(sum)

$Q_1$: (1) Identity: $a_1+b_1\sqrt{2}+a_2+b_2\sqrt{2}=(a_1+a_2)+(b_1+b_2)\sqrt{2}$

let $a_2=0,\ b_2=0$ $\quad =(a_1+0)+(b_1+0)\sqrt{2}=a_1+b_1\sqrt{2}$

sum(+) have Identity cause 0 is a rational number.

Associativity: $(a_1+b_1\sqrt{2}+a_2+b_2\sqrt{2})+a_3+b_3\sqrt{2}=(a_1+a_2)+(b_1+b_2)\sqrt{2}+a_3+b_3\sqrt{2}$

$=(a_1+a_2+a_3)+(b_1+b_2+b_3)\sqrt{2}=a_1+b_1\sqrt{2}+(a_2+a_3)+(b_2+b_3)\sqrt{2}=a_1+b_1\sqrt{2}+(a_2+b_2\sqrt{2}+a_3+b_3\sqrt{2})$

Inverse: since $a, b$ are rational numbers, $-a, -b$ are rational numbers.

$a+b\sqrt{2}+(-a)+(-b)\sqrt{2}=(a-a)+(b-b)\sqrt{2}=(-a)+(-b)\sqrt{2}+a+b\sqrt{2}=0$

Commutativity: $a_1+b_1\sqrt{2}+a_2+b_2\sqrt{2}=(a_1+a_2)+(b_1+b_2)\sqrt{2}=a_2+b_2\sqrt{2}+a_1+b_1\sqrt{2}$

(multiplicative)

Identity: $(a_1+b_1\sqrt{2})\cdot(a_2+b_2\sqrt{2})=a_1a_2+b_1a_2\sqrt{2}+b_2a_1\sqrt{2}+b_1b_2(\sqrt{2})^2$

let $a_2=1$ $b_2=0$ $\quad =a_1\cdot 1+b_1\cdot 1\cdot\sqrt{2}=a_1+b_1\sqrt{2}$

So $1+0\sqrt{2}$ is Identity.

Associative: $((a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2}))(a_3+b_3\sqrt{2})=(a_1a_2+b_1a_2\sqrt{2}+a_1b_2\sqrt{2}+b_1b_2(\sqrt{2})^2)(a_3+b_3\sqrt{2})$

$=a_1a_2a_3+b_1a_2a_3\sqrt{2}+a_1a_3b_2\sqrt{2}+b_1b_2a_3(\sqrt{2})^2+a_1a_2b_3(\sqrt{2})^2+b_1b_3a_2(\sqrt{2})^2+a_1b_2b_3(\sqrt{2})^2+b_1b_2b_3(\sqrt{2})^3$

$=(a_1+b_1\sqrt{2})(a_2a_3+b_2a_3\sqrt{2}+a_2b_3\sqrt{2}+b_2b_3(\sqrt{2})^2)$

$=(a_1+b_1\sqrt{2})((a_2+b_2\sqrt{2})(a_3+b_3\sqrt{2}))$

So $\times$ operation is associative

Distributivity: $(a_1+b_1\sqrt{2})\times((a_2+b_2\sqrt{2})+a_3+b_3\sqrt{2})=a_1(a_2+a_3)+a_1(b_2+b_3)\sqrt{2}+b_1(a_2+a_3)\sqrt{2}+b_1(b_2+b_3)(\sqrt{2})^2$

$=(a_1a_2+a_1b_2\sqrt{2}+b_1a_2\sqrt{2}+b_1b_2(\sqrt{2})^2)+(a_1a_3+a_1b_3\sqrt{2}+b_1a_3\sqrt{2}+b_1b_3(\sqrt{2})^2)$

$=(a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2})+(a_1+b_1\sqrt{2})(a_3+b_3\sqrt{2})$

