MATH 4570 - Homework 1

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Question 1

1.) F has 2 operations: $+, \times$

Identity for sum: $\exists e \in F, s.t. e + x = x + e = x$ $(e = 0)$

$0 + a + b\sqrt{2} = a + b\sqrt{2} + 0 = a + b\sqrt{2}$

Associativity for sum: $(a + b) + c = a + (b + c)$

$(a + b\sqrt{2} + c + d\sqrt{2}) + e + f\sqrt{2} = a + b\sqrt{2} + (c + d\sqrt{2} + e + f\sqrt{2})$

$= (a + b + c) + (d + e + f)\sqrt{2}$

Inverse for sum: $\exists h \in F s.t. a + h = h + a = 0$ $(h = -a - b\sqrt{2})$

$(a + b\sqrt{2} + (-a - b\sqrt{2}) = -a - b\sqrt{2} + a + b\sqrt{2} = 0$

Commutativity for sum: $a + b = b + a$

$a + b\sqrt{2} + c + d\sqrt{2} = c + d\sqrt{2} + a + b\sqrt{2}$

Multiplicative identity: $\exists e' \in R s.t. \forall a \in F, a \times e' = a$ $(e' = 1)$

$1 \times (a + b\sqrt{2}) = a + b\sqrt{2}$

Associativity for product: $(a \times b) \times c = a \times (b \times c)$

$((a + b\sqrt{2}) \times (c + d\sqrt{2})) \times (e + f\sqrt{2}) = (a + b\sqrt{2}) \times ((c + d\sqrt{2}) \times (e + f\sqrt{2}))$

Distributivity for product: $(a \times (b + c) = a \times b + a \times c$

$(a + b\sqrt{2}) \times ((c + d\sqrt{2}) \times (e + f\sqrt{2})) = (a + b\sqrt{2}) \times (c + d\sqrt{2}) + (a + b\sqrt{2}) \times (e + f\sqrt{2})$

Commutativity for product: $a \times b = b \times a$

$(a + b\sqrt{2}) \times (c + d\sqrt{2}) = (c + d\sqrt{2}) \times (a + b\sqrt{2}) = ac + 2bd + (ad + bc)\sqrt{2}$

Inverse for product: $\forall a \neq 0 \in F, \exists x \in F s.t. ax = e$

$(a + b\sqrt{2}) \times (x + y\sqrt{2}) = 1$

$= (ax + 2by) + (ay + bx)\sqrt{2} = 1$

$\text{rref}(\begin{bmatrix} a & 2b & 1 \\ b & a & 0 \end{bmatrix}) = \begin{bmatrix} 1 & 0 & a/(a^2 - 2b^2) \\ 0 & 1 & -b/(a^2 - 2b^2) \end{bmatrix}$

$(a+b\sqrt{2})^{-1} = a/(a^2-2b^2) - b/(a^2-2b^2)\sqrt{2}$

Since F satisfies all of these axioms, F is a field.

2.) F has 2 operations: $+, \times$

Identity for sum: $\exists e \in F, s.t. e + x = x + e = x$ $(e = 0)$

$0 + a + bi = a + bi + 0 = a + bi$

Associativity for sum: $(a + b) + c = a + (b + c)$

$(a + bi + c + di) + e + fi = a + bi + (c + di + e + fi)$

$= (a + b + c) + (d + e + f)i$

Inverse for sum: $\exists h \in F s.t. a + h = h + a = 0$ $(h = -a - bi)$

$(a + bi + (-a - bi) = -a - bi + a + bi = 0$

Commutativity for sum: $a + b = b + a$

$a + bi + c + di = c + di + a + bi$

Multiplicative identity: $\exists e' \in R s.t. \forall a \in F, a \times e' = a$ $(e' = 1)$

$1 \times (a + bi) = a + bi$

Associativity for product: $(a \times b) \times c = a \times (b \times c)$

$((a + bi \times (c + di)) \times (e + fi) = (a + bi) \times ((c + di) \times (e + fi))$

Distributivity for product: $(a \times (b + c) = a \times b + a \times c$

$(a + bi) \times ((c + di) \times (e + fi)) = (a + bi) \times (c + di) + (a + bi) \times (e + fi)$

Commutativity for product: $a \times b = b \times a$

$(a + bi) \times (c + di) = (c + di) \times (a + bi) = ac - bd + (ad + bc)i$

Inverse for product: $\forall a \neq 0 \in F, \exists x \in F s.t. ax = e$

$(a + bi) \times (x + yi) = 1$

$= (ax - by) + (ay + bx)i = 1$

$\text{rref}(\begin{bmatrix} a & -b & 1 \\ b & a & 0 \end{bmatrix}) = \begin{bmatrix} 1 & 0 & a/(a^2-2b^2) \\ 0 & 1 & -b/(a^2-2b^2) \end{bmatrix}$

$(a+b\sqrt{2})^{-1} = a/(a^2+b^2) - b/(a^2+b^2)i$

Since F satisfies all of these axioms, F is a field.

