MATH 4570 HW 1 Hannah Marks

1 (1) sum is defined as $(a+b\sqrt{2})+(c+d\sqrt{2}) = a+c+(b+d)\sqrt{2}$

product is defined as $(a+b\sqrt{2})(c+d\sqrt{2}) = ac+2bd+(ad+bc)\sqrt{2}$

identity for sum: $a+b\sqrt{2}+0 = 0+a+b\sqrt{2} = a+b\sqrt{2}$

associativity for sum: $(a+b\sqrt{2}+c+d\sqrt{2})+f+g\sqrt{2} = a+b\sqrt{2}+(c+d\sqrt{2}+f+g\sqrt{2})$

$= a+c+f+(b+d+g)\sqrt{2}$

inverse for sum: $\forall\, a+b\sqrt{2} \in F,\ \exists\ -(a+b\sqrt{2}) = -a-b\sqrt{2}$

$a+b\sqrt{2}-a-b\sqrt{2} = 0$

commutativity for sum: $a+b\sqrt{2}+c+d\sqrt{2} = c+d\sqrt{2}+a+b\sqrt{2} = (a+c)+(b+d)\sqrt{2}$

$= (c+a)+(d+b)\sqrt{2}$

since regular addition of ($a+c$ and $b+d$) is commutative

multiplicative identity: $(a+b\sqrt{2})\cdot 1 = 1\cdot(a+b\sqrt{2}) = a+b\sqrt{2}$

associativity for product: $[(a+b\sqrt{2})(c+d\sqrt{2})](f+g\sqrt{2}) = (a+b\sqrt{2})[(c+d\sqrt{2})(f+g\sqrt{2})]$

$(ac+2bd+(ad+bc)\sqrt{2})(f+g\sqrt{2}) = (a+b\sqrt{2})(cf+2dg+(cg+df)\sqrt{2})$

$acf+2bdf+2adg+2bcg+(acg+2adg+adf+bcf)\sqrt{2})$

$= acf+2adg+2bcg+2bdf+(acg+adf+bcf+2bdg)\sqrt{2}$

distributivity: $(a+b\sqrt{2})\cdot[(c+d\sqrt{2})+(f+g\sqrt{2})] = (a+b\sqrt{2})(c+d\sqrt{2})+(a+b\sqrt{2})(f+g\sqrt{2})$

$(a+b\sqrt{2})[c+f+(d+g)\sqrt{2})$ $= ac+2bd+(ad+bc)\sqrt{2}+af+2bg+(ag+bf)\sqrt{2}$

$ac+af+(ad+ag+bc+bf)\sqrt{2}+2(bd+bg) = ac+af+2(bd+bg)+(ad+ag+bc+bf)\sqrt{2}$

commutativity for product: $(a+b\sqrt{2})(c+d\sqrt{2}) = (c+d\sqrt{2})(a+b\sqrt{2})$

$ac+2bd+(ad+bc)\sqrt{2} = ca+2db+(cb+da)\sqrt{2}$

since regular multiplication is commutative, $ac=ca$, $bd=db$, $ad=da$, $bc=cb$

inverse for product: $(a+b\sqrt{2})^{-1} = x+y\sqrt{2}$

$(a+b\sqrt{2})(x+y\sqrt{2}) = 1$

$ax+2by+(ay+bx)\sqrt{2} = 1$

so $ax+2by = 1$, $ay+bx = 0$

$$\left[\begin{array}{cc|c} a & 2b & 1 \\ b & a & 0 \end{array}\right] \xrightarrow{\frac{1}{a}\cdot R1} \left[\begin{array}{cc|c} 1 & \frac{2b}{a} & 1/a \\ b & a & 0 \end{array}\right] \xrightarrow{-bR1+R2} \left[\begin{array}{cc|c} 1 & 2b/a & 1/a \\ 0 & -2b^2 & -b/a \end{array}\right]$$

$$\xrightarrow{1/-2b^2\cdot R2} \left[\begin{array}{cc|c} 1 & 2b/a & 1/a \\ 0 & 1 & 1/2ab \end{array}\right] \xrightarrow{-2b/a\, R2+R1} \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{a}-\frac{1}{a^2} \\ 0 & 1 & 1/2ab \end{array}\right]$$

