Wednesday, July 29, 2020 12:50 PM

TODAY: HIGHER ORDER EQUATIONS AND BOUNDARY CONDITIONS

RECALL:

* SOLVING LINEAR SECOND (HIGHER) ORDER ODES WITH CONST. COEFFS

HOMOGENEOUS:

$$(*) \quad a_2 x'' + a_1 x' + a_0 x = 0, \qquad a_i \in \mathbb{R}.$$

WE SAW LAST TIME THE SOLUTIONS ARE (GENERICALLY) OF THE FORM

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

TO FIND r_i, WE CAN USE A TRICK:

SET

$$x = e^{rt}$$

AND PLUG INTO $(*)$ AND SOLVE FOR r:

r:

$$a_2 r^2 e^{rt} + a_1 r e^{rt} + a_0 e^{rt} = 0$$

$$\Rightarrow \quad a_2 r^2 + a_1 r + a_0 = 0$$

CHARACTERISTIC POLYNOMIAL

GIVES r.

Ex: FIND GENERAL SOLUTION TO

$$x'' - x' - 2x = 0, \quad x(0) = 0, \; x'(0) = 1$$

ANS:

$$\boxed{x = c_1 e^{2t} + c_2 e^{-t}}$$

TO FIND:

SOLVE CHAR. POLY:

$$r^2 - r - 2 = 0$$

$$(r+1)(r-2) = 0$$

SO $r = -1, 2$.

WRITE GENERAL SOLUTION

WRITE GENERAL SOLUTION

$$X = C_1 e^{r_1 t} + C_2 e^{r_2 t} = C_1 e^{-t} + C_2 e^{2t}$$

FIND PARTICULAR SOLUTION, GIVEN BOUNDARY CONDITIONS

$$x(0) = C_1 + C_2 = 0$$

$$x'(0) = -C_1 + 2C_2 = 1$$

SOLVE: $C_1 = -C_2$, SO $3C_2 = 1$,

$C_2 = 1/3$, $C_1 = -1/3$.

$$X = -\frac{1}{3} e^{-t} + \frac{1}{3} e^{2t}$$

$\square$

SUPER POSITION FOR LINEAR HOMOGENEOUS DIFF. EQS:

IF x_1 AND x_2 ARE SOLUTIONS TO

$$a_2 x'' + a_1 x' + a_0 x = 0$$

THEN SO $c_1 x_1 + c_2 x_2$:

$$a_2 (c_1 x_1 + c_2 x_2)'' + a_1 (c_1 x_1 + c_2 x_2)'$$

;k

$$+ a_0 (c_1 x_1 + c_2 x_2)$$

$$= c_1 (a_2 x_1'' + a_1 x_1' + a_0 x) + c_2 (a_2 x_2'' + a_1 x' + a_0 x)$$

$$= c_1 \cdot 0 + c_2 \cdot 0$$

$$= 0$$

$\square$

GENERAL FORM FOR COMPLEX ROOTS:

IF CHAR POLY $a_2 r^2 + a_1 r + a_0 = 0$ IS SOLVED BY $r = \alpha + i\beta$, THEN THE GENERAL SOLUTION IS

$$x = e^{\alpha t} (c_1 \cos \beta t + c_2 \sin \beta t)$$

EXC: SOLVE $x'' + x = 0$

EQUAL ROOTS:

SO FAR, WE'VE ALWAYS ASSUMED A BASIS OF EIGENVECTORS TO A IN $\vec{x}' = A\vec{x}$.

ASSUME CHAR. POLY.

$$a_2 r^2 + a_1 r + a_0 = 0$$

HAS DOUBLE ROOT $r = r_0$. THEN GENERAL SOLUTION IS

$$x = (c_0 + c_1 t)e^{r_0 t}$$

INHOMOGENEOUS LINEAR DIFF EQ'S WITH CONSTANT COEFF.'S

Ex:

HOMO. PART INHOMOGENEOUS PART

$$x'' + 3x' + 2x = 2 \qquad (*)$$

WHAT SHOULD WE "GUESS" FOR A SOLUTION?

