} 5/2 & 3 & 3/2 \\ 2 & 4 & 2 \\ 3/2 & 3 & 5/2 \end{pmatrix}$$

$$Nc = \begin{pmatrix} 5/2 + 3 + 3/2 \\ 2 + 4 + 2 \\ 3/2 + 3 + 5/2 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 7 \end{pmatrix}$$

$$NR = \begin{pmatrix} 5/2 & 3 & 3/2 \\ 2 & 4 & 2 \\ 3/2 & 3 & 5/2 \end{pmatrix} \begin{pmatrix} 1/4 & 0 \\ 0 & 0 \\ 0 & 1/4 \end{pmatrix} = \begin{pmatrix} 5/8 & 3/8 \\ 1/2 & 1/2 \\ 3/8 & 5/8 \end{pmatrix}$$

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(4)

$$P = \begin{pmatrix} 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \end{pmatrix}$$

$$\therefore \quad Q = \begin{pmatrix} 0 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(I-Q) = \begin{pmatrix} 1 & -1/4 & -1/4 & -1/4 \\ 0 & 1 & -1/3 & -1/3 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$N = (I-Q)^{-1} = \begin{pmatrix} 1 & 1/4 & 1/3 & 1/2 \\ 0 & 1 & 1/3 & 1/2 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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$$N_C = \begin{pmatrix} 1 + 1/4 + 1/3 + 1/2 \\ 0 + 1 + 1/3 + 1/2 \\ 0 + 0 + 1 + 1/2 \\ 0 + 0 + 0 + 1 \end{pmatrix} = \begin{pmatrix} 25/12 \\ 11/6 \\ 3/2 \\ 1 \end{pmatrix}$$

Since we can only start in state 1, we only care about the first entry in the column

$\therefore$ Expected number of steps to reach stage 5 $\Big\} = \boxed{\dfrac{25}{12}}$

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(5) $w_x = P(\text{reech } T \mid X_0 = x)$

Condition on X_1 :

$$w_x = P(\text{reech } T \mid X_0 = x, X_1 = x+1) \times p$$
$$+ P(\text{reech } T \mid X_0 = x, X_1 = x-1) \times q$$

$$\boxed{w_x = p\, w_{x+1} + q\, w_{x-1}} \quad ----(1)$$

We have, $w_0 = 0$, $w_T = 1$ as terminal conditions

$p = q = \frac{1}{2} \Rightarrow 2w_x = w_{x+1} + w_{x-1}$

$\Rightarrow w_x - w_{x-1} = w_{x+1} - w_x$

$\vdots$

$w_1 - w_0 = w_2 - w_1$

Common difference is same $\Rightarrow$ Arithmetic Progression

$\therefore w_x = ax + b$

Substituting terminal conditions,

we get,

$$a = \frac{1}{T}, \quad b = 0$$

$$\therefore \omega_x = \frac{x}{T}$$

let, $p \neq q$

$$\therefore (1) \Rightarrow \omega_{x+1} - \omega_x = \frac{q}{p}(\omega_x - \omega_{x-1})$$

$$= \left(\frac{q}{p}\right)^x (\omega_1 - \omega_0)$$

$$= \left(\frac{q}{p}\right)^x \omega_1$$

$$\therefore \omega_{x+1} - \omega_1 = \sum_{k=1}^{x} (\omega_{k+1} - \omega_k)$$

$$= \sum_{k=1}^{x} \left(\frac{q}{p}\right)^k \omega_1.$$

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$$\omega_{x+1} = \omega_1 + \omega_1 \sum_{k1}^{x} \left(\frac{q}{p}\right)^k$$

$$\therefore \omega_{x+1} = \omega_1 \left(\frac{1-\left(\frac{q}{p}\right)^{x+1}}{1-\left(\frac{q}{p}\right)} \right), \text{ as } p \neq q$$

$$\therefore \omega_x = \omega_1 \left(\frac{1-\left(\frac{q}{p}\right)^{x}}{1-\left(\frac{q}{p}\right)} \right)$$

$$1 = \omega_T = \omega_1 \left(\frac{1-\left(\frac{q}{p}\right)^{T}}{1-\left(\frac{q}{p}\right)} \right)$$

$$\therefore \omega_1 = \frac{1-\left(\frac{q}{p}\right)}{1-\left(\frac{q}{p}\right)^{T}}$$

$$\therefore \boxed{\omega_x = \frac{1-\left(\frac{q}{p}\right)^{x}}{1-\left(\frac{q}{p}\right)^{T}}}$$

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(6) $P_{i,i+1} = a_i \,,\; P_{i,0} = 1 - a_i$

(a) $f_{00} = P(X_n = 0 \text{ for some } n \geq 1 \mid X_0 = 0)$

$= P(C_0 < \infty \mid X_0 = 0) = 1$

M.C. moves to

$$X_{n+1} = \begin{cases} X_n + 1, & \text{with } P_{i,i+1} = a_i \\ 0, & \text{with } P_{i,0} = 1 - a_i \end{cases}$$

$\therefore P(T_0 > n \mid X_0 = 0) = P$

$= P(X_1 = 1, X_2 = 2, \ldots, X_n = n \mid X_0 = 0)$

$= a_0 a_1 a_2 \cdots a_{n-1}$

$= \prod_{i=1}^{n-1} a_i = b_{n-1}$

Also, $b_0 = 1$

$$\therefore f_{00}(1) = 1 - a_0 = 1 - b_1$$

$$f_{00}(2) = a_0(1 - a_1) = b_1 - b_2$$

$$f_{00}(3) = a_0 a_1 (1 - a_2) = b_2 - b_3$$

$$\Rightarrow f_{00}(k) = b_{k-1} - b_k.$$

$$\therefore \sum_{k=1}^{\infty} f_{00}(k) = \lim_{n \to \infty} (1 - b_n)$$

$$\therefore f_{00} = 1 \iff \lim_{n \to \infty} b_n = 0$$

$\therefore$ Chain is persistent iff $\lim_{i \to \infty} b_i \to 0$

(b) A chain is positive persistent if,

$$\mu_i = E[N] = \sum_{n=1}^{\infty} n f_{ii}^{(n)} < \infty$$

Consider,

$$\sum_i b_i = \sum_{i=0}^{\infty} P(N > k)$$

$$= \sum_{i=0}^{\infty} \sum_{k=i+1}^{\infty} P(X = k)$$

$$= \sum_{k=1}^{\infty} \sum_{i=0}^{k} P(X = k)$$

$$= \sum_{k=1}^{\infty} k\, P(X = k) = E[N]$$

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For stationary distribution,

$$\pi_1 = a_0 \pi_0 = b_1 \pi_0$$

$$\pi_2 = a_1 \pi_1 = a_1 a_0 \pi_0 = b_2 \pi_0$$

$$\vdots$$

$$\pi_k = b_k \pi_0$$

$$\sum \pi_i = 1$$

$$\Rightarrow \pi_0 (1 + b_1 + b_2 + \cdots) = 1$$

$$\therefore \pi_0 = \frac{1}{\sum_{k=0}^{\infty} b_k} \quad (\text{as } b_0 = 1)$$

$$\therefore \boxed{\pi_i = \frac{b_i}{\sum_{k=0}^{\infty} b_k}}$$