Homework #1 MATH 4570

Question 1:

1) Set of all numbers of the form $a+b\sqrt{2}$ where a and b are rational numbers

Identity for sum $(e=0)$

$\exists e \in F$, s.t. $e + a+b\sqrt{2} = a+b\sqrt{2} + e = a+b\sqrt{2}$

Identity for product $(e'=1)$

$\exists e' \in R$, s.t. $e' \times (a+b\sqrt{2}) = (a+b\sqrt{2}) \times e' = a+b\sqrt{2}$, $\forall a+b\sqrt{2}$

Inverse for sum $(f = -(a+b\sqrt{2}) = -a-b\sqrt{2})$

$\forall a+b\sqrt{2} \in F$, $a+b\sqrt{2} + f = f + a+b\sqrt{2} = 0$

$a+b\sqrt{2} - a - b\sqrt{2} = -a-b\sqrt{2} + a + b\sqrt{2} = 0$

Inverse for product

$(a+b\sqrt{2})(x+y\sqrt{2}) = 1$

$(x+y\sqrt{2}) = (a+b\sqrt{2})^{-1}$

$ax + ay\sqrt{2} + bx\sqrt{2} + 2by = 1$

$(ax+2by) + (ay+bx)\sqrt{2} = 1$

$$\begin{vmatrix} a & 2b \\ b & a \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$$

$A\vec{x} = \vec{b}$

$$A^{-1} \underline{A \vec{x}} = A^{-1} b$$

$$\vec{x} = A^{-1} b$$

$$\vec{x} = \frac{1}{\det(A)} \begin{vmatrix} a & -2b \\ -b & a \end{vmatrix} \begin{vmatrix} 1 \\ 0 \end{vmatrix}$$

$$\vec{x} = \frac{1}{a^2 - 2b^2} \begin{vmatrix} a \\ -b \end{vmatrix}$$

$$\begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} \frac{a}{a^2-2b^2} \\ \frac{-b}{a^2-2b^2} \end{vmatrix}$$

check: $(a + b\sqrt{2})(x + y\sqrt{2}) = 1$

$$(a + b\sqrt{2})\left(\frac{a}{a^2-2b^2} \frac{-b\sqrt{2}}{a^2-2b^2}\right)$$

$$\frac{a^2}{a^2-2b^2} \cancel{\frac{-ab\sqrt{2}}{a^2-2b^2}} \cancel{+\frac{ab}{a^2-2b^2}} + \left(\frac{-2b^2}{a^2-2b^2}\right)$$

$$\frac{a^2-2b^2}{a^2-2b^2} = 1$$

$$1 = 1$$

Question 1:

2) Set of all numbers of the form $a+b\sqrt{-1}$ where a and b are rational numbers

Identity for sum $(e=0)$

$\exists e \in F$, s.t. $e + a + b\sqrt{-1} = a + b\sqrt{-1} + e = a + b\sqrt{-1}$

Identity for product $(e'=1)$

$\exists e' \in R$, s.t $e' \times (a+b\sqrt{-1}) = (a+b\sqrt{-1}) \times e' = a + b\sqrt{-1}$ for $\forall\ a+b\sqrt{-1}$

Inverse for ~~product~~ sum $(f = -(a+b\sqrt{-1}) = -a - b\sqrt{-1})$

$\forall\ a+b\sqrt{-1} \in F$, $a+b\sqrt{-1} + f = f + a + b\sqrt{-1} = 0$

$a + b\sqrt{-1} - a - b\sqrt{-1} =$

$-a - b\sqrt{-1} + a + b\sqrt{-1} = 0$

Inverse for Product

$(x + y\sqrt{-1}) = (a + b\sqrt{-1})^{-1}$

$(a + b\sqrt{-1})(x + y\sqrt{-1}) = 1$

$ax + ay\sqrt{-1} + bx\sqrt{-1} - by = 1$

$(ax - by) + (ay + bx)\sqrt{-1} = 1$

$A \vec{x} = \vec{b}$

$ax - by = 1$

$ay + bx = 0$

$\begin{vmatrix} a & -b \\ b & a \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$

$A^{-1} = \frac{1}{\det(A)} \begin{vmatrix} a & -b \\ b & a \end{vmatrix} = \frac{1}{a^2 + b^2} \begin{vmatrix} a & +b \\ -b & a \end{vmatrix}$

