Math 4570 Matrix Methods - HW1 Calvin Cai

Answer 1a:

Let $A = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$. We want to prove that A is a field.

We know that:

1. Rationals are included in the real numbers, $\mathbb{Q} \subseteq \mathbb{R}$
2. $\sqrt{2}$ is a real number, $\sqrt{2} \in \mathbb{R}$
3. Addition is a bin-op mapping reals to reals, $(+ : (\mathbb{R} \times \mathbb{R}) \to \mathbb{R})$
4. Multiplication is a bin-op mapping reals to reals, $(* : (\mathbb{R} \times \mathbb{R}) \to \mathbb{R})$

Then, $a + b\sqrt{2} \in \mathbb{R}$, so $A \subseteq \mathbb{R}$

Notice that $(+ : (A \times A) \to A)$, since $a + b\sqrt{2} + c + d\sqrt{2} = (a + b) + (c + d)\sqrt{2}$.

1. (Add. identity) $0 \in A$ and 0 is the additive identity for reals, so 0 is the additive identity for A.
2. (Add. associativity) The real numbers are associative under addition, so A is associative under addition.
3. (Add. inverses) The real numbers have inverses under addition ($a^{-1} = -a$). For $x = a+b\sqrt{2} \in A$, $-x = -a - b\sqrt{2} \in A$. So, A has inverses under addition.
4. (Add. commutativity) The real numbers are commutative, so A is commutative.
5. (Mult. identity). 1 is the multiplicative identity in $\mathbb{R}$. $1 \in A$, so 1 is the identity in A.
6. (Mult. associativity). The real numbers are associative under multiplication, so A is too.
7. (Mult. distributivity). The real numbers are distributive under multiplication, so A is too.
8. (Mult. commutativity). The real numbers are commutative under multiplication, so A is too.
9. (Mult. inverses for non-zero elements). The multiplicative inverse for $r \in \mathbb{R}$ is $\frac{1}{r}$, where $r \neq 0$. For $x \in A$, is $\frac{1}{x} \in A$?

   Let $x = a + b\sqrt{2}$, $x \neq 0$.
   $$\frac{1}{x} = \frac{1}{a + b\sqrt{2}}$$
   Multiplying by the identity.
   $$\frac{1}{x} = \frac{1}{a + b\sqrt{2}} * \frac{a - b\sqrt{2}}{a - b\sqrt{2}}$$
   Distribute and simplify.
   $$\frac{1}{x} = \frac{a - b\sqrt{2}}{a^2 - 2b^2} = \frac{a}{a^2 - 2b^2} - \frac{b}{a^2 - 2b^2}\sqrt{2}$$
   $\frac{1}{x} \in A$ because $\frac{a}{a^2-2b^2}, \frac{b}{a^2-2b^2} \in \mathbb{Q}$. Therefore, $\frac{1}{x}$ is the multiplicative inverse in A.

Since A satisfies these 9 properties, A is a field.

Answer 1b:

Let $B = \{a + b\sqrt{-1} \mid a, b \in \mathbb{R}\}$. Let $x_i = a_i + b_i\sqrt{-1}$. Define the following operations:

- $x_1 + x_2 := (a_1 + a_2) + (b_1 + b_2)\sqrt{-1}$
- $x_1 * x_2 := (a_1 * a_2 - b_1 * b_2) + (a_1 * b_2 + b_1 * a_2)\sqrt{-1}$

We want to prove that $(B, +, *)$ is a field. We will do this by showing that B satisfies the 8 axioms for fields.

