Math 4570 - Homework 1 - Maxwell Arnold

Question 1. Show that the following sets of numbers are fields if the usual addition and multiplication of arithmetic are used:

(1) the set of all numbers of the form $a + b\sqrt{2}$ where a and b are rational numbers.

A field is a set F on which the operations addition and multiplication are defined such that $\forall\ (a+b\sqrt{2}), (c+d\sqrt{2}) \in F$ there are unique elements $(a+b\sqrt{2})+(c+d\sqrt{2})$ and $(a+b\sqrt{2})*(c+d\sqrt{2})$ in F such that the following conditions hold $\forall\ (a+b\sqrt{2}), (c+d\sqrt{2}), (x+y\sqrt{2}) \in F$:

(i.) Commutativity of Addition:

$$(a+b\sqrt{2})+(c+d\sqrt{2}) = (c+d\sqrt{2})+(a+b\sqrt{2}) \checkmark$$

(ii.) Associativity of Addition:

$$(a+b\sqrt{2}+c+d\sqrt{2})+x+y\sqrt{2} = c+d\sqrt{2}+(a+b\sqrt{2}+x+y\sqrt{2}) \checkmark$$

(iii.) Identity of Addition:

$$(0+0*\sqrt{2})+(a+b\sqrt{2}) = (a+b\sqrt{2})+(0+0*\sqrt{2}) = (a+b\sqrt{2})$$

$$\therefore \exists\ e \in F\ s.t.\ e = \text{Identity of Addition} = (0+0*\sqrt{2}) = 0 \checkmark$$

(iv.) Inverse of Addition:

$$(a+b\sqrt{2})+(-a-b\sqrt{2}) = 0$$

$$\therefore \forall\ a+b\sqrt{2} \in F \text{ the inverse of addition is } -a-b\sqrt{2} \checkmark$$

(v.) Commutativity of Multiplication:

$$(a+b\sqrt{2})*(c+d\sqrt{2}) = (c+d\sqrt{2})*(a+b\sqrt{2}) \checkmark$$

(vi.) Associativity of Multiplication:

$$((a+b\sqrt{2})*(c+d\sqrt{2}))*x+y\sqrt{2} = c+d\sqrt{2}*((a+b\sqrt{2})*(x+y\sqrt{2}))$$

$$(a+b\sqrt{2})*(c+d\sqrt{2})*(x+y\sqrt{2}) = (c+d\sqrt{2})*(a+b\sqrt{2})*(x+y\sqrt{2}) \checkmark$$

(vii.) Identity of Multiplication:

$$(a+b\sqrt{2})*(1+0*\sqrt{2}) = (1+0*\sqrt{2})*(a+b\sqrt{2}) = (a+b\sqrt{2})*1 = (a+b\sqrt{2})$$

$$\therefore \forall\ (a+b\sqrt{2}) \in F \text{ the identity of multiplication is } (1+0*\sqrt{2}) = 1 \checkmark$$

(viii.) Inverse of Multiplication:

$$(a+b\sqrt{2})^{-1} := (x+y\sqrt{2})$$

$$(a+b\sqrt{2})*(x+y\sqrt{2}) = 1$$

$$(ax+2by)+(ay+bx)\sqrt{2} = 1$$

$$\therefore \text{ we need to solve the linear system } ax+2by = 1 \text{ and } ay+bx = 0 \text{ for } x, y$$

