Math 4570- MM in Data Analysis and ML -Homework 1

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I have attached pdf for my handwriting problems and for python codes and its corresponding results after this latex part. I apologize that some of the questions have different orders as I am not familiar to insert a picture in latex.

Question 1

1. If a and b are rational numbers, then a and b can be expressed as

$$a = \frac{m}{n}, b = \frac{o}{p}$$

$$x = a + b\sqrt{2}$$

where m, n, o,and p are real numbers. $\forall a, b, m, n, o, px \in F$, I am going to show that the following axioms apply to x, which leads to show that F is a field.

(a) Identity
There exists $e \in F$ such that

$$e + x = x + e = x$$

$$0 + x = x + 0 = x$$

Thus, $e = 0$ under the addition.

Similarly, for multiplication,

$$ex = xe = x \times 1 = x$$

Thus, $e = 1$ under the multiplication.

(b) Associativity for sum
Because

$$\begin{aligned} x + (y + z) &= a + b\sqrt{2} + (c + d\sqrt{2} + e + f\sqrt{2}) \\ &= (a + c + e) + (b + d + f)\sqrt{2} \\ (x + y) + z &= (a + b\sqrt{2} + c + d\sqrt{2}) + e + f\sqrt{2} \\ &= (a + c + e) + (b + d + f)\sqrt{2} \end{aligned} \tag{1}$$

Thus,

$$x + (y + z) = (x + y) + z$$

(c) Inverse for sum
There exists f such that $x + f = 0$. Denoted as $f = -x$,

$$x + f = f + x = -x + x = 0$$

Thus, $f = -x$ is an inverse for sum.

(d) Commutativity for sum
$A + B = B + A$ holds for addition.
For $x, y \in F$,

$$\begin{aligned} x + y &= (a + b\sqrt{2}) + (c + d\sqrt{2}) \\ &= (c + d\sqrt{2}) + (a + b\sqrt{2}) \\ &= y + x \end{aligned} \tag{2}$$

Thus, commutativity holds for sum.

(e) Associativity for multiplication
For $x, y, z \in F$,

$$\begin{aligned}(xy)z &= \{(a+b\sqrt{2})(c+d\sqrt{2})\}(e+f\sqrt{2}) \\ &= \{ac+2bd+(ad+bc)\sqrt{2}\}(e+f\sqrt{2}) \\ &= (ace+2adf+2bcf+2bde)+\sqrt{2}(acf+ade+bce+2bdf) \\ x(yz) &= (a+b\sqrt{2})\{(c+d\sqrt{2})(e+f\sqrt{2})\} \\ &= (ace+2adf+2bcf+2bde)+\sqrt{2}(acf+ade+bce+2bdf)\end{aligned} \tag{3}$$

Thus, $x(yz) = (xy)z$, so associativity holds for multiplication.

(f) Distributivity
For $x, y, z \in F$,

$$\begin{aligned}x(y+z) &= (a+b\sqrt{2})\{(c+d\sqrt{2})+(e+f\sqrt{2})\} \\ &= a(c+e)+(adf+bc+be)\sqrt{2}+2bdf \\ xy+xz &= (a+b\sqrt{2})(c+d\sqrt{2})+(a+b\sqrt{2})(e+f\sqrt{2}) \\ &= \{ac+(ad+bc)\sqrt{2}+2bd\}+\{ae+(af+be)\sqrt{2}+2bf\} \\ &= a(c+e)+(adf+bc+be)\sqrt{2}+2bdf\end{aligned} \tag{4}$$

Thus $x(y+z) = xy + xz$ holds, which verified the distributivity.

(g) Commutativity for product
$AB = BA$ holds for product.
For $x, y \in F$,

$$\begin{aligned}xy &= (a+b\sqrt{2})(c+d\sqrt{2}) \\ &= (ac+2bd)+\sqrt{2}(ad+bc) \\ yx &= (c+d\sqrt{2})(a+b\sqrt{2}) \\ &= (ac+2bd)+\sqrt{2}(ad+bc)\end{aligned} \tag{5}$$

Because $xy = yx$, commutativity holds for product.

