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(1)

$$\underset{0}{\cdot} \quad \cdot \quad \cdot \quad \underset{k}{\cdot} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \underset{N}{\cdot}$$

$$R_k = \mathbb{P}\left(\text{reach } N \text{ without returning to } k \mid X_0 = k\right)$$

$$= \underbrace{\mathbb{P}\left(\text{reach } N \text{ without returning to } k \mid X_0 = k, X_1 = k-1\right)}_{=0} \times q$$

$$+ \mathbb{P}\left(\text{reach } N \text{ without returning to } k \mid X_0 = k, X_1 = k+1\right) \times p$$

$$= p \times \mathbb{P}\left(\text{reach } N \text{ without returning to } k \mid X_0 = k+1\right)$$

$$= p \times P_{k+1}$$

$$= \begin{cases} p\left(\dfrac{1-\left(\frac{q}{p}\right)^{k+1}}{1-\left(\frac{q}{p}\right)^{N}}\right), & q \neq p, 0 \leq k \leq N-1 \\ p\,\dfrac{(k+1)}{N}, & q = p, 0 \leq k \leq N-1 \\ 1, & k = N \end{cases}$$

2

$$Q_k = \begin{cases} \dfrac{\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^k}{\left(\frac{q}{p}\right)^N - 1}, & q \neq p \\[2ex] 1 - \dfrac{k}{N}, & p = q \end{cases}$$

i) $q < p$

$$\lim_{N \to \infty} Q_k = \frac{0 - \left(\frac{q}{p}\right)^k}{0 - 1} = \boxed{\left(\frac{q}{p}\right)^k}$$

ii) $q > p$

$$\lim_{N \to \infty} Q_k = \lim_{N \to \infty} \frac{1 - \left(\frac{q}{p}\right)^{k-N}}{1 - \left(\frac{q}{p}\right)^{-N}} = \frac{1-0}{1-0}$$

$$= \boxed{1}$$

iii) $q = p$

$$\lim_{N \to \infty} Q_k = 1 - 0 = \boxed{1}$$

4

$$\phi(t) = E(e^{tx}) = \frac{4}{(2-t)^2}, \quad t<2$$

$$\phi'(t) = \frac{d}{dt}\left(E[e^{tx}]\right) = E[x e^{tx}]$$

a)

$$E[X] = \phi'(0)$$

$$\phi'(t) = \frac{8}{(2-t)^3}$$

$$\therefore E[X] = \frac{8}{(2-0)^3} = \boxed{1}$$

b) According to Cramer's theorem,

$$\lim_{n\to\infty} \frac{1}{n} \ln P(Y_n > x) = -\Lambda^*(x)$$

where, $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

$x > E[X]$

$$\Lambda^*(x) = \sup_{t\in\mathbb{R}} \left(tx - \Lambda(t)\right),$$ where, $\Lambda(t) = \ln[E(e^{tx})]$

$$= \sup_{t\in\mathbb{R}} \left(tx - \ln \frac{4}{(2-t)^2}\right)$$

$$= \sup_{t \in \mathbb{R}} \left(tx + 2 \ln\left(\frac{2-t}{2} \right) \right)$$

$$\frac{d}{dt}\left(tx + 2\ln\left(\frac{2-t}{2}\right)\right) = x - \frac{2}{2-t} = 0$$

$$x = \frac{2}{2-t} \Rightarrow 2-t = \frac{2}{x}$$

$$\Rightarrow \boxed{t = \left(2 - \frac{2}{x}\right)}$$

$$\therefore \Lambda^*(x) = x\left(2 - \frac{2}{x}\right) + 2\ln\left(\frac{1}{x}\right)$$

$$= 2x - 2 - 2\ln(x)$$

$$\therefore \lim_{n\to\infty} \frac{1}{n}\ln\left(P(Y_n > x)\right) = -\Lambda^*(x)$$

$$= \boxed{2 + 2\ln(x) - 2x}$$

(5)