We need to solve the augmented matrix $A = \begin{bmatrix} a & 2b & 1 \\ b & a & 0 \end{bmatrix}$ to find the values of x and y :

$$R_1 = \frac{R_1}{a} \Rightarrow \begin{bmatrix} 1 & \frac{2b}{a} & \frac{1}{a} \\ b & a & 0 \end{bmatrix}$$

$$R_2 = R_2 - R_1 * b \Rightarrow \begin{bmatrix} 1 & \frac{2b}{a} & \frac{1}{a} \\ 0 & a - \frac{2b^2}{a} & -\frac{b}{a} \end{bmatrix}$$

$$R_2 = R_2 * \frac{1}{a - \frac{2b^2}{a}} \Rightarrow \begin{bmatrix} 1 & \frac{2b}{a} & \frac{1}{a} \\ 0 & 1 & -\frac{b}{a^2-2b^2} \end{bmatrix}$$

$$R_1 = R_1 - R_2 * \frac{2b}{a} \Rightarrow \begin{bmatrix} 1 & 0 & \frac{a}{a^2-2b^2} \\ 0 & 1 & -\frac{b}{a^2-2b^2} \end{bmatrix} = \text{RREF}(A)$$

$$\therefore x = \frac{a}{a^2 - 2b^2} \text{ and } y = -\frac{b}{a^2 - 2b^2}$$

$$(a + b\sqrt{2})^{-1} = (\frac{a}{a^2 - 2b^2} - \frac{b}{a^2 - 2b^2}\sqrt{2}) \checkmark$$

(ix.) Distributivity of Multiplication:

$$(a + b\sqrt{2}) * (c + d\sqrt{2} + x + y\sqrt{2}) = ((a + b\sqrt{2}) * (c + d\sqrt{2})) + ((a + b\sqrt{2}) * (x + y\sqrt{2}))$$

$$ac+ad\sqrt{2}+ax+ay\sqrt{2}+bc\sqrt{2}+2bd+bx\sqrt{2}+2by = (ac+ad\sqrt{2}+bc\sqrt{2}+2bd)+(ax+ay\sqrt{2}+bx\sqrt{2}+2by) \checkmark$$

$\therefore$ the set of all numbers of the form $a + b\sqrt{2}$ where a and b are rational numbers is a field on multiplication and addition $\square$

(2) the set of all numbers of the form $a+b\sqrt{-1}$ where a and b are real numbers. What is this field?

This particular field is the set of all complex numbers, $\mathbb{C}$. Complex numbers are denoted $a + bi$, where $i = \sqrt{-1}$. A field is a set F on which the operations addition and multiplication are defined such that $\forall$ $(a + bi), (c + di) \in F$ there are unique elements $(a + bi) + (c + di)$ and $(a + bi) * (c + di)$ in F such that the following conditions hold $\forall$ $(a + bi), (c + di), (x + yi) \in F$:

(i.) Commutativity of Addition:

$$(a + bi) + (c + di) = (c + di) + (a + bi) \checkmark$$

(ii.) Associativity of Addition:

$$(a + bi + c + di) + x + yi = c + di + (a + bi + x + yi) \checkmark$$

(iii.) Identity of Addition:

$$(0 + 0 * i) + (a + bi) = (a + bi) + (0 + 0 * i) = (a + bi)$$

$$\therefore \exists\ e \in F\ s.t.\ e = \text{Identity of Addition} = (0 + 0 * i) = 0 \checkmark$$

(iv.) Inverse of Addition:

$$(a + bi) + (-a - bi) = 0$$

$$\therefore \forall\ a + bi \in F \text{ the inverse of addition is } -a - bi\ \checkmark$$

(v.) Commutativity of Multiplication:

$$(a + bi) * (c + di) = (c + di) * (a + bi)\ \checkmark$$

(vi.) Associativity of Multiplication:

$$((a + bi) * (c + di)) * x + yi = c + di * ((a + bi) * (x + yi))$$

$$(a + bi) * (c + di) * (x + yi) = (c + di) * (a + bi) * (x + yi)\ \checkmark$$

(vii.) Identity of Multiplication:

$$(a + bi) * (1 + 0 * i) = (1 + 0 * i) * (a + bi) = (a + bi) * 1 = (a + bi)$$

$$\therefore \forall\ (a + bi) \in F \text{ the identity of multiplication is } (1 + 0 * i) = 1\ \checkmark$$

