Math 4570- Matrix methods for Data Analysis and Machine Learning - Homework 1
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Answer to Question 1:

(1):
Associativity through Addition-
$(a+b\sqrt{2})+((c+d\sqrt{2})+(e+f\sqrt{2}) = a+c+e+(b+d+f)\sqrt{2} = ((a+b\sqrt{2})+(c+d\sqrt{2}))+(e+f\sqrt{2})$
Associativity through Multipcation-
$(a+b\sqrt{2})\times((c+d\sqrt{2})\times(e+f\sqrt{2})) =$
$ace+acf\sqrt{2}+aed\sqrt{2}+2afd+ceb\sqrt{2}+2bef+2bed+2bfd\sqrt{2} = ((a+b\sqrt{2})\times(c+d\sqrt{2}))\times(e+f\sqrt{2})$
Commutativity through Addition-
$(a+b\sqrt{2})+(c+d\sqrt{2}) = a+c+(b+d)\sqrt{2} = (c+d\sqrt{2})+(a+b\sqrt{2})$
Commutativity through Multiplication-
$(a+b\sqrt{2})\times(c+d\sqrt{2}) = ac+ad\sqrt{2}+cb\sqrt{2}+2bd = (c+d\sqrt{2})\times(a+b\sqrt{2})$
Identity for Sum-
$(a+b\sqrt{2})+\mathbf{0} = (a+b\sqrt{2})$
Identity for Multiplication-
$(a+b\sqrt{2})\times 1 = (a+b\sqrt{2})$
Inverse for Sum-
$(a+b\sqrt{2})-f = \mathbf{0}$ where $f = a+b\sqrt{2}$
Inverse for Multiplication-
If $(a+b\sqrt{2})\neq\mathbf{0}, f=(a+b\sqrt{2})^{-1}, (a+b\sqrt{2})\times f = 1$
Distributivity-
$(a+b\sqrt{2})\times((c+d\sqrt{2})+(e+f\sqrt{2})) = (a+b\sqrt{2})(c+d\sqrt{2})+(a+b\sqrt{2})(e+f\sqrt{2})$

$(F,+,\times)$ is a field.

(2): Where $\sqrt{-1} = i$
Associativity through Addition-
$(a+bi)+((c+di)+(e+fi)) = a+c+e+(b+d+f)i = ((a+bi)+(c+di))+(e+fi)$
Associativity through Multipcation-
$(a+bi)\times((c+di)\times(e+fi)) =$
$ace+acfi+aedi+cebi-afd-bef-bed-bfdi = ((a+bi)\times(c+di))\times(e+fi)$
Commutativity through Addition-
$(a+bi)+(c+di) = a+c+(b+d)i = (c+di)+(a+bi)$
Commutativity through Multiplication-
$(a+bi)\times(c+di) = ac+adi+cbi-bd = (c+di)\times(a+bi)$
Identity for Sum-
$(a+bi)+\mathbf{0} = (a+bi)$
Identity for Multiplication-
$(a+bi)\times 1 = (a+bi)$
Inverse for Sum-
$(a+bi)-f = \mathbf{0}$ where $f = (a+bi)$
Inverse for Multiplication-
If $(a+bi)\neq\mathbf{0}, f=(a+bi)^{-1}, (a+bi)\times f = 1$
Distributivity-
$(a+bi)\times((c+di)\times(e+fi)) = (a+bi)(c+di)+(a+bi)(e+fi)$

$(F, +, \times)$ is a field.

Answer to Question 2:

If n >1 then the set of all n×n matrices $\mathbb{R}^{n \times n}$ cannot be a field due to the commutativity of multiplication. For example, if $n = 2$ and $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 11 & 6 \\ 3 & 2 \end{bmatrix}$, $AB = \begin{bmatrix} 0 & 0 \\ 3 & 2 \end{bmatrix}$ while $BA = \begin{bmatrix} 0 & 6 \\ 0 & 2 \end{bmatrix}$ so that $AB \neq BA$.

