HW 1 - Max Goott

Tuesday, September 21, 2021 3:37 PM

Question 1. Show that the following sets of numbers are fields if the usual addition and multiplication of arithmetic are used:

(1) the set of all numbers of the form $a + b\sqrt{2}$ where a and b are rational numbers.

(2) the set of all numbers of the form $a + b\sqrt{-1}$ where a and b are real numbers. What is this field?

1. Mult inverse

$(a+b\sqrt{2})(x+y\sqrt{2})=1$

$ax+ay\sqrt{2}+bx\sqrt{2}+2by=1$

$(ax+2by)+\sqrt{2}(ay+bx)=1$

$\downarrow 1 \qquad \downarrow 0$

$\begin{vmatrix} a & 2b \\ b & a \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$

A

$\frac{1}{\det(A)} \underbrace{\begin{vmatrix} a & -2b \\ -b & a \end{vmatrix}}_{}^{A^{-1}} \begin{vmatrix} 1 \\ 0 \end{vmatrix}$

$A\vec{x}=\vec{b}$

$A^{-1}A\vec{x}=A^{-1}B$

$\vec{x}=A^{-1}\vec{b}$

$\frac{1}{a^2-2b^2}\begin{vmatrix} a \\ -b \end{vmatrix} = \begin{vmatrix} \frac{a}{a^2-2b^2} \\ \frac{-b}{a^2-2b^2} \end{vmatrix}$

$(a+b\sqrt{2})\left(\frac{a}{a^2-2b^2}+\frac{b}{a^2-2b^2}\sqrt{2}\right)=1$

$\left(\frac{a^2}{a^2-2b^2}\ \frac{-2b^2}{a^2 2b^2}\right)\left(\cancel{\frac{ab\sqrt{2}}{a^2-2b^2}-\frac{ab\sqrt{2}}{a^2-2b^2}}\right)$

A^{-1}

$\frac{a^2-2b^2}{a^2-2b^2}=1$

Sum identity

$(a+b\sqrt{2})+0=a+b\sqrt{2}$

$a+b\sqrt{2}=a+b\sqrt{2}$

$(a+b\sqrt{-1})+0=a+b\sqrt{-1}$

$a+b\sqrt{-1}=a+\sqrt{-1}$

Sum Inverse

$a+b\sqrt{2}+(-a-b\sqrt{2})=0$

$a+b\sqrt{2}-a-b\sqrt{2}=0$

$a-a+b\sqrt{2}-b\sqrt{2}=0$

$0+0=0$

$a+b\sqrt{-1}+(-a-b\sqrt{-1})=0$

$a+b\sqrt{-1}-a-b\sqrt{-1}=0$

$a-a+b\sqrt{-1}-b\sqrt{-1}=0$

$0+0=0$

b. $(a+b\sqrt{-1})(x+y\sqrt{-1})=1$

$ax+ay\sqrt{-1}+bx\sqrt{-1}-by=1$

$(ax-by)+(ay+bx)\sqrt{-1}=1$

$1 \qquad 0$

$\begin{vmatrix} a & -b \\ b & a \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$

$\frac{1}{\det(A)}=\begin{vmatrix} a & b \\ -b & a \end{vmatrix}\begin{vmatrix} 1 \\ 0 \end{vmatrix}$

$\frac{1}{a^2+b^2}\begin{vmatrix} a & b \\ -b & a \end{vmatrix}\begin{vmatrix} 1 \\ 0 \end{vmatrix}$

$\frac{1}{a^2+b^2}\begin{vmatrix} a \\ -b \end{vmatrix} = \begin{vmatrix} \frac{a}{a^2+b^2} \\ \frac{-b}{a^2+b^2} \end{vmatrix} = \begin{vmatrix} x \\ y \end{vmatrix}$

$A\vec{x}=\vec{b}$

$A^{-1}A\vec{x}=A^{-1}\vec{b}$

$\vec{x}=A^{-1}\vec{b}$

$(a+b\sqrt{-1})\left(\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}\sqrt{-1}\right)=1$

$\left(\frac{a^2}{a^2+b^2}+\frac{b^2}{a^2+b^2}\right)\left(\cancel{\frac{ab\sqrt{-1}}{a^2+b^2}-\frac{ab\sqrt{-1}}{a^2+b^2}}\right) \quad \frac{a^2+b^2}{a^2+b^2}=1$

Question 2. Show that the set of all $n \times n$ matrices $\mathbb{R}^{n\times n}$ with the usual matrix addition and multiplication is not a field if $n > 1$.

