Worksheet solutions

1/1 Is there a graph where the degrees are exactly

(a) 7,7,7,6,6,6,5,5,5 (9 vertices)?

(b) 6,6,5,4,4,3,2,2,1 (9 vertices)?

(c) 6,6,6,6,3,3,2,2 (8 vertices)?

Solution:

(a) Yes, there is. Try to draw one!

(b) No, there is not, since the sum of the degrees is any graph is equal to twice the number of edges. So in any graph the sum of the degrees is even. Here the sum is odd, so there cannot be such a graph.

(c) No, there is not. To prove this, assume such a graph exists. Separate the vertices into two group: group A containing all the degree 6 vertices, group B the rest. Let's count the number of edges between the two groups, and call it x. Each vertex in group A should have degree 6, but at most 3 of those edges can end in group A, so at least 3 has to end in group B. Thus $x \geq 12$. On the other hand the number of edges ending in group B is at most the sum of the degrees there, which is 10, so we get $x \leq 10$. This is a contradiction, so this graph cannot exist.

□

1/2 Show that at a party there have to be two participants who know exactly the same amount of other people. (Acquaintance is mutual.)

Solution: We use the pigeonhole principle: if there are n people, then the possible number of acquaintances range from 0 to $n-1$. The only way everyone could know a different number of people would be to have one person know 0 others, one person know 1 other, one person know 2 others, etc., one person know $n-1$ others. But this is not possible, since a person who doesn't know anyone and a person who knows everyone cannot be present at the same time. □

1/3 There are two airlines on Middle-Earth: ElvenAir and AirDwarf. Between any two settlements there is a single non-stop flight operated by one of the two airlines.

I want to use only one of the two airlines because I want to collect miles in a single loyalty program. At the same time I want to be able to travel from anywhere to anywhere - allowing multiple layovers if necessary.

Am I asking too much? Or is this always possible?

Solution: This is always possible. We prove it using induction on the number of towns. When there are only 2 towns, the problem is trivial. Suppose the statement is true for n towns, and let's show it for $n+1$ towns. Let's consider the $n+1$ towns, and pick one of them, say Boston.

- If both airlines have flights out of Boston, let's use the inductive hypothesis for the remaining n towns. By induction one of the airlines provides full connectivity in those n towns, and it also provides at least one flight from Boston to one of the other towns, so it does provide full connectivity in the whole system.
- If only one airline flies from Boston, then it has to fly to every other town. So this airline provides full connectivity with Boston as the hub.

□

1/4 Suppose all 30 students in the Graph Theory class have at least 5 friends in the class. Again, friendship is mutual. I have to create a long line of students so everyone is standing next to their friends. (So people in the middle would be between two of their friends, people on the ends only next to one friend.) Let g denote the longest line I can make. What is the smallest possible value of g?

Solution: Take the longest path in the graph. Denote its vertices in order by $v_1, v_2, \ldots, v_k$. The degree of v_k is at least 5, and all its edges have to go to different vertices within the longest path, otherwise there would be a longer path. So there need to be 5 different vertices preceding v_k, meaning that $k \geq 6$. So the longest path has length at least 6.

The graph K_6 (or disjoint unions of many K_6s) has no path of length longer than 6, but every degree is 5. □

1/5 In last year's Graph Theory class, it turned out that each student had exactly 3 friends there. Furthermore, when two students weren't friends, they had exactly one mutual friend. However, no 3 students were all friends with each other. How many people were there in the class?

Solution: Let there be n people (aka nodes), and e friendships (aka edges). First, as each node has degree 3, we have $2e = 3n$. On the other hand, let us count cherries. Each node is the stem of $\binom{3}{2} = 3$ cherries, so $ch(G) = 3n$. However, each pair of nodes that is not an edge is the fruit-pair of exactly 1 cherry, and each pair of nodes that is an edge is a fruit-pair of 0 cherries. So

$$3n = ch(G) = \binom{n}{2} - e = \frac{n(n-1) - 3n}{2} = \frac{n^2 - 4n}{2},$$

from which $10 = n^2$ so $n = 0$ or $n = 10$.