(viii.) Inverse of Multiplication:

$$(a + bi)^{-1} := (x + yi)$$

$$(a + bi) * (x + yi) = 1$$

$$(ax - by) + (ay + bx)i = 1$$

$\therefore$ we need to solve the linear system $ax - by = 1$ and $ay + bx = 0$ for x, y

We need to solve the augmented matrix $A = \begin{bmatrix} a & -b & 1 \\ b & a & 0 \end{bmatrix}$ to find the values of x and y :

$$R_1 = \frac{R_1}{a} \Rightarrow \begin{bmatrix} 1 & \frac{-b}{a} & \frac{1}{a} \\ b & a & 0 \end{bmatrix}$$

$$R_2 = R_2 - R_1 * b \Rightarrow \begin{bmatrix} 1 & \frac{-b}{a} & \frac{1}{a} \\ 0 & a + \frac{b^2}{a} & -\frac{b}{a} \end{bmatrix}$$

$$R_2 = R_2 * \frac{1}{a + \frac{b^2}{a}} \Rightarrow \begin{bmatrix} 1 & \frac{-b}{a} & \frac{1}{a} \\ 0 & 1 & -\frac{b}{a^2+b^2} \end{bmatrix}$$

$$R_1 = R_1 + R_2 * \frac{b}{a} \Rightarrow \begin{bmatrix} 1 & 0 & \frac{a}{a^2+b^2} \\ 0 & 1 & -\frac{b}{a^2+b^2} \end{bmatrix} = \text{RREF}(A)$$

$$\therefore x = \frac{a}{a^2 + b^2} \text{ and } y = -\frac{b}{a^2 + b^2}$$

$$(a + bi)^{-1} = (\frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i)\ \checkmark$$

(ix.) Distributivity of Multiplication:

$$(a + bi) * (c + di + x + yi) = ((a + bi) * (c + di)) + ((a + bi) * (x + yi))$$

$$ac + adi + ax + ayi + bci - bd + bxi - by = (ac + adi + bci - bd) + (ax + ayi + bxi - by) \checkmark$$

$\therefore$ the set of all numbers of the form $a + b\sqrt{-1}$ where a and b are rational numbers is a field on multiplication and addition $\square$

Question 2. Show that the set of all $n \times n$ matrices $\mathbb{R}^{n\times n}$ with the usual matrix addition and multiplication is not a field if $n > 1$.

The set of all $n \times n$ matrices is not a field with matrix addition and multiplication because the axiom of commutativity of multiplication does not hold true. Proof by counterexample. Suppose we have the following 2×2 matrices:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$AB \stackrel{?}{=} BA$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix} \neq \begin{bmatrix} ae + cf & be + df \\ ag + ch & gb + dh \end{bmatrix} \checkmark$$

Question 3. Write down the two operations on field $\mathbb{Z}_3$.

$+$	[0]	[1]	[2]
[0]	0	1	2
[1]	1	2	0
[2]	2	0	1

$\times$	[0]	[1]	[2]
[0]	0	0	0
[1]	0	1	2
[2]	0	2	1

Question 4. Show that $\mathbb{C}$ is a field with the usual sum, scalar product and product.

This question has already been solved in **Question 1** - (2)

Question 5. Determine which of the matrices below are in reduced row-echelon form.

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}; B = \begin{bmatrix} 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; C = \begin{bmatrix} 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 5 \end{bmatrix}; D = \begin{bmatrix} 0 & 1 & 0 & 2 & 4 \end{bmatrix}; E = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}$$

(A): Yes, (B): Yes, (C): No, (D): Yes, (E): No

Question 6. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ be two matrices over the field $\mathbb{Z}_2$. Compute $A + B$, A^2, and $A \times B$ over the field $\mathbb{Z}_2$.

