MTH 7241: Fall 2020

Practice Problems for Test 1

1). Four balls are chosen at random from a box which contains two Red balls and some number of Green balls. The probability that both Red balls are chosen is twice the probability that neither Red ball is chosen. How many Green balls are in the box?

$$\boxed{\begin{array}{ll} 2 & \text{Red} \\ n & \text{Green} \end{array}}$$

$$\mathbb{P}(\text{both Red}) = \binom{4}{2} P(RRGG)$$

$$= 6 \, \frac{2}{n+2} \, \frac{1}{n+1}$$

$$\mathbb{P}(\text{neith Red}) = P(GGGG)$$

$$= \frac{n}{n+2} \, \frac{n-1}{n+1} \, \frac{n-2}{n} \, \frac{n-3}{n-1}$$

$$\Rightarrow \quad \frac{12}{(n+2)(n+1)} = 2 \, \frac{(n-2)(n-3)}{(n+2)(n+1)}$$

$$\Rightarrow n = 5$$

2). A maze for rats is constructed with two doors; door 1 immediately leads to the exit, door 2 leads back to the maze after 1 minute. Assume that a rat is equally likely to choose either door at all times, and that if several rats are in the maze then they choose independently.
a). A rat is put in the maze. Find the expected time until it escapes.
b). Two rats are put in the maze. Find the expected time until the first escape occurs, and find the expected time until both escape.
c). Suppose n rats are put in the maze. Find the expected time until the first escape occurs. [Hint: you may want to condition on the first choices made by all the rats].

a) T_1 = time to escape

Condition on first choice:

$$\mathbb{E}[T_1] = 0 \cdot \tfrac{1}{2} + (1 + \mathbb{E}[T_1])\tfrac{1}{2}$$

$$\Rightarrow \mathbb{E}[T_1] = 1.$$

b) $T_{min}^{(2)}$ = time for first escape

$T_{max}^{(2)}$ = time for second escape

Condition on first choice of both rats:

$$\mathbb{E}[T_{min}^{(2)}] = \mathbb{E}[T_{min}^{(2)} \mid \text{at least one escapes on first try}] \cdot \left(\tfrac{3}{4}\right)$$

$$+ \mathbb{E}[T_{min}^{(2)} \mid \text{neither escape first try}] \cdot \left(\tfrac{1}{4}\right)$$

$$= 0 \cdot \tfrac{3}{4} + (1 + \mathbb{E}[T_{min}^{(2)}]) \cdot \tfrac{1}{4}$$

$$\Rightarrow \mathbb{E}[T_{min}^{(2)}] = \tfrac{1}{3}$$

Now $T_{min}^{(2)} + T_{max}^{(2)} = \underbrace{T_1}_{\text{time for Rat \#1}} + \underbrace{T_2}_{\text{time for Rat \#2}}$

$$\Rightarrow \quad \mathbb{E}[T_{min}^{(2)}] + \mathbb{E}[T_{max}^{(2)}] = \mathbb{E}[T_1] + \mathbb{E}[T_2]$$

$$\frac{1}{3} + \mathbb{E}[T_{max}^{(2)}] = 1 + 1$$

$$\Rightarrow \quad \mathbb{E}[T_{max}^{(2)}] = \frac{5}{3}$$

c) T_{min} = time to first escape

$$\mathbb{E}[T_{min}] = \mathbb{E}[T_{min} \mid \text{n one chose door 1}] \cdot \left(\frac{1}{2}\right)^n$$
$$+ \mathbb{E}[T_{min} \mid \text{at least one chose door 1}] \cdot \left(1 - \left(\frac{1}{2}\right)^n\right)$$

$$= \left(1 + \mathbb{E}[T_{min}]\right) 2^{-n}$$

$$\Rightarrow \mathbb{E}[T_{min}] = \frac{1}{2^n - 1}$$

3). A biased coin has probability p of coming up Heads. The coin is tossed repeatedly. Let N_2 be the number of tosses until the first occurrence of the sequence (Heads, Tails). Use the conditional expectation method to compute $\mathrm{E}[N_2]$. [Hint: follow the methodology used in class, when we computed the expected number of tosses until the first occurrence of Heads. You will find it useful to first separately compute $\mathrm{E}[N_2|H_1]$ where H_1 is the event that the first toss comes up Heads].

$$\begin{aligned}\mathbb{E}[N_2 | H_1] &= \mathbb{E}[N_2 | H_1, H_2]\, \mathbb{P}(H_2 | H_1) \\ &\quad + \mathbb{E}[N_2 | H_1, T_2]\, \mathbb{P}(T_2 | H_1) \\ &= (1 + \mathbb{E}[N_2 | H_1])p + 2.(1-p)\end{aligned}$$

$$\Rightarrow \mathbb{E}[N_2 | H_1] = \frac{2-p}{1-p}$$

Now

$$\begin{aligned}\mathbb{E}[N_2] &= \mathbb{E}[N_2 | H_1]\, P(H_1) + \mathbb{E}[N_2 | T_1]\, \mathbb{P}(T_1) \\ &= \frac{2-p}{1-p}\, p + (1 + \mathbb{E}[N_2])(1-p)\end{aligned}$$

$$\Rightarrow \mathbb{E}[N_2] = \frac{1}{p(1-p)}$$

4). Mary's bowl of spaghetti contains n strands. She selects two ends at random and joins them together. She does this until there are no ends left. What is the expected number of spaghetti hoops in the bowl?