$\vec{x} = A^{-1}b$

$$\vec{x} = \frac{1}{a^2+b^2}\begin{vmatrix} a & +b \\ -b & a \end{vmatrix}\begin{vmatrix} 1 \\ 0 \end{vmatrix} = \frac{1}{a^2+b^2}\begin{vmatrix} a \\ -b \end{vmatrix} = \begin{vmatrix} \frac{a}{a^2+b^2} \\ \frac{-b}{a^2+b^2} \end{vmatrix}$$

Check: $(a+b\sqrt{-1})(x+y\sqrt{-1}) = 1$

$$(a+b\sqrt{-1})\left(\frac{a}{a^2+b^2} \quad \frac{-b}{a^2+b^2}\right) = 1$$

$$\left(\frac{a^2}{a^2+b^2} \quad \cancel{\frac{ab\sqrt{-1}}{a^2+b^2}} \quad \cancel{-\frac{ab\sqrt{-1}}{a^2+b^2}} \quad \frac{+b^2}{a^2+b^2}\right) = 1$$

$$\frac{a^2+b^2}{a^2+b^2} = 1 \qquad 1 = 1$$

Thus, $a+b\sqrt{-1}$ is the field of Complex numbers $\mathbb{C}$

Question 2

The set of all $n \times n$ matrices $\mathbb{R}$ is not a field

Proof: Let A be a $n \times n$ matrix

by definition $A^{-1} = \frac{1}{\det(A)}$ adjugate (A)

$\det(A) = |A|$

if the determinant of $A = 0$, then A^{-1} d.n.e

and there exists non-zero $n \times n$ matrices whose $||A| = 0|$ or determinant $= 0$,

↓ Thus, $\exists$ $n \times n$ matrices (non-zero) whose inverse do not exist

For a set to be a field, it must have inverses and all $n \times n$ matrices over $\mathbb{R}^n$ do not have inverses →

Therefore, the set of all $n \times n$ matrices over $\mathbb{R}^n$ is not a field

Question 3

+	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

x	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

Question 4

As proved in 1.2 $a+bi = a+b\sqrt{-1}$, as $\sqrt{-1}=i$, therefore $a+bi$ is $\mathbb{C}$ which is a field as it has a sum inverse, multiplicative inverse, sum identity, and product identity over all its elements

Question 5

$$B = \begin{vmatrix} 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix} \quad D = [0\ 1\ 0\ 2\ 4]$$

are in rref

Question 6

$$A+B = \begin{vmatrix} 1+0 & 1+1 & 1+1 \\ 0+1 & 1+1 & 1+1 \\ 0+1 & 1+1 & 0+0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

$$A^2 = AA = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$AB = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

Question 7

$$A = \begin{vmatrix} 6 & -1 & 1 \\ t & 0 & 1 \\ 0 & 1 & t \end{vmatrix}$$

$\det(A) = 0$

$6(0-1) - (-1)(t^2) + (1)(t-0) = 0$

$-6 + t^2 + t = 0$

$t^2 + t - 6 = 0$

$(t+3)(t-2) = 0$

$t = -3, 2$

In order for A to not have an inverse, ~~$t=3$ or $t=2$~~

$t = -3$ or 2

Question 8

$$\left[\begin{array}{cc|c} 1 & h & 4 \\ 3 & 6 & 8 \end{array}\right] \xrightarrow{R_2 = R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & h & 4 \\ 0 & 6-3h & -4 \end{array}\right]$$

For the matrix to be consistent

$h \neq 2 \rightarrow h: (-\infty, 2) \cup (2, \infty)$

$$\left[\begin{array}{cc|c} -4 & 12 & h \\ 2 & -6 & -3 \end{array}\right] \rightarrow \cancel{R_2 = -2R_2}$$

$R_2 = R_2 + \frac{1}{2}R_1$

$$\cancel{\left[\begin{array}{cc|c} -4 & 12 & h \\ -4 & 12 & 6 \end{array}\right]} \quad \left[\begin{array}{cc|c} -4 & 12 & h \\ 0 & 0 & -3 + \frac{h}{2} \end{array}\right]$$

$h = 6$, for the matrix to be consistent

$\frac{h}{2} - 3 = 0$

$h = 6$

Question 9

3×2

1) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

2) 2×3

$\begin{bmatrix} 1 & 0 & * \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

rank=1

$\begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$

3) 4×1 matrix

2 matrices

$[\cancel{1} *]$ $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Question 10

$a = *$, $b = *$, $c = 1,0$, $d = *$, $e = 0$

Question 11

1) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{vmatrix} \xrightarrow{R_2 = R_2 - R_1} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -2 \\ 2 & 0 & 1 & 2 \end{vmatrix} \xrightarrow{R_3 = R_3 - 2R_1}$

$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -2 \\ 0 & -4 & -5 & -6 \end{vmatrix} \xrightarrow{R_2 = -R_2} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & -4 & -5 & -6 \end{vmatrix} \xrightarrow{R_1 = R_1 - 2R_2}$

$\begin{vmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 7 & 2 \end{vmatrix} \xrightarrow{R_3 = \frac{R_3}{7}} \begin{vmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & \frac{2}{7} \end{vmatrix} \xrightarrow{R_1 = R_1 + 3R_2}$

$\begin{vmatrix} 1 & 0 & 0 & \frac{6}{7} \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & \frac{2}{7} \end{vmatrix} \xrightarrow{R_2 = R_2 - 3R_3} \begin{vmatrix} 1 & 0 & 0 & \frac{6}{7} \\ 0 & 1 & 0 & \frac{8}{7} \\ 0 & 0 & 1 & \frac{2}{7} \end{vmatrix} =$

$\underset{A}{\begin{vmatrix} 1 & 0 & 0 & 6/7 \\ 0 & 1 & 0 & 8/7 \\ 0 & 0 & 1 & 2/7 \end{vmatrix}}$ $\cancel{\begin{vmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & t_2 \\ 0 & 0 & 1 & t_3 \end{vmatrix}}$

$\begin{vmatrix} 1 & 0 & 0 & 6/7 \\ 0 & 1 & 0 & 8/7 \\ 0 & 0 & 1 & 2/7 \end{vmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = 0 \quad \begin{vmatrix} x + 6/7t \\ y + 8/7t \\ z + 2/7t \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\begin{bmatrix} -6/7 \\ -8/7 \\ -2/7 \end{bmatrix} t$

Question 11

1) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{vmatrix} \xrightarrow{R_2 \div R_1} \begin{vmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \end{vmatrix} \; R_2 = R_2 - R_1$

$\begin{vmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 3 & 2 \\ 2 & 0 & 1 & 2 \end{vmatrix} \xrightarrow{R_3 - 2R_1} \begin{vmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & -2 & 1 & -2 \end{vmatrix} \xrightarrow{R_3 = R_3 + 2R_2}$

$\begin{vmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 7 & 2 \end{vmatrix} \xrightarrow{R_3 = \frac{1}{7}R_3} \begin{vmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & \frac{2}{7} \end{vmatrix} \xrightarrow{R_2 = R_2 - 3R_3}$

$\begin{vmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 8/7 \\ 0 & 0 & 1 & 2/7 \end{vmatrix} \xrightarrow{R_1 - R_2} \begin{vmatrix} 1 & 0 & 0 & 6/7 \\ 0 & 1 & 0 & 8/7 \\ 0 & 0 & 1 & 2/7 \end{vmatrix} \quad \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 6/7 \\ 8/7 \\ 2/7 \end{vmatrix}$

2) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{vmatrix} \xleftrightarrow{R_2 = R_2 + 6R_1} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 6 & 4 & 5 \\ 2 & 0 & 1 & 2 \end{vmatrix} \; R_2 = 6R_2$

$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 2 & 0 & 1 & 2 \end{vmatrix} \xrightarrow{R_3 = R_3 + 5R_1} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 3 & 12 & 1 \end{vmatrix} \xleftrightarrow{R_3 = R_3 + 4R_2}$

$\begin{vmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{vmatrix} \longrightarrow \begin{vmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$

```
def question_12_check():
    GF7 = galois.GF(7)
    A = GF7([[3, 1, 4, 1],[5, 2, 6, 5], [0, 5, 2, 1]])
    print(GF7.row_reduce(A))

question_12_check()
```

```
GF([[1, 0, 0, 4],
    [0, 1, 0, 3],
    [0, 0, 1, 0]], order=7)
```

```
def question_13():
    x =[[3, 11, 18], [7, 23, 39], [-4, -3, -2]]
    y = [-2, 10, 6]
    solutions = np.linalg.inv(x).dot(y)
    parametric = np.array(solutions)
    answer = parametric.reshape(3,1)
    print('x = {}'.format(answer))

question_13()
```

```
x = [[  5.23943662]
 [-14.05633803]
 [  7.6056338 ]]
```

```
def question_14():
    C = sym.Matrix([[3,6,9, 5, 25, 53], [7, 14, 21, 9, 53, 105], [-4, -8, -12, 5, -10, 11]])
    print(C.rref())

question_14()
```

```
(Matrix([
[1, 2, 3, 0, 5, 6],
[0, 0, 0, 1, 2, 7],
[0, 0, 0, 0, 0, 0]]), (0, 3))
```

```
def question_15():
    x = [[2, 4, 3, 5, 6], [4, 8, 7, 5, 2], [-2, -4, 3, 4, -5], [1, 2, 2, -1, 2], [5, -10, 4, 6, 4]]
    y = [37, 74, 20, 26, 24]
    solutions = np.linalg.inv(x).dot(y)
    parametric = np.array(solutions)
    answer = parametric.reshape(5,1)
    print('x = {}'.format(answer))

question_15()
```

```
x = [[-1.89423963]
 [ 0.98974654]
 [10.81797235]
 [-1.05760369]
 [ 1.61059908]]
```


```
def question_15():
    x = [[2, 4, 3, 5, 6], [4, 8, 7, 5, 2], [-2, -4, 3, 4, -5], [1, 2, 2, -1, 2], [5, -10, 4, 6, 4]]
    y = [37, 74, 20, 26, 24]
    solutions = np.linalg.inv(x).dot(y)
    parametric = np.array(solutions)
    answer = parametric.reshape(5,1)
    print('x = {}'.format(answer))

question_15()
```

```
x = [[-1.89423963]
 [ 0.98974654]
 [10.81797235]
 [-1.05760369]
 [ 1.61059908]]
```

17. ~~2x2 matrices~~

16. i) $ABC = I_n$ $\qquad$ $A^{-1}ABC = A^{-1}I_n$

$BC = A^{-1}I_n$

$\underline{A^{-1} = BC}$ $\qquad$ $BC = I_n$

$C^{-1}ABC = C^{-1}I_n$ $\qquad$ $B^{-1}ABC = B^{-1}I_n$

$\underline{AB = C^{-1}}$ $\qquad$ $\underline{AC = B^{-1}}$

ii) Let C and D be nxn matrices
$C = B(AB)^{-1}$ and $D = (AB)^{-1}A$

$AC = A(B(AB)^{-1}) = AB(AB)^{-1} = I_n$

$DB = B(AB)^{-1}A = AB(AB)^{-1} = I_n$

Thus $C = A^{-1}$ and $D = B^{-1}$;
~~thus~~ A and B are both invertible

17.

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ -3 & -5 \end{bmatrix}$$

$$AB = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & -5 \end{bmatrix} = \begin{bmatrix} -10 & -17 \\ 2 & 4 \end{bmatrix}$$

$$A^2B^2 = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & -5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} -6 & -8 \\ 12 & 19 \end{bmatrix}$$

Only if $AB = BA$, $A^2B^2 = (AB)^2$

otherwise

$$(AB)^2 = ABAB$$

11.

3)

4) Yes, it is possible as proven in (3)

18. $A = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$ (rotation matrix)

$A^T = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$

$A^{-1} = \dfrac{\cancel{\#}1}{\sin^2 x + \cos^2 x} \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$

$A^{-1} = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$

$A^T = A^{-1}$ for $A = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$

Question 19 Symmetric ~~&~~ ~~Skew~~ symmetric

a) 2x2

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

3x3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

4x4

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

b) it must all be zeros (0s) zero

c) any ~~null m~~ null-matrix or zero matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

d)

i) if A is symmetric

$$\boxed{\begin{aligned} (A+A^T)^T &= A+A^T \\ A^T+A &= A+A^T \\ A+A^T &= A+A^T \end{aligned}}$$

iii) Same as 2d

ii) $AA^T = (AA^T)^T$

$AA^T = A^TA$, since $A^T = A$

$A^TA = A^TA$

iv) $(A-A^T)^T = -(A-A^T)$

$A^T - A = -A + A^T$

$A^T - A = A^T - A$ ✓

19.

e) symmetric skew-symmetric

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

$$A = \frac{1}{2}A + \frac{1}{2}A + \frac{1}{2}A^T - \frac{1}{2}A^T$$

$$A = 2\left(\frac{1}{2}A\right) + 0$$

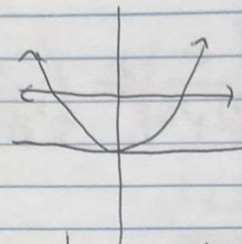
$$A = A$$

20.)