1. (Add. identity) 0 is an element in B, ie $0 = (0 + 0\sqrt{-1}) \in B$. If we add 0 to some $x \in B$, we get: $0 + (a + b\sqrt{-1}) = (0 + a) + (0 + b)\sqrt{-1} = a + b\sqrt{-1}$. We know that the identity is unique (shown in class), so 0 is the additive identity in B.
2. (Add. associativity) We want to show that $x_1 + (x_2 + x_3) = (x_1 + x_2) + x_3$. Given the LHS expression, we can evaluate the terms like so: $(a_1+a_2+a_3)+(b_1+b_2+b_2)\sqrt{-1}$, using the definition for addition. Because the real numbers are associative/commutative, we can re-group the terms and apply our definition of addition in reverse to get $((a_1 + a_2) + (b_1 + b_2)\sqrt{-1}) + (a_3 + b_3\sqrt{-1})$ which is our original RHS $(x_1 + x_2) + x_3$.
3. (Add. inverses) $x^{-1} = -x$ because $x + (-x) = (a - a) + (b - b)\sqrt{-1} = 0$, our additive identity.
4. (Add. commutativity) We want to show that $x_1 + x_2 = x_2 + x_1$. Give the LHS, using our definition for addition gives $(a_1 + a_2) + (b_1 + b_2)\sqrt{-1}$. Because the reals are commutative, we can flip the order of terms in the parentheses, and apply the definition for addition in reverse to get the RHS: $x_2 + x_1$.
5. (Mult. identity) $1 = 1 + 0\sqrt{-1}$ is the multi. identity. Using our definition of multiplication, $1 * (a + b\sqrt{-1}) = (1 * a - 0 * b) + (1 * b + 0 * a)\sqrt{-1} = a + b\sqrt{-1}$.
6. (Mult. associativity) We want to show that $x_1 * (x_2 * x_3) = (x_1 * x_2) * x_3$. Evaluating the LHS using the definition of multiplication as well as the distributive property of reals gives:

   $LHS = (a_1 + b_1\sqrt{-1})((a_2a_3 - b_2b_3) + (a_2b_3 + b_2a_3)\sqrt{-1})$

   $LHS = (a_1(a_2a_3 - b_2b_3) - b_1(a_2b_3 + b_2a_3)) + (a_1(a_2b_3 + b_2a_3) + b_1(a_2a_3 - b_2b_3))\sqrt{-1}$

   $LHS = (a_1a_2a_3 - a_1b_2b_3 - b_1a_2b_3 + b_1b_2a_3) + (a_1a_2b_3 + a_1b_2a_3 + b_1a_2a_3 - b_1b_2b_3)\sqrt{-1}$

   Evaluating the RHS gives:

   $RHS = ((a_1a_2 - b_1b_2) + (a_1b_2 + b_1a_2)\sqrt{-1})(a_3 + b_3\sqrt{-1})$

   $RHS = ((a_1a_2 - b_1b_2)a_3 - (a_1b_2 + b_1a_2)b_3) + ((a_1a_2 - b_1b_2)b_3 + (a_1b_2 + b_1a_2)a_3)\sqrt{-1}$

   $RHS = (a_1a_2a_3 - b_1b_2a_3 - a_1b_2b_3 + b_1a_2b_3) + (a_1a_2b_3 - b_1b_2b_3 + a_1b_2a_3 + b_1a_2a_3)\sqrt{-1}$

   Rearranging the terms (by commutativity of reals over addition), gives:

   $RHS = (a_1a_2a_3 - a_1b_2b_3 - b_1a_2b_3 + b_1b_2a_3) + (a_1a_2b_3 + a_1b_2a_3 + b_1a_2a_3 - b_1b_2b_3)\sqrt{-1} = LHS$
7. (Mult. distributivity) We want to show that $(x_1 + x_2)x_3 = x_1x_3 + x_2x_3$.

   Evaluating and rearranging terms gives:

   $LHS = ((a_1a_3 - b_1b_3) + (a_2a_3 - b_2b_3)) + ((a_1b_3 + b_1a_3) + (a_2b_3 + b_2a_3))\sqrt{-1}$

   Using the reverse of the definition for addition gives:

   $LHS = ((a_1a_3 - b_1b_3) + (a_1b_3 + b_1a_3)\sqrt{-1}) + ((a_2a_3 - b_2b_3) + (a_2b_3 + b_2a_3))\sqrt{-1}$

   Using the reverse of the definition of multiplication gives:

$LHS = x_1x_3 + x_2x_3 = RHS$

8. (Mult. commutativity) We want to show that $x_1x_2 = x_2x_1$.

