<u>MATH 7241: Probability 1</u> FALL 2020 TEST 2

Important:

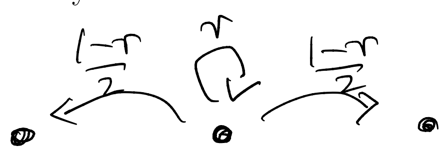
- This Test will be available at **6pm on Tuesday December 1. You must start the Test at 6pm**.
- This Test must be **completed within 2 hours** – you will not be able to upload your answer after that time.
- You must **upload your answer as a pdf file**. Photos, jpg files etc will not be accepted. You may wish to install and use a **scanner app on your phone**.
- You must put your **full name and student ID** at the top of your answer.
- Send me an email if you have any questions or encounter any problems.
- You may use any material from the class, including notes, problem sets and recordings. **You may not access material from any other source, and you may not discuss these problems with anyone until they have been submitted**.

Questions:

1) Consider the following extension of the Gambler's Ruin Problem: a random walk on the integers $\{0, 1 \ldots, N\}$ with, at every step, probability r to remain at the same position and probability $(1-r)/2$ to jump right and $(1-r)/2$ to jump left, and with absorbing states at 0 and N. For each $k \in \{0, 1, \ldots, N\}$, let M_k be the expected number of steps, starting at $X_0 = k$, until the walk reaches either 0 or N. Assume that $0 \le r < 1$.

a). By conditioning on X_1, derive a recursion formula for M_k.

b). The general solution of the recursion formula from part (a) is $g(k) = A + Bk + Ck^2$ where A, B, C are constants. Find the unique values of A, B, C for which $M_k = g(k)$. [Hint: use the boundary conditions at $k = 0$ and $k = N$. Your answer will depend on r].



$$M_k = \mathbb{E}[\#\text{ steps to absorb} \mid X_0 = k]$$

a) Condition on X_1:

$$M_k = r(1+M_k) + \frac{1-r}{2}(1+M_{k+1}) + \frac{1-r}{2}(1+M_{k-1})$$

$$\Rightarrow M_k = \frac{1}{1-r} + \frac{1}{2}M_{k+1} + \frac{1}{2}M_{k-1}$$

b) Try $M_k = A + Bk + Ck^2$.

B.C. $M_0 = 0 \Rightarrow 0 = A$

$M_N = 0 \Rightarrow 0 = BN + CN^2$

$\Rightarrow B = -NC.$

Substitute in (a):

$$RHS = \frac{1}{1-r} + \frac{1}{2}\big(B(k+1) + C(k+1)^2 + B(k-1) + C(k-1)^2\big)$$

$$= \frac{1}{1-r} + \frac{1}{2}\left(2Bk + 2Ck^2 + 2C\right)$$

$$= \frac{1}{1-r} + Bk + Ck^2 + C$$

$$LHS = Bk + Ck^2$$

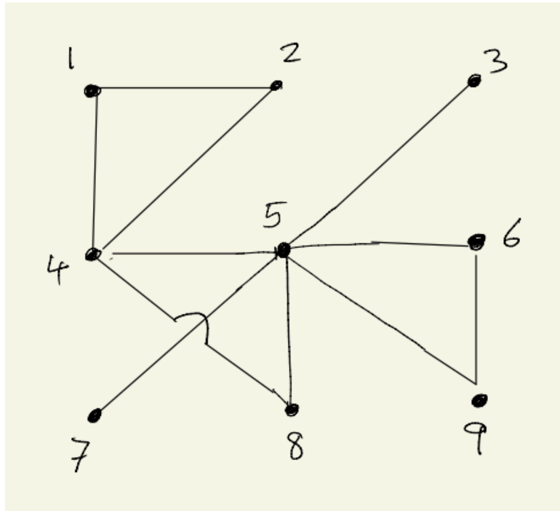
$\Rightarrow$ choose $C = -\frac{1}{1-r}$

$$\Rightarrow \quad B = \frac{N}{1-r}.$$

$$\Rightarrow \quad M_k = \frac{N}{1-r}k - \frac{1}{1-r}k^2$$

$$= \frac{1}{1-r}k(N-k).$$

2)



