Sai Nikhil
NUID: 001564864

1

(1) $V_n = \{ A \in R^{n \times n} \mid tr(A) = 0 \}$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$tr(A) = 0 \Rightarrow \sum_{i=1}^{n} a_{ii} = 0$$

$$\Rightarrow \boxed{a_{11} = -\sum_{j=2}^{n} a_{jj}}$$

$$A = \begin{bmatrix} -(a_{22} + a_{33} + \cdots a_{nn}) & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} a_{12} + \cdots + \begin{bmatrix} 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} a_{1n}$$

$$+ \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} a_{21} + \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} a_{22} + \cdots + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} a_{2n}$$

$$\vdots$$

$$+ \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} a_{n1} + \cdots + \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} a_{nn}$$

$$\boxed{V_n = \left\{ E_{ij} : i \neq j \right\} \cup \left\{ \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \cdots, \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \right\}}$$

Standard basis

(2)

$$\text{Dimension} = \left(n^2 - n\right) + (n-1)$$

$$= \boxed{n^2 - 1}$$

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(2)

$$S = \{\vec{u}, \vec{v}, \vec{w}\}$$

$$T = \{\vec{u}+\vec{v}+\vec{w}, \vec{v}+2\vec{w}, 3\vec{v}+4\vec{w}\}$$

Linear Combination of "S" can be written as,

$$f(S) = a_1\vec{u} + a_2\vec{v} + a_3\vec{w} \quad \text{— (1)}$$

Linear Combination of "T" can be written as,

$$g(T) = c_1(\vec{u}+\vec{v}+\vec{w}) + c_2(\vec{v}+2\vec{w}) + c_3(3\vec{v}+4\vec{w})$$

$$= c_1\vec{u} + (c_1 + c_2 + 3c_3)\vec{v} + \underbrace{(c_1 + 2c_2 + 4c_3)\vec{w}}_{(2)}$$

From (1) & (2),

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ clearly,

$$\boxed{a_1 = a_2 = a_3 = 0 \iff c_1 = c_2 = c_3 = 0}$$

③

(1)

"A" is diagonalizable

$$A = PDP^{-1}$$

$$A - \lambda I = PDP^{-1} - \lambda PP^{-1}$$

$$= P(D - \lambda I)P^{-1}$$

$$(A - \lambda I)^2 = P(D - \lambda I)^2 P^{-1}$$

$$\text{rank}(A - \lambda I) = \text{rank}(D - \lambda I)$$

$\left(\because \text{"}B\text{" is invertible} \ni \text{rank}(BA) = \text{rank}(AB) = \text{rank}(A) \right)$

$$= \text{rank}\left(\begin{bmatrix} \lambda_1 - \lambda & 0 & \cdots & 0 \\ 0 & \lambda_2 - \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n - \lambda \end{bmatrix} \right) \quad \text{— (i)}$$

$$\operatorname{rank}\left((A-\lambda I)^2\right) = \operatorname{rank}\left((D-\lambda I)^2\right)$$

$$= \operatorname{rank}\left(\underbrace{\begin{bmatrix} (\lambda_1-\lambda)^2 & 0 & \cdots & 0 \\ 0 & (\lambda_2-\lambda)^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (\lambda_n-\lambda)^2 \end{bmatrix}}_{(ii)}\right)$$

Number of non-zero rows in (i)

= Number of non-zero rows in (ii)

$$\therefore \operatorname{rank}(A-\lambda I) = \operatorname{rank}\left((A-\lambda I)^2\right)$$

(2)

$$A = P \underbrace{J}_{\text{Jordan Normal Form}} P^{-1}$$

$$\operatorname{rank}(A-\lambda I) = \operatorname{rank}\left(P(J-\lambda I)P^{-1}\right)$$

$$= \operatorname{rank}(J-\lambda I)$$

$$\text{rank}\,(A-\lambda I)^2 = \text{rank}\left(P(J-\lambda I)^2 P^{-1}\right)$$

$$= \text{rank}\,(J-\lambda I)^2$$

If, "J" is in "Jordan-normal form"

"J" has 1 on off-diagonal (the diagonal above main diagonal)

"$J-\lambda I$" has 1 on off-diagonal

It is eesy to prove by a counter example

if,

$$J-\lambda I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow (J-\lambda I)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ $\text{rank}(J-\lambda I) = 1$ & $\text{rank}\,(J-\lambda I)^2 = 0$

$\Rightarrow$ J - cannot have 1's on off-diagonal

$\Rightarrow$ $J = D$ $\left(\text{where } D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \right)$