$x = \frac{1}{a}-\frac{1}{a^2}$, $y = \frac{1}{2ab}$

1 (2) sum is defined as $(a+b\sqrt{-1}) + (c+d\sqrt{-1}) = a+c+(b+d)\sqrt{-1} = a+c+(b+d)i$

product is defined as $(a+bi)(c+di) = ac-bd+(ad+bc)i$

identity for sum: $a+bi+0 = 0+a+bi = a+bi$

associativity for sum: $(a+bi+c+di)+f+gi = a+bi+(c+di+f+gi)$

$a+c+f+(b+d+g)i = (a+c+f)+(b+d+g)i$

inverse for sum: $\forall a+bi \in F, \exists -(a+bi) = -a-bi$

$a+bi-a-bi = 0$

commutativity for sum: $(a+bi)+(c+di) = (c+di)+(a+bi)$

$a+c+(b+d)i = c+a+(d+b)i$

multiplicative identity: $(a+bi)\cdot 1 = 1\cdot(a+bi) = a+bi$

associativity for product: $[(a+bi)(c+di)](f+gi) = (a+bi)[(c+di)(f+gi)]$

$(ac-bd+(ad+bc)i)(f+gi) = (a+bi)(cf-dg+(cg+df)i)$

$acf-bdf-adg-bcg+(adf+bcf+acg+bdg)i =$

$= acf-adg-bcg-bdf+(adf+bcf+acg-bdg)i$

distributivity: $(a+bi)[(c+di)+(f+gi)] = (a+bi)(c+di)+(a+bi)(f+gi)$

$(a+bi)(c+f+(d+g)i) = (ac-bd+(ad+bc)i)+(af-bg+(ag+bf)i)$

$ac+af-bd-bg+(ad+ag+bc+bf)i = ac+af-bd-bg+(ad+ag+bc+bf)i$

commutativity for product: $(a+bi)(c+di) = (c+di)(a+bi)$

$ac-bd+(ad+bc)i = ca-db+(cb+da)i$

multiplicative inverse: $(a+bi)^{-1} = x+yi$

$(a+bi)(x+yi) = 1$

$(ax-by)+(ay+bx)i = 1$

$ax-by = 1, \quad ay+bx = 0$

$$\left[\begin{array}{cc|c} a & -b & 1 \\ b & a & 0 \end{array}\right] \xrightarrow{1/a R1} \left[\begin{array}{cc|c} 1 & -b/a & 1/a \\ b & a & 0 \end{array}\right] \xrightarrow{-bR1+R2} \left[\begin{array}{cc|c} 1 & -b/a & 1/a \\ 0 & -b^2 & -b/a \end{array}\right]$$

$$\xrightarrow{-1/b^2 R2} \left[\begin{array}{cc|c} 1 & -b/a & 1/a \\ 0 & 1 & 1/ab \end{array}\right] \xrightarrow{b/a R2+R1} \left[\begin{array}{cc|c} 1 & 0 & 1/a + 1/a^2 \\ 0 & 1 & 1/ab \end{array}\right]$$

$x = \frac{1}{a} + \frac{1}{a^2}, \quad y = \frac{1}{ab}$

a+bi is the set of complex numbers!

2 let $n = 2 > 1$

let $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$

then $AB = \begin{bmatrix} 2 & 4 \\ 4 & 6 \end{bmatrix}$ and $BA = \begin{bmatrix} 8 & 2 \\ 2 & 0 \end{bmatrix}$

thus the product of two $n \times n$ matrices is not commutative and $\therefore$ not a field

3

+	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

x	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

4 see #1(2)

5 B and D are in row-echelon form

6 $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$A + B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $A^2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

7 $\begin{bmatrix} 6 & -1 & 1 \\ t & 0 & 1 \\ 0 & 1 & t \end{bmatrix} = A$

if $t = 0$, A DOES have an inverse

if $t \neq 0$, A DOESN'T have an inverse

8 $\left[\begin{array}{cc|c} 1 & h & 4 \\ 3 & 6 & 8 \end{array}\right] \xrightarrow{-3R1+R2} \left[\begin{array}{cc|c} 1 & h & 4 \\ 0 & 6-3h & -4 \end{array}\right] \xrightarrow{1/3 R2 + R1} \left[\begin{array}{cc|} 1 & 2 \\ 0 & 6-3h \end{array}\right.$

consistent if $h \neq 2$

$\left[\begin{array}{cc|c} -4 & 12 & h \\ 2 & -6 & -3 \end{array}\right]$

consistent if $h = 6$.