2. The set of all $n\times n$ matrices over $\mathbb{R}$ is not a field

Let A be a matrix $n\times n$

inverse $A^{-1}=\frac{1}{|A|}\operatorname{adj} A$

$|A|=\det(A)$

If $|A|=0 \rightarrow A^{-1}$ d.n.e

$\exists$ $n\times n$ matrices where $\boxed{|A|=0}$

and the determinant is 0

$\exists$ an $n \times n$ matrix where its inverse d.n.e.
Therefore, the set of all $n \times n$ matrices is not a field

Question 3. Write down the two operations on field $\mathbb{Z}_3$.

+	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

×	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

Question 4. Some basic knowledge of complex numbers.

- Just as $\mathbb{R}$ denotes the set of real numbers, we will use $\mathbb{C}$ to denote the set of complex numbers $z = a+ib$. Here $i = \sqrt{-1}$, and a and b are real numbers called/denoted

$$a = Re(z) = \textbf{real part of } z$$
$$b = Im(z) = \textbf{imaginary part of } z$$

- The **complex conjugate** of $z = a + bi \in \mathbb{C}$ is $\bar{z} := a - bi$
- The **absolute value** of z is $|z| = \sqrt{a^2 + b^2}$.
- $z\bar{z} = |z|^2$

Show that $\mathbb{C}$ is a **field** with the usual sum, scalar product and product.

(4). As shown in Question 1.2,
I proved that $\mathbb{C}$ has a sum inverse,
multiplicative inverse, sum and product identity
over all its outlets, therefore making it a field

Question 5. Determine which of the matrices below are in reduced row-echelon form.

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}; B = \begin{bmatrix} 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; C = \begin{bmatrix} 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 5 \end{bmatrix}; D = \begin{bmatrix} 0 & 1 & 0 & 2 & 4 \end{bmatrix}; E = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}$$

Question 6. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ be two matrices over the field $\mathbb{Z}_2$. Compute $A + B$, A^2 and AB over the field $\mathbb{Z}_2$.

$A + B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

A^2

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

AB

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Question 7. For which values of t does the matrix $A = \begin{bmatrix} 6 & -1 & 1 \\ t & 0 & 1 \\ 0 & 1 & t \end{bmatrix}$ NOT have an inverse?

$\det(A) = 0$

$6(0-1) - (-1)(t^2) + 1(t-0) = 0$

$-6 - t^2 + t = 0$

$t^2 - t + 6 = 0$

$(t+3)(t-2) = 0$

$t = -3, 2$

Question 8. Find all values of h that make the following matrices **consistent**, i.e., at least has one solution.

a) $\begin{bmatrix} 1 & h & | & 4 \\ 3 & 6 & | & 8 \end{bmatrix}$

b) $\begin{bmatrix} -4 & 12 & | & h \\ 2 & -6 & | & -3 \end{bmatrix}$

a: $\begin{bmatrix} 1 & h & | & 4 \\ 3 & 6 & | & 8 \end{bmatrix}$ $R_2 = R_2 - 3R_1$ $\begin{bmatrix} 1 & 0 & | & 9 \\ 0 & 1 & | & 2 \end{bmatrix}$

$[2 \quad -6 \mid -3]$

a. $\begin{bmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{bmatrix}$ $R_2 = R_2 - 3R_1$ $\quad \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \end{bmatrix}$

