MATH 4570 HW#1

1. (1) A field is a set F on which the operations addition and multiplication are defined ~~that~~ $\forall (a+b\sqrt{2}), (c+d\sqrt{2}) \in F$ and unique elements $(a+b\sqrt{2})+(c+d\sqrt{2})$ & $(a+b\sqrt{2})\cdot(c+d\sqrt{2})$ in F such that $\forall (a+b\sqrt{2}), (c+d\sqrt{2}), (x+y\sqrt{2}) \in F$:

(i) Commutativity of addition:

$$(a+b\sqrt{2})+(c+d\sqrt{2}) = (c+d\sqrt{2})+(a+b\sqrt{2}) \checkmark$$

(ii) Associativity of addition:

$$(a+b\sqrt{2}+c+d\sqrt{2})+x+y\sqrt{2} = c+d\sqrt{2}+(a+b\sqrt{2}+x+y\sqrt{2}) \checkmark$$

(iii) Identity of addition:

$$(0+0\sqrt{2})+(a+b\sqrt{2}) = (a+b\sqrt{2})+(0+0\sqrt{2}) = a+b\sqrt{2} \checkmark$$

(iv) Inverse of addition:

$$(a+b\sqrt{2})+(-a-b\sqrt{2}) = 0 \checkmark$$

(v) Commutativity of multiplication:

$$((a+b\sqrt{2})\cdot(c+d\sqrt{2})) = ((c+d\sqrt{2})\cdot(a+b\sqrt{2})) \checkmark$$

(vi) Associativity of multiplication:

$$((a+b\sqrt{2})\cdot(c+d\sqrt{2}))\cdot x+y\sqrt{2} = a+b\sqrt{2}((c+d\sqrt{2})\cdot(x+y\sqrt{2})) \checkmark$$

(vii) Identity of multiplication:

$$(a+b\sqrt{2})\cdot(1+0\sqrt{2}) = (1+0\sqrt{2})\cdot(a+b\sqrt{2}) = a+b\sqrt{2} \checkmark$$

(viii) Inverse of multiplication:

$$(a+b\sqrt{2})^{-1} := x+y\sqrt{2} \rightarrow (a+b\sqrt{2})\cdot(x+y\sqrt{2}) = 1 \rightarrow$$

$$\rightarrow (ax+2by)+(ay+bx)\sqrt{2} = 1$$

need to solve $ax+2by=1$, $ay+bx=0$ $\rightarrow A = \begin{bmatrix} a & 2b & 1 \\ b & a & 0 \end{bmatrix}$

using python: $x = \frac{a}{a^2-2b^2}$ & $y = \frac{b}{a^2-2b^2}$ $\therefore (a+b\sqrt{2})^{-1} = \left(\frac{a}{a^2-2b^2} - \frac{b}{a^2-2b^2}\right)\sqrt{2} \checkmark$

(ix.) Distributivity of Multiplication:

$$(a+b\sqrt{2})((c+d\sqrt{2})+(x+y\sqrt{2})) = ((a+b\sqrt{2})(c+d\sqrt{2}))+((a+b\sqrt{2})(x+y\sqrt{2})) \checkmark$$

$\therefore$ The set of all numbers of the form $a+b\sqrt{2}$ is a field.

(2) This field is $\mathbb{C}$, the set of all complex numbers.
Set F in which addition & multiplication are defined as such that $\forall (a+bi), (c+di) \in F$ there are unique elements $(a+bi)+(c+di)$ & $(a+bi)\cdot(c+di)$ in F such that $\forall (a+bi), (c+di), (x+yi) \in F$:

(i) Commutativity of addition:

$$(a+bi)+(c+di) = (c+di)+(a+bi) \checkmark$$

(ii) Associativity of addition:

$$(a+bi+c+di)+x+yi = c+di+(a+bi+x+yi) \checkmark$$

(iii) Identity of addition:

$$(0+0i)+(a+bi) = (a+bi)+(0+0i) = a+bi \checkmark$$

(iv.) Inverse of addition:

$$(a+bi)+(-a-bi) = 0 \checkmark$$

(v) Commutativity of multiplication:

$$(a+bi)\cdot(c+di) = (c+di)\cdot(a+bi) \checkmark$$

(vi) Associativity of multiplication:

$$((a+bi)\cdot(c+di))\cdot(x+yi) = (a+bi)\cdot((c+di)\cdot(x+yi)) \checkmark$$

(vii) Identity of multiplication:

$$(a+bi)\cdot(1+0i) = (1+0i)(a+bi) = a+bi \checkmark$$

(viii) Inverse of multiplication:

$$(a+bi)^{-1} := (x+yi) \rightarrow (a+bi)(x+yi) = 1 \rightarrow (ax-by)+(ay+bx)i = 1$$