(h) Inverse for multiplication
There exists an inverse, denoted as $f = x^{-1}$ such that $xf = fx = 1$. For $x \in F$,

$$xf = fx = \frac{(a+b\sqrt{2})}{(a+b\sqrt{2})} = 1 \tag{6}$$

Thus, $f = x^{-1}$, and it is an inverse for multiplication.

As shown above, the set of all numbers of the form $a + b\sqrt{2}$ are fields under the usual addition and multiplication.

2. Let $x = a + bi, y = c + di$ where $a, b, c, d \in \mathbb{R}$. Noted that $\sqrt{-1} = i$. To show that the set of complex numbers is a field with the usual sum, scalar product and product, I will go over all axioms for a field.

(a) Identity
There exists $e \in F$ such that

$$e + x = x + e = x : (e = 0)$$

$$ex = xe = 1 : (e = 1)$$

Thus, there exists $e \in F$ such that $e = 0$ for sum and $e = 1$ for multiplication.

(b) Distributivity
Distributivity $A(B+C) = AB + AC$ holds under complex numbers and is shown as the similar procedure as what I have shown in (1).

(c) Associativity for sum and multiplication
Associativity for sum is obvious. $x + y = y + x$ holds in F.
For multiplication,

$$\begin{aligned} (xy)z &= \{(a+bi)(c+di)\}(e+fi) \\ &= \{(ac-bd)+(ad+bc)i\}(e+fi) \\ &= (ace-adf-bcf-bdf)+i(acf+ade+bce-bdf) \\ x(yz) &= (a+bi)\{(c+di)(e+fi)\} \\ &= (ace-adf-bcf-bdf)+i(acf+ade+bce-bdf) \end{aligned} \tag{7}$$

Thus, $x(yz) = (xy)z$, so the associativity holds for complex multiplication as well as sum.

(d) Inverse of sum
There exists an inverse $f \in F$ such that $x + f = f + x = 0$. For summation, the inverse is $f = -x$:

$$x + (-x) = (-x) + x = 0$$

(e) Inverse of multiplication
There exists an inverse $f \in F$ such that $xf = fx = 1$. For multiplication, the inverse is $f = x^{-1}$:

$$\begin{aligned} xx^{-1} &= x^{-1}x = 1 \\ (a+bi)\frac{1}{a+bi} &= \frac{1}{a+bi}(a+bi) = 1 \end{aligned}$$

(f) Commutativity for sum and multiplication
This is redundant work as shown in Q1 (1), so I will skip this. Commutativity holds for sum and multiplication under complex number as well.

Question 2
Commutativity does not hold for matrices
Generally speaking, the multiplication of matrices is not commutative. That is, $AB \neq BA$ in most cases.
For $n = 1$, $A \in \mathbb{R}$, meaning that A is just a real number, so $ab = ba$ for $a, b \in \mathbb{R}$. Therefore, for $n > 1$, the multiplication of matrices is not a field because it violates the condition of a field, commutativity.
Specifically, let's take an example of 2×2 matrices. Let A and B be 2×2 square matrices. Then,

$$\begin{aligned} \text{AB} &= \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 9 & 1 \end{pmatrix} \\ \text{BA} &= \begin{pmatrix} 2 & 2 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 2 \\ 18 & 5 \end{pmatrix} \end{aligned}$$

(8)In this way, $AB \neq BA$ in 2×2 matrices. This violates one of the conditions being a field, commutativity. Therefore, the set of square matrices is not a field.

Question 3 See the attached pdf
Question 4
Let $r = a + bi, s = c + di, t = e + fi$. where $a, b, c, d, e, f \in \mathbb{R}$

1. addition

(a) identity There exists an identity e such that

$$e + r = r + e = r$$

$$0 + r = r + 0 = r$$

Thus, $e = 0$ for addition.