According to Cramer's theorem,

$$\lim_{n\to\infty} \frac{1}{n} \ln P(S_n > an)$$

$$= \lim_{n\to\infty} \frac{1}{n} P(Y_n > a) = -\Lambda^*(a)$$

$$Y_n = \frac{1}{n}\sum_{i=1}^{n} x_i \quad , \quad \sum_{i=1}^{n} x_i = S_n$$

$$= N_H - N_T$$

$$\therefore X_i = \begin{cases} +1, & \text{if } i^{th} \text{ toss is H} \\ -1, & \text{if } i^{th} \text{ toss is T} \end{cases}$$

$$\therefore X_i = 2W_i - 1, \text{ where } W_i = \begin{cases} 1, & \text{if } i^{th} \text{ toss is H} \\ 0, & \text{if } i^{th} \text{ toss is T} \end{cases}$$

$$\therefore \phi(t) = E\left(e^{tX}\right) = E\left(e^{2tW - t}\right)$$

$$\phi(t) = e^{-t} E\left(e^{2tW}\right)$$

$$= e^{-t} E\left(e^{2t(W)}\right)$$

$$= e^{-t}\left[\frac{1}{2}e^{2t} + \left(1-\frac{1}{2}\right)\right]^{n}$$

$$= e^{-t}\left[\frac{e^{2t}+1}{2}\right]^{n}$$

$$\therefore \lim_{n\to\infty} \frac{1}{n}\ln\left(P\left(Y_n > a\right)\right) = -\Lambda^*(a)$$

$$\Lambda^*(a) = \sup_{t\in\mathbb{R}}\left(ta - \ln\left[e^{-t}\left[\frac{1+e^{2t}}{2}\right]^{n}\right]\right)$$

$$= \sup_{t\in\mathbb{R}}\left(ta - \left(\ln e^{-t}\right) - \ln\left(\frac{1+e^{2t}}{2}\right)^{n}\right)$$

$$= \sup_{t\in\mathbb{R}}\left(ta + t - n\ln\left(\frac{1+e^{2t}}{2}\right)\right)$$

$$\frac{d}{dt}\left[ta + t - n\ln\left(\frac{1+e^{2t}}{2}\right)\right]$$

$$= \left(a + 1 - n\frac{1}{1+e^{2t}} \times 2e^{2t}\right)$$

$$= a + 1 - \frac{2ne^{2t}}{(1+e^{2t})} = 0$$

$$\therefore \quad 2ne^{2t} = (1+e^{2t})(a+1)$$

$$e^{2t}(2n - a - 1) = (a+1)$$

$$e^{2t} = \frac{a+1}{2n-a-1}$$

$$t = \frac{1}{2}\ln\left(\frac{a+1}{2n-a-1}\right)$$

$$\therefore \Lambda^{*}(a) = \left(\frac{1}{2}a\ln\left(\frac{a+1}{2n-a-1}\right) + \frac{1}{2}\ln\left(\frac{a+1}{2n-a-1}\right) - n\ln\left(\frac{1+\frac{a+1}{2n-a-1}}{2}\right)\right)$$

$$= \left(\frac{1}{2}\ln\left(\frac{a+1}{2n-a-1}\right)(a+1) - n\ln\left(\frac{n}{2n-a-1}\right)\right)$$

$$= \left(\frac{1}{2}\ln\left(\frac{a+1}{2n-a-1}\right)(a+1) - \ln\left(2-\left(\frac{a+1}{n}\right)\right)^{-n}\right)$$

$$\therefore \lim_{n\to\infty} \Lambda^{*}(a) = \lim_{n\to\infty} (\text{above expression})$$

$$= 0$$

$$\therefore \lim_{n\to\infty} \frac{1}{n}\ln(P(Y_n > a)) = 0$$

$$\Rightarrow \lim_{n\to\infty} (P(Y_n > a))^{1/n} = -\infty$$

Not sure where it went wrong.

