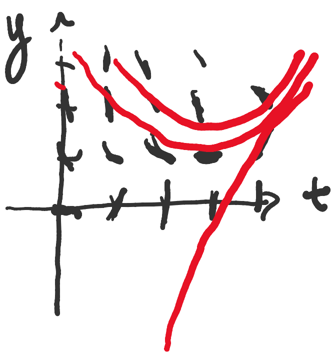
Wednesday, July 8, 2020 12:57 PM

POPULATION MODELS AND PHASES

RECALL: WE WANT TO UNDERSTAND SOLUTIONS OF DIFF EQ'S, AND HOW THE CHANGE BASED ON PARAMETERS. SO FAR, WE'VE LOOK (MOSTLY) AT INITIAL CONDITIONS: $y(0)=y_0$

$$\frac{dy}{dt} = t - 3y \quad \Rightarrow \quad y(t) = \frac{t}{3} - \frac{1}{9} + \underset{\in}{C}\, e^{-3t}$$



For $y(0) = y_0$

$$\Rightarrow y = \underbrace{\frac{t}{3} - \frac{1}{9}}_{\text{ASYMPT.}} + \underbrace{\left(\frac{1}{9} + y_0\right)e^{-3t}}_{\text{BECOMES SMALL}}.$$

THIS PLOT SUMMERIZES BEHAVIOR OF SOLUTIONS AS WE VARY y_0.

OTHER PARAMETERS:

RECALL: POPULATION

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right), \quad r, K > 0.$$

r - UNCONSTRAINED GROWTH

K - CARRYING CAPACITY.

WE LOOK AT

K - CARRYING CAPACITY.

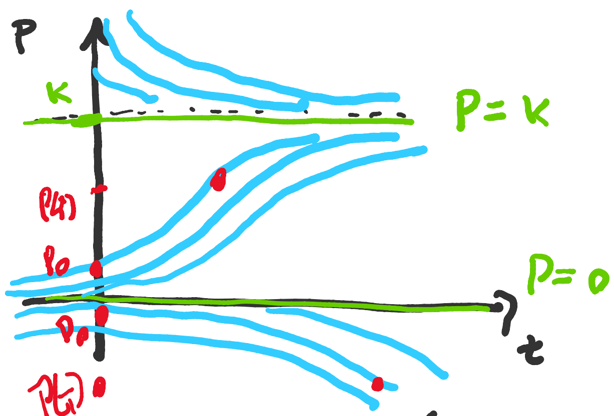
WE LOOK AT

① EQUILIBRIUM SOLUTIONS, $\frac{dP}{dt} = 0$

OR $P(t) = C$

② CLASSIFY, AT LEAST ASYMPTOTICALLY, NON-EQUILIBRIUM BEHAVIOR.

EQ SOLUTIONS: $P(t) = 0, K$.



THREE GROUPS OF SOLUTIONS, OR PHASES, DEPENDING ON $P(0) = P_0$:

- $P < 0$, $P' < 0$, SOLUTIONS DEC.
- $0 < P < K$, $P' > 0$, " INC
- $P > K$, $P' < 0$, " DEC

SOL: FOR $0 < P < K$ IS

$$P(t) = \frac{KCe^{-rt}}{1 + Ce^{-rt}}, \quad C > 0$$

TURNS OUT IF $C > 0$, $0 < P_0 < K$;

AND IF $C < 0$ $P(t)$ IS DISCONTINUOUS.

RECALL:

RECALL:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) \Rightarrow \frac{1}{P(1 - P/K)}\frac{dP}{dt} = r, \quad P \neq 0, K$$

$$\Rightarrow \frac{1}{P} + \frac{1}{K-P}\frac{dP}{dt} = r \qquad *$$

UPPER

$$\Rightarrow \int_{P_0}^{P(T)} \frac{1}{P} + \frac{1}{K-P}\, dP = \int_0^T r\, dt$$

LOWER

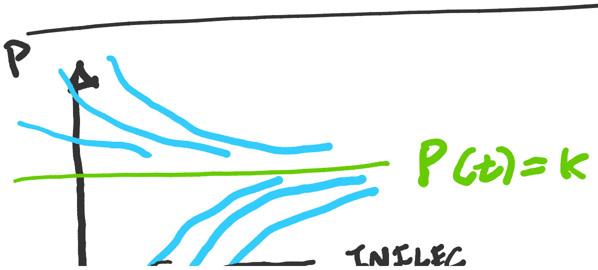
$$\Rightarrow \operatorname{Log}\left|\frac{P}{K-P}\right| - \underbrace{\operatorname{Log}\left|\frac{P_0}{K-P_0}\right|}_{C} = rT$$

IF $P < 0$ OR $P > K$, THEN WE SHOULD HAVE

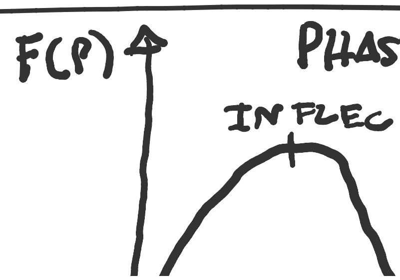
$$* \Rightarrow \int_{P(T)}^{P_0} \frac{1}{P} + \frac{1}{K-P}\, dP = \int_0^T r\, dt$$

