Homework 1.

Q. 1

(1) Sum: $a_1+b_1\sqrt{2}+a_2+b_2\sqrt{2} = a_1+a_2+(b_1+b_2)\sqrt{2}$

Product: $(a_1+b_1\sqrt{2})\cdot(a_2+b_2\sqrt{2}) = a_1a_2+a_1b_2\sqrt{2}+a_2b_1\sqrt{2}+2b_1b_2$

① identity for sum. $\exists x \in F$ s.t $x+e=e+x, \forall x \in F$

when $e=0$ $0+a+b\sqrt{2} = a+b\sqrt{2}$

② $(a_1+b_1\sqrt{2}+a_2+b_2\sqrt{2})+a_3+b_3\sqrt{2}$

$= a_1 b_1\sqrt{2}+(a_2+b_2\sqrt{2}+a_3+b_3\sqrt{2})$

which satisfies associativity for sum

③ $a_1+b_1\sqrt{2}+a_2+b_2\sqrt{2}=0$ is possible

when $a_1+a_2=0$ $b_1+b_2=0$.

④ $a_1+b_1\sqrt{2}+a_2+b_2\sqrt{2} = a_2+b_2\sqrt{2}+a_1+b_1\sqrt{2}$

for any a_1 b_1 a_2 b_2, since the same rule applies to

a_1+a_2 and b_1+b_2 (definition of sum)

⑤ $1\times(a+b\sqrt{2}) = a+b\sqrt{2}$ which satisfies the multiplicative

⑥ $(a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2})(a_3+b_3\sqrt{2})$

$=(a_1a_2+a_1b_2\sqrt{2}+a_2b_1\sqrt{2}+2b_1b_2)(a_3+b_3\sqrt{2})$

$= a_1a_2a_3+a_1b_2a_3\sqrt{2}+a_2a_3b_1\sqrt{2}+2b_1b_2a_3+a_1a_2b_3\sqrt{2}$

$+2a_1b_2b_3+2a_2b_1b_3+2\sqrt{2}b_1b_2b_3$

$= (a_1+b_1\sqrt{2})(a_2a_3 + a_2b_3\sqrt{2} + a_3b_2\sqrt{2} + 2b_2b_3)$

which satisfies the associative rule.

⑦ $(a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2}+a_3+b_3\sqrt{2})$

$= a_1a_2 + a_1b_2\sqrt{2} + a_1a_3 + a_1b_3\sqrt{2} + a_2b_1\sqrt{2} + 2b_1b_2 + a_3b_1\sqrt{2}$

$+ 2b_1b_3$

$(a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2}) + (a_1+b_1\sqrt{2})(a_3+b_3\sqrt{2})$

$= a_1a_2 + a_1b_2\sqrt{2} + a_2b_1\sqrt{2} + 2b_1b_2 + a_1a_3 + a_1b_3\sqrt{2} + a_3b_1\sqrt{2}$

$+ 2b_1b_3$

So $(a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2}+a_3+b_3\sqrt{2}) = (a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2})$

$+ (a_1+b_1\sqrt{2})(a_3+b_3\sqrt{2})$ which satisfies the distributivity

⑧ $(a_2+b_2\sqrt{2})(a_1+b_1\sqrt{2})$.

$= a_1a_2 + a_2b_1\sqrt{2} + a_1b_2\sqrt{2} + 2b_1b_2 = (a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2})$

which satisfies Commutative for product.

⑨ $(a+b\sqrt{2})^{-1} = x+y\sqrt{2}$

$(a+b\sqrt{2}) \times (x+y\sqrt{2}) = 1$

$ax+2by+(ay+bx)\sqrt{2} = 1$

So $ax+2by = 1$

$(ay+bx)\sqrt{2} = 0$.

solve $\begin{bmatrix} a & 2b & 1 \\ b & a & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2b/a & 1/a \\ 1 & a/b & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2b/a & 1/a \\ 0 & a/b - 2b/a & -1/a \end{bmatrix}$

So we get x & y in the form of a & b. which satisfies the inverse of product.

