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(1) $$V = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\vec{u}_1 = \vec{V}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{e}_1 = \frac{\vec{u}_1}{\|\vec{u}_1\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{u}_2 = \vec{v}_2 - \frac{\vec{u}_1 \cdot \vec{v}_2}{\|\vec{u}_1\|^2}\vec{u}_1 = \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{e}_2 = \frac{\vec{u}_2}{\|\vec{u}_2\|} = \sqrt{\frac{3}{5}} \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{u}_3 = \vec{v}_3 - \frac{\vec{u}_1 \cdot \vec{v}_3}{\|\vec{u}_1\|^2}\vec{u}_1 - \frac{\vec{u}_2 \cdot \vec{v}_3}{\|\vec{u}_2\|^2}\vec{u}_2$$

$$= \begin{bmatrix} 1/5 \\ -3/5 \\ 2/5 \\ 1/5 \\ 1 \end{bmatrix}$$

$$\vec{e}_3 = \frac{\vec{u}_3}{\|\vec{u}_3\|} = \sqrt{\frac{5}{8}} \begin{bmatrix} 1/5 \\ -3/5 \\ 2/5 \\ 1/5 \\ 1 \end{bmatrix}$$

$\therefore$ orthonormal basis $= \left\{ \vec{e}_1 , \vec{e}_2 , \vec{e}_3 \right\}$

(2)

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\left\{ \begin{array}{l} \text{Basis of} \\ \text{orthogonal complement} \end{array} \right\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\vec{u}_4 = \vec{v}_4 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \; ; \; \vec{e}_4 = \frac{\vec{u}_4}{\|\vec{u}_4\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{u}_5 = \vec{v}_5 - \frac{(\vec{u}_4 \cdot \vec{v}_5)}{\|\vec{u}_4\|^2} \vec{u}_4$$

$$= \vec{v}_5 - \frac{1}{3} \vec{u}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1/3 \\ 1 \\ -2/3 \\ -1/3 \\ 1 \end{bmatrix} \Rightarrow \vec{e}_5 = \sqrt{\frac{3}{8}} \begin{bmatrix} -1/3 \\ 1 \\ -2/3 \\ -1/3 \\ 1 \end{bmatrix}$$

$$\left.\begin{array}{l} \text{orthonormal basis of} \\ \text{orthogonal complement} \\ \text{of } V \end{array}\right\} = \left\{ \vec{e}_4 , \vec{e}_5 \right\}$$

(3) $\vec{e}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \sqrt{\frac{3}{5}} \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \\ 1 \\ 0 \end{bmatrix}$, $\vec{e}_3 = \sqrt{\frac{5}{8}} \begin{bmatrix} 1/5 \\ -3/5 \\ 2/5 \\ 1/5 \\ 1 \end{bmatrix}$

$$\text{proj}_V \vec{y} = (\vec{e}_1 \cdot \vec{y})\vec{e}_1 + (\vec{e}_2 \cdot \vec{y})\vec{e}_2 + (\vec{e}_3 \cdot \vec{y})\vec{e}_3$$

$$= \frac{1}{3} \times 3 \times \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{3}{5} \times 0 + \frac{5}{8} \times 0$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

(4) $\vec{y}_1$ $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ (answer in part 3)

$$\vec{e}_4 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{e}_5 = \sqrt{\frac{3}{8}} \begin{bmatrix} -1/3 \\ 1 \\ -2/3 \\ -1/3 \\ 1 \end{bmatrix}$$

$$y_2 = \text{Proj}_{V1} \vec{y} = (\vec{e}_4 \cdot \vec{y})\vec{e}_4 + (\vec{e}_5 \cdot \vec{y})\vec{e}_5$$

$$= \frac{1}{3} \times 4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \frac{3}{8} \times \frac{8}{3} \begin{bmatrix} -1/3 \\ 1 \\ -2/3 \\ -1/3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore \; y = y_1 + y_2$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

2

$$\vec{u}_i \cdot \vec{u}_j = 0 \quad \forall \; i \neq j$$

$$\text{rank}\left([\vec{u}_1 \; \vec{u}_2 \; \vec{u}_3 \; \vec{u}_4] \right) = 4$$

$\Rightarrow \{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ form orthogonal basis of $\mathbb{R}^4$

(b) $\vec{e}_1 = \frac{1}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}$, $\vec{e}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}$