(b) associativity

$$\begin{aligned} r + (s + t) &= a + bi + (c + di) + (e + fi) \\ &= (a + c + e) + (b + d + f)i \end{aligned} \tag{9}$$

$$\begin{aligned} (r + s) + t &= (a + bi + c + di) + e + fi \\ &= (a + c + e) + (b + d + f)i \end{aligned} \tag{10}$$

Because $r + (s + t) = (r + s) + t$, the associativity holds for complex numbers.

(c) commutativity: $A + B = B + A$

$$\begin{aligned} r + s &= (a + bi) + (c + di) \\ &= (a + c) + (b + d)i \\ s + r &= (c + di) + (a + bi) \\ &= (a + c) + (b + d)i = r + s \end{aligned} \tag{11}$$

Do the same things to $r + t = t + r, t + s = s + t$. Therefore, commutativity holds for complex addition.

(d) inverse

$$r + (-r) = (a + bi) + (-a - bi) = 0 \tag{12}$$

Similarly for s and t, there exists inverse for complex addition.

2. Multiplication

(a) identity There exists an identity e such that

$$er = re = r$$

$$(1)r = r(1) = r$$

$e = 1$ for multiplication.

(b) Associativity: A(BC) = (AB)C

$$\begin{aligned} r(st) &= (a + bi)(c + di)(e + fi) \\ &= (a + bi)ce + (cf + de)i - df \\ &= (ace - adf - bcf - bde) + (acf + ade + bce - bdf)i \\ (rs)t &= (a + bi)(c + di)(e + fi) \\ &= ac + (ad + bd)i - bd(e + fi) \\ &= (ace - adf - bcf - bde) + (acf + ade + bce - bdf)i \end{aligned} \tag{13}$$

Thus, associativity holds for complex multiplications.

(c) Commutativity: AB = BA

$$\begin{aligned} rs &= (a+bi)(c+di) \\ &= (ac-bd)+(ad+cd)i \\ sr &= (c+di)(a+bi) \\ &= (ac-bd)+(ad+cd)i \end{aligned} \tag{14}$$

Since $rs = sr$, the same procedures apply for other combinations. Thus, commutativity holds for complex multiplication.

(d) Inverse

$$r(\frac{1}{r}) = \frac{a+bi}{a+bi} = 1 \tag{15}$$

There exists an inverse for any complex numbers which results in the complex numbers to be 1 after multiplying with it.

3. Multiplication and addition

(a) Additive and multiplicative identity
Without confirming, it is obvious that

$$\begin{aligned} r+0=r, s+0=s, t+0=t \\ r*0=s*0=t*0=0 \end{aligned} \tag{16}$$

(b) Distributivity: A(B+C) = AB+AC

$$\begin{aligned} r(s+t) &= (a+bi)(c+di+e+fi) \\ &= (a+bi)(e+di+e+fi) \\ &= a(c+e)-b(d+f)+b(c+e)+a(d+f)i \\ rs+rt &= (a+bi)(c+di)+(a+bi)(e+fi) \\ &= a(c+e)-b(d+f)+b(c+e)+a(d+f)i \end{aligned} \tag{17}$$

Thus, distributivity holds for complex addition and multiplication.

4. Scalar product
As shown above, the same axioms also holds for scalar products. I will show some of them here, but it applies to all combination of scalar products for complex numbers. Let $C \in \mathbb{R}$ be a constant.

$$Cr = C(a+bi) = (a+bi)C = rC \tag{18}$$

Question 5
B, D
Both matrices have leading 1s, and there is no entries under the leading 1s.

