Math 4570-Matrix Methods-Fall 2021

Instructor: He Wang

Test 1. (1:35-2:40pm)

Student Name: ____________________ __________/**50**

Rules and Instructions for Exams:

1. Unless otherwise specified, to receive full credits you must show **all** necessary work. The grading is based on your work shown. Only a final result from calculator will receive zero point.

2. You need to finish the exam yourself. Any discussions with the other people will be considered as **academic dishonesty**. **Cheating, Unauthorized Collaboration, and Facilitating Academic Dishonesty are not allowed.** You can read a description of each here http://www.northeastern.edu/osccr/academic-integrity-policy/

3. You are allowed to bring one lecture notes or a textbook.

4. However, you are **not** allowed to bring the homework or practice questions.

5. However, you are **not** allowed to use any electronic devices.

Notation: $\vec{x} \in \mathbb{R}^n$ means a column vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

1. (2 points) Consider the finite field $\mathbb{Z}_5$. Answer the following questions:
(1) Which of the following is the **additive inverse** of $[2] \in \mathbb{Z}_5$? Answer: ___________

A. [1] B. [2] C. [3] D. [4] E. [0]

(2) Which of the following is the **multiplicative inverse** of $[2] \in \mathbb{Z}_5$? Answer: ___________

A. [1] B. [2] C. [3] D. [4] E. [0]

(1) C (2) C

2. (2 points) Let F be the field contains all numbers of the form $a+b\sqrt{3}$ where a and b are rational numbers, with the usual addition and multiplication of arithmetic

$$(a+b\sqrt{3})+(c+d\sqrt{3}) := a+c+(b+d)\sqrt{3}$$

$$(a+b\sqrt{3})\times(c+d\sqrt{3}) := ac+3bd+(ad+bc)\sqrt{3}$$

(1) What is the **additive inverse** of $2+\sqrt{3}$? Answer: ___________

A. $2+\sqrt{3}$ B. $2-\sqrt{3}$ C. $1+\sqrt{3}$ D. $-1-\sqrt{3}$ E. $-2-\sqrt{3}$

(2) What is the **multiplicative inverse** of $2+\sqrt{3}$? Answer: ___________

A. $2+\sqrt{3}$ B. $2-\sqrt{3}$ C. $1+\sqrt{3}$ D. $-1-\sqrt{3}$ E. $-2-\sqrt{3}$

(1) E (2) B

3. (4 points) Determine whether or not the following set a **subspace** of $\mathbb{R}^3$. Explain your reason.
(1) $S = \{\vec{x} \in \mathbb{R}^3 \mid x_1 + x_2 = 1\}$.

No. Since S does not contains zero vector.

(2) $S = \{\vec{x} \in \mathbb{R}^3 \mid x_1 \geq 0\}$.

No. Since S does not contains zero vector.

4. (4 points) Let $V :=$ { all funcitons $f(x) : \mathbb{R} \to \mathbb{R}$}be the vector space of functions.

Let $W = \text{Span}\{e^x, x^4, \sin x\}$ be the subset of V. **(1)** Is W a subspace of V? **(2)** Write down a basis for W? **(3)** What is the dimension of W? (No proof needed. But some explanation can receive partial credits.)

Yes. Since the Span of any subset of V is always a subspace.
$\dim(W) = 3$, since a basis is given by $\{e^x, x^4, \sin x\}$

5. Let $\mathbb{R}^{n\times n}$ be the vector space of all $n \times n$ matrices? Let S_n be the set of all $n \times n$ skew-symmetric matrices with real entries. That is $S_n := \{A \in \mathbb{R}^{n\times n} \mid A^T = -A\}$.

(1) (3 points) Is S_n a subspace of $\mathbb{R}^{n\times n}$? Prove your result.

(1) zero matrix is skew symmetric
(2) sum is closed. If A and B are skew-symmetric, then $(A+B)^T = A^T + B^T = -A - B = -(A+B)$. So $A+B$ is skew-symmetric.
(3) scalar product is closed. If A is skew-symmetric, then $(cA)^T = cA^T = -cA$. So cA is skew-symmetric.

(2) (2 point) Write a basis for S_3?

Any 3×3 skew-symmetric matrix has the form $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$

So a basis is given by $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

(3) (1 point) What is the dimension of S_3?

3

6. (6 points) Recall that elementary matrices are obtained from identity matrix I_n by only one elementary row operation, i.e., $I_n \xrightarrow{R_i \leftrightarrow R_j} E_{ij}$, $I_n \xrightarrow{rR_i} E_i(r)$, and $I_n \xrightarrow{R_i + kR_j} E_{ij}(k)$.
(1) Which elementary matrices are symmetric? E_{ij}, $E_i(r)$, or $E_{ij}(k)$.

E_{ij}, $E_i(r)$

(2) Which elementary matrices are invertible? E_{ij}, $E_i(r)$, or $E_{ij}(k)$.

all

(3) Is it possible to write **any** matrix as a product of elementary matrices? Reason.

No. Only invertible matrices, since product of elementary matrices are invertible.

(4) Is it possible to write any matrix A as a product of elementary matrices and $\mathbf{rref}(A)$?