(1) $A + B = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 1 & 2 & 0 \end{bmatrix}$

(2) $A^2 = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

(3) $A \times B = \begin{bmatrix} 2 & 3 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Question 7. For which values of t does the matrix $A = \begin{bmatrix} 6 & -1 & 1 \\ t & 0 & 1 \\ 0 & 1 & t \end{bmatrix}$ NOT have an inverse?

The determinant of $A = (t-2)(t+3)$. In order for A to have an inverse, the determinant cannot be equal to zero. Therefore, A does not have an inverse for $t = 2$ and $t = -3$.

Question 8. Find all values of h that make the following matrices **consistent**, i.e., has at least one solution.

(a) $\begin{bmatrix} 1 & h & | & 4 \\ 3 & 6 & | & 8 \end{bmatrix} \to x_1 = \frac{8h-24}{3h-6}, x_2 = \frac{-4}{6-3h} \to$ the matrix is consistent when $h \neq 2$

(b) $\begin{bmatrix} -4 & 12 & | & h \\ 2 & -6 & | & -3 \end{bmatrix} \to$ The RREF of the matrix is $\begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so the matrix is consistent when $x_1 - 3x_2 = 0$

Question 9. We say that two $m \times n$ matrices in reduced row-echelon form are the same type if they have the same number of leading 1's in the same position.

(1) How many types of 3×2 matrices in reduced row-echelon form? $\to 4$

(2) How many types of 2×3 matrices in reduced row-echelon form? $\to 5$

(3) Find all 4×1 matrices in reduced row-echelon form:

$$\text{Rank 0: } \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{Rank 1: } \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{Rank 2: } \begin{bmatrix} * \\ 1 \\ 0 \\ 0 \end{bmatrix}, \text{Rank 3: } \begin{bmatrix} * \\ * \\ 1 \\ 0 \end{bmatrix}, \text{Rank 4: } \begin{bmatrix} * \\ * \\ * \\ 1 \end{bmatrix}$$

Question 10. For which values of a, b, c, d, and e is the following matrix in reduced row-echelon form?

$$A = \begin{bmatrix} 1 & a & b & 3 & 0 & -2 \\ 0 & 0 & c & 1 & d & 3 \\ 0 & e & 0 & 0 & 1 & 1 \end{bmatrix} \to a = \text{any value}, b = \text{any value}, c = 1 \text{ or } 0, d = \text{any value}, e = 0$$

Question 11. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix}$

(1) Calculate **rref**(A) over $\mathbb{R}$. Solve $A\vec{x} = \vec{0}$ and write all solutions in parametric vector forms.

$$\text{RREF}(A)=\begin{bmatrix}1&0&0&|&\frac{6}{7}\\0&1&0&|&\frac{8}{7}\\0&0&1&|&\frac{2}{7}\end{bmatrix},\vec{x}=\begin{bmatrix}\frac{6}{7}\\\frac{8}{7}\\\frac{2}{7}\end{bmatrix}$$

(2) Calculate **rref**(A) over field $\mathbb{Z}_7$.

$$\text{RREF}(A)\text{ in }\mathbb{Z}_7=\begin{bmatrix}1&0&4&|&0\\0&1&3&|&0\\0&0&0&|&1\end{bmatrix},\vec{x}=\begin{bmatrix}\frac{2}{3}\\\frac{1}{2}\\-\frac{1}{6}\end{bmatrix}$$

(3 Calculate **rref**(A) over field $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

$$\text{RREF}(A)\text{ in }\mathbb{Z}_2=\begin{bmatrix}1&0&0&|&0\\0&1&0&|&0\\0&0&1&|&0\end{bmatrix},\vec{x}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\text{RREF}(A)\text{ in }\mathbb{Z}_3=\begin{bmatrix}1&0&0&|&0\\0&1&0&|&2\\0&0&1&|&2\end{bmatrix},\vec{x}=\begin{bmatrix}0\\2\\2\end{bmatrix}$$

(4) Is it possible that a matrix M has a different rank over different fields $\mathbb{Z}_p$?

