Q1 (1) $a+b\sqrt{2}$ where a and b are rational numbers

- $(F,+)$ is an abelian group.
  - (Identity): let $e = 0+0\sqrt{2} = 0$ and $x = a+b\sqrt{2} \in F$

    $e+x \stackrel{?}{=} x \quad x+e \stackrel{?}{=} x$

    $\Rightarrow 0+a+b\sqrt{2} \stackrel{?}{=} a+b\sqrt{2} \quad a+b\sqrt{2}+0 \stackrel{?}{=} a+b\sqrt{2}$ (substitution)

    $\Rightarrow a+b\sqrt{2} \stackrel{\checkmark}{=} a+b\sqrt{2} \quad a+b\sqrt{2} \stackrel{\checkmark}{=} a+b\sqrt{2}$ (simplify)

    $\Rightarrow e+x = x+e = x$ (equality)

    $\therefore e=0$ is the identity in $(F,+)$
  - (Associativity): Let $x = a+b\sqrt{2}$, $y = c+d\sqrt{2}$, $z = e+f\sqrt{2} \in F$

    $x+(y+z) \stackrel{?}{=} (x+y)+z$

    $\Rightarrow a+b\sqrt{2}+(c+d\sqrt{2}+e+f\sqrt{2}) \stackrel{?}{=} (a+b\sqrt{2}+c+d\sqrt{2})+e+f\sqrt{2}$ (substitution)

    $\Rightarrow a+b\sqrt{2}+c+e+(d+f)\sqrt{2} \stackrel{?}{=} a+c+(b+d)\sqrt{2}+e+f\sqrt{2}$ (simplify)

    $\Rightarrow a+c+e+(b+d+f)\sqrt{2} \stackrel{\checkmark}{=} a+c+e+(b+d+f)\sqrt{2}$ (simplify)

    Let $a' = a+c+e$ and $b' = b+d+f$

    $\Rightarrow a'+b'\sqrt{2} = a+c+e+(b+d+f)\sqrt{2}$ (substitution)

    $\Rightarrow a'+b'\sqrt{2} \in F$

    $\therefore F$ is associative under addition
  - (Inverse): Let $x = a+b\sqrt{2}$, $y = -a-b\sqrt{2} \in F$

    $x+y \stackrel{?}{=} 0$

    $\Rightarrow a+b\sqrt{2}-a-b\sqrt{2} \stackrel{?}{=} 0$ (substitution)

    $\Rightarrow 0 \stackrel{\checkmark}{=} 0$ (simplify)

    $\Rightarrow \forall x \in F\ \exists\ y \in F$ s.t. $x+y=0$

    $\therefore$ The inverse exists in $(F,+)$
  - (Commutativity): Let $x = a+b\sqrt{2}$, $y = c+d\sqrt{2} \in F$

    $x+y \stackrel{?}{=} y+x$

    $\Rightarrow a+b\sqrt{2}+c+d\sqrt{2} \stackrel{?}{=} c+d\sqrt{2}+a+b\sqrt{2}$ (substitution)

    $\Rightarrow a+c+(b+d)\sqrt{2} \stackrel{\checkmark}{=} a+c+(b+d)\sqrt{2}$ (simplification)

    Let $a' = a+c$ and $b' = b+d$

    $\Rightarrow a'+b'\sqrt{2} = a+c+(b+d)\sqrt{2}$

    $\Rightarrow a'+b'\sqrt{2} \in F$

    $\therefore F$ is commutative under addition
- Multiplicative identity: Let $e = 1+0\sqrt{2} = 1$ and $x = a+b\sqrt{2} \in F$

  $ex \stackrel{?}{=} x \quad xe \stackrel{?}{=} x$

  $\Rightarrow 1(a+b\sqrt{2}) \stackrel{?}{=} a+b\sqrt{2} \quad (a+b\sqrt{2})1 \stackrel{?}{=} a+b\sqrt{2}$ (substitution)

  $\Rightarrow a+b\sqrt{2} \stackrel{\checkmark}{=} a+b\sqrt{2} \quad a+b\sqrt{2} \stackrel{\checkmark}{=} a+b\sqrt{2}$ (simplification)

  $\Rightarrow ex = xe = x$ (equality)

  $\therefore e=1$ is the identity in $(F,\cdot)$

$\longrightarrow$

- Multiplicative associative: Let $x = a+b\sqrt{2},\ y = c+d\sqrt{2},\ z = e+f\sqrt{2} \in F$

$x(yz) \stackrel{?}{=} (xy)z$

$\Rightarrow (a+b\sqrt{2})\big((c+d\sqrt{2})(e+f\sqrt{2})\big) \stackrel{?}{=} \big((a+b\sqrt{2})(c+d\sqrt{2})\big)(e+f\sqrt{2})$ (substitution)

$\Rightarrow a+b\sqrt{2}\,(ce+(cf+de)\sqrt{2}+2df) \stackrel{?}{=} (ac+(ad+bc)\sqrt{2}+2bd)(e+f\sqrt{2})$ (simplify)

Let $a' = ce+2df \quad b' = cf+de \quad a'' = ac+2bd \quad b'' = ad+bc$

$\Rightarrow (a+b\sqrt{2})(a'+b'\sqrt{2}) \stackrel{?}{=} (a''+b''\sqrt{2})(e+f\sqrt{2})$ (substitution)

$\Rightarrow aa'+(ab'+a'b)\sqrt{2}+2bb' \stackrel{?}{=} a''e+(a''f+b''e)\sqrt{2}+2b''f$ (simplify)

Let $a''' = aa'+2bb',\ b''' = ab'+a'b,\ a'''' = a''e+2b''f,\ b'''' = a''f+b''e$

$\Rightarrow a'''+b'''\sqrt{2} \stackrel{?}{=} a''''+b''''\sqrt{2}$ (substitution)

