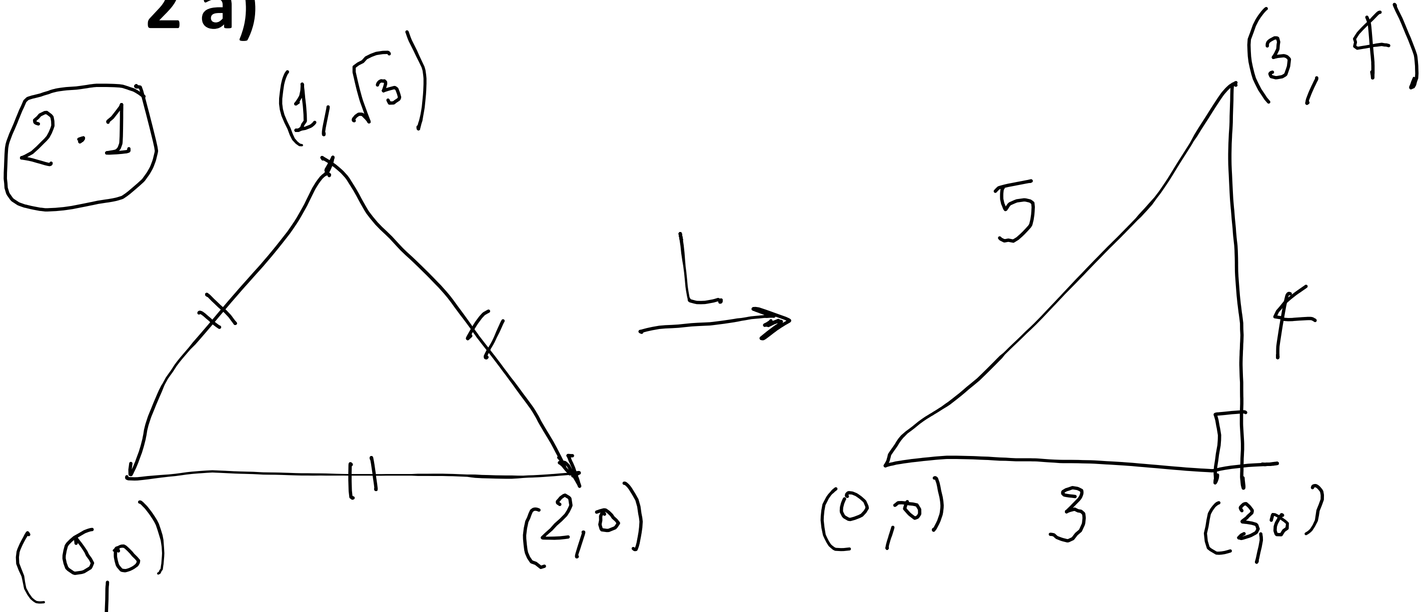
2 a)

2.1



$(0,0) \longrightarrow (0,0)$

$(2,0) \longrightarrow (3,0)$

$(1,\sqrt{3}) \longrightarrow (3,4)$

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & \sqrt{3} \end{bmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{bmatrix} 3 & 3 \\ 0 & 4 \end{bmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & 4 \end{bmatrix}^{-1}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{12}\begin{pmatrix} 8 & -3 \\ 0 & 3\sqrt{3} \end{pmatrix}$$

2.2

$$\left.\begin{matrix}\text{Area of} \\ \text{largest circle} \\ \text{inside equilateral} \\ \text{triangle}\end{matrix}\right\} = \begin{matrix}\text{Area of} \\ \text{incircle}\end{matrix}$$

we know that,

$$\text{Area of triangle} = r \times s$$

$$= r \times \left(\frac{a+b+c}{2}\right)$$

$$= \frac{3ar}{2} \quad (\because a=b=c)$$

$$\therefore r = \frac{2 \times \Delta}{3a}$$

$$= \frac{2 \times \frac{\sqrt{3}}{4} \times a^{\cancel{2}}}{3 \times \cancel{a}}$$