   Using definition of multiplication:

   $LHS = (a_1a_2 - b_1b_2) + (a_1b_2 + b_1a_2)\sqrt{-1}$

   Rearrange terms because real numbers are commutative over addition/multiplication

   $LHS = (a_2a_1 - b_2b_1) + (a_2b_1 + b_2a_1)\sqrt{-1}$

   Apply definition of multiplication:

   $LHS = (a_2 + b_2\sqrt{-1})(a_1 + b_1\sqrt{-1}) = x_2x_1 = RHS$

9. (Mult. inverses for non-zero elements) Let $x^{-1} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}\sqrt{-1}, x \in B, x \neq 0$. Want to show $xx^{-1} = 1$

   $LHS = (a + b\sqrt{-1}) * (\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}\sqrt{-1})$

   Using our definition for multiplication

   $LHS = (\frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2}) + (\frac{ab}{a^2+b^2} - \frac{ab}{a^2+b^2})\sqrt{-1}$

   Simplifying real numbers

   $LHS = \frac{a^2+b^2}{a^2+b^2} = 1 = RHS$

Since B satisfies these 9 properties, B is a field. B is identical to the set of complex numbers, $B = \mathbb{C}$.

Answer 2:

$(\mathbb{R}^{nxn}, +, *)$, for $n > 1$ is not a field because it does not have multiplicative inverses for all non-zero elements. For example take the nxn matrix M given by $M_{ij} = 1$. M is non-zero, but it has determinant 0, so M has no inverse.

Answer 3:

Addition: (([0], [1], [2]), ([1], [2], [0]), ([2], [0], [1]))

Multiplication: (([0], [0], [0]), ([0], [1], [2]), ([0], [2], [1]))

Answer 4:

See 1b.

Answer 5:

B and D are in RREF.

Answer 6:

$$A + B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, A^2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Answer 7:

Use the fact that $det(A) = 0$ iff A is not invertible. Using expansion on the last column, we get:

$det(A) = 1(t - 0) - 1(6 - 0) + t(t - 0) = t^2 + t - 6 = 0$

Solving for t, we get $t = -3, 2$

Answer 8:

a) The solution space for $x + hy = 4$ is not coincident with the solution space for $3x + 6y = 8$ because there is no scalar multiple that would take us from one equation to another. Therefore, to make the matrix consistent, we should find h such that the matrix has exactly 1 solution. Using determinant: $det(M) = 6 - 3h \neq 0$, so $h \neq 2$.

b) The solution space for $-4x + 12y = h$ is parallel to the solution space for $2x - 6y = -3$. The equations are inconsistent when the lines are not coincident. Adding the second row multiplied by two to the first row gives: $0x + 0y = h - 6$, so $h = 6$ produces infinitely many solutions. Therefore, $h \neq 6$ is inconsistent.

Answer 9:

1) 3 rows, 2 column matrix. Select the first pivot in the first row (either column 1, column 2, or no pivot). If C1, then there is either a pivot in (R2, C2) or no pivot (2 options). If C2, then there cannot be any more pivots (1 option). If there are no pivots, there are no pivots (1 option). Total 4 options.

2) 2 rows, 3 columns. Select the first pivot in the first row (either C1, C2, C3, or none). If C1, 3 options for R2 (C2 pivot, C3, pivot, or no pivot). If C2, 2 options for R2 (C3 pivot, or no pivot). If C3, only 1 option. Only 1 option if no pivot (no pivot). Total 7 options.

3) Only one 4x1 matrix in RREF: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Answer 10:

$e = 0$, since pivots must go top left to bottom right. a can be any value, since it's not above or below any pivot. $c = 1$, because the 1 in column 4 cannot be a pivot (there's a non-zero number above it). $b = 0$ because it's above a pivot. $d = 0$ because it's above a pivot.

Answer 11:

1) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix}$,

$R2 = R1 - R2, R3 = 2 * R1 - R3 \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 4 & 5 & 6 \end{bmatrix}$

$R1 = R1 - 2 * R2, R3 = 7^{-1}(4 * R2 - R3) \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix}$

$R1 = R1 + 3 * R3, R2 = R2 - 3 * R3 \begin{bmatrix} 1 & 0 & 0 & \frac{6}{7} \\ 0 & 1 & 0 & \frac{8}{7} \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix}$

$rref(A) = \begin{bmatrix} 1 & 0 & 0 & \frac{6}{7} \\ 0 & 1 & 0 & \frac{8}{7} \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix}$

$Ax = 0$ Gives: $x = x_4 \begin{bmatrix} -\frac{6}{7} \\ -\frac{8}{7} \\ -\frac{2}{7} \\ 1 \end{bmatrix}$

2) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix}$, $A_{ij} \in \mathbb{Z}_7$

$R2 = R1 - R2, R3 = 2 * R1 - R3 \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 4 & 5 & 6 \end{bmatrix}$

$R1 = R1 - 2 * R2, R3 = 4 * R2 - R3 \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

$R2 = R2 - R3, R3 = 2^{-1} * R3 \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$Ax = 0$ Gives: $\begin{bmatrix} x_1 + 4x_3 = 0 \\ x_2 + 3x_3 = 0 \\ x_4 = 0 \end{bmatrix}$, so $x = \begin{bmatrix} -4x_3 \\ -3x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix}$

Answer 11 - 24:

See handwritten work. I didn't have enough time to write all of the solutions in LaTeX!

HW 1 - ML Calvin Cm

12)

① $\text{rref}(A \mid b) = \text{rref}\left(\begin{array}{ccc|c} 3 & 1 & 4 & 1 \\ 5 & 2 & 6 & 5 \\ 0 & 5 & 2 & 1 \end{array}\right)$, elements $\in \mathbb{Z}_7$.

$$\left(\begin{array}{ccc|c} 3 & 1 & 4 & 1 \\ 5 & 2 & 6 & 5 \\ 0 & 5 & 2 & 1 \end{array}\right) \xrightarrow[R_3 = 5^{-1}R_1 = 3R_1]{R_1 = 3^{-1} \cdot R_1 = 5R_1} \left(\begin{array}{ccc|c} 1 & 5 & 6 & 5 \\ 5 & 2 & 6 & 5 \\ 0 & 1 & 6 & 3 \end{array}\right) \xrightarrow{R_1 = R_1 - 5R_3} \left(\begin{array}{ccc|c} 1 & 0 & 4 & 4 \\ 5 & 2 & 6 & 5 \\ 0 & 1 & 6 & 3 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 - 5R_1} \left(\begin{array}{ccc|c} 1 & 0 & 4 & 4 \\ 0 & 2 & 0 & 6 \\ 0 & 1 & 6 & 3 \end{array}\right) \xrightarrow[R_3 = R_3 - 4R_2]{R_2 = 4R_2} \left(\begin{array}{ccc|c} 1 & 0 & 4 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 6 & 0 \end{array}\right) \xrightarrow{R_1,\ R_3 = 6^{-1}R_3} \boxed{\left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right)}$$

② $A\vec{x} = \vec{b}$, $\vec{x} = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$

13-15) see next page.

16) $A, B, C \in \mathbb{R}^{n \times n}$. $ABC = I_n$. Yes

① $\boxed{A^{-1} = BC \quad C^{-1} = AB}$ B^{-1}?

$\hookrightarrow ABC = I_n \Rightarrow BC = A^{-1} \Rightarrow B = A^{-1}C^{-1} \Rightarrow \boxed{B^{-1} = CA}$

② AB. Suppose $(AB)^{-1}$ exists. Do A^{-1}, B^{-1} exist?

$$AB(AB)^{-1} = (AB)^{-1}AB = I_n$$

Yes. $A^{-1} = B(AB)^{-1}$, $B^{-1} = (AB)^{-1}A$.

17) $(AB)^2 \neq A^2B^2$.

$$\overset{A}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} \overset{B}{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad (AB)^2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

A^2

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

B^2

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

$$A^2B^2 = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$$

$$\boxed{(AB)^2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} = A^2B^2}$$

13) rref. $\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \rightarrow 0=1$, system is inconsistent

no solutions

14) rref: $\left(\begin{array}{ccccc|c} 1 & 2 & 3 & 0 & 5 & 6 \\ 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) = \begin{array}{r} x_1 + 2x_2 + 3x_3 + 5x_5 = 6 \\ x_4 + 2x_5 = 7 \end{array}$

$$\Rightarrow \begin{pmatrix} 6 - 2x_2 - 3x_3 - 5x_5 \\ x_2 \\ x_3 \\ 7 - 2x_5 \\ x_5 \end{pmatrix} =$$

$$\vec{x} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -5 \\ 0 \\ 0 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 6 \\ 0 \\ 0 \\ 7 \\ 0 \end{pmatrix}$$

15)

$$\vec{x} = \begin{pmatrix} -8221/4340 \\ 859/8680 \\ 4695/434 \\ -459/434 \\ 679/434 \end{pmatrix}$$

18) Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $AA^T = A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

19) ①

Symmetric: $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 2 & 3 & 4 \\ 0 & 3 & 4 & 0 \\ 3 & 4 & 0 & 1 \end{pmatrix}$

Skew symmetric: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 1 & 2 \\ -1 & 0 & 0 & -5 \\ -1 & 0 & 0 & 3 \\ -2 & 5 & -3 & 0 \end{pmatrix}$

② they're always 0.