④ (1) Let, $f(x), g(x) \in P_3$

$$T(f(x)+g(x)) = (f(x)+g(x)) + \frac{d}{dx}(f(x)+g(x))$$

$$= \left(f(x) + \frac{d}{dx}(f(x))\right) + \left(g(x) + \frac{d}{dx}(g(x))\right)$$

$$= T(f(x)) + T(g(x))$$

Let, $\alpha \in \mathbb{R}$

$$T(\alpha f(x)) = \alpha f(x) + \frac{d}{dx}(\alpha f(x))$$

$$= \alpha\left(f(x) + \frac{d}{dx}(f(x))\right)$$

$$= \alpha T(x)$$

$\therefore$ "T" is a lineer transformation

(2)

$$T(1) = 1 + \frac{d}{dx}(1) = 1$$

$$T(x) = x + \frac{d}{dx}(x) = x+1$$

$$T(x^2) = x^2 + \frac{d}{dx}(x^2) = x^2+2x$$

$$T(x^3) = x^3 + \frac{d}{dx}(x^3) = x^3+3x^2$$

$$\therefore [T(1)]_M = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [T(x)]_M = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$[T(x^2)]_M = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad [T(x^3)]_M = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \end{bmatrix}$$

$$\therefore M = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(3) $\det(M) = 1 \times 1 \times 1 \times 1$ ($\because$ upper triangular matrix)

$= 1$

(4)
$$\begin{vmatrix} 1-\lambda & 1 & 0 & 0 \\ 0 & 1-\lambda & 2 & 0 \\ 0 & 0 & 1-\lambda & 3 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = (\lambda-1)^4$$

"$\lambda = 1$" is the only eigenvalue

$$A - \lambda I = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X = \begin{pmatrix} x_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \therefore v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$ is the only eigen vector

(5) $\quad p, q \in P_3$

let, $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$q(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$

$$(p(x)) = (a_0 + a_1 x + a_2 x^2 + a_3 x^3) + (a_1 + 2a_2 x + 3a_3 x^2)$$

$$= (a_0 + a_1) + (a_1 + 2a_2)x + (a_2 + 3a_3)x^2 + a_3 x^3$$

$\therefore \; T((x)) = q(x)$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{M} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$\det(M) \neq 0$

$\Rightarrow$ Exactly one Solution

5

$W = \text{Span}\{I_n, A, A^2, A^3, ---\}$

Let, $f(\lambda) = 0$ is the characteristic quetion

$\Rightarrow f(A) = 0$ (By Cayley-Hamiltion theorem)

Highest power of "λ" is $f(\lambda) = n$

$\Rightarrow \{I_n, A, A^2, A^3, ---- A^n\}$ is lineerly dependent

$\therefore \{I_n, A, A^2, \text{---} A^{n-1}\}$ is the highest possible independent set of vectors

$$\therefore \dim\{I_n, A, A^2, \text{----}\} = \dim\{I_n, A, A^2, \text{---} A^{n-1}\}$$

$\leq n$ (Cardinality of set of highest possible independent vectors)

$$\therefore \boxed{\dim W \leq n}$$

6

(1) clearly,

$\det(A) = 0 \Rightarrow$ product of eigen values $= 0$

$\therefore$ one eigen value is, $\lambda_1 = 0$

$\dim(\ker(A - 0I)) = \dim(\ker(A)) = 4$

"A" is symmetric $\Rightarrow$ "A" is diagonalizable

$\Rightarrow$ Algebraic multiplicity = geometric multiplicity

$\Rightarrow$ Algebraic multiplicity of "0" = 4

$\Rightarrow$ only one more eigen value is present

Sum of eigenvalues = $tr(A) = 5$

$\therefore \; 4(0) + \lambda_2 = 5$

$\Rightarrow \lambda_2 = 5$

$\therefore \; \lambda_1 = 0$ (algebraic multiplicity = 4)

$\lambda_2 = 5$ (algebraic multiplicity = 1)

(2)

$$B = A + I = PDP^{-1} + PIP^{-1}$$

$$= P(D+I)P^{-1}$$

$\therefore \; \lambda_1 = 0 + 1 = 1$ (algebraic multiplicity = 4)

$\lambda_2 = 5 + 1 = 6$ (algebraic multiplicity = 1)

(3)

$$M = b \underbrace{\begin{bmatrix} a/b & 1 & 1 & 1 & 1 \\ 1 & a/b & 1 & 1 & 1 \\ 1 & 1 & a/b & 1 & 1 \\ 1 & 1 & 1 & a/b & 1 \\ 1 & 1 & 1 & 1 & a/b \end{bmatrix}}_{C}$$

$$\det(M) = b^5 \times \det(C)$$

From above questions,
eigen values of "C" are,

$$\lambda_1 = 0 + (a/b - 1) \quad \left[\text{algebraic multiplicity} = 4\right]$$

$$\lambda_2 = 5 + (a/b - 1) \quad \left(\text{algebraic multiplicity} = 1\right)$$

$$= 4 + a/b$$

$$\therefore \det(M) = b^5 \times (a/b - 1)^4 \times \left(\frac{a}{b} + 4\right) = \boxed{(a-b)^4 (a+4b)}$$

