MTH 7241: Fall 2020

Practice Problems for Test 2

1). Consider an irreducible chain on 3 states. Prove or give a counterexample: $p_{jj}(3) > 0$ for every state j.

Counterexample:

Chain is irreducible and

$p_{jj}(3) = 0$ for every state j.

2). Recall the Gambler's Ruin Problem: a random walk on the integers $\{0, 1 \ldots, N\}$ with probability p to jump right and $q = 1 - p$ to jump left at every step, and absorbing states at 0 and N. Starting at $X_0 = k$, let M_k be the expected number of steps until the walk reaches either 0 or N.

a). By conditioning on X_1, derive a recursion formula for M_k.

b). Compute the boundary conditions M_0 and M_N.

c). Show that the recursion formula from (a) is satisfied by the special solution $M_k = c\,k$ and compute the value of the constant c (for this part you should ignore the boundary conditions in (b)).

d). Show that the recursion formula from (a) is satisfied by the solution $M_k = c\,k + A + B(q/p)^k$ where c was computed in (c), and A, B are constants (again ignore the boundary conditions in (b)).

e). Use the boundary conditions from (b) to find the values of A and B in (d).

a) $$M_k = p(1 + M_{k+1}) + q(1 + M_{k-1})$$

$$M_k = 1 + pM_{k+1} + qM_{k-1}$$

b) $$M_0 = M_N = 0$$

c) let $M_k = ck$

$$\Rightarrow \text{RHS} = 1 + p\,c(k+1) + q\,c(k-1)$$

$$= 1 + c(p-q) + ck$$

so $$\text{RHS} = M_k = ck$$

$$\Longleftrightarrow 1 + c(p-q) = 0 \Longleftrightarrow \boxed{c = \frac{1}{q-p}}$$

[Note: this assumes $q \neq p$.
If $q = p$, the special solution is
ck^2]

d) let $A + B\left(\frac{q}{p}\right)^k = P_k$

can check this satisfies the homogeneous equation

$$P_k = p\,P_{k+1} + q\,P_{k-1}$$

$\Rightarrow$ $ck + P_k$ satisfies eqn. (a).

e) Apply B.C.

$$M_k = \frac{k}{q-p} + A + B\left(\frac{q}{p}\right)^k$$

$k=0$: $\quad 0 = A + B$

$k=N$: $\quad 0 = \dfrac{N}{q-p} + A + B\left(\dfrac{q}{p}\right)^N$

$$\Rightarrow \quad 0 = \frac{N}{q-p} + B\left[\left(\frac{q}{p}\right)^N - 1\right]$$

$$\Rightarrow \quad B = \frac{N}{q-p} \cdot \frac{1}{1-\left(q/p\right)^N}$$

$$A = -B.$$

3). A reduced 4×4 chessboard has 16 squares, labeled $i = 1, \ldots, 16$. Let $d(i)$ denote the number of available nearest neighbors to the square i, where we allow vertical, horizontal and diagonal moves (so for example the corner has 3 neighbors; a square in the middle has 8 neighbors). A random walk is constructed on the board as follows:

$$p_{i,j} = \begin{cases} \dfrac{1}{d(i)} & \text{if } j \text{ is an available nearest neighbor to } i \\ 0 & \text{otherwise} \end{cases}$$

a). Find the stationary distribution of the chain. [Hint: write down and solve the time reversible equations for the chain].

b). Starting in a corner, find the expected number of steps until the first return to the same corner.

3	5	5	3
5	8	8	5
5	8	8	5
3	5	5	3

$d(i)$ for each square,

a) RW on graph with random jump along edges $\Rightarrow$ chain is reversible and stationary distribution is

$$W_i = \frac{d(i)}{\sum_j d(j)}$$

b)

Mean return time to corner

$$= \frac{1}{w_{corner}}$$

Now $\quad d(corner) = 3$

$$\sum_j d(j) = 84$$

$$\Rightarrow \mu_{corner} = \frac{3}{84} = 28$$

4). An irreducible persistent Markov chain X_n is defined on the infinite state space $S = \{1, 2, 3, \ldots\}$. The chain has a stationary positive vector $v = (v_1, v_2, \ldots)$ where

$$v_k = \frac{1}{1+k}$$

a). Suppose $u = (u_1, u_2, \ldots)$ is another stationary positive vector for the chain. Calculate u_1/u_2.

b). Determine whether the chain X_n is null persistent or positive persistent.

a) Chain is irreducible persistent

$\Rightarrow$ there is unique stationary positive vector, up to positive multiples.

$\Rightarrow \quad u_k = c v_k \quad , \quad$ some $c > 0$.