$\Rightarrow x(yz) \in F,\quad (xy)z \in F$

$\Rightarrow a(ce+2df)+2b(cf+de)+\big(a(cf+de)+(ce+2df)b\big)\sqrt{2} \stackrel{?}{=}$ (substitution)

$\Rightarrow ace+2adf+2bcf+2bde+(acf+ade+bce+2bdf)\sqrt{2} \stackrel{?}{=}$ (simplify)

$\Rightarrow \stackrel{?}{=} (ac+2bd)e+2(ad+bc)f+\big((ac+2bd)f+(ad+bc)e\big)\sqrt{2}$ (substitution)

$\Rightarrow \stackrel{?}{=} ace+2bde+2adf+2bcf+(acf+2bdf+ade+bce)\sqrt{2}$ (simplify)

$\Rightarrow a'''+b'''\sqrt{2} \stackrel{\checkmark}{=} a''''+b''''\sqrt{2}$

$\therefore$ F is associative under multiplication

- Distributivity: Let $x = a+b\sqrt{2},\ y = c+d\sqrt{2},\ z = e+f\sqrt{2} \in F$

$x(y+z) \stackrel{?}{=} xy+xz$

$\Rightarrow (a+b\sqrt{2})(c+d\sqrt{2}+e+f\sqrt{2}) \stackrel{?}{=} (a+b\sqrt{2})(c+d\sqrt{2})+(a+b\sqrt{2})(e+f\sqrt{2})$ (substitution)

$\Rightarrow (a+b\sqrt{2})(c+e+(d+f)\sqrt{2}) \stackrel{?}{=} (ac+(ad+bc)\sqrt{2}+2bd)+(ae+(af+be)\sqrt{2}+2bf)$ (simplify)

$\Rightarrow ac+ae+(ad+af+bc+be)\sqrt{2}+2bd+2bf \stackrel{\checkmark}{=} ac+ae+(ad+af+bc+be)\sqrt{2}+2bd+2bf$ (simplify)

Let $a' = ac+ae+2bd+2bf$ and $b' = ad+af+bc+be$

$\Rightarrow a'+b'\sqrt{2} = ac+ae+(ad+af+bc+be)\sqrt{2}+2bd+2bf$ (substitution)

$\Rightarrow a'+b'\sqrt{2} \in F$

$(y+z)x \stackrel{?}{=} yx+zx$

$\Rightarrow (c+d\sqrt{2}+e+f\sqrt{2})(a+b\sqrt{2}) \stackrel{?}{=} (c+d\sqrt{2})(a+b\sqrt{2})+(e+f\sqrt{2})(a+b\sqrt{2})$ (substitution)

$\Rightarrow (c+e+(d+f)\sqrt{2})(a+b\sqrt{2}) \stackrel{?}{=} (ac+(ad+bc)\sqrt{2}+2db)+(ae+(af+be)\sqrt{2}+2bf)$ (simplify)

$\Rightarrow ac+ae+(ad+af+bc+be)\sqrt{2}+2bd+2bf \stackrel{\checkmark}{=} ac+ae+(ad+af+bc+be)\sqrt{2}+2bd+2bf$ (simplify)

$\Rightarrow x(y+z) = xy+xz$ and $(y+z)x = yx+zx$

$\therefore (F,+,\cdot)$ is distributive, $\therefore (F,+,\cdot)$ is a ring

- Commutative: Let $x = a+b\sqrt{2},\ y = c+d\sqrt{2} \in F$

$xy \stackrel{?}{=} yx$

$\Rightarrow (a+b\sqrt{2})(c+d\sqrt{2}) \stackrel{?}{=} (c+d\sqrt{2})(a+b\sqrt{2})$ (substitution)

$\Rightarrow ac+(ad+bc)\sqrt{2}+2bd \stackrel{\checkmark}{=} ac+(ad+bc)\sqrt{2}+2bd$ (simplify)

Let $a' = ac+2bd \quad b' = (ad+bc)$

$\Rightarrow a'+b'\sqrt{2} = ac+(ad+bc)\sqrt{2}+2bd$ (substitution)

$\Rightarrow a'+b'\sqrt{2} \in F$

$\therefore (F,+,\cdot)$ is a commutative ring

- Multiplicative inverse: Let $x = a+b\sqrt{2},\ y = c+d\sqrt{2} \in F$ and $x \neq 0$

$xy \stackrel{?}{=} e$

$\Rightarrow (a+b\sqrt{2})(c+d\sqrt{2}) \stackrel{?}{=} 1$ (substitution)

$\Rightarrow (ac+2bd)+(ad+bc)\sqrt{2} \stackrel{?}{=} 1$ (simplify)

$\Rightarrow \begin{cases} ac+2bd = 1 \\ bc+ad = 0 \end{cases}$ (linear equations)

$$\Rightarrow \left[\begin{array}{cc|c} a & 2b & 1 \\ b & a & 0 \end{array}\right] \stackrel{R_1/a}{=} \left[\begin{array}{cc|c} 1 & \frac{2b}{a} & \frac{1}{a} \\ b & a & 0 \end{array}\right] \underset{-R_1(b)}{=} \left[\begin{array}{cc|c} 1 & \frac{2b}{a} & \frac{1}{a} \\ 0 & \frac{a^2-2b^2}{a} & 0 \end{array}\right] \stackrel{R_2\frac{a}{a^2-2b^2}}{=} \left[\begin{array}{cc|c} 1 & \frac{2b}{a} & \frac{1}{a} \\ 0 & 1 & -\frac{b}{a^2-2b^2} \end{array}\right] \stackrel{-R_2\frac{2b}{a}}{=} \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{a}+\frac{2b^2}{a(a^2-2b^2)} \\ 0 & 1 & -\frac{b}{a^2-2b^2} \end{array}\right]$$

$$\Rightarrow c = \frac{1}{a}+\frac{2b^2}{a(a^2-2b^2)},\quad d = -\frac{b}{a^2-2b^2}$$

$\Rightarrow \forall x \neq 0 \in F,\ \exists y$ s.t. $xy = e = 1$

$\therefore$ The multiplicative inverse exists for $(F,+,\cdot)$

$\therefore (F,+,\cdot)$ is a field

Q1 (2) $a + b\sqrt{-1}$ where a and b are real numbers. The field is $(\mathbb{C}, +, \cdot)$, the set of complex numbers.