Since product have commutativity (prove below) $=(a_2+b_2\sqrt{2})(a_1+b_1\sqrt{2})+(a_3+b_3\sqrt{2})(a_1+b_1\sqrt{2})$

$=((a_2+b_2\sqrt{2})+(a_3+b_3\sqrt{2}))(a_1+b_1\sqrt{2})$

Commutativity: $(a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2})=a_1a_2+b_1b_2(\sqrt{2})^2+a_1b_2\sqrt{2}+a_2b_1\sqrt{2}$

$=(a_2+b_2\sqrt{2})(a_1+b_1\sqrt{2})$

Inverse: $(a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2})=(1+0\sqrt{2})$

let $a_2=\frac{a_1}{a_1^2-2b_1^2}$, $b_2=\frac{b_1}{2b_1^2-a^2}$, $a_2, b_2$ are rational numbers

since $a_1, b_1$ are rational numbers.

Therefore, we prove it is a field.

(2) Sum:

Identity. $(a+b\sqrt{-1})+(0+0\sqrt{-1})=a+b\sqrt{-1}=(0+0\sqrt{-1})+(a+b\sqrt{-1})$

since 0 is a real number, $e=0+0\sqrt{-1}$

associativity: $((a_1+b_1\sqrt{-1})+(a_2+b_2\sqrt{-1}))+(a_3+b_3\sqrt{-1})=(a_1+a_2+(b_1+b_2)\sqrt{-1})+(a_3+b_3\sqrt{-1})$

$=a_1+a_2+a_3+(b_1+b_2+b_3)\sqrt{-1}=a_1+b_1\sqrt{-1}+(a_2+a_3+(b_2+b_3)\sqrt{-1})$

$=a_1+b_1\sqrt{-1}+((a_2+b_2\sqrt{-1})+(a_3+b_3\sqrt{-1}))$

Inverse: $(a_1+b_1\sqrt{-1})+(-a_1+(-b_1)\sqrt{-1})=0+0\sqrt{-1}=e$

since $a_1, b_1$ are real numbers. $-a_1, -b$ also the real number.

So $-a+(-b)\sqrt{-1}$ in the set if $a+b\sqrt{-1}$ in the set.

multiplicative

Identity: $(a_1+b_1\sqrt{-1})(a_2+b_2\sqrt{-1})$ let $a_2=1$ $b_2=0$

$=(a_1+b_1\sqrt{-1})(1+0\sqrt{-1})=a_1+b_1\sqrt{-1}=(1+0\sqrt{-1})(a_1+b_1\sqrt{-1})$ So $e=(1+0\sqrt{-1})$

Associativity: $((a_1+b_1\sqrt{-1})(a_2+b_2\sqrt{-1}))(a_3+b_3\sqrt{-1})=(a_1a_2-b_1b_2+a_1b_2\sqrt{-1}+a_2b_1\sqrt{-1})(a_3+b_3\sqrt{-1})$

$=(a_1+b_1\sqrt{-1})(a_2a_3-b_2b_3+a_2b_3\sqrt{-1}+a_3b_2\sqrt{-1})$

$=(a_1+b_1\sqrt{-1})((a_2+b_2\sqrt{-1})(a_3+b_3\sqrt{-1}))$

distributivity: $(a_1+b_1\sqrt{-1})((a_2+b_2\sqrt{-1})+(a_3+b_3\sqrt{-1}))=$

$(a_1(a_2+a_3)+b_1(b_2+b_3)(-1)+a_1(b_2+b_3)\sqrt{-1}+b_1(a_2+a_3)\sqrt{-1}$

$=(a_1a_2+b_1b_2(-1)+a_1b_2\sqrt{-1}+a_2b_1\sqrt{-1})+$

$(a_1a_3+b_1b_3(-1)+a_1b_3\sqrt{-1}+b_1a_3\sqrt{-1})$

$=(a_1+b_1\sqrt{-1})(a_2+b_2\sqrt{-1})+(a_1+b_1\sqrt{-1})(a_3+b_3\sqrt{-1})$

Commutativity $(a_1+b_1\sqrt{-1})(a_2+b_2\sqrt{-1})=a_1a_2+b_1b_2(-1)+a_1b_2\sqrt{-1}+b_1a_2\sqrt{-1}$