Let $L_k = \begin{cases} 1 & \text{if create loop at } k^{\text{th}} \text{ step} \\ 0 & \text{else} \end{cases}$

$$N_n = \text{total number of loops at end}$$

$$= L_n + L_{n-1} + L_{n-2} + \cdots + L_1$$

$$\Rightarrow \mathbb{E}[N_n] = \mathbb{E}[L_n] + \mathbb{E}[L_{n-1}] + \cdots + \mathbb{E}[L_1]$$

Now $\mathbb{E}[L_1] = \mathbb{P}(L_1 = 1)$

$= \mathbb{P}(\text{create loop at step } 1)$

$= \dfrac{1}{2n-1}$ ← after picking first end, must pick other end of that string out of $(2n-1)$ ends

and $\mathbb{E}[L_k] = \mathbb{P}(L_k = 1)$

$= \mathbb{P}(\text{create loop at step } k)$

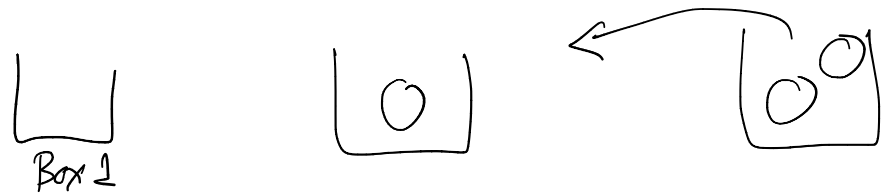
$= \dfrac{1}{2n-2k+1}$ ← have $2n-2k+2$ ends, pick one, then next pick must be its other end.

$$\Rightarrow \mathbb{E}[N_n] = \frac{1}{2n-1} + \frac{1}{2n-3} + \cdots + \frac{1}{3} + 1.$$

5). 3 balls are distributed in 3 boxes. At each step, one of the balls is selected at random, taken out of whichever box it is in, and moved at random to one of the other boxes. Let X_n be the number of balls in the first box, after n steps.

a). Find the transition matrix of the chain $X_0, X_1, \ldots$.

b). Find the stationary distribution of the chain.



a) $X_n \in \{0, 1, 2, 3\}$

$$P = \begin{array}{cc} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 1 & 0 \end{pmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \end{array}$$

b) $W = \left(\frac{8}{27}, \frac{4}{9}, \frac{2}{9}, \frac{1}{27}\right)$

6). Consider the following transition probability matrix for a Markov chain on 4 states:

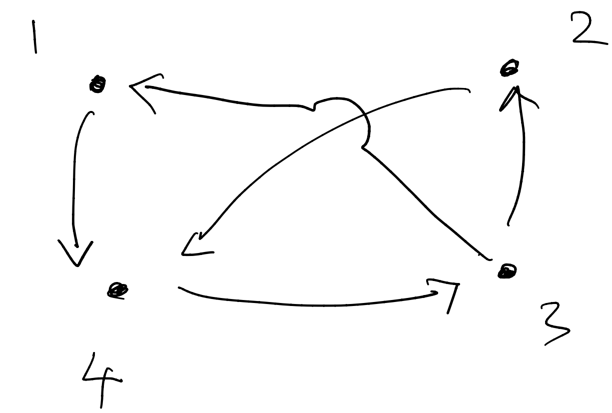
$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Number the states $\{1, 2, 3, 4\}$ in the order presented.

a). Find and classify the equivalence classes of the states (irreducible and transient).

b). Find a stationary distribution for the chain.

a)



All intercommunicate $\Rightarrow$ irreducible, all states persistent.

b) $w = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3}\right)$

7). Suppose that coin 1 has probability 0.7 of coming up Heads, and coin 2 has probability 0.4 of coming up Heads. If the coin tossed today comes up Heads, then we select coin 1 to toss tomorrow, and if it comes up Tails, then we select coin 2 to toss tomorrow. If the coin initially tossed is equally likely to be coin 1 or coin 2, then what is the probability that the coin tossed on the third day after the initial toss is coin 1?

Two states $\{1, 2\}$ $\quad$ 1 = coin 1, $\quad 2$ = coin 2

Transition matrix

$$P = \begin{array}{c} \\ \\ \end{array} \underbrace{\begin{array}{cc} 1 & 2 \end{array}}_{\text{coin used tomorrow}}$$

$$P = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} \begin{array}{c} 1 \\ 2 \end{array} \Big\} \text{coin used today}$$

Initial state: $\alpha^T = \left(\frac{1}{2} \quad \frac{1}{2}\right)$

$$\mathbb{P}(X_3 = j) = (\alpha P^3)_j$$

$$\alpha P^3 = \left(\tfrac{1}{2} \quad \tfrac{1}{2}\right)\begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}^3 = (0.57 \quad 0.43)$$

$$\Rightarrow \mathbb{P}(X_3 = 1) = 0.57$$