1) $F = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ $(F, +, \cdot)$

(i) Definitions of + and · (standard for polynomials)

→ Closed under addition

$a_1 + b_1\sqrt{2},\ a_2 + b_2\sqrt{2}$

$(a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2}) = (a_1 + a_2) + (b_1 + b_2)\sqrt{2}$

→ Closed under multiplication

$(a_1 + b_1\sqrt{2})(a_2 + b_2\sqrt{2}) = a_1a_2 + a_1b_2\sqrt{2} + a_2b_1\sqrt{2} + 2b_1b_2$

- Identity for sum: $0 + a + b\sqrt{2} = a + b\sqrt{2}$
- Associativity for sum: $(a_1 + b_1\sqrt{2}) + ((a_2 + b_2\sqrt{2}) + (a_3 + b_3\sqrt{2}))$

$$= a_1 + b_1\sqrt{2} + ((a_2 + a_3) + (b_2 + b_3)\sqrt{2})$$
$$= (a_1 + (a_2 + a_3)) + (b_1 + (b_2 + b_3))\sqrt{2}$$
$$= ((a_1 + a_2) + a_3) + ((b_1 + b_2) + b_3)\sqrt{2}$$
$$= [(a_1 + a_2) + (b_1 + b_2)\sqrt{2}] + a_3 + b_3\sqrt{2}$$
$$= ((a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2})) + (a_3 + b_3\sqrt{2})$$
$$\boxed{A + (B + C) = (A + B) + C}$$

- Inverse for sum: $a_1 + b_1\sqrt{2} + (-a_1 - b_1\sqrt{2}) = 0$
- Commutativity for sum: $a_1 + b_1\sqrt{2} + a_2 + b_2\sqrt{2} = a_2 + b_2\sqrt{2} + a_1 + b_1\sqrt{2}$

$$= (a_1 + a_2) + (b_1 + b_2)\sqrt{2} = (a_1 + a_2) + (b_1 + b_2)\sqrt{2}$$

1)(a) continued

Multiplicative identity: $(a+b\sqrt{2})\cdot 1 = a+b\sqrt{2}$

Associativity for mult:

$\to (a_1+b_1\sqrt{2})\left((a_2+b_2\sqrt{2})(a_3+b_3\sqrt{2})\right) = \left((a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2})\right)(a_3+b_3\sqrt{2})$

$(a_1+b_1\sqrt{2})(a_2a_3 + a_2b_3\sqrt{2} + a_3b_2\sqrt{2} + 2b_2b_3)$

$(a_3+b_3\sqrt{2})(a_1a_2 + a_1b_2\sqrt{2} + a_2b_1\sqrt{2} + 2b_1b_2)$

$= (a_1a_2a_3 + 2a_1b_2b_3 + 2a_2b_1b_3 + 2a_3b_1b_2)$
$+ (a_1a_2b_3 + 2b_1b_2b_3 + a_1a_3b_2 + a_2a_3b_1)\cdot\sqrt{2}$

$= (a_1a_2a_3 + 2a_1b_2b_3 + 2a_2b_1b_3 + 2a_3b_1b_2)$
$+ (a_1a_2b_3 + 2b_1b_2b_3 + a_1a_3b_2 + a_2a_3b_1)\cdot\sqrt{2}$ ✓

$\therefore \underline{A(BC) = (AB)C}$

Commutativity for mult

$(a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2}) = (a_1a_2 + 2b_1b_2) + (a_1b_2 + a_2b_1)\sqrt{2}$

$= (a_2a_1 + 2b_2b_1) + (a_2b_1 + b_2a_1)\sqrt{2} = (a_2+b_2\sqrt{2})(a_1+b_1\sqrt{2})$

$\therefore \underline{AB = BA}$

Multiplicative Inverse

$(a+b\sqrt{2})\cdot(x+y\sqrt{2}) = 1 \to (a+b\sqrt{2})\cdot(a+b\sqrt{2})^{-1} = 1$

$(a+b\sqrt{2})^{-1} = \frac{1}{a+b\sqrt{2}} = \frac{a-b\sqrt{2}}{a^2-2b^2} = \overbrace{\left(\frac{a}{a^2-2b^2}\right)}^{x} + \overbrace{\left(\frac{-b}{a^2-2b^2}\right)}^{y}\sqrt{2}$