$=(a_2+b_2\sqrt{-1})(a_1+b_1\sqrt{-1})$

inverse: $(a+b\sqrt{-1})(x+y\sqrt{-1})=(1+0\sqrt{-1})$

$x=\frac{a}{a^2+b^2}$, $y=-\frac{b}{b^2+a^2}$

since $a, b\in\mathbb{R}$, $x, y\in\mathbb{R}$ $(a+b\sqrt{-1})^{-1}=(\frac{a}{a^2+b^2}-\frac{b}{b^2+a^2}\sqrt{-1})$

So it is a field

Q12:

(1) $\left[\begin{array}{ccc|c} 3 & 1 & 4 & 1 \\ 5 & 2 & 6 & 5 \\ 0 & 5 & 2 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 3 & 1 & 4 & 1 \\ 0 & \frac{1}{3} & -\frac{2}{3} & \frac{10}{3} \\ 0 & 5 & 2 & 1 \end{array}\right]$

$= \left[\begin{array}{ccc|c} 3 & 1 & 4 & 1 \\ 0 & \frac{1}{3} & -\frac{2}{3} & \frac{10}{3} \\ 0 & 0 & 12 & -49 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & & & \frac{31}{6} \\ & 1 & & \frac{11}{6} \\ & & 1 & -\frac{49}{12} \end{array}\right]$

(2) $x_1 = \frac{31}{6} \quad x_2 = \frac{11}{6} \quad x_3 = -\frac{49}{12}$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Q14 (1) symmetric: 2x2 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 3x3 $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 3 \end{bmatrix}$

4x4 $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

skew-symmetric: 2x2: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ 3x3: $\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$

4x4 $\begin{bmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & 2 & 3 \\ -2 & -2 & 0 & 3 \\ -3 & -3 & -3 & 0 \end{bmatrix}$

(2) all element on the main diagonal is 0

(3) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(4) $A + A^t = \begin{bmatrix} a_{11}+a_{11} & \cdots & a_{1n}+a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n}+a_{n1} & \cdots & a_{nn}+a_{nn} \end{bmatrix}$ for any $i, j \le n$ the element at $(i,j) = a_{ij} + a_{ji}$

So it is symmetric

$$AA^T = \begin{bmatrix} \sum_{i=1}^{n} a_{1i}a_{i1} & \cdots & \sum_{i=1}^{n} a_{ni}a_{i1} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{n} a_{ni}a_{i2} & \cdots & \sum_{i=1}^{n} a_{ni}a_{in} \end{bmatrix}$$

$A^TA$. $AA^T$ is symmetric $= A^TA$

Since $(AA^T)_{ij} = (AA^T)_{ji}$, $(A^TA)_{ij} = (A^TA)_{ij}$

$$A - A^T = \begin{bmatrix} a_{11}-a_{11} & \cdots & a_{1n}-a_{n1} \\ \vdots & & \\ a_{n1}-a_{1n} & & a_{nn}-a_{nn} \end{bmatrix}$$

since all elements on the main diagonal is 0 and $(A-A^T)_{ij} = -(A-A^T)_{ji}$

So it is skew-symmetric

(5) Since $A+A^T$ is a symmetric matrix

and $A-A^T$ is a skew-symmetric matrix

any $n\times n$ matrix $A$ can generate those matrices.