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RIGHT HAND SIDE IS <u>CONSTANT</u>, SO "GUESS"

$X = C$.

FOR $X = C$, $X'' = X' = 0$

$2C = 2$ SO $C = 1$

AND

$X = 1$ IS A PARTICULAR SOLUTION.

TO FIND A GENERAL SOLUTION, NOTICE THAT IF X_p IS A SOLUTION TO $(*)$ AND X_h IS A SOLUTION TO THE HOMOGENEOUS PART OF $(*)$:

$$X'' + 3X' + 2X = 0$$

THEN $X = X_h + X_p$ IS A SOLUTION TO $(*)$:

$$x'' + 3x' + 2x = (x_h'' + x_p'') + 3(x_h' + x_p') + 2(x_h + x_p)$$

$$= X_h'' + 3X_h' + 2X_h + X_p'' + 3X_p' + 2X_p$$

$$-\ddot{x}_h + 3\dot{x}_h + 2x_h + \ddot{x}_p + 3\dot{x}_p + 2x_p$$

$$\underbrace{\ddot{x}_h + 3\dot{x}_h + 2x_h}_{0} + \underbrace{\ddot{x}_p + 3\dot{x}_p + 2x_p}_{2}$$

$$= 2$$

AS CLAIMED. SO THE GENERAL SOLUTION IS

$$x = c_1 e^{-t} + c_2 e^{-2t} + 1$$

$$= x_h + x_p$$

FINDING PARTICULAR SOLUTIONS

GIVEN A LINEAR CONSTANT COEFF. ODE

$$a_2 x'' + a_1 x' + a_0 x = f(t)$$

WE SHOULD GUESS THE FOLLOWING FOR x_p:

NOTE: THIS DOES NOT CONVERGE

$$x_p = b_0 f(t) + b_1 f'(t) + b_2 f''(t) + \ldots$$

$$y_p = b_0 g^{(0)} + b_1 g^{(1)} + b_2 g^{(2)} + \dots$$

FOR MOST COMMON FUNCTIONS, THIS TERMINATES AND HAS A COMPACT FORM.

Ex: $$y'' - 2y' - 3y = \underbrace{3t^2 + 4t - 5}_{g(t)}$$

PARTICULAR SOLUTION:

$$y_p = \sum_{i=0}^{\infty} b_i g^{(i)}$$ ← THIS IS JUST A DEGREE 2 POLYNOMIAL!

$$= d_2 t^2 + d_1 t + d_0$$

SOLVE FOR d_i:

$$y' = 2d_2 t + d_1$$

$$y'' = 2d_2$$

PLUG IN:

$$2d_2 - 4d_2 t - 2d_1 - 3d_2 t^2 - 3d_1 t - 3d_0 = 3t^2 + 4t - 5$$

$$= 3t^2 + 4t - 5$$

NOW: FOR POLYNOMIALS TO BE EQUAL FOR ALL t, MUST BE EQUAL ORDER BY ORDER:

$$-3d_2 t^2 = 3t^2$$

$$(-4d_2 - 3d_1)t = 4t$$

$$2d_2 - 2d_1 - 3d_0 = -5$$

SO

$d_2 = -1$

$d_1 = 0$ AND $\boxed{y_p = -t^2 + 1}$ ✓

$d_0 = 1$

$$y_G = y_p + y_h = -t^2 + 1 + C_1 e^{3t} + C_2 e^{-t}$$

QUESTION:

WHAT SHOULD WE GUESS FOR X_p:

WHAT SHOULD WE GUESS FOR X_p?

