MATH4570 HW1

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1 Problem 1

1.1 Part 1: $\mathbb{Q} + \mathbb{Q}\sqrt{2}$ is a field

I prove below that $\mathbb{Q} + \mathbb{Q}\sqrt{2}$ is a ring with 1 since it is an abelian group with addition, has a multiplicative identity, is associative with multiplication, and is distributive.
To see that every non-zero element has a multiplicative inverse, we say that element $a + b\sqrt{2}$ has inverse $\frac{1}{a+b\sqrt{2}}$. Multiplying top and bottom by $a - b\sqrt{2}$, we get $\frac{a-b\sqrt{2}}{a^2-2b^2} = \frac{a}{a^2-2b^2} + \frac{-b}{a^2-2b^2}\sqrt{2}$.
We know $c = a^2 - 2b^2$ is rational since it is a combination of rational numbers using addition and multiplication. Thus we know $\frac{a}{a^2-2b^2}$ and $\frac{-b}{a^2-2b^2}$ are rational. So $(a + b\sqrt{2})^{-1}$ is in the set. So every element has a multiplicative inverse, and $\mathbb{Q} + \mathbb{Q}\sqrt{2}$ is a field.

1.1.1 $(\mathbb{Q} + \mathbb{Q}\sqrt{2}, +)$ is an abelian group

First, the set is a monoid since it has an identity and is associative. 0 is the identity, as we can trivially see $0 + a + b\sqrt{2} = a + b\sqrt{2} + 0 = a + b\sqrt{2}$, as 0 is also an identity for the rational numbers and real numbers. For associativity, we can see that
$(a + b\sqrt{2} + c + d\sqrt{2}) + e + f\sqrt{2}$
$= (a + c) + (b + d)\sqrt{2} + e + f\sqrt{2}$
$= (a + c + e) + (b + d + f)\sqrt{2}$
$= a + b\sqrt{2} + (c + e) + (d + f)\sqrt{2}$
$= a + b\sqrt{2} + (c + d\sqrt{2} + e + f\sqrt{2})$
since $\mathbb{Q}$ and $\mathbb{R}$ are also associative and distributive with multiplication. So $(\mathbb{Q} + \mathbb{Q}\sqrt{2}, +)$ is a monoid.
Every member $a + b\sqrt{2}$ in the set has an additive inverse $-a - b\sqrt{2}$, as it is its inverse in $\mathbb{R}$ as well. $-a$ and $-b$ are also rational, so the inverse is also in the set. So $(\mathbb{Q} + \mathbb{Q}\sqrt{2}, +)$ is a group. As addition is commutative in $\mathbb{R}$ and the set is a subset of $\mathbb{R}$, addition is also commutative in the set. So $(\mathbb{Q} + \mathbb{Q}\sqrt{2}, +)$ is an abelian group.

1.1.2 Multiplicative identity

1 is the multiplicative identity in $(\mathbb{Q}+\mathbb{Q}\sqrt{2},+)$ since it is also the multiplicative identity in $\mathbb{R}$, the superset of $(\mathbb{Q}+\mathbb{Q}\sqrt{2},+)$. 1 is in $(\mathbb{Q}+\mathbb{Q}\sqrt{2},+)$ since $1 = 1+0\sqrt{2}$

1.1.3 Multiplicative associativity

Since multiplication is associative in $\mathbb{R}$, the superset of $(\mathbb{Q}+\mathbb{Q}\sqrt{2},+)$, it is also associative in $(\mathbb{Q}+\mathbb{Q}\sqrt{2},+)$.

1.1.4 Distributivity

Since multiplication and addition are distributive in $\mathbb{R}$, the superset of $(\mathbb{Q}+\mathbb{Q}\sqrt{2},+)$, they are also distributive in $(\mathbb{Q}+\mathbb{Q}\sqrt{2},+)$.