Answer to Question 3:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	0

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Answer to Question 4:

Associativity through Addition-
$(a + bi) + ((c + di) + (e + fi)) = a + c + e + (b + d + f)i = ((a + bi) + (c + di)) + (e + fi)$
Associativity through Multipcation-
$(a + bi) \times ((c + di) \times (e + fi)) =$
$ace + acfi + aedi + cebi - afd - bef - bed - bfdi = ((a + bi) \times (c + di)) \times (e + fi)$
Commutativity through Addition-
$(a + bi) + (c + di) = a + c + (b + d)i = (c + di) + (a + bi)$
Commutativity through Multiplication-
$(a + bi) \times (c + di) = ac + adi + cbi - bd = (c + di) \times (a + bi)$
Identity for Sum-
$(a + bi) + \mathbf{0} = (a + bi)$
Identity for Multiplication-
$(a + bi) \times 1 = (a + bi)$
Inverse for Sum-
$(a + bi) - f = \mathbf{0}$ where $f = (a + bi)$
Inverse for Multiplication-
If $(a + bi) \neq \mathbf{0}, f = (a + bi)^{-1}, (a + bi) \times f = 1$
Distributivity-
$(a + bi) \times ((c + di) \times (e + fi)) = (a + bi)(c + di) + (a + bi)(e + fi)$

$(F, +, \times)$ is a field.

Answer to Question 5:

Only matrices B = $\begin{bmatrix} 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and D = $\begin{bmatrix} 0 & 1 & 0 & 2 & 4 \end{bmatrix}$ are in reduced row-echelon form.

Answer to Question 6:

If A = $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and B = $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ over the field $\mathbb{Z}_2 \Longrightarrow$

$$A + B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, A^2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Answer to Question 7:

If a matrix's determinant is zero, then the matrix is singular and does not have an inverse.

A = $\begin{bmatrix} 6 & -1 & 1 \\ t & 0 & 1 \\ 0 & 1 & t \end{bmatrix} \Longrightarrow$

$\det(A) = 6{\cdot}\det\begin{bmatrix} 0 & 1 \\ 1 & t \end{bmatrix} - t{\cdot}\det\begin{bmatrix} -1 & 1 \\ 1 & t \end{bmatrix} + 0 =$ 6[0 - 1] - t[-t - 1]= t$^2 + t - 6 = (t-2)(t+3)$

For $\det(A) = 0$, $t =$ -3,2

Answer to Question 8:

a). For $\begin{bmatrix} 1 & h & | & 4 \\ 3 & 6 & | & 8 \end{bmatrix}$ the matrix is only consistent for when $h \neq 2$.

b). For $\begin{bmatrix} -4 & 12 & | & h \\ 2 & -6 & | & -3 \end{bmatrix}$ the matrix is only consistent for when $h = 6$.

Answer to Question 9:

(1):
There are 4 types of 3×2 matrices in reduced row-echelon form. They are:

Rank 0 - $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Rank 1 - $\begin{bmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Rank 2 - $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

(2):
There are 7 types of 2×3 matrices in reduced row-echelon form. They are:

Rank 0 - $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Rank 1 - $\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Rank 2 - $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(3):
There are 2 types of 4×1 matrices in reduced roe-echelon form. They are:

Rank 0 - $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Rank 1 - $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Answer to Question 10:

For matrix $A = \begin{bmatrix} 1 & a & b & 3 & 0 & -2 \\ 0 & 0 & c & 1 & d & 3 \\ 0 & e & 0 & 0 & 1 & 1 \end{bmatrix}$ to be in reduced row-echelon form:
$a = *, b = 0, c = 1, d = 0, e = 0$ where $*$ represents any real number.