$a, b \neq 0$

Multiplicative $\longleftarrow$
inverse of form $a+b\sqrt{2}$ for $a+b\sqrt{2} \neq 0$

1) (1) continued

Distributive

$\rightarrow (a_1 + b_1\sqrt{2})(a_2 + a_3 + b_2\sqrt{2} + b_3\sqrt{2}) = A(B+C)$

$= (a_1a_2 + a_1a_3 + 2b_1b_2 + 2b_1b_3) + (a_1b_2 + a_1b_3 + a_2b_1 + a_3b_1)\sqrt{2}$

$AB+AC = (a_1a_2 + 2b_1b_2 + (a_1b_2 + a_2b_1)\sqrt{2}) + (a_1a_3 + 2b_1b_3 + (a_1b_3 + a_3b_1)\sqrt{2})$

$A(B+C) = AB+AC \implies$ Therefore a field, all conditions met

(2) (Same as question 4) $F = \{a + b\sqrt{-1} : a, b \in \mathbb{R}\}$

Field of Complex #s $\sqrt{-1} = i$

Definitions of $+$ and $\cdot$ (standard polynomial)

Closed under $+$: $(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2) i$

Closed under $\cdot$: $(a_1 + b_1 i) \cdot (a_2 + b_2 i) = a_1a_2 + a_1b_2 i + a_2b_1 i - b_1b_2$

$= a_1a_2 - b_1b_2 + (a_1b_2 + a_2b_1) i$

Identity for sum: $0 + a + bi = a + bi$

Associativity for sum: $(a_1 + b_1 i) + ((a_2 + b_2 i) + (a_3 + b_3 i))$

$= (a_1 + a_2 + a_3) + (b_1 + b_2 + b_3) i$

$= ((a_1 + b_1 i) + (a_2 + b_2 i)) + (a_3 + b_3 i)$

$A + (B + C) = (A + B) + C$

Inverse for sum $a_1 + b_1 i + -(a_1 + b_1 i) = 0$

Commutativity for sum: $a_1 + b_1 i + a_2 + b_2 i = a_2 + b_2 i + a_1 + b_1 i$

$= (a_1 + a_2) + (b_1 + b_2) i = (a_1 + a_2) + (b_1 + b_2) i$

(1) (a) Continued

Multiplicative identity: $(a+bi)\cdot 1 = a+bi$

Associativity for mult

$$\overset{A}{(a_1+b_1 i)}\left(\overset{B}{(a_2+b_2 i)}\overset{C}{(a_3+b_3 i)}\right)$$

$$= a_1a_2a_3 - a_1b_2b_3 - a_2b_1b_3 - a_3b_1b_2$$
$$+ (a_1a_2b_3 + a_1a_3b_2 + a_2a_3b_1 - b_1b_2b_3)\cdot i = A\cdot(BC)$$

$$\left((a_1+b_1 i)(a_2+b_2 i)\right)(a_3+b_3 i)$$

$$= (a_1a_2a_3 - b_1b_2a_3 - a_1b_2b_3 - a_3b_1b_3)$$
$$+ (a_1a_2b_3 - b_1b_2b_3 + a_1a_3b_2 + a_2a_3b_1)\cdot i = (AB)\cdot C$$

$$\Rightarrow A\cdot(BC) = (AB)\cdot C$$

Commutativity for mult

$$(a_1+b_1 i)(a_2+b_2 i) = (a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1)i$$

$$= (a_2a_1 - b_2b_1) + (a_2b_1 - b_2a_1)i = (a_2+b_2 i)(a_1+b_1 i)$$

$$\therefore AB = BA$$

Multiplicative Inverse

$$(a+bi)\cdot(x+yi) = 1 \qquad (a+b\sqrt{2})\cdot(a-b\sqrt{2})^{-1} = 1$$

$$(a+bi)^{-1} = \frac{1}{a+bi} = \frac{a-bi}{a^2+b^2} = \overbrace{\frac{a}{a^2+b^2}}^{x} + \overbrace{\frac{-b}{a^2+b^2}}^{y}\cdot i$$

$a, b \neq 0$

Multiplicative inverse of form $a+bi$ for $a+bi \neq 0$

1) (2) Continued

Distributive

$\Rightarrow (a_1 + b_1 i)(a_2 + a_3 + b_2 i + b_3 i) = A(B+C)$

$= (a_1 a_2 + a_1 a_3 - b_1 b_2 - b_1 b_3) + (a_1 b_2 + a_1 b_3 + a_2 b_1 + a_3 b_1) i$