We are not done yet, we still need to show that the conditions can actually be satisfied for $n = 0$ and $n = 10$. The former is trivial. For the latter case, the Petersen graph is a good example. □

1/6 Show that, given any 6 integers, there are either 3 of them that are pairwise coprime, or 3 of them that are pairwise not coprime. (Two integers are *coprime* if they have no common primes in their factorization.) Is the same true for 5 integers? **Solution:** We create a graph on 6 nodes representing the 6 integers, and draw edges between pairs of coprimes. Then we need to show that either the graph or its complement has a triangle. To do so, let us fix a node v. Since there are 5 other nodes, the degree of v will be at least 3 in either G or G^c. Without loss of generality we can assume $\deg(v) \geq 3$. Let w_1, w_2, w_3 be three neighbors of v. If any of the $w_i w_j$ pairs form an edge, then they together with w form a triangle. Otherwise the triple $w_1 w_2 w_3$ forms a triangle in the complement. □

1/7 It can be shown by induction that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

where $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ is the Fibonacci sequence.

2/1 **Solution:** $(A^2)_{i,i} = \deg(i)$ and $(A^3)_{i,i} = 2\times$ the number of triangles the node i is contained in. □

2/2 Show that any graph in which every degree is at least 2 contains a cycle.

Solution: If all the vertices are at least 2, then consider the longest path in the graph. Its endpoint has another edge leaving it, but that has to end in one of the previous vertices of the path, otherwise it would create and even longer path. Thus the graph has to have a cycle. □

2/3 Show that a tree on n vertices has exactly $n - 1$ edges! (Hint: use induction on n.)

Solution: Use induction on n. If $n = 1$ this is clear. Otherwise take a tree T on $n + 1$ vertices. It has to have a degree 1 vertex otherwise it would have a cycle or an isolated vertex. Call this vertex v, and look at $T \setminus v$. This is a graph on n vertices, it is still cycle free, and it is still connected, since removing a vertex of degree 1 cannot disconnect a graph. So $T \setminus v$ is a tree on n vertices, so it has $n - 1$ edges by induction. But then T has n edges, and we are done. □

2/4 How many trees are there up to isomorphism that have 3 vertices? And 4? And 5?

Solution: One, two, and three, respectively. (Sort them according to the longest path, for example.) □

2/5 Show that any connected graph has a *spanning tree*: a subgraph that is a tree and has the same number of vertices as the original graph. (Hint: are there any edges you can throw away without disconnecting the graph?)

Solution: Proof by algorithm: throw away edges from the graph 1-by-1 as follows. If the graph has a cycle, throw away one edge from the cycle. This way the graph remains connected. Stop when the remaining graph has no cycles. At this moment we have a connected, but cycle free, subgraph. That's exactly a spanning tree. □

2/6 Show that if the number of edges is one less than the number of vertices in a connected graph, then it's a tree. (Hint: use the previous problem!)

Solution: Let G be a connected graph with n vertices and $n-1$ edges. We know it has a spanning tree $T \subset G$, ie a spanning subgraph that is a tree. But we know that T has exactly $n-1$ edges, so the only way this is possible is that $T = G$. So G itself is a tree. □

2/7 Show that every tree has at least 2 nodes of degree exactly 1.

Solution: Both endpoints of a longest path must have degree 1, otherwise there would be a cycle in the graph. □

2/8 Show that if G as at least 5 nodes, then either G or G^c must contain a cycle.

Solution: Suppose neither G nor G^c have cycles. Then they are both forests, so $e(G) \le n-1$ and $e(G^c) \le n-1$. Thus $\binom{n}{2} = e(G)+e(G^c) \le 2n-2$, so $n^2-n \le 4n-4$, so $n^2 < 5n$ so $n < 5$. □

