Fifth Worksheet, MATH 7233

October 13, 2021

1. Let G be a tournament where no player beat everyone else. Show that G has at least 2 pseudo champions.

2. Show that in a graph that has no cycles of length 4 (may have longer or shorter cycles),
$$e \leq \frac{1}{2}n^{3/2} + \frac{n}{4}.$$

3. Is there a number R such that if G is any graph on at least R nodes then G or G^c contains a K_4 subgraph?

4. Show that $\chi(G) \leq 2$ if and only if G does not contain any odd cycles.

5. Let $\Delta = \Delta(G)$ denote the largest degree in a graph. Show that $\chi(G) \leq \Delta(G) + 1$.

6. Let L_G be the Laplace matrix associated to a graph G. Find the smallest eigenvalue of L_G. Describe all the eigenfunctions corresponding to this eigenvalue.

7. Let $S \subset V(G)$, and define the vector $\chi_S \in \mathbb{R}^{V(G)}$ to be $\chi_S(v) = 1$ if $v \in S$ and 0 otherwise.

   (a) Compute $\chi_S^T \cdot L_G \cdot \chi_S$ in combinatorial terms.

   (b) Let $\mathbb{1} = \chi_{V(G)}$ be the constant 1 vector. Find $\alpha, \beta \in \mathbb{R}$ and $g \in \mathbb{R}^{V(G)}$ such that $\langle g, \mathbb{1} \rangle = 0$ and
   $$\chi_S = \alpha \mathbb{1} + \beta g$$