9 (1) 3 types

$$\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix},\begin{bmatrix}1&0\\0&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix}$$

(2) 6 types

$$\begin{bmatrix}1&0&*\\0&1&*\end{bmatrix}\begin{bmatrix}1&*&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&*&*\\0&0&0\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&*\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}$$

(3) 1 type

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

10 a = any number

$b=0$ if $c=1$, b = any number if $c=0$

$d=0$

$e=0$

11 (1) $$\begin{bmatrix}1&2&3&4\\1&1&0&2\\2&0&1&2\end{bmatrix}\xrightarrow{-R1+R2}\begin{bmatrix}1&2&3&4\\0&-1&-3&-2\\2&0&1&2\end{bmatrix}\xrightarrow[-R2]{2R2+R1}\begin{bmatrix}1&0&-3&0\\0&1&3&2\\2&0&1&2\end{bmatrix}\xrightarrow{-2R1+R3}$$

$$\begin{bmatrix}1&0&-3&0\\0&1&3&2\\0&0&7&2\end{bmatrix}\xrightarrow{-R3+R2}\begin{bmatrix}1&0&-3&0\\0&1&-4&0\\0&0&7&2\end{bmatrix}\xrightarrow{1/7R3}\begin{bmatrix}1&0&-3&0\\0&1&-4&0\\0&0&1&2/7\end{bmatrix}\xrightarrow[4R3+R2]{3R3+R1}\begin{bmatrix}1&0&0&8/7\\0&1&0&8/7\\0&0&1&2/7\end{bmatrix}$$

(2)

	[0]	[1]	[2]	[3]	[4]	[5]	[6]
additive inverse		[6]	[5]	[4]	[3]	[2]	[1]
multiplicative inverse		[1]	[4]	[5]	[2]	[3]	[6]

$$\begin{bmatrix}1&2&3&4\\1&1&0&2\\2&0&1&2\end{bmatrix}\xrightarrow{-R1+R2}\begin{bmatrix}1&2&3&4\\0&6&4&5\\2&0&1&2\end{bmatrix}\xrightarrow{2R2+R1}\begin{bmatrix}1&0&4&0\\0&6&4&5\\2&0&1&2\end{bmatrix}\xrightarrow{6R2}\begin{bmatrix}1&0&4&0\\0&1&3&2\\2&0&1&2\end{bmatrix}$$

$$\xrightarrow{-2R1+R3}\begin{bmatrix}1&0&4&0\\0&1&3&2\\0&0&0&2\end{bmatrix}\xrightarrow[4R3]{-R3+R2}\begin{bmatrix}1&0&4&0\\0&1&3&0\\0&0&0&1\end{bmatrix}$$

(3) verified on Python

rref(A) over $\mathbb{Z}_2$: $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}$

rref(A) over $\mathbb{Z}_3$ = $\begin{bmatrix}1&0&0&0\\0&1&0&2\\0&0&1&2\end{bmatrix}$

(4) yes

12 (1) using Python, rref $(A|\vec{b})$ over $\mathbb{Z}_7$ = $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$

(2) $A\vec{x} = \vec{b} \bmod 7$

$$\begin{bmatrix} 3 & 1 & 4 \\ 5 & 2 & 6 \\ 0 & 5 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} \bmod 7$$

$$\boxed{\vec{x} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}}$$

13 solved using Python: $x_1 = x_3$, $x_2 = -2x_3 = -2x_1$

14 solved using Python: $x_1 + 2x_2 + 3x_3 + 5x_5 = 6$

$x_4 + 2x_5 = 7$

15 solved using Python: $x_1 = -1.894$, $x_2 = .9897$, $x_3 = 10.818$, $x_4 = -1.0576$, $x_5 = 1.6106$

16 (1) yes $(AB)(C) = I_n$ so $(AB)^{-1} = C$ and $C^{-1} = AB$

then since AB is invertible, A and B are invertible and

$A^{-1} = BC$ and $B^{-1} = A^{-1}C^{-1}$

(2) yes

17 $$\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\right)^2 = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 10 & 3 \\ 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2 \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 4 \\ 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 10 & 3 \\ 3 & 0 \end{bmatrix} \neq \begin{bmatrix} 9 & 4 \\ 2 & 1 \end{bmatrix}$$