$\begin{bmatrix} 1 & h & 4 \\ 0 & 6-3h & -4 \end{bmatrix}$ $h \neq 2$

0

$h = (-\infty, 2) \cup (2, \infty)$

b. $\begin{bmatrix} -4 & 12 & h \\ 2 & -6 & -3 \end{bmatrix}$ $R_2 = R_2 + \frac{1}{2}R_1$

$\begin{bmatrix} -4 & 12 & h \\ 0 & 0 & -3 + \frac{1}{2}h \end{bmatrix}$

$h = 6$

Question 9. We says that two $m \times n$ matrices in reduced row-echelon form are of the same type if they have the same number of leading 1's in the same position.

(1) How many types of 3×2 matrices in reduced row-echelon form.

(2) How many types of 2×3 matrices in reduced row-echelon form.

(3) Find all 4×1 matrices in reduced row-echelon form.

List all of them. (Use $*$ to denote any real number. Group them by rank)

9. 3×2

$\begin{bmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & * \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

2×3

$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & * & * \\ 0 & 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$

4×1

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Question 10. For which values of a, b, c, d, and e is the following matrix in reduced row-echelon form?

$$A = \begin{bmatrix} 1 & a & b & 3 & 0 & -2 \\ 0 & 0 & c & 1 & d & 3 \\ 0 & e & 0 & 0 & 1 & 1 \end{bmatrix}$$

$a = *$ $\quad b = *$ $\quad c = 0, 1$

$d = *$ $\quad e, 0$

Question 11. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix}$.

(1) Calculation **rref**(A) over $\mathbb{R}$ by hand. Solve $A\vec{x} = \vec{0}$ and write all solutions in parametric vector forms.

(2) Calculation **rref**(A) over field $\mathbb{Z}_7$ by hand.

(3) Using Python verify your result and calculation **rref**(A) over field $\mathbb{Z}_2$ and $\mathbb{Z}_3$. (Python suggestion is uploaded on Canvas.)

(4) Is it possible that a matrix M has different rank over different fields $\mathbb{Z}_p$? (By calculation in (3))

1. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix}$ $R_2 = R_2 - R_1$, $R_3 = R_3 - 2R_1$ $\quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -2 \\ 0 & -4 & -5 & -6 \end{bmatrix}$ $R_2 = -R_2$

$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & -4 & -5 & -6 \end{bmatrix}$ $R_1 = R_1 - 2R_2$, $R_3 = R_3 + 4R_2$ $\quad \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 7 & 2 \end{bmatrix}$ $R_3 = \frac{1}{7}R_3$

$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix}$ $R_1 = R_1 + 3R_3$ $\quad \begin{bmatrix} 1 & 0 & 0 & \frac{6}{7} \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix}$ $R_2 = R_2 - 3R_3$

$\begin{bmatrix} 1 & 0 & 0 & \frac{6}{7} \\ 0 & 1 & 0 & \frac{8}{7} \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix}$ $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ $\quad z + \frac{2}{7}w = b_3$

$z = b_3 - \frac{2}{7}w$

$y + \frac{8}{7}w = b_2$

4. it is not possible, since Rank

2

```
In [44]:  def question_11():
              M = Matrix ([[1, 2, 3, 4], [1, 1, 0, 2], [2, 0, 1, 2]])
              print ("Matrix : {} ".format(M))
              M_rref = M.rref()
              print ("The Row ecehlon for of Matrix M and Pivot Columns : {}".format(M_rref))

              GF7 = galois.GF(7)
              A = GF7([[1, 2, 3, 4], [1, 1, 0, 2], [2, 0, 1, 2]])
              print("RREF in Z7", GF7.row_reduce(A))
          question_11()

          Matrix : Matrix([[1, 2, 3, 4], [1, 1, 0, 2], [2, 0, 1, 2]])
          The Row ecehlon for of Matrix M and the Pivot Columns : (Matrix([
          [1, 0, 0, 6/7],
          [0, 1, 0, 8/7],
          [0, 0, 1, 2/7]]), (0, 1, 2))
          RREF in Z7 GF([[1, 0, 4, 0],
              [0, 1, 3, 0],
              [0, 0, 0, 1]], order=7)
```