Yes, the rank of RREF(A) in $\mathbb{Z}_7$ is 2, but rank of RREF(A) is 3.

Question 12. (Solve a linear system over field $\mathbb{Z}_7$.) Let $A=\begin{bmatrix}3&1&4\\5&2&6\\0&5&2\end{bmatrix}$ and $\vec{b}=\begin{bmatrix}1\\5\\1\end{bmatrix}$.

(1) Calculate **rref**$(A|\vec{b})$ over field $\mathbb{Z}_7$.

The following python code will give us **rref**$(A|\vec{b})$ over field $\mathbb{Z}_7$:

GF7 = galois.GF(7) # Define the $\mathbb{Z}_7$ field
A = GF7([[3, 1, 4, 1], [5, 2, 6, 5], [0, 5, 2, 1]]) # Define the augmented matrix

$$\text{GF7.row\_reduce(A)}=\begin{bmatrix}1&0&0&4\\0&1&0&3\\0&0&1&0\end{bmatrix}$$

(2) Find the solution to the linear system $A\vec{x}=\vec{b}$ mod 7.

From part (1) we know that $\vec{x}=[4,3,0]$

Question 13. (Use python) Solve the linear system and write solutions in parametric vector forms:

$$\begin{bmatrix}3&11&19&|&-2\\7&23&39&|&10\\-4&-3&-2&|&6\end{bmatrix}\to\text{RREF}(A)=\begin{bmatrix}1&0&-1&|&0\\0&1&2&|&0\\0&0&0&|&1\end{bmatrix}\to\ \vec{x}=\begin{bmatrix}t\\-2t\\t\end{bmatrix}$$

Question 14. (Use python) Solve the linear system and write solutions in parametric vector forms:

$$\begin{bmatrix} 3 & 6 & 9 & 5 & 25 & | & 53 \\ 7 & 14 & 21 & 9 & 53 & | & 105 \\ -4 & -8 & -12 & 5 & -10 & | & 11 \end{bmatrix} \to \mathrm{RREF}(A) = \begin{bmatrix} 1 & 2 & 3 & 0 & 5 & | & 6 \\ 0 & 0 & 0 & 1 & 2 & | & 7 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} \to \vec{x} = \begin{bmatrix} t + 11u - 20 \\ 13 - 4u \\ u \\ 7 - 2t \\ t \end{bmatrix}$$

Question 15. (Use python) Solve the linear system and write solutions in parametric vector forms:

$$\begin{bmatrix} 2 & 4 & 3 & 5 & 6 & | & 37 \\ 4 & 8 & 7 & 5 & 2 & | & 74 \\ -2 & -4 & 3 & 4 & -5 & | & 20 \\ 1 & 2 & 2 & -1 & 2 & | & 126 \\ 5 & -10 & 4 & 6 & 4 & | & 24 \end{bmatrix} \to \mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & | & -\frac{126,221}{4,340} \\ 0 & 1 & 0 & 0 & 0 & | & -\frac{57,409}{8,680} \\ 0 & 0 & 1 & 0 & 0 & | & \frac{21,695}{434} \\ 0 & 0 & 0 & 1 & 0 & | & -\frac{12,659}{434} \\ 0 & 0 & 0 & 0 & 1 & | & \frac{8,499}{434} \end{bmatrix} \vec{x} = \begin{bmatrix} -\frac{126,221}{4,340} \\ -\frac{57,409}{8,680} \\ \frac{21,695}{434} \\ -\frac{12,659}{434} \\ \frac{8,499}{434} \end{bmatrix}$$

Question 16. **(1)** If A, B, C are $n \times n$ matrices and $ABC = I_n$ is each of the matrices invertible? What are their inverses?