$AB + AC = (a_1 a_2 - b_1 b_2 + (a_1 b_2 + a_2 b_1) i) + (a_1 a_3 - b_1 b_3 + (a_1 b_3 + a_3 b_1) i$

$A(B+C) = AB + AC \Rightarrow$ Therefore a field, all conditions met

5) On $\mathbb{Z}_3$

+	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

×	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

2) $\overset{A}{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}} \cdot \overset{B}{\begin{bmatrix} 3 & 3 \\ 1 & 3 \end{bmatrix}} = \begin{bmatrix} 3+2 & 3+6 \\ 6+1 & 6+3 \end{bmatrix} = \begin{bmatrix} 5 & 9 \\ 7 & 9 \end{bmatrix}$

$B \cdot A = \begin{bmatrix} 3+6 & 6+3 \\ 1+6 & 2+3 \end{bmatrix} = \begin{bmatrix} 9 & 9 \\ 7 & 5 \end{bmatrix}$

$\begin{bmatrix} 3 & 3 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$AB = BA$, not commutative, ∴ not a field

5) A ← No (Pivot not only nonzero in column)
B ← Yes
C ← No (row of 0s)
D ← Yes
E ← No (Pivot not only nonzero in column

All conditions met

6) $A = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix}$ $B = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$ $A+B = \begin{vmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 1 & 2 & 0 \end{vmatrix}$

$A^2 = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1+1+1 & 1+1 \\ 0 & 1+1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$

$AB = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1+1 & 1+1+1 & 1+1 \\ 1+1 & 1+1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix}$

7) $A = \begin{bmatrix} 6 & -1 & 1 \\ t & 0 & 1 \\ 0 & 1 & t \end{bmatrix}$ det=0, no inverse

$\det A = -1 \cdot \begin{vmatrix} t & 1 \\ 0 & t \end{vmatrix} + 1 \cdot \begin{vmatrix} 6 & 1 \\ t & 1 \end{vmatrix} = -t^2 + 6 - t$

$-t^2 + 6 - t = 0 \rightarrow t^2 + t - 6 \quad (t+3)(t-2) = 0$

$\boxed{t = -3, 2}$

8) a. $\left[\begin{array}{cc|c} 1 & h & 4 \\ 3 & 6 & 8 \end{array}\right]$ $h : \mathbb{R}, h \neq 2$

b. $\left[\begin{array}{cc|c} -4 & 12 & h \\ 2 & -6 & -3 \end{array}\right]$ $h = \frac{3}{2}$

9)

(1) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ One type

(2) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Two types

(3) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ Rank=1

10) $A = \begin{bmatrix} 1 & a & b & 3 & 0 & -2 \\ 0 & 0 & c & 1 & d & 3 \\ 0 & e & 0 & 0 & 1 & 1 \end{bmatrix}$

$a = *$ $\quad d = *$
$b = 0$
$c = 1$
$e = 0$

1)

(1) $\begin{bmatrix}1&2&3&4\\1&1&0&2\\2&0&1&2\end{bmatrix} \xrightarrow[R_3-2R_1]{R_2-R_1} \begin{bmatrix}1&2&3&4\\0&-1&-3&-2\\0&-4&-5&-6\end{bmatrix} \xrightarrow{R_2\cdot -1} \begin{bmatrix}1&2&3&4\\0&1&3&2\\0&-4&-5&-6\end{bmatrix}$

$\xrightarrow{R_3+4R_2} \begin{bmatrix}1&2&3&4\\0&1&3&2\\0&0&7&2\end{bmatrix} \xrightarrow{\frac{1}{7}R_3} \begin{bmatrix}1&2&3&4\\0&1&3&2\\0&0&1&\frac{2}{7}\end{bmatrix}$

$\xrightarrow{R_1-2R_2} \begin{bmatrix}1&0&-3&0\\0&1&3&2\\0&0&1&\frac{2}{7}\end{bmatrix} \xrightarrow[R_2-3R_3]{R_1+3R_3} \begin{bmatrix}1&0&0&\frac{6}{7}\\0&1&0&\frac{8}{7}\\0&0&1&\frac{2}{7}\end{bmatrix} \leftarrow$ RREF