18 $$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\shortparallel$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{T} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

19 (1)

	symmetric	skew-symmetric
2x2	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
3x3	$\begin{bmatrix} 7 & 1 & 1 \\ 1 & 8 & 2 \\ 1 & 2 & 9 \end{bmatrix}$	$\begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix}$
4x4	$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 4 \\ 3 & 3 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix}$	$\begin{bmatrix} 0 & -7 & 2 & 3 \\ 7 & 0 & -5 & -6 \\ -2 & 5 & 0 & 1 \\ -3 & 6 & -1 & 0 \end{bmatrix}$

(2) the main diagonal of a skew-symmetric matrix is all zeroes because it has to be the negative of itself and zero is the only number for which this is possible

(3) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(4) $$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = A \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

$$A + A^T = \begin{bmatrix} a_{11}+a_{11} & a_{12}+a_{21} & \cdots & a_{1n}+a_{n1} \\ a_{21}+a_{12} & a_{22}+a_{22} & \cdots & a_{2n}+a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{n1}+a_{1n} & a_{n2}+a_{2n} & \cdots & a_{nn}+a_{nn} \end{bmatrix}$$

since addition is commutative, $a_{12}+a_{21} = a_{21}+a_{12}$, $a_{1n}+a_{n1} = a_{n1}+a_{1n}$, etc, so $A+A^T$ is symmetric

$$AA^T = \begin{bmatrix} a_{11}^2+a_{12}^2+\cdots+a_{1n}^2 & a_{11}a_{21}+a_{12}a_{22}+\cdots+a_{1n}a_{2n} & \cdots & a_{11}a_{n1}+a_{12}a_{n2}+\cdots+a_{1n}a_{nn} \\ a_{21}a_{11}+a_{22}a_{12}+\cdots+a_{n1}a_{1n} & a_{21}^2+a_{22}^2+\cdots+a_{2n}^2 & \cdots & a_{21}a_{n1}+a_{22}a_{n2}+\cdots+a_{2n}a_{nn} \\ \vdots & \vdots & & \vdots \\ a_{n1}a_{11}+a_{n2}a_{12}+\cdots+a_{nn}a_{1n} & a_{n1}a_{21}+a_{n2}a_{22}+\cdots+a_{nn}a_{2n} & \cdots & a_{n1}^2+a_{n2}^2+\cdots+a_{nn}^2 \end{bmatrix}$$

since multiplication is commutative, $a_{11}a_{21} = a_{21}a_{11}$, $a_{12}a_{22} = a_{22}a_{12}$ etc, so AA^T is symmetric

$$A^TA = \begin{bmatrix} a_{11}^2+\cdots+a_{n1}^2 & \cdots & a_{11}a_{1n}+\cdots+a_{n1}a_{nn} \\ \vdots & \vdots & \vdots \\ a_{1n}a_{11}+\cdots+a_{nn}a_{n1} & \cdots & a_{1n}^2+\cdots+a_{nn}^2 \end{bmatrix}$$

as w/ AA^T, multiplication is commutative, so $a_{11}a_{1n} = a_{1n}a_{11}$, etc. and A^TA is symmetric

19 (4 cont.)

$$A - A^T = \begin{bmatrix} 0 & a_{12}-a_{21} & \cdots & a_{1n}-a_{n1} \\ a_{21}-a_{12} & 0 & \cdots & a_{2n}-a_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}-a_{1n} & a_{n2}-a_{2n} & \cdots & 0 \end{bmatrix}$$

$A - A^T$ has a zero diagonal and $-(a_{12}-a_{21}) = a_{21}-a_{12}$, $-(a_{1n}-a_{n1}) = a_{n1}-a_{1n}$, etc, so $A-A^T$ is skew-symmetric

(5) if A is a square matrix, we can write it as
$A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$
as shown above, we know that $A+A^T$ is symmetric, and $A-A^T$ is skew-symmetric.
Since for any matrix A, $(cA)^T = cA^T$, $\frac{1}{2}(A+A^T)$ is symmetric and $\frac{1}{2}(A-A^T)$ is skew-symmetric
$\therefore$ any square matrix can be written as the sum of symmetric + skew-symmetric matrices $\square$