Question 6-10 See the attached pdf.
Question 11

1. See the attached pdf for rref calculations.

$$\begin{aligned} [0] &= \{0, \pm 7, \pm 14...\} \\ [1] &= \{1, 1\pm 7, 1\pm 14...\} \\ [2] &= \{2, 2\pm 7, 2\pm 14...\} \\ [3] &= \{3, 3\pm 7, 3\pm 14...\} \\ [4] &= \{4, 4\pm 7, 4\pm 14...\} \\ [5] &= \{5, 5\pm 7, 5\pm 14...\} \\ [6] &= \{6, 6\pm 7, 6\pm 14...\} \end{aligned} \tag{19}$$

2. See the attached pdf for rref calculations.

3. See the attached pdf

4. It is possible to have a different rank in $\mathbb{Z}_p$ As $\mathbb{Z}_2$ of the matrix has the rank of 3.

Question 12 - 15 See the attached pdf
Question 16

1. Yes. All matrices, A, B, C, are invertible because

$$\begin{aligned} det(I) &= 1 \\ &= det(ABC) \\ &= det(A)det(B)det(C) \neq 0 \end{aligned} \tag{20}$$

   Thus, $det(A), det(B), det(C) \neq 0$, meaning that all matrices A, B, C are invertible.

2. If AB is invertible, then $det(AB) \neq 0$. Because $det(A)det(B) \neq 0$, A, B are invertible.

Question 17
When $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$,

$$(AB)^2 = \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}$$

$$A^2B^2 = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} \neq (AB)^2$$

Question 18

1. $A^{-1} = A^T$ is a property of orthogonal matrix.

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$A^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = A^T$$

Question 19

1. (a) symmetric matrices

$$\begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 5 \end{pmatrix}, \begin{pmatrix} 9 & 13 & 3 & 6 \\ 13 & 11 & 7 & 6 \\ 3 & 7 & 4 & 7 \\ 6 & 6 & 7 & 10 \end{pmatrix}$$

   (b) skew-symmetric $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 3 & 6 \\ 1 & 0 & 2 & -5 \\ -3 & -2 & 0 & 4 \\ -6 & 5 & -4 & 0 \end{pmatrix}$

2. The diagonal entries of skew-symmetric matrices are zero.

3. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

4. (a) **If A is symmetric, then $A + A^T$ is symmetric.**
   If A is symmetric, then $A = A^T$.
   Thus, $A + A = A + A^T$ is also symmetric.

   If $A + A^T$ is symmetric, then A is symmetric.
   If $A + A^T$ is symmetric, then

$$A + A^T = (A + A^T)^T = A^T + A$$

$$A = A^T$$

   Thus, A is symmetric.

   (b) Because A is symmetric, $A = A^T$,

$$(AA^T)^T = A^T A = AA^T$$

   (c) **If A is skew symmetric, then $A - A^T$ is skew-symmetric.**

$$(A - A^T)^T = A^T - A = -A + A^T = -(A - A^T)$$

   If $A - A^T$ is skew-symmetric, then A is skew symmetric.

$$(A - A^T)^T = -(A - A^T) = -A + A^T$$

$$A^T = -A$$

5. Let M be a $n \times n$ square matrix, A be a symmetric matrix, B be a skew-matrix.
   Suppose $M = A + B$.
   Because $A = A^T$ and $B^T = -B$,

$$M = A + B$$

$$M^T = (A + B)^T = A^T + B^T = A - B$$

   So,

$$A = \frac{M + M^T}{2}, B = \frac{M - M^T}{2}$$

   Thus,

$$M = A + B = \frac{M + M^T}{2} + \frac{M - M^T}{2} = M$$

Question 20

1. $F(x) = x^2$ is surjective
   $F : \mathbb{R} \longmapsto [0, \infty)$

2. $F(x) = \frac{x^3}{x^2+1}$ is bijective
   $F : \mathbb{R} \longmapsto \mathbb{R}$

3. $F(x) = x(x - 1)(x - 2)$ is surjective
   $F : \mathbb{R} \longmapsto \mathbb{R}$

4. $F(x) = e^x + 2$ is injective $F : \mathbb{R} \longmapsto [2, \infty)$

Question 21

From Question 22, I found a relation

$$\{p_i, q_i, r_i\} = \{d_i l_i, u_{i-1} l_{i-1} + di, u_i\}$$

.