(2) Sum: $a_1+b_1\sqrt{-1}+a_2+b_2\sqrt{-1} = a_1+a_2+(b_1+b_2)\sqrt{-1}$

product, $(a_1+b_1\sqrt{-1})(a_2+b_2\sqrt{-1}) = a_1a_2+a_1b_2\sqrt{-1}+a_2b_1\sqrt{-1}-b_1b_2$

① $0+a_1+b_1\sqrt{-1} = a_1+b_1\sqrt{-1}$

② $(a_1+b_1\sqrt{-1}+a_2+b_2\sqrt{-1})+a_3+b_3\sqrt{-1} = a_1b_1\sqrt{-1}+(a_2+b_2\sqrt{-1}+a_3+b_3\sqrt{-1})$

$=(a_1+a_2+a_3)+(b_1+b_2+b_3)\sqrt{-1}$

③ $a_1+a_2=0$ $b_1+b_2=0$ is possible.

So it satisfies inverse for sum.

④ $a_1+b_1\sqrt{-1}+a_2+b_2\sqrt{-1} = a_2+b_2\sqrt{-1}+a_1+b_1\sqrt{-1} = (a_1a_2)+(b_1+b_2)\sqrt{-1}$

⑤ $1\times(a_1+b_1\sqrt{-1}) = a_1+b_1\sqrt{-1}$

⑥ $(a_1+b_1\sqrt{-1})(a_2+b_2\sqrt{-1})(a_3+b_3\sqrt{-1})$

$=(a_1a_2+a_1b_2\sqrt{-1}+a_2b_1\sqrt{-1}-b_1b_2)(a_3+b_3\sqrt{-1})$

$=a_1a_2a_3+a_1b_2a_3\sqrt{-1}+a_2a_3b_1\sqrt{-1}-a_3b_1b_2+a_1a_2b_3\sqrt{-1}$
$-a_1b_2b_3-a_2b_1b_3-b_1b_2b_3\sqrt{-1}$

$=(a_1+b_1\sqrt{-1})(a_2a_3+a_2b_3\sqrt{-1}+a_3b_2\sqrt{-1}-b_2b_3)$

which satisfies associative.

⑦ $(a_1+b_1\sqrt{-1})(a_2+b_2\sqrt{-1}+a_3+b_3\sqrt{-1})$

$=a_1a_2+a_1b_2\sqrt{-1}+a_1a_3+a_1b_3\sqrt{-1}+a_2b_1\sqrt{-1}-b_1b_2+a_3b_1\sqrt{-1}$
$-b_1b_3$

$=(a_1+b_1\sqrt{-1}+a_2+b_2\sqrt{-1})(a_3+b_3\sqrt{-1})$

which satisfies the distributivity

⑧ $(a_1+b_1\sqrt{-1})(a_2+b_2\sqrt{-1}) = a_1a_2 + a_1b_2\sqrt{-1} + a_2b_1\sqrt{-1} - b_1b_2$

$= (a_2+b_2\sqrt{-1})(a_1+b_1\sqrt{-1})$

which satisfies the commutativity

⑨ $(a+b\sqrt{-1})^{-1} = x + y\sqrt{-1}$

$ax+2by + (ay+bx)\sqrt{-1} = 1$

So $ax+2by = 1$

$ay+bx = 0$

which happens to be the same x & y solution as the previous question.

Q2

1. Matrix multiplication is not commutative. $AB \neq BA$
2. Only determinant could determine whether the matrix is invertible.

Q3

+	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

−	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

Q4.

sum: $a_1+a_2+(b_1+b_2)\sqrt{-1}$

product: $a_1a_2+a_1b_2\sqrt{-1}+a_2b_1\sqrt{-1}-b_1b_2$

① $0+a_1+b_1\sqrt{-1} = a_1+b_1\sqrt{-1}$

② $(a_1+b_1\sqrt{-1}+a_2+b_2\sqrt{-1})+a_3+b_3\sqrt{-1} = a_1b_1\sqrt{-1}+(a_2+b_2\sqrt{-1}+a_3+b_3\sqrt{-1})$

$=(a_1+a_2+a_3)+(b_1+b_2+b_3)\sqrt{-1}$

③ $a_1+a_2=0$ $b_1+b_2=0$ is possible.