- $(F, +)$ is an abelian group:
  - (Identity): Let $e = 0 + 0\sqrt{-1} = 0$ and $x = a+b\sqrt{-1} \in F$

    $e + x \overset{?}{=} x \qquad x + e \overset{?}{=} x$

    $\Rightarrow 0 + a + b\sqrt{-1} \overset{?}{=} a + b\sqrt{-1} \qquad a + b\sqrt{-1} + 0 \overset{?}{=} a + b\sqrt{-1}$ (substitution)

    $\Rightarrow a + b\sqrt{-1} \overset{\checkmark}{=} a + b\sqrt{-1} \qquad a + b\sqrt{-1} \overset{\checkmark}{=} a + b\sqrt{-1}$ (simplify)

    $\Rightarrow e + x = x + e = x$ (equality)

    $\therefore$ $e = 0$ is the identity in $(F, +)$
  - (Associativity): Let $x = a + b\sqrt{-1}$, $y = c + d\sqrt{-1}$, $z = e + f\sqrt{-1} \in F$

    $x + (y + z) \overset{?}{=} (x + y) + z$

    $\Rightarrow a + b\sqrt{-1} + (c + d\sqrt{-1} + e + f\sqrt{-1}) \overset{?}{=} (a + b\sqrt{-1} + c + d\sqrt{-1}) + e + f\sqrt{-1}$ (substitution)

    $\Rightarrow a + b\sqrt{-1} + (c + e + (d+f)\sqrt{-1}) \overset{?}{=} (a + c + (b+d)\sqrt{-1}) + e + f\sqrt{-1}$ (simplify)

    $\Rightarrow a + c + e + (b + d + f)\sqrt{-1} \overset{\checkmark}{=} a + c + e + (b + d + f)\sqrt{-1}$ (simplify)

    Let $a' = a + c + e \quad b' = b + d + f$

    $\Rightarrow a' + b'\sqrt{-1} = a + c + e + (b + d + f)\sqrt{-1}$

    $\Rightarrow a' + b'\sqrt{-1} \in F$

    $\therefore$ F is associative under addition
  - (Inverse): Let $x = a + b\sqrt{-1}$, $y = -a - b\sqrt{-1} \in F$

    $x + y \overset{?}{=} 0$

    $\Rightarrow a + b\sqrt{-1} - a - b\sqrt{-1} \overset{?}{=} 0$ (substitution)

    $\Rightarrow 0 \overset{\checkmark}{=} 0$ (simplify)

    $\Rightarrow \forall x \in F \ \exists y$ s.t. $x + y = 0$

    $\therefore$ The inverse exists in $(F, +)$
  - (Commutativity): Let $x = a + b\sqrt{-1}$, $y = c + d\sqrt{-1} \in F$

    $x + y \overset{?}{=} y + x$

    $\Rightarrow a + b\sqrt{-1} + c + d\sqrt{-1} \overset{?}{=} c + d\sqrt{-1} + a + b\sqrt{-1}$ (substitution)

    $\Rightarrow a + c + (b+d)\sqrt{-1} \overset{\checkmark}{=} a + c + (b+d)\sqrt{-1}$ (simplify)

    Let $a' = a + c \quad b' = b + d$

    $\Rightarrow a' + b'\sqrt{-1} = a + c + (b+d)\sqrt{-1}$

    $\Rightarrow a' + b'\sqrt{-1} \in F$

    $\therefore$ F is commutative under addition
- Multiplicative identity: Let $e = 1 + 0\sqrt{-1} = 1$ and $x = a + b\sqrt{-1} \in F$

  $ex \overset{?}{=} x \qquad xe \overset{?}{=} x$

  $\Rightarrow 1(a + b\sqrt{-1}) \overset{?}{=} a + b\sqrt{-1} \qquad (a + b\sqrt{-1})1 \overset{?}{=} a + b\sqrt{-1}$ (substitution)

  $\Rightarrow a + b\sqrt{-1} \overset{\checkmark}{=} a + b\sqrt{-1} \qquad a + b\sqrt{-1} \overset{\checkmark}{=} a + b\sqrt{-1}$ (simplify)

  $\Rightarrow ex = xe = x$ (equality)

  $\therefore$ $e = 1$ is the identity $(F, \cdot)$

- Multiplicative associative: let $x = a+b\sqrt{-1}$, $y = c+d\sqrt{-1}$, $z = e+f\sqrt{-1} \in F$

$x(yz) \stackrel{?}{=} (xy)z$

$\Rightarrow (a+b\sqrt{-1})((c+d\sqrt{-1})(e+f\sqrt{-1})) \stackrel{?}{=} ((a+b\sqrt{-1})(c+d\sqrt{-1}))(e+f\sqrt{-1})$ (substitution)

$\Rightarrow (a+b\sqrt{-1})(ce+(cf+de)\sqrt{-1}-df) \stackrel{?}{=} (ac+(ad+bc)\sqrt{-1}-bd)(e+f\sqrt{-1})$ (simplify)

let $a' = ce-df$ $\quad b' = cf+de$ $\quad a'' = ac-bd$ $\quad b'' = ad+bc$

$\Rightarrow (a+b\sqrt{-1})(a'+b'\sqrt{-1}) \stackrel{?}{=} (a''+b''\sqrt{-1})(e+f\sqrt{-1})$ (substitution)

