Seventh Worksheet, MATH 7233

November 3, 2021

1. Construct tournaments for any $n \geq 5$ where everybody is a pseudo champion.

2. Let G be a k-regular (ie every degree is equal to k) bipartite graph.

   (a) Show that it has a perfect matching.

   (b) Show that the edges can be colored using k colors so that the all the edges adjacent to any single vertex have different colors.

3. In a bipartite graph G with $V = A \cup B$, let us call a set $H \subset A$ *tight* if $|H| = |N(H)|$.

   Suppose G has *no obstacles* and $H \subset A$ is tight. Let $G_1 = G[H \cup N(H)]$ and $G_2 = G[(A \setminus H) \cup (B \setminus N(H))]$ be the subgraphs of G restricted to these subsets. Show that neither G_1, nor G_2 has obstacles.

4. Give a proof of Hall's theorem by induction on the size of A. Hint: separate into two cases based on whether there is a tight set $H \subsetneq A$ or not. If there is a tight set, use the previous problem. If there is no tight set, try simply adding any edge to the matching.

5. We draw all diagonals of a convex n-gon. What is the largest number of intersection points we can create this way inside the polygon?

6. What could the "face-shake" lemma for planar graphs possibly state? Try to figure it out, and then prove it!

7. What is the expected number of cherries in $G(n, p)$?

8. Fix $0 < p < 1$. Let x_n denote the probability that $G(n, p)$ is disconnected. Show that $x_n \to 0$ as $n \to \infty$.