20 a) bijective
b) injective
c) surjective
d) injective

21

$$\begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{-\frac{1}{4}R1 + R2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 15/4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{-\frac{4}{15}R2 + R3}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 0 & 4/15 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 15/4 & 1 & 0 \\ 0 & 0 & 56/15 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{-\frac{15}{56}R3 + R4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 0 & 4/15 & 1 & 0 \\ 0 & 0 & \frac{15}{56} & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 15/4 & 1 & 0 \\ 0 & 0 & 56/15 & 1 \\ 0 & 0 & 0 & 209/56 \end{bmatrix}$$

22 $LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_1 & 1 & 0 & 0 \\ 0 & l_2 & 1 & 0 \\ 0 & 0 & l_3 & 1 \end{bmatrix} \begin{bmatrix} d_1 & u_1 & 0 & 0 \\ 0 & d_2 & u_2 & 0 \\ 0 & 0 & d_3 & u_3 \\ 0 & 0 & 0 & d_4 \end{bmatrix} = \begin{bmatrix} d_1 & u_1 & 0 & 0 \\ l_1 d_1 & l_1 u_1 + d_2 & u_2 & 0 \\ 0 & l_2 d_2 & l_2 u_2 + d_3 & u_3 \\ 0 & 0 & l_3 d_3 & l_3 u_3 + d_4 \end{bmatrix}$

since $A = \begin{bmatrix} q_1 & r_1 & 0 & 0 \\ p_1 & q_2 & r_2 & 0 \\ 0 & p_2 & q_3 & r_3 \\ 0 & 0 & p_3 & q_4 \end{bmatrix}$, we can see that

$r_i = u_i$ (i.e. dummy/scale by 1)

$p_i = l_i d_i$ (i.e. scaling)

$q_i = l_{i-1} u_{i-1} + d_i$ (i.e. replacement)

23 based on answers to 22,

$$A = LU = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1/4 & 1 & 0 & \cdots & 0 \\ 0 & 4/15 & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & l_i d_i & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 & \cdots & 0 \\ 0 & 15/4 & \ddots & 0 & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & 0 & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & l_{i-1} u_{i-1} + d_i \end{bmatrix}$$

24(1) $H_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} [u_1 \; u_2 \cdots u_n]$

$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} u_1^2 & u_1u_2 & \cdots & u_1u_n \\ u_2u_1 & u_2^2 & \cdots & u_2u_n \\ \vdots & \vdots & & \vdots \\ u_nu_1 & u_nu_2 & \cdots & u_n^2 \end{bmatrix} = \begin{bmatrix} 1-2u_1^2 & -2u_1u_2 & \cdots & -2u_1u_n \\ -2u_2u_1 & 1-2u_2^2 & \cdots & -2u_2u_n \\ \vdots & & & \vdots \\ -2u_nu_1 & -2u_nu_2 & \cdots & 1-2u_n^2 \end{bmatrix}$

H_n is a symmetric matrix, as shown here. Additionally, we proved in A19 that AA^T is symmetric, so $2uu^T$ is symmetric, and we know that I_n is symmetric, so $H_n = I_n - 2uu^T$ is symmetric.

(2) Since H_n is symmetric, we know that $H_n = H_n^T$

$I = H_n^T H_n = H_n H_n$

$= \left(I_n - \frac{2uu^T}{u^Tu}\right)\left(I_n - \frac{2uu^T}{u^Tu}\right)$ ← where $u^Tu = 1$

$= I_n - \frac{4uu^T}{u^Tu} + \frac{4u(u^Tu)u^T}{(u^Tu)^2}$

$= I_n - \frac{4uu^T}{u^Tu} + \frac{4uu^T}{u^Tu^2}$

$= I_n$

$\therefore$ H_n is orthogonal

(3) $H_n^2 = I_n$ as shown above

(4) $H_n \vec{u} = \left(I_n - \frac{2uu^T}{u^Tu}\right)u$

$= u - \frac{2uu^Tu}{u^Tu}$

$= u - 2u = -u$

(5) $H_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} [1/\sqrt{3} \;\; 1/\sqrt{3} \;\; 1/\sqrt{3}]$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$

$= \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix}$

$H_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} [1/2 \;\; 1/2 \;\; 1/2 \;\; 1/2]$

$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$

$= \begin{bmatrix} 1/2 & -1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & -1/2 \\ -1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$