Sixth Worksheet, MATH 7233

October 25, 2021

Reminder:

- the eigenvalues of the adjacency matrix A_G are denoted $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$,
- the eigenvalues of the Laplace matrix L_G are denoted $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.
- For any symmetric matrix M, it's largest eigenvalue is equal to

$$\max_{v \neq 0} \frac{v^T M v}{v^T v}$$

1. Show that for any graph
$$\frac{2|E|}{|V|} \leq \mu_1 \leq \Delta(G).$$

2. Let $S \subset V(G)$ be an independent set (ie there are no edges between any two nodes in S). Show that
$$|S| \leq n \left(1 - \frac{\deg_{ave}(S)}{\lambda_n}\right)$$
where $\deg_{ave}(S) = \frac{1}{|S|} \sum_{v \in S} \deg(v)$.

3. Show that in a tournament there cannot be exactly 2 pseudo champions.

4. Let H be a set of n points in the plane. Let $D = \{(x, y) \in H^2 : d(x, y) = 1\}$ be the set of (ordered) pairs whose distance is exactly 1 unit. Show that $|D| \leq n/2 + \sqrt{2} n^{3/2}$.

5. Suppose that in a bipartite graph G, each vertex in A has degree 7 while each vertex in B has degree 9. What can be said about $|A|/|B|$?

6. The plane is divided into regions by straight lines. Show that you can color the regions black or white such that edge-wise adjacent regions have different color.

7. Is it true that for any $k, m \geq 2$ integers there is a number $R(k, m)$ such that if G is any graph on at least $R(k, m)$ nodes then G contains a K_k subgraph or G^c contains a K_m subgraph?