$\vec{e}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\0\\1\end{bmatrix}$, $\vec{e}_4 = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

$$\vec{v} = (\vec{e}_1 \cdot \vec{v})\vec{e}_1 + (\vec{e}_2 \cdot \vec{v})\vec{e}_2 + (\vec{e}_3 \cdot \vec{v})\vec{e}_3 + (\vec{e}_4 \cdot \vec{v})\vec{e}_4$$

$$= \frac{1}{2}(-2)\vec{e}_1 + \frac{1}{\sqrt{2}}(-2\ \vec{e}_2 + \frac{1}{\sqrt{2}}(2)\vec{e}_3 + \frac{1}{2}(10)\vec{e}_4$$

$$= -\vec{e}_1 - \sqrt{2}\,\vec{e}_2 + \sqrt{2}\,\vec{e}_3 + 5\,\vec{e}_4$$

(3)

(1) $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 1 \\ -1 & 2 & 1 & 0 \end{bmatrix}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & -1/3 \end{bmatrix}$$

$$\therefore \vec{v}_4 = \begin{bmatrix} -1/3 \\ -1/3 \\ 1/3 \\ 1 \end{bmatrix}$$

(2)

$$\mathrm{Proj}_V \vec{y} = \frac{(\vec{v}_1 \cdot \vec{y})}{\|\vec{v}_1\|^2}\vec{v}_1 + \frac{(\vec{v}_2 \cdot \vec{y})}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$+ \frac{(\vec{v}_3 \cdot \vec{y})}{\|\vec{v}_3\|^2}\vec{v}_3$$

$$= \frac{1}{2}(2)\vec{v}_1 + \frac{1}{4}(2)\vec{v}_2$$

$$+ \frac{1}{6}(2)\vec{v}_3$$

$$= \frac{1}{6}\begin{bmatrix} 7 \\ 7 \\ 5 \\ 3 \end{bmatrix}$$

4 $\vec{u} = \begin{bmatrix} a \\ d \\ f \end{bmatrix}, \vec{v} = \begin{bmatrix} b \\ 1 \\ g \end{bmatrix}, \vec{w} = \begin{bmatrix} c \\ e \\ 1/2 \end{bmatrix}$

$\|\vec{v}\| = 1 \Rightarrow b = g = 0$

$\vec{v} \cdot \vec{w} = 0 \Rightarrow e = 0$

$\|\vec{w}\| = 1 \Rightarrow c = \frac{\sqrt{3}}{2}$

$\vec{u} \cdot \vec{v} = 0 \Rightarrow d = 0$

$\vec{u} \cdot \vec{w} = 0 \Rightarrow \frac{\sqrt{3}}{2} a + \frac{f}{2} = 0$

$f = -\sqrt{3}\, a$

$\|\vec{u}\| = 1 \Rightarrow a^2 + 3a^2 = 1$

$\therefore \; a = \pm \frac{1}{2}$

$\therefore \quad f = \mp \dfrac{\sqrt{3}}{2}$

$\therefore (a, b, c, d, e, f, g) = \left(\dfrac{1}{2}, 0, \dfrac{\sqrt{3}}{2}, 0, 0, -\dfrac{\sqrt{3}}{2}, 0 \right)$

(or)

$\left(-\dfrac{1}{2}, 0, \dfrac{\sqrt{3}}{2}, 0, 0, +\dfrac{\sqrt{3}}{2}, 0 \right)$

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(5)

(a) $\quad -3x + y + z = 0$

$z = 3x - y$

One possible basis is, $\left\{ \underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}} \right\}$

$$\vec{u}_1 = \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\vec{u}_2 = \vec{v}_2 - \frac{(\vec{v}_2 \cdot \vec{u}_1)}{\|\vec{u}_1\|^2} \left(\vec{u}_1 \right)$$

$$= \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \frac{3}{10} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3/10 \\ 1 \\ -1/10 \end{bmatrix}$$

orthogonal basis $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3/10 \\ 1 \\ -1/10 \end{bmatrix} \right\}$

(b)

$$\text{proj}_V \vec{y} = \frac{(\vec{y} \cdot \vec{u}_1)}{\|\vec{u}_1\|^2} \vec{u}_1 + \frac{(\vec{y} \cdot \vec{u}_2)}{\|\vec{u}_2\|} \vec{u}_2$$

$$= \begin{bmatrix} 8/11 \\ 12/11 \\ 12/11 \end{bmatrix}$$

$$\text{proj}_{V^\perp} \vec{y} = \vec{y} - \text{proj}_V \vec{y}$$

$$= \begin{bmatrix} 3/11 \\ -1/11 \\ -1/11 \end{bmatrix}$$

$\therefore$ Shortest distance $= \|\text{proj}_{V^\perp} \vec{y}\| = \frac{\sqrt{11}}{11} = \boxed{\frac{1}{\sqrt{11}}}$