(6) a) $\phi(t) = E\left[e^{tx}\right] = \dfrac{\lambda}{\lambda - t}$

$$\overset{*}{\Lambda}(x) = \sup_{t \in \mathbb{R}} \left(tx - \ln\left(\frac{\lambda}{\lambda - t}\right)\right)$$

$$= \sup_{t \in \mathbb{R}} \left(tx + \ln\left(\frac{\lambda - t}{\lambda}\right)\right)$$

$$\frac{d}{dt}\left(tx + \ln\left(\frac{\lambda - t}{\lambda}\right)\right)$$

$$= x - \frac{1}{\lambda - t} = 0$$

$$\lambda - t = \frac{1}{x}$$

$$t = \lambda - \frac{1}{x}$$

$$\therefore \Lambda^*(t) = \left(\lambda - \frac{1}{x}\right)x + \ln \frac{1}{\lambda x}$$

$$= \lambda x - 1 - \ln(\lambda x)$$

$$\therefore \lim_{n \to \infty} \frac{1}{n} P(Y_n > x) = \boxed{\ln(\lambda x) - \lambda x + 1}$$

b)

$$\phi(t) = E\left(e^{tx}\right) = \frac{e^{tb} - e^{ta}}{t\,(b-a)}$$

$$\Lambda^*(x) = \sup_{t \in \mathbb{R}} \left(tx - \ln\left(\frac{e^{tb} - e^{ta}}{t\,(b-a)} \right)\right)$$

$$\frac{d}{dt}\left(tx - \ln\left(e^{tb} - e^{ta}\right) - \ln(t) - \ln(b-a)\right)$$

$$= x - \frac{\left(be^{tb} - ae^{ta}\right)}{e^{tb} - e^{ta}} + \frac{1}{t}$$

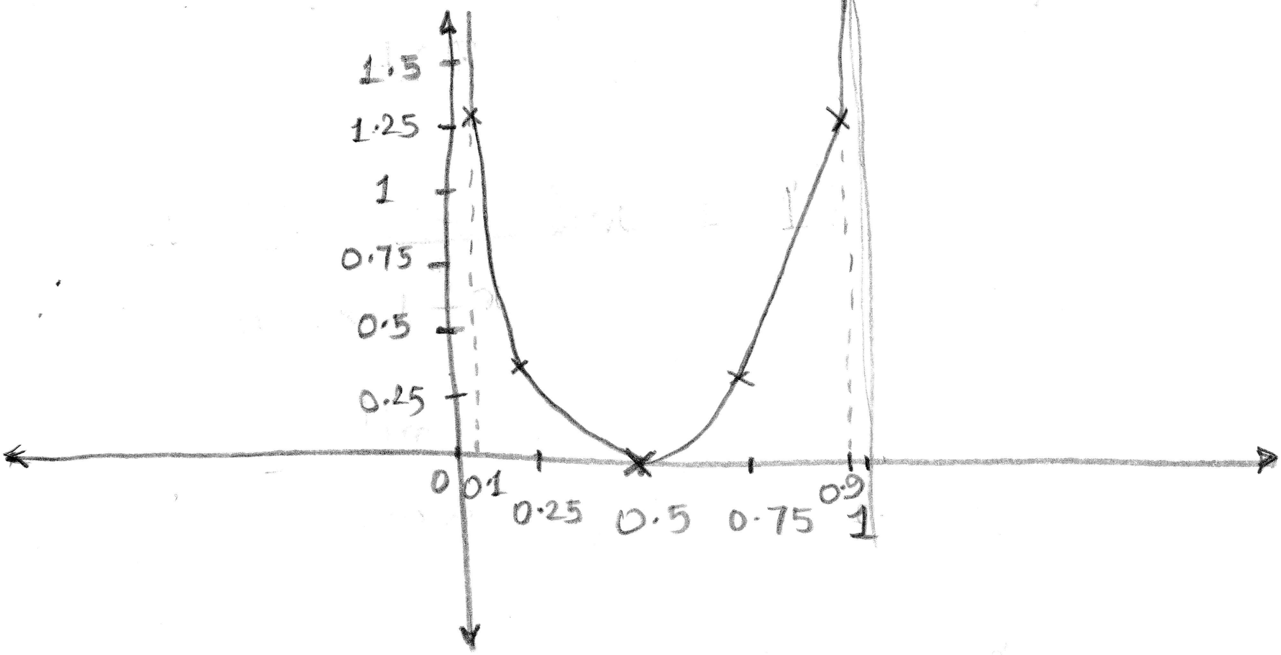
Here, $a=0$, $b=1$

$$\therefore = x - \frac{e^t}{e^t - 1} + \frac{1}{t} = 0$$

This equation is implicit

$$\therefore \Lambda^*(x) = \sup_{t \in \mathbb{R}} \left(tx - \ln\left(\frac{e^t - 1}{t}\right)\right)$$