$\left[\begin{array}{cccc|c}1&0&0&\frac{6}{7}&0\\0&1&0&\frac{8}{7}&0\\0&0&1&\frac{2}{7}&0\end{array}\right]$

$x_1 + \frac{6}{7}x_4 = 0 \qquad x_1 = -\frac{6}{7}x_4$

$x_2 + \frac{8}{7}x_4 = 0 \qquad x_2 = -\frac{8}{7}x_4$

$x_3 + \frac{2}{7}x_4 = 0 \qquad x_3 = -\frac{2}{7}x_4$

$x_4 = t$

$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = t\begin{bmatrix}-\frac{6}{7}\\-8/7\\-2/7\\1\end{bmatrix}$

(2) The same until here $\xrightarrow{R_3+4R_2} \begin{bmatrix}1&2&3&4\\0&1&3&2\\0&0&0&-5\end{bmatrix} \xrightarrow[R_3\cdot -\frac{1}{5}]{R_1-2R_2} \begin{bmatrix}1&0&4&0\\0&1&3&2\\0&0&0&1\end{bmatrix}$

$\xrightarrow{R_2-2R_3} \begin{bmatrix}1&0&4&0\\0&1&3&0\\0&0&0&1\end{bmatrix}$

(3) rref(A) over $\mathbb{Z}_2 = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}$

over $\mathbb{Z}_3 = \begin{bmatrix}1&0&0&0\\0&1&0&2\\0&0&1&2\end{bmatrix}$

(4) Yes, different rank is possible

for $B = \begin{bmatrix}1&2&3\\1&1&0\\2&0&1\end{bmatrix}$ over $\mathbb{Z}_7$ rank $= 2$

over $\mathbb{Z}_2$ rank $= 3$

Negatives placeholders for value in $\mathbb{Z}_7$, ie $[-5]=[2]$

12) $A = \left[\begin{array}{ccc|c} 3 & 1 & 4 & 1 \\ 5 & 2 & 6 & 5 \\ 0 & 5 & 2 & 1 \end{array}\right] \xrightarrow{2 \cdot R_1} \left[\begin{array}{ccc|c} 6 & 2 & 1 & 2 \\ 5 & 2 & 6 & 5 \\ 0 & 5 & 2 & 1 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & -5 & -3 \\ 5 & 2 & 6 & 5 \\ 0 & 5 & 2 & 1 \end{array}\right]$

$\xrightarrow{R_2 - 5R_1} \left[\begin{array}{ccc|c} 1 & 0 & -5 & -3 \\ 0 & 2 & 3 & 6 \\ 0 & 5 & 2 & 1 \end{array}\right] \xrightarrow{3 \cdot R_2} \left[\begin{array}{ccc|c} 1 & 0 & -5 & -3 \\ 0 & 6 & 2 & 4 \\ 0 & 5 & 2 & 1 \end{array}\right]$

$\xrightarrow{R_2 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & -5 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 5 & 2 & 1 \end{array}\right] \xrightarrow{R_3 - 5R_2} \left[\begin{array}{ccc|c} 1 & 0 & -5 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -5 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\xrightarrow{R_1 + 5R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{[-3]=4} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$ (1), order = 7

(2) $A\vec{x} = \vec{b}$

$$\begin{bmatrix} 3 & 1 & 4 \\ 5 & 2 & 6 \\ 0 & 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$$

$\rightarrow$ From matrix above, $x_1 = 4$, $x_2 = 3$, $x_3 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$$

13) RREF from python: $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

$0x_3 = 1$ $\therefore$ inconsistent and no solution

14) RREF from python: $\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 0 & 5 & 6 \\ 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$x_4 + 2x_5 = 7$ $\quad x_1 + 2x_2 + 3x_3 + 5x_5 = 6$

$x_4 = 7 - 2x_5$ $\quad x_1 = 6 - 2x_2 - 3x_3 - 5x_5$

$x_5 = t$ $\quad x_3 = s$ $\quad x_2 = r$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 7 \\ 0 \end{bmatrix} + r\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

15) RREF from python: $\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -8221/4340 \\ 0 & 1 & 0 & 0 & 0 & 8591/8680 \\ 0 & 0 & 1 & 0 & 0 & 4695/434 \\ 0 & 0 & 0 & 1 & 0 & -459/434 \\ 0 & 0 & 0 & 0 & 1 & 699/434 \end{array}\right]$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -8221/4340 \\ 8591/8680 \\ 4695/434 \\ -459/434 \\ 699/434 \end{bmatrix}$$

