Math 4570 Matrix methods for DA and ML -Fall 2021.

He Wang

he.wang@northeastern.edu

Homework 1.

Notations of **column** vectors: $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = (v_1, v_2, v_3)$. The right-side used in our book is a point notation. It is different from 1×3 matrix $[v_1\ v_2\ v_3]$.

2. Questions:

Rules of answering the questions: (1.) Write reason or proof for each conclusion of your answer.

(2.) For calculation "by hand" questions, write down all steps of calculations. For calculation by Python questions write down (copy) the input and useful output.

(3.) You can scan and submit your handwriting answers. However, it is highly recommended that you use **LaTex** to write your answers. (At least for some homework.) You can either use the online version https://www.overleaf.com/ or download the local disc version https://www.latex-project.org/get/ on Mac or PC. *Warning:* Texmaker or Texworks are just editors. You need to download the full tex first. I recommend to use Texmaker.

A basic template can be (copy the following text and run tex.) There are many packages for tex. For example, using "tikz" you can draw many beautiful pictures. A template I used for lecture notes is also on Canvas.

```
\documentclass[11pt]{paper}
\usepackage{amssymb,amscd,amsmath}
\usepackage[all]{xy}

\textwidth=17cm \textheight=23cm
\voffset=-0.4in
\hoffset=-0.9in

\begin{document}
\begin{center}
\textbf{Math 5110- Applied Linear Algebra-Homework 1 }

\textbf{Name:  Your name}
\end{center}

Write your answers Here. For example

\textbf{Answer of Question 1:}
If you don't know how to write formulas in Latex, just Google: ''Latex  ...."

\end{document}
```

For all questions, if there is no particular instruction, the field is real number field $\mathbb{R}$.

Question 1. Show that the following sets of numbers are fields if the usual addition and multiplication of arithmetic are used:

(1) the set of all numbers of the form $a + b\sqrt{2}$ where a and b are rational numbers.

(2) the set of all numbers of the form $a + b\sqrt{-1}$ where a and b are real numbers. What is this field?

(1) The definition of **sum** and **product** is by standard sum and product of polynomials

$$(a + b\sqrt{2}) + (c + d\sqrt{2}) := a + c + (b + d)\sqrt{2}$$

$$(a + b\sqrt{2}) \times (c + d\sqrt{2}) := ac + 2bd + (ad + bc)\sqrt{2}$$

First note that the set of all numbers of the form $a + b\sqrt{2}$ is closed with respect to addition, multiplication, taking negatives, and forming inverses (if not zero).
We need **identity for sum** is 0, since $0 + a + b\sqrt{2} = a + b\sqrt{2}$,
The **identity for product** is 1 to satisfy the condition $1 \times (a + b\sqrt{2}) = a + b\sqrt{2}$.
For each element $a + b\sqrt{2} \in F$, the **inverse for sum** is $-a - b\sqrt{2}$, since $a + b\sqrt{2} + (-a - b\sqrt{2}) = 0$
The **multiplicative inverse** $(a + b\sqrt{2})^{-1} := x + y\sqrt{2}$ of $a + b\sqrt{2} \in F$ is calculated the condition

$$(a + b\sqrt{2}) \times (x + y\sqrt{2}) = 1$$

That is

$$x + y\sqrt{2} = \frac{1}{a^2 - 2b^2}(a - b\sqrt{2})$$

The usual rules of arithmetic guarantee that the field axioms hold for the set numbers.
(2) The definition of **sum** and **product** is by standard sum and product of polynomials