$\Rightarrow \quad \dfrac{u_1}{u_2} = \dfrac{v_1}{v_2} = \dfrac{1/2}{1/3} = \dfrac{3}{2}$

b) Chain is positive persistent if and only if stationary vector is summable

$$\sum_{k=1}^{\infty} v_k = \sum_{k=1}^{\infty} \frac{1}{1+k} = \infty$$

(divergent harmonic series)

$\Rightarrow$ chain is null persistent.

5). Let $X_1, X_2, \ldots$ be IID random variables, where the moment generating function is

$$\mathrm{E}[e^{tX}] = e^{t^2-3t}$$

a). Find the mean $\mu = \mathrm{E}[X]$.

b). Let $Y_n = (1/n)\sum_{i=1}^{n} X_i$. Use Cramer's Theorem to compute

$$\lim_{n\to\infty} \frac{1}{n} \log \mathrm{P}(Y_n > -2)$$

a) $\mu = \mathbb{E}[X] = \frac{d}{dt}\mathbb{E}[e^{tX}]\Big|_{t=0}$

$= e^{t^2-3t}(2t-3)\Big|_{t=0}$

$= -3$

b) mgf $\quad M(t) = e^{t^2-3t}$

$\Rightarrow \quad \Lambda(t) = t^2 - 3t$

$\Rightarrow \quad \Lambda^*(x) = \sup_t \{xt - t^2 + 3t\}$.

Find max.: $\quad \frac{d}{dt}(xt - t^2 + 3t) = 0$

$\Rightarrow \quad x - 2t + 3 = 0$

$\Rightarrow \quad t^* = \frac{x+3}{2}$

$$\Rightarrow \Lambda^*(x) = x t^* - (t^*)^2 + 3 t^*$$

$$= (x+3) t^* - (t^*)^2$$

$$= 2 (t^*)^2 - (t^*)^2$$

$$= \left(\frac{x+3}{2}\right)^2$$

Cramer's Theorem:

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{P}(Y_n > -2)$$

$$= - \Lambda^*(-2)$$

$$= - \left(\frac{-2+3}{2}\right)^2$$

$$= -\frac{1}{4}$$

6). For a branching process, calculate the probability of extinction when

$$p_0 = \mathrm{P}(Z=0) = 1/12, \quad p_1 = \mathrm{P}(Z=1) = 1/2, \quad p_2 = \mathrm{P}(Z=2) = 5/12.$$

To find ρ = prob. of extinction, solve

$$s = \phi(s)$$

$$= p_0 + s p_1 + s^2 p_2$$

$$\Leftrightarrow \quad s = \frac{1}{12} + s\frac{1}{2} + s^2\frac{5}{12}$$

$$\Leftrightarrow \quad 12s = 1 + 6s + 5s^2$$

$$\Leftrightarrow \quad 5s^2 - 6s + 1 = 0$$

$$\Leftrightarrow \quad (s-1)(5s-1) = 0$$

so ρ = smallest positive root of $(s = \phi(s))$

$$= \frac{1}{5}.$$

7). Suppose that the distribution of Z for a branching process is geometric, so that

$$p_k = \mathrm{P}(Z = k) = x^k (1 - x) \quad \text{for} \quad k = 0, 1, 2, \ldots$$

where $0 < x < 1$.

a). Find the ~~smallest~~ largest value of x for which extinction is guaranteed. Call this value x_m.

b). For $x > x_m$ compute the probability of extinction.

a) Extinction is guaranteed if $m \leq 1$.

$$m = \mathbb{E}[Z] = \sum_{k=0}^{\infty} k\, x^k (1-x)$$

$$= x(1-x) \sum_{k=1}^{\infty} k x^{k-1}$$

$$= x(1-x) \frac{1}{(1-x)^2}$$

$$= \frac{x}{1-x}.$$

so $m \leq 1 \iff \dfrac{x}{1-x} \leq 1 \iff x \leq 1-x$

$$\iff x \leq \tfrac{1}{2}$$

$\Rightarrow x_m = \frac{1}{2}$ is largest value of x where extinction is guaranteed.

b) Find $\rho =$ prob. of extinction

Solve $s = \phi(s)$

$$= \mathbb{E}[s^{Z}]$$

$$= \sum_{k=0}^{\infty} s^k x^k (1-x)$$

$$= (1-x) \sum_{k=0}^{\infty} (sx)^k$$

$$= \frac{1-x}{1-sx}.$$

$$\Leftrightarrow \quad s(1-sx) = 1-x$$

$$\Leftrightarrow \quad s - s^2 x = 1-x$$

$$\Leftrightarrow \quad s^2 x - s + 1 - x = 0$$

$$\Leftrightarrow \quad (s-1)(sx-1+x) = 0$$

ρ = smallest positive root

$$= \frac{1-x}{x} < 1 \quad \text{for} \quad x > x_c = \frac{1}{2}$$