- $x' + 3x = 4e^{3t} \;\} g$

$$X_p = b_0 g + b_1 g' + b_2 g'' + \dots$$

$$= 4b_0 e^{3t} + 4\cdot 3 b_1 e^{3t} + \dots$$

$$= \underbrace{d_0 e^{3t}}_{\text{GUESS.}}$$

- $X' - 3X = 3\cos 2t$

$$X_p = d_0 \cos 2t + d_1 \sin 2t.$$

- $X' - 3X = \dfrac{4}{X}$

$$X_p = \frac{b_0}{X} + \frac{b_1}{X^2} + \frac{b_2}{X^3} + \dots$$

THEN GET TERM BY TERM RELATIONSHIP BETWEEN b_i's.

RETURN TO HEAT FIN EQ:

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FOUND

$$\frac{d^2U}{dx^2} = \beta(U - U_s)$$, AT EQUILIBRIUM.

REWRITE:

$$U'' - \beta U = -\beta U_s \quad , \quad \beta, U_s \in \mathbb{R}$$

FIND PARTICULAR SOLUTION:

$$U_p = b_0 \quad \Rightarrow \quad -\beta b_0 = -\beta U_s \quad \text{so}$$

$$U_p = U_s$$

GENERAL SOLUTION:

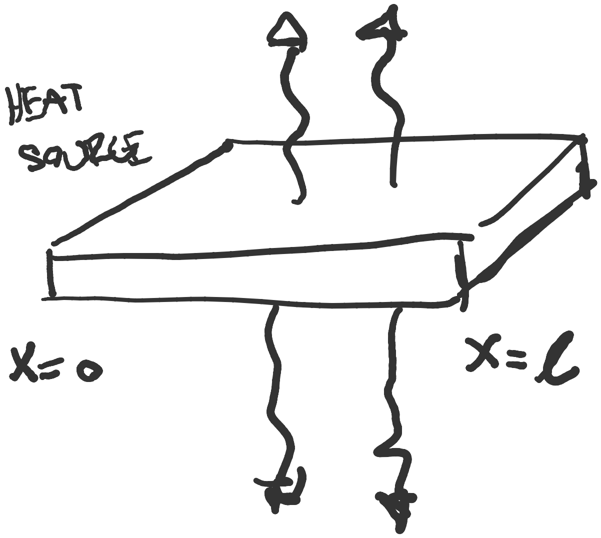
HOMOGENEOUS PART:

$$U'' - \beta U = 0 \quad , \quad r^2 - \beta = 0$$

SO $r = \pm\sqrt{\beta}$.

$$U = C_1 e^{\sqrt{\beta}x} + C_2 e^{-\sqrt{\beta}x} + U_s$$

$$U = C_1 e^{\sqrt{\beta} x} + C_2 e^{-\sqrt{\beta} x} + U_s .$$



BOUNDARY CONDITIONS

ASSUME $U(x=0) = U_*$, $U(x=l) = U_s$, THEN

$$U(0) = C_1 + C_2 + U_s = U_*$$

$$U(l) = C_1 e^{\sqrt{\beta} l} + C_2 e^{-\sqrt{\beta} l} + U_s = U_s$$

GET:

$$U = (U_* - U_s)\left[\frac{-e^{\sqrt{\beta} l x}}{1 - e^{2\sqrt{\beta} l}} + \frac{e^{-\sqrt{\beta} l x}}{1 - e^{-2\sqrt{\beta} l}}\right]$$

$$+ U_s$$

$$= \frac{U_s + (U_* - U_s)\sinh\left(-\sqrt{\beta}x + \sqrt{\beta}\,l\right)}{\sinh(\sqrt{\beta}\,l)}$$

$\square$

HEAT EQUATIONS SUMMARY:

- HEAT - TEMPERATURE EQUATION:

  Q - RATE OF CHANGE OF HEAT, U - TEMP, C - SPECIFIC HEAT, M - THERMAL MASS

$$Q = cm\frac{dU}{dt}$$

- NEWTONS LAW OF COOLING (NLC):

  h - NEWTON COOLING COEFF, S - SURFACE AREA

$$Q_1 = -hS(U_1 - U_2)$$

COOLING BETWEEN TWO DIFFERENT MATERIALS.