$$(a + b\sqrt{-1}) + (c + d\sqrt{-1}) := a + c + (b + d)\sqrt{-1}$$

$$(a + b\sqrt{-1}) \times (c + d\sqrt{-1}) := ac - bd + (ad + bc)\sqrt{-1}$$

First note that the set of all numbers of the form $a + b\sqrt{-1}$ is closed with respect to addition, multiplication, taking negatives, and forming inverses (if not zero).
We need **identity for sum** is 0, since $0 + a + b\sqrt{-1} = a + b\sqrt{-1}$,
The **identity for product** is 1 to satisfy the condition $1 \times (a + b\sqrt{-1}) = a + b\sqrt{-1}$.
For each element $a+b\sqrt{-1} \in F$, the **inverse for sum** is $-a-b\sqrt{-1}$, since $a+b\sqrt{-1}+(-a-b\sqrt{-1}) = 0$
The **multiplicative inverse** $(a + b\sqrt{-1})^{-1} := x + y\sqrt{-1}$ of $a + b\sqrt{-1} \in F$ is calculated the condition

$$(a + b\sqrt{-1}) \times (x + y\sqrt{-1}) = 1$$

That is

$$x + y\sqrt{-1} = \frac{1}{a^2 + b^2}(a - b\sqrt{-1})$$

The usual rules of arithmetic guarantee that the field axioms hold for the set numbers.
Denote $i = \sqrt{-1}$. This is complex filed, which is the same as question 4.

Question 2. Show that the set of all $n \times n$ matrices $\mathbb{R}^{n\times n}$ with the usual matrix addition and multiplication is not a field if $n > 1$.

If $n > 1$, there are non-zero matrices in $\mathbb{R}^{n\times n}$ which are not invertible, thus, $\mathbb{R}^{n\times n}$ is not a field.

Question 3. Write down the two operations on field $\mathbb{Z}_3$.

+	[0]	[1]	[2]
[0]			
[1]			
[2]			

×	[0]	[1]	[2]
[0]			
[1]			
[2]			

+	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

×	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

Question 4. Some basic knowledge of complex numbers.

- Just as $\mathbb{R}$ denotes the set of real numbers, we will use $\mathbb{C}$ to denote the set of complex numbers $z = a+ib$. Here $i = \sqrt{-1}$, and a and b are real numbers called/denoted

$$a = Re(z) = \textbf{real part of } z$$
$$b = Im(z) = \textbf{imaginary part of } z$$

- The **complex conjugate** of $z = a + bi \in \mathbb{C}$ is $\bar{z} := a - bi$
- The **absolute value** of z is $|z| = \sqrt{a^2 + b^2}$.
- $z\bar{z} = |z|^2$

Show that $\mathbb{C}$ is a **field** with the usual sum, scalar product and product.

Same as question 1.

Question 5. Determine which of the matrices below are in reduced row-echelon form.

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}; B = \begin{bmatrix} 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; C = \begin{bmatrix} 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 5 \end{bmatrix}; D = \begin{bmatrix} 0 & 1 & 0 & 2 & 4 \end{bmatrix}; E = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}$$

B, D

Question 6. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ be two matrices over the field $\mathbb{Z}_2$. Compute $A + B$, A^2 and AB over the field $\mathbb{Z}_2$.

$A+B=\begin{bmatrix}1&0&0\\1&0&0\\1&0&0\end{bmatrix}$, $A^2=\begin{bmatrix}1&1&0\\0&0&1\\0&1&1\end{bmatrix}$ and $AB=\begin{bmatrix}0&1&0\\0&0&1\\1&1&1\end{bmatrix}$

Question 7. For which values of t does the matrix $A=\begin{bmatrix}6&-1&1\\t&0&1\\0&1&t\end{bmatrix}$ NOT have an inverse?

Adjoin the matrix I_3, and put the resulting matrix $[A|I_3]$ in its reduced row echelon form R . The original matrix will fail to be invertible if and only if the first three columns of R do not form the matrix I_3. This occurs precisely when $t^2+t-6=0$, that is, $t=2$ or -3.
Method 2: Using determinant $\det(A)=0$.

Question 8. Find all values of h that make the following matrices **consistent**, i.e., at least has one solution.

a) $\begin{bmatrix}1&h&|&4\\3&6&|&8\end{bmatrix}$

$R_2-3R_1=\begin{bmatrix}1&h&|&4\\0&-3h+6&|&-4\end{bmatrix}$
$\therefore$ the augmented matrix is consistent if $h\neq 2$, as that will make $R_2, 0=-4$ which is inconsistent.

b) $\begin{bmatrix}-4&12&|&h\\2&-6&|&-3\end{bmatrix}$

$R_2+\frac{1}{2}R_1=\begin{bmatrix}-4&12&|&h\\0&0&|&h-3\end{bmatrix}$
$\therefore$ the augmented matrix is consistent if and only if $h=3$, otherwise it's inconsistent.