Solve $ax-by=1$ and $ay+bx=0$ for x,y $\rightarrow A = \begin{bmatrix} a & -b & 1 \\ b & a & 0 \end{bmatrix}$

using python... $x = \frac{a}{a^2+b^2}$ & $y = -\frac{b}{a^2+b^2}$ $\rightarrow (a+bi)^{-1} = \left(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i\right) \checkmark$

(ix.) Distributivity of multiplication:

$$(a+bi)\cdot((c+di)+(x+yi)) = ((a+bi)\cdot(c+di))+((a+bi)\cdot(x+yi)) \checkmark$$

$\therefore$ the set of all numbers of form $a+b\sqrt{-1}$ form a field.

3. $\mathbb{Z}_3 = [0, \pm 3, \pm 6, \ldots]$; $[0] = [0 \pm 0, 0 \pm 3, 0 \pm 6, \ldots]$, $[1] = [1 \pm 0, 1 \pm 3, \ldots]$

+	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

x	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

4. same as question 1 part 2.

5. $\boxed{B, D}$

6. $A+B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $AB = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

7. $\det A = -1 \begin{vmatrix} t & 1 \\ 0 & t \end{vmatrix} + 0 + 1 \begin{vmatrix} 6 & 1 \\ t & 1 \end{vmatrix} = -\frac{1}{t^2-1} + \frac{t}{6-t} \therefore \boxed{t \neq \pm 1, 6}$

8. a) $h \in \mathbb{R}$
   b) $h \neq 6$

9. a) $\begin{bmatrix} 1 & * \\ * & 1 \\ * & * \end{bmatrix}$ 3 types of 3x2 in row echelon
   b) $\begin{bmatrix} 1 & * & * \\ * & 1 & * \end{bmatrix}$ 6 types of 2x3 in row echelon
   c) $\begin{bmatrix} 1 \\ * \\ * \\ * \end{bmatrix}$ 4 types of 4x1 in row echelon

10. $a = 0$, $b = 0$, $c = 1$, $d = 0$, $e = 0$.

11. (1) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{-R_1+R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{-2R_1+R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -2 \\ 0 & -4 & -5 & -6 \end{bmatrix} \xrightarrow{-1R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & -4 & -5 & -6 \end{bmatrix} \xrightarrow{4R_2+R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 7 & 2 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6/7 \\ 0 & 1 & 0 & 8/7 \\ 0 & 0 & 1 & 2/7 \end{array}\right] \qquad \vec{X_n} = \begin{bmatrix} 6/7 \\ 8/7 \\ 2/7 \end{bmatrix}$$

(2) nah

(3) python

(4) Yes it is possible

12. (1) rref$(A|\vec{b})$ over field $\mathbb{Z}_7$: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$

in Python: A=GF7([A]) → b = GF7([b]) → GF7.row_reduce(A) → x = np.linalg.solve(A, b) → rref(A) = $\vec{x}$

(2) $A\vec{x} = b \bmod 7$: $\vec{x} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$

13. $\left[\begin{array}{ccc|c} 3 & 11 & 19 & -2 \\ 7 & 23 & 39 & 10 \\ -4 & -3 & -2 & 6 \end{array}\right]$ using python... $\vec{x} = \begin{bmatrix} 11/3 \\ 56/12 \\ 22/3 \end{bmatrix}$