(4) "M" is symmetric

$\therefore$ positive definite $\Rightarrow \lambda_i > 0$

$\therefore \lambda_1 > 0$	$\lambda_2 > 0$
$b\left(\frac{a}{b} - 1\right) > 0$	$b\left(\frac{a}{b} + 4\right) > 0$
$a - b > 0$	$a + 4b > 0$
$a > b$ — (i)	$a > -4b$ — (ii)

From (i) & (ii),

Case-1: $b \geq 0$

$a > b$

Case-2: $b < 0$

$a > -4b$

Refer to "problems_7_to_10.m" for code

7

(1)

basis of $(\text{im} A)^{\perp} = \ker(A^T)$

$$A^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 0 \end{bmatrix}$$

$$\text{rref}(A^T) = \begin{bmatrix} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \text{basis} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(2)

$$U = \begin{bmatrix} 0.37 & 0.24 & -0.71 & 0.03 & -0.55 \\ 0.45 & 0.14 & -0.28 & 0.36 & 0.75 \\ 0.53 & 0.03 & 0.14 & -0.83 & 0.13 \\ 0.61 & -0.08 & 0.57 & 0.43 & -0.34 \\ 0.12 & -0.96 & -0.27 & 0 & 0 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 16.29 & 0 & 0 \\ 0 & 4.93 & 0 \\ 0 & 0 & 0.56 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V^T = \begin{bmatrix} 0.13 & 0.36 & 0.92 \\ -0.13 & -0.92 & 0.38 \\ -0.98 & 0.17 & 0.07 \end{bmatrix}$$

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(8)

$$B = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

$\lambda_1 = 1$ (Algebraic multiplicity $= 4$)

$\lambda_2 = 6$ (Algebraic multiplicity $= 1$)

Eigen vector matrix,

$$U = \begin{bmatrix} -1 & -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Clearly Column vectors are not orthogonal

$\therefore$ Applying Gram-Schmidt,

we get,

$$P = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{6} & -\frac{\sqrt{5}}{10} & \frac{\sqrt{5}}{5} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{6} & -\frac{\sqrt{5}}{10} & \frac{\sqrt{5}}{5} \\ 0 & \frac{\sqrt{6}}{3} & -\frac{\sqrt{3}}{6} & -\frac{\sqrt{5}}{10} & \frac{\sqrt{5}}{5} \\ 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{5}}{10} & \frac{\sqrt{5}}{5} \\ 0 & 0 & 0 & \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

$$\therefore \boxed{B = PDP^T}$$

⑨ $M = \frac{1}{7}(A + 2I)$ problem 6

eigen values of M are,

$\lambda_1 = \frac{0+2}{7} = \frac{2}{7}$ (algebraic multiplicity $= 4$)

$\lambda_2 = \frac{5+2}{7} = 1$ (algebraic multiplicity $= 1$)

M is symmetric

$\Rightarrow$ M is diagonalizable

$\Rightarrow M = PDP^{-1}$

$$D = \begin{pmatrix} 2/7 & 0 & 0 & 0 & 0 \\ 0 & 2/7 & 0 & 0 & 0 \\ 0 & 0 & 2/7 & 0 & 0 \\ 0 & 0 & 0 & 2/7 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\lambda_1 = \frac{2}{7} \Rightarrow$ eigen vectors are $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$\lambda_2 = 1 \Rightarrow$ eigen vectors are $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

$$\therefore P = \begin{pmatrix} -1 & -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\therefore \quad P^{-1} = \frac{1}{5}\begin{pmatrix} -1 & -4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\therefore \lim_{n\to\infty} M^n = \lim_{n\to\infty} P\, D^n\, P^{-1}$$

$$= P \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} P^{-1}$$

$$= \frac{1}{5}\, \mathbf{1} \times \mathbf{1}^T \qquad \left(\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right)$$

$$= \frac{1}{5}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

(10)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 3 & 4 \\ 1 & 3 & 2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 3 \\ 6 \\ 9 \\ 3 \\ 0 \end{bmatrix}$$

$$A \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\vec{w}} = \vec{v}$$

$$\vec{w} = A \backslash \vec{v}$$

(Solution to normal equation $A^T A \vec{\omega} = A^T \vec{v}$)

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 7.4 \\ -1.2 \\ -0.4 \end{bmatrix}$$

$$\therefore \quad z = 7.4 - 1.2x - 0.4y$$

(2)

$$\|\vec{v} - A\vec{w}\|^2 = \text{norm}(\vec{v} - A\vec{w})^2$$

$$= \boxed{37.2}$$