Consider a random walk on the graph shown above. At each step the walker randomly jumps along an edge to a neighboring vertex. Let $d(i)$ denote the number of edges at vertex i, then the transition matrix is

$$p_{i,j} = \begin{cases} \dfrac{1}{d(i)} & \text{if there is an edge from } i \text{ to } j \\ \\ 0 & \text{otherwise} \end{cases}$$

a). Find the stationary distribution of the chain. [Hint: use the same method as the chessboard example].

b). Find the mean return time to vertex number 1.

c). You can add one extra edge from vertex number 5 to any other vertex (double edges are allowed). Which choice of new edge will cause the largest reduction in the mean return time to vertex 1?

a) Random walk on graph

$$\Rightarrow \quad w_j = \frac{d_j}{\sum_k d_k} = \frac{d_j}{22}$$

b) $$\mu_1 = \frac{1}{w_1} = \frac{22}{2} = 11$$

Case 1: add edge from 5 to another vertex (not 1).

$$\Rightarrow \mu_1' = \frac{24}{2} = 12$$

Case 2: add edge between 5 and 1

$$\Rightarrow \mu_1'' = \frac{24}{3} = 8$$

$\Rightarrow$ largest reduction is case 2.

3). Let $X_1, X_2, \ldots$ be IID random variables, where the moment generating function is

$$\mathrm{E}[e^{tX}] = e^{2e^{3t}-2}$$

a). Find the mean $\mu = \mathrm{E}[X]$.
b). Let $Y_n = (1/n)\sum_{i=1}^n X_i$. Use Cramer's Theorem to compute

$$\lim_{n\to\infty} \frac{1}{n} \log \mathrm{P}(Y_n > 12)$$

a) $\mu = \mathbb{E}[X] = \frac{d}{dt}\mathbb{E}[e^{tX}]\Big|_{t=0}$

$= e^{2e^{3t}-2}\left(6e^{3t}\right)\Big|_{t=0}$

$= 6$

b) $\Lambda(t) = \ln \mathbb{E}[e^{tX}] = 2e^{3t}-2$

$\Lambda^*(x) = \sup_t \{xt - \Lambda(t)\}$.

Solve $\frac{d}{dt}(xt - \Lambda(t)) = 0$

$\Leftrightarrow \quad x = \Lambda'(t) = 6e^{3t}$

$\Leftrightarrow \quad t^* = \frac{1}{3}\ln\left(\frac{x}{6}\right)$.

$\cancel{4}$

$\Rightarrow \Lambda^*(x) = xt^* - \Lambda(t^*)$

$$= \frac{x}{3}\ln\left(\frac{x}{6}\right) - 2\left(\frac{x}{6}\right) + 2$$

$$= \frac{x}{3}\ln\left(\frac{x}{6}\right) - \frac{x}{3} + 2.$$

$$\Rightarrow \Lambda^*(12) = \frac{12}{3}\ln\left(\frac{12}{6}\right) - \frac{12}{3} + 2$$

$$= 4\ln 2 - 2$$

Cramer's Theorem

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n}\ln \mathbb{P}(Y_n > 12) = -\Lambda^*(12)$$

$$= 2 - 4\ln 2.$$

4). The number of bacteria in an experiment is described by a branching process with the following distribution for number of offspring:

$$p_0 = \mathrm{P}(Z=0) = 1/6, \quad p_1 = \mathrm{P}(Z=1) = 5/12, \quad p_2 = \mathrm{P}(Z=2) = 5/12.$$

a). Find the mean number of offpsring.

b). Calculate the probability of extinction.

c). Three independent copies of this experiment are run. Find the probability that at least one of the populations does not become extinct.

a) $m = \mathbb{E}[Z] = \frac{5}{12} + 2\left(\frac{5}{12}\right) = \frac{15}{12} = \frac{5}{4}$

b) solve $\quad s = \phi(s)$

$\Leftrightarrow \quad s = \frac{1}{6} + \frac{5}{12}s + \frac{5}{12}s^2$

$\Leftrightarrow \quad 5s^2 + 5s - 12s + 2 = 0$

$\Leftrightarrow \quad 5s^2 - 7s + 2 = 0$

$\Leftrightarrow \quad (s-1)(5s-2) = 0$

$\Rightarrow \quad \rho = \frac{2}{5}.$

c) Independent, each has prob. $\rho = \frac{2}{5}$ of extinction.