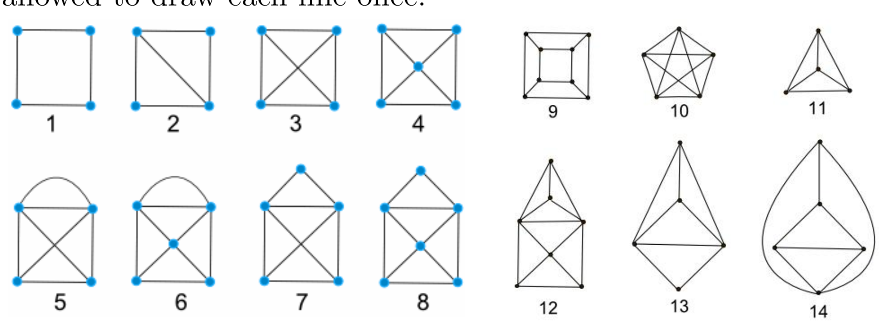
3/1 Show that any group of people can be arranged in two rooms such that each person has at most as many friends in their own room group than in the other room.

Solution: Start with any splitting of the people. If a person has more friends in their current room than the other one, move them to the other room. Keep doing this. Each such move increases the number of "cross-room friendships". Hence the procedure has to end in finitely many moves, at which moment we have the desired splitting. □

3/2 In a tennis tournament everybody played against everybody. Show that you can arrange the players in one long line such that everybody beat the person behind them.

Solution: Proceed by induction on n. Base case is trivial. Inductive step: take a tournament on $n+1$ players, but let one player sit down. Arrange the remaining players using the inductive hypothesis. Then try to insert the last player, starting at the end of the line, and always moving 1 if that's not a correct place. It is not hard to see, that eventually one of the places will be good, or they can stand at the front. □

3/3 Which of the following pictures can you draw without lifting your pencil? You're only allowed to draw each line once!



Solution: If there are more than 2 points where an odd number of lines meet, you can't draw the picture, because you are guaranteed to get stuck at one of these points before finishing. Otherwise you can draw them. □

3/4 We want to move a knight around the regular chessboard so that it visits every square exactly once, and then arrives back at the starting position. Can this be done? And what about on a 7×7 chessboard?

Solution: For a solution in the 8×8 case see the "knights tour problem" on wikipedia. For the 7×7 case, the knight can only move from black to white and from white to black squares. So any closed tour must have even length. However the board has 49 squares, so a tour that visits each square once would have length 49, so this is impossible. □

3/5 A cherry in a graph is simply a node s and a pair of nodes f_1, f_2 such that s is adjacent to both f_1 and f_2. (The stem is s, and the fs are the fruits of the cherry....) We will use $ch = ch(G)$ to denote the number of cherries in a graph.

Let G be a graph that has n vertices and e edges, and let $d_1, d_2, \ldots, d_n$ denote the degrees of its vertices. Show that

$$ch(G) = \sum_i \binom{d_i}{2}.$$

Using this, show that

$$ch(G) \geq \frac{2e^2}{n} - e.$$

(Hint: use inequality of arithmetic and quadratic means.)

Solution: The number of cherries whose stem is node v is exactly $\binom{\deg(v)}{2}$. So the total cherries are given by

$$ch(G) = \sum_i \binom{d_i}{2} = \frac{1}{2}\sum_i d_i^2 - \frac{1}{2}\sum_i d_i$$

Then by the arithmetic-quadratic inequality, and using that $2e = \sum d_i$, we get that

$$ch(G) \geq \frac{1}{2}\frac{\left(\sum d_i\right)^2}{n} - e = \frac{2e^2}{n} - e.$$

□

3/6 Let G be a graph on 10 vertices that has no triangles. Show that G^c must have a K_4 subgraph.

Solution: Let's take a node v. If $\deg(v) \geq 4$, then let $w_1, \ldots, w_4$ be four neighbors. Neither $w_i w_j$ pair can be connected, otherwise we'd have a triangle in G with nodes v, w_i, w_j. Hence all 6 pairs are connected in the complement, so $w_1, \ldots, w_4$ forms a K_4 in G^c. If $\deg(v) \leq 3$, then $\deg^c(v) \geq 6$. Let $w_1, \ldots, w_6$ denote 6 neighbors. The subgraph spanned by these nodes doesn't have a triangle, so by Problem 1/6 there must be 3 of them (w_i, w_j, w_k) that form a triangle in G^c. But then (v, w_i, w_j, w_k) is a K_4 in G^c. □

Definition (Chromatic number)**.** Let $G(V, E)$ be a graph, and S be a finite set. A function $c : V \to S$ is a *proper coloring* if for all edges $xy \in E$ one has $c(x) \neq c(y)$. The set S is usually called the set of colors.