So it satisfies inverse for sum.

④ $a_1+b_1\sqrt{-1}+a_2+b_2\sqrt{-1} = a_2+b_2\sqrt{-1}+a_1+b_1\sqrt{-1} = (a_1a_2)+(b_1+b_2)\sqrt{-1}$

⑤ $1\times(a_1+b_1\sqrt{-1}) = a_1+b_1\sqrt{-1}$

⑥ $(a_1a_2+a_1b_2\sqrt{-1}+a_2b_1\sqrt{-1}-b_1b_2)(a_3+b_3\sqrt{-1})$

$=a_1a_2a_3+a_1b_2a_3\sqrt{-1}+a_2a_3b_1\sqrt{-1}-a_3b_1b_2+a_1a_2b_3\sqrt{-1}$
$-a_1b_2b_3-a_2b_1b_3-b_1b_2b_3\sqrt{-1}$

$=(a_1+b_1\sqrt{-1})(a_2a_3+a_2b_3\sqrt{-1}+a_3b_2\sqrt{-1}-b_2b_3)$

So $(a_1+b_1\sqrt{-1})(a_2+b_2\sqrt{-1})(a_3+b_3\sqrt{-1}) = (a_1+b_1\sqrt{-1})$
$(a_2+b_2\sqrt{-1})(a_3+b_3\sqrt{-1})$

⑦ $(a_1+b_1\sqrt{-1})(a_2+{}_2\sqrt{-1}+a_3+b_3\sqrt{-1})$

$= a_1a_2 + a_1b_2\sqrt{-1} + a_1a_3 + a_1b_3\sqrt{-1} + a_2b_1\sqrt{-1} - b_1b_2 + a_3b_1\sqrt{-1}$
$- b_1b_3$

$= (a_1 + b_1\sqrt{-1} + a_2 + b_2\sqrt{-1})(a_3 + b_3\sqrt{-1})$

So $(a_1+b_1\sqrt{-1} + a_2+b_2\sqrt{-1})(a_3+b_3\sqrt{-1}) = (a_1+b_1\sqrt{-1})$
$(a_2+b_2\sqrt{-1} + a_3+b_3\sqrt{-1})$

8. $(a_1+b_1\sqrt{-1})(a_2+b_2\sqrt{-1}) = a_1a_2 + a_1b_2\sqrt{-1} + a_2b_1\sqrt{-1} - b_1b_2$
$= (a_2+b_2\sqrt{-1})(a_1+b_1\sqrt{-1})$

which satisfies the commutativity

9. $(a+b\sqrt{-1})^{-1} = x + y\sqrt{-1}$

$ax + 2by + (ay+bx)\sqrt{-1} = 1$

So $ax + 2by = 1$
$ay + bx = 0$

which happens to be the same x & y solution as the previous question.

Q5

RREF: B, C, D.

Q6. $A+B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \Leftarrow [0] = [2]$

$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ in $\mathbb{Z}_2$

$$\begin{matrix} 6 & -1 & 1 \\ 0 & 0 & 6/t \end{matrix}$$

Q7

$$\det(A) = 6\cdot(0-1) + t^2 + t$$

$$= t^2 + t - 6$$

$$= (t-2)(t+3)$$

so $t \neq 2$ & -3

Q8

a) $\xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & h & 4 \\ 0 & 6-3h & -4 \end{bmatrix}$ $\xrightarrow[\frac{1}{6-3h}R_2]{R_1 - \frac{h}{6-3h}R_2} \begin{bmatrix} 1 & 0 & 4+\frac{4h}{6-3h} \\ 0 & 1 & -\frac{4}{6-3h} \end{bmatrix}$

$h \neq 2$.

b) $\xrightarrow[\frac{1}{2}R_2]{R_1 + 2R_2} \begin{bmatrix} 0 & 0 & h-6 \\ 1 & -3 & -\frac{3}{2} \end{bmatrix}$

$h = 6$.