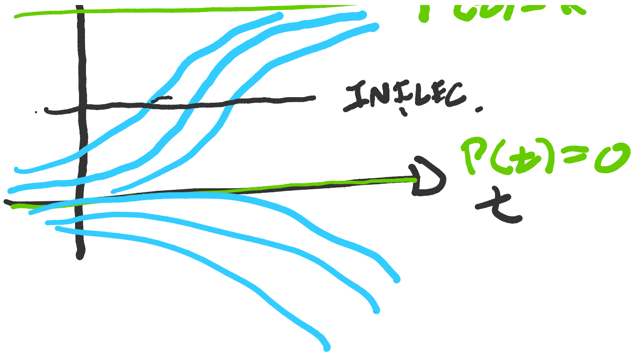
$$\Rightarrow \boxed{\operatorname{Log}\left|\frac{P}{K-P}\right| = -rt + C.}$$

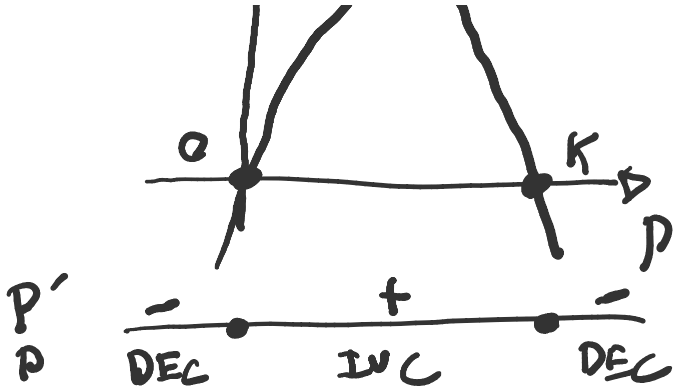
WHEN IN DOUBT, CAREFULLY USE DEF. INTEGRAL.



PHASE PLOT







$$\frac{dP}{dt} = F(P) = rP\left(1 - \frac{P}{K}\right)$$

DEF: AN AUTONOMOUS 1ST ORDER DIFF. EQ IS ONE OF THE FORM

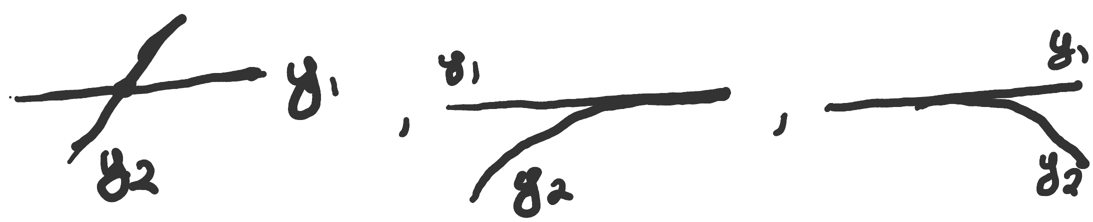
$$\frac{dy}{dt} = F(y).$$

- AN EQUILIBRUM SOLUTION TO AN AUTO. DIFF. EQ. CAN BE FOUND BY SOLVING $F(y) = 0$.

THRM:

- SOLUTIONS TO AUTO. DIFF. EQ'S DO NOT CROSS OR MERGE OR BIFERCATE.

NO!



- SO A PARTICULAR SOLUTION IS ALWAYS UNIFORMLY INCREASING, DECREASING OR CONSTANT

CONSTANT.

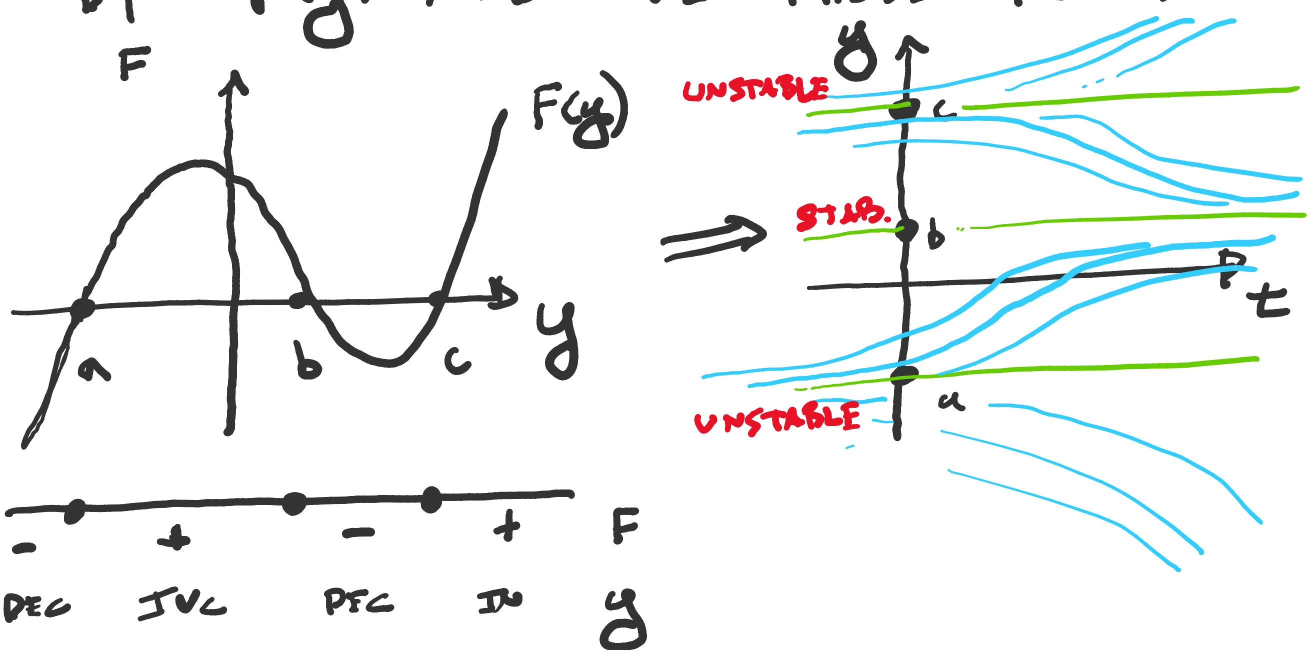
DEF: STABILITY:

- AN EQUILIBRIUM SOLUTION $y = C$ IS **STABLE** IF $\forall \varepsilon > 0$, $\exists \delta > 0$ S.T. IF $|y(0) - c| < \delta$, THEN $|y(t) - c| < \varepsilon$ $\forall x \geq 0$.

  IE, THERE IS A INTERVAL OF SOLUTIONS AROUND C THAT ARE ASYMPTOTIC TO C.

- AN EQ. SOLUTION IS **UNSTABLE** IF ITS NOT STABLE.

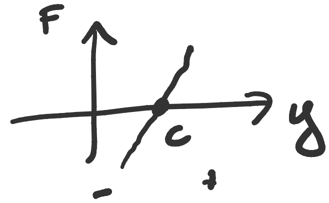
FOR AUTO. DIFF. EQ.'S $y' = F(y)$, STABILITY IS COMPLETELY CONTROLLED BY $F(y)$ AND THE PHASE PLOT.



DEC IVC PFC IN y

STABLE AND UNSTABLE SOLUTIONS ARE COMPLETELY DEFINED BY PHASE PLOT

$F'(c) > 0$,



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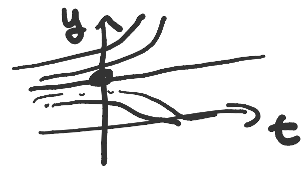
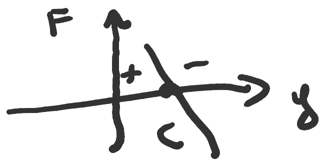
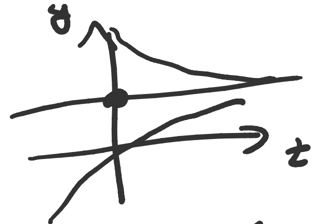