1.2 Part 2: $\mathbb{R}+\mathbb{R}i$ is a field

1.2.1 $(\mathbb{R}+\mathbb{R}i,+)$ is an abelian group

First, the set is a monoid since it has an identity and is associative. 0 is the identity, as $0+a+bi = a+bi+0 = a+bi$. For associativity, we can see that
$(a+bi+c+di)+e+fi$
$= (a+c)+(b+d)i+e+fi$
$= (a+c+e)+(b+d+f)i$
$= a+bi+(c+e)+(d+f)i$
$= a+bi+(c+di+e+fi)$
since addition and multiplication are distributive. So $\mathbb{R}+\mathbb{R}i$ is a monoid.
Every member $a+bi$ has an additive inverse $-a-bi$, as it simply uses the inverses of the real parts. The inverses of the real parts are also real, so the inverse is also in $\mathbb{R}+\mathbb{R}i$. So $\mathbb{R}+\mathbb{R}i$ is a group. For commutativity, we see that
$a+bi+c+di$
$= a+c+(b+d)i$
$= c+a+(d+b)i$
$= c+di+a+bi.$
So $\mathbb{R}+\mathbb{R}i$ is an abelian group.

1.3 Multiplicative identity

1 or $(1+0i)$ is the multiplicative identity in $\mathbb{R}+\mathbb{R}i$, as we can see that
$1(a+bi) = 1a+1bi = a+bi$.

1.4 Multiplicative associativity

$((a+bi)(c+di))(e+fi)$
$= (ac-bd+(ad+bc)i)(e+fi)$
$= ace-bde-adf-bcf+(ade+bce+acf-bdf)i$

$= (a+bi)(ce-df+(cf+de)i)$
$= (a+bi)((c+di)(e+fi))$

1.5 Distributivity

$(a+bi+c+di)(e+fi)$
$= (a+c+(b+d)i)(e+fi)$
$= (a+c)e-(b+d)f+((b+d)e+(a+c)f)i$
$= ae+ce-bf-df+bei+dei+afi+cfi$
$= ae+afi+bei-bf+ce+cfi+dei-df$
$= (a+bi)(e+fi)+(c+di)(e+fi)$

2 Problem 2

3 Problem 3

+	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

×	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

4 Problem 4

5 Problem 5

A: no
B: yes
C: no
D: yes
E: no

6 Problem 6

$$A+B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

7 Problem 7

$\det(A) = 0(-1-0) - 1(6-t) + t(0+t)$
$= t^2 + t - 6$
$= (t+3)(t-2) = 0$
$t = -3, 2$

8 Problem 8

8.1 Part a

$$\left[\begin{array}{cc|c} 1 & h & 4 \\ 3 & 6 & 8 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & h & 4 \\ 0 & 6-3h & -4 \end{array}\right]$$

$6 - 3h \neq 0$
$3h \neq 6$
$h \neq 2$

8.2 Part b

$$\left[\begin{array}{cc|c} -4 & 12 & h \\ 2 & -6 & -3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -3 & -\frac{h}{4} \\ 2 & -6 & -3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -3 & -\frac{h}{4} \\ 0 & 0 & -3+\frac{h}{2} \end{array}\right]$$

$-3 + \frac{h}{2} = 0$
$\frac{h}{2} = 3$
$h = 6$

9 Problem 9

9.1 3×2

rank 2:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

rank 1:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

rank 0:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

9.2 2×3

rank 2:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

rank 1:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

rank 0:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

9.3 4×1

rank 1:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

rank 0:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

10 Problem 10

$a \in \mathbb{R}$
$b = 0$ if $c = 1$ else $b \in \mathbb{R}$
$c \in \{0, 1\}$
$d = 0$
$e = 0$

11 Problem 11

11.1 rref(A) over $\mathbb{R}$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & -4 & -5 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 7 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & \frac{8}{7} \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & \frac{12}{7} \\ 0 & 1 & 0 & \frac{8}{7} \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \frac{6}{7} \\ 0 & 1 & 0 & \frac{8}{7} \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix}$$

$$\vec{x} = k \begin{bmatrix} -6 \\ -8 \\ -2 \\ 7 \end{bmatrix} \text{ for } k \in \mathbb{R}$$

11.2 rref(A) over $\mathbb{Z}_7$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 6 & 4 & 5 \\ 2 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 3 & 2 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

11.3 Python stuff

In:

```
M = [[1,2,3,4],[1,1,0,2],[2,0,1,2]]
GF2 = galois.GF(2)
GF3 = galois.GF(3)
M2 = GF2([[y % 2 for y in x] for x in M])
GF2.row_reduce(M2)
```

