Monday, July 20, 2020 12:58 PM

HW 4:

$$S' = -a\frac{B}{K+B}S + dR = 0 \quad (1)$$

$$I' = a\frac{B}{K+B}S - rI = 0 \quad (2)$$

$$B' = Bn_b + eI = 0 \quad (3)$$

$$R' = rI - dR = 0 \quad (4)$$

WANT EQ. SOLUTIONS

$(3) \Rightarrow B = -\frac{eI}{n_b}$, $(4) \Rightarrow R = \frac{rI}{d}$.

THEN

$(2) \Rightarrow$

$$a\frac{\frac{-eI}{n_b}}{K+\frac{eI}{n_b}}S - rI = 0$$

$$-a\frac{e}{m_0}SI - rI\left(k - \frac{cI}{m_0}\right) = 0$$

$I = 0$, $S = S_0$.

FACTOR OUT I.

$$S = AI_0 + D$$

SYSTEMS OF NONLINEAR EQUATIONS

SOLUTION SPACE AND TRAJECTORY DIAGRAMS.

L

Ex.:

$$\frac{dx}{dt} = -ay$$

$$\frac{dy}{dt} = ax$$

ROUGH ANALYSIS:

1. FIND EQUILIBRIUM POINTS $x'=0$ AND $y'=0$

2. FIND NULLCLINES: CURVES WHERE $x'=0$ OR $y'=0$.

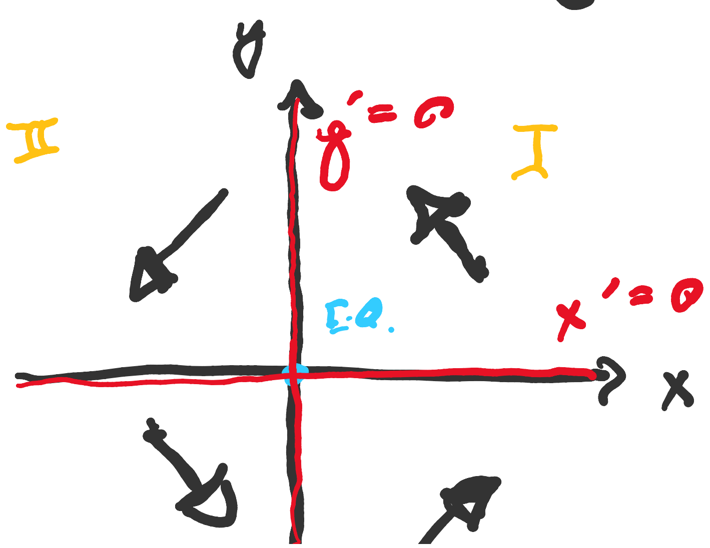
3. BETWEEN EACH PAIR OF NULLCLINES, DETERMINE ROUGH DIRECTION OF SOLUTIONS.

IN EX

EQ. POINT: $(x,y) = (0,0)$

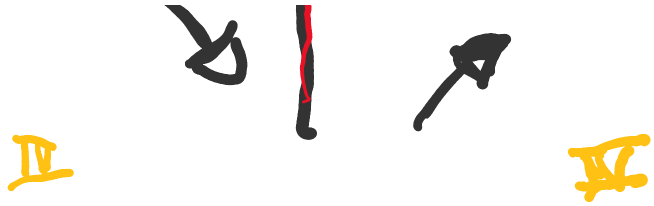
NULLCLINES: $x'=0 \Rightarrow y=0$

$y'=0 \Rightarrow x=0$



FOR EACH REGION, IS $x', y' >$ OR < 0?

