Second Worksheet, MATH 7233

September 15, 2021

Definition. A *cycle* in a graph is a sequence of vertices $v_1, v_2, \ldots, v_k$, such that all of these are distinct, and there is an edge between v_i and v_{i+1} for all $i = 1, 2, \ldots, k-1$, as well as between v_k and v_1.

Definition. A graph is a "Connected, Cycle Free" graph (*CCF* for short) if it's connected and doesn't have any cycles.

Definition. A graph $G_1(V_1, E_1)$ is a *subgraph* of a graph $G(V, E)$ if $V_1 \subset V$ and $E_1 \subset E$. So a subgraph of G is anything that can be obtained from G by removing some (possibly zero) vertices and edges. When removing a vertex, you have to remove all the edges incident to that vertex, of course.

1. Let A denote the adjacency matrix of a graph G. What is the entry of A^2 in the ith column of the ith row? And how about the same entry of A^3?

2. Show that any graph in which every degree is at least 2 contains a cycle.

3. Show that a CCF graph on n vertices has exactly $n - 1$ edges! (Hint: use induction on n.)

* 4. How many CCF graphs are there up to isomorphism that have 3 vertices? And 4? And 5?

5. Show that any connected graph contains a *connecting CCF subgraph*: a subgraph that is CCF and has the same number of vertices as the original graph. (Hint: are there any edges you can throw away without disconnecting the graph?)

6. Show that if the number of edges is one less than the number of vertices in a connected graph, then it has no cycles. (Hint: use the previous problem!)

7. Show that every CCF graph has at least 2 nodes of degree exactly 1.

8. Show that if G as at least 5 nodes, then either G or G^c must contain a cycle.