UNSTABLE

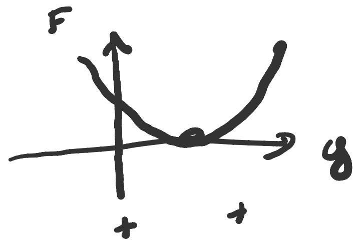
$F'(c) < 0$,

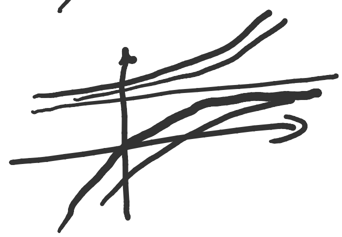




STABLE

$F'(c) = 0$, COULD BE STABLE OR UNSTABLE

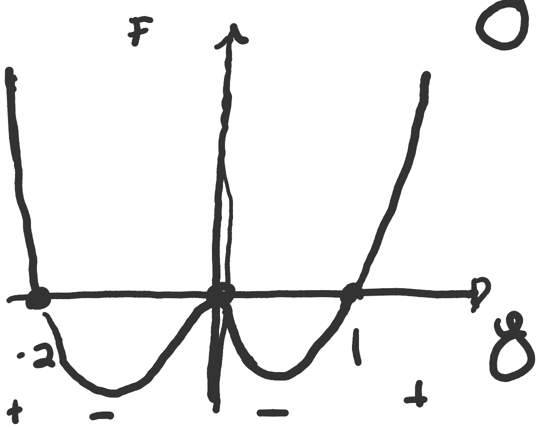




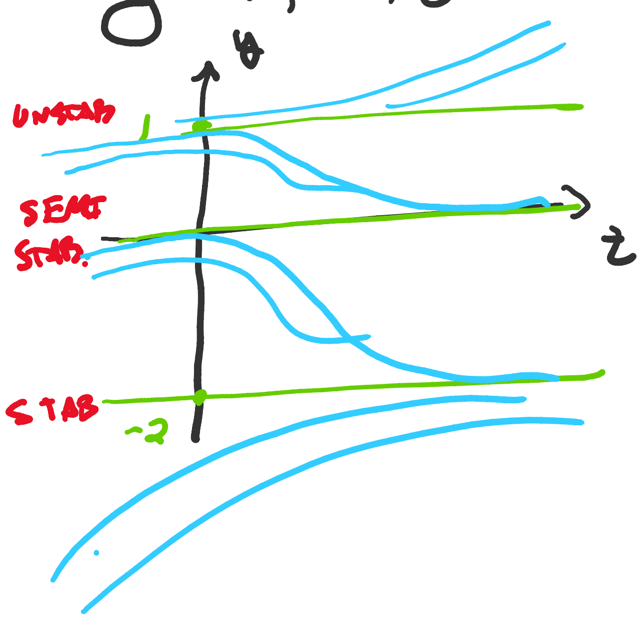
SEMI-STABLE

Ex: $y' = (y-1)(y+2)y^2$

EQ SOLUTIONS: $y' = 0 \Rightarrow y = 1, -2, 0$



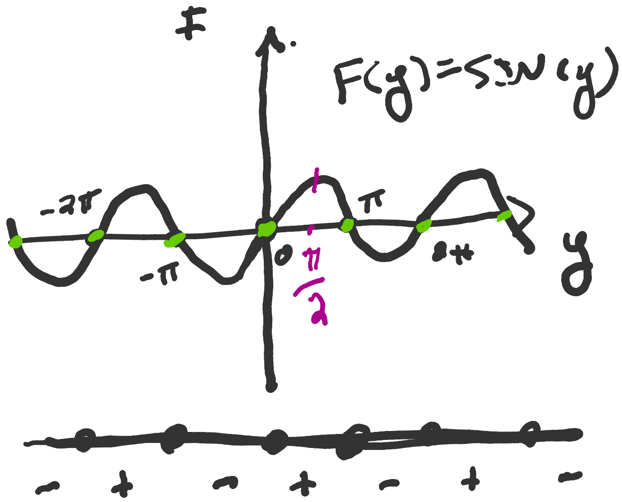
$\Rightarrow$

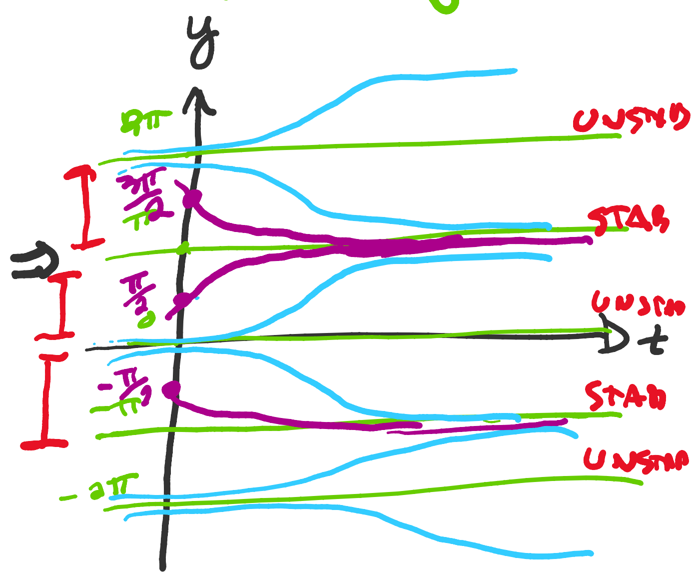


Ex: $y' = \sin y$, $y(0) = \pi/2$, WHAT

Ex: $y' = \sin y$, $y(0) = \pi/2$, WHAT IS $\lim_{t \to \infty} y(t)$?

$0 = y' = \sin y$



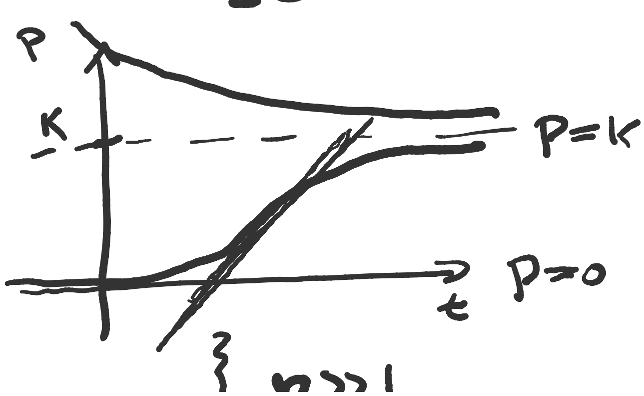


SO $y(0) = \frac{\pi}{2} \Rightarrow \lim_{t \to \infty} y(t) = \pi$.

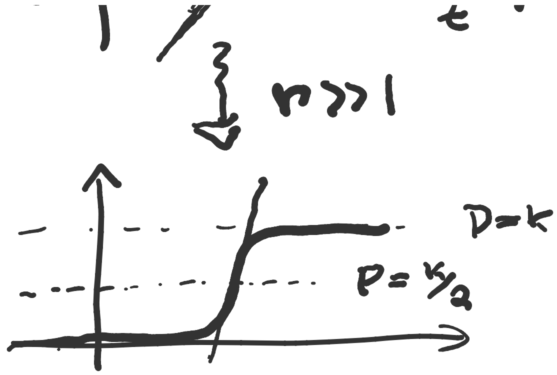
PHASE DIAGRAMS FOR OTHER PARAMETERS

WANT TO SUMMERIZE / UNDERSTAND DIFFERENT FAMLIES OF SOLUTIONS (PHASES) AND HOW THEY DEPEND ON PARAMETERS.