$x' = -4y$



$$\boxed{\begin{aligned} x' &= -ay \\ y' &= ax, \quad a>0 \end{aligned}}$$

I: $x'<0$, $y'>0$, $(-,+)$, SO TRAJ. IS ↖

II: $x'<0$, $y'<0$, $(-,-)$, ↙

III: $x'>0$, $y'<0$, $(+,-)$, ↘

IV: $x'>0$, $y'>0$, $(+,+)$, ↗

IF SIGN OF DER. CHANGES FOR X OR Y, <u>MUST</u> CROSS A NULL-CLINE. MOREOVER, CROSSING, SAY, A $y'=0$ NULLCLINE OFTEN CHANGES THE SIGN OF y'.

NOTE: WHEN YOU HAVE HIGHER ORDER U'S A NULLCLINE MAY NOT

U'S, A NULL CLINE MAY NOT CHANGE DER. SIGN:

$$x' = -y^2 \qquad y' = x^2 \, , \quad x' \leq 0, \quad y' \geq 0$$

FOR ALL x, y.

Ex: SIR MODEL:

$$S' = -\beta S I$$

$$I' = \beta S I - \gamma I$$

LETS DRAW A TRAJECTORY DIAGRAM!

EQ: $I = 0, \quad S = S_0$

NULL CLINES: $I' = 0 \Rightarrow I = 0, \quad S = \dfrac{\gamma}{\beta}$.

$S' = 0 \Rightarrow I = 0, \quad S = 0$

TRAJECTORY DIAGRAM:

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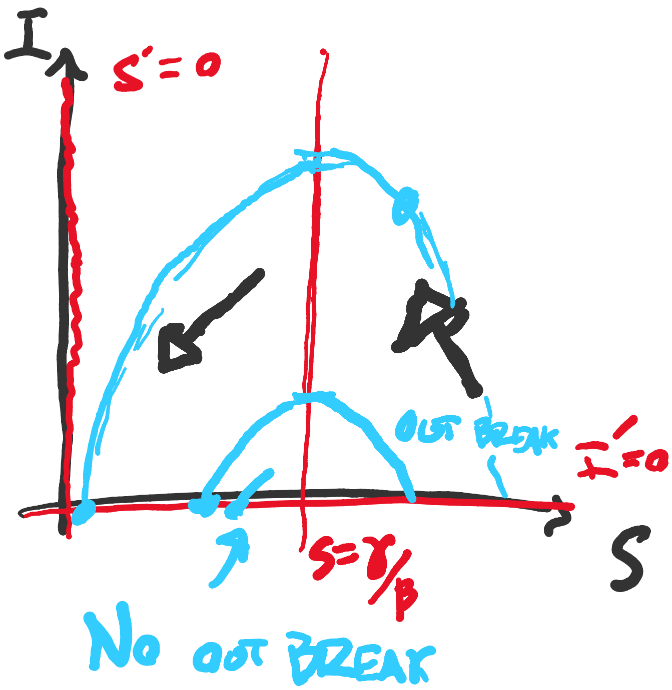


REGIONS:

- $S < \gamma/\beta$: $(-,-)$

  $S' < 0,\ I' < 0$

- $S > \gamma/\beta$: $(-,+)$

WHAT WE'RE SEEING HERE IS CRITICAL TOWN SIZE. NOTE: CLEAR HERE THAT CLOSER TO γ/β S_0 IS, THE FEWER TOTAL PEOPLE WILL GET SICK, (GIVEN FIXED I_0).

SIR MODIFIED:

EX: SIR WITH INCOMING POPULATION

$$S' = -\beta S I + d$$
$$I' = \beta S I - \gamma I$$

$$I' = \beta S I - \gamma I$$

SOLUTION SPACE ANALYSIS:

NULLCLINES: $S' = 0 \Rightarrow d = \beta S I$

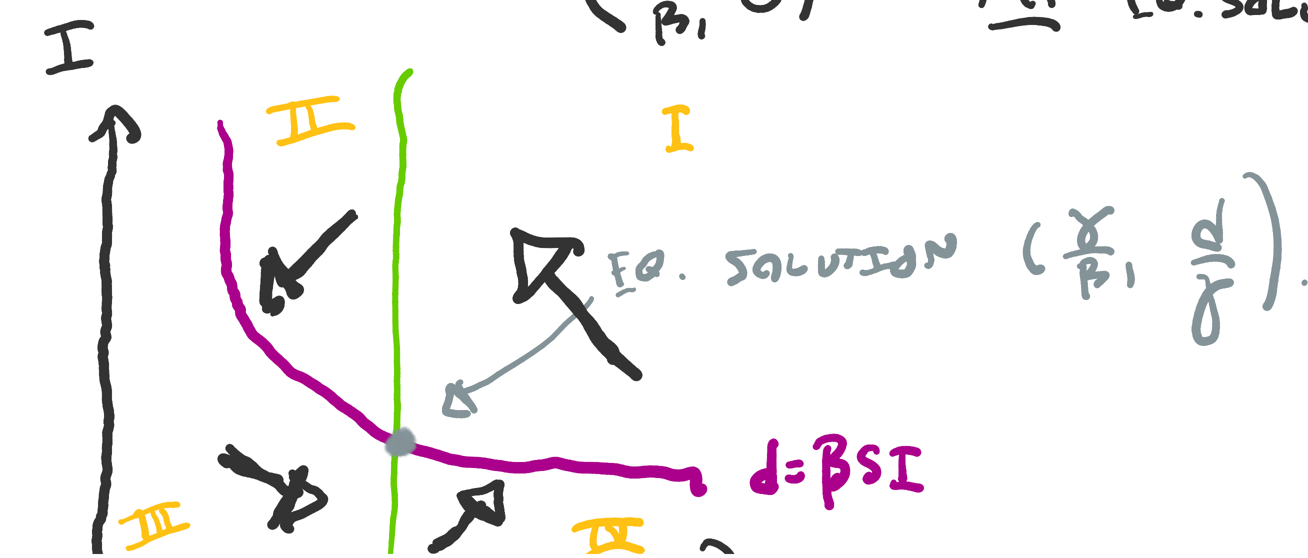
$I' = 0 \Rightarrow I = 0, \; S = \frac{\gamma}{\beta}$

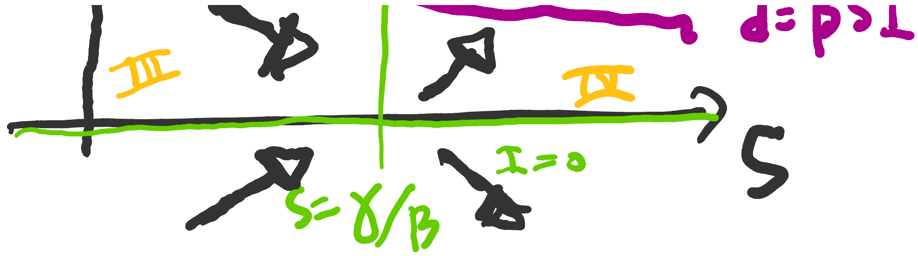
- EQUILIBRIUM SOLUTIONS ALWAY OCCURE AT THE INTERSECTION OF NULL CLINES FROM N DIFFERENT EQUATIONS.

HERE:

$(S, I) = \left(\frac{\gamma}{\beta}, \frac{d}{\gamma}\right)$ EQ. SOLUTION

$= \left(\frac{\gamma}{\beta}, 0\right)$ NOT EQ. SOLUTION





ROUGH TRAJECTORIES:

I: $S > \gamma/\beta$, $I > \frac{d}{\beta S}$, $(-,+)$

II: $(-,-)$

III: $(+,-)$

IV: $(+,+)$

$$S' = -\beta IS + d$$
$$I' = \beta IS - \gamma I = I(\beta S - \gamma)$$

IN REGION I, ASSUME $S \approx I \gg 1$. THEN $\beta IS \gg d$, SO $S' < 0$. SIM. $\beta IS \gg \gamma I$, SO $I' > 0$.