Given a matrix A= $\begin{pmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}$, LU factorization will be

L= $\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ 0 & \frac{4}{3} & 1 & 0 \\ 0 & 0 & \frac{3}{8} & 1 \end{pmatrix}$, U= $\begin{pmatrix} 4 & 1 & 0 & 0 \\ 0 & \frac{3}{4} & 1 & 0 \\ 0 & 0 & \frac{8}{3} & 1 \\ 0 & 0 & 0 & \frac{29}{8} \end{pmatrix}$

Question 22

Let A= $\begin{pmatrix} q_1 & r_1 & 0 & 0 \\ p_1 & q_2 & r_2 & 0 \\ 0 & p_2 & q_3 & r_3 \\ 0 & 0 & p_3 & q_4 \end{pmatrix}$, L= $\begin{pmatrix} 1 & 0 & 0 & 0 \\ l_1 & 1 & 0 & 0 \\ 0 & l_2 & 1 & 0 \\ 0 & 0 & l_3 & 1 \end{pmatrix}$, U= $\begin{pmatrix} d_1 & u_1 & 0 & 0 \\ 0 & d_2 & u_2 & 0 \\ 0 & 0 & d_3 & u_3 \\ 0 & 0 & 0 & d_4 \end{pmatrix}$

And $A = LU$. By multiplying LU, I get

A = LU = $\begin{pmatrix} d_1 & u_1 & 0 & 0 \\ d_1 l_1 & u_1 l_1 + d_2 & u_2 & 0 \\ 0 & d_2 l_2 & u_2 l_2 + d_3 & u_3 \\ 0 & 0 & d_3 l_3 & u_3 l_3 + d_4 \end{pmatrix}$

Thus, $\{p_i, q_i, r_i\} = \{d_i l_i, u_{i-1} l_{i-1} + di, u_i\}$.

Question 23

Given the $n \times n$ matrix, LU factorization will follow the rules such that:

$$\{p_i, q_i, r_i\} = \{d_i l_i, u_{i-1} l_{i-1} + di, u_i\}$$

Specifically,

$$\{d_i = 4 - l_{i-1}, l_i = \frac{1}{d_i}, u_i = 1\}$$

When i=1, $d_1 = 4$.

Question 24

1. If H_n is a symmetric matrix, then ${H_n}^T = H_n$.

$$\begin{aligned} \{H_n\}^T &= (I_n - 2uu^T)^T \\ &= I_n^T - 2(uu^T)^T \\ &= I_n - 2(u^T)^T(u^T) \\ &= I_n - 2(uu^T) \\ &= H_n \end{aligned} \tag{21}$$

   Hence, H_n is a symmetric matrix.

2. If H_n is an orthogonal matrix, then $H_n^T H_n = I_n$.

$$\begin{aligned} H_n^T H_n &= H_n^2 \\ &= (I_n - 2uu^T)(I_n - 2uu^T) \\ &= I_n^2 - 2uu^T - 2uu^T + 4(uu^T)^2 \\ &= I_n - 4 + 4 \\ &= I_n \end{aligned} \tag{22}$$

   Since $H_n^T H_n = I_n$, H_n is an orthogonal matrix.

3. $H_n^2 = H_n H_n = H_n^T H_n = I_n$ (The derivation has shown above.)