Answer to Question 11:
(1):

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 = R_2 - R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 = R_3 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -2 \\ 0 & -4 & -5 & -6 \end{bmatrix}$$

$$\xrightarrow{R_2 = -R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & -4 & -5 & -6 \end{bmatrix} \xrightarrow{R_1 = R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & -4 & -5 & -6 \end{bmatrix} \xrightarrow{R_3 = R_3 + 4R_2} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 7 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 = \frac{R_3}{7}} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix} \xrightarrow{R_1 = R_1 + 3R_3} \begin{bmatrix} 1 & 0 & 0 & \frac{6}{7} \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix} \xrightarrow{R_2 = R_2 - 3R_3} \begin{bmatrix} 1 & 0 & 0 & \frac{6}{7} \\ 0 & 1 & 0 & \frac{8}{7} \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix}$$

rref(A)= $\begin{bmatrix} 1 & 0 & 0 & \frac{6}{7} \\ 0 & 1 & 0 & \frac{8}{7} \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix}$

(2): Within $\mathbb{Z}_7$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 = R_2 - R_1} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 3 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 = R_3 - 2R_1} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 5 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow{R_2 = R_2 - 5R_2} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_1 = R_1 - R_2} \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_2 = R_2 - R_3} \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 = \frac{R_3}{2}} \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

rref(A)= $\begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ in $\mathbb{Z}_7$

(3):

Listing 1: RREF of (1) using Python

```
IN [1]:
import numpy as np
from sympy import Matrix, pprint
import galois
M = Matrix([[1, 2, 3, 4], [1, 1, 0, 2], [2, 0, 1, 2]])
M.rref()

OUT [1]:
(Matrix([
 [1, 0, 0, 6/7],
 [0, 1, 0, 8/7],
 [0, 0, 1, 2/7]]),
 (0, 1, 2))
```

Listing 2: RREF of (2) using Python

```
IN [2]:
import numpy as np
from sympy import Matrix, pprint
import galois
GF7 = galois.GF(7)
M = GF7([[1, 2, 3, 4], [1, 1, 0, 2], [2, 0, 1, 2]])
GF7.row_reduce(M)

OUT [2]:
GF([[1, 0, 4, 0],
    [0, 1, 3, 0],
    [0, 0, 0, 1]], order=7)
```

Listing 3: RREF(A) in $\mathbb{Z}_2$

```
IN [3]:
import numpy as np
from sympy import Matrix, pprint
import galois
GF2 = galois.GF(2)
M = GF2([[1, 0, 1, 0], [1, 1, 0, 0], [0, 0, 1, 0]])
GF2.row_reduce(M)
```

```
OUT [3]:
GF([[1, 0, 0, 0],
    [0, 1, 0, 0],
    [0, 0, 1, 0]], order=2)
```

Listing 4: RREF(A) in $\mathbb{Z}_3$

```
IN [4]:
import numpy as np
from sympy import Matrix, pprint
import galois
GF3 = galois.GF(3)
N = GF3([[1, 2, 0, 1], [1, 1, 0, 2], [2, 0, 1, 2]])
GF3.row_reduce(N)

OUT [4]:
GF([[1, 0, 0, 0],
    [0, 1, 0, 2],
    [0, 0, 1, 2]], order=3)
```

Answer to Question 12:

(1):

$$M = \begin{bmatrix} 3 & 1 & 4 & 1 \\ 5 & 2 & 6 & 5 \\ 0 & 5 & 2 & 1 \end{bmatrix}$$

$$\text{rref}(M) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \text{ in } \mathbb{Z}_7$$

(2):

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

Listing 5: RREF(M) in $\mathbb{Z}_7$

```
IN [5]:
import numpy as np
import sympy as sym
from sympy import Matrix, pprint
import galois

GF7 = galois.GF(7)
M = GF7([[3, 1, 4, 1], [5, 2, 6, 5], [0, 5, 1, 2]])
GF7.row_reduce(M)

OUT [5]:
GF([[1, 0, 0, 0],
    [0, 1, 0, 0],
    [0, 0, 1, 2]], order=7)
```

Answer to Question 13:

For $\begin{cases} 3x_1 + 11x_2 + 19x_3 = -2 \\ 7x_1 + 23x_2 + 39x_3 = 10 \\ -4x_1 - \ \ 3x_2 - \ \ 2x_3 = 6 \end{cases} \Longrightarrow$ no solutions exist

Listing 6: Solving Systems of Linear Equations

```
IN [6]:
import numpy as np
import sympy as sym

x = sym.Symbol('x')
y = sym.Symbol('y')
z = sym.Symbol('z')

solution = sym.solve((3*x + 11*y + 19*z + 2, 7*x + 23*y + 39*z - 10, -4*x - 3*y - 2*z - 6), (x, y, z
solution

OUT [6]:
[]
```

Answer to Question 14:

$$\begin{cases} 3x_1 + \ \ 6x_2 + \ \ 9x_3 + 5x_4 + 25x_5 = 53 \\ 7x_1 + 14x_2 + 21x_3 + 9x_4 + 53x_5 = 105 \\ -4x_1 - \ \ 8x_2 - 12x_3 + 5x_4 - 10x_5 = 11 \end{cases} \Longrightarrow \begin{bmatrix} 1 & 2 & 3 & 0 & 5 & 6 \\ 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 7 - 2x_5 \\ x_5 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7 \\ 0 \end{bmatrix}$$

Listing 7: Solving Systems of Linear Equations

```
IN [7]:
import numpy as np
import sympy as sym
from sympy import Matrix, pprint

M = Matrix([[3, 6, 9, 5, 25, 53], [7, 14, 21, 9, 53, 105], [-4, -8, -12, 5, -10, 11]])
M.rref()

OUT [7]:
(Matrix([
 [1, 2, 3, 0, 5, 6],
 [0, 0, 0, 1, 2, 7],
 [0, 0, 0, 0, 0, 0]]),
 (0, 3))
```

Answer to Question 15:

$$\begin{cases} 2x_1 + \ \ 4x_2 + 3x_3 + 5x_4 + 6x_5 = 37 \\ 4x_1 + \ \ 8x_2 + 7x_3 + 5x_4 + 2x_5 = 74 \\ -2x_1 - \ \ 4x_2 + 3x_3 + 4x_4 - 5x_5 = 20 \\ x_1 + \ \ 2x_2 + 2x_3 - \ \ x_4 + 2x_5 = 26 \\ 5x_1 - 10x_2 + 4x_3 + 6x_4 + 4x_5 = 24 \end{cases} \implies X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1.894239 \\ 0.989746 \\ 10.817972 \\ -1.057603 \\ 1.610599 \end{bmatrix}$$

Listing 8: Solving Systems of Linear Equations

```
IN [8]:
import numpy as np
import sympy as sym

x = sym.Symbol('x')
y = sym.Symbol('y')
z = sym.Symbol('z')
r = sym.Symbol('r')
t = sym.Symbol('t')

solution = sym.solve((2*x + 4*y + 3*z + 5*r + 6*t - 37, 4*x + 8*y + 7*z + 5*r + 2*t - 74, -2*x - 4*y
solution[x], solution[y], solution[z], solution[r], solution[t]

OUT [8]:
(-8221/4340, 8591/8680, 4695/434, -459/434, 699/434)
```

Answer to Question 16:

A matrix is invertible if its determinant $\neq 0$. Since ABC is an identity matrix I_n and all identity matrices are invertible the $det(ABC) \neq 0$. Furthermore, this can prove each matrix (A, B, and C) are invertible because the $det(ABC) = det(A)det(B)det(C)$ so none of them can have a determinant of 0.

Additionally, each matrix (A, B, and C) has an inverse that is equal to the product of the other two such that $A^{-1} = BC$ and $C^{-1} = AB$ because $I_n = A(BC) = (AB)C$.