<u>MORE CAREFULLY</u>: $S > \gamma/\beta$, $I > d/\beta S$.

<u>NOW</u>: NULLCLINES ARE FIRST ORDER SO CROSSING I NULLCLINE CHANGES SIGN OF I.

WHAT IS ACTUAL BEHAVIOR AT EQ SOLUTION?

EQ SOLUTION?

JACOBEAN ANALYSIS:

WANT HIGHER COMPLEXITY ANALOG OF

$$\vec{x}' = A\vec{x}.$$

FOR NONLINEAR SYSTEMS,

$$\begin{aligned} x_1' &= f_1(\vec{x}) \\ &\vdots \\ x_N' &= f_N(\vec{x}) \end{aligned} \implies \vec{x}' = F(\vec{x}), \quad F = (f_1, \ldots, f_N)$$

SUPPOSE A SOLUTION $\vec{x}(t)$ IS "CLOSE" TO AN EQUILIBRIUM SOLUTION $\vec{x}_e$ FOR t IN SOME INTERVAL.

TAYLOR EXPAND $F(\vec{x})$ AT $\vec{x}_e$:

$$x_1' = f_1(\vec{x}_e) + \sum_{i=1}^{N} \frac{\partial f_1}{\partial x_i}(x_i - x_{e_i}) + \ldots \text{ HIGHER ORDER TERMS}$$

$$\vdots$$

$$x_N' = \quad \text{=} \quad \text{BUT } N \text{ NOT } 1.$$

WRITE THIS AS

$$\vec{x}' = F(\vec{x}_e) + \left.\frac{\partial F}{\partial \vec{x}}\right|_{\vec{x}=\vec{x}_e} (\vec{x} - \vec{x}_e) + \dots$$

WHERE

$$\frac{\partial F}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \vdots & \vdots & & \\ \frac{\partial f_N}{\partial x_1} & \frac{\partial f_N}{\partial x_2} & \cdots & \frac{\partial f_N}{\partial x_N} \end{bmatrix} =: J(\vec{x}).$$

<u>JACOBIAN</u>

<u>NOW</u>: $F(\vec{x}_e) = 0$, SINCE $\vec{x}_e$ EQUIL. SOLUTION, SO, SETTING $\delta x = \vec{x} - \vec{x}_e$.

WE HAVE

$$\delta x' = \vec{x}' \approx \cancelto{0}{F(\vec{x}_e)} + \overbrace{\left.\frac{\partial F}{\partial \vec{x}}\right|_{\vec{x}=x_e}}^{\text{CONST. MAT.}} \underbrace{(\vec{x} - \vec{x}_e)}_{\delta x} + \dots$$

$$\delta x' \approx J(\vec{x}_e)\, \delta x$$

THIS IS A FIRST ORDER LINEAR

THIS IS A FIRST ORDER LINEAR EQUATION, SO CAN APPROXIMATE SOLUTIONS WITH EIGEN VALUES / VECTORS

RETURN TO EX:

HAD

$$\begin{aligned} S' &= -\beta S I + d \\ I' &= \beta S I - \gamma I \end{aligned} \Rightarrow F(S,I) = \begin{bmatrix} -\beta S I + d \\ \beta S I - \gamma I \end{bmatrix}$$

$$J(S,I) = \begin{bmatrix} \frac{\partial f_1}{\partial S} & \frac{\partial f_1}{\partial I} \\ \frac{\partial f_2}{\partial S} & \frac{\partial f_2}{\partial I} \end{bmatrix} = \begin{bmatrix} -\beta I & -\beta S \\ \beta I & \beta S - \gamma \end{bmatrix}$$

AT EQ POINT $S_e = \frac{\gamma}{\beta}$, $I_e = \frac{d}{\gamma}$,

$$J(S_e, I_e) = \begin{bmatrix} -\frac{\beta d}{\gamma} & -\gamma \\ \frac{\beta d}{\gamma} & 0 \end{bmatrix}.$$