16) (1) Each matrix is invertible, with the product of the other 2 matrices as the inverse

$ABC = I_n \quad A \cdot A^{-1} = I_n \quad A^{-1} = BC$

$\longrightarrow B^{-1} = CA$

$\longrightarrow C^{-1} = AB$

(2) Yes, both matrices are invertible

$C(AB) = I \quad C = (AB)^{-1}$

$\therefore (CA)B = I$

$(CB)A = I \longrightarrow$ both A and B are invertible

17) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1+6 & 2+8 \\ 3+12 & 6+16 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} = A^2$

$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} = B^2$

$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1+2 & 1+4 \\ 3+4 & 3+8 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 7 & 11 \end{bmatrix}$

$\begin{bmatrix} 3 & 5 \\ 7 & 11 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 7 & 11 \end{bmatrix} = \begin{bmatrix} 9+35 & 15+55 \\ 21+77 & 35+121 \end{bmatrix} = \begin{bmatrix} 44 & 70 \\ 98 & 156 \end{bmatrix}$

$A^2B^2 = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} = \begin{bmatrix} 14+45 & 20+66 \\ 21+75 & 30+110 \end{bmatrix} = \begin{bmatrix} 59 & 86 \\ 76 & 140 \end{bmatrix}$

$\hookrightarrow \neq \hookleftarrow$

18) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = I_2$

$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

19) (1) Symmetric matrices

2x2 3x3 4x4

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 5 & 4 & 3 \\ 4 & 5 & 4 \\ 3 & 4 & 5 \end{bmatrix} \quad \begin{bmatrix} 7 & 1 & 2 & 3 \\ 1 & 6 & 9 & 2 \\ 2 & 9 & 5 & 1 \\ 3 & 2 & 1 & 4 \end{bmatrix}$$

Skew-symmetric

2x2 3x3 4x4

$$\begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -4 & -5 \\ 2 & 4 & 0 & -6 \\ 3 & 5 & 6 & 0 \end{bmatrix}$$

(2) The main diagonal of a skew-symmetric matrix must be composed of all zeroes.

(3) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ← both symmetric and skew symmetric

(4) $\longrightarrow A + A^T = (A^T)^T + A^T = (A^T + A)^T = (A + A^T)^T$ ← Symmetric by def

Properties used: (Because $A = (A^T)^T$) (Because $A^T + B^T = (A+B)^T$)

Because $A+B = B+A$

$\longrightarrow AA^T = (A^T)^T A^T = (AA^T)^T$ ← Symmetric by def

Properties used

Because $B^T A^T = (AB)^T$

19 continued) $\longrightarrow A^TA = A^T(A^T)^T = (A^TA)^T$ ← Symmetric by def

Properties: $(A^T)^T = A$ $\quad B^TA^T = (AB)^T$

$\longrightarrow A - A^T = (A^T)^T + (-A^T) = (-A + A^T)^T = -(A - A^T)^T$ ← skew-symmetric by def

Properties: $A^T + B^T = (A+B)^T$ $\quad A+B = B+A$

(5) $A = \frac{A + A^T}{2} + \frac{A - A^T}{2} = \frac{2A}{2} = A$

$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

Scaling by $\frac{1}{2}$ does not change symmetry of matrices

symmetric as shown previously

skew-symmetric as shown previously

$\longrightarrow A = B + C$

$B = \frac{1}{2}(A + A^T)$ ← symmetric

$C = \frac{1}{2}(A - A^T)$ ← skew-symmetric

20) (a) $F(x)=x^2$ is surjective

(b) $F(x)=x^3/(x^2+1)$ is bijective

(c) $F(x)=x(x-1)(x-2)$ is surjective

(d) $F(x)=e^x+2$ is injective

21) $A=\begin{bmatrix}4&1&0&0\\1&4&1&0\\0&1&4&1\\0&0&1&4\end{bmatrix}$ $L=\begin{bmatrix}1&0&0&0\\\frac{1}{4}&1&0&0\\0&\frac{4}{15}&1&0\\0&0&\frac{15}{56}&1\end{bmatrix}$