Q9

(1) 4 :

1 leading 1 0 leading 1 2 leading 1

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

(2) 7 :

2 leading 1 1 leading 1 0 leading 1

$\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}$ $\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & * & 0 \\ 0 & * & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(3) a. $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ $\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

Q10. $a \in \mathbb{R}$ $b=0$ $c=1$ $d=0$ $e=0$

Q11

(1) $A \Rightarrow \begin{array}{c} \\ R_2-R_1 \\ R_3-2R_1\end{array}\begin{bmatrix}1 & 2 & 3 & 4\\0 & -1 & -3 & -2\\0 & -4 & -5 & -6\end{bmatrix} \begin{array}{c} R_1+2R_2 \\ -R_2 \\ -4R_2+R_3\end{array}\begin{bmatrix}1 & 0 & -3 & 8\\0 & 1 & 3 & 2\\0 & 0 & 7 & 2\end{bmatrix}$

$\Rightarrow \begin{array}{c} \\ \\ \frac{1}{7}R_3\end{array}\begin{bmatrix}1 & 0 & -3 & 8\\0 & 1 & 3 & -2\\0 & 0 & 1 & 2/7\end{bmatrix} \Rightarrow \begin{array}{c} R_1+3R_3 \\ R_2-3R_3 \\ \\ \end{array}\begin{bmatrix}1 & 0 & 0 & 6/7\\0 & 1 & 0 & 8/7\\0 & 0 & 1 & -2\end{bmatrix}$

$x_1 = -2x_4$
$x_2 = -4x_4$
$x_3 = 2x_4$

$\vec{x} = \begin{pmatrix}-6/7\\-8/7\\2\\1\end{pmatrix} x_4$

(2) $A \Rightarrow \begin{array}{c} \\ R_2+6R_1 \\ R_2+7R_1\end{array}\begin{bmatrix}1 & 2 & 3 & 4\\0 & 6 & 4 & 5\\0 & 3 & 2 & 1\end{bmatrix} R_2+3R_3 \begin{bmatrix}1 & 2 & 3 & 4\\0 & 1 & 3 & 1\\0 & 3 & 2 & 1\end{bmatrix} \begin{array}{c} R_1+11R_2 \\ \\ R_3+4R_2\end{array}\begin{bmatrix}1 & 0 & 4 & 2\\0 & 1 & 3 & 1\\0 & 0 & 0 & 5\end{bmatrix}$

$\begin{array}{c} R_1+R_3 \\ R_2+11R_3 \\ 10R_3\end{array}\begin{bmatrix}1 & 0 & 4 & 0\\0 & 1 & 3 & 0\\0 & 0 & 0 & 1\end{bmatrix}$

(3) Input: $\Rightarrow \mathbb{Z}_2$

```
GF2 = galois.GF(2)
GF2.row_reduce(GF2([[1,0,1,0],
                    [1,1,0,0],
                    [0,0,1,0]]))
```

```
GF2 = galois.GF(2)
B= GF2([[1,0,1,0],
    [1, 1, 0,0],
    [0,0, 1, 0]]);
GF2.row_reduce(B)
```

```
GF([[1, 0, 0, 0],
    [0, 1, 0, 0],
    [0, 0, 1, 0]], order=2)
```

Output:

GF([[1,0,0,0],
[0,1,0,0],
[0,0,1,0]], order=2)

$\mathbb{Z}_3$:

Input:

GF3 = galois.GF(3)

GF3.row_reduce(GF3([[1,2,0,1],
[1,1,0,2],
[2,0,1,2]])

Output: GF([[1,0,0,0],
[0,1,0,2],
[0,0,1,2]], order=3)

```
GF3 = galois.GF(3)
B= GF3([[1,2,0,1],
    [1, 1, 0,2],
    [2,0, 1, 2]]);
GF3.row_reduce(B)
```

```
GF([[1, 0, 0, 0],
    [0, 1, 0, 2],
    [0, 0, 1, 2]], order=3)
```

(4) NO, the rank should be the same

Q12.