Yes

7. Let $A = [\vec{a}_1\ \vec{a}_2\ \vec{a}_3\ \vec{a}_4\ \vec{a}_5] = \begin{bmatrix} 1 & 3 & 3 & 5 & 4 \\ 1 & 4 & 5 & 5 & 6 \\ 2 & 6 & 6 & 8 & 6 \end{bmatrix}$ with $\textbf{rref}(A) = \begin{bmatrix} 1 & 0 & -3 & 0 & -7 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$. Answer the following questions.

(1) (5 points) Find a basis for the kernel (null) space $\ker A$.

So, $\begin{cases} x_1 + 4x_3 - 7x_5 \\ x_2 - 2x_5 \\ x_4 + x_5 = 0 \\ x_3, x_5 \text{ is a free variable} \end{cases}$

The **vector form** is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -4x_3 + 7x_5 \\ -2x_5 \\ x_3 \\ -x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 7 \\ -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

So, a basis for $\ker A$ is $\{ \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \}$

(2) (2 points) Find **two bases** for the column subspace $\text{im}(A)$.

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 8 \end{bmatrix} \right\}$

Standard basis for $\text{im}(A)=\mathbb{R}^3$

(3) (1 point) What is the rank of A?

rank(A)=3

(4) (2 points) Is $\vec{a}_3$ a linear combination of $\vec{a}_1$ and $\vec{a}_2$? If Yes, write it done. If No, explain the reason.

Yes. Solve augmented matrix $[\vec{a}_1\ \vec{a}_2 \mid \vec{a}_3]$, we find solution (-3,2) So, $\vec{a}_3 = -3\vec{a}_1 + 2\vec{a}_2$

(5) (2 points) Is $\vec{a}_4$ a linear combination of $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$? If Yes, write it done. If No, explain the reason.

No. Solve augmented matrix $[\vec{a}_1\ \vec{a}_2\ \vec{a}_3 \mid \vec{a}_4]$ there is a contradiction.

8. (2 points) Consider $\mathbb{R}^2$ with $\langle\cdot,\cdot\rangle$ defined for all $\vec{x}, \vec{y} \in \mathbb{R}^2$ as

$$\langle\vec{x},\vec{y}\rangle := x_1x_2 + y_1y_2$$

Is $\langle\cdot,\cdot\rangle$ an inner product?

No. The following axiom is not satisfied:
Axiom $\langle c\vec{u},\vec{v}\rangle = c\langle\vec{u},\vec{v}\rangle$.

9. (6 points) Consider the inner product space $P_3(\mathbb{R})$where $\langle f,g\rangle$ is defined to be $\langle f,g\rangle = \int_0^1 f(x)g(x)dx$.

Let polynomials $f = 1 + x$ and $g = 5 - 9x$.

Find the norm $||f||$, $||g||$ and inner product $\langle f,g\rangle$. Find the angle between f and g.

$\langle f,g\rangle = \int_0^1 (1+x)(5-9x)dx = 0$
$||f|| = \sqrt{\langle f,f\rangle} = \sqrt{\int_0^1 (1+x)^2dx} = \sqrt{7/3}$
$||g|| = \sqrt{\langle g,g\rangle} = \sqrt{\int_0^1 (5-9x)^2dx} = \sqrt{7}$
$\cos\theta = \dfrac{\langle f,g\rangle}{||f||||g||} = 0.$

10. Let $A = [\vec{a}_1\ \vec{a}_2\ \vec{a}_3\ \vec{a}_4\ \vec{a}_5\ \vec{a}_6] = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 4 & 4 & 5 & 1 \\ 2 & 4 & 6 & 9 & 4 & 2 \\ 2 & 4 & 6 & 9 & 4 & 3 \end{bmatrix}$. Let $U = \text{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ and $V = \text{Span}\{\vec{a}_4, \vec{a}_5, \vec{a}_6\}$.

Suppose $\mathbf{rref}(A) = \begin{bmatrix} \mathbf{1} & 0 & 1 & 0 & 29 & 0 \\ 0 & \mathbf{1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$ and $\mathbf{rref}([\vec{a}_4\ \vec{a}_5\ \vec{a}_6]) = \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 \end{bmatrix}$

(1) (4 points) What are the dimensions of U, V, $U + V$, $\mathbb{R}^4/U$?

$\dim(U) = 2$
$\dim(V) = 3$
$\dim(U + V) = 4$
$\dim(\mathbb{R}^4/U) = 4 - \dim U = 2$

(2) (1 point) What is the dimension of $U \cap V$?

Using formula: $\dim(U + V) = \dim U + \dim V - \dim U \cap V$
So, $\dim U \cap V = 1$

(3) (1 point) Is $U \cup V$ a subspace of $\mathbb{R}^4$? (No reason needed.)

No

(4) (Bonus 2 points) Find a basis for the subspace $U \cap V$? (Explain your reason.)

$$\begin{cases} x_1 = -x_3 - 29x_5 \\ x_2 = -x_3 \\ x_4 = 6x_5 \\ x_6 = 0 \end{cases}$$

So, solution for kernel is given by

$$\vec{x} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 29 \\ 0 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix}$$

So, $U \cap V$ is spanned by the following two elements: $-\vec{a}_1 - \vec{a}_2 + \vec{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $29\vec{a}_1 = 29 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$

So a basis for $U \cap V$ is given by $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$