Eigth Worksheet, MATH 7233

November 19, 2021

1. Show that in a random tournament, with high probability everybody is a pseudo champion. That is, the probability of the existence of a non-pseudo champion goes to 0 as $n \to \infty$.

2. Show that $K_{3,3}$ is not planar.

3. Show that an n-vertex planar graph can not have more than $3n - 6$ edges.

4. (a) Show that any planar graph has a node whose degree is at most 5.
   (b) Show that if G is planar then $\chi(G) \le 6$. (Hint: use induction on n.)

5. Draw n points on a circle, and connect each pair with a straight line. What is the largest possible number of pieces this cuts the inside of circle into? (Just to make sure you interpret this correctly, the answer should be $1, 2, 4, 8, 16$ for $n = 1, 2, 3, 4, 5$ respectively.)

6. We need to fill an 8×8 table with the letters A,B,C,D,E,F,G,H in a way that each row and each column contains exactly one of each. However, someone already wrote all the As, Bs, Cs, Ds, and, Es in (following the rule), leaving only 24 empty spaces. Can we fill these **for sure** with the remaining 8 Fs, Gs, and Hs to complete the task, no matter how the first 40 letters were written?

7. Let A be a symmetric matrix and let B denote an $n-1 \times n-1$ principal minor of A. Show that $\mu_1(A) \ge \mu_1(B)$. Hint: find a vector $\psi \in \mathbb{R}^n$ such that $R_A(\psi) = \mu_1(B)$.

8. For a subset of nodes $S \subset V$ of a graph $G(V, E)$ we define it's isoperimetric ratio as

$$\theta(S) = \frac{|E(S, V \setminus S)|}{|S|}$$

   and let $\theta_G = \min\{\theta(S) : S \subset V, |S| \le |V|/2\}$ the isoperimetric constant of G.

   (a) Show that $\theta(S) \ge \lambda_2(1 - s)$ where $s = |S|/|V|$.
   (b) Show that $\theta_G \ge \lambda_2/2$.