$$= \frac{a}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\therefore \left.\begin{array}{l}\text{Area of} \\ \text{Incircle}\end{array}\right\} = \pi \times \left(\frac{1}{\sqrt{3}}\right)^2$$

$$= \frac{\pi}{3}.$$

2.3

$$\text{Area}(L(C)) = |\text{Det}(A)|\,\text{Area}(C)$$

$$\text{Det}(A) = \left|\frac{1}{12}\begin{pmatrix} 8 & -3 \\ 0 & 3\sqrt{3} \end{pmatrix}\right|$$

$$= \frac{1}{2\sqrt{3}}$$

$$\therefore \text{Area}(D) = \frac{1}{2\sqrt{3}} \times \frac{\pi}{3}$$

$$= \frac{\pi}{6\sqrt{3}}$$

2.4

Center of circle $= \left(1, \frac{1}{\sqrt{3}}\right)$

radius $= \frac{1}{\sqrt{3}}$

Parametric form of a point on the circle is,

$$S = \left\{ \left(1 + \frac{1}{\sqrt{3}}\cos(t), \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\sin(t)\right) : 0 \le t \le 2\pi \right\}$$

image of S under L $= \frac{1}{12}\begin{bmatrix} 8 & -3 \\ 0 & 3\sqrt{3} \end{bmatrix} \begin{bmatrix} 1 + \frac{1}{\sqrt{3}}\cos(t) \\ \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\sin(t) \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{4}\left(\frac{-\sin(t)}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right) + \frac{2}{3}\left(\frac{\cos(t)}{\sqrt{3}} + 1\right) \\ \frac{1}{4}\left(\sin(t) + 1\right) \end{bmatrix}$$

$$= \begin{bmatrix} \left(-\frac{1}{4\sqrt{3}} + \frac{2}{3\sqrt{3}}\right) + \left(\frac{2}{3\sqrt{3}}\cos(t) - \frac{1}{4\sqrt{3}}\sin(t)\right) \\ \frac{1}{4} + \frac{1}{4}\left(\sin(t)\right) \end{bmatrix}$$

$\therefore$ Parametric equation of ellipse is,

$$E = \left\{ \begin{bmatrix} \left(-\frac{1}{4\sqrt{3}} + \frac{2}{3\sqrt{3}}\right) + \left(\frac{2}{3\sqrt{3}}\cos(t) - \frac{1}{4\sqrt{3}}\sin(t)\right), \\ \frac{1}{4} + \frac{1}{4}\sin(t) \end{bmatrix} : 0 \leq t \leq 2\pi \right\}$$

2 b) (Part 1)

2.1

$$A = \frac{1}{8}\begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ 2 & 3 & 4 \end{pmatrix}$$

Eigen values are,

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \simeq \begin{bmatrix} 0.8786 \\ -0.0457 \\ 0.2920 \end{bmatrix}$$

$|\lambda_i| < 1 \Rightarrow$ System with matrix A is asymptotically stable

2.2

From MATLAB code,

for $s=1$,

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 1.8071 \\ -0.0569 \\ 0.1748 \end{bmatrix}$$

$|\lambda_1| > 1 \Rightarrow$ system is unstable

Assumption: "m" is positive integer.

using MATLAB code,
it is found that smallest positive integer "m" such that $|\lambda_1|>1$ for $A-mB$ is, $\boxed{m=3}$

2.3

Using MATLAB code,

Values of "a" and "b" are,

$a = 2.9200$

$b = 0.1150$

$\therefore (a, b) = (-2.92, 0.115)$

Accurate upto 3 decimal places

Please refer to "Task2.m" file for codes.