Ex: $\frac{dP}{dt} = rP(1 - P/K)$, HOW DOES THIS CHANGE W/ K AND r ? HERE $K, r > 0$



- K - ASYMPT. LIMIT
- r - OVERALL SCALE OF SLOPE



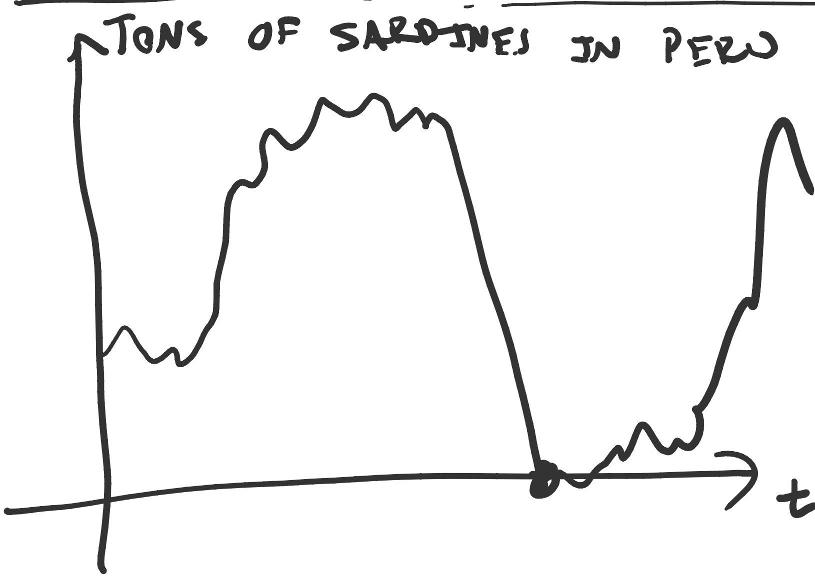
SLOPE

SOMETHING ABOUT r:

WHEN $P=K/2$, $\frac{dP}{dt}=\frac{rK}{4}$.

SO r CAN BE DESCRIBED IN TERMS OF SLOPE AT INFLECTION POINT, WHICH HERE IS <u>MAX</u> SLOPE OF SOLUTIONS BETWEEN 0 AND K.

<u>CASE STUDY</u>: <u>SARDINE FISHING IN PERU</u>



BOOK: 3.4

MODELING POPULATIONS FOR NATURAL RESOURCES BECAME VERY IMPORTANT

<u>CAULKINS ET AL</u>: MODIFY LOGISTIC

CAULKINS ET AL: MODIFY LOGISTIC MODEL TO ADD IN (1) PREDATION BY BIRDS

(2) FISHING.

ASSUME BIRDS EAT CONSTANT AMOUNT OF FISH PER YEAR, AND ASSUME FISHING IS ALSO CONSTANT.

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - B - F$$

$$= RP - \gamma P^2 - B - F$$

r - GROWTH, B - BIRD PREDATION, F - FISHING PREDATION, K - CARRYING CAP.

CAULKINS ET AL ESTIMATE PARAM'S AT

$$R = \begin{pmatrix}\text{AVG. \# OF} \\ \text{EGGS PER FISH}\end{pmatrix} \times \begin{pmatrix}\text{\% SURVIVE TO} \\ \text{ADULTHOOD}\end{pmatrix}$$

$$= \left(10{,}000 \frac{\text{EGGS}}{\text{FISH}\cdot\text{YEAR}}\right)\left(.002 \frac{\text{FISH}}{\text{EGG}}\right) = 20 \frac{\text{FISH}}{\text{FISH}\cdot\text{YEAR}}$$

$$B = (\text{\# BIRDS})\begin{pmatrix}\text{FISH EATEN} \\ \text{PER BIRD}\end{pmatrix}\begin{pmatrix}\text{TONS EACH} \\ \text{FISH WEIGHS}\end{pmatrix}$$

$$= (25 \text{ MILL.})\left(35{,}000 \frac{\text{FISH}}{\text{BIRD}}\right)\left(4.95 \times 10^{-5} \frac{\text{TON}}{\text{FISH}}\right)$$