③ $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is both

④ Let $B = A + A^T$, by def of addition and transpose

$$\Rightarrow \left.\begin{array}{l} \text{entry } b_{xy} = a_{xy} + a_{yx} \\ \text{entry } b_{yx} = a_{yx} + a_{xy} \end{array}\right\} \Rightarrow b_{xy} = b_{yx} \Rightarrow B \text{ is symmetric}$$

Let $C = AA^T$

$$\Rightarrow \begin{array}{l} c_{xy} = a_{x1}a_{y1} + a_{x2}a_{y2} + \dots + a_{xn}a_{yn} \\ c_{yx} = a_{y1}a_{x1} + a_{y2}a_{x2} + \dots + a_{yn}a_{xn} \end{array} \Rightarrow c_{xy} = c_{yx} \Rightarrow C \text{ is symmetric}$$

Let $D = A^TA$

$$\Rightarrow \begin{array}{l} d_{xy} = a_{1x}a_{1y} + a_{2x}a_{2y} + \dots + a_{nx}a_{ny} \\ d_{yx} = a_{1y}a_{1x} + a_{2y}a_{2x} + \dots + a_{ny}a_{nx} \end{array} \Rightarrow d_{xy} = d_{yx} \Rightarrow D \text{ symmetric}$$

Let $E = A - A^T$

$$\Rightarrow \begin{array}{l} e_{xy} = a_{xy} - a_{yx} \\ e_{yx} = a_{yx} - a_{xy} \end{array} \Rightarrow -e_{yx} = a_{xy} - a_{yx} \Rightarrow e_{xy} = -e_{yx} \Rightarrow E \text{ skew symmetric}$$

⑤ Yes. $\frac{1}{2}(A + A^T)$ is symmetric. $\frac{1}{2}(A - A^T)$ is skew symmetric (using previous results) $\Rightarrow A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

20) (a) injective, (b) bijective, (c) surjective, (d) injective

21)

$$A = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix} \quad R_2 = R_2 - \tfrac{1}{4}R_1 \quad \begin{pmatrix} 4 & 1 & 0 & 0 \\ 0 & \frac{15}{4} & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix} \quad R_3 = R_3 - \tfrac{4}{15}R_2 \quad \begin{pmatrix} 4 & 1 & 0 & 0 \\ 0 & \frac{15}{4} & 1 & 0 \\ 0 & 0 & \frac{56}{15} & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$$R_4 = R_4 - \tfrac{15}{56}R_3 \quad \begin{pmatrix} 4 & 1 & 0 & 0 \\ 0 & 15/4 & 1 & 0 \\ 0 & 0 & 56/15 & 1 \\ 0 & 0 & 0 & \frac{209}{56} \end{pmatrix} = U \qquad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 0 & 4/15 & 1 & 0 \\ 0 & 0 & 15/56 & 1 \end{pmatrix}$$

22)

$$LU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ l_1 & 1 & 0 & 0 \\ 0 & l_2 & 1 & 0 \\ 0 & 0 & l_3 & 1 \end{pmatrix} \begin{pmatrix} d_1 & u_1 & 0 & 0 \\ 0 & d_2 & u_2 & 0 \\ 0 & 0 & d_3 & u_3 \\ 0 & 0 & 0 & d_4 \end{pmatrix} = \begin{pmatrix} d_1 & u_1 & 0 & 0 \\ d_1 l_1 & l_1 u_1 + d_2 & u_2 & 0 \\ 0 & d_2 l_2 & l_2 u_2 + d_3 & u_3 \\ 0 & 0 & d_3 l_3 & l_3 u_3 + d_4 \end{pmatrix}$$

