MATH 4570 PSET #1 9/23/21

$\boxed{\text{GIVEN EXTENSION FOR NON ACCURACY-GRADED PROBLEMS}}$

1. a. $a+b\sqrt{2}$, $a,b \in \mathbb{Q}$

Sum: $(a+b\sqrt{2})+(c+d\sqrt{2}) = (a+c)+(b+d)\sqrt{2}$

Product: $(a+b\sqrt{2})(c+d\sqrt{2}) = ac+2bd+(ad+bc)\sqrt{2}$

Identity:

sum: $0+a+b\sqrt{2} = a+b\sqrt{2}$, id $=0$

product: $1(a+b\sqrt{2}) = (a+b\sqrt{2})1 = a+b\sqrt{2}$, id $=1$

Associativity

sum: $(a+b\sqrt{2}+c+d\sqrt{2})+e+f\sqrt{2} = a+b\sqrt{2}+(c+d\sqrt{2}+e+f\sqrt{2})$

$= (a+c+e)+(b+d+f)\sqrt{2}$ ✓

Product $=$ $[(a+b\sqrt{2})(c+d\sqrt{2})](e+f\sqrt{2}) = (ac+ad\sqrt{2}+cb\sqrt{2}+2bd)(e+f\sqrt{2})$

$a+b\sqrt{2}[(c+d\sqrt{2})(e+f\sqrt{2})] = a+b\sqrt{2}(ce+cf\sqrt{2}+de\sqrt{2}+2df)$

① $\Rightarrow [(ac+2bd)+(ad+bc)\sqrt{2}](e+f\sqrt{2})$

② $\Rightarrow (a+b\sqrt{2})[(ce+2df)+(cf+de)\sqrt{2}]$

① $\Rightarrow e(ac+2bd)+f\sqrt{2}(ac+2bd)+e\sqrt{2}(ad+bc)$
$+2f(ad+bc)$

② $\Rightarrow a(ce+2df)+a\sqrt{2}(cf+de)+b\sqrt{2}(ce+2df)$
$+2b(cf+de)$

① $\Rightarrow ace+2bde+(acf+2bd)\sqrt{2}+(ade+bce)\sqrt{2}$
$+2adf+2bcf$

② $\Rightarrow ace+2adf+(acf+ade)\sqrt{2}+(bce+2bdf)\sqrt{2}$
$+2bcf+2bde$

$\Rightarrow$ Both equal, $\therefore$ associativity holds for the product operator

Inverse

sum: $(a+b\sqrt{2})+(-a-b\sqrt{2})=0$, inv $= -a-b\sqrt{2}$

Product: $(a+b\sqrt{2})(x+y\sqrt{2})=1 \Rightarrow (ax+2by)+(ay+bx)\sqrt{2}=1$

$\Rightarrow ax+2by=1$, $ay+bx=0$

$$\left[\begin{array}{cc|c} a & 2b & 1 \\ b & a & 0 \end{array}\right] \xrightarrow{\text{Python, sympy}} \left[\begin{array}{cc|c} 1 & 0 & \frac{a}{a^2-2b^2} \\ 0 & 1 & -\frac{b}{a^2-2b^2} \end{array}\right]$$

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$$\Rightarrow (a+b\sqrt{2})^{-1} = \left(\frac{a}{a^2-2b^2} - \frac{b}{a^2-2b^2}\sqrt{2}\right)$$

Commutivity

$(a+b\sqrt{2}) + (c+d\sqrt{2}) = (c+d\sqrt{2}) + (a+b\sqrt{2})$ ✓ $(+)$

$(a+b\sqrt{2})(c+d\sqrt{2}) = (c+d\sqrt{2})(a+b\sqrt{2})$ ✓ $(\times)$

Distributivity $(\times)$ only

$$(a+b\sqrt{2})(c+d\sqrt{2}+x+y\sqrt{2}) = ((a+b\sqrt{2})\cdot(c+d\sqrt{2})) + ((a+b\sqrt{2})\cdot(x+y\sqrt{2}))$$

$$= (ac + ad\sqrt{2} + ax + ay\sqrt{2} + bc\sqrt{2} + 2bd + bx\sqrt{2} + 2by)$$

$$= (ac + ad\sqrt{2} + bc\sqrt{2} + 2bd) + (ax + ay\sqrt{2} + bx\sqrt{2} + 2by)$$ ✓

b. This is the set of all complex numbers, $\mathbb{C}$, of the form $a+bi$; $+, \times$ are defined

Addition

Commutativity: $(a+bi)+(c+di) = (c+di)+(a+bi)$ ✓

Associativity: $(a+bi+c+di) + (x+yi) = (c+di)+(a+bi+x+yi)$ ✓

Identity: $(0+0i)+(a+bi) = a+bi$, $I_d = 0$

Inverse: $(a+bi)+(-a-bi)=0$, $I_{nv} = -a-bi$

Multiplication

Commutativity: $(a+bi)(c+di) = (c+di)(a+bi)$ ✓

Associativity: $[(a+bi)(c+di)](x+yi) = (c+di)\cdot[(a+bi)(x+yi)]$ ✓

$\Rightarrow (a+bi)(c+di)(x+yi) = (c+di)(a+bi)(x+yi)$ ✓

Identity: $(a+bi)(1+0i) = (1+0i)(a+bi) = a+bi \Rightarrow I_d = 1+0i = 1$

Inverse: $(a+bi)^{-1} \Rightarrow x+yi$, $(a+bi)(x+yi)=1$

$\Rightarrow (ax-by)+(ay+bx)i = 1 \Rightarrow$ $ax-by=1$, $ay+bx=0$

$\Rightarrow \left[\begin{array}{cc|c} a & -b & 1 \\ b & a & 0 \end{array}\right]$