$\to C^{-1} = AB$ and $A^{-1} = BC$
$\to ABC = (BC)^{-1}B(AB)^{-1}$
$\to ABC = C^{-1}B^{-1}BB^{-1}A$
$\to ABC = C^{-1}B^{-1}A$
$\to CABC = B^{-1}A$
$\to CABCA^{-1} = B^{-1}$

(2) Suppose A and B are $n \times n$ matrices. If AB is invertible, are both A and B invertible?

Any $n \times n$ matrix M is invertible if and only if $\det(M) \neq 0$. Note that the $\det(AB) = \det(A)$*$\det(B)$. Since AB is invertible, the $\det(AB) \neq 0$. Therefore, the $\det(A) \neq 0$ and the $\det(B) \neq 0$.

Question 17. Provide a counter-example to the statement: for any $n \times n$ matrices A and B, $(AB)^2 = A^2B^2$.

$$A = \begin{bmatrix} 1 & 10 \\ -3 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 4 \\ 7 & 3 \end{bmatrix}$$

$$(AB)^2 = \begin{bmatrix} 5456 & 2244 \\ 528 & 308 \end{bmatrix} \text{ and } A^2B^2 = \begin{bmatrix} 122 & 530 \\ -1198 & -1142 \end{bmatrix}$$

Question 18. Find an example of a 2×2 nonidentity matrix whose transpose is its inverse.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Question 19. Here are a couple of new definitions: An $n \times n$ matrix is *symmetric* provided $A^T = A$ and *skew-symmetric* provided $A^T = -A$.

(1) Give examples of symmetric and skew-symmetric 2×2, 3×3, and 4×4 matrices.

(**a**) Symmetric: $\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$, Skew-Symmetric: $\begin{bmatrix} 0 & 8 \\ -8 & 0 \end{bmatrix}$

(**b**) Symmetric: $\begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$, Skew-Symmetric: $\begin{bmatrix} 0 & 4 & 3 \\ -4 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$

(**c**) Symmetric: $\begin{bmatrix} 0 & 3 & 2 & 1 \\ 3 & 0 & 3 & 2 \\ 2 & 3 & 0 & 3 \\ 1 & 2 & 3 & 0 \end{bmatrix}$, Skew-Symmetric: $\begin{bmatrix} 0 & 3 & 2 & 1 \\ -3 & 0 & 3 & 2 \\ -2 & -3 & 0 & 3 \\ -1 & -2 & -3 & 0 \end{bmatrix}$

(2) What can you say about the main diagonal of a skew-symmetric matrix?

The main diagonal of a skew-symmetric matrix must be all zeroes. This is because transposing a matrix does not change the value of its diagonal but negating a matrix will change the value. Therefore, we need a value in the diagonals where $x = -x$, and this is only true for $x = 0$.

(3) Give an example of a matrix that is both symmetric and skew-symmetric.

Symmetric and Skew-Symmetric: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(4) Prove that for any $n \times n$ matrix, the matrices $A + A^T$, AA^T, and A^TA are symmetric and $A - A^T$ is skew-symmetric.

(**a**) $A + A^T$ is symmetric if and only if $A + A^T = (A + A^T)^T$.
The properties of transpose tell us that for any $n \times n$ matrices A, B:

$$(A + B)^T = A^T + B^T \text{ and } (B^T)^T = B$$

$\therefore (A + A^T)^T = A^T + (A^T)^T = A^T + A$
$\rightarrow A + A^T = A^T + A\checkmark$

(**b**) $A \times A^T$ is symmetric if and only if $A \times A^T = (A \times A^T)^T$.
The properties of transpose tell us that for any $n \times n$ matrices A, B:

$$(A \times B)^T = B^T \times A^T \text{ and } (B^T)^T = B$$

$\therefore (A \times A^T)^T = (A^T)^T \times A^T = A \times A^T$
$\rightarrow A \times A^T = A \times A^T\checkmark$