$\frac{1}{2}((A+A^T)+(A-A^T)) = A.$

So we just add $\frac{1}{2}(A+A^T)$ and $\frac{1}{2}(A-A^T)$

Q23:

$$A = \left(\begin{bmatrix} 1 & & & \\ -\frac{1}{4} & \ddots & & \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & \frac{1}{4} & 1 \end{bmatrix}\right)^{-1} \begin{bmatrix} 4 & 1 & \cdots & 0 \\ & \frac{15}{4} & \ddots & \vdots \\ & & \ddots & 1 \\ & & & \frac{15}{4} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & & & \\ \frac{1}{4} & \ddots & & \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & \frac{1}{4} & 1 \end{bmatrix} \quad U = \begin{bmatrix} 4 & 1 & \cdots & 0 \\ & \frac{15}{4} & \ddots & \vdots \\ & & \ddots & 1 \\ & & & \frac{15}{4} \end{bmatrix}$$

$Q_2$: Let matrices $A, B \in R^{n\times n}$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix} = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} a_{ij} b_{ji} \right)$$

$$\neq \sum_{i=1}^{n} \left( \sum_{j=1}^{n} a_{ji} b_{ij} \right) = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

Therefore, the multiplication have not commutativity

so $(R^{n\times n}, +, \cdot)$ is not a field by field definition

Q3:

| + | [0] | [1] | [2] |
|---|---|---|---|
| [0] | [0] | [1] | [2] |
| [1] | [1] | [2] | [0] |
| [2] | [2] | [0] | [1] |

| × | [0] | [1] | [2] |
|---|---|---|---|
| [0] | [0] | [0] | [0] |
| [1] | [0] | [1] | [2] |
| [2] | [0] | [2] | [1] |

Q4: Sum  Let $(a_i + ib_i) \in C$, $i = 1, 2, 3$

Identity: $(a_1 + ib_1) + (0 + i0) = a_1 + ib_1$

So $e = (0 + 0i) \in C$

associativity: $((a_1 + ib_1) + (a_2 + ib_2)) + (a_3 + ib_3)$

$= (a_1 + a_2) + i(b_1 + b_2) + (a_3 + ib_3)$

$= (a_1 + a_2 + a_3) + i(b_1 + b_2 + b_3)$

$= (a_1 + ib_1) + (a_2 + a_3 + i(b_2 + b_3))$

$= (a_1 + ib_1) + ((a_2 + ib_2) + (a_3 + ib_3))$

inverse: $(a + ib) + (-a + i(-b)) = 0 = (-a + i(-b)) + (a + ib)$

$(-a + i(-b)) \in C$ since $(a + ib) \in C$  $(a + ib)^{-1} = -a + i(-b)$

commutativity: $(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$

$= (a_2 + ib_2) + (a_1 + ib_1)$

multiplicative:

identity  $(a_1 + ib_1)(1 + i0) = a_1 + ib_1 = (1 + i0)(a_1 + ib_1)$

so $e = (1 + i0)$

associative: $((a_1 + ib_1)(a_2 + ib_2))(a_3 + ib_3) = (a_1a_2 - b_1b_2 + a_1b_2i +$

$a_2b_1i)(a_3 + ib_3) = (a_1 + ib_1)(a_2a_3 - b_2b_3 + a_2b_3i + a_3b_2i$

$= (a_1+ib_1)((a_2+ib_2)(a_3+ib_3))$

distributivity: $(a_1+ib_1)((a_2+ib_2)+(a_3+ib_3)) = (a_1+ib_1)((a_2+a_3)+i(b_2+b_3))$

$= (a_1a_2+a_1b_2i+a_2b_1i-b_1b_2)+(a_1a_3+a_1b_3i+a_3b_1i-b_1b_3)$

$= (a_1+ib_1)(a_2+ib_2)+(a_1+ib_1)(a_3+ib_3)$

Commutativity: $(a_1+ib_1)(a_2+ib_2) = a_1a_2+ib_1a_2+ia_1b_2-b_1b_2$

$= (a_2+ib_2)(a_1+ib_1)$

inverse: $(a+ib)(x+iy) = 1+i0$

$x = \frac{a}{a^2+b^2}$ $y = -\frac{b}{b^2+a^2}$

so $(a+ib)^{-1} = \left(\frac{a}{a^2+b^2}+i\frac{b}{b^2+a^2}\right)$

Q5: B D

Q6: $A+B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Q7: $A = \begin{bmatrix} 6 & -1 & 1 \\ 0 & 1 & t \\ t & 0 & 1 \end{bmatrix}$ if A have an inverse then A is full rank