(1) $r_i = u_i$, (2) $p_i = d_i l_i$, (3) $q_i = \begin{cases} d_i, & i = 1 \\ l_{i-1} u_{i-1} + d_i, & i > 1 \end{cases}$

23) $d_1 = 4$, $d_2 = \frac{15}{4}$, $d_3 = \frac{56}{15}$, $\boxed{d_i = d_1 - \frac{1}{d_{i-1}} \quad l_i = \frac{1}{d_i}}$

$l_1 = \frac{1}{4}$, $l_2 = \frac{4}{15}$, $l_3 = \frac{15}{56}$

$$L = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ l_1 & 1 & 0 & \\ 0 & l_2 & 1 & \\ \vdots & & & \ddots \end{pmatrix} \qquad U = \begin{pmatrix} d_1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & d_2 & 1 & 0 & 0 & \\ 0 & 0 & d_3 & 1 & 0 & \\ \vdots & & & & \ddots & \\ \vdots & & & & & \ddots \end{pmatrix}$$

24) ① Lemma: sum of symmetric matrices is still symmetric

↳ A, B symmetric $n \times n$ matrices.

$A+B = C$, s.t $c_{ij} = a_{ij} + b_{ij}$

$= a_{ji} + b_{ji}$, since A, B symmetric

$= c_{ji}$ ✓

Is $-2\vec{u}\vec{u}^T$ symmetric? Yes

$$-2\vec{u}\vec{u}^T = \begin{pmatrix} -2u_1u_1 & -2u_1u_2 & \cdots \\ -2u_2u_1 & -2u_2u_2 & \cdots \\ \vdots & & \ddots \end{pmatrix} \quad \text{entry}_{ij} = -2u_iu_j = -2u_ju_i$$

$= \text{entry}_{ji}$ ✓

Since I_n symmetric, $I_n - 2\vec{u}\vec{u}^T$ also symmetric by Lemma.

② $H_n^T H_n = (I_n - 2\vec{u}\vec{u}^T)^T(I_n - 2\vec{u}\vec{u}^T)$

Since H_n symmetric, $H_n^T = H_n$

$\Rightarrow H_n^T H_n = H_n H_n = (I_n - 2\vec{u}\vec{u}^T)(I_n - 2\vec{u}\vec{u}^T)$

$= I_n - 4\vec{u}\vec{u}^T + 4(\vec{u}\vec{u}^T)^2 = I_n - 4\vec{u}\vec{u}^T + 4\vec{u}\vec{u}^T\vec{u}\vec{u}^T$

Note: $u^Tu = u_1^2 + u_2^2 + \dots + u_n^2$, also the dot product.

Since u is a unit vector $\|u\| = 1 = \sqrt{u_1^2+u_2^2+\dots+u_n^2} \Rightarrow u^Tu = 1$.

$\Rightarrow H_n^T H_n = I_n - 4\vec{u}\vec{u}^T + 4\vec{u}\vec{u}^T = I_n$. So H_n is orthogonal.

③ $H_n^2 = H_n^T H_n$ because H_n is symmetric

$= I_n$

④ $H_n\vec{u} = (I_n - 2uu^T)\vec{u}$

$= I_n\vec{u} - 2\vec{u}\vec{u}^T\vec{u} = \vec{u} - 2\vec{u}(\vec{u}^T\vec{u}) = \vec{u} - 2\vec{u} = -\vec{u}$

⑤ $H_3 = I_3 - 2\vec{u}\vec{u}^T \quad , \quad \vec{u} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$

$$= I_3 - 2\left(\tfrac{1}{3}\right)\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}1 & 1 & 1\end{pmatrix}$$

$$= I_3 - \frac{2}{3}\begin{pmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1 & -2 & -2\\-2 & 1 & -2\\-2 & -2 & 1\end{pmatrix}$$

$H_4 = I_4 - 2\vec{u}\vec{u}^T \quad , \quad \vec{u} = \frac{1}{\sqrt{4}}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$

$$= I_3 - \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\1 & 1 & 1 & 1\\1 & 1 & 1 & 1\\1 & 1 & 1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -1 & -1 & -1\\-1 & 1 & -1 & -1\\-1 & -1 & 1 & -1\\-1 & -1 & -1 & 1\end{pmatrix}$$