Question 2

The set of all matrices in $F : R^{nxn}$ with usual operations is not a field if $n > 1$. Fields require that multiplication is commutative, i.e. $ab = ba$ for all a, b in F. The matrices $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ in F do not satisfy this axiom of fields. $AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $BA = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.

Question 3

+	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

×	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

Question 4

See question 1, part 2.)

Question 5

Matrices B and D are in RREF.

Question 6

$$A + B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}; A^2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}; AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Question 7

A is invertible if and only if $det(A) \neq 0$.

$det(A) = 6(-1) - t(-t - 1) = t^2 + t - 6$

Set $= 0$: $t2 + t - 6 = 0$

$t = 3; t = -2$

A does not have an inverse for $t = -2$ and $t = 3$.

Question 8

a.) $\begin{bmatrix} 1 & h & 4 \\ 3 & 6 & -8 \end{bmatrix} R_2 - 3R_1 \begin{bmatrix} 1 & h & 4 \\ 0 & 6-3h & -20 \end{bmatrix}$

Matrix is consistent where $h \neq 2$.

b.) $\begin{bmatrix} -4 & 12 & h \\ 2 & -6 & -3 \end{bmatrix} R_1/-4 \begin{bmatrix} 1 & -3 & -h/4 \\ 2 & -6 & -3 \end{bmatrix} R_2 - 2R_1 \begin{bmatrix} 1 & -3 & -h/4 \\ 0 & 0 & -3+h/2 \end{bmatrix}$

Matrix is consistent where $h = 6$.

Question 9

1.)

Rank 0: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Rank 1: $\begin{bmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Rank 2: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Total = 3

2.)

Rank 0: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Rank 1: $\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Rank 2: $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}$ $\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Total = 7

3.)

Rank 0: $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Rank 1: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Total = 2

Question 10

The possible combinations (a, b, c, d, e) that make A in RREF are:

$(*, 0, 1, 0, 0)$

$(*, *, 0, 0, 0)$

Question 11

1.) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix}$

$R_2 - R_1, R_3 - 2R_1$ $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -2 \\ 0 & -4 & -5 & -6 \end{bmatrix}$

$R_2 * -1, R_3 - 4R_2$ $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 7 & 2 \end{bmatrix}$

$R_3 * 1/7$ $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 2/7 \end{bmatrix}$

$$R_2 - 3R_3, R_1 - 2R_2 \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & 8/7 \\ 0 & 0 & 1 & 2/7 \end{bmatrix}$$

$$R_1 + 3R_3 \begin{bmatrix} 1 & 0 & 0 & 6/7 \\ 0 & 1 & 0 & 8/7 \\ 0 & 0 & 1 & 2/7 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6/7 \\ 8/7 \\ 2/7 \end{bmatrix}$$

2.) $\begin{bmatrix} [1] & [2] & [3] & [4] \\ [1] & [1] & [0] & [2] \\ [2] & [0] & [1] & [2] \end{bmatrix}$

$$R_2 + [6]R_1, R_3 + [5]R_1 \begin{bmatrix} [1] & [2] & [3] & [4] \\ [0] & [6] & [4] & [6] \\ [0] & [3] & [2] & [1] \end{bmatrix}$$

$$R_3 + [3]R_3 \begin{bmatrix} [1] & [2] & [3] & [4] \\ [0] & [6] & [4] & [6] \\ [0] & [0] & [0] & [6] \end{bmatrix}$$

$$R_1 + [9]R_2 R_2 + [6]R_3 R_3 * [6] \begin{bmatrix} [1] & [0] & [4] & [0] \\ [0] & [6] & [4] & [0] \\ [0] & [0] & [0] & [1] \end{bmatrix}$$

$$R_2 - R_3, R_3/2 \begin{bmatrix} [1] & [0] & [3] & [0] \\ [0] & [1] & [3] & [0] \\ [0] & [0] & [0] & [1] \end{bmatrix}$$

3.) rref(A) over Z_2:

Python code:

```
GF2 = galois.GF(2)
GF2.row_reduce(GF2([
    [1, 0, 1, 0],
    [1, 1, 0, 0],
    [0, 0, 1, 0]
]))
```

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

rref(A) over Z_3:

Python code:

```
GF3 = galois.GF(3)
GF3.row_reduce(GF3([
        [1, 2, 0, 1],
        [1, 1, 0, 2],
        [2, 0, 1, 2]
]))
```

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

4.) It is not possible for a matrix to have different rank over different fields.