$\Rightarrow aa'+(ab'+a'b)\sqrt{-1}-bb' \stackrel{?}{=} a''e+(a''f+b''e)\sqrt{-1}-b''f$ (simplify)

let $a''' = aa'-bb'$ $\quad b''' = ab'+a'b$ $\quad a'''' = a''e-b''f$ $\quad b'''' = a''f+b''e$

$\Rightarrow a'''+b'''\sqrt{-1} \stackrel{?}{=} a''''+b''''\sqrt{-1}$ (substitution)

$\Rightarrow x(yz) \in F$, $(xy)z \in F$

$\Rightarrow a(ce-df)-b(cf+de)+(a(cf+de)+(ce-df)b)\sqrt{-1} \stackrel{?}{=}$ (substitution)

$\Rightarrow ace-adf-bcf-bde+(acf+ade+bce-bdf)\sqrt{-1} \stackrel{?}{=}$ (simplify)

$\Rightarrow \stackrel{?}{=} (ac-bd)e+(ad+bc)f+((ac-bd)f+(ad+bc)e)\sqrt{-1}$ (substitution)

$\Rightarrow \stackrel{?}{=} ace-bde-adf-bcf+(acf-bdf+ade+bce)\sqrt{-1}$ (simplify)

$\Rightarrow a'''+b'''\sqrt{-1} \stackrel{\checkmark}{=} a''''-b''''\sqrt{-1}$

$\therefore$ F is associative under multiplication

- Distributivity: let $x = a+b\sqrt{-1}$, $y = c+d\sqrt{-1}$, $z = e+f\sqrt{-1} \in F$

$x(y+z) \stackrel{?}{=} xy+xz$

$\Rightarrow (a+b\sqrt{-1})(c+d\sqrt{-1}+e+f\sqrt{-1}) \stackrel{?}{=} (a+b\sqrt{-1})(c+d\sqrt{-1})+(a+b\sqrt{-1})(e+f\sqrt{-1})$ (substitution)

$\Rightarrow (a+b\sqrt{-1})(c+e+(d+f)\sqrt{-1}) = (ac+(ad+bc)\sqrt{-1}-bd)+(ae+(af+be)\sqrt{-1}-bf)$ (simplify)

$\Rightarrow ac+ae+(ad+af+bc+be)\sqrt{-1}-bd-bf \stackrel{\checkmark}{=} ac+ae+(ad+af+bc+be)\sqrt{-1}-bd-bf$ (simplify)

let $a' = ac+ae-bd-bf$ $\quad b' = ad+af+bc+be$

$\Rightarrow a'+b'\sqrt{-1} = ac+ae+(ad+af+bc+be)\sqrt{-1}-bd-bf$ (substitution)

$\Rightarrow a'+b'\sqrt{-1} \in F$

$(y+z)x \stackrel{?}{=} yx+zx$

$\Rightarrow (c+d\sqrt{-1}+e+f\sqrt{-1})(a+b\sqrt{-1}) \stackrel{?}{=} (c+d\sqrt{-1})(a+b\sqrt{-1})+(e+f\sqrt{-1})(a+b\sqrt{-1})$ (substitution)

$\Rightarrow (c+e+(d+f)\sqrt{-1})(a+b\sqrt{-1}) \stackrel{?}{=} (ac+(ad+bc)\sqrt{-1}-db)+(ae+(af+be)\sqrt{-1}-bf)$ (simplify)

$\Rightarrow ac+ae+(ad+af+bc+be)\sqrt{-1}-bd-bf \stackrel{\checkmark}{=} ac+ae+(ad+af+bc+be)\sqrt{-1}-bd-bf$ (simplify)

$\Rightarrow x(y+z) = xy+xz$ and $(y+z)x = yx+zx$

$\therefore (F,+,\cdot)$ is distributive $\quad \therefore (F,+,\cdot)$ is a ring

- Commutative: let $x = a+b\sqrt{-1}$, $y = c+d\sqrt{-1} \in F$

$xy \stackrel{?}{=} yx$

$\Rightarrow (a+b\sqrt{-1})(c+d\sqrt{-1}) \stackrel{?}{=} (c+d\sqrt{-1})(a+b\sqrt{-1})$ (substitution)

$\Rightarrow ac+(ad+bc)\sqrt{-1}-bd \stackrel{?}{=} ac+(ad+bc)\sqrt{-1}-bd$ (simplify)

let $a' = ac-bd$ $\quad b' = ad+bc$

$\Rightarrow a'+b'\sqrt{-1} = ac+(ad+bc)\sqrt{-1}-bd$ (substitution)

$\Rightarrow a'+b'\sqrt{-1} \in F$

$\therefore (F,+,\cdot)$ is a commutative ring

- Multiplicative inverse: let $x = a+b\sqrt{-1}$, $y = c+d\sqrt{-1} \in F$ and $x \neq 0$

$xy \stackrel{?}{=} e$

$\rightarrow (a+b\sqrt{-1})(c+d\sqrt{-1}) \stackrel{?}{=} 1$ (substitution)

$\rightarrow (ac-bd)+(ad+bc)\sqrt{-1} \stackrel{?}{=} 1$ (simplify)

$\Rightarrow \begin{cases} ac-bd = 1 \\ bc+ad = 0 \end{cases}$ (linear equations)

$$\Rightarrow \left[\begin{array}{cc|c} a & -b & 1 \\ b & a & 0 \end{array}\right] \xrightarrow[= \; -R_1 b]{\frac{R_1}{a}} \left[\begin{array}{cc|c} 1 & -\frac{b}{a} & \frac{1}{a} \\ b & a & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & -\frac{b}{a} & \frac{1}{a} \\ 0 & \frac{a^2+b^2}{a} & -\frac{b}{a} \end{array}\right] \xrightarrow{\frac{R_2 a}{a^2+b^2}} \left[\begin{array}{cc|c} 1 & -\frac{b}{a} & \frac{1}{a} \\ 0 & 1 & -\frac{b}{a^2+b^2} \end{array}\right] \xrightarrow[=]{+R_2 \frac{b}{a}} \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{a} - \frac{b^2}{a(a^2+b^2)} \\ 0 & 1 & -\frac{b}{a^2+b^2} \end{array}\right]$$

$\Rightarrow c = \frac{1}{a} - \frac{b^2}{a(a^2+b^2)}$, $d = -\frac{b}{a^2+b^2}$

$\Rightarrow \forall x \neq 0 \,\forall F$, $\exists y$ s.t $xy = e = 1$

The multiplicative inverse exists for $(F,+,\cdot)$

$(F,+,\cdot)$ is a field

Q2 In order to be a field, all non-zero elements in F must have a multiplicative inverse. For the set of all $n \times n$ matrices $\mathbb{R}^{n \times n}$ if $n > 1$, not all elements have a multiplicative inverse. The condition for a matrix to be invertible is if its determinant is not equal to zero.

Example of matrix that is not invertible : $\begin{bmatrix} 1 & 5 \\ 1 & 5 \end{bmatrix} = A$ where $n = 2$

$\det(A) = 5 - 5 = 0 \Rightarrow$ non-invertible

Q3.

+	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

x	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

Q4. $z = a + bi$ where $i = \sqrt{-1}$ and a and b are real numbers

- $(F, +)$ is an abelian group: see Q1(2) for proof.
- Multiplicative identity: see Q1(2) for proof that $e = 1$ is the identity.
- Multiplicative associative: see Q1(2) for proof.
- Distributivity: see Q1(2) for proof.
- Commutative: see Q1(2) for proof.
- Multiplicative inverse see Q1(2) for proof

Q5 $B = \begin{bmatrix} 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Q6. $A + B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ in $\mathbb{Z}_2$

$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ in $\mathbb{Z}_2$

$AB = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ in $\mathbb{Z}_2$

Q7. $A = \begin{bmatrix} 6 & -1 & 1 \\ t & 0 & 1 \\ 0 & 1 & t \end{bmatrix} = \begin{bmatrix} t & 0 & 1 \\ 6 & -1 & 1 \\ 0 & 1 & t \end{bmatrix} \xrightarrow{+R_1(-\frac{6}{t})} = \begin{bmatrix} t & 0 & 1 \\ 0 & -1 & 1-\frac{6}{t} \\ 0 & 1 & t \end{bmatrix} \xrightarrow{\cdot(-1)} = \begin{bmatrix} t & 0 & 1 \\ 0 & 1 & \frac{6}{t}-1 \\ 0 & 1 & t \end{bmatrix} \xrightarrow{+R_2(-1)} = \begin{bmatrix} t & 0 & 1 \\ 0 & 1 & \frac{6}{t}-1 \\ 0 & 0 & \frac{t^2+t-6}{t} \end{bmatrix}$

$t + 1 - \frac{6}{t}$

$\frac{t^2}{t} + \frac{t}{t} - \frac{6}{t}$

$\det(A) = 0 = t \cdot 1 \cdot \frac{t^2+t-6}{t}$

$0 = t^2 + t - 6$

$0 = (t+3)(t-2)$

$\boxed{t = -3, 2}$

Q8 a $\left[\begin{array}{cc|c} 1 & h & 4 \\ 3 & 6 & 8 \end{array}\right] \underset{+R_1(-3)}{=} \left[\begin{array}{cc|c} 1 & h & 4 \\ 0 & 6-3h & -4 \end{array}\right] \Rightarrow \exists$ a solution when $6 - 3h \neq 0$

$6 \neq 3h$

$\boxed{h \neq 2}$

b $\left[\begin{array}{cc|c} -4 & 12 & h \\ 2 & -6 & -3 \end{array}\right] \underset{=}{\frac{R_1}{-4}} \left[\begin{array}{cc|c} 1 & -3 & -\frac{h}{4} \\ 2 & -6 & -3 \end{array}\right] \underset{+R_1(-2)}{=} \left[\begin{array}{cc|c} 1 & -3 & -\frac{h}{4} \\ 0 & 0 & -3+\frac{h}{2} \end{array}\right] \Rightarrow \exists$ a solution when $-3 + \frac{h}{2} = 0$

$\frac{h}{2} = 3$

$\boxed{h = 6}$

Q9 (1) 3x2 matrices: $\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\text{Rank 2}}, \underbrace{\begin{bmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\text{Rank 1}}$ 3 types

(2) 2x3 matrices: $\underbrace{\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Rank 2}}, \underbrace{\begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Rank 1}}$ 5 types

(3) 4x1 matrices $\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{Rank 1}}$ 1 type

Q10. $A = \begin{bmatrix} 1 & a & b & 3 & 0 & -2 \\ 0 & 0 & c & 1 & d & 3 \\ 0 & e & 0 & 0 & 1 & 1 \end{bmatrix}$ $a = *$, $b = 0$, $c = 1$, $d = 0$, $e = 0$ $\Rightarrow \begin{bmatrix} 1 & * & 0 & 3 & 0 & -2 \\ 0 & 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

Q11 (1) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \underset{-R_1}{=} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \underset{\substack{(-1)R_2 \\ -2R_1}}{=} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & -4 & -5 & -6 \end{bmatrix} \underset{-2R_2}{=} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & -4 & -5 & -6 \end{bmatrix} \underset{+4R_2}{=} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 7 & 2 \end{bmatrix} \underset{\frac{R_3}{7}}{=} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix} \begin{matrix} +3R_3 \\ -3R_3 \\ \end{matrix} =$

$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & \frac{6}{7} \\ 0 & 1 & 0 & \frac{8}{7} \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix}$