- FOURIERS LAW OF HEAT FLUX:

  GIVEN HEAT FLUX $J(x)$ AT X, K - THERMAL CONDUCTIVITY,

THERMAL CONDUCTIVITY,

$$J(x) = -K\frac{dU}{dx}.$$

WE'VE BEEN TALKING ABOUT HEAT <u>CONDUCTION</u>, IE HEAT TRAVELING BETWEEN STATIC OBJECTS IN CONTACT

<u>OTHER KINDS:</u>

- ADVECTION
- CONVECTION
- RADIATION

<u>OTHER HEATING LAWS?</u>

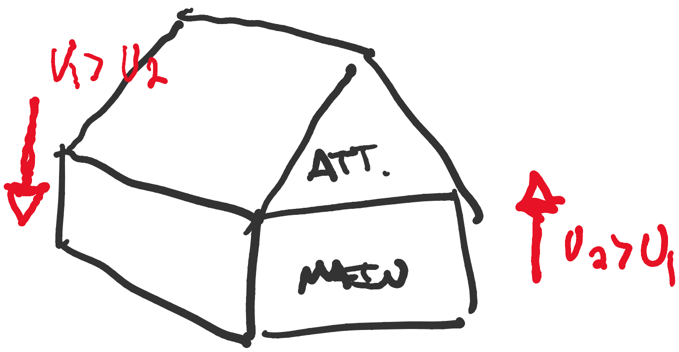
YES. FOR EXAMPLE, IN OUR COFFEE CUP CASE, CUP MAY SLIGHTLY WARM AIR AROUND IT. WE MAY USE MODIFIED EQ'S:

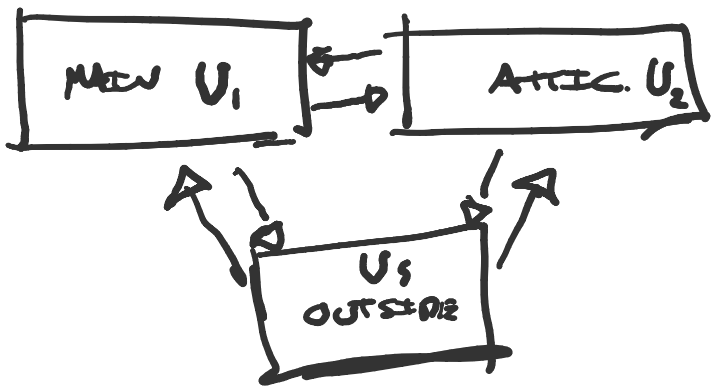
$$\frac{dU}{dt} = -\frac{hS}{cm}\left(U - U_s\right)^{5/4}$$

STATIC AIR CORRECTION.

HEAT MODELING IN A HOUSE:

Ex:





$$M_1 \frac{dU_1}{dt} = \frac{S_{12}}{R_{12}}(U_2 - U_1) + \frac{S_{10}}{R_{10}}(U_S - U_1)$$

$$M_2 \frac{dU_2}{dt} = \frac{S_{12}}{R_{12}}(U_1 - U_2) + \frac{S_{20}}{R_{20}}(U_S - U_2)$$

HERE: M_i - THERMAL MASS

R_{ij} - HEAT RESISTANCE BETWEEN FLOORS/ BETWEEN FLOOR AND OUTSIDE, $\frac{h}{C}$

Q) WHAT "KIND" OF MODEL IS THIS?

HOW WOULD YOU SOLVE THIS?

NOTE:

... SOLVE THIS?

NOTE: EVERYTHING HERE EXCEPT U_1 AND U_2 IS CONSTANT. UP TO A CONSTANT TERM, THIS IS A LINEAR SYSTEM OF EQ'S.