3

$$\begin{bmatrix} 0 & 1 & 0 & \frac{8}{7} \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix} \begin{bmatrix} y \\ z \\ w \end{bmatrix} = \begin{bmatrix} b_2 \\ b_3 \end{bmatrix}$$

$z = b_3 - \frac{2}{7} w$

$y + \frac{8}{7} w = b_2$

$y = b_2 - \frac{8}{7} w$

4. it is not possible, since Rank cannot exceed # of Rows or Columns

Question 12. (Solve a linear system over field $\mathbb{Z}_7$.) Let $A = \begin{bmatrix} 3 & 1 & 4 \\ 5 & 2 & 6 \\ 0 & 5 & 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$.

(1) Calculation **rref**$(A|\vec{b})$ over field $\mathbb{Z}_7$.

(2) Find solution of the linear system $A\vec{x} = \vec{b}$ mod 7.

```
In [11]:  def question_12():
              M = Matrix ([[3, 1, 4, 1], [5, 2, 6, 5], [0, 5, 2, 1]])
              M_rref = M.rref()
              print(M_rref)
          question_12()

          (Matrix([
          [1, 0, 0,   31/6],
          [0, 1, 0,   11/6],
          [0, 0, 1, -49/12]]), (0, 1, 2))
```

```
In [13]:  pip install galois

          Collecting galoisNote: you may need to restart the kernel to use updated packages.

            Downloading galois-0.0.21-py3-none-any.whl (783 kB)
          Requirement already satisfied: numpy<1.21,>=1.17.3 in c:\users\maxgo\anaconda3\lib\site-packages (from galois) (1.20.1)
          Requirement already satisfied: numba<0.54,>=0.53 in c:\users\maxgo\anaconda3\lib\site-packages (from galois) (0.53.1)
          Requirement already satisfied: setuptools in c:\users\maxgo\anaconda3\lib\site-packages (from numba<0.54,>=0.53->galois) (5
          2.0.0.post20210125)
          Requirement already satisfied: llvmlite<0.37,>=0.36.0rc1 in c:\users\maxgo\anaconda3\lib\site-packages (from numba<0.54,>=0.
          53->galois) (0.36.0)
          Installing collected packages: galois
          Successfully installed galois-0.0.21

In [14]:  import galois

In [18]:  GF7 = galois.GF(7)
          A = GF7([[3, 1, 4, 1], [5, 2, 6, 5], [0, 5, 2, 1]])
          GF7.row_reduce(A)

Out[18]:  GF([[1, 0, 0, 4],
              [0, 1, 0, 3],
              [0, 0, 1, 0]], order=7)
```

Question 13. (Use Python) Solve the linear system

$$\begin{cases} 3x_1 + 11x_2 + 19x_3 & = -2 \\ 7x_1 + 23x_2 + 39x_3 & = 10 \\ -4x_1 - 3x_2 - 2x_3 & = 6 \end{cases}$$

and write solutions in parametric vector forms.

```
In [31]:  def question_13():
              x = [[3, 11, 18], [7, 23, 39], [-4, -3, -2]]
              y = [-2, 10, 6]
              solutions = np.linalg.inv(x).dot(y)
              parametric = np.array(solutions)
              answer = parametric.reshape(3,1)
              print('parametric_vector = {}'.format(answer))
          question_13()

          parametric_vector = [[  5.23943662]
           [-14.05633803]
           [  7.6056338 ]]
```