4.

$$\begin{aligned} H_n u &= (I_n - 2uu^T)u \\ &= I_n u - 2uu^T u \\ &= u - 2u \\ &= -u \end{aligned} \tag{23}$$

5. (a) $u_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

   Then,

$$\begin{aligned} H_3 &= I_3 - 2uu^T \\ &= I_3 - \frac{2}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \end{aligned} \tag{24}$$

   (b) $u_4 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

   Then,

$$\begin{aligned} H_4 &= I_4 - 2uu^T \\ &= I_4 - \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{aligned} \tag{25}$$

Q3

$\mathbb{Z}_3$

+	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

$[0] = \{0, \pm 3, \pm 6 \cdots\}$

$[1] = \{1, 1 \pm 3, 1 \pm 6 \cdots\}$

$[2] = \{2, 2 \pm 3, 2 \pm 6 \cdots\}$

X	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

Q6

$\underset{\sim\sim}{\mathbb{Z}_2}$ $\quad [0] = \{0, \pm 2, \pm 4 \cdots\}$

$[1] = \{1, 1\pm 2, 1\pm 4 \cdots\}$

$$A+B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & {}^{2}0 & {}^{2}0 \\ 1 & {}^{2}0 & {}^{2}0 \\ 1 & {}^{2}0 & 0 \end{bmatrix} \xrightarrow{\text{ref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Q7 $A = \begin{bmatrix} 6 & -1 & 1 \\ t & 0 & 1 \\ 0 & 1 & t \end{bmatrix}$

$\det A \neq 0 \longrightarrow A$ is invertible.

$$\det A = 6\begin{vmatrix} 0 & 1 \\ 1 & t \end{vmatrix} + \begin{vmatrix} t & 1 \\ 0 & t \end{vmatrix} + \begin{vmatrix} t & 0 \\ 0 & 1 \end{vmatrix}$$

$$= 6(-1) + (t^2) + t$$

$$= -6 + t^2 + t$$

$$= t^2 + t - 6$$

$$= (t+3)(t-2) = 0$$

$$t = -3, 2.$$

when $t = -3, 2$, A is not invertible, meaning A does not have an inverse.

Q8

a) $\left[\begin{array}{cc|c} 1 & h & 4 \\ 3 & 6 & 8 \end{array}\right] \xrightarrow{r_2 - 3r_1} \left[\begin{array}{cc|c} 1 & h & 4 \\ 0 & 6-3h & -4 \end{array}\right]$

To be inconsistent,

$$6 - 3h = 0.$$

$$3h = 6 \quad \underline{h = 2}$$

b) $\left[\begin{array}{cc|c} -4 & 12 & h \\ 2 & -6 & -3 \end{array}\right] \xrightarrow{\div 2} \left[\begin{array}{cc|c} -2 & 6 & h \\ 1 & -3 & -\frac{3}{2} \end{array}\right]$

$\xrightarrow{r_1 + 2r_2} \left[\begin{array}{cc|c} 0 & 0 & h-3 \\ 1 & -3 & -\frac{3}{2} \end{array}\right]$

$\xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & -3 & -\frac{3}{2} \\ 0 & 0 & h-3 \end{bmatrix}$

$h-3 \neq 0 \longrightarrow h \neq 3$ for being inconsistent.

Q9.

1) $\begin{bmatrix} 1 & * \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ 2 types. $(* = 0, 1, \cdots)$

2) rank 0 : $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

rank 1 : $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}$

7 types

rank 2 : $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & * \end{bmatrix}$

3) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ 1 type.

Q10

$$A = \begin{bmatrix} 1 & a & b & 3 & 0 & -2 \\ 0 & 0 & c & 1 & d & 3 \\ 0 & e & 0 & 0 & 1 & 1 \end{bmatrix}$$

a $*$ d $*$

b $*$ e 0

c 1

Q11 $$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix}$$

1) $$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[r_3 - 2r_2]{r_1 - r_2} \begin{bmatrix} 0 & 1 & 3 & 2 \\ 1 & 1 & 0 & 2 \\ 0 & -2 & 1 & -2 \end{bmatrix} \xrightarrow[r_3 + 2r_1]{r_2 - r_1} \begin{bmatrix} 0 & 1 & 3 & 2 \\ 1 & 0 & -3 & 0 \\ 0 & 0 & 7 & 2 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 7 & 2 \end{bmatrix} \xrightarrow{\frac{1}{7}r_3} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix}$$