Question 9. We says that two $m\times n$ matrices in reduced row-echelon form are of the same type if they have the same number of leading 1's in the same position.

(1) How many types of 3×2 matrices in reduced row-echelon form.

(2) How many types of 2×3 matrices in reduced row-echelon form.

(3) Find all 4×1 matrices in reduced row-echelon form.

List all of them. (Use $*$ to denote any real number. Group them by rank)

(1) 3×2 **rref**:

$$\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix},\quad \begin{bmatrix}1&*\\0&0\\0&0\end{bmatrix},\quad \begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix},\quad \begin{bmatrix}0&0\\0&0\\0&0\end{bmatrix}.$$

(2) 2×3 **rref**:

$$\begin{bmatrix}1&0&*\\0&1&*\end{bmatrix},\quad \begin{bmatrix}1&*&0\\0&0&1\end{bmatrix},\quad \begin{bmatrix}0&1&0\\0&0&1\end{bmatrix},$$
$$\begin{bmatrix}1&*&*\\0&0&0\end{bmatrix},\quad \begin{bmatrix}0&1&*\\0&0&0\end{bmatrix},\quad \begin{bmatrix}0&0&1\\0&0&0\end{bmatrix},\quad \begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}.$$

(3) $\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$; and $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$

Question 10. For which values of a, b, c, d, and e is the following matrix in reduced row-echelon form? $A = \begin{bmatrix}1&a&b&3&0&-2\\0&0&c&1&d&3\\0&e&0&0&1&1\end{bmatrix}$

$e=0$; $c=1$; $d=0$; $b=0$; a any real number

Question 11. Let $A = \begin{bmatrix}1&2&3&4\\1&1&0&2\\2&0&1&2\end{bmatrix}$.

(1) Calculation **rref**(A) over $\mathbb{R}$ by hand. Solve $A\vec{x} = \vec{0}$ and write all solutions in parametric vector forms.

(2) Calculation **rref**(A) over field $\mathbb{Z}_7$ by hand.

(3) Using Python verify your result and calculation **rref**(A) over field $\mathbb{Z}_2$ and $\mathbb{Z}_3$. (Python suggestion is uploaded on Canvas.)

(4) Is it possible that a matrix M has different rank over different fields $\mathbb{Z}_p$? (By calculation in (3))

$$A = \begin{bmatrix}1&2&3&4\\1&1&0&2\\2&0&1&2\end{bmatrix} \xrightarrow[R_2-R_1]{R_3-2R_1} \begin{bmatrix}1&2&3&4\\0&1&3&2\\0&-4&-5&-6\end{bmatrix} \xrightarrow{R_3+4R_2} \begin{bmatrix}1&2&3&4\\0&1&3&2\\0&0&7&2\end{bmatrix} \xrightarrow{R_3/7} \begin{bmatrix}1&2&3&4\\0&1&3&2\\0&0&1&2/7\end{bmatrix} \xrightarrow[R_1-3R_3]{R_2-3R_3}$$
$$\begin{bmatrix}1&2&0&22/7\\0&1&0&8/7\\0&0&1&2/7\end{bmatrix} \xrightarrow{R_1-2R_2} \begin{bmatrix}1&0&0&6/7\\0&1&0&8/7\\0&0&1&2/7\end{bmatrix} = \textbf{rref}(A)$$

(2) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[R_2 - R_1]{R_3 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 3 & 2 & 1 \end{bmatrix} \xrightarrow{R_3 - 3R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_3/2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 - 4R_3]{R_2 - 2R_3}$
$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{rref}(A)$

(3) See Python out put:

(4) Compare Ar2 and Ar7, we can see that , the first three columns different rank. So $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ has different rank over $\mathbb{Z}_2$ and $\mathbb{Z}_7$.