Question 14. (Use Python) Solve the linear system

$$\begin{cases} 3x_1 + 6x_2 + 9x_3 + 5x_4 + 25x_5 & = 53 \\ 7x_1 + 14x_2 + 21x_3 + 9x_4 + 53x_5 & = 105 \\ -4x_1 - 8x_2 - 12x_3 + 5x_4 - 10x_5 & = 11 \end{cases}$$

```
In [63]:  def question_14():
              A = Matrix ([[3, 6, 9, 5, 25, 53], [7, 14, 21, 9, 53, 105], [-4, -8, -12, 5, -10, 11]])
              A_rref = A.rref()
              print(A.rref())

          question_14()

          (Matrix([
          [1, 2, 3, 0, 5, 6],
          [0, 0, 0, 1, 2, 7],
          [0, 0, 0, 0, 0, 0]]), (0, 3))
```

Question 15. (Use Python) Solve the linear system

$$\begin{cases} 2x_1 + 4x_2 + 3x_3 + 5x_4 + 6x_5 & = 37 \\ 4x_1 + 8x_2 + 7x_3 + 5x_4 + 2x_5 & = 74 \\ -2x_1 - 4x_2 + 3x_3 + 4x_4 - 5x_5 & = 20 \\ x_1 + 2x_2 + 2x_3 - x_4 + 2x_5 & = 26 \\ 5x_1 - 10x_2 + 4x_3 + 6x_4 + 4x_5 & = 24 \end{cases}$$

and write solutions in parametric vector forms. (Hint: In Python, if you want precise value, use symbolic calculation A=sym(A))

```
In [34]:  def question_15():
              x = [[2, 4, 3, 5, 6], [4, 8, 7, 5, 2], [-2, -4, 3, 4, -5], [1, 2, 2, -1, 2], [5, -10, 4, 6, 4]]
              y = [37, 74, 20, 26, 24]
              solutions = np.linalg.inv(x).dot(y)
              parametric = np.array(solutions)
              answer = parametric.reshape(5,1)
              print('parametric_vector = {}'.format(answer))
          question_15()

          parametric_vector = [[-1.89423963]
           [ 0.98974654]
           [10.81797235]
           [-1.05760369]
           [ 1.61059908]]
```

Question 16. (1) If A , B and C are $n \times n$ matrices and $ABC = I_n$, is each of the matrices invertible? What are their inverses?

(2) Suppose A and B are $n \times n$ matrices. If AB is invertible, are both A and B are invertible?

1. We know that BC is invertible because since ABC = I_n, then $A^{-1}ABC = A^{-1}I_n$. Therefore, BC = $A^{-1}I_n$, and BC = I_n
2. If AB is invertible, both A and B are invertible by the following proof

$C = B(AB)^{-1}$ and $D = (AB)^{-1}A$

$AC = A(B(AB)^{-1}) = AB(AB)^{-1} = I$

$DB = B(AB)^{-1}A = AB(AB)^{-1} = I$

$\therefore C = A^{-1}$ and $D = B^{-1}$

Question 17. Provide a counter-example to the statement: For any 2×2 matrices A and B, $(AB)^2 = A^2B^2$.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $AB = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$ $(AB)^2 = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 20 & 13 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$ $B^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^2B^2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} \neq (AB)^2$

Question 18. Find an example of a 2×2 nonidentity matrix whose transpose is its inverse.

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$A^{-1} = \frac{\mathrm{adj}(A)}{|A|}$$

$$A^{-1} = \frac{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}{\cos\theta + \sin\theta}$$

$$A^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Therefore, $A^{-1} = A^T$

Question 19. Here are a couple of new definitions: An $n \times n$ matrix A is *symmetric* provided $A^T = A$ and *skew-symmetric* provided $A^T = -A$.

(1) Give examples of symmetric and skew-symmetric 2×2, 3×3, and 4×4 matrices.
(2) What can you say about the main diagonal of a skew-symmetric matrix?
(3) Give an example of a matrix that is both symmetric and skew-symmetric.
(4) Prove that for any $n \times n$ matrix A, the matrices $A + A^T$, AA^T, and $A^T A$ are symmetric and $A - A^T$ is skew-symmetric.
(5) Prove that any $n \times n$ can be written as the sum of a symmetric and skew-symmetric matrices. Hint: Did you do part (4) yet?