2 b) (Part 2)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1+\alpha_B-\beta_B & -\gamma_F \\ \rho_B & 1-\beta_F \end{pmatrix}$$

eigen values are

$$\lambda_1 = \frac{(a+d)+\sqrt{(a-d)^2+4bc}}{2}$$

$$\lambda_2 = \frac{(a+d)-\sqrt{(a-d)^2+4bc}}{2}$$

In most cases in real world,

for population dynamics,

We have,

$$\alpha_B > \beta_B \Rightarrow a > 1$$

$$\gamma_F > 0 \Rightarrow b < 0$$

$$\beta_B > 0 \not\Rightarrow c > 0$$

$$\beta_F < 1 \Rightarrow d > 0$$

However,

in order to analyze different scanarios, we can mathematically try different values for a, b, c, d that may not make much sense in the real world.

Case-(i)

Both frog and beetle die

Let A is diagonalizable.

$$A = P D P^{-1}$$

$$A^n = P D^n P^{-1}$$

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

If $\lambda_1 < 0$ and $\lambda_2 < 0$

$$\lim_{n\to\infty} A^n = \lim_{n\to\infty} P D^n P^{-1}$$

$$= \lim_{n\to\infty} P \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} P^{-1}$$

$$= 0$$

The values chosen for this scenario are as follows:

$$\alpha_B = 0.5 \quad , \quad \beta_B = 1$$

$$f_F = 0.5$$

$$\rho_B = 0.01$$

$$\beta_F = 0.1$$

which yields

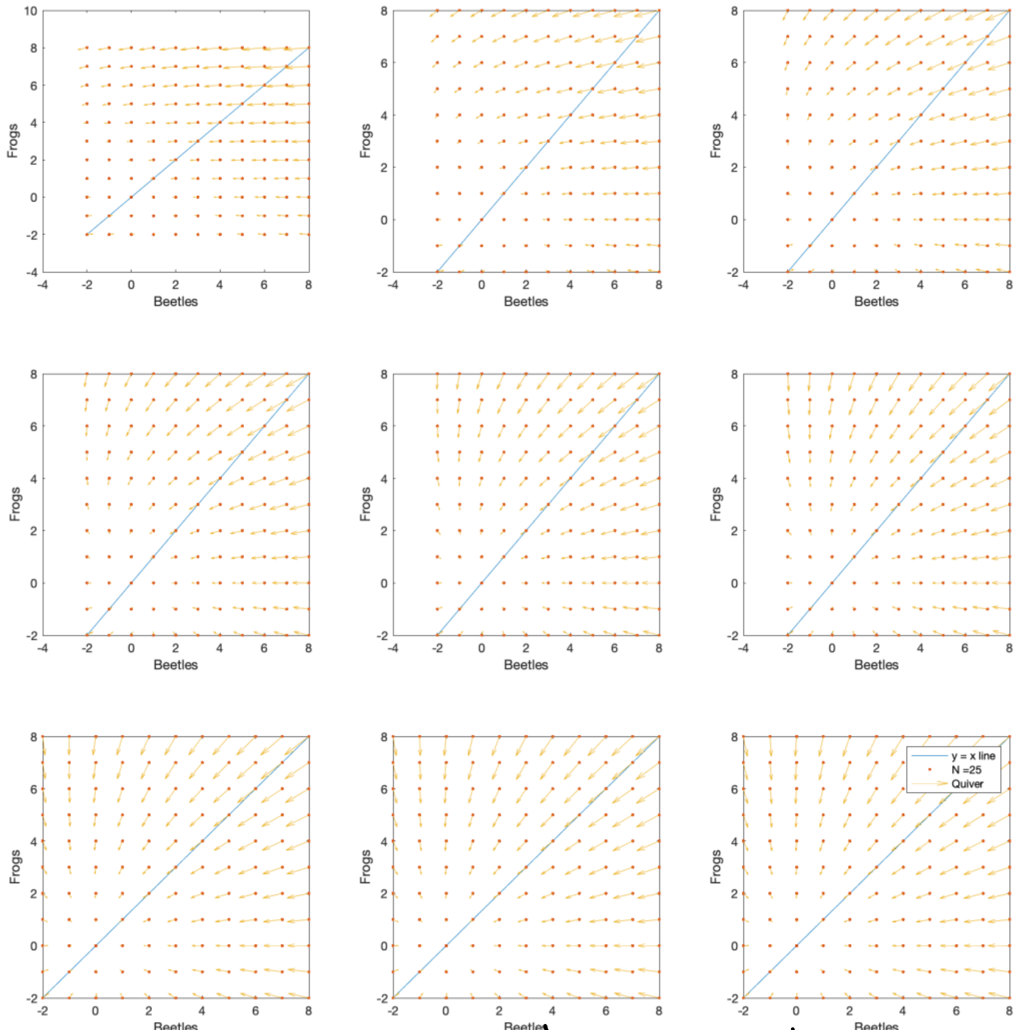
$$a = 0.5, \quad b = -0.5$$

$$c = 0.01, \quad d = 0.9$$

The eigen values are,

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0.5129 \\ 0.8871 \end{bmatrix}$$

The graph looks like,



The initial populations chosen from range b/w -2 to $+8$ in a grid fashion.
Negative values are just for experimentation.
The analysis is done from following 9 years:

1, 4, 7, 10, 13, 16, 19, 22, 25

It can be seen that population finally goes to zero.

Case-(ii):

Both Frog and Beetle grow exponentially

Let $\lambda_1 > 1$ and $\lambda_2 < 1$

In this case,

$$\lim_{n\to\infty} A^n = \lim_{n\to\infty} P D^n P^{-1}$$

$$= \lim_{n\to\infty} P \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} P^{-1}$$

$\lambda_1^n \to \infty$; $\lambda_2^n \to 0$

$\therefore$ Rank$(A^n) = 1$ (two zeroes in last row)

$$\Rightarrow \begin{bmatrix} B_{n+1} \\ F_{n+1} \end{bmatrix} = A^n \begin{bmatrix} B_0 \\ F_0 \end{bmatrix}$$

$\therefore$ Depending on initial conditions, either both increase or both decrease.