The chromatic number of G, denoted $\chi(G)$, is the smallest integer k such that there is a proper coloring $c : V \to S$ with $|S| = k$.

4/1 Compute $\chi(K_n), \chi(C_n), \chi(S_n)$. What can we say about $\chi(T)$ is T is a tree?

Solution: $\chi(K_n) = n, \chi(S_n) = 2, \chi(C_{2n}) = 2, \chi(C_{2n+1} = 3$. For any tree on at least 2 nodes $\chi(T) = 2$. □

4/2 Evil Elf: see separate file under "pages".

4/3 TBA

4/4 Show that a graph has an Euler tour if and only if it is connected and all its degrees are even.

Solution: Let us start from a vertex v, and try to visit new edges as long as we can. Since every degree is even, if we enter a vertex other than v, there has to be an unused edge leaving that vertex. This means that we can only get stuck in v. At this moment we have a closed walk. If this covers all edges, we are done. Otherwise there has to be a vertex w on this walk that is adjacent to some unvisited edge. (At this point we exploit that the graph is connected.) Now start the same from w on the remaining set of edges. This yields a closed walk starting from w. Finally merge the two walks, to have one looong closed walk from v.

Iterate this process, until all edges are covered. The only if direction is trivial. □

4/5 Let G be a tournament (aka complete graph with all edges oriented). We say a node $v \in V(G)$ is a *pseudo champion* if for any $y \neq x$ there is an oriented path of length 1 or 2 from x to y. (In tournament language x beat y or x beat some z who beat y.)

Prove that any tournament has a pseudo champion!

Solution: Proof by induction: for $n = 1$ is trivial. If we know if for all tournaments of size less than n, then let's take a tournament on n nodes and let x be any node. If all edges incident to x point away from x, then x is a pseudo champion and we are done. Otherwise let $W = \{y : y \to x \in E\}$, which is non-empty by assumption, but clearly has fewer than n nodes. So by the inductive hypothesis there is a node $y \in W$ that is a pseudo champion of W. But then y is a pseudo champion of the whole tournament, since $y \to x \in E$ and for all $z \in V \setminus (x \cup W)$ we have $y \to x \to z$. □

4/6 Let $k \geq 3$. Clearly, if a graph has K_k as a subgraph, then $\chi(G) \geq k$. Show an example of a graph that does not contain K_k as a subgraph, yet $\chi(G) = k$. (Hint: start with $k = 3$.) **Solution:** Let G have $5 + (k - 3)$ nodes, called $a, b, c, d, e, x_1, x_2, \ldots, x_{k-3}$. Let the edges be ab, bc, cd, de, ea, and also connect x_j to every node, for all values of j. □

4/7 Show that if a simple graph has no triangles, then $e \le n^2/4$. (Hint: count cherries!)

Solution: We know that $ch(G) \ge 2e^2/n - e$. On the other hand, each edge can be part of at most $n-2$ cherries, because otherwise there'd be a triangle. Hence $ch(G) \le e(n-2)/2$, since there are e edges, at most $n-2$ cherries per edges, but every cherry is counted twice. Putting these together we get $e(n-2)/2 \ge 2e^2/n - e$, so $n-2 \ge 4e/n - 2$ or $n \ge 4e/n$ and we are done. □

4/8 Let G be a graph on 9 vertices. Show that either G contains a triangle, or G^c contains a K_4 subgraph. **Solution:** We proceed the same way as in Problem 3/6. We are done if any node has $\deg(v) \ge 4$ or $\deg(v) \le 2$. The only bad case is if all nodes have degree 3. However with 9 nodes this is impossible. □

5/1 Let G be a tournament where no player beat everyone else. Show that G has at least 2 pseudo champions.