$$\xrightarrow[r_2 - 3r_3]{r_1 + 3r_3} \begin{bmatrix} 1 & 0 & 0 & \frac{6}{7} \\ 0 & 1 & 0 & \frac{8}{7} \\ 0 & 0 & 1 & \frac{2}{7} \end{bmatrix}$$

$$\therefore \vec{x} = -\frac{2}{7} s \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} : s \in \mathbb{R}$$

//

2) $\mathbb{Z}_7$

$$[0] = \{0,\ \pm 7,\ \pm 14 \cdots\}$$
$$[1] = \{1,\ 1\pm 7,\ 1\pm 14 \cdots\}$$
$$[2] = \{2,\ 2\pm 7,\ 2\pm 14 \cdots\}$$
$$[3] = \{3,\ 3\pm 7,\ 3\pm 14 \cdots\}$$
$$[4] = \{4,\ 4\pm 7,\ 4\pm 14 \cdots\}$$
$$[5] = \{5,\ 5\pm 7,\ 5\pm 14 \cdots\}$$
$$[6] = \{6,\ 6\pm 7,\ 6\pm 14 \cdots\}$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[r_3+2r_1]{r_2-r_1} \begin{bmatrix} [0] & [1] & [3] & [2] \\ [1] & [0] & [4] & [0] \\ [0] & [0] & [0] & [2] \end{bmatrix}$$

$$\xrightarrow[r_1 \Leftrightarrow r_2]{\frac{1}{2}r_3} \begin{bmatrix} [1] & [0] & [4] & [0] \\ [0] & [1] & [3] & [2] \\ [0] & [0] & [0] & [1] \end{bmatrix} \xrightarrow{r_2-2r_3} \begin{bmatrix} [1] & [0] & [4] & [0] \\ [0] & [1] & [3] & [0] \\ [0] & [0] & [0] & [1] \end{bmatrix}$$

//

MATH 4570 Matrix methods for DA and ML

HW 1 Manami Kanemura

In [108]:
```
import numpy as np
from sympy import Matrix, pprint
import galois
import sympy as sym
from sympy.interactive.printing import init_printing
from sympy.matrices import Matrix, eye, zeros, ones, diag, GramSchmidt
from sympy import *
from numpy import shape
```

Q11-3

(a) Calculate rref(A)

In [109]:
```
A = Matrix([[1, 2, 3, 4], [1, 1, 0, 2], [2, 0, 1, 2]])
A
```

Out[109]:
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 2 \end{bmatrix}$$

In [110]:
```
rrefA = A.rref()
rrefA
```

Out[110]:
```
(Matrix([
 [1, 0, 0, 6/7],
 [0, 1, 0, 8/7],
 [0, 0, 1, 2/7]]),
 (0, 1, 2))
```

This result agrees with my calculation.

(b) Calculate rref(A) over Z2 and Z3 fields.

In [111]:
```
def get_rref (A, n):
    An = np.mod(A, n)
    pprint(An)
    print("")
    GFn = galois.GF(n)
    pprint(GFn.row_reduce(An))

    return
```

In [112]:
```
## Matrix A over Z2 and its rref
get_rref(A, 2)
```

```
[[1 0 1 0]
 [1 1 0 0]
[0 0 1 0]]

[[1 0 0 0]
 [0 1 0 0]
[0 0 1 0]]
```

In [113]:
```
## Matrix A over Z3 and its rref
get_rref(A, 3)
```

```
[[1 2 0 1]
 [1 1 0 2]
[2 0 1 2]]

[[1 0 0 3]
[0 1 0 -1]
[0 0 1 -4]]
```