Sample values for this scenario are,

$\alpha_B = 1$, $\beta_B = 0.5$

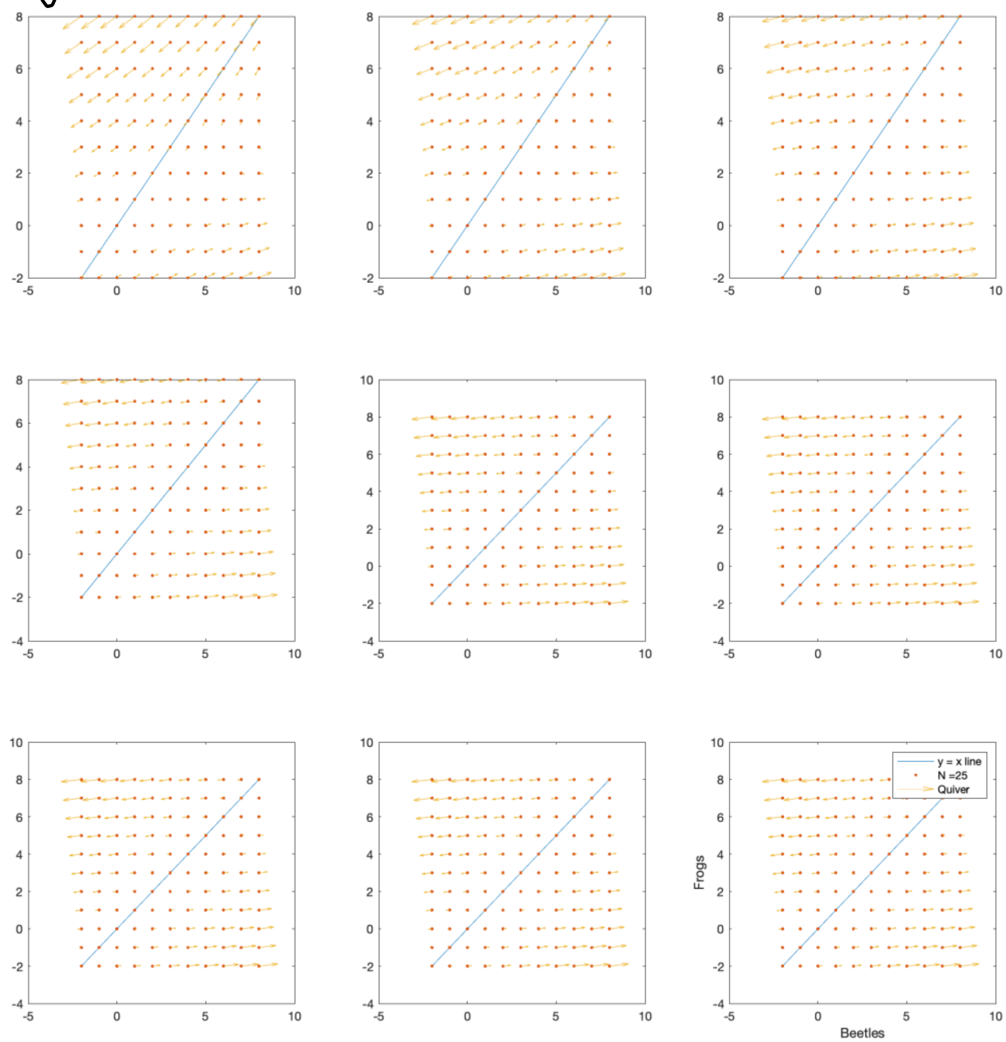
$\delta_F = 1$

$\rho_B = 0.1$

$\beta_F = 0.5$

$$\therefore A = \begin{pmatrix} 1.5 & -1 \\ 0.1 & 0.5 \end{pmatrix}; \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 1.3873 \\ 0.6127 \end{bmatrix}$$

The graph is shown as below:



From graph, it is evident that, for higher values of initial frog population, both tend to perish which makes complete sense because if more and more frogs eat beetles the both die as prey is only source for predator survival.

On Contrary,

when we have initial high beetle population, they grow exponentially and help frogs grow exponentially.

This scenario is described by points closer to x-axis

Case-(iii):

Beetles grow exponentially, but frogs die out.

This looks like an impossible scenario. Because if beetles grow exponentially, frogs (predator) gets more and more Prey (beetles) thus supporting for survival.

But mathematically it is possible to achieve a solution which can make sense in real world, if following happens.

Suppose beetles are infected by usage of pesticides on near by farms. Therefore consuming more and more beetle (prey) can cause frog (predator) to die out as it is assumed that prey is only source for predator survival.

The parameters that justify this explanation are as follows:

$\alpha_B = 5.5$, $\beta_B = 0.5$

$\gamma_F = 2$

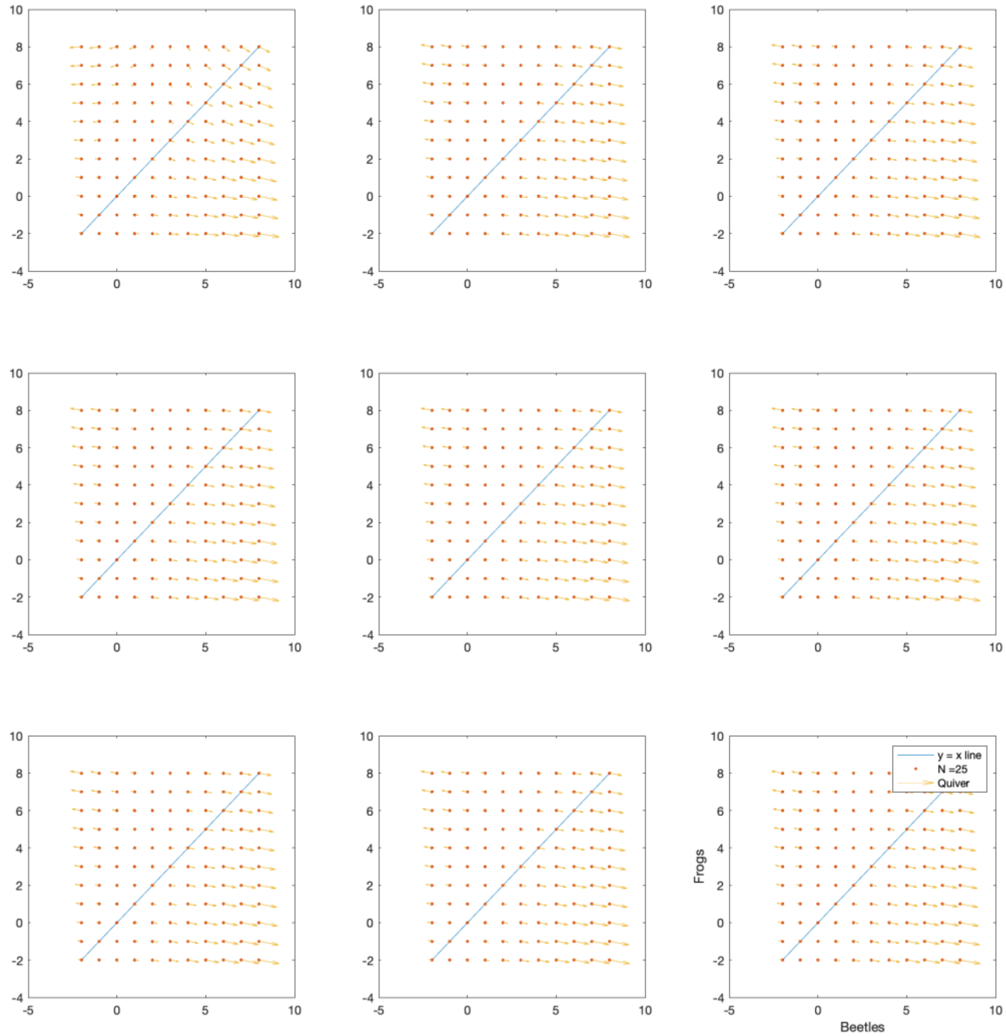
$\rho_B = -1$

$\beta_F = 0.5$

Negative value for ρ_B means eating beetle (prey) kills frog (predator).

$$A = \begin{pmatrix} 6 & -2 \\ -1 & 0.5 \end{pmatrix} \quad ; \quad \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 6.3423 \\ 0.1577 \end{bmatrix}$$

The graph looks like:



It can be seen that, for high initial values of beetle population, frogs eat more and more beetle and eventually perish, which is supported by direction of arrow.

Once the population of frogs become zero, we need to remove it from the equation

Case -(iv)

Frogs grow exponentially but beetle dies out

It is known that predator (frog) survives only because of prey (beetle). Therefore, if prey dies out then predator also should die. However, if $\beta_F < 0$, then mathematically we have a solution. This is not possible in real world.

The parameters for this scenario are:

$\alpha_B = 3$, $\beta_B = 1$

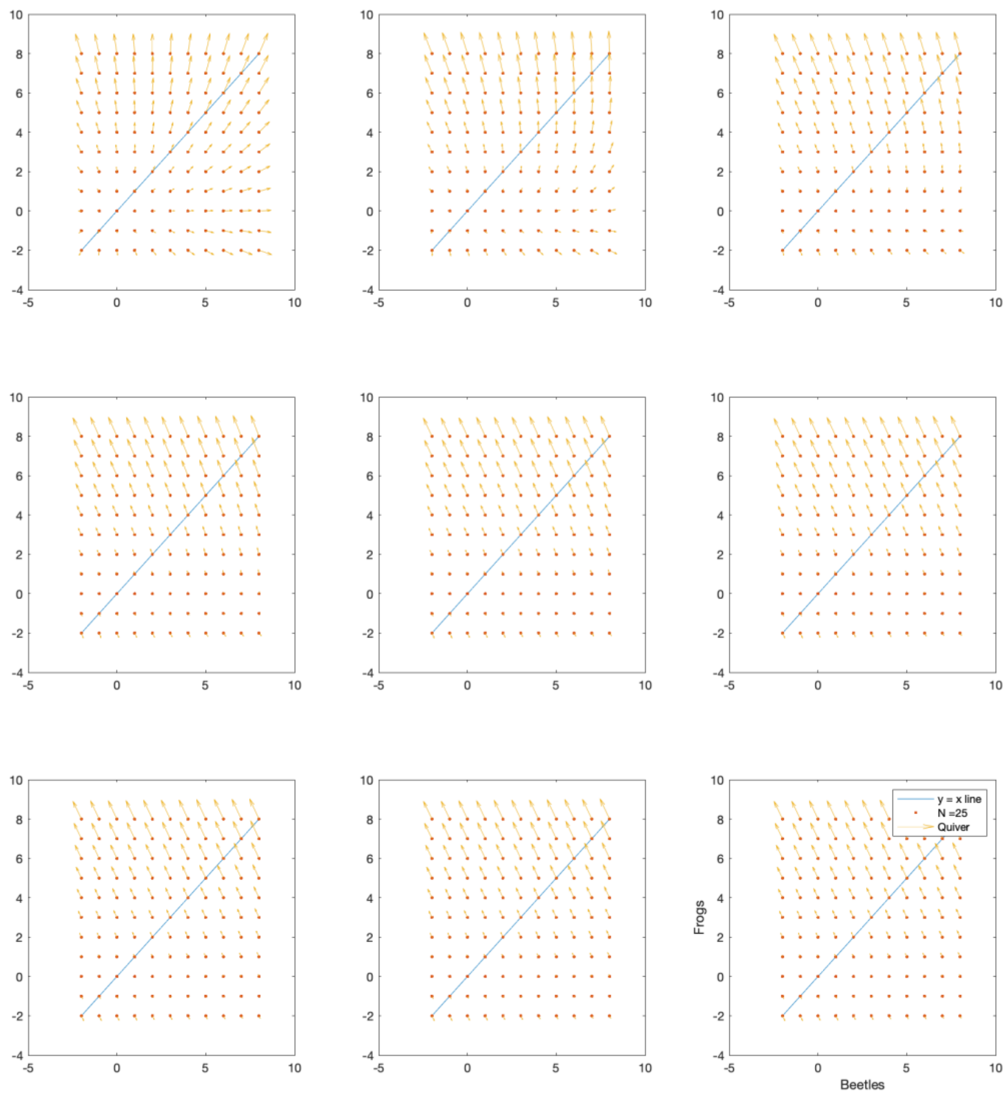
$\gamma_f = 0.5$

$\rho_B = 0.1$

$\beta_f = -3$

$$A = \begin{bmatrix} 3 & -0.5 \\ 0.1 & 4 \end{bmatrix}, \quad \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 3.0528 \\ 3.9472 \end{bmatrix}$$

The graph looks like the following:



It Can be Seen that Population of frogs increase over time. Beetles die out.

Case-(V)

Oscillating population.

It is quite possible in realworld that at times population of frog can increase over and decrease at other times. Same case with beetle. The parameters that support this scenario are:

$\alpha_B = 3$, $\beta_B = 1$

$\gamma_F = 2$

$\rho_B = 3$

$\beta = 0.5$

$$A = \begin{pmatrix} 3 & -2 \\ 3 & 0.5 \end{pmatrix} \; ; \; \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 1.75 + i\,2.1065 \\ 1.75 - i\,2.1065 \end{bmatrix}$$