Solution: Let x be a pseudo champion. We already showed that if x didn't beat everyone, then among the players who beat x, there is a pseudo champion y. But $y \ne x$ so we then have at least two. □

5/2 Show that in a graph that has no cycles of length 4 (may have longer or shorter cycles),

$$e \le \frac{1}{2}n^{3/2} + n/4.$$

Solution: If there are no 4-cycles then any pair of nodes can be the fruit-pair of at most 2 cherries. So $ch(G) \le n(n-1)$. Combining this with $ch(G) \ge 2e^2/n - e$ we get $2e^2/n - e \le n(n-1)$. Solving this as a quadratic inequality in e yields the desired formula. □

5/3 Is there a number R such that if G is any graph on at least R nodes then G or G^c contains a K_4 subgraph?

Solution: Yes, $R = 18$ works. If v is any node, then either $\deg(v) \ge 9$ or $\deg^c(v) \ge 9$. Without loss of generality we can assume the first. Then among the 9 neighbors of v, by Problem 4/5, we can find a triangle - in which case adding v makes it a K_4, or we can find a K_4 in the complement. In both cases we are done. □

5/4 Show that $\chi(G) \le 2$ if and only if G does not contain any odd cycles.

Solution: If $\chi(G) \le 2$, then any cycle alternates between the two colors, so can only return to the starting point in an even number of steps.

For the converse direction, take a spanning tree of G, and fix a vertex v. From this vertex as "root", color the tree red and blue alternating towards the leaves. Claim: this is a good coloring. Suppose there are two, say red, vertices in G that are adjacent. Let x, y denote these vertices. The edge xy cannot be in the tree since the tree was colored correctly. So this edge is outside of the tree. But since it's a spanning tree, there is a unique path from x to y. This path is alternating (since the tree is colored correctly) and hence has even length. But then this path, together with the xy edge would be an odd cycle, which we assumed does not exist.

We got a contradiction, so no xy edge can exist between same-colored vertices and we have a proper coloring. $\square$

5/5 Let $\Delta = \Delta(G)$ denote the largest degree in a graph. Show that $\chi(G) \leq \Delta(G) + 1$.

Solution: We can greedily color the nodes one-by-one. Just make sure when coloring node v to avoid colors that have been used on its neighbors. Since the number of neighbors is less than the number of available colors, we can always achieve this. $\square$

5/6 Find the smallest eigenvalue of L_G. Describe all corresponding eigenvectors. **Solution:** Let $L_G v = \lambda v$. Then by [4/3] we know that $0 \leq \sum_{i \sim j}(v(i) - v(j))^2 = v^T L_G v = \lambda v^T v$, hence $\lambda \geq 0$. Indeed $\lambda = 0$ is an eigenvalue, so it must be the smallest. Clearly for any eigenvector v it has to hold that $v(i) = v(j)$ whenever $i \sim j$, and thus v has to be constant over each connected component. Such vectors are indeed eigenvectors, as is easily verified by carrying out the multiplication. $\square$

5/7 Compute $\chi_S^T L \chi$ in combinatorial terms. Find s such that $\chi_S - s \cdot 1$ is orthogonal to $\mathbb{1}$. **Solution:** From [4/3] we see that $\chi_S^T L \chi = \sum_{i \sim j}(\chi(i) - \chi(j))^2$. But $\chi(i) - \chi(j) = 0$ is $i, j \in S$ or $i, j \in V \setminus S$. Otherwise $\chi(i) - \chi(j) = \pm 1$. Thus $\chi_S^T L \chi = |E(S, V \setminus S)|$ the number of edges between S and $V \setminus S$.

To find s, compute $0 = \langle \chi_S - s \cdot \mathbb{1}, \mathbb{1} \rangle = |S| - sn$, so $s = |S|/n$. $\square$

6/1 Show that for any graph

$$\frac{2|E|}{|V|} \leq \mu_1 \leq \Delta(G)$$

Solution: For the left hand side consider the vector $v = \mathbb{1}$. Then $R_A(v) \leq \mu_1$ from the basic properties of the Rayleigh quotient. However a simple computation shows that $R_A(v) = 2|E|/|V|$, the average degree of the graph.