$x_1 + \frac{6}{7}x_4 = 0 \Rightarrow x_1 = -\frac{6}{7}x_4$

$x_2 + \frac{8}{7}x_4 = 0 \qquad x_2 = -\frac{8}{7}x_4$

$x_3 + \frac{2}{7}x_4 = 0 \qquad x_3 = -\frac{2}{7}x_4$

$\vec{x} = x_4 \begin{bmatrix} -\frac{6}{7} \\ -\frac{8}{7} \\ -\frac{2}{7} \\ 1 \end{bmatrix}$

(2) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \underset{+6R_1}{=} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 6 & 4 & 5 \\ 2 & 0 & 1 & 2 \end{bmatrix} \underset{+5R_1}{=} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 6 & 4 & 5 \\ 0 & 3 & 2 & 1 \end{bmatrix} \underset{6R_2}{=} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 3 & 2 & 1 \end{bmatrix} \underset{\substack{+5R_2 \\ +4R_2}}{=} \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \underset{4R_3}{=} \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} +5R_3$

$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(3) Verified (2) using: GF7 = galois.GF(7)
GF7.row-reduce(A)

Using similar method I converted A to be in $\mathbb{Z}_2$ and $\mathbb{Z}_3$ and got

$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ for $\mathbb{Z}_2$ and $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ for $\mathbb{Z}_3$

(4) No, by calculation in (3) we saw $\mathbb{Z}_7$, $\mathbb{Z}_2$, and $\mathbb{Z}_3$ share the same rank for A

Q12 (1) using Python, rref $(A|\vec{b})$ over field $\mathbb{Z}_7$ = $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$

(2) $\vec{x} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$

Q13 $\begin{cases} 3x_1 + 11x_2 + 19x_3 = -2 \\ 7x_1 + 23x_2 + 39x_3 = 10 \\ -4x_1 - 3x_2 - 2x_3 = 6 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 3 & 11 & 19 & -2 \\ 7 & 23 & 39 & 10 \\ -4 & -3 & -2 & 6 \end{array}\right] \xrightarrow[\text{python}]{\text{using}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow$ Row 3 is inconsistent. $\therefore$ No solution exists.

Q14 $\begin{cases} 3x_1 + 6x_2 + 9x_3 + 5x_4 + 25x_5 = 53 \\ 7x_1 + 14x_2 + 21x_3 + 9x_4 + 53x_5 = 105 \\ -4x_1 - 8x_2 - 12x_3 + 5x_4 - 10x_5 = 11 \end{cases} \Rightarrow \left[\begin{array}{ccccc|c} 3 & 6 & 9 & 5 & 25 & 53 \\ 7 & 14 & 21 & 9 & 53 & 105 \\ -4 & -8 & -12 & 5 & -10 & 11 \end{array}\right] \xrightarrow[\text{python}]{\text{using}} \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 0 & 5 & 6 \\ 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 + 2x_2 + 3x_3 + 5x_5 = 6$
$x_4 + 2x_5 = 7$
$\Rightarrow$
$x_1 = 6 - 2x_2 - 3x_3 - 5x_5$
$x_2 = x_2$
$x_3 = x_3$
$x_4 = \frac{7}{2} x_5$
$x_5 = x_5$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ 0 \\ 0 \\ \frac{7}{2} \\ 1 \end{bmatrix}$$

Q15. $$\left[\begin{array}{ccccc|c} 2 & 4 & 3 & 5 & 6 & 37 \\ 4 & 8 & 7 & 5 & 2 & 74 \\ -2 & -4 & 3 & 4 & -5 & 20 \\ 1 & 2 & 2 & -1 & 2 & 26 \\ 5 & -10 & 4 & 6 & 4 & 24 \end{array}\right] \Rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -8221/4340 \\ 0 & 1 & 0 & 0 & 0 & 8591/8680 \\ 0 & 0 & 1 & 0 & 0 & 4695/434 \\ 0 & 0 & 0 & 1 & 0 & -459/434 \\ 0 & 0 & 0 & 0 & 1 & 699/434 \end{array}\right] \Rightarrow \vec{x} = \begin{bmatrix} -8221/4340 \\ 8591/8680 \\ 4695/434 \\ -459/434 \\ 699/434 \end{bmatrix}$$

Q16 (1) Yes, since they are square matrices, we know that $AB = X$, X is a square matrix as well. Therefore, if $XC = I$, then $(AB)^{-1} = C$. Since they are square matrices, $CX = I$, or $C(AB)^{-1} = I$. So, it follows that if $ABC = I$, then $A^{-1} = BC$, $C^{-1} = (AB)^{-1}$ and $B^{-1} = (CA)^{-1}$ by using associativity of matrix multiplication.

(2) Yes, since AB is invertible, $\exists$ C s.t. $C(AB) = I$, and applying the associativity of matrix multiplication, we see that $(CA)B = I$, and that B is invertible. We can say the same for A since these are square matrices and are right-invertible and left-invertible.

Q17. $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ $AB = \begin{bmatrix} 5 & 5 \\ 5 & 10 \end{bmatrix}$ $A^2 = \begin{bmatrix} 7 & 4 \\ 6 & 7 \end{bmatrix}$ $B^2 = \begin{bmatrix} 7 & 6 \\ 4 & 7 \end{bmatrix}$ $A^2B^2 = \begin{bmatrix} 65 & 70 \\ 70 & 85 \end{bmatrix}$

$(AB)^2 = \begin{bmatrix} 50 & 75 \\ 75 & 125 \end{bmatrix}$

$\Rightarrow (AB)^2 \neq A^2B^2$

Q18. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\begin{cases} ad-bc = 1 \\ a = d \\ c = -b \end{cases} \Rightarrow a^2+b^2 = 1 = \cos^2\theta + \sin^2\theta \quad \Rightarrow \boxed{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$$