Out:

```
GF([[1, 0, 0, 0],
    [0, 1, 0, 0],
    [0, 0, 1, 0]], order=2)
```

In:

```
M3 = GF3([[y % 3 for y in x] for x in M])
GF3.row_reduce(M3)
```

Out:

```
GF([[1, 0, 0, 0],
    [0, 1, 0, 2],
    [0, 0, 1, 2]], order=3)
```

11.4 Part 4

Yes

12 Problem 12

12.1 Part 1: rref($A|\vec{b}$)

$$\left[\begin{array}{ccc|c} 3 & 1 & 4 & 1 \\ 5 & 2 & 6 & 5 \\ 0 & 5 & 2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 3 & 6 \\ 5 & 2 & 6 & 5 \\ 0 & 5 & 2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 3 & 6 \\ 0 & 1 & 5 & 3 \\ 0 & 5 & 2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 3 & 6 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 5 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 3 & 3 & 6 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 3 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 0 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

12.2 Part 2

$$\vec{x} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$$

13 Problem 13

In:

```
M = sympy.Matrix([[3,11,19, -2], [7,23,39, 10], [-4,-3,-2, 6]])
M.rref()
```

Out:

```
(Matrix([
 [1, 0, -1, 0],
 [0, 1,  2, 0],
 [0, 0,  0, 1]]),
 (0, 1, 3))
```

No solution

14 Problem 14

In:

```
sympy.solve((3*x1 + 6*x2 + 9*x3 + 5*x4 + 25*x5 - 53,
             7*x1 + 14*x2 + 21*x3 + 9*x4 + 53*x5 - 105,
             -4*x1 - 8*x2 - 12*x3 + 5*x4 - 10*x5 - 11),
            (x1, x2, x3, x4, x5))
```

Out:

```
{x1: -2*x2 - 3*x3 - 5*x5 + 6, x4: 7 - 2*x5}
```

$$\vec{x} = \begin{bmatrix} -2x_2 - 3x_3 - 5x_5 + 6 \\ x_2 \\ x_3 \\ 7 - 2x_5 \\ x_5 \end{bmatrix}$$

15 Problem 15

In:

```
sympy.solve((2*x1 + 4*x2 + 3*x3 + 5*x4 + 6*x5 - 37,
             4*x1 + 8*x2 + 7*x3 + 5*x4 + 2*x5 - 74,
             -2*x1 - 4*x2 + 3*x3 + 4*x4 - 5*x5 - 20,
             x1 + 2*x2 + 2*x3 - x4 + 2*x5 - 26,
             5*x1 - 10*x2 + 4*x3 + 6*x4 + 4*x5 - 24),
            (x1, x2, x3, x4, x5))
```

Out:

```
{x1: -8221/4340, x2: 8591/8680, x3: 4695/434, x4: -459/434, x5: 699/434}
```

$$\vec{x} = \begin{bmatrix} -\frac{8221}{4340} \\ \frac{8591}{8680} \\ \frac{4695}{434} \\ -\frac{459}{434} \\ \frac{699}{434} \end{bmatrix}$$

16 Problem 16

16.1 Part 1

Yes.
$A^{-1} = BC$
$B^{-1} = CA$
$C^{-1} = AB$

16.2 Part 2

Yes
$AB(AB)^{-1} = I$
$A^{-1} = B(AB)^{-1}$
$(AB)^{-1}AB = I$
$B^{-1} = (AB)^{-1}A$

17 Problem 17

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$(AB)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^2B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

18 Problem 18

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

19 Problem 19

19.1 Part 1

Symmetric:

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 6 \\ 3 & 5 & 7 & 8 \\ 4 & 6 & 8 & 9 \end{bmatrix}$$

Skew-symmetric:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & -2 & 3 \\ -1 & 0 & -4 & 5 \\ 2 & 4 & 0 & -6 \\ -3 & -5 & 6 & 0 \end{bmatrix}$$

19.2 Part 2

The main diagonal of skew-symmetric matrices must consist of 0s.