(**c**) $A^T \times A$ is symmetric if and only if $A^T \times A = (A^T \times A)^T$.
The properties of transpose tell us that for any $n \times n$ matrices A, B:

$$(A \times B)^T = B^T \times A^T \text{and} (B^T)^T = B$$

$\therefore (A^T \times A)^T = A^T \times (A^T)^T = A^T \times A$
$\rightarrow A^T \times A = A^T \times A\checkmark$

(d) $A - A^T$ is skew-symmetric if and only if $(A - A^T)^T = -(A - A^T)$.
The properties of transpose tell us that for any $n \times n$ matrices A, B:

$$(A - B)^T = A^T - B^T \text{and} (B^T)^T = B$$

$\therefore (A - A^T)^T = A^T - (A^T)^T = A^T - A$

The properties of matrices tell us that for any scalar k and any $n \times n$ matrices A, B:

$$k(A + B) = kA + kB$$

$\therefore -(A - A^T) = -A + A^T = A^T - A$
$\rightarrow A^T - A = A^T - A\checkmark$

(5) Prove that any $n \times n$ can be written as the sum of a symmetric and skew-symmetric matrices.

From the previous problem, we know that $A + A^T$ is symmetric and $A - A^T$ is skew-symmetric. Also, we know that multiplying a matrix by a positive and non-zero scalar will not change whether a matrix is symmetric or skew-symmetric. Therefore, if the following statement is true then it must also be true that any square matrix can be written as the sum of a symmetric and skew-symmetric matrices.

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$
$$A = \frac{1}{2}A + \frac{1}{2}A^T + \frac{1}{2}A - \frac{1}{2}A^T$$
$$A = \frac{1}{2}A + \frac{1}{2}A$$
$$A = A\checkmark$$

Question 20. Mark each of the following functions $F : \mathbb{R} \rightarrow \mathbb{R}$ injective, surjective, or bijective, as is most appropriate.

(a) $F(x) = x^2 \rightarrow$ injective
(b) $F(x) = \frac{x^3}{x^2+1} \rightarrow$ bijective
(c) $F(x) = x(x-1)(x-2) \rightarrow$ injective
(d) $F(x) = e^x + 2 \rightarrow$ bijective

Question 21. Skipped Question (1/3)

Question 22. Skipped Question (2/3)

Question 23. Skipped Question (3/3)

Question 24. Let I_n be the $n \times n$ identity matrix. Let $\vec{u}$ be a **unit** vector in $\mathbb{R}^n$. Define $H_n = I_n - 2\,\vec{u}\vec{u}^T$. Here a unit vector $\vec{u}$ means that norm $\left\|\vec{u}\right\| = 1$

(1) Is H_n a symmetric matrix? Prove your result.

H_n is symmetric if and only if $H = H^T$:

$$H^T = (I - 2uu^T)^T = I^T - 2(uu^T)^T = I - 2(u^T)^T u^T = I - 2u^T = H\checkmark$$

(2) Is H_n an orthogonal matrix? (i.e. is $H_n^T H_n = I_n$).

H_n is orthogonal if and only if $H^T H = I$:

$$\begin{aligned} H^T H = HH = (I - 2uu^T)(I - 2uu^T) &= I - 2uu^T - 2uu^T + 4(uu^T)(uu^T) \\ &= I - 4uu^T + 4u(u^T u)u^T = I - 4uu^T + 4uu^T = I\checkmark \end{aligned}$$

(3) What is H_n^2?

$$H^2 = HH = I \text{ (From (2))}$$

(4) What is $H_n \vec{u}$?

$$Hu = (I - 2uu^T)u = u - (2uu^T)u = u - 2u(u^T u) = u - 2u = -u$$

(5) Suppose $\vec{u} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ Write down H_3 and H_4.

$$H_3 = I_3 - 2uu^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\left(\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$H_4 = I_4 - 2uu^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - 2\left(\frac{1}{\sqrt{4}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \times \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$