$\uparrow$ orthogonal matrix

Q19 (1) Symmetric

2×2 : $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

Skew-Symmetric

2×2 : $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = A$ $\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = A^T = -A$

3×3 : $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 4 \end{bmatrix}$ 3×3 : $\begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 3 \\ -1 & -3 & 0 \end{bmatrix} = A$ $\begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & -3 \\ 1 & 3 & 0 \end{bmatrix} = A^T = -A$ Generally, $\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = A^T$

4×4 : $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$ 4×4 : $\begin{bmatrix} 0 & 1 & -2 & -3 \\ -1 & 0 & -4 & 5 \\ 2 & 4 & 0 & -6 \\ 3 & -5 & 6 & 0 \end{bmatrix} = A$ $\begin{bmatrix} 0 & -1 & 2 & 3 \\ 1 & 0 & 4 & -5 \\ -2 & -4 & 0 & 6 \\ -3 & 5 & -6 & 0 \end{bmatrix} = A^T = -A$ Generally, $\begin{bmatrix} a & e & i & m \\ b & f & j & n \\ c & g & k & o \\ d & h & l & p \end{bmatrix} = A^T$

(2) The main diagonal of a skew-symmetric matrix is made up of zeroes.

(3) Any $n\times n$ matrix made up of zeroes:

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ since $A = A^T = -A$
$\Rightarrow A = -A$

(4) $(A+A^T)^T \stackrel{?}{=} A + A^T$

$\Rightarrow A^T + (A^T)^T \stackrel{?}{=} A + A^T$ (Property 2 of Matrix Transposition)

$\Rightarrow A^T + A \stackrel{\checkmark}{=} A + A^T$ (Property 1 of Matrix Transposition and commutative property)

$\therefore$ Since $(A+A^T)^T = A+A^T$, $A+A^T$ is a symmetric matrix

$(AA^T)^T \stackrel{?}{=} AA^T$

$\Rightarrow (A^T)^T A^T \stackrel{?}{=} AA^T$ (Property 4 of Matrix Transposition)

$\Rightarrow AA^T \stackrel{\checkmark}{=} AA^T$ (Property 1 of Matrix Transposition)

$\therefore$ Since $(AA^T)^T = AA^T$, AA^T is a symmetric matrix

$(A^TA)^T \stackrel{?}{=} A^TA$

$\Rightarrow A^T(A^T)^T \stackrel{?}{=} A^TA$ (Property 4 of Matrix Transposition)

$\Rightarrow A^TA \stackrel{\checkmark}{=} A^TA$ (Property 1 of Matrix Transposition)

$\therefore$ Since $(A^TA)^T = A^TA$, A^TA is a symmetric matrix

$\Rightarrow (A-A^T)^T \stackrel{?}{=} -(A-A^T)$

$A^T - (A^T)^T \stackrel{?}{=} -A + A^T$ (Property 2 of Matrix Transposition and Distributivity)

$A^T - A \stackrel{\checkmark}{=} -A + A^T$ (Property 1 of Matrix Transposition and commutative property)

$\therefore$ Since $(A-A^T)^T = -(A-A^T)$, $A-A^T$ is a skew-symmetric matrix

(5) Let X be the symmetric matrix $A+A^T$ (proved above) and Y be the skew-symmetric matrix $A-A^T$ (proved above).

Want to show that $X+Y$ yields an $n\times n$ matrix

$X+Y$

$\Rightarrow A + A^T + A - A^T$ (substitution)

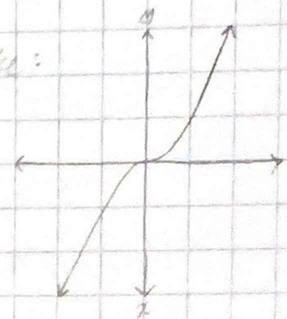
$\Rightarrow 2A$ (simplify)

Let B be the $n\times n$ matrix $2A$, since addition holds the matrices dimensions.

$\therefore$ An $n\times n$ matrix B can be written as $2A$, the sum of a symmetric matrix $(A+A^T)$ and a skew-symmetric matrix $(A-A^T)$.

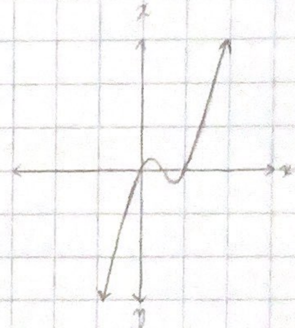
Q20 (a) $F(x) = x^2$ is neither injective (because $f(2) = 4$ and $f(-2) = 4$, so it is not one-to-one), surjective (because the codomain $\mathbb{R}$ includes negative numbers, so $f(x) = -1$ wouldn't include a value for x in the domain $\mathbb{R}$), and therefore is also not bijective.

(b) $F(x) = x^3/(x^2+1)$ is the continuous function that looks like:



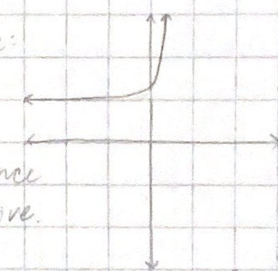
It is clear from the graph that the function is injective because each value corresponds to one value from the domain. $F(x)$ is also surjective since we see there is a solution across the whole co-domain that exists in the domain. Thus, $F(x) = x^3/(x^2+1)$ is bijective.

(c) $F(x) = x(x-1)(x-2)$ is the continuous function that looks like:



It is clear from the graph that this function is not injective $f(0) = f(1) = f(2) = 0$. The function is surjective since we see the solutions to $F(x)$ span the domain $\mathbb{R}$. However, since it is not injective, it is not bijective.

(d) $F(x) = e^x + 2$ is the continuous function that looks like:



It is clear from the graph that $F(x)$ is injective. However, it is not surjective because there is no x in the domain that could satisfy $F(x) = 0$ since the left side approaches 2. Therefore, it isn't bijective.