WRITE

$V_1 = U_1 - U_s$, $V_2 = U_2 - U_s$, THIS IS

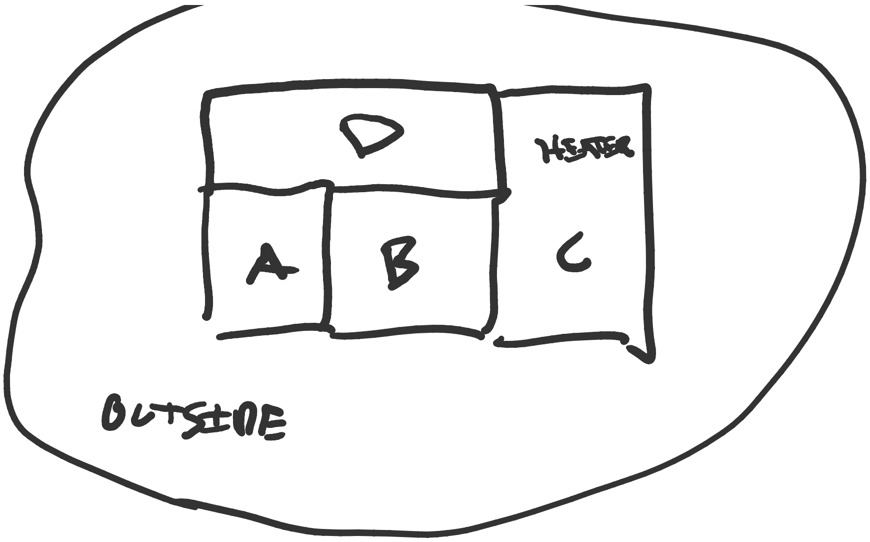
$$M_1 V_1' = \frac{S_{12}}{R_{12}}(V_2 - V_1) + \frac{S_{10}}{R_{10}} V_1$$

$$M_2 V_2' = \frac{S_{12}}{R_{12}}(V_1 - V_2) + \frac{S_{20}}{R_{20}} V_2$$

OR

$\vec{V}' = A \vec{V}$, FOR A PROPERLY DEFINED.

SO WE CAN WRITE (AND SOLVE) VERY GENERAL MODELS IF WE CAN WRITE A: FLOOR PLAN

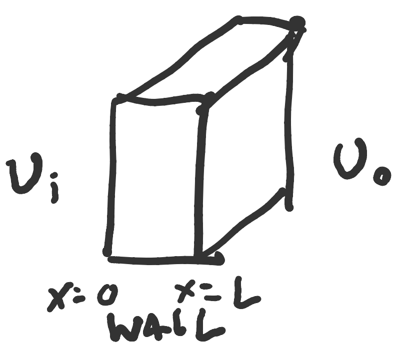


CAN MODEL THIS COMPLETLY.

HEAT LOSS THROUGH A WALL:

ASSUME WE HAVE TEMP. SEPERATING WALL.

WE GOT THE DIFFEQ:



$$\frac{d^2U}{dx^2} = 0$$

SO $U = C_1X + C_2$. WANT TO SAY SOMETHING ABOUT HEAT FLUX $J(x)$ THROUGH WALL.

BOUNDRY CONDITIONS:

- $U(0) = U_i$

- EXTERNAL WALL USES N.L.C.

$$J(L) = h(U(L) - U_0)$$

LETS SOLVE:

- $U(0) = C_2 = U_i$, SO $C_2 = U_i$.

- $J(x) = -k \frac{dU}{dx} = -k C_1$

SO

$$J(L) = -kC_1 = h(C_1 L + U_i - U_0)$$

RE ARRANGE:

$$C_1 = -\frac{h(U_i - U_0)}{hl + k}$$

PLUGGING IN:

$$U(x) = U_i - \frac{h(U_i - U_0)}{hl + k} x$$

SO HEAT LOSS IS

SO HEAT LOSS IS

$$J(x) = -k \cdot \frac{h(U_0 - U_i)}{hl + k} = \frac{U_i - U_0}{\frac{l}{k} + \frac{1}{h}}$$

LENGTH OF WALL + MATERIAL INSIDE

INTERFACE BETWEEN EDGE OF WALL AND OUTSIDE AIR

WE CAN THINK ABOUT

$$R_{WALL} = \frac{l}{k}, \quad \frac{1}{h} = R_{NLC}$$

AS <u>THERMAL</u> <u>RESISTENCE</u>, AND JUST LIKE IN ELECRICITY WE SEE THEY'RE <u>ADDITIVE</u>.