For the right hand side, consider $L = D - A$ and notice that for any w we have $0 = \lambda_1 \leq R_L(w) = R_D(w) - R_A(w)$. Hence $R_A(w) \leq R_D(w) = \frac{\sum d_i w_i^2}{\sum w_i^2} \leq \Delta(G)$. Thus $\mu_1 = \max_w R_A(w) \leq \Delta(G)$ as well.

$\square$

6/2 **Solution:** Let S be an independent set. Then by [4/3] we know $R_L(\chi_S) = |E(S, V \setminus S)|/|S| = \deg_{ave}(S)$, since all edges starting in S must end in $V \setminus S$ by the independence of S.

Now let $v = \chi_S - s \cdot \mathbb{1}$ as in [4/3]. Then $v^T L v = \chi_S^T L \chi_S$ because v is a shift of χ_S by a constant. However $v^T v = |S|(1 - s)$ by a simple computation. Thus $R_L(v) = R_L(\chi_S)/(1 - s) = \deg_{ave}(S)/(1 - s)$. From this we know that $\lambda_n = \max_w R_L(w) \geq R_L(v) = \deg_{ave}(S)/(1 - s)$ and the inequality follows from simple algebraic manipulation. $\square$

6/3 Show that in a tournament there cannot be exactly 2 pseudo champions.

Solution: We know from [5/1] that if there is no absolute champion, then every pseudo champion loses to some other pseudo champion. So in this second case if x is a pseudo champion, there must be another pseudo champion y that beat x, and there must be another pseudo champion z who beat y. And clearly z and x can't be the same, because one beat x the other did not. $\square$

6/4 Let H be a set of n points in the plane. Let $D = \{(x, y) \in H^2 : d(x, y) = 1\}$ be the set of (ordered) pairs whose distance is exactly 1 unit. Show that $|D| \leq n/2 + \sqrt{2}n^{3/2}$.

Solution: Any pair of points x, y can be the fruits of only up to 2 cherries (there are only up to 2 points in the plane that are 1 distance from two given points, because the two circles can only intersect in up to 2 points). From here it's essentially an identical computation to [5/2]. $\square$

6/5 Suppose that in a bipartite graph G, each vertex in A has degree 7 while each vertex in B has degree 9. What can be said about $|A|/|B|$?

Solution: Count the edges of the graph 2 ways: counting via endpoints in A gives $e = 7|A|$, and counting via endpoints in B gives $e = 9|B|$. So $|A|/|B| = 9/7$. $\square$

6/6 The plane is divided into regions by straight lines. Show that you can color the regions black or white such that edge-wise adjacent regions have different color.

Solution: **1st approach:** We need to show that the "dual graph", formed by the regions, is bipartite. Hence it's enough to show that it has only even cycles. But any cycle crosses any line on the picture an even number of times ("river crossing"), so the total number of crossings is also even: the cycle has to be even.

2nd approach: Use induction on the number of lines. If there is 1 line, it's clearly true.

Now when you already have $n + 1$ lines, take one of them away temporarily. Then you'll see a picture consisting of n lines. Color this picture correctly using the inductive hypothesis. Now add back the last line. The only line segments where the picture right now is colored incorrectly are the segments on this last line (both sides have the same color at each such segment). To fix this, swap colors on every region south of this line.

Claim: the new coloring will be correct everywhere. Proof: Each line segment where two regions meet are either "south of the new line", "north of the new line", or "on the new line". If a line segment was south or north of the new line, then it was already good before the swap, and it remains good after the swap. If a line segment was on the new line, it was bad before the swap, but it becomes good after the swap. So after the swap there are no segments where both sides would get the same color, so this is now a correct coloring. $\square$

6/7 Is it true that for any $k, m \geq 2$ integers there is a number $R(k, m)$ such that if G is any graph on at least $R(k, m)$ nodes then G contains a K_k subgraph or G^c contains a K_m subgraph?

Solution: We proceed by a double induction on k and m. It's easy to see that $R(k, 2) = R(2, k) = k$ works. Then we show that if $R(k - 1, m)$ and $R(k, m - 1)$ are known, then $R(k, m)$ exists and can be chosen to be $R(k - 1, m) + R(k, m - 1)$.