Python Output

```
Ar2 =
       1        0        0        0
       0        1        0        0
       0        0        1        0

Ar3 =
       1        0        0        0
       0        1        0        2
       0        0        1        2

Ar7 =
       1        0        4        0
       0        1        3        0
       0        0        0        1
```

Question 12. (Solve a linear system over field $\mathbb{Z}_7$.) Let $A = \begin{bmatrix} 3 & 1 & 4 \\ 5 & 2 & 6 \\ 0 & 5 & 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$.

(1) Calculation $\mathbf{rref}(A|\vec{b})$ over field $\mathbb{Z}_7$.

(2) Find solution of the linear system $A\vec{x} = \vec{b} \mod 7$.

$\mathbf{rref}(A|\vec{b}) = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

So solutions over $\mathbb{Z}_7$ is $\vec{x} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} \mod 7$.

Question 13. (Use Python) Solve the linear system

$$\begin{cases} 3x_1 + 11x_2 + 19x_3 & = -2 \\ 7x_1 + 23x_2 + 39x_3 & = 10 \\ -4x_1 - 3x_2 - 2x_3 & = 6 \end{cases}$$

and write solutions in parametric vector forms.

$\mathbf{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. No solution.

Question 14. (Use Python) Solve the linear system

$$\begin{cases} 3x_1 + 6x_2 + 9x_3 + 5x_4 + 25x_5 & = 53 \\ 7x_1 + 14x_2 + 21x_3 + 9x_4 + 53x_5 & = 105 \\ -4x_1 - 8x_2 - 12x_3 + 5x_4 - 10x_5 & = 11 \end{cases}$$

and write solutions in parametric vector forms.

Let A be the augmented matrix.

$$\mathbf{rref}(A) = \begin{bmatrix} 1 & 2 & 3 & 0 & 5 & 6 \\ 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So, $\begin{cases} x_1 = 6 - 2x_2 - 3x_3 - 5x_5 \\ x_4 = 7 - 2x_5 \end{cases}$ and

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 - 2x_2 - 3x_3 - 5x_5 \\ x_2 \\ x_3 \\ 7 - 2x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 7 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$, where x_2, x_3, x_5 are any real numbers.

Question 15. (Use Python) Solve the linear system

$$\begin{cases} 2x_1 + 4x_2 + 3x_3 + 5x_4 + 6x_5 & = 37 \\ 4x_1 + 8x_2 + 7x_3 + 5x_4 + 2x_5 & = 74 \\ -2x_1 - 4x_2 + 3x_3 + 4x_4 - 5x_5 & = 20 \\ x_1 + 2x_2 + 2x_3 - x_4 + 2x_5 & = 26 \\ 5x_1 - 10x_2 + 4x_3 + 6x_4 + 4x_5 & = 24 \end{cases}$$

and write solutions in parametric vector forms. (Hint: In Python, if you want precise value, use symbolic calculation A=sym(A))

Let A be the augmented matrix and calculate rref(A) in Python.

$$\begin{cases} x_1 = -8221/4340 \approx -1.89 \\ x_2 = 8591/8680 \approx 0.99 \\ x_3 = 4695/434 \approx 10.82 \\ x_4 = -459/434 \approx -1.06 \\ x_5 = 699/434 \approx 1.61 \end{cases}$$

Question 16. (1) If A , B and C are $n \times n$ matrices and $ABC = I_n$, is each of the matrices invertible? What are their inverses?

(2) Suppose A and B are $n \times n$ matrices. If AB is invertible, are both A and B are invertible?

(1) By invertible theorem, A and C are invertible and $A^{-1} = BC$ and $C^{-1} = AB$. Then $AB = C^{-1}$, then $CAB = I$. So, B is invertible and $B^{-1} = CA$.
(2) If AB is invertible, then there exist a matrix C such that $ABC = I$. Then by (1) each matrix is invertible.