$$= 44.0 \text{ MILL. TONS PER YEAR.}$$

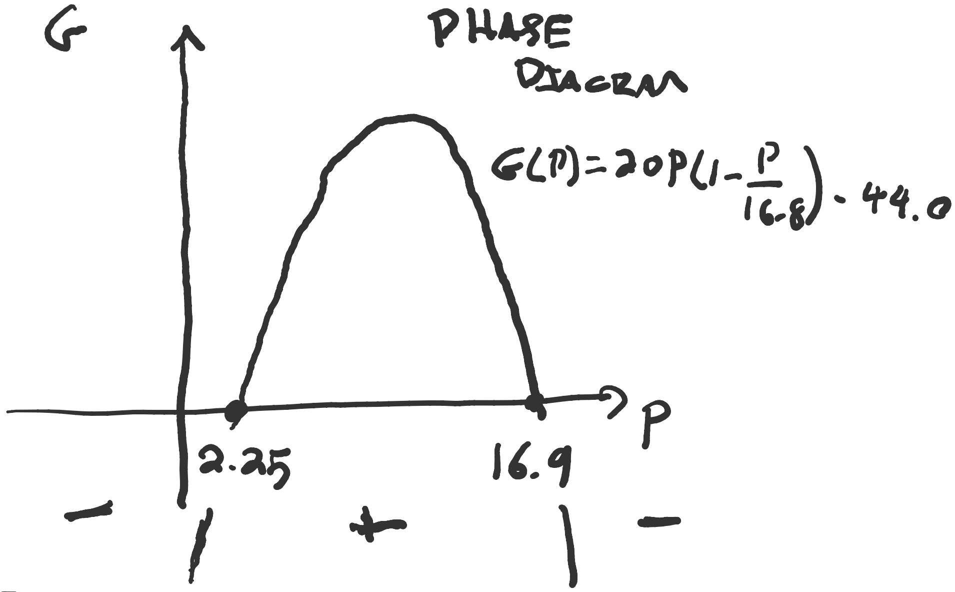
FOR K/γ, FISH STOCK WAS ESTIMATED

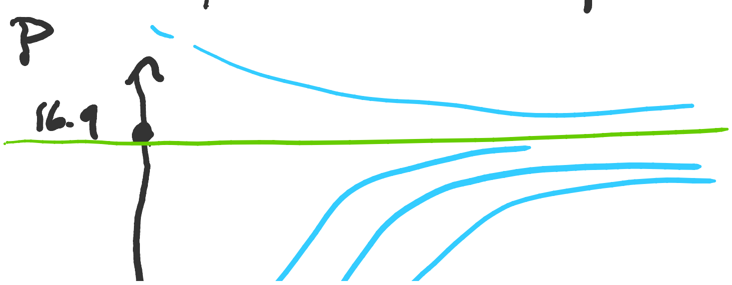
For K/γ, fish stock was estimated at 13.9 – 19.7 mill. tons over 1960's, so can estimate K at half way between. Here $K = 16.8$. So

$$\frac{dP}{dt} = 20P\left(1 - \frac{P}{16.8}\right) - 44.0 - F = G(P)$$

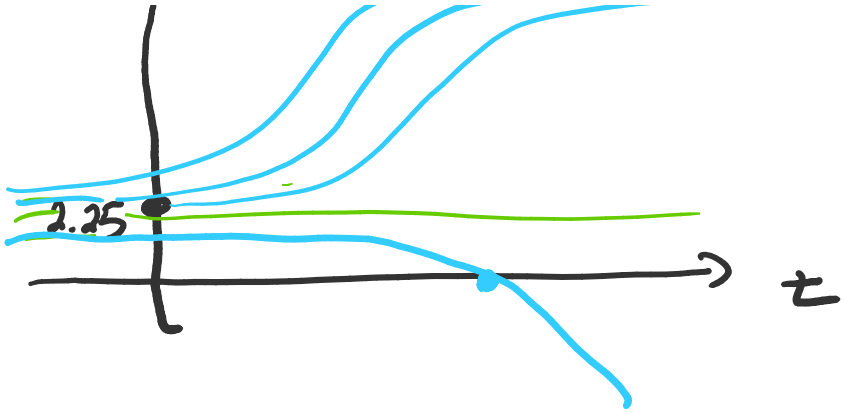
Want to understand how solutions depend on F.

Since autonomous: set $F = 0$. Get





• Note: Now a min stable



I

A MIN STABLE POPULATION OF $P_0 = 2.25$.

SO FAR POP TO BE STABLE IT MUST BE ABOVE 2.25 MILL TONS?

LET COMPUTE THE EQ. SOLUTIONS FOR $\frac{dP}{dt} = G(P)$ FOR NONZERO F:

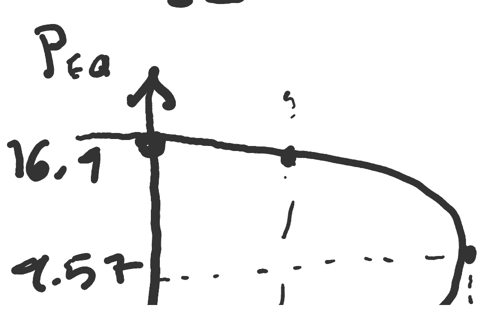
$$0 = 20P\left(1 - \frac{P}{16.8}\right) - 44.0 - F$$

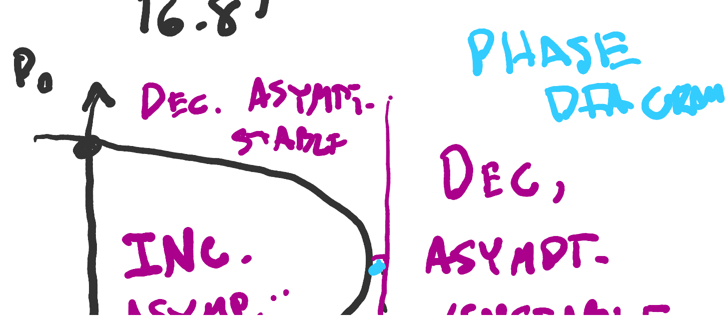
SOLVE:

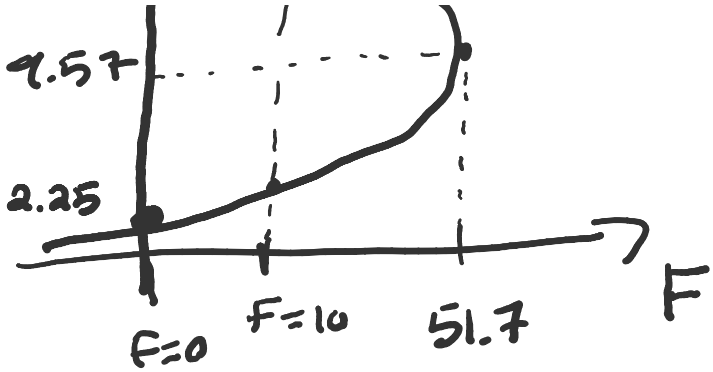
$$P = \frac{20 \pm \sqrt{(20)^2 - 4\,\frac{20}{16.8}(44+F)}}{2 \cdot \frac{20}{16.8}}$$

