Linear Algebra Practice

Let A be an $n \times n$ matrix. Assume that there exists a linearly independent set of n dimensional column vectors $v_1, \ldots, v_n$ and numbers $\lambda_1, \ldots, \lambda_n$ such that

$$Av_i = \lambda_i v_i \ (1 \leq i \leq n).$$

That is, v_i is an eigenvector for A with eigenvalue λ_i. Form the $n \times n$ matrix

$$S = [v_1, \ldots, v_n].$$

Let e_j denote the column vector whose entries are all 0, except the jth entry is 1.

1. Compute the matrix $D = S^{-1}AS$. (Hint: first compute Se_j. Then compute $S^{-1}v_j$. Then compute the columns of AS. Then try D.)

2. Show that $A^n = SD^nS^{-1}$ for all $n \geq 0$.

3. Let
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
with $\det A = ad - bc = 1$.

   Compute A^{-1}, and find the eigenvalues of A.

4. Let
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$
What are the eigenvalues of A? Find two independent eigenvectors for A and define S as above.

   Express A^n in closed form. Express $A^n(1,1)$ in closed form.

5. The Rabbit numbers r_n are defined recursively as follows. Let $r_1 = r_2 = 1$ and for $n > 2$ let $r_n = r_{n-1} + r_{n-2}$. So, the sequence looks like: $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$

   Express r_n in closed form. (Hint: look at the entries of A^n.)

6. Let A be an $n \times n$ matrix such that the rows of A form an orthonormal system of vectors. Show that the columns of A also do. (Hint: compute AA^T. Then look at A^TA.)