$\Rightarrow \mathbb{P}(\text{at least one is not extinct})$

$= 1 - \mathbb{P}(\text{all extinct}) = 1 - \left(\frac{2}{5}\right)^3$

5). For a branching process it is known that the number of offspring of each individual can be either 0, 1 or 3 (no other values are possible). Let p_0, p_1, p_3 be the probabilities of $0, 1, 3$ offspring. It is also known that the probability of extinction is $\rho = 1/4$.

a). These probabilities satisfy the linear equation $p_0 + p_1 + p_3 = 1$. Use the value of ρ to find another linear equation satisfied by these probabilities.

b). By eliminating p_3 you can use the two equations from part (a) to find a linear equation satisfied by p_0 and p_1. Use this equation to find the largest possible value of p_0.

a) $\rho = \phi(\rho)$

$$\Rightarrow \frac{1}{4} = p_0 + \frac{1}{4} p_1 + \left(\frac{1}{4}\right)^3 p_3$$

b) $1 = p_0 + p_1 + p_3$

$$\Rightarrow p_3 = 1 - p_0 - p_1$$

$$\Rightarrow \frac{1}{4} = p_0 + \frac{1}{4} p_1 + \left(\frac{1}{4}\right)^3 - \left(\frac{1}{4}\right)^3 p_0 - \left(\frac{1}{4}\right)^3 p_1$$

$$\Rightarrow \frac{1}{4} - \left(\frac{1}{4}\right)^3 = \left(1 - \left(\frac{1}{4}\right)^3\right) p_0 + \left(\frac{1}{4} - \left(\frac{1}{4}\right)^3\right) p_1$$

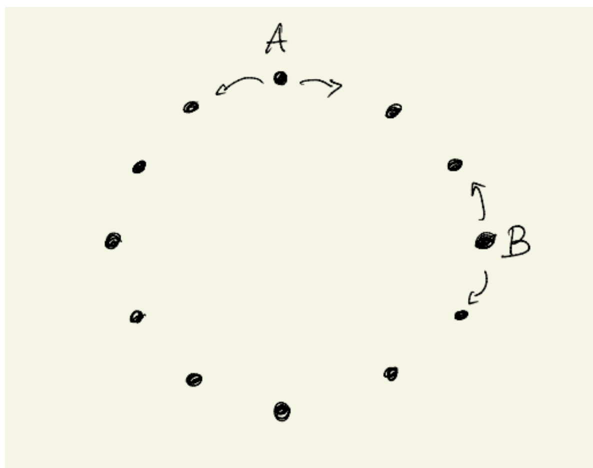
$$\Rightarrow p_0 = \frac{\left(\frac{1}{4} - \left(\frac{1}{4}\right)^3\right)(1 - p_1)}{1 - \left(\frac{1}{4}\right)^3}$$

Max. value $\Leftrightarrow p_1 = 0$

$$\Leftrightarrow p_0^{(max)} = \frac{\frac{1}{4} - \left(\frac{1}{4}\right)^3}{1 - \left(\frac{1}{4}\right)^3}$$

EXTRA CREDIT CHALLENGE: only attempt this if you are bored!!

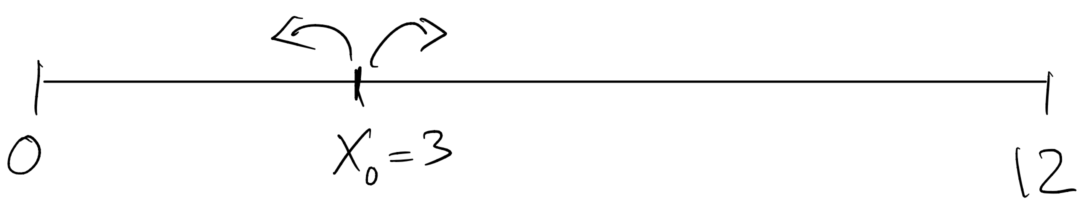
5) Two random walkers start at the points labeled A and B on the circle of 12 points shown below. At each step one of the walkers randomly jumps to one of its two neightboring points, each with probability 1/2, while the other walker remains in its position. Find the expected number of steps until the two walkers meet.



Map onto RW on line:

X_n = separation between A & B

$X_0 = 3$



Absorbing states at 0 and 12

From Q.1, $r = 0$, $N = 12$

$k = 3$

$$\Rightarrow M_3 = 3(12-3) = 27.$$