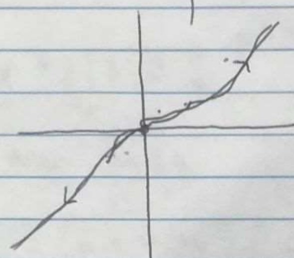
a) $F(x) = x^2$

20.

$F(x) = x^2$ surjective

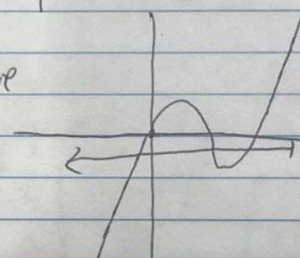


$F(x) = \frac{x^3}{x^2+1}$

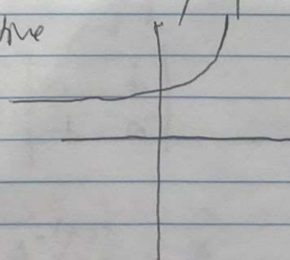


Bijective

$F(x) = x(x-1)(x-2)$ surjective



d($F(x) = e^x + 2$ injective



24

1.) Let u be a unit vector in R^n and $H_n = I_n - 2\vec{v}\vec{v}^T$

$H_n^T = H_n$ to be symmetric

$H_n^T = (I_n - 2\vec{u}\vec{u}^T)^T$

$H_n^T = I_n^T - 2(\vec{u}\vec{u}^T)^T$

$H_n^T = I_n - 2(\vec{u}^T)^T u^T$

$H_n^T = I_n - 2\vec{u}\vec{u}^T$

$H_n^T \stackrel{\checkmark}{=} H_n$; there H_n is symmetric

2.)

$H_n^T H_n = I_n$ means H_n is orthogonal by definition

$H_n^T H_n = I_n$

$$\begin{aligned} H_n^T H_n &= H_n H_n \\ &= (I_n - 2\vec{u}\vec{u}^T)(I - 2\vec{u}\vec{u}^T) \\ &= I_n - 2u\vec{u}^T - 2\vec{u}\vec{u}^T + 4(uu^T)(uu^T) \\ &= I_n - 4u\vec{u}^T + 4u(u^Tu)u^T \\ &= I_n - 4\vec{u}\vec{u}^T + 4\vec{u}\vec{u}^T \\ H_n^T H_n &= I_n \end{aligned}$$

$\therefore$ H_n is orthogonal

3.) Same as above

$H_n^T H_n = H_n^2 = H_n H_n = I_n$

4.) $H_n\vec{u} = (I_n - 2\vec{u}\vec{u}^T)\vec{u}$

$= I_n\vec{u} - 2\vec{u}(\vec{u}^T\vec{u})$

$= I_n\vec{u} - 2\vec{u}$

$= \vec{u}(I_n - 2)$

5. $H_3 =$

$$\vec{u}_3 = \frac{1}{\sqrt{3}}\begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{vmatrix} \quad 3\times 1$$

$$u_3^T = \begin{vmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{vmatrix} \quad 1\times 3$$

$$\begin{vmatrix} \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right) & \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right) & \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right) \\ \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right) & \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right) & \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right) \\ \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right) & \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right) & \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right) \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{vmatrix}$$

for any $n\times n$ matrix $u_n u_n^T = \begin{vmatrix} \frac{1}{n} & \frac{1}{n} & \\ \vdots & & \\ \vdots & & \frac{1}{n} \end{vmatrix}$ H_3

$H_3 = I_3 - 2u_n u_n^T$

$$\boxed{H_3 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} - 2\begin{vmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{vmatrix} = \begin{vmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{vmatrix}} = \boxed{\begin{vmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{vmatrix}}$$