HW 1 Work

Monday, September 20, 2021 10:08 PM

Question 1. Show that the following sets of numbers are fields if the usual addition and multiplication of arithmetic are used:

(1) the set of all numbers of the form $a + b\sqrt{2}$ where a and b are rational numbers.

(2) the set of all numbers of the form $a + b\sqrt{-1}$ where a and b are real numbers. What is this field?

1) 1) identity Sum

$(a+b\sqrt{2})+0 = a+b\sqrt{2}$

inverse Sum

$(a+b\sqrt{2})+(-a-b\sqrt{2}) = 0$

inverse multiplication

$(a+b\sqrt{2})(x+y\sqrt{2})=1$

$ax+ay\sqrt{2}+bx\sqrt{2}+2by$

$(ax+2by)+(ay+bx)\sqrt{2}=1$

$\begin{vmatrix} a & 2b \\ b & a \end{vmatrix}\begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$

$A\vec{x} = \vec{b}$

$A^{-1}A\vec{x} = A^{-1}\vec{b}$

$\vec{x} = A^{-1}\vec{b}$

$A^{-1} = \frac{1}{\det(A)}\begin{vmatrix} a & -2b \\ -b & a \end{vmatrix}\begin{vmatrix} 1 \\ 0 \end{vmatrix}$

$\frac{1}{a^2-2b^2}\begin{vmatrix} a & -2b \\ -b & a \end{vmatrix}\begin{vmatrix} 1 \\ 0 \end{vmatrix} =$

$\frac{1}{a^2-2b^2}\begin{vmatrix} a \\ -b \end{vmatrix} = \begin{vmatrix} \frac{a}{a^2-2b^2} \\ \frac{-b}{a^2-2b^2} \end{vmatrix}$

1) 2) identity Sum

$(a+b\sqrt{-1})+0 = a+b\sqrt{-1}$

inverse Sum

$(a+b\sqrt{-1})+(-a-b\sqrt{-1}) = 0$

Inverse Multiplication

$(a+b\sqrt{-1})(x+y\sqrt{-1})=1$

$ax+ay\sqrt{-1}+bx\sqrt{-1}-by=1$

$(ax-by)+(ay+bx)\sqrt{-1}=1$

$\begin{vmatrix} a & -b \\ b & a \end{vmatrix}\begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$

$A\vec{x} = \vec{b}$

$A^{-1}A\vec{x} = A^{-1}b$

$\vec{x} = A^{-1}\vec{b}$

$\frac{1}{\det A}\begin{vmatrix} a & b \\ -b & a \end{vmatrix}\begin{vmatrix} 1 \\ 0 \end{vmatrix}$

$\frac{1}{a^2+b^2}\begin{vmatrix} a & b \\ -b & a \end{vmatrix}\begin{vmatrix} 1 \\ 0 \end{vmatrix}$

$\frac{1}{a^2+b^2}\begin{vmatrix} a \\ -b \end{vmatrix} = \begin{vmatrix} \frac{a}{a^2+b^2} \\ \frac{-b}{a^2+b^2} \end{vmatrix}$

Question 2. Show that the set of all $n \times n$ matrices $\mathbb{R}^{n\times n}$ with the usual matrix addition and multiplication is not a field if $n > 1$.

Let A be an nxn matrix

then its inverse is given

$\rightarrow A^{-1} = \frac{1}{|A|}\operatorname{adj} A$

where $|A|$ is det A

if $|A| = 0 \nRightarrow A^{-1}$ DNE

$\exists$ there exist non-0 nxn metrics $[|A| = 0]$

determinent $= 0$

There exist nxn matrix whose inverse DNE

$\Rightarrow$ set of all nxn matrices is not field

$\rightarrow$ There must be an inverse of an element for a field to exist

Question 3. Write down the two operations on field $\mathbb{Z}_3$.

+	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

×	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

Question 4. Some basic knowledge of complex numbers.