Take a graph on $R(k,m) = R(k-1,m) + R(k,m-1)$ nodes, and let v be a fixed node. If $\deg(v) \geq R(k-1,m)$, then among the neighbors of v we can either find a K_m in the complement (and then we're done) or find a K_{k-1} in the graph, in which case together with v they form a K_k and we're done again. If $\deg^c(v) \geq R(k,m-1)$ we proceed in the same fashion.

But one of these two must hold, because otherwise the graph would have at most $1 + R(k-1,m) - 1 + R(k,m-1) - 1 < R(k,m)$ nodes. □

7/1 Construct tournaments for any $n \geq 5$ where everybody is a pseudo champion.

Solution: If T is a tournament where everyone is a pseudo champion, and $T' = T \cup \{x,y\}$ where $x \to y \to T \to x$ gives the direction of the new edges in T', then it's easy to see that everyone is a pseudo champion in T' as well. Hence it is enough to show examples for $n = 3$ and $n = 6$ participants where everyone is a pseudo champion, and then apply the previous extension trick repeatedly. The $n = 3$ case is trivial, and $n = 6$ can be done but we omit the description here. □

7/2 Let G be a k-regular (ie every degree is equal to k) bipartite graph.

(a) Show that it has a perfect matching.

(b) Show that the edges can be colored using k colors so that the all the edges adjacent to any single vertex have different colors.

Solution: HW

7/3 In a bipartite graph G with $V = A \cup B$, let us call a set $H \subset A$ *tight* if $|H| = |N(H)|$. Suppose G has *no obstacles* and $H \subset A$ is tight. Let $G_1 = G[H \cup N(H)]$ and $G_2 = G[(A \setminus H) \cup (B \setminus N(H))]$ be the subgraphs of G restricted to these subsets. Show that neither G_1, nor G_2 has obstacles.

Solution: HW

7/4 Give a proof of Hall's theorem by induction on the size of A. Hint: separate into two cases based on whether there is a tight set $H \subsetneq A$ or not. If there is a tight set, use part (a). If there is no tight set, try simply adding any edge to the matching.

Solution: HW

7/5 We draw all diagonals of a convex n-gon. What is the largest number of intersection points we can create this way inside the polygon?

Solution: Any intersection point determines 4 corners of the polygon. Vice versa, and 4 corners of the polygon uniquely determine an intersection point inside the polygon (because of convexity). So the number of intersection points equals the number of ways one can choose 4 corners. That's exactly the binomial coefficient $\binom{n}{4}$. □

7/6 State and proof the "face-shake" lemma for planar graphs.

Solution: Let the faces of a planar drawing of G be $F_1, \ldots, F_f$. Then $\sum_j \deg(F_j) = 2e$. The reason is that each edge is a side of exactly two faces (or possibly the same face twice), so adding the number of sides of each face counts every edge exactly twice. $\square$

7/7 What is the expected number of cherries in $G(n,p)$?

Solution: For any three nodes i,j,k let $X_{ijk} = 1$ if ij, jk are edges, and 0 otherwise. Then $ch(G) = \frac{1}{2}\sum_{i,j,k} X_{ijk}$ (because $X_{ijk} = X_{kji}$). Thus $E(ch(G(n,p))) = \frac{1}{2}\sum_{ijk} E(X_{ijk})$. But $E(X_{ijk}) = P(X_{ijk} = 1) = p^2$. So $E(ch(G(n,p))) = \frac{n(n-1)(n-2)}{2}p^2$. $\square$

7/8 Show that $G(n,p)$ is connected with probability tending to 1 as $n \to \infty$.

Solution: HW

8/1 Show that in a random tournament, with probability 1 all players are pseudochampions at the same time.

Solution: The probability that i didn't pseudo-beat j is $\frac{1}{2} \times \left(\frac{3}{4}\right)^{n-2}$. Thus the probability that there are such i,j is at most $\binom{n}{2}\frac{1}{2} \times \left(\frac{3}{4}\right)^{n-2}$, which clearly goes to 0. $\square$

8/2 Show that $K_{3,3}$, the complete bipartite graph on $3+3$ vertices, is not planar.