1. $A: \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $A = A^T$ Symmetric

2x2 $A: \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = A^{-1}$ $A^T = -A$ skew symmetric

3x3 $A: \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $A^T: \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $A^T = A$ Sym

$A: \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ $A^T: \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$ $A^T = -A$ Skew Sym

4x4
$A: \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $A^T: \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $A^T = A$ sym

$A: \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$ $A^T: \begin{bmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ $A^T = -A$ skew sym

2. has to be 0
3. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
4 $A + A^T$

$$(A + A^T)^T = A^T + (A^T)^T = A^T + A$$

Thus $(A + A^T)^T = A + A^T$

$$AA^T = (AA^T)^T = A^T \cdot (A^T)^T = A^T \cdot A = AA^T$$

$$(AA^T)^T = AA^T$$

$$A^T A = (A^T)^T \cdot A^T = A \cdot A^T = A^T A$$

$$A^T A = (A^T A)^T$$

$$A - A^T = (A - A^T)^T = A^T - (A^T)^T = A^T - A$$

$A^T A = (A^T A)$

$A - A^T = (A - A^T)^T = A^T - (A^T)^T = A^T - A$

This is skew symmetric since $A - A^T = -(A - A^T)$

skew definition

5. $\underbrace{A = \frac{1}{2}(A + A^T)}_{\text{symmetric}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{skew sym}}$

$A = \frac{1}{2}A + \frac{1}{2}A^T + \frac{1}{2}A - \frac{1}{2}A^T$

$A = \frac{1}{2}(2A) + 0$

$A = A$

$\therefore$ A can be written as a sum of a symmetric and skew-symmetric matrix

Question 20. Mark each of the following functions $F : \mathbb{R} \to \mathbb{R}$ injective, surjective or bijective, as is most appropriate. (You may wish to draw the graph of the function in some cases.)

(a) $F(x) = x^2$; injective

(b) $F(x) = x^3/(x^2 + 1)$; bijective

(c) $F(x) = x(x - 1)(x - 2)$; surjective

(d) $F(x) = e^x + 2$. surjective

Question 24. Let I_n be the $n \times n$ identity matrix. Let $\vec{u}$ be a **unit** vector in $\mathbb{R}^n$. Define $H_n = I_n - 2\vec{u}\vec{u}^T$. Here a unit vector $\vec{u}$ means that norm $\|\vec{u}\| = 1$ or equivalently $\vec{u}^T\vec{u} = \vec{u} \cdot \vec{u} = 1$.

(1) Is H_n an symmetric matrix? Prove your result.

(2) Is H_n an orthogonal matrix? (i.e. is $H_n^T H_n = I_n$?)

(3) What is H_n^2?

(4) What is $H_n\vec{u}$?

(5) Suppose $\vec{u} = \frac{1}{\sqrt{n}}\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}$. Write down H_3 and H_4?

1. $H^T = [I_n - 2\vec{u}\vec{u}^T]^T$

$= I^T - 2(\vec{u}\vec{u}^T)^T$

$= I - 2(\vec{u}^T)^T \vec{u}^T$

$= I - 2\vec{u}\vec{u}^T$

$= H$ $\therefore$ H_n is symmetric

2. $H^T H = HH$

$= (I - 2uu^T)^2$

$= I - 2uu^T - 2uu^T + 4(uu^T)(uu^T)$

$= I - 4uu^T + 4u(u^Tu)u^T$

$= I - 4uu^T + 4uu^T$

$= I$

$\therefore$ H_n is orthagonal

3. $H^2_n = I$ (see problem 2)

4. $H_n \vec{u} = (I_n - 2\vec{u}\vec{u}^T)\vec{u}$

$= I_n\vec{u} - 2\vec{u}$

$= \vec{u}(I_n - 2)$

5. $H_n = I_n - 2\vec{u}\vec{u}^T$

$$H_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$H_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{\sqrt{4}} \\ \frac{2}{\sqrt{4}} \\ \frac{2}{\sqrt{4}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$