Question 12

1.) Python code:

```
GF7 = galois.GF(7)
GF7.row_reduce(GF7([
        [3, 1, 4, 1],
        [5, 2, 6, 5],
        [0, 5, 2, 1]
]))
```

$$= \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

2.) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$

Question 13

Python code:

```
Matrix([
    [3, 11, 19, −2],
    [7, 23, 39, 10],
    [−4, −3, −2, 6]
]).rref()[0]
```

$$= \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

There is no solution for the linear system.

Question 14

Python code:

```
Matrix([
    [3, 6, 9, 5, 25, 53],
    [7, 14, 21, 9, 53, 105],
    [−4, −8, −12, 5, −10, 11]
]).rref()[0]
```

$$= \begin{bmatrix} 1 & 2 & 3 & 0 & 5 & 6 \\ 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 - 2x_2 - 3x_3 - 5x_5 \\ * \\ * \\ 7 - 2x_5 \\ * \end{bmatrix}$$

Question 15

Python code:

```
Matrix([
    [2, 4, 3, 5, 6, 37],
    [4, 8, 7, 5, 2, 74],
    [−2, −4, 3, 4, −5, 20],
    [1, 2, 2, −1, 2, 26],
    [5, −10, 4, 6, 4, 24]
]).rref()[0]
```

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -8221/4340 \\ 0 & 1 & 0 & 0 & 0 & 8591/8680 \\ 0 & 0 & 1 & 0 & 0 & 4695/434 \\ 0 & 0 & 0 & 1 & 0 & -459/434 \\ 0 & 0 & 0 & 0 & 1 & 699/434 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -8221/4340 \\ 8591/8680 \\ 4695/434 \\ -459/434 \\ 699/434 \end{bmatrix}$$

Question 16

1.) Since $det(ABC) = det(A)det(B)det(C)$ and $det(I_n) = 1$, then none of $det(A)$, $det(B)$ and $det(C)$ are zero (otherwise the product of the three would be 0, not 1.) Because a matrix is invertible if the determinant is not zero, A, B, and C are invertible.

$A^{-1} = BC$ since $A(BC) = I_n$ and $AA^{-1} = I_n$.

$C^{-1} = AB$ since $(AB)C = I_n$ and therefore $C = (AB)^{-1}$; it follows that $C^{-1} = ((AB)^{-1})^{-1}) = AB$

$B^{-1} = CA$ since if $A(BC) = I_n$ then $BC(A) = I_n$; since $B(CA) = I_n$ then we know $B^{-1} = CA$

2.) By similar logic, if $det(AB) \neq 0$ (i.e. AB is invertible) then both A and B must be invertible because $det(AB) = det(A)det(B)$, and neither can be zero if their product is not zero.

Question 17

The 2x2 matrices $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ do not satisfy this statement. $(AB)^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ and $A^2B^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Question 18

This is an orthogonal matrix; example is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Question 19

1.)

2x2 symmetric: $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

2x2 skew-symmetric: $\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$

3x3 symmetric: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

3x3 skew-symmetric: $\begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & -1 \\ 3 & 1 & 0 \end{bmatrix}$

4x4 symmetric: $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$

4x4 skew-symmetric: $\begin{bmatrix} 0 & -2 & -3 & -4 \\ 2 & 0 & -4 & -5 \\ 3 & 4 & 0 & -6 \\ 4 & 5 & 6 & 0 \end{bmatrix}$

2.) The main diagonal of a skew-symmetric matrix contains only 0 (this is the only solution for $x = -x$).

3.) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

4.) $A + A^T$ is symmetric:

Any matrix A is symmetric if $A^T = A$. Thus $A + A^T$ is symmetric if $(A + A^T)^T = (A + A^T)$.