- Just as $\mathbb{R}$ denotes the set of real numbers, we will use $\mathbb{C}$ to denote the set of complex numbers $z = a+ib$. Here $i = \sqrt{-1}$, and a and b are real numbers called/denoted

$$a = Re(z) = \textbf{real part of } z$$
$$b = Im(z) = \textbf{imaginary part of } z$$

- The **complex conjugate** of $z = a + bi \in \mathbb{C}$ is $\bar{z} := a - bi$
- The **absolute value** of z is $|z| = \sqrt{a^2 + b^2}$.
- $z\bar{z} = |z|^2$

a+bi

Show that $\mathbb{C}$ is a **field** with the usual sum, scalar product and product.

as shown in problem 1 pt 2 field of complex #'s

Question 5. Determine which of the matrices below are in reduced row-echelon form.

$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$; $C = \begin{bmatrix} 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 5 \end{bmatrix}$; $D = \begin{bmatrix} 0 & 1 & 0 & 2 & 4 \end{bmatrix}$; $E = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}$

Question 6. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ be two matrices over the field $\mathbb{Z}_2$. Compute $A + B$, A^2 and AB over the field $\mathbb{Z}_2$.

$A+B = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}$

$A \times A = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix}$

$\boxed{= \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}}$

$A \times B = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix}\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$

$= \boxed{\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix}}$

Question 7. For which values of t does the matrix $A = \begin{bmatrix} 6 & -1 & 1 \\ t & 0 & 1 \\ 0 & 1 & t \end{bmatrix}$ NOT have an inverse?

$\det A = 0$

$6(0-1) - 1(t^2-0) + 1(6-0)$

$-6 - t^2 + t$

$t^2 - t + 6$

$(t+3)(t-2)$

$t = 2, -3$.

Question 8. Find all values of h that make the following matrices **consistent**, i.e., at least has one solution.

a) $\begin{bmatrix} 1 & h & | & 4 \\ 3 & 6 & | & 8 \end{bmatrix}$

b) $\begin{bmatrix} -4 & 12 & | & h \\ 2 & -6 & | & -3 \end{bmatrix}$

a) $\begin{bmatrix} 1 & h & | & 4 \\ 3 & 6 & | & 8 \end{bmatrix}$ $6 - 3h = 0$

$h \neq 2$

$\boxed{(-\infty, 2)(2, \infty)}$

b) $\begin{bmatrix} -4 & 12 & h \\ 2 & -6 & -3 \end{bmatrix}$ $R_2 = R_2 + \frac{1}{2}R_1$

$0 = -3 + \frac{1}{2}h$

$h = 6$

Question 9. We says that two $m \times n$ matrices in reduced row-echelon form are of the same type if they have the same number of leading 1's in the same position.

(1) How many types of 3×2 matrices in reduced row-echelon form.

(2) How many types of 2×3 matrices in reduced row-echelon form.

(3) Find all 4×1 matrices in reduced row-echelon form.

1) $\begin{bmatrix} 1 & * \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

2) $\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}$ - $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Question 10. For which values of a, b, c, d, and e is the following matrix in reduced row-echelon form?

$$A = \begin{bmatrix} 1 & a & b & 3 & 0 & -2 \\ 0 & 0 & c & 1 & d & 3 \\ 0 & e & 0 & 0 & 1 & 1 \end{bmatrix}$$

$a = *$ $b = *$ $c = 0, 1$ $d = *$ $e = 0$

Question 11. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix}$.

(1) Calculation **rref**(A) over $\mathbb{R}$ by hand. Solve $A\vec{x} = \vec{0}$ and write all solutions in parametric vector forms.

(2) Calculation **rref**(A) over field $\mathbb{Z}_7$ by hand.

(3) Using Python verify your result and calculation **rref**(A) over field $\mathbb{Z}_2$ and $\mathbb{Z}_3$. (Python suggestion is uploaded on Canvas.)

(4) Is it possible that a matrix M has different rank over different fields $\mathbb{Z}_p$? (By calculation in (3))

1) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix}$ $R_2 - R_1$ $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -2 \\ 2 & 0 & 1 & 2 \end{bmatrix}$ $R_3 - 2R_1$ $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -2 \\ 0 & -4 & -5 & -6 \end{bmatrix}$

$R_2 = -R_2$ $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & -4 & -5 & -6 \end{bmatrix}$ $R_1 - 2R_2$ $\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & -4 & -5 & -6 \end{bmatrix}$ $R_3 + 4R_2$ $\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 7 & 2 \end{bmatrix}$

$\frac{R_3}{7}$ $\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 2/7 \end{bmatrix}$ $R_1 + 3R_3$ $\begin{bmatrix} 1 & 0 & 0 & 6/7 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 2/7 \end{bmatrix}$ $R_2 - 3R_3$ $\begin{bmatrix} 1 & 0 & 0 & 6/7 \\ 0 & 1 & 0 & 8/7 \\ 0 & 0 & 1 & 2/7 \end{bmatrix}$

Question 12. (Solve a linear system over field $\mathbb{Z}_7$.) Let $A = \begin{bmatrix} 3 & 1 & 4 \\ 5 & 2 & 6 \\ 0 & 5 & 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$.