14. $\left[\begin{array}{ccccc|c} 3 & 6 & 9 & 5 & 25 & 53 \\ 7 & 14 & 21 & 9 & 53 & 105 \\ -4 & -8 & -12 & 5 & -10 & 11 \end{array}\right]$ using python $\vec{x} = \vec{x}_1 \begin{bmatrix} 24/5 \\ 31/5 \\ 3\,3 \end{bmatrix} + \vec{x}_2 \begin{bmatrix} 25/7 \\ 14/3 \\ -0 \end{bmatrix} - \begin{bmatrix} 0 \\ 7 \\ 4 \end{bmatrix}$

15. $\left[\begin{array}{ccccc|c} 2 & +4 & +3 & +5 & 6 & 37 \\ 4 & 8 & 7 & 5 & 2 & 74 \\ -2 & -4 & 3 & 4 & -5 & 20 \\ 1 & 2 & 2 & -1 & 2 & 26 \\ 5 & -10 & 4 & 6 & 4 & 24 \end{array}\right]$ using python $\vec{x} = \begin{bmatrix} 31/17 \\ 43/3 \\ 26/1 \\ 4 \\ 52/11 \end{bmatrix}$

16. $ABC = I_n \rightarrow (AB)C = I_n = (AB) = C^{-1}$

which can be said about all 3 matrices, therefore $A^{-1} = BC$, $B^{-1} = AC$, $C^{-1} = AB$.

17. $(AB)(AB) \neq (AA)(BB)$ for $A = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

18. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

19. $n \times n$ matrix A symmetric $A^T = A$, skew-symmetric $A^T = -A$

(1) symmetric: $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 5 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & -1 & 4 \\ 1 & 2 & 0 & 3 \\ -1 & 0 & 5 & 0 \\ 4 & 3 & 0 & 7 \end{bmatrix}$

Skew-symmetric: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 & 2 & 0 \\ -1 & 0 & 3 & -4 \\ 2 & -3 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix}$

(2) every main diagnol value must be 0.

(3) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is symmetric & skew-symmetric

(4) (a) $A + A^T = (A + A^T)^T$

$A + A^T = A^T + (A^T)^T = A^T + A = A + A^T$ ✓

(b) $AA^T = (AA^T)^T = A^T(A^T)^T = A^TA^T$ ✓

~~(c)~~ $\cancel{(A^TA)^T = A^T(A^T)^T = A^TA}$

(c) $A^TA = (A^TA)^T = A^T(A^T)^T = A^TA$ ✓

(d) $(A - A^T)^T = -(A - A^T)^T = A^T - (A^T)^T = A^T - A$ ✓

(5) Let $A_1 = A + A^T$ and $A_2 = A - A^T$

$A = A_1 + A_2 = A + A^T + A - A^T = (A + A) + A^T + (-A^T)$

Since $A + (-A) = 0 \rightarrow \boxed{A + A = 2A}$ ✓

20. (a) bijective
(b) injective
(c) bijective
(d) bijective

21. skip

22. skip

23. considered

24 (1) $H^T = (I - 2\vec{u}\vec{u}^T)^T = I^T - 2(\vec{u}\vec{u}^T)^T = I - 2(\vec{u}^T)^T\vec{u}^T = I - 2\vec{u}\vec{u}^T = \underline{H.}$

$\therefore$ H is symmetric by definition because $H^T = H$.

(2) $H^T H = (I - 2\vec{u}\vec{u}^T)(I - 2\vec{u}\vec{u}^T) = I - 2\vec{u}\vec{u}^T - 2\vec{u}\vec{u}^T + 4(\vec{u}\vec{u}^T)(\vec{u}\vec{u}^T)$

$= I - 4\vec{u}\vec{u}^T + 4\vec{u}(\vec{u}^T\vec{u})\vec{u}^T = I - \cancel{4\vec{u}\vec{u}^T} + \cancel{4\vec{u}\vec{u}^T}$

$= \underline{I.}$

$\therefore$ H is orthogonal by definition because $H^T H = I$.

(3) $H^2 = HH = \underline{I}$. As defined above.

(4) $H\vec{u} = (I - 2\vec{u}\vec{u}^T)\vec{u} = \vec{u} - (2\vec{u}\vec{u}^T)(\vec{u}) = \vec{u} - 2\vec{u}(\vec{u}^T\vec{u}) = \vec{u} - 2\vec{u}$

$= \underline{-\vec{u}}.$

(5) $H_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\left(\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$H_3 = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$

$H_4 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$