(6)

$$L(\vec{u}) = A\vec{u}$$

"L" is orthogonal in $R^n \to R^n$ $\Rightarrow$ $\boxed{AA^T = I}$

$$\therefore \langle L(\vec{v}), L(\vec{w}) \rangle = L(\vec{v}) \cdot L(\vec{w})$$

$$= A\vec{v} \cdot A\vec{w}$$

$$= (A\vec{v})^T A\vec{w}$$

$$= \vec{v}^T \underbrace{A^T A}_{\longrightarrow I_n} \vec{w}$$

$$= \vec{v}^T \vec{w}$$

$$= \langle \vec{v}, \vec{w} \rangle$$

$$\| L(\vec{v}) \|^2 = L(\vec{v})^T L(\vec{v})$$

$$= (A\vec{v})^T A\vec{v}$$

$$= \vec{v}^T \underbrace{A^T A}_{\longrightarrow I_n} \vec{v}$$

$$= \vec{v}^T \vec{v}$$

$$= \|\vec{v}\|^2$$

$\therefore$ Linear transformations preserve products and angles

$$\therefore \theta(\vec{v}, \vec{w}) = \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\| \times \|\vec{w}\|}$$

$\therefore$ angle is also preserved

Let, $L : R^2 \to R^2$ such that

$$L(\vec{u}) = 3\vec{u}$$

$$\theta_{(L(\vec{v}), L(\vec{w}))} = \frac{3\vec{v} \cdot 3\vec{w}}{3\|\vec{v}\| \cdot 3\|\vec{w}\|}$$

$$= \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|} = \theta_{(\vec{v}, \vec{w})}$$

$\therefore$ angle is preserved but "L" is not orthogonal

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(1) $$(A^2)^T A^2 = A^T \underline{A^T A} A \xrightarrow{} I$$

True

$$= A^T A = I$$

(2) $(A^T)^T A^T = AA^T = I$

True

(3) A is orthogonal $\Rightarrow$ A^T is orthogonal

True

$\therefore$ Rows of "A" are orthogonal

(4) $\left\langle T\begin{bmatrix}1\\2\\3\end{bmatrix}, T\begin{bmatrix}1\\1\\1\end{bmatrix}\right\rangle = \left\langle\begin{bmatrix}3\\2\\1\end{bmatrix}, \begin{bmatrix}0\\3\\1\end{bmatrix}\right\rangle$

$= \boxed{7}$ — (1)

$\left\langle\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\rangle = \boxed{6}$ — (2)

(1) $\neq$ (2) $\rightarrow$ False

(5) $(A^T A)^T = A^T \times (A^T)^T = A^T A$

$(AA^T)^T = (A^T)^T \times A^T = AA^T$

$\therefore$ Both are symmetric

True

(6) $(AB)^T = B^T A^T = BA$

$\neq AB$ (mostly)

Counter example

$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$

$AB = \begin{pmatrix} 8 & 11 \\ 13 & 18 \end{pmatrix}$; $BA = \begin{pmatrix} 8 & 13 \\ 11 & 18 \end{pmatrix}$