Question 17. Provide a counter-example to the statement: For any 2×2 matrices A and B, $(AB)^2 = A^2B^2$.

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
$(AB)^2 = \begin{bmatrix} 0 & 0 \\ 12 & 16 \end{bmatrix}$
$A^2B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Question 18. Find an example of a 2×2 nonidentity matrix whose transpose is its inverse.

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Then, $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ We want $A^{-1} = A^T$. We may set $ad - bc = 1$, then we need $a = d$ and $b = -c$.
So, examples will be $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ such that $a^2 + b^2 = 1$.
For example $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ or $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$. Later, we will see more examples like this for $n \times n$ matrices, called orthogonal matrix.

Question 19. Here are a couple of new definitions: An $n \times n$ matrix A is *symmetric* provided $A^T = A$ and *skew-symmetric* provided $A^T = -A$.

(1) Give examples of symmetric and skew-symmetric 2×2, 3×3, and 4×4 matrices.
(2) What can you say about the main diagonal of a skew-symmetric matrix?
(3) Give an example of a matrix that is both symmetric and skew-symmetric.
(4) Prove that for any $n \times n$ matrix A, the matrices $A + A^T$, AA^T, and A^TA are symmetric and $A - A^T$ is skew-symmetric.
(5) Prove that any $n \times n$ can be written as the sum of a symmetric and skew-symmetric matrices. Hint: Did you do part (4) yet?

(1) Write some examples. e.g., for symmetric matrices $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 6 \end{bmatrix}$

For skew-symmetric matrices, we should notice that the diagonal elements are 0. For example, $\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$

(2) All zeros. Since $a_{ii} = -a_{ii}$.

(3) zero matrix.

(4) $(A + A^T)^T = A^T + (A^T)^T = A^T + A$.

$(AA^T)^T = (A^T)^T A^T = AA^T$

$(A^T A)^T = A^T (A^T)^T = A^T A$.

So, all above three are symmetric.

$(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$.

(5) $A = \dfrac{(A + A^T)}{2} + \dfrac{(A - A^T)}{2}$

Question 20. Mark each of the following functions $F : \mathbb{R} \to \mathbb{R}$ injective, surjective or bijective, as is most appropriate. (You may wish to draw the graph of the function in some cases.)

(a) $F(x) = x^2$;

(b) $F(x) = x^3/(x^2 + 1)$;

(c) $F(x) = x(x - l)(x - 2)$;

(d) $F(x) = e^x + 2$.

For F to be injective each line drawn parallel to the x -axis should meet the graph of y = F (x) at most once. For F to be surjective each such line should meet the graph at least once. Draw the graphs of the four functions and apply these tests.

We conclude that

(a) is neither injective nor surjective,

(b) is bijective,

(c) is surjective but not injective, and

(d) is injective but not surjective.

Question 21. Find a LU-factorization for the matrix $A = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$. (Hint: compute directly. Or do it together with the following two questions.)

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_1 & 1 & 0 & 0 \\ 0 & l_2 & 1 & 0 \\ 0 & 0 & l_3 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} d_1 & u_1 & 0 & 0 \\ 0 & d_2 & u_2 & 0 \\ 0 & 0 & d_3 & u_3 \\ 0 & 0 & 0 & d_4 \end{bmatrix}.$$

$u_i = 1$
$d_1 = 4,$
$l_j = 1/d_j,$
$d_{j+1} = 4 - l_j$, for $j = 1, 2, 3$.

Question 22. Find a LU-factorization for the tridiagonal matrix $A = \begin{bmatrix} q_1 & r_1 & 0 & 0 \\ p_1 & q_2 & r_2 & 0 \\ 0 & p_2 & q_3 & r_3 \\ 0 & 0 & p_3 & q_4 \end{bmatrix}$ as $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_1 & 1 & 0 & 0 \\ 0 & l_2 & 1 & 0 \\ 0 & 0 & l_3 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} d_1 & u_1 & 0 & 0 \\ 0 & d_2 & u_2 & 0 \\ 0 & 0 & d_3 & u_3 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$. Find relations between $\{q_i, p_i, r_i\}$ and $\{l_i, d_i, u_i\}$. (Think about the general situation for $n \times n$ tridiagonal matrices.)