WANT TO CLASSIFY SOLUTIONS. SO

WANT TO CLASSIFY SOLUTIONS, SO WANT TO UNDERSTAND EIG. VALUES:

$$\det(J-\lambda I) = -\lambda\left(-\frac{\beta d}{\gamma}-\lambda\right) + \beta d$$

$$= \lambda^2 + \frac{\beta d}{\gamma}\lambda + \beta d$$

SO

$$\lambda = \frac{-\frac{\beta d}{\gamma} \pm \sqrt{\left(\frac{\beta d}{\gamma}\right)^2 - 4\beta d}}{2}$$

FIRST: SINCE $-\frac{\beta d}{2\gamma} < 0$ MUST BE AT LEAST STABLE.

SECOND: CONDITION FOR FOCUS VS NODE:

$$\Delta = \left(\frac{\beta d}{\gamma}\right)^2 - 4\beta d > 0 \qquad \text{NODE}$$

SO HAVE AN EXACT CLASSIFICATION OF LOCAL BEHAVIOR:

$\beta d > 4\gamma$ NODE

$\beta d > 4\gamma$ NODE.

$\beta d < 4\gamma$ FOCUS.

PREDITOR - PREY MODEL (LOTKA - VOLTERRA)

ASSUMPTIONS :

- TWO POPULATIONS: PREDITORS Y AND PREY X.
- WITHOUT PRED., PREY WOULD INCREASE AS $X' = aX$, BIRTH RATE > DEATH RATE
- WITH NO PREY PRED. DIE OFF EXP.

$$Y' = -bY$$

, DEATH RATE > BIRTH RATE

- PREDATION: SOME PERCENT CHANCE p FOR ANY PRED TO MEET ANY PREY, SOME PERCENT CHANCE h HUNT IS SUCCESSFUL, AND THAT INCREASE PRED. BIRTH RATE BY SOME

PRED. BIRTH RATE BY SOME AMOUNT r:

$$X' = aX - phXY = aX - c_1XY$$

$$Y' = rphXY - bY = c_2XY - bY$$

ANALYZE:

NULLCLINES: $X' = 0$: $Y = \frac{a}{c_1}$, $X = 0$

$Y' = 0$: $Y = 0$, $X = \frac{b}{c_2}$

EQUILIBRIUM SOLUTIONS: $(X, Y) = (0, 0)$, $\left(\frac{b}{c_2}, \frac{a}{c_1}\right)$

WHY IS $\left(0, \frac{a}{c_1}\right)$ NOT AN EQ. SOLUTION?

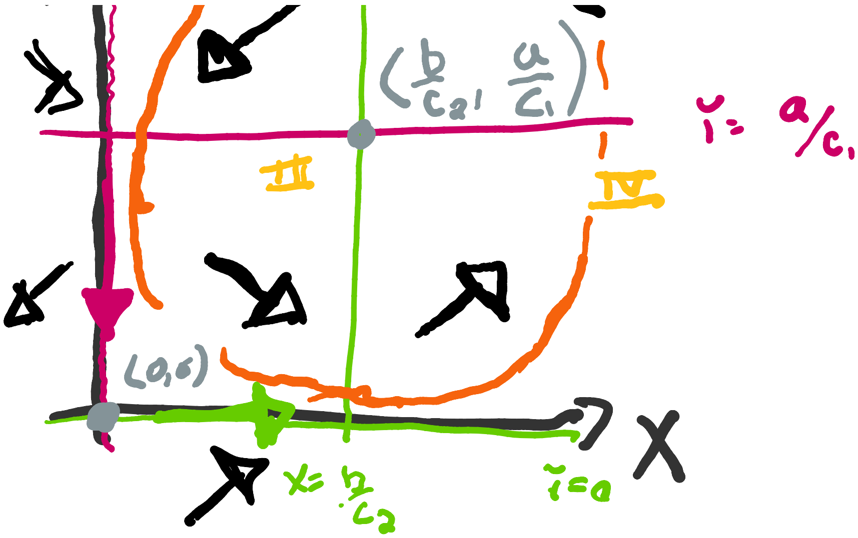
BECAUSE $Y' \neq 0$.