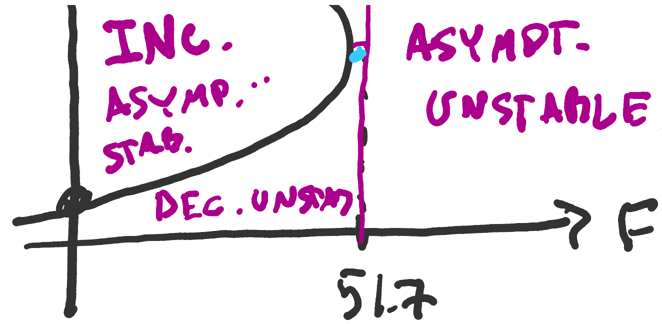
ALT:

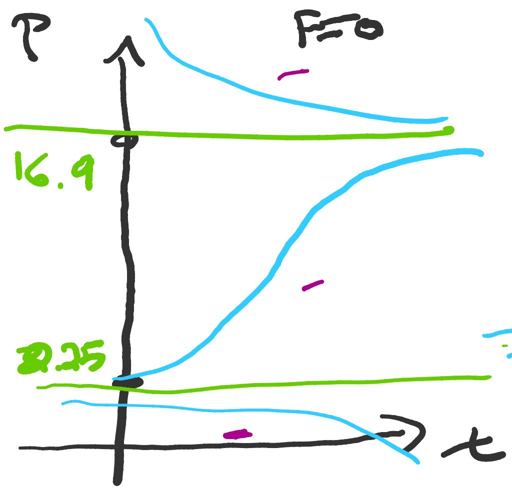
$$0 = \frac{dP}{dt}, \qquad F = 20P\left(1 - \frac{P}{16.8}\right) - 44.0$$

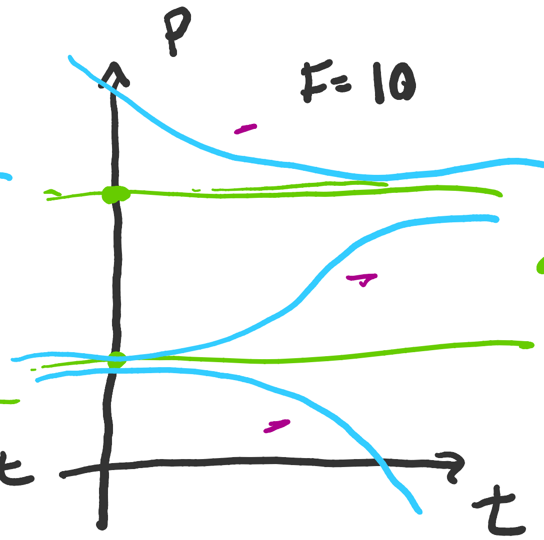


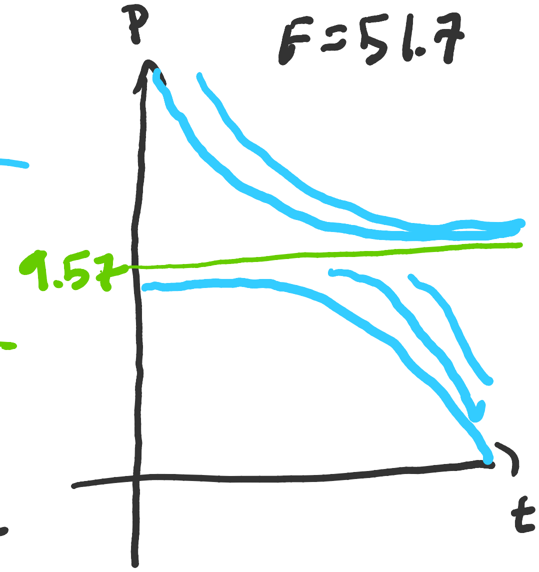


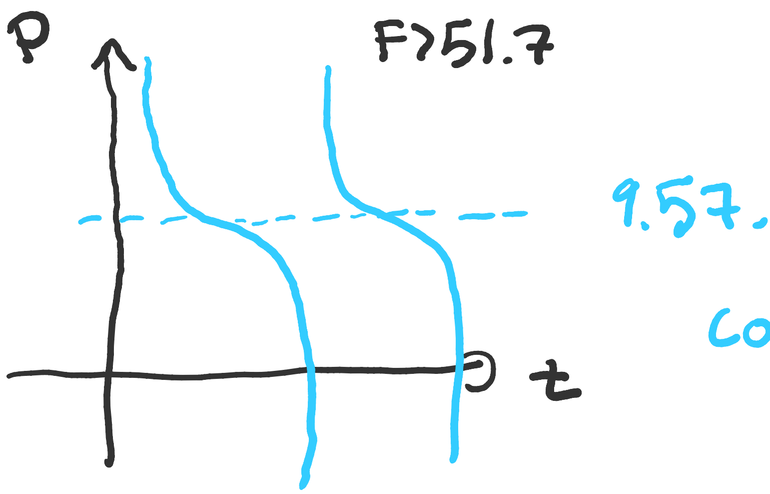












COMPLETE
COLLAPSE.