Solution: A planar drawing of $K_{3,3}$ would have $9+2-6 = 5$ faces by Euler's formula. But each of these faces would have at least 4 sides, since the shortest cycle in $K_{3,3}$ has 4 edges. So the number of edges would be at least $(5 \times 4)/2 = 10 > 9$, a contradiction. So $K_{3,3}$ is not planar. $\square$

8/3 Let G be a simple planar graph on n vertices. Show that G has at most $3n-6$ edges!

Solution: HW

8/4 (a) Show that G has at least one vertex whose degree is at most 5.

(b) Show that the vertices of any planar graph can be properly colored using 6 colors.

Solution: HW

8/5 We draw n points on a circle, and connect all pairs with straight lines. At most how many regions are formed inside the circle?

Solution: Think of this as a planar graph, where intersection points are the vertices, and line segments (and the arcs of the circle) are the edges. Then this graph has $v = n + \binom{n}{4}$ vertices. Each interior vertex has degree 4, while the outer vertices have degree $n+1$ each. So the sum of all degrees is

$$2e = 4\binom{n}{4} + n(n+1),$$

so

$$e = 2\binom{n}{4} + \frac{n(n+1)}{2}.$$

Then the number of faces, by Euler's formula, is

$$f = e - n + 2 = \binom{n}{4} + \frac{n(n+1)}{2} - n + 2 = \binom{n}{4} + \binom{n}{2} + 2,$$

so the number of regions inside the circle is

$$\binom{n}{4} + \binom{n}{2} + 1.$$

□

8/6 We need to fill an 8×8 table with the letters A,B,C,D,E,F,G,H in a way that each row and each column contains exactly one of each. However, someone already wrote all the As, Bs, Cs, Ds, and, Es in (following the rule), leaving only 24 empty spaces. Can we fill these **for sure** with the remaining 8 Fs, Gs, and Hs to complete the task, no matter how the first 40 letters were written?

Solution: We build a bipartite graph on $8 + 8$ nodes. The nodes in part A will represent columns, the node in part B will represent nodes. We put an edge between two nodes if the square at the intersection of the corresponding row and column free. Thus we get a 3-regular bipartite graph. This graph has a 3-edge-coloring by Problem 7/2b. Each color can be used as the location of one of the missing letters. So one color for all Fs, one for all Gs, and one for all Hs. □

8/7 Let A be a symmetric matrix and let B denote an $n-1 \times n-1$ principal minor of A. Show that $\mu_1(A) \geq \mu_1(B)$. Hint: find a vector $\psi \in \mathbb{R}^n$ such that $R_A(\psi) = \mu_1(B)$.

Solution: HW

8/8 For a subset of nodes $S \subset V$ of a graph $G(V, E)$ we define it's isoperimetric ratio as

$$\theta(S) = \frac{|E(S, V \setminus S)|}{|S|}$$

and let $\theta_G = \min\{\theta(S) : S \subset V, |S| \leq |V|/2\}$ the isoperimetric constant of G.

(a) Show that $\theta(S) \geq \lambda_2(1-s)$ where $s = |S|/|V|$.

(b) Show that $\theta_G \geq \lambda_2/2$.

Solution: Let χ_S as before, $s = |S|/n$, and let $v_S = \chi_S - s \cdot \mathbb{1}$. Then, as we have computed in problem 5/7 and 6/2, we have $\langle v_S, \mathbb{1}\rangle = 0$, and we have $R_L(v_S) = \frac{E(S,V\setminus S)|}{|S|(1-s)} = \theta(S)/(1-s)$. From the properties of the Rayleigh quotient we get $\lambda_2 = \min_{w : w \perp \mathbb{1}} R_L(w) \leq R_L(v_S) = \theta(S)/(1-s)$ as claimed.

If $|S| \leq |V|/2$ then $1 - s \geq 1/2$. Thus $\theta_G = \min_{S : s \leq 1/2} \theta(S) \geq \frac{\lambda_2}{2}$.

□