19.3 Part 3

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

19.4 Part 4

19.4.1 $A + A^T$ symmetric

Call the resulting matric B. The value of every element b_{ij} will be $a_{ij} + a_{ji}$. The value of b_{ji} would be $a_{ji} + a_{ij} = b_{ij}$. Thus, the value at every position (i, j) of B^T would be $b_{ji} = b_{ij}$. So B is symmetric.

19.4.2 $AA^T symmetric$

Call the resulting matrix B. The value of every element b_{ij} will be
$a_{i1}a^T_{1j} + a_{i2}a^T_{2j} + \ldots + a_{in}a^T_{nj} = a_{i1}a_{j1} + a_{i2}a_{j2} + \ldots + a_{in}a_{jn}$
We can see that the corresponding value b_{ji} will be
$a_{j1}a^T_{1i} + a_{j2}a^T_{2i} + \ldots + a_{jn}a^T_{ni} = a_{j1}a_{i1} + a_{j2}a_{i2} + \ldots + a_{jn}a_{in} = b_{ij}$
So B is symmetric.

19.4.3 $A^T A$ symmetric

We know that $A = (A^T)^T$. Thus this problem is in the same form as the previous, and we know that the result must be symmetric.

19.4.4 $A - A^T$ skew-symmetric

Call the resulting matrix B. Every element b_{ij} will have value $a_{ij} - a_{ji}$, and element b_{ji} will have value $a_{ji} - a_{ij} = -b_{ij}$. So every element in B^T is the additive inverse of its corresponding element in B. So B is skew-symmetric.

19.5 Part 5

We know from previous problems that matrix $A + A^T$ is symmetric and $A - A^T$ is skew-symmetric for any square matrix A. We can see that their sum is $2A$. Thus, any matrix B can be written as the sum of symmetric matrix $\frac{1}{2}B + \frac{1}{2}B^T$ and skew-symmetric matrix $\frac{1}{2}B - \frac{1}{2}B^T$ as we can see by setting A as $\frac{1}{2}B$.

20 Problem 20

a. none
b. bijective
c. surjective
d. injective

21 Problem 21

22 Problem 22

$$\begin{bmatrix} q_1 & r_1 & 0 & 0 \\ p_1 & q_2 & r_2 & 0 \\ 0 & p_2 & q_3 & r_3 \\ 0 & 0 & p_3 & q_4 \end{bmatrix} \rightarrow \begin{bmatrix} q_1 & r_1 & 0 & 0 \\ 0 & q_2 - \frac{p_1 r_1}{q_1} & r_2 & 0 \\ 0 & 0 & q_3 - \frac{p_2 r_2}{q_2 - \frac{p_1 r_1}{q_1}} & r_3 \\ 0 & 0 & p_3 & q_4 \end{bmatrix}$$

23 Problem 23

24 Problem 24

24.1 H_n symmetric

Let $B = 2\vec{u}\vec{u}^T$. The value of every element b_{ij} will be
$u_{i1}u_{1j}^T = u_{i1}u_{j1}$
We can see that the corresponding value b_{ji} will be
$u_{j1}u_{1i}^T = u_{j1}u_{i1} = u_{i1}u_{j1} = b_{ij}$
So B is symmetric.
The sum of symmetric matrices is symmetric. Say $C = A + B$, where A and B are symmetric. Then for every c_{ij} in C, $c_{ij} = a_{ij} + b_{ij} = a_{ji} + b_{ji} = c_{ji}$. So C is also symmetric.
So H_n is symmetric, since I_n and $2\vec{u}\vec{u}^T$ are symmetric.

24.2 H_n orthogonal

$H^T H = HH$
$= (I_n - 2\vec{u}\vec{u}^T)(I_n - 2\vec{u}\vec{u}^T)$
$= I_n - 4\vec{u}\vec{u}^T + 4\vec{u}\vec{u}^T\vec{u}\vec{u}^T$
$= I_n - 4\vec{u}\vec{u}^T + 4\vec{u}(\vec{u}^T\vec{u})\vec{u}^T$
$= I_n - 4\vec{u}\vec{u}^T + 4\vec{u}\vec{u}^T$
$= I_n$

24.3 H_n^2

$H_n^2 = H^T H = I_n$

24.4 H_3 and H_4

$$H_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$H_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$