i.e. $\begin{cases} t-6x=0 \\ 1-x \neq 0 \end{cases}$ $t \neq 6$

Q8: (1) $\left[\begin{array}{cc|c} 1 & h & 4 \\ 3 & 6 & 8 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & h & 4 \\ 0 & 6-3h & -4 \end{array}\right]$

If matrix consistent then it is full rank

i.e. $6-3h \neq 0 \quad h \neq 2$

(2) $\left[\begin{array}{cc|c} -4 & 12 & h \\ 2 & -6 & -3 \end{array}\right] = \left(\begin{array}{cc|c} 2 & -6 & -3 \\ 0 & 0 & h-6 \end{array}\right)$

$h-6=0 \qquad h=6$

Q9: (1) 3 types, leading 1's at: [(1,1) (2,2)], [(1,1)], [(1,2)]

(2) 7 types, leading 1's at [(1,1), (2,2)], [(1,1), (2,3)], [(1,1)], [(1,2), (2,3)], [(1,2)], [(1,3)] [(1,3)]

(3) 1 type, [(1,1)]

Q10 $a=b=c=d=e=0$

Q11: (1) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -2 \\ 0 & -4 & -5 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -2 \\ 0 & 0 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & \frac{22}{7} \\ 0 & -1 & 0 & -\frac{8}{7} \\ 0 & 0 & 7 & 2 \end{bmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 & \frac{6}{7} \\ 0 & 1 & 0 & \frac{8}{7} \\ 0 & 0 & 1 & \frac{2}{7} \end{pmatrix} \qquad \vec{X} = \begin{bmatrix} \frac{6}{7}x \\ \frac{8}{7}x \\ \frac{2}{7}x \end{bmatrix}$

(2) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Q11

(3). M = Matrix([[1, 0, 1], [2, -1, 3], [4, 3, 2]])

M.rref()

M % 3

Out: 1 2 0 1

1 1 0 2

2 0 1 2

M = Matrix([[1, 0, 1], [2, -1, 3], [4, 3, 2]])

M.rref()

M % 2

Out: 1 0 1 0

1 1 0 0

0 0 1 0

Q13

M = Matrix([[3, 11, 19, -2], [7, 23, 39, 10], [-4, -3, -2, 6]])

M.rref()

Out: 1 0 -1 0

0 1 2 0

0 0 0 1

So x = [x3, -2*x3, x3]

Q14

M = Matrix([[3, 6, 9, 5, 25, 53], [7, 14, 21, 9, 53, 105], [-4, -8, -12, 5, -10, 11]])

M.rref()

Out: 1 2 3 0 5 6

0 0 0 1 2 7

0 0 0 0 0 0

So X =[6 – 2*x2 -3 * x3 – 5 *x5, x2, 7 -2 * x5, x4, x5]

Q15

M = Matrix([[2, 4, 3, 5, 6, 37], [4, 8, 7, 5, 2, 74], [-2, -4, 3, 4, -5, 20], [1, 2, 2, -1, 2, 26], [5, -10, 4, 6, 4, 24]])

M.rref()

Out:

```
[1, 0, 0, 0, 0, -8221/4340],
 [0, 1, 0, 0, 0,  8591/8680],
 [0, 0, 1, 0, 0,   4695/434],
 [0, 0, 0, 1, 0,   -459/434],
 [0, 0, 0, 0, 1,    699/434]
```

X = [-8221/4340, 8591/868, 4695/434, -459/434, 699/434]