(1) Calculation **rref**($A|\vec{b}$) over field $\mathbb{Z}_7$.

(2) Find solution of the linear system $A\vec{x} = \vec{b}$ mod 7.

1)

```
def question_12():
    M = Matrix([[3, 1, 4, 1], [5, 2, 6, 5], [0, 5, 2, 1]])
    print("Matrix : {} ".format(M))
    M_rref = M.rref()
    print("The Row echelon form of matrix M and the pivot columns : {}".format(M_rref))
question_12()
```

```
Matrix : Matrix([[3, 1, 4, 1], [5, 2, 6, 5], [0, 5, 2, 1]])
The Row echelon form of matrix M and the pivot columns : (Matrix([
[1, 0, 0,   31/6],
[0, 1, 0,   11/6],
[0, 0, 1, -49/12]]), (0, 1, 2))
```

→ RREF

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 31/6 \\ 0 & 1 & 0 & 11/6 \\ 0 & 0 & 1 & 49/12 \end{array}\right]$

2)

```
In [14]:  GF7 = galois.GF(7)
          A = GF7([[3, 1, 4, 1], [5, 2, 6, 5], [0, 5, 2, 1]])
          GF7.row_reduce(A)
```

```
Out[14]: GF([[1, 0, 0, 4],
             [0, 1, 0, 3],
             [0, 0, 1, 0]], order=7)
```

Question 19. Here are a couple of new definitions: An $n \times n$ matrix A is *symmetric* provided $A^T = A$ and *skew-symmetric* provided $A^T = -A$.

(1) Give examples of symmetric and skew-symmetric 2×2, 3×3, and 4×4 matrices.
(2) What can you say about the main diagonal of a skew-symmetric matrix?
(3) Give an example of a matrix that is both symmetric and skew-symmetric.
(4) Prove that for any $n \times n$ matrix A, the matrices $A + A^T$, AA^T, and $A^T A$ are symmetric and $A - A^T$ is skew-symmetric.
(5) Prove that any $n \times n$ can be written as the sum of a symmetric and skew-symmetric matrices. Hint: Did you do part (4) yet?

1) $\boxed{2 \times 2}$

Symmetric $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^T = A$

Skew Symmetric $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = A^{-1}$

$A^T = -A$

$\boxed{3 \times 3}$

Symmetric $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A^T = A$

Skew Symmetric

$A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ $A^T = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$

$A^T = -A$

$\boxed{4 \times 4}$

Symmetric $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$A^T = A$

Skew Symmetric

$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \end{bmatrix}$ $A^T = \begin{bmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ & & 0 & -1 \end{bmatrix}$

$A^T = A$ $\quad \cdots 0\,0\,0\,1] \quad [0\,0\,0\,1]$

Skew! Symmetric

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix} \quad A^T = \begin{bmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$A^T = -A$

2) The main diagonal has to be 0 for a skew symmetric matrix

3) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is symmetric & Skew symmetric

4) $\Rightarrow A + A^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A$

$\Rightarrow (A + A^T)^T = A + A^T$

$\Rightarrow AA^T = (AA^T)^T = A^T \cdot (A)^T = A^T \cdot A = AA^T$

$\Rightarrow (AA^T)^T = AA^T$

$\Rightarrow A^TA = (A^TA)^T = (A^T)^T \cdot A^T = A \cdot A^T = AA^T$

$\neq AA^T$

$\Rightarrow A - A^T = (A - A^T)^T = A^T - A = -(A - A^T)$

$\Rightarrow$ Skew symmetric b/c

$A - A^T = -(A - A^T)$

by def of skew symmetric

5) $(A + A^T) + (A - A^T)$

$\frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

$\frac{1}{2}A + \frac{1}{2}A^T + \frac{1}{2}A - \frac{1}{2}A^T$

$A = A + 0$

$\Rightarrow$ $n \times n$ can be written as the sum of symmetric & skew-symmetric matrices

Question 24. Let I_n be the $n \times n$ identity matrix. Let $\vec{u}$ be a **unit** vector in $\mathbb{R}^n$. Define $H_n = I_n - 2\vec{u}\vec{u}^T$.