(1)

```
GF7 = galois.GF(7)
B= GF7([[3,1,4,1],
    [5, 2, 6,5],
    [0,5, 2, 1]]);
GF7.row_reduce(B)
```

```
GF([[1, 0, 0, 4],
    [0, 1, 0, 3],
    [0, 0, 1, 0]], order=7)
```

$$RREF = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(2) Solution: $\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$

Q13

Input: Matrix([[3, 11, 19, -2],
[7, 23, 39, 10],
[-4, -3, -2, 6]]).rref()

output: Matrix([[1, 0, -1, 0]
[0, 1, 2, 0]
[0, 0, 0, 1]]),
(0, 1, 3))

rref: $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Solution: no solution.

Q14.

Input: Matrix([[3, 6, 9, 5, 25, 53],
[7, 14, 21, 9, 53, 105],
[-4, -8, -12, 5, -10, 11]]).rref()

Output: Matrix([[1, 2, 3, 0, 5, 6],
[0, 0, 0, 1, 2, 7],
[0, 0, 0, 0, 0, 0]]),
(0, 3)]

Q15. Input: Matrix([[2, 4, 3, 5, 6, 37],
[4, 8, 7, 5, 2, 74],
[-2, -4, 3, 4, -5, 20],
[1, 2, 2, -1, 2, 26],
[5, -10, 4, 6, 4, 24]]).rref()

Output: $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -8221/4340 \\ 0 & 1 & 0 & 0 & 0 & 8591/8680 \\ 0 & 0 & 1 & 0 & 0 & 4195/434 \\ 0 & 0 & 0 & 1 & 0 & -459/434 \\ 0 & 0 & 0 & 0 & 1 & 699/434 \end{bmatrix}$

Solution: $\begin{bmatrix} -8224/4340 \\ 8591/8680 \\ 4695/434 \\ -409/434 \\ 699/434 \end{bmatrix}$

Q16.

(1) Yes.

$A^{-1} = BC$. $B^{-1} = AC$ $C^{-1} = AB$.

(2) Yes,

$(AB)^{-1} = A^{-1}B^{-1}$

Q17.

$ABAB = AABB$

Since $BA \neq AB$ in some case.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

Q18. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Q19.

(1) Symmetric $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 2 & 3 \\ 4 & 3 & 2 \end{bmatrix}$. $\begin{bmatrix} 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 2 \\ 5 & 2 & 3 & 2 \\ 6 & 2 & 2 & 3 \end{bmatrix}$

Skew-symmetric $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -2 \\ 1 & 2 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$

(2) the element on the diagonal is zero

(3) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(4) $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ $A^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$

$$A + A^T = \begin{bmatrix} 2a & b+d & g+c \\ b+d & 2e & f+h \\ g+c & f+h & 2i \end{bmatrix}$$

Because a_{ij} becomes a_{ji} after transposing.
if the element of $A+A^T$ is b,
$b_{ij} = a_{ij} + a_{ji}$ always. which makes it symmetric.

Same rule applies to $\overset{\Rightarrow c}{AA^T}$ $\overset{\Rightarrow d}{A^TA}$.
$c_{ij} = a_{ij} a_{ji}$.
$d_{ij} = a_{ji} a_{ij}$

Element in $A - A^T$ would be $a_{ij} - a_{ji}$ for uppertriangle
and $a_{ji} - a_{ij}$ for lower triangle; which makes
the upper triangle plus the lower triangle is zero.

$$(A+A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$$

$$(AA^T)^T = (A^T)^T A^T = AA^T$$

$$(A^TA)^T = A^T(A^T)^T = A^TA$$

$$A-A^T = \begin{pmatrix} 2a & b-c \\ c-b & 2d \end{pmatrix} = -(A-A^T)$$

(5) $A = \frac{1}{2}A + \frac{1}{2}A + \frac{1}{2}A^t - \frac{1}{2}A^t$

$$= \underbrace{\frac{1}{2}(A+A^t)}_{\text{Symmetric}} + \underbrace{\frac{1}{2}(A-A^t)}_{\text{Skew-symmetric.}}$$

Q20.

(a) surjective $\rightarrow$ at least one solution.

$y = \pm x$

(b) bijective $\rightarrow$ at least one solution.

$x^2+1 \neq 0$.

(c) surjective $\rightarrow$ at least one solution.

there are second order x

(d) injective $\rightarrow$ exactly one solution

it's one to one

$x = \ln(y-3)$

Q21

$$L=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$A \Rightarrow R_2-R_1/4\begin{bmatrix}4&1&0&0\\0&15/4&1&0\\0&1&4&1\\0&0&1&4\end{bmatrix} \quad R_3-\frac{4R_2}{15}\begin{bmatrix}4&1&0&0\\0&15/4&1&0\\0&0&56/15&1\\0&0&1&4\end{bmatrix} \quad R_4-\frac{15R_3}{56}\begin{bmatrix}4&1&0&0\\0&15/4&1&0\\0&0&56/15&1\\0&0&0&209/56\end{bmatrix}$$

$$L=\begin{bmatrix}1&0&0&0\\1/4&1&0&0\\0&4/15&1&0\\0&0&15/56&1\end{bmatrix} \quad U=\begin{bmatrix}4&1&0&0\\0&15/4&1&0\\0&0&56/15&1\\0&0&0&209/56\end{bmatrix}$$

Q22

$$d_n = l_n^{-1}$$

$$l_1=\frac{1}{q_1-\frac{p_1}{q_1}} \quad l_2=\frac{1}{q_2-\frac{p_2}{q_1-\frac{p_1}{q_1}}} \quad l_3=\frac{1}{q_3-\frac{p_3}{q_2-\frac{p_2}{q_2}}}$$

$$r_n = u_n$$

Q23.

$$L=\begin{bmatrix}1&0&0&\cdots\\ \frac{1}{4}&1&0&\cdots\\ \vdots&\ddots&\ddots&\cdots\\ \vdots&&\frac{1}{q_n-\frac{p_n}{q_{n-1}-\frac{p_{n-1}}{q_{n-1}}}}&1\end{bmatrix} \quad U=\begin{bmatrix}4&1&0&\cdots&\cdots\\0&\frac{15}{4}&1&0&\cdots\\ \vdots&&\ddots&\ddots&\\ \vdots&&&0&q_n-\frac{p_n}{q_{n-1}-\frac{p_{n-1}}{q_{n-1}}}\end{bmatrix}$$

Q24

(1) $H_n^T = I_n + (-2\vec{u}\vec{u}^T)^T = I_n - 2(\vec{u}^T)^T(\vec{u})^T$

$= I_n - 2\vec{u}\vec{u}^T$

so H_n is symmetric.

(2) $H_n^T = H_n \quad H_n^T H_n = H_n^2 = (I_n - 2\vec{u}\vec{u}^T)^2$

$= I_n + 4(\vec{u}\vec{u}^T)^2 - 4\vec{u}\vec{u}^T$

$= I_n + 4\vec{u}\vec{u}^T\vec{u}\vec{u}^T - 4\vec{u}\vec{u}^T$

So H_n is an orthogonal matrix $= I_n + 4\vec{u}\cdot 1\cdot\vec{u}^T - 4\vec{u}\vec{u}^T$

$= I_n$

(3) $H_n^2 = I_n$

(4) $H_n\vec{u} = (I_n - 2\vec{u}\vec{u}^T)\vec{u} = I_n - 2\vec{u}\vec{u}^T\vec{u} = I_n - 2\vec{u}$

(5) $H_3 = I_3 - 2\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\end{bmatrix}$

$= \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix} = \begin{bmatrix}1/3 & -2/3 & -2/3\\ -2/3 & 1/3 & -2/3\\ -2/3 & -2/3 & 1/3\end{bmatrix}$

$H_4 = I_4 - 2\cdot\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\cdot\frac{1}{2}\begin{bmatrix}1&1&1&1\end{bmatrix}$

$= \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix} = \begin{bmatrix}1/2 & -1/2 & -1/2 & -1/2\\ -1/2 & 1/2 & -1/2 & -1/2\\ -1/2 & -1/2 & 1/2 & -1/2\\ -1/2 & -1/2 & -1/2 & 1/2\end{bmatrix}$