Q16: (1) $n = \text{rank}(ABC) \leq \min(\text{rank}(A), \text{rank}(B), \text{rank}(C))$

So $A, B, C$ are invertible

$A^{-1} = CB^{-1}$ $\quad B^{-1} = A^{-1}C^{-1}$ $\quad C^{-1} = BA^{-1}$

(2) $A, B$ are invertible, since $\text{rank}(AB) = n$

$= \min(\text{rank}(A), \text{rank}(B))$ so $\text{rank}(A) = \text{rank}(B) = n$

i.e. $A, B$ are full rank. So $A, B$ are invertible

Q17: $\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right)^2 = \left(\begin{bmatrix} 3 & 0 \\ 7 & 0 \end{bmatrix}\right)^2 = \begin{bmatrix} 9 & 0 \\ 21 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^2 \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 17 & 0 \\ 37 & 0 \end{bmatrix}$

Q18: $\begin{bmatrix} 1 & 0 & 1 \\ & 1 & 0 \\ & & 1 \end{bmatrix}^T \rightarrow \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 1 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$

Q20. injective (a) $x \geq 0$ (b) $x \in \mathbb{R}$ (c) $x \in \mathbb{R}$ (d) $x \in \mathbb{R}$

surjective: (a) $x \in \mathbb{R}$ (b) $x \in \mathbb{R}$ (c) $x \in \mathbb{R}$ (d) $x \in \mathbb{R}$

bijective: (a) $x \geq 0$ (b) $x \in \mathbb{R}$ (c) $x \in \mathbb{R}$ (d) $x \in \mathbb{R}$

Q21 $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ 0 & \frac{1}{4} & 1 & 0 \\ 0 & 0 & \frac{1}{4} & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 & 0 & 0 \\ & \frac{15}{4} & 1 & 0 \\ & & \frac{15}{4} & 1 \\ & & & \frac{15}{4} \end{bmatrix}$ $\quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ 0 & \frac{1}{4} & 1 & 0 \\ 0 & 0 & \frac{1}{4} & 1 \end{bmatrix}$

$R = \begin{bmatrix} 4 & 1 & 0 & 0 \\ & \frac{15}{4} & 1 & 0 \\ & & \frac{15}{4} & 1 \\ & & & \frac{15}{4} \end{bmatrix}$

Q22 $L \cdot U = \begin{bmatrix} d_1 & u_1 & 0 & 0 \\ l_1 d_1 & u_1 l_1 + d_2 & u_2 & 0 \\ 0 & l_2 d_2 & l_2 u_2 + d_3 & u_3 \\ 0 & 0 & l_3 d_3 & l_3 u_3 + d_4 \end{bmatrix} = \begin{bmatrix} q_1 & r_1 & 0 & 0 \\ p_1 & q_2 & r_2 & 0 \\ 0 & p_2 & q_3 & r_3 \\ 0 & 0 & p_3 & q_4 \end{bmatrix}$

$p_n = l_n d_n \quad q_n = d_n + u_{n-1} l_{n-1}$

$u_n = r_n$

Q24: (1) $H^T = (I_n - 2\vec{v}\vec{v}^T)^T = I_n^T - 2\vec{v}^{T\,T}\vec{v}^T$

$= I_n - 2\vec{v}\vec{v}^T$

So $H$ is symmetric

(2) $H_n^T H_n = (I_n - 2\vec{v}\vec{v}^T)^2$

$= I_n - 4\vec{v}\vec{v}^T + 4(\vec{v}\vec{v}^T)^2 = I_n$

(3) $H_n^2 = H_n^T H = I_n$

(4) $H_n = I_n - 2\vec{v}\vec{v}^T$

$H_n\vec{v} = I_n - 2\vec{v}$

(5) $H_3 = I_3 - \frac{2}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}(1\ 1\ 1) = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$

$H_4 = I_4 - \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}(1\ 1\ 1\ 1) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$