$(AB)^T = BA \neq AB$ False

(7) $(A^K)^T = (A^{K-1}A)^T = A^T(A^{K-1})^T$

$= A\,(A^{K-1})^T$

$f(K) = Af(K-1)\,,\quad f(1) = A$

$\Rightarrow f(K) = A^K$

True

(8) $A^{-1}A = I_n = I_n^T = (AA^{-1})^T = (A^{-1})^T A^T$

$= (A^{-1})^T A$

Right multiply by A^{-1}

$$A^{-1}\underbrace{A\,A^{-1}}_{I_n} = (A^{-1})^T\underbrace{A\,A^{-1}}_{I_n}$$

$$\Rightarrow \quad A^{-1} = (A^{-1})^T$$

True

(g) $\quad A^T = A^{-1} \quad ; \quad S^T = S^{-1}$

Assumption
A, S are $n \times n$

$$(S^{-1}AS)^T = S^T A^T (S^{-1})^T$$

$$= S^{-1} A^T S$$

$$\therefore \quad (S^{-1}AS)(S^{-1}AS)^T = S^{-1}A\underbrace{S\,S^{-1}}A^T S$$

$$= I_n$$

True

(10) $A = [\vec{a}_1 \ \vec{a}_2 \ --- \ \vec{a}_n]$

"A" is orthogonal $\Rightarrow \|\vec{a}_i\| = 1$

$\therefore a_{i1}^2 + a_{i2}^2 + --- a_{in}^2 = 1$

If $a_{ij} > 1$, $\sum_{k=1, k \neq j}^{n} a_{ik}^2 = 1 - a_{ij}^2 < 0$

which is a contradiction True

$\therefore a_{ij} \leq 1$

(11) $A = \begin{bmatrix} 0 & a \\ 1/a & 0 \end{bmatrix}$, $a \notin \{-1, 0, 1\}$

$A^{-1} = \dfrac{1}{-1} \begin{bmatrix} 0 & -a \\ -1/a & 0 \end{bmatrix} = A$

$A^T = \begin{bmatrix} 0 & 1/a \\ a & 0 \end{bmatrix} \neq A^{-1} \Rightarrow$ "A" is not orthogonal

False

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$c_i > 0$

$$\langle \vec{v}, \vec{w} \rangle = \sum_{i=1}^{n} c_i v_i w_i$$

1) Linearity check:

$$\langle a\vec{u} + b\vec{v}, \vec{w} \rangle = \sum_{i=1}^{n} c_i (a u_i + b v_i) w_i$$

$$= \sum_{i=1}^{n} (a\, c_i u_i w_i + b\, c_i v_i w_i)$$

$$= a \sum_{i=1}^{n} c_i u_i w_i + b \sum_{i=1}^{n} c_i v_i w_i$$

$$= a \langle \vec{u}, \vec{w} \rangle + b \langle \vec{v}, \vec{w} \rangle \checkmark$$

2) <u>Symmetry</u> check:

$$\langle \vec{v}, \vec{w} \rangle = \sum_{i=1}^{n} c_i v_i w_i$$

$$= \sum_{i=1}^{n} c_i w_i v_i$$

$$= \langle \vec{w}, \vec{v} \rangle \checkmark$$

3) Positive <u>Definite</u> check :-

$$\langle \vec{u}, \vec{u} \rangle$$

$$= \sum_{i=1}^{n} c_i u_i u_i$$

$$= \sum_{i=1}^{n} c_i u_i^2 \geq 0 \quad \left[\begin{array}{l} \text{as } c_i > 0 \\ = 0 \text{ only when} \\ u_i = 0 \end{array}\right]$$

$\checkmark$

(9)

$$\|\vec{x}\| := \sum_{i=1}^{n} x_i^2$$

$$\|\alpha \vec{x}\| := \sum_{i=1}^{n} \alpha^2 x_i^2$$

$$:= \alpha^2 \sum_{i=1}^{n} x_i^2$$

$$:= \alpha^2 \|\vec{x}\| \neq |\alpha| \|\vec{x}\|$$

$\therefore \|\vec{x}\|$ does not define a norm

(10)

$x_1 + 2x_2 + 3x_3 + 4x_4$ is minimum

and $|x_i| \leq 1$

$\Rightarrow$ x_4^* has to be minimum and

$|x_4^*| = 1$

$$\Rightarrow \begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\therefore \min\left(x_1 + 2x_2 + 3x_3 + 4x_4\right) = -4$$

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(11)

1) Linearity check :-

$$\langle xA + yC, B \rangle = tr\left((xA+yC)^T B\right)$$

$$= tr\left(xA^T B + yC^T B\right)$$

$$= x\,tr\left(A^T B\right) + y\,tr\left(C^T B\right)$$

$$= x\langle A, B\rangle + y\langle C, B\rangle \checkmark$$

2) Symmetry Check :-

$$\langle A, B\rangle = tr(A^T B)$$

$$= tr((A^T B)^T)$$

$$= tr(B^T A) = \langle B, A\rangle \checkmark$$

3) Positive Definite check :-

$$\langle A, A\rangle = tr(A^T A)$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij}^2$$

$$\geq 0 \quad \left[= 0 \Rightarrow a_{ij} = 0 \; \forall \, i, j \right]$$

$\checkmark$