$(A + A^T)^T = A^T + (A^T)^T = A + A^T$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix}$$
$$\begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix}^T = \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix}$$

AA^T is symmetric:

For any two matrices A, B, $(AB)^T = B^T A^T$. Therefore $(AA^T)^T = (A^T)^T A^T = AA^T$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & a^2+b^2 \end{bmatrix}$$
$$\begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & a^2+b^2 \end{bmatrix}^T = \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & a^2+b^2 \end{bmatrix}$$

$A^T A$ is symmetric:

$((A^T)A)^T = A^T(A^T)^T$

$= A^T A$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & a^2+c^2 \end{bmatrix}$$
$$\begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & a^2+c^2 \end{bmatrix}^T = \begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & a^2+c^2 \end{bmatrix}$$

$A - A^T$ is skew-symmetric if $(A - A^T)^T = -(A - A^T)$:

$(A - A^T)^T = A^T - (A^T)^T$

$= -(A - A^T)$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & b-c \\ c-b & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & b-c \\ c-b & 0 \end{bmatrix}^T = -\begin{bmatrix} 0 & b-c \\ c-b & 0 \end{bmatrix}$$

5.) Any matrix A can be rewritten as $A = 1/2(A + A^T) + 1/2(A - A^T)$. The first term is a symmetric matrix, and the second term is skew-symmetric.

Question 20

a.) neither

b.) bijective

c.) surjective

d.) injective

Question 21

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 0 & 4/15 & 1 & 0 \\ 0 & 0 & 15/56 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 15/4 & 1 & 0 \\ 0 & 0 & 56/15 & 0 \\ 0 & 0 & 0 & 209/56 \end{bmatrix}$$

Question 22

$$LU = \begin{bmatrix} d_1 & u_1 & 0 & 0 \\ l_1 d_1 & l_1 u_1 + d_2 & u_2 & 0 \\ 0 & l_2 d_2 & l_2 u_2 + d3 & u_3 \\ 0 & 0 & l_3 d_3 & l_3 u_3 + d_4 \end{bmatrix}$$

$p_i = l_i d_i$

$q_i = d_i + l_{i-1} u_{i-1}$ for $i > 1$; otherwise d_i

$r_i = u_i$

Question 23

By the above equations:

$$L = \begin{bmatrix} 1 & 0 & .. & 0 & 0 \\ 1/4 & 1 & .. & 0 & 0 \\ .. & .. & .. & .. & .. \\ 0 & .. & .. & 1 & 0 \\ 0 & 0 & .. & 1/d_n & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 4 & 1 & .. & 0 & 0 \\ 0 & 4-1/4 & 1 & .. & 0 \\ .. & .. & .. & .. & .. \\ 0 & .. & .. & d_{n-1} & 1 \\ 0 & 0 & .. & 0 & d_n \end{bmatrix}$$

Question 24

1.) Yes;

$(H_n)^T = (I_n - 2\vec{u}\vec{u}^T)^T.$

$= (I_n)^T - 2(\vec{u}\vec{u}^T)^T.$

$= I_n - 2(\vec{u}^T)^T(\vec{u}^T).$

$= I_n - 2\vec{u}\vec{u}^T.$

$= H$

2.) Yes; since $H^T = H$ then $H^T H = H^2$.

$= (I_n - 2\vec{u}\vec{u}^T)(I - 2\vec{u}\vec{u}^T)$

$= I_n - 4\vec{u}\vec{u}^T + 4(\vec{u}\vec{u}^T)^2$

$= I_n - 4\vec{u}\vec{u}^T + 4\vec{u}(\vec{u}^T\vec{u})\vec{u}^T$

$= I_n - 4\vec{u}\vec{u}^T + 4\vec{u}\vec{u}^T \qquad (\vec{u}^T\vec{u}) = 1$

$= I_n$

3.)I_n by the above.

4.) $= (I_n - 2\vec{u}\vec{u}^T)(\vec{u})$

$= I_n\vec{u} - 2\vec{u}\vec{u}^T\vec{u}$

$= \vec{u} - 2\vec{u}$

$= -\vec{u}$

5.) $H_3 = I_3 - 2\vec{u}\vec{u}^T = I_3 - 2(1/\sqrt{3})^2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

$$= I_3 - 2/3 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix}$$

H_4 is the same general calculation but using $1/\sqrt{4}$ inplace of $1/\sqrt{3}$.

$$H_4 = \begin{bmatrix} 1/2 & -1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & -1/2 \\ -1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$$