This is the key question, which can be used for question 21 and 23.
Multiply LU and compare with A we have three classes for elementary algebraic equations.

$$LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_1 & 1 & 0 & 0 \\ 0 & l_2 & 1 & 0 \\ 0 & 0 & l_3 & 1 \end{bmatrix} \begin{bmatrix} d_1 & u_1 & 0 & 0 \\ 0 & d_2 & u_2 & 0 \\ 0 & 0 & d_3 & u_3 \\ 0 & 0 & 0 & d_4 \end{bmatrix} = \begin{bmatrix} d_1 & u_1 & 0 & 0 \\ l_1 d_1 & l_1 u_1 + d_2 & u_2 & 0 \\ 0 & l_2 d_2 & l_2 u_2 + d_3 & u_3 \\ 0 & 0 & l_3 d_3 & l_3 u_3 + d_4 \end{bmatrix}.$$

Compare LU with the given matrix $A = \begin{bmatrix} q_1 & r_1 & 0 & 0 \\ p_1 & q_2 & r_2 & 0 \\ 0 & p_2 & q_3 & r_3 \\ 0 & 0 & p_3 & q_4 \end{bmatrix}$. (Reminder: A is known. Find L and U.)

We quickly get

$$u_i = r_i \text{ for } i = 1, 2, 3 \text{ and } q_1 = d_1.$$

Next use $l_1 d_1 = p_1$ to get $l_1 = p_1/d_1$.
Use $l_1 u_1 + d_2 = q_2$ to get $d_2 = q_2 - l_1 u_1$.
Keep going, we get l_2, d_3, l_3, d_4 in order. White as one recurrence formula,

$$l_t = \frac{p_t}{d_t} \text{ for } t = 1, 2, 3$$

and

$$d_{t+1} = q_{t+1} - l_t u_t \text{ for } t = 1, 2, 3$$

Here, we write 4×4 matrix. Even for $n \times n$ matrix. We already see the patten. Just change 3 to $n - 1$, we have the same result using the recurrence relation.

Question 23. Consider LU factorization of the $n \times n$ matrices $A = \begin{bmatrix} 4 & 1 & \cdots & 0 & 0 \\ 1 & 4 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots\ddots & 4 & 1 \\ 0 & 0 & \cdots & 1 & 4 \end{bmatrix}$

Using the above two questions

$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_1 & \ddots & 0 & 0 \\ 0 & \ddots & 1 & 0 \\ 0 & 0 & l_{n-1} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} d_1 & u_1 & 0 & 0 \\ 0 & d_2 & \ddots & 0 \\ 0 & 0 & \ddots & u_{n-1} \\ 0 & 0 & 0 & d_n \end{bmatrix}$.

$u_i = 1$

$d_1 = 4,$

$l_j = 1/d_j,$

$d_{j+1} = 4 - l_j$, for $j = 1, 2, 3, ..., n-1$.

Question 24. Let I_n be the $n \times n$ identity matrix. Let $\vec{u}$ be a **unit** vector in $\mathbb{R}^n$. Define $H_n = I_n - 2\vec{u}\vec{u}^T$.

Here a unit vector $\vec{u}$ means that norm $||\vec{u}|| = 1$ or equivalently $\vec{u}^T\vec{u} = \vec{u} \cdot \vec{u} = 1$.

(1) Is H_n an symmetric matrix? Prove your result.

(2) Is H_n an orthogonal matrix? (i.e. is $H_n^T H_n = I_n$?)

(3) What is H_n^2?

(4) What is $H_n\vec{u}$?

(5) Suppose $\vec{u} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$. Write down H_3 and H_4?

(1) Yes. Check $H_n^T = H_n$

(2) Yes. Check $H_n^T H_n = I_n$

(3) $H_n^2 = I_n$

(4) $H_n\vec{u} = -\vec{u}$

(5) $H_3 = \frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$ and $H_4 = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$