For different values of x, plot of $\Lambda^*(x)$ is,

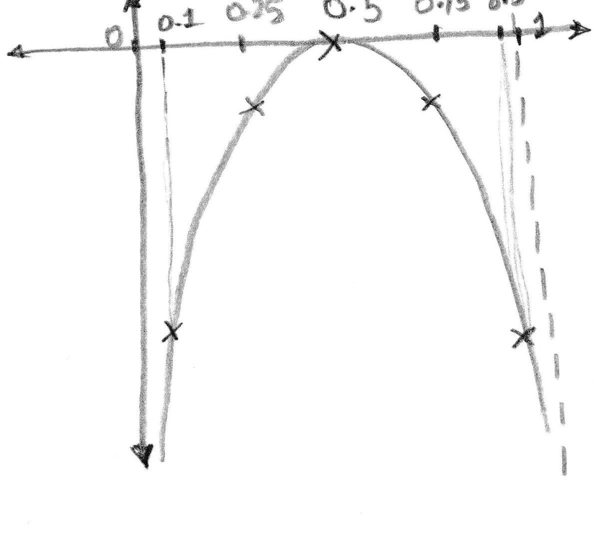


The function is defined between $[0, 1]$

The curve is symmetric about $x = 0.5$

$$\therefore \lim_{n \to \infty} \frac{1}{n} \ln P(Y_n > x) = -\Lambda^*(x),$$

The plot is reflection w.r.t. X-axis



⑦

$$\phi(s) = s^2 P_2 + s P_1 + P_0$$

$$s = s^2 \times \frac{1}{3} + s \times \frac{1}{2} + \frac{1}{6}$$

$$\Rightarrow \boxed{s = \frac{1}{2}} \quad \text{or} \quad \cancel{s = 1}$$

⑧

$$P_0 = p, \quad P_1 = q, \quad P_2 = 2p - \frac{1}{6}$$

$$P(z > 2) = 0$$

a)

$$P(z \leq 2) = 1$$

$$\therefore \; P(z=2) + P(z=1) + P(z=0) = 1$$

$$\left(2p - \frac{1}{6}\right) + q + p = 1$$

$$\boxed{q = \frac{7}{6} - 3p} \quad \text{—} (1)$$

b)

$$E(z) = 0 \times p + 1 \times q + 2\left(2p - \frac{1}{6}\right)$$

$$= p + \frac{5}{6} \quad \left(\because \text{from (1) get "q"} \right.$$

c)

$$\mu = E[X]$$

$$= p + \frac{5}{6} < 1 \Rightarrow \text{extinction is certain}$$

$$\boxed{p = \frac{1}{6}}$$

d)

$$\phi(s) = s^2 P_2 + s P_1 + P_0$$

$$s = s^2\left(2p - \frac{1}{6}\right) + s\left(\frac{7}{6} - 3p\right) + p$$

$$s = \frac{(18p-1) \pm \sqrt{(18p-1)^2 - 24p(12p-1)}}{2(12p-1)}$$

$$s = \boxed{\frac{6p}{12p-1}} \quad \text{or} \quad \cancel{12p}$$

③

$$X = \{X_n\} \qquad Y = \{Y_n\}$$

$$X_0 = i \ , \quad Y_0 = j$$

$$T = \min\{n \geq 1 \mid Z_n = (s,s)\}$$

$$\pi_k = \frac{P_k(k)}{\sum_j P_j(k)} = \frac{1}{\omega_k} \quad (\because P_k(k) = 1)$$

$$E(N_k) = \sum_{n=0}^{\infty} P_{i,i}^{\ n}$$

$$= \sum_{n=0}^{\infty} f_i^{\ n-1}$$

$$= \frac{1}{1-f_i}$$