$\Rightarrow$ Python.sympy $\Rightarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{a}{a^2+b^2} \\ 0 & 1 & \frac{-b}{a^2+b^2} \end{array}\right]$

$\Rightarrow (a+bi)^{-1} = (x+yi) = \boxed{\left(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i\right) = (a+bi)^{-1}}$

Distributivity: $(a+bi)(c+di+x+yi) = [(a+bi)(c+di)] + [(a+bi)(x+yi)]$

$\Rightarrow ac+adi+ax+ayi+bci-bd+bxi-by$

$= (ac+adi+bci-bd)+(ax+ayi+bxi-by)$ ✓

3. $\mathbb{Z}^3$

$+$	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

$\times$	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

$[0] = 0, \pm 3, \pm 6\ldots$

$[1] = 1, 1\pm 3, 1\pm 6\ldots$

$[2] = 2, 2\pm 3, 2\pm 6\ldots$

2. a. using python: sympy.galois

$$\left[\begin{array}{ccc|c} 3 & 1 & 4 & 1 \\ 5 & 2 & 6 & 5 \\ 0 & 5 & 2 & 1 \end{array}\right]$$

$$RREF(\mathbb{Z}_7) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

b. $\vec{x}_{\mathbb{Z}_7} = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$, $\boxed{\begin{array}{l} x_1 = 4 \\ x_2 = 3 \\ x_3 = 0 \end{array}}$

19. a. 2x2: symmetric, $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, Skew symmetric, $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

3x3: S, $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 5 \end{bmatrix}$, SS $\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$

4x4: S, $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 7 \\ 3 & 1 & 3 & 0 \\ 4 & 7 & 0 & 4 \end{bmatrix}$, SS = $\begin{bmatrix} 0 & 4 & -3 & 5 \\ -4 & 0 & 22 & -34 \\ 3 & -22 & 0 & \cancel{22} \\ -5 & 34 & -2 & 0 \end{bmatrix}$

b. They are all 0

c. S & SS: A = $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

d. i. $(A+B)^T = A^T + B^T$, $(B^T)^T = B$

$\Rightarrow (A+A^T)^T = A^T + A$

$\Rightarrow A + A^T = A^T + A$ ✓, since $A+A^T$ is symm. iff $A+A^T = (A+A^T)^T$

ii. $A \cdot A^T$ is symm iff $A \cdot A^T = (A \cdot A^T)^T$; $(AB)^T = B^T A^T$

$\Rightarrow (A \cdot A^T)^T = A^{T^T} A^T = A \cdot A^T$

$\Rightarrow (A \cdot A^T)^T = A \cdot A^T$, $\therefore$ AA^T is symmetric

iii. $(A^T A)$ is symm. iff $A^T A = (A^T A)^T$

$\Rightarrow (A^T A)^T = (A^T A^{TT}) = A^T A$

$\Rightarrow (A^T A)^T = A^T A$, $\therefore$ $A^T A$ is symmetric

iv. $A - A^T$ is skew symm iff $(A-A^T)^T = -(A-A^T)$

$(A-A^T)^T = A^T - A$, $-(A^T - A) = -A^T + A = A - A^T$

~~$\therefore (A^T - A) = (A - A^T)^T$~~

$\therefore$ $(A - A^T)$ is skew symm.

e. Let A be any nxn matrix:

$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$; $A + A^T$ = symm

$= \frac{1}{2}A + \frac{1}{2}A + \frac{1}{2}A^T - \frac{1}{2}A^T = A$ $\quad$ $A - A^T$ = skew symm.

If we scale A by any scalar, a, the transpose is just $(aA)^T = aA^T$, $\therefore$ Any nxn matrix A can be expressed by the sum of symm. & skew symm matrices

24. a. $H^T = (I - 2\vec{u}\vec{u}^T)^T = I^T - 2(\vec{u}\vec{u}^T)^T = I - 2(u^T)^T u^T = I - 2\vec{u}\vec{u}^T = H$

$\therefore$ H is symmetric b/c $H = H^T$

b. $H^T H = (I - 2\vec{u}\vec{u}^T)(I - 2\vec{u}\vec{u}^T) = I - 2\vec{u}\vec{u}^T - 2\vec{u}\vec{u}^T + 4\vec{u}\vec{u}^T(\vec{u}\vec{u}^T)$

$\Rightarrow I - 4uu^T + 4\vec{u}\underbrace{\vec{u}^T\vec{u}}_{1}u^T = I \underbrace{- 4\vec{u}\vec{u}^T + 4\vec{u}\vec{u}^T}_{0} = I$

$\therefore$ H is orthogonal b/c $H^T H = I$

c. $H^2 = H^T H = 1$ since $H^T = H$

$\therefore H^2 = 1$

d. $H\vec{u} = (I - 2\vec{u}\vec{u}^T)\vec{u} = \vec{u} - 2\vec{u}\underbrace{\vec{u}^T\vec{u}}_{1} = \vec{u} - 2\vec{u} = -\vec{u}$

$\Rightarrow H\vec{u} = -\vec{u}$

e. $H_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\left(\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$H_3 = \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix}$

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$d = 0$

$e = 0$

$$H: \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - 2\left(\frac{1}{\sqrt{4}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & -1/2 \\ -1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix} = H$$