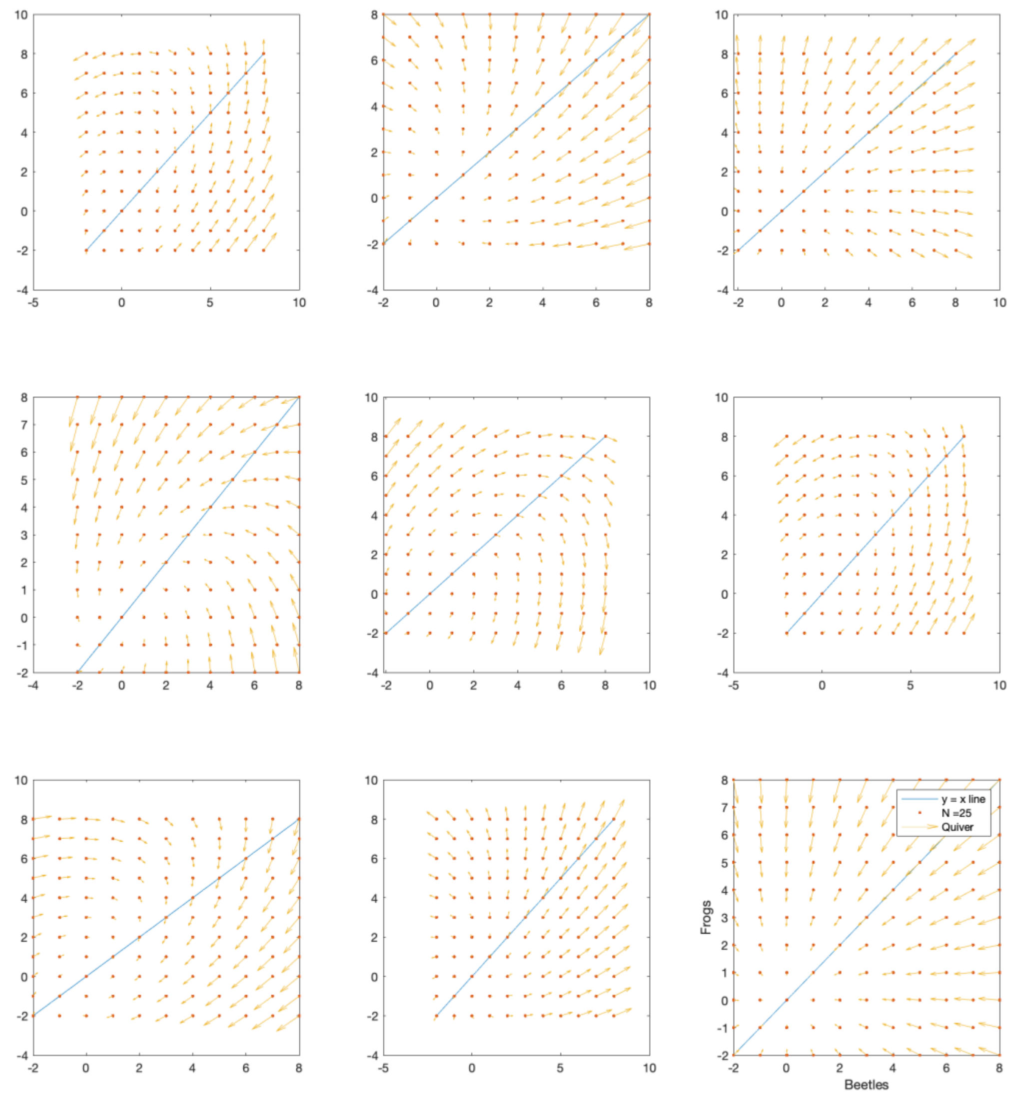
Here the eigen values are Complex. This is the reason for oscillating behaviour of population.

Because Solution Comes in form of,

$$f\left(e^{a+ib}\right) = f\left[e^{a}\left(\cos b + i \sin b\right)\right]$$

The "sin" and "cos" in above function are responsible for the oscillating behavior.

The graph looks like the following:



Let, initial beetle population = 8 million

initial frog population = 4000

After year,

y=1, the population of both beetle and frog is increasing

$y=4$, the population of both beetle and frog is decreasing

$y=7$, the population of both increases

$y=10$, the population of beetle decreases and frog increases

$y=13$, the population of beetle increases and frog decreases

$y=16$, the population of both increases

$y=19$, the population of both decreases

$y=22$, the population of both increases

$y=25$, the population of both decreases

--- and so on.