Answer to Question 17:

$If\,\mathrm{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathrm{B} = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$ $then$ $(\mathrm{AB})^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $A^2B^2 = \begin{bmatrix} 0 & 0 \\ 3 & 2 \end{bmatrix}$. $(AB)^2 \neq A^2B^2$

Answer to Question 18:

If A $= \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix}$ and $A^T = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix}$ so $A^{-1} = A^T$

Answer to Question 19:

(1):

Symmetric- $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 5 & 8 & 0 \\ 5 & 1 & 1 & 1 \\ 8 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

Skew-Symmetric- $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 5 & -3 \\ -5 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -5 & 8 & 2 \\ 5 & 0 & -1 & 1 \\ -8 & 1 & 0 & -7 \\ -2 & -1 & 7 & 0 \end{bmatrix}$

(2):
The diagonal of a skew-symmetric matrix is always made of zeros. This is so that nonzero eigenvalues of a skew-symmetric matrix are non-real.

(3):
The only type of matrix that can be both symmetric and skew-symmetric are zero matrices. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(4):
If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$A + A^T = \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix}$ ✓Symmetric

$A \times A^T = \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix}$ ✓Symmetric

$A^T \times A = \begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{bmatrix}$ ✓Symmetric

$A - A^T = \begin{bmatrix} 0 & b-c \\ c-b & 0 \end{bmatrix}$ ✓Skew-Symmetric

(5):
As long as A is a square matrix it can be defined by $A = 1/2(A + A^T) + 1/2(A - A^T)$. Knowing that $A + A^T$ is Symmetric and $A - A^T$ is Skew-Symmetric, A is simply a sum of the two.

Answer to Question 20:
a). $F(x) = x^2$ is surjective.
b). $F(x) = x^3/(x^2 + 1)$ is bijective.
c). $F(x) = x(x - 1)(x - 2)$ is surjective.
d). $F(x) = e^x + 2$ is injective.

Answer to Question 21:

$$A = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_2 - \frac{1}{4}R_1} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & \frac{15}{4} & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_3 - \frac{4}{15}R_2} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & \frac{15}{4} & 1 & 0 \\ 0 & 0 & \frac{56}{15} & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_4 - \frac{15}{56}R_3} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & \frac{15}{4} & 1 & 0 \\ 0 & 0 & \frac{56}{15} & 1 \\ 0 & 0 & 0 & \frac{209}{56} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ 0 & \frac{4}{15} & 1 & 0 \\ 0 & 0 & \frac{15}{56} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & \frac{15}{4} & 1 & 0 \\ 0 & 0 & \frac{56}{15} & 1 \\ 0 & 0 & 0 & \frac{209}{56} \end{bmatrix}$$

Answer to Question 22:

$$\text{For } A = \begin{bmatrix} q_1 & r_1 & 0 & 0 \\ p_1 & q_2 & r_2 & 0 \\ 0 & p_2 & q_3 & r_3 \\ 0 & 0 & p_3 & q_4 \end{bmatrix} \text{ as } L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_1 & 1 & 0 & 0 \\ 0 & l_2 & 1 & 0 \\ 0 & 0 & l_3 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} d_1 & u_1 & 0 & 0 \\ 0 & d_2 & u_2 & 0 \\ 0 & 0 & d_3 & u_3 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$$

$u_i = r_i$, $d_i = q_i$, and $l_i = \frac{p_i}{q_i}$ after each row operation.

Answer to Question 23:

$$\text{For } n \times n \text{ matrix } A = \begin{bmatrix} 4 & 1 & \dots & 0 & 0 \\ 1 & 4 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 4 & 1 \\ 0 & 0 & \dots & 1 & 4 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ U^{-1}_{i-1,i} & 1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \dots & U^{-1}_{i-1,i} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 4 & 1 & \dots & 0 & 0 \\ 0 & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 1 \\ 0 & 0 & \dots & 0 & U^{-1}_{n-1,n-1} \end{bmatrix}$$

Answer to Question 24:

(1):
Since, $H_n^T = (I_n - 2uu^T)^T = I_n^T - 2(uu^T)^T = I_n - 2uu^T = H_n$, H_n is symmetric.
(2):
As u is a unit vector, $HH^T = HH = (I - 2uu^T)(I - 2uu^T)$
$= I - 4uu^T + 4u(uu^T)u^T = I - 4uu^T + 4uu^T = T$, therefore H is orthogonal.