Q) HOW SHOULD WE EXPECT J(x) CHANGE IF WE ADD N.L.C. INSIDE?

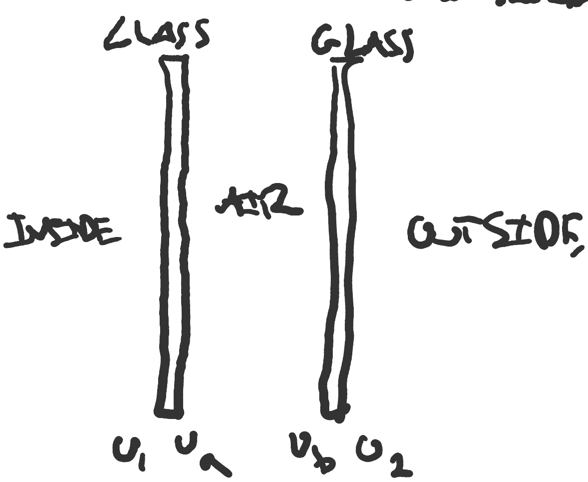
$$J(x) = \frac{U_i - U_0}{R_{WALL} + R_{INSIDE\ NLC} + R_{OUTSIDE\ NLC}}$$

$$R_{wall} + \frac{R_{inside}}{NLC} + \frac{R_{outside}}{NLC}$$

$$= \frac{U_i - U_o}{\frac{l}{k} + \frac{1}{h_{in}} + \frac{1}{h_{out}}} \qquad \square$$

APPLICATION: DOES DOUBLE GLAZING WINDOWS WORK?

DOUBLE GLAZING



JUST CAL. HEAT FLOW TO BE

$$J = \frac{U_1 - U_2}{R}$$

R - TOTAL RESISTENCE.

ASSUME THERMAL EQ, SO HEAT FLOW $J(x)$ MUST BE CONSTANT AND THE SAME THROUGH EACH MEDIUM.

IN PARTICULAR

$$J = \frac{U_1 - U_a}{R_g} = \frac{U_a - U_b}{R_a} = \frac{U_b - U_2}{R_g}$$

$K_g \quad R_a \quad R_g$

(ASSUMING GLASS THERMAL RES. IS SAME).

NO AIR GAP:

$$J_s = \frac{U_1 - U_2}{2R_g}$$

ASSUME TEMPS ALL CLOSE, DEFINE RELITIVE HEAT FLUX:

$$\Delta = \frac{J_s - J}{J_s} \longrightarrow \text{PERCENTAGE IMPROVEMENT IN HEAT FLUX}$$

Now: $R_g = \frac{l_g}{K_g}$, $R_a = \frac{l_a}{K_a}$,

$$\Delta = \frac{R_a}{2R_g + R_a} = \frac{\frac{K_g l_a}{K_a l_g}}{2 + \frac{K_g l_a}{K_a l_g}}$$

FOR GLASS AND AIR, $\frac{K_g}{K_a} \approx 16$, SET $r = \frac{l_a}{l_g}$. GET

$r = \frac{\text{[illegible]}}{\ell g}$. GET

$$\Delta = \frac{16r}{16r+2} = \frac{r}{r+\frac{1}{8}}$$

$r=0 \Rightarrow \Delta = 0$, $r \to \infty$ $\Delta \to 1$,

SO 100% IMPROVEMENT. WHEN $r=1$

$\Delta(1) \approx .89$

SO 89% IMPROVEMENT. BY $r=4$
GET 97% IMPROVEMENT.

EXC) MODEL HOW THIS WOULD WORK
FOR PLASTIC WRAP.