Here a unit vector $\vec{u}$ means that norm $\|\vec{u}\| = 1$ or equivalently $\vec{u}^T\vec{u} = \vec{u} \cdot \vec{u} = 1$.

(1) Is H_n an symmetric matrix? Prove your result.

(2) Is H_n an orthogonal matrix? (i.e. is $H_n^T H_n = I_n$?)

(3) What is H_n^2?

(4) What is $H_n\vec{u}$?

(5) Suppose $\vec{u} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$. Write down H_3 and H_4?

1) $H^T = (I - 2\vec{u}\vec{u}^T)^T$

$= I^T - 2(\vec{u}\vec{u}^T)^T$

$= I - 2(\vec{u}^T)^T(\vec{u}^T)$

$= I - 2\vec{u}\vec{u}^T$

$= H^T$

2) $H_n^T H_n = HH$

$= (I - 2\vec{u}\vec{u}^T)(I - 2\vec{u}\vec{u}^T)$

$= I - 2\vec{u}\vec{u}^T - 2uu^T + 4(uu^T)(uu^T)$

$= I - 4\vec{u}\vec{u}^T + 4u(u^Tu)u^T$

$= I - 4\vec{u}\vec{u}^T + 4uu^T$

$= I$

3) $H_n^2 = I \Rightarrow$ b/c of 24 (2)

4) $H_n\vec{u} = (I_n - 2\vec{u}\vec{u}^T)\vec{u}$

$= I_n\vec{u} - 2(1)\vec{u}$

$= I_n\vec{u} - 2\vec{u}$

5) $H_3 = I_3 - 2\vec{u}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$

$H_3 = \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$

$H_4 = I_4 - 2\vec{u}$

$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 1/\sqrt{4} \\ 1/\sqrt{4} \\ 1/\sqrt{4} \\ 1/\sqrt{4} \end{bmatrix} \begin{bmatrix} 1/\sqrt{4} & 1/\sqrt{4} & 1/\sqrt{4} & 1/\sqrt{4} \end{bmatrix}$

$= \begin{bmatrix} 1/2 & -1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & -1/2 \\ -1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$

Question 20. Mark each of the following functions $F : \mathbb{R} \to \mathbb{R}$ injective, surjective or bijective, as is most appropriate. (You may wish to draw the graph of the function in some cases.)

(a) $F(x) = x^2$; Surjective

(b) $F(x) = x^3/(x^2 + 1)$; bijective

(c) $F(x) = x(x - 1)(x - 2)$; Surjective

(d) $F(x) = e^x + 2$. Surjective

Question 16. (1) If A , B and C are $n \times n$ matrices and $ABC = I_n$, is each of the matrices invertible? What are their inverses?

(2) Suppose A and B are $n \times n$ matrices. If AB is invertible, are both A and B are invertible?

1) Yes they are all invertible

$ABC = I_n$

$A^{-1} \quad ABC = I_n A^{-1}$

$I_n BC = A^{-1}$

$A^{-1} = BC$

$\Rightarrow B^{-1} = AC$

$\Rightarrow C^{-1} = AB$

2) If AB is invertible $\Rightarrow$ both $A \& B$ have to be invertible

Question 17. Provide a counter-example to the statement: For any 2×2 matrices A and B, $(AB)^2 = A^2B^2$.

$A = \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $AB = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$ $(AB)^2 = \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix}$

$A^2 = \begin{bmatrix} 16 & 24 \\ 40 & 64 \end{bmatrix}$ $B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A^2B^2 \begin{bmatrix} 16 & 24 \\ 40 & 64 \end{bmatrix} \Rightarrow (AB)^2 \neq A^2B^2$

Question 18. Find an example of a 2×2 nonidentity matrix whose transpose is its inverse.