DIAGRAM:

ROUGH TRAJECTORIES

I: $(-, +)$

II: $(-, -)$



II: $(-,-)$
III: $(+,-)$
IV: $(+,+)$

$$X' = aX - c_1 XY$$
$$Y' = c_2 XY - bY$$

Q) WHAT IS STABILITY TYPE OF $(0,0)$?

SEMI-STABLE (?)

STABILITY OF EQUILIBRIA:

JACOBEAN:

$$\bullet\ J(X,Y) = \begin{bmatrix} a - c_1 Y & -c_1 X \\ c_2 Y & c_2 X - b \end{bmatrix}$$

$\bullet\ J(0,0) = \begin{bmatrix} a & 0 \end{bmatrix}$ …

- $J(0,0) = \begin{bmatrix} a & 0 \\ 0 & -b \end{bmatrix}$, $\lambda_1 = a$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

  $\lambda_2 = -b$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- $J\left(\frac{b}{c_2}, \frac{a}{c_1}\right) = \begin{bmatrix} 0 & -\frac{bc_1}{c_2} \\ \frac{c_2 a}{c_1} & 0 \end{bmatrix}$

$\Rightarrow \lambda^2 + ab = 0$ OR $\lambda = \pm\sqrt{-ab}$.

SO $\lambda \in i\mathbb{R}$ <u>PURELY IMAGINARY</u>

SO STABILITY TYPE IS CONCENTRIC CIRCLES.

<u>NOTE</u>: SHOULDN'T ACTUALLY EXPECT CLOSED SOLUTIONS, SINCE STABILITY IS APPROXIMATE, AND ORBITS CLOSING IS <u>EXACT</u>.

IN THIS CASE, THERE'S A TRICK:

WRITE

WRITE

$$\frac{\frac{dY}{dt}}{\frac{dX}{dt}} = \frac{dY}{dX} = \frac{c_2 X Y - bY}{aX - c_1 X Y}$$

$$= \frac{Y(c_2 X - b)}{X(a - c_1 Y)}$$

SO CAN SEP. VARS:

$$\frac{a - c_2 Y}{Y}\, dY = \frac{c_2 X - b}{X}\, dX$$

SOLVE:

$$\underline{\log Y^a X^b = c_1 Y + c_2 X + K}.$$

?

EXC: **PROVE:** FOR THE PRED. PREY EQS, THE SOLUTIONS DO <u>ACTUALLY CLOSE</u>.

<u>EX</u>: <u>DDT AND PEST CONTROL</u>

EX: DDT AND PEST CONTROL

ASSUME WE HAVE A CROP EATING PEST, BUT WE ALSO HAVE PREDITOR KEEPING IT SOMEWHAT IN CHECK.

WANT TO SIMULATE PEST CONTROL VIA A CHEMICAL AGENT LIKE DDT WHICH MAY EFFECT BOTH POPULATIONS

ASSUME DDT KILLS A PERCENT OF EACH POP. WE MODIFY THE EQUATIONS TO

$$\frac{dX}{dt} = aX - c_1 XY - p_1 X = (a-p_1)X - c_1 XY$$

$$\frac{dY}{dt} = c_2 XY - bY - p_2 Y = c_2 XY - (b+p_2)Y.$$

EQ SOLUTIONS: $(0,0)$, $\left(\frac{b+p_2}{c_2}, \frac{a-p_1}{c_1}\right)$

SO, HOW DOES EQ. POP CHANGE?

ASSUME $p_2 = 0$. HOW DOES

ASSUME $\gamma_2 = 0$. HOW DOES EQ. POP FOR PRED CHANGE?

NOTHING, IN FACT EQ. POP. FOR PRED. GOES DOWN.