Q21. $A = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

$d_1 = 4$, $u_1 = 1$ $\Rightarrow p_1 = 1 = 4l_1 \Rightarrow l_1 = \frac{1}{4}$ $\Rightarrow q_2 = \frac{1}{4}(1) + d_2 = 4 \Rightarrow d_2 = \frac{15}{4}$

$p_2 = \frac{15}{4} l_2 = 1 \Rightarrow l_2 = \frac{4}{15}$ $\quad q_3 = \frac{4}{15}(1) + d_3 = 4 \Rightarrow d_3 = \frac{56}{15}$

$p_3 = \frac{56}{15} l_3 = 1 \Rightarrow l_3 = \frac{15}{56}$ $\quad q_4 = \frac{15}{56}(1) + d_4 = 4 \Rightarrow d_4 = \frac{209}{56}$

$\Rightarrow L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ 0 & \frac{4}{15} & 1 & 0 \\ 0 & 0 & \frac{15}{56} & 1 \end{bmatrix}$ $\quad U = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & \frac{15}{4} & 1 & 0 \\ 0 & 0 & \frac{56}{15} & 1 \\ 0 & 0 & 0 & \frac{209}{56} \end{bmatrix}$

Q22. $LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_1 & 1 & 0 & 0 \\ 0 & l_2 & 1 & 0 \\ 0 & 0 & l_3 & 1 \end{bmatrix} \begin{bmatrix} d_1 & u_1 & 0 & 0 \\ 0 & d_2 & u_2 & 0 \\ 0 & 0 & d_3 & u_3 \\ 0 & 0 & & d_4 \end{bmatrix} = \begin{bmatrix} d_1 & u_1 & 0 & 0 \\ d_1 l_1 & l_1 u_1 + d_2 & u_2 & 0 \\ 0 & d_2 l_2 & l_2 u_2 + d_3 & u_3 \\ 0 & 0 & d_3 l_3 & l_3 u_3 + d_4 \end{bmatrix}$

$$\boxed{\begin{aligned} q_i &= l_{i-1} u_{i-1} + d_i \\ r_i &= u_i \\ p_i &= d_i l_i \end{aligned}}$$

Q23 $L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & & & & 0 \\ 0 & \frac{1}{d_2} & & & 0 \\ 0 & & & & 0 \\ 0 & 0 & 0 & \frac{1}{d_i} & 1 \end{bmatrix}$ $\quad U = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4-l_1 & u_2 & & 0 \\ 0 & & & & 0 \\ 0 & & & & u_i = 1 \\ 0 & 0 & 0 & 0 & 4-l_{i-1} \end{bmatrix}$

$\Rightarrow l_i = \frac{1}{d_i}$

$\Rightarrow d_i = 4 - l_{i-1}$

$\Rightarrow u_i = 1$

Q24. (1) $(H_n)^T \overset{?}{=} H_n$

$\Rightarrow (I_n - 2\vec{u}\vec{u}^T)^T \overset{?}{=} I_n - 2\vec{u}\vec{u}^T$ (substitution)

$\Rightarrow I_n^T - (2\vec{u}\vec{u}^T)^T \overset{?}{=} I_n - 2\vec{u}\vec{u}^T$ (Property 2 of Matrix Transposition)

$\Rightarrow I_n - 2(\vec{u}^T)^T\vec{u}^T \overset{?}{=} I_n - 2\vec{u}\vec{u}^T$ (Identity matrix is symmetric, and Property 4 of Matrix Transposition)

$\Rightarrow I_n - 2\vec{u}\vec{u}^T \overset{\checkmark}{=} I_n - 2\vec{u}\vec{u}^T$ (Property 1 of Matrix Transposition)

$\Rightarrow (H_n)^T = H_n$

$\therefore$ H_n is a symmetric matrix

(2) $H_n^T H_n \overset{?}{=} I_n$

$\Rightarrow (I_n - 2\vec{u}\vec{u}^T)(I_n - 2\vec{u}\vec{u}^T) \overset{?}{=} I_n$ (substitution and $H_n^T = H_n$ proved above)

$\Rightarrow (I_n)^2 - 4\vec{u}\vec{u}^T + 4(\vec{u}\vec{u}^T)(\vec{u}\vec{u}^T) \overset{?}{=} I_n$ (Distributivity)

$\Rightarrow I_n - 4\vec{u}\vec{u}^T + 4\vec{u}(\vec{u}^T\vec{u})\vec{u}^T \overset{?}{=} I_n$ (Associativity in vector space)

$\Rightarrow I_n - 4\vec{u}\vec{u}^T + 4\vec{u}\vec{u}^T \overset{?}{=} I_n$ (using $\vec{u}^T = \vec{u}$ and $\vec{u} \cdot \vec{u} = 1$)

$\Rightarrow I_n \overset{\checkmark}{=} I_n$ (simplify)

$\Rightarrow H_n^T H_n = I_n$

$\therefore$ H_n is an orthogonal matrix

(3) $H_n^2 = I_n$ (calculation shown above using $H_n^T = H_n$)

(4)
$$\begin{aligned} H_n\vec{u} &= (I_n - 2\vec{u}\vec{u}^T)\vec{u} \\ &= I_n\vec{u} - 2\vec{u}\vec{u}^T\vec{u} \\ &= \vec{u} - 2\vec{u} \\ H_n\vec{u} &= -\vec{u} \end{aligned}$$

(5)
$$\begin{aligned} H_3 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\left(\tfrac{1}{\sqrt{3}}\right)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\left(\tfrac{1}{\sqrt{3}}\right)\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \tfrac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \tfrac{2}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} H_4 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - 2\left(\tfrac{1}{\sqrt{4}}\right)\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\left(\tfrac{1}{\sqrt{4}}\right)\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \tfrac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \tfrac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ H_4 &= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{aligned}$$