$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$A^{-1} = \dfrac{1}{\cos^2\theta + \sin^2\theta} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \dfrac{1}{1} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Question 12. (Solve a linear system over field $\mathbb{Z}_7$.) Let $A = \begin{bmatrix} 3 & 1 & 4 \\ 5 & 2 & 6 \\ 0 & 5 & 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$.

(1) Calculation $\mathbf{rref}(A|\vec{b})$ over field $\mathbb{Z}_7$.

(2) Find solution of the linear system $A\vec{x} = \vec{b} \mod 7$.

Question 13. (Use Python) Solve the linear system

$$\begin{cases} 3x_1 + 11x_2 + 19x_3 & = -2 \\ 7x_1 + 23x_2 + 39x_3 & = 10 \\ -4x_1 - 3x_2 - 2x_3 & = 6 \end{cases}$$

and write solutions in parametric vector forms.

Question 14. (Use Python) Solve the linear system

$$\begin{cases} 3x_1 + 6x_2 + 9x_3 + 5x_4 + 25x_5 & = 53 \\ 7x_1 + 14x_2 + 21x_3 + 9x_4 + 53x_5 & = 105 \\ -4x_1 - 8x_2 - 12x_3 + 5x_4 - 10x_5 & = 11 \end{cases}$$

4

and write solutions in parametric vector forms.

Question 15. (Use Python) Solve the linear system

$$\begin{cases} 2x_1 + 4x_2 + 3x_3 + 5x_4 + 6x_5 & = 37 \\ 4x_1 + 8x_2 + 7x_3 + 5x_4 + 2x_5 & = 74 \\ -2x_1 - 4x_2 + 3x_3 + 4x_4 - 5x_5 & = 20 \\ x_1 + 2x_2 + 2x_3 - x_4 + 2x_5 & = 26 \\ 5x_1 - 10x_2 + 4x_3 + 6x_4 + 4x_5 & = 24 \end{cases}$$

and write solutions in parametric vector forms. (Hint: In Python, if you want precise value, use symbolic

```
def question_12():
    M = Matrix([[3, 1, 4, 1], [5, 2, 6, 5], [0, 5, 2, 1]])
    M_rref = M.rref()
    print(M_rref)
question_12()
```

```
(Matrix([
[1, 0, 0,   31/6],
[0, 1, 0,   11/6],
[0, 0, 1, -49/12]]), (0, 1, 2))
```

In [14]:
```
GF7 = galois.GF(7)
A = GF7([[3, 1, 4, 1], [5, 2, 6, 5], [0, 5, 2, 1]])
GF7.row_reduce(A)
```

Out[14]:
```
GF([[1, 0, 0, 4],
    [0, 1, 0, 3],
    [0, 0, 1, 0]], order=7)
```

In [5]:
```
def question_13():
    x =[[3, 11, 18], [7, 23, 39], [-4, -3, -2]]
    y = [-2, 10, 6]
    solutions = np.linalg.inv(x).dot(y)
    parametric = np.array(solutions)
    answer = parametric.reshape(3,1)
    print('x = {}'.format(answer))

question_13()
```

```
x = [[  5.23943662]
 [-14.05633803]
 [  7.6056338 ]]
```

In [2]:
```
def question_14():
    M = sym.Matrix([[3, 6, 9, 5, 25, 53], [7, 14, 21, 9, 53, 105], [-4, -8, -12, 5, -10, 11]])
    print(M.rref())

question_14()
```

```
(Matrix([
[1, 2, 3, 0, 5, 6],
[0, 0, 0, 1, 2, 7],
[0, 0, 0, 0, 0, 0]]), (0, 3))
```

In [6]:
```
def question_15():
    x = [[2, 4, 3, 5, 6], [4, 8, 7, 5, 2], [-2, -4, 3, 4, -5], [1, 2, 2, -1, 2], [5, -10, 4, 6, 4]]
    y = [37, 74, 20, 26, 24]
    solutions = np.linalg.inv(x).dot(y)
    parametric = np.array(solutions)
    answer = parametric.reshape(5,1)
    print('x = {}'.format(answer))

question_15()
```

```
x = [[-1.89423963]
 [ 0.98974654]
 [10.81797235]
 [-1.05760369]
 [ 1.61059908]]
```