$\xrightarrow{R_2-\frac{1}{4}R_1}\begin{bmatrix}4&1&0&0\\0&\frac{15}{4}&1&0\\0&1&4&1\\0&0&1&4\end{bmatrix}\xrightarrow{R_3-\frac{4}{15}R_2}\begin{bmatrix}4&1&0&0\\0&\frac{15}{4}&1&0\\0&0&\frac{56}{15}&1\\0&0&1&4\end{bmatrix}$

$\xrightarrow{R_4-\frac{15}{56}R_3}\begin{bmatrix}4&1&0&0\\0&\frac{15}{4}&1&0\\0&0&\frac{56}{15}&1\\0&0&0&\frac{209}{56}\end{bmatrix}$ $A=\begin{bmatrix}1&0&0&0\\\frac{1}{4}&1&0&0\\0&\frac{4}{15}&1&0\\0&0&\frac{15}{56}&1\end{bmatrix}\cdot\begin{bmatrix}4&1&0&0\\0&\frac{15}{4}&1&0\\0&0&\frac{56}{15}&1\\0&0&0&\frac{209}{56}\end{bmatrix}$

$A = L \cdot U$

22) $A=\begin{bmatrix}q_1&r_1&0&0\\p_1&q_2&r_2&0\\0&p_2&q_3&r_3\\0&0&p_3&q_4\end{bmatrix}$ $L=\begin{bmatrix}1&0&0&0\\l_1&1&0&0\\0&l_2&1&0\\0&0&l_3&1\end{bmatrix}$ $U=\begin{bmatrix}d_1&u_1&0&0\\0&d_2&u_2&0\\0&0&d_3&u_3\\0&0&0&d_4\end{bmatrix}$

$l_1=\frac{p_1}{q_1}$ $d_1=q_1$ $u_1=r_1$ $u_2=r_2$ $u_3=r_3$

$l_2=(q_2-l_1)^{-1}$ $d_2=(l_1)^{-1}$

$l_3=(q_3-l_2)^{-1}$ $d_3=(l_2)^{-1}$

$d_4=q_4-l_3$

23) $l_1=\frac{p_1}{q_1}$ $d_1=q_1$ $u_n=r_n$

$l_n=(q_n-l_{n-1})^{-1}$ $d_2\rightarrow d_{m-1}=(l_{n-1})^{-1}$ $d_m=q_m-l_{m-1}$

24) (1) H_n is symmetric

$$H_n^T = (I - 2\vec{u}\vec{u}^T)^T = I^T - 2(\vec{u}\vec{u}^T)^T$$

$$= I - 2(\vec{u}^T)^T \vec{u}^T = I - 2\vec{u}\vec{u}^T = H$$

$\therefore H_n^T = H_n$, symmetric

(2) $H_n^T \cdot H_n = H_n \cdot H_n = (I_n - 2\vec{u}\vec{u}^T)(I_n - 2\vec{u}\vec{u}^T)$

$$= I_n - 2\vec{u}\vec{u}^T - 2\vec{u}\vec{u}^T + 4(\vec{u})\underbrace{(\vec{u}^T\vec{u})}_{1}\vec{u}^T)$$

$$= I_n - \cancel{4\vec{u}\vec{u}^T} + \cancel{4(\vec{u}\vec{u}^T)}$$

$= I_n$ $\therefore H_n$ is orthogonal

(3) $H_n^2 = H_n^T \cdot H_n = I$

(4) $H_n = I_n - 2\vec{u}\vec{u}^T \rightarrow H_n\vec{u} = \vec{u}(I_n - 2\vec{u}\vec{u}^T)$

$$= I_n\vec{u} - 2\vec{u}\vec{u}^T\vec{u}$$

$$= \vec{u} - 2\vec{u}\vec{u}^T\vec{u}$$

$$= \vec{u} - 2\vec{u} = -\vec{u}$$

(5) H_3 $\quad n=3$

For $n=3$, $H_3 = I_3 - 2\vec{u}\vec{u}^T$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \vec{u} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \cdot \underbrace{\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}} \qquad = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$\boxed{H_3 = \frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}}$$

$$H_4 = I_4 - 2\vec{u}\vec{u}^T$$

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \vec{u} = \frac{1}{\sqrt{4}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$2\vec{u}\cdot\vec{u}^T = 2\cdot\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\cdot\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$H_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$H_4 = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$