Q 12

In [130]:
```
M12 = Matrix([[3, 1, 4, 1], [5, 2, 6, 5], [0, 5, 2, 1]])
get_rref(M12, 7)
```

```
[[3 1 4 1]
 [5 2 6 5]
[0 5 2 1]]

[[1 0 0 31/6]
 [0 1 0 11/6]
[0 0 1 -49/12]]
```

In [132]:
```
a12 = M12.col([0, 1, 2])
b12 = M12.col(3)
```

In [155]:
```
def solve_linsys_param (a, b):
    num_equations, num_variables = a.shape
    x = sym.symarray('x', num_variables)
    solution = sym.solve([sym.Eq(ax-b) for ax, b in zip(np.dot(a, x), b)])
    print(solution)
    return
```

In [156]:
```
solution = solve_linsys_param(a12, b12)
```

```
{x_0: 31/6, x_1: 11/6, x_2: -49/12}
```

Q 13: Solve for a linear system

In [114]:
```
M = Matrix([[3, 6, 9, 5, 25, 53], [7, 14, 21, 9, 53, 105], [-4, -8, -12, 5, -10, 11]])
M_rref = M.rref()[0]## this returns tuple
M_rref
```

Out[114]:
$$\begin{bmatrix} 1 & 2 & 3 & 0 & 5 & 6 \\ 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In [115]:
```
#a = M_rref.col_del(1)

#b = M_rref.col(4)
#b
a_13 = M_rref.col([0, 1, 2, 3, 4])

b_13 = M_rref.col(5)
print(a, b)
```

```
Matrix([[1, 2, 3, 0, 5], [0, 0, 0, 1, 2], [0, 0, 0, 0, 0]]) Matrix([[6], [7], [0]])
```

In [117]:
```
solve_linsys_param(a_13, b_13)
```

```
{x_3: 7 - 2*x_4, x_0: -2*x_1 - 3*x_2 - 5*x_4 + 6}
```

Q 14: Solve the linear system

In [118]:
```
M14 = Matrix([[3, 6, 9, 5, 25, 53], [7, 14, 21, 9, 53, 105], [-4, -8, -12, 5, -10, 11]])
M14_rref = M14.rref()
a_14 = M14_rref[0].col([0, 1, 2, 3, 4])
b_14 = M14_rref[0].col(5)
```

In [119]:
```
a_14
```

Out[119]:
$$\begin{bmatrix} 1 & 2 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In [120]:
```
b_14
```

Out[120]:
$$\begin{bmatrix} 6 \\ 7 \\ 0 \end{bmatrix}$$

In [121]:
```
solve_linsys_param(a_14, b_14)
```

```
{x_3: 7 - 2*x_4, x_0: -2*x_1 - 3*x_2 - 5*x_4 + 6}
```

Q15: Solve the linear system

In [122]:
```
M15 = Matrix([[2, 4, 3, 5, 6, 37],
              [4, 8, 7, 5, 2, 74],
              [-2, -4, 3, 4, -5, 20],
              [1, 2, 2, -1, 2, 26],
              [5, -10, 4, 6, 4, 24]
             ])
#M15_rref = M15.rref()
#M15_rref[0]
M15
```

Out[122]:
$$\begin{bmatrix} 2 & 4 & 3 & 5 & 6 & 37 \\ 4 & 8 & 7 & 5 & 2 & 74 \\ -2 & -4 & 3 & 4 & -5 & 20 \\ 1 & 2 & 2 & -1 & 2 & 26 \\ 5 & -10 & 4 & 6 & 4 & 24 \end{bmatrix}$$

In [126]:
```
a_15 = M15.col([0, 1, 2, 3, 4])
b_15 = M15.col(5)
```

In [128]:
```
solve_linsys_param(a_15, b_15)
```

```
{x_0: -8221/4340, x_1: 8591/8680, x_2: 4695/434, x_3: -459/434, x_4: 699/434}
```

In [ ]: