1. a.

(1) Identity for sum: $0 \in a+b\sqrt{2}$ when $a=0, b=0$

$0 + a + b\sqrt{2} = a + b\sqrt{2}$

(2) Associativity of sum: $(a_1+b_1\sqrt{2} + a_2+b_2\sqrt{2}) + a_3+b_3\sqrt{2}$

$= a_1+b_1\sqrt{2} + (a_2+b_2\sqrt{2} + a_3+b_3\sqrt{2})$

(3) $a_1+b_1\sqrt{2} + a_2+b_2\sqrt{2} = 0$ when $a_1 = -a_2$, $b_1 = -b_2$

(4) $a_1+b_1\sqrt{2} + a_2+b_2\sqrt{2} = a_2+b_2\sqrt{2} + a_1+b_1\sqrt{2}$

(5) when $a=1, b=0$, $a+b\sqrt{2} = 1$ and $1 \cdot (a+b\sqrt{2}) = a+b\sqrt{2}$

(6) $((a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2}))(a_3+b_3\sqrt{2}) = (a_1+b_1\sqrt{2})((a_2+b_2\sqrt{2}) \times (a_3+b_3\sqrt{2}))$

(7) $(a_1+b_1\sqrt{2}) \times [(a_2+b_2\sqrt{2}) + (a_3+b_3\sqrt{2})]$

$= (a_1+b_1\sqrt{2}) \times (a_2+b_2\sqrt{2}) + (a_1+b_1\sqrt{2}) \times (a_3+b_3\sqrt{2})$

(8) $(a_1+b_1\sqrt{2}) \times (a_2+b_2\sqrt{2}) = (a_2+b_2\sqrt{2})(a_1+b_1\sqrt{2})$

(9) $a_1+b_1\sqrt{2} + a_2+b_2\sqrt{2} = (a_1+a_2) + (b_1+b_2)\sqrt{2}$ for sum

$(a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2}) = (a_1a_2 + 2b_1b_2) + (a_1b_2 + a_2b_1)\sqrt{2}$ for multi

$\Rightarrow$ inverse in multi, $(a_1+b_1\sqrt{2}) \cdot (a_2+b_2\sqrt{2}) = 1$

when $a_2 + b_2\sqrt{2} = \frac{1}{a_1+b_1\sqrt{2}}$

$a_2 = \frac{a_1}{a_1^2 - 2b_1^2}$, $b_2 = -\frac{b_1}{a_1^2 - 2b_1^2}$

(b) For sum: $a_1+b_1\sqrt{-1} + a_2+b_2\sqrt{-1} = (a_1+a_2) + (b_1+b_2)\sqrt{-1}$ ✓

For multi: $(a_1+b_1\sqrt{-1})(a_2+b_2\sqrt{-1}) = (a_1a_2 - b_1b_2) + (a_1b_2 + b_1a_2)\sqrt{-1}$ ✓

(1) $0 \in a+b\sqrt{-1}$

(2) $((a_1+b_1\sqrt{-1}) + (a_2+b_2\sqrt{-1})) + (a_3+b_3\sqrt{-1}) = (a_1+b_1\sqrt{-1}) + (a_2+b_2\sqrt{-1} + a_3+b_3\sqrt{-1})$

(3) $a_1+b_1\sqrt{-1} + a_2+b_2\sqrt{-1} = 0$ if $a_1 = -a_2$, $b_1 = -b_2$

(4) $(a_1+b_1\sqrt{-1}) + (a_2+b_2\sqrt{-1}) = (a_2+b_2\sqrt{-1}) + (a_1+b_1\sqrt{-1})$

(5) when $a_1 = 1, b_1 = 0$, there is $1 \in R$ for

$1 \cdot (a_1+b_1\sqrt{-1}) = a_1+b_1\sqrt{-1}$

(6) $((a_1+b_1\sqrt{-1})(a_2+b_2\sqrt{-1}))(a_3+b_3\sqrt{-1}) = (a_1+b_1\sqrt{-1})((a_2+b_2\sqrt{-1})(a_3+b_3\sqrt{-1}))$

(7) $(a_1+b_1\sqrt{-1})((a_2+b_2\sqrt{-1})+(a_2+b_3\sqrt{-1})) = (a_1+b_1\sqrt{-1})(a_2+b_2\sqrt{-1})$
$+ (a_1+b_1\sqrt{-1})(a_3+b_3\sqrt{-1})$

(8) $(a_1+b_1\sqrt{-1})(a_2+b_2\sqrt{-1}) = (a_2+b_2\sqrt{-1})(a_1+b_1\sqrt{-1})$

(9) $(a_1+b_1\sqrt{-1})\cdot(a_2+b_2\sqrt{-1}) = 1$

$$\Rightarrow a_2+b_2\sqrt{-1} = \frac{1}{a_1+b_1\sqrt{-1}} = \frac{a_1-b_1\sqrt{-1}}{a_1^2+b_1^2}$$

$$= \frac{a_1}{a_1^2+b_1^2} - \frac{b_1}{a_1^2+b_1^2}\cdot\sqrt{-1}$$

$$\Rightarrow a_2 = \frac{a_1}{a_1^2+b_1^2}, \quad b_2 = -\frac{b_1}{a_1^2+b_1^2}$$

2. $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ has no inverse in multiplication so it is not a field in $n \times n$ matrix when $n > 1$.

3. $\mathbb{Z}_3$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

4. In sum: $a_1+a_2+b_1i+b_2i \Rightarrow (a_1+a_2)+(b_1+b_2)i$ and
when $a_1=-a_2$, $b_1=-b_2$, $(a_1+b_1i)^{-1}=(a_2+b_2i)$

In scaler product: $(a_1+b_1i)\cdot(a_2+b_2i)$
$= a_1a_2 + b_1b_2 i$
If $a_1 = \cancel{\ } \frac{-1}{a_2}$, $b_1 = -\frac{2i}{b_2}$
$\Rightarrow (a_1+b_1i)^{-1} = (a_2+b_2i)$

In product: $(a_1+b_1i)\times(a_2+b_2i) = a_1a_2+a_1b_2i+a_2b_1i+(-b_1b_2)$
$= (a_1a_2-b_1b_2)+(a_1b_2+a_2b_1)i$

make $a_2+b_2i = \frac{1}{a_1+b_1i} = \frac{a_1-b_1i}{a_1^2+b_1^2} = \frac{a_1}{a_1^2+b_1^2} - \frac{b_1}{a_1^2+b_1^2}i$

Thus, it has inverse in product

So it is a field.

5. A is not because of the second and third row
B is reduced row echelon form
C is not because its second row is all zero element.
D is reduced row-echelon form
E is not because first row and third row are all zero element.

6. $A+B$: $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

A^2: $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

AB $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

~~8.~~ If $t=6$ or $t=-1$

when $t=6$, $\left[\begin{array}{ccc|ccc} 6 & -1 & 1 & 1 & 0 & 0 \\ 6 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 6 & 0 & 0 & 1 \end{array}\right]$

row 1 $-$ row 2 $\Rightarrow$ $\left[\begin{array}{ccc|ccc} 6 & -1 & 1 & 1 & 0 & 0 \\ 0 & \cancel{0}-1 & 0 & \cancel{0}1 & 1 & 0 \\ 0 & 1 & 6 & 0 & 0 & 1 \end{array}\right]$

7. $\left[\begin{array}{ccc|ccc} 6 & -1 & 1 & 1 & 0 & 0 \\ t & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & t & 0 & 0 & 1 \end{array}\right]$

If row 1 $-$ row 2: $\left[\begin{array}{ccc|ccc} 6 & -1 & 1 & 1 & 0 & 0 \\ 6-t & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & t & 0 & 0 & 1 \end{array}\right]$

row 1 + row 3: $\left[\begin{array}{ccc|ccc} 6 & -1 & 1 & 1 & 0 & 0 \\ 6-t & -1 & 0 & 1 & -1 & 0 \\ 6 & 0 & 1+t & 1 & 0 & 1 \end{array}\right]$

We make $t=\frac{1}{n}\cdot 6$ for $n \in \mathbb{Z}^+$ $\Rightarrow$ we should row 1 $-$ $n\cdot$ row 2

$\Rightarrow \begin{bmatrix} 6 & -1 & 1 \\ 0 & -1 & 1-n \\ 0 & 1 & \frac{1}{n}\cdot 6 \end{bmatrix}$ Then, if $1-n = -\frac{1}{n}\cdot 6$

$n-n^2=-6$

$n^2-n-6=0$

$\Rightarrow n=3$ or $n=-2$

Thus, $t=2$ or -3 to make the matrix has no inverse

8. a) $\left[\begin{array}{cc|c} 1 & h & 4 \\ 3 & 6 & 8 \end{array}\right]$

$1x+hy=4$ ①

$3x+6y=8$ ②

$3\times① = 3x+3hy=12$ ③

$③-② = 3x+3hy-3x-6y=4 = (3h-6)y=4$

$\Rightarrow h\neq 2$

Thus it is not consistent.

b) $\left[\begin{array}{cc|c} -4 & 12 & h \\ 2 & -6 & -3 \end{array}\right]$

$-4x+12y=h$ ①

$2x-6y=-3$ ②

$②\times 2+① = -4x+\cancel{12h}\,12y+4x-12y=\cancel{2h}-3\ h-6$

$\Rightarrow h=\cancel{3}\,6$

It is consistent

9.(1) $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ~~6~~ 7 types

⇒ second one
2×3

(2) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ ~~3~~ 4 types ⇒ first one
3×2

(3) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ~~4~~ 5 types

10. a, b, c are in reduced row-echelon form
$e, c = 0$

17.

(1) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \xRightarrow[\text{row3+2row1}]{\text{row2+row1}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 7 & 2 \end{bmatrix}$

$\xRightarrow{\frac{1}{7}\cdot\text{row3}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix} \xRightarrow{\text{row1-row2}} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix} \xRightarrow{\text{row2-3row3}} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & \frac{8}{7} \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix}$

$\xRightarrow{\text{row1-row2}} \begin{bmatrix} 1 & 0 & 0 & \frac{6}{7} \\ 0 & 1 & 0 & \frac{8}{7} \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix} = rref(A)$

$\Rightarrow \begin{cases} x + \frac{6}{7}a = 0 \\ y + \frac{8}{7}a = 0 \\ z + \frac{2}{7}a = 0 \end{cases} \Rightarrow \begin{bmatrix} x \\ y \\ z \\ a \end{bmatrix} = \begin{bmatrix} -\frac{6}{7} \\ -\frac{8}{7} \\ -\frac{2}{7} \\ 1 \end{bmatrix} \cdot t$

$\Rightarrow \vec{A} \cdot t \begin{bmatrix} -\frac{6}{7} \\ -\frac{8}{7} \\ -\frac{2}{7} \\ 1 \end{bmatrix} = \vec{0}$

(2) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xRightarrow{\text{row1-2row3}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xRightarrow{\text{row2-2row3}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$= rref(A)$

(4). The M will have same rank over different field $\mathbb{Z}_p$.

2. (1) $\left[\begin{array}{ccc|c} 3 & 1 & 4 & 1 \\ 5 & 2 & 6 & 5 \\ 0 & 5 & 2 & 1 \end{array}\right] \xrightarrow{-\text{row2}\cdot\frac{3}{5}+\text{row1}} \left[\begin{array}{ccc|c} 3 & 1 & 4 & 1 \\ 0 & \cancel{\ }\frac{1}{5} & \frac{2}{5} & -2 \\ 0 & 5 & 2 & 1 \end{array}\right]$

$\overset{\text{row2}\times 5}{\Longrightarrow} \left.\begin{array}{ccc|c} 3 & 1 & 4 & 1 \\ 0 & 1 & 2 & \cancel{\ } \\ 0 & 5 & 2 & 1 \end{array}\right. \xrightarrow{-\text{row3}+5\cdot\text{row2}} \left[\begin{array}{ccc|c} 3 & 1 & 4 & 1 \\ 0 & 1 & 2 & \cancel{-10}\ 4 \\ 0 & 0 & \cancel{\ }1 & 0 \end{array}\right]$

$\overset{\frac{1}{3}\text{row1}}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 5 & 6 & 5 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cccc} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$

(2) $\left.\begin{array}{r} x = 4 \\ y = 3 \\ z = 0 \end{array}\right\} \Rightarrow \cancel{t \cdot \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} \cdot \vec{x} = \vec{b}}$

$\vec{x} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$

~~15.~~ 13. $\left[\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

$x - z = 0$

$y + 2z = 0$

no solution

14. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} -5 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \\ 7 \\ 0 \end{bmatrix}$

15. $\begin{bmatrix} x \\ y \\ z \\ a \\ b \end{bmatrix} = \begin{bmatrix} -\frac{8221}{4346} \\ -\frac{3541}{860} \\ \frac{4695}{434} \\ -\frac{4454}{434} \\ \frac{699}{434} \end{bmatrix}$

16. (1) $A^{-1} = BC$

$C^{-1} = AB$

$B^{-1} = CA$

(2) $(AB)(AB)^{-1} = I$

$\Rightarrow A \cdot B \cdot B^{-1} A^{-1} = I$

Thus, A and B are invertible

17. $(AB)^2 = ABAB$ $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = A$, $\begin{bmatrix} 1 & 1 \\ 4 & 0 \end{bmatrix} = B$

$(AB)^2 = \begin{bmatrix} 6 & 2 \\ 2 & 1 \end{bmatrix}$

$\neq$

$A^2 B^2 = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$

18. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ inverse $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$$\begin{bmatrix} a^2+b^2=1 & ac+bd=0 \\ ac+bd=0 & c^2+d^2=1 \end{bmatrix}$$

$$\begin{cases} a=\frac{1}{2},\ b=\frac{\sqrt{3}}{2} \Rightarrow b^2+a^2=1 \\ c=-\frac{\sqrt{3}}{2},\ d=\frac{1}{2} \Rightarrow ac+bd=0 \\ \qquad c^2+d^2=1 \end{cases}$$

$$\Rightarrow \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

19. (1) symmetric: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

skew-symmetric: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -2 \\ 1 & 2 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -2 & -4 \\ 2 & 2 & 0 & 3 \\ 3 & 4 & -3 & 0 \end{bmatrix}$

(2) In symmetric, the main diagonal can be any number, but other elements should be symmetric to the diagonal.

In skew-symmetric, the main diagonal should be all 0 and the negative of one side elements should be ~~same~~ symmetric to the other side from the main diagonal.

(3) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(4) $(A+A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$

$(AA^T)^T = (A^T)^T \cdot A^T = AA^T$
$(A^TA)^T = A^T (A^T)^T = A^TA$
because it is $n \times n$ matrix so A^T and AA^T are also $n \times n$ matrix.

$(A-A^T)^T = A^T - A = \cancel{(A-A^T)} -(A-A^T)$

$\Rightarrow$ Thus, $A+A^T$, AA^T, A^TA are symmetric and $A-A^T$ is skew symmetric

(5) Let A be any $n \times n$ matrix, $(A+A^T)$, which is symmetric, plus $(A-A^T)$ which is not symmetric, $(A+A^T)+(A-A^T) = 2A$, so $2A$ can also be any $n \times n$ matrix

20 (a) $\Rightarrow$ surjective

(b) ~~surjective~~ bijective

(c) ~~surjective~~ bijective

(d) bijective

21: $A=\begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & -15 & -4 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & -15 & -4 & 0 \\ 0 & 0 & 56 & -15 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

$U \Rightarrow \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & -15 & -4 & 0 \\ 0 & 0 & 56 & -15 \\ 0 & 0 & 0 & -223 \end{bmatrix}$

$L=\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ 0 & -\frac{1}{15} & 1 & 0 \\ 0 & 0 & \frac{1}{56} & 1 \end{bmatrix}$

22: $d_1=q_1$, $u_1=r_1$, $d_2=-\frac{a_1}{p_1}\cdot q_2+r_1$

$d_3=-\frac{a_2}{p_2}\cdot d_2+r_2$ $\quad u_2=-\frac{a_1}{p_1}\cdot(-d_1)\,\cancel{r_2}$, $d_4=-\frac{a_3}{p_3}\cdot d_3+r_3$

$u_3=\cancel{r_3}-d_2$ $\quad l_1=\frac{1}{d_1}$, $l_2=\frac{1}{d_2}$, $l_3=\frac{1}{d_3}$

23. $U=\begin{bmatrix} 4 & 1 & \cdots & \\ 0 & -15 & -4 & \cdots \\ \vdots & & \ddots & -15 \\ 0 & \cdots & & \end{bmatrix}$ $\quad L=\begin{bmatrix} 1 & 0 & 0 & \cdots \\ \frac{1}{4} & 1 & 0 & 0 \\ 0 & \frac{1}{15} & 1 & 0 \\ 0 & 0 & \frac{1}{56} & 1 \\ 0 & 0 & 0 & -\frac{1}{223} & \ddots \end{bmatrix}$

24. (1) $H_n = I_n - 2\vec{u}\vec{u}^T$

$H_n^T = (I_n - 2\vec{u}\vec{u}^T)^T = I_n^T - 2(\vec{u}^T)^T\vec{u}^T$

$= I_n - 2\vec{u}\vec{u}^T = H_n$

It is symmetric matrix

(2) $H_n^T H_n \Rightarrow$ we have $H_n^T = H_n$

~~(3)~~

~~[illegible]~~ we have

$H_n^2 = H_n^T H_n = (I_n - 2\vec{u}\vec{u}^T)^2$

$= I_n - 4\vec{u}\vec{u}^T + 4(\vec{u}\vec{u}^T)^2$

$= I_n - 4\vec{u}\vec{u}^T + 4\vec{u}(\vec{u}^T\vec{u})\vec{u}^T$

$= I_n - 4\vec{u}\vec{u}^T + 4\vec{u}\vec{u}^T$

$= I_n$ so it is orthogonal

(3) $H_n^2 = I_n$

(4) $H_n\vec{u} = I_n \cdot \vec{u} - 2\vec{u}\vec{u}^T \cdot \vec{u}$

$= I_n \cdot \vec{u} - 2\vec{u}$

$= -\vec{u}$

(5) $H_3 = I_3 - 2\vec{u}\vec{u}^T$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} & \dots \\ \frac{2}{3} & \dots \\ \frac{2}{3} & \dots \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$H_4 = I_4 - 2\vec{u}\vec{u}^T$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$