SYSTEMS OF DIFFERENTIAL EQ-s:

Ex:

$x' = $ RATE IN $-$ RATE OUT

$y' = $ RATE IN $-$ RATE OUT

$y' = $ RATE IN - RATE OUT

PUT SIMPLE NUMBERS IN:

$$x' = -\tfrac{1}{5}x$$
$$y' = \tfrac{1}{5}x - \tfrac{1}{2}y$$

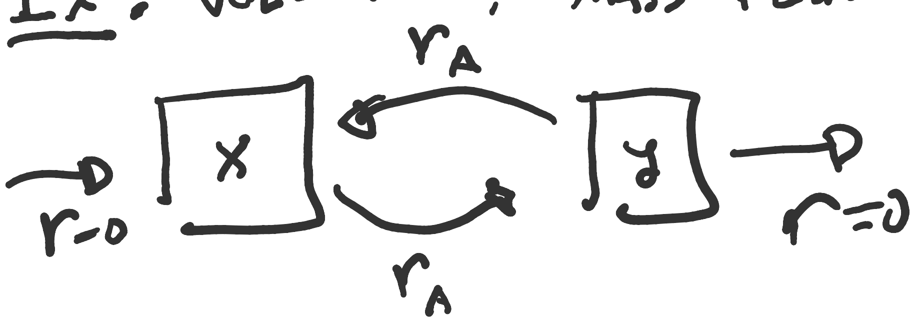
SOLVE ONE AT A TIME

$x = x_0 e^{-t/5}$ SO $y' = \frac{1}{5}x_0 e^{-t/5} - \frac{1}{2}y$

SO

$$y = \tfrac{2}{3}x_0 e^{-\frac{1}{5}t} + \left(y_0 - \tfrac{2}{3}x_0\right)e^{-\frac{1}{2}t}.$$

<u>Ex</u>: VOL: V, MASS FLOW, $C_{IN} = 0$



$$x' = \frac{r_A y}{V} - \frac{r_A x}{V}$$
$$y' = \frac{r_A x}{V} - \frac{r_A y}{V}$$

$\Leftarrow$ HOW DO WE SOLVE THIS???

THIS IS A SYSTEM OF DIFF. EQS.

WANT TO USE LINEAR ALG. TO UNDER-

WANT TO USE LINEAR ALG. TO UNDERSTAND IT.

LINEAR ALGEBRA:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \left(\text{OR } \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} \right)$$

VECTORS WILL ALWAY BE COLUMN VECTORS UNLESS EXPLICITY TRANSPOSED.

SO

$$A\vec{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}; \quad \frac{d\vec{x}}{dt} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \vec{x}'.$$

THIS MEANS WE CAN WRITE IN VECTOR NOTATION:

FOR $A = \begin{bmatrix} -\frac{r_a}{V} & \frac{r_a}{V} \\ \frac{r_a}{V} & -\frac{r_a}{V} \end{bmatrix}$, CAN WRITE

$$\boxed{x'(t) = A\,x(t)} \quad \text{IS} \quad \begin{aligned} x' &= -\frac{r_a}{V}x + \frac{r_a}{V}y \\ y' &= \frac{r_a}{V}x - \frac{r_a}{V}y \end{aligned}$$

HOW DO WE SOLVE? $\quad A = \frac{r_a}{V}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

IDEA: IS IT $x = e^{At}c$?

IDEA: IS IT $X = e^{\cdots} C$?

SORT OF YES?

BETTER IDEA:

WE CAN SOLVE SYSTEMS LIKE THIS, OF THE FORM

$$\begin{aligned} x_1' &= a_1 x \\ x_2' &= a_2 x \\ &\vdots \\ x_N' &= a_N x \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= c_1 e^{a_1 t} \\ &\vdots \\ x_N &= c_N e^{a_N t} \end{aligned}$$

CAN WE TRANSFORM $\vec{x}' = A\vec{x}$ TO THE SIMPLER FORM?

ASSUME A HAS A BASIS OF DISTINCT EIGEN-VECTORS $\vec{v}_1, \ldots, \vec{v}_N$ W/ EIG. VALUES $\lambda_1, \ldots, \lambda_N$. THEN WE CAN DIAGONALIZE A. LET

$$U = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_N \\ | & & | \end{bmatrix}$$ BE MATRIX W/ COLUMNS $\vec{v}_1, \ldots, \vec{v}_N$

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_N \end{bmatrix},$$ THEN

$$A = UDU^{-1} \quad \text{OR} \quad D = U^{-1}AU.$$

NOW, LET $\vec{X} = U\vec{y}$, THEN

$$\frac{d\vec{X}}{dt} = \frac{d}{dt}U\vec{y} = \frac{d}{dt}\begin{bmatrix} U_{11}y_1 + \dots + U_{1N}y_N \\ \vdots \\ \end{bmatrix}$$

$$= \begin{bmatrix} U_{11}y_1' + \dots + U_{1N}y_N' \\ \vdots \\ \end{bmatrix}$$

$$= U\vec{y}'$$

SO NOW SOLVE $\vec{x}' = A\vec{x}$

$$\vec{X}' = Uy' = AU\vec{y}$$

$$\Rightarrow \vec{y}' = U^{-1}AU\vec{y} = Dy$$

$$\Rightarrow \quad y_1' = \lambda_1 y_1$$

$$\vdots$$

$$y_N' = \lambda_N y_N$$

SO

$\lambda_1 t$ $\lambda_N t$

SO

$$y_1 = c_1 e^{\lambda_1 t} \quad , \cdots , \quad y_N = c_N e^{\lambda_N t}$$

THERE ARE $\vec{y}(t)$ SOLUTIONS. BUT

$$\vec{x} = U\vec{y} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_N \\ | & & | \end{bmatrix} \begin{bmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_N e^{\lambda_N t} \end{bmatrix}$$

$$\vec{x} = c_1 e^{\lambda_1 t} \vec{v}_1 + \cdots + c_N e^{\lambda_N t} \vec{v}_N \checkmark$$