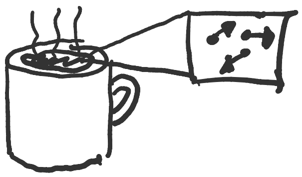
Monday, July 27, 2020 12:55 PM

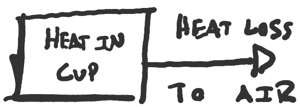
TODAY: HEAT EQ'S - TOWARDS SPACIAL SYSTEMS

WE'LL BE FOLLOWING CONVENTIONS IN BOOK CHAPTERS 9-11.

BASICS OF HEAT TRANSFER:

SIMPLE COOLING EXAMPLE:





PICTURE: HEAT IS KENETIC ENERGY OF TINY MOLICULES. HEAT CONTENT IS ROUGHLY TOTAL KINETIC ENERGY OF PART. MAKING OF THE MASS.

ASSUMPTIONS:

- HEAT IS A QUANTITY
- TEMPERATURE U IS PROP. TO AVERAGE KINETIC ENERGY.
- FOR A SYSTEM WITH FIXED MASS, IF HEAT LEAVES TEMP GOES DOWN AND VISA VERSA.

SO WE HAVE FOLLOWING EQUATION:

$$Q = cm \frac{dU}{dt}$$

THIS IS NOT STANDARD! USUALLY $\dot{Q}$

WHERE

Q - RATE OF CHANGE OF HEAT, J/s

c - SPECIFIC HEAT J/kg°C.

m - MASS IN KG

U - TEMPERATURE.

NOTE: C DEPENDS ON MATERIAL AND OTHER PARAMETERS, MEASURED EXPARIMENTALLY.

NEWTONS LAW OF COOLING

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THE RATE OF CHANGE OF HEAT LOSS TO SURROUNDINGS IS PROPORTIONAL TO THE TEMP. DIFFERENCE.

$$\left\{\begin{array}{l}\text{RATE HEAT IS}\\ \text{EXCHANGED WITH}\\ \text{SURROUNDINGS}\end{array}\right\} = \pm h S \Delta U$$

WHERE

h - NEWTONS COOLING COEFF.

S - SURFACE AREA OF INTERFACE

$\Delta U = U - U_s$, U_s TEMP OF SURROUNDINGS

NOTE: SIGN $\pm$ DEPENDS ON IF WE USE $\Delta U = U - U_s$ OR $\Delta U = U_s - U$

WANT: IF $U_s > U$ THEN WANT TO BE LOSING HEAT FROM SYSTEM. SO NEED TO MAKE SURE SIGN IS FIXED TO MAKE THAT TRUE.

EX:

$$\left\{\begin{array}{l}\text{RATE OF}\\ \text{CHANGE OF}\\ \text{HEAT IN}\\ \text{COFFEE}\end{array}\right\} = \underbrace{\left\{\begin{array}{l}\text{RATE HEAT}\\ \text{LOST TO}\\ \text{SURROUNDINGS}\end{array}\right\}}_{\text{NEWTON LAW OF COOLING}}$$

$$c m \frac{dU}{dt} = \pm h S (U - U_s)$$

NEED TO PICK SIGN: IF $U_s < U$, EXPECT COOLING SO $U' < 0$.

$$c m \frac{dU}{dt} = -h S (U - U_s) = h S (U_s - U)$$

OR

$$U' = -\frac{hS}{cm}(U - U_s).$$

WHAT IS U, WHEN U_s IS CONSTANT?

$$\boxed{U = U_s + (U_0 - U_s)e^{-\frac{hS}{cm}t}}$$

IF ROOM TEMP U_s CHANGES:

$$c m U' = -h S (U - U_s)$$

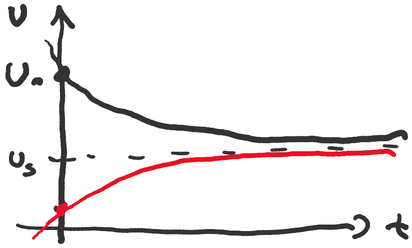
$$c_s m_s U_s' = h S (U - U_s)$$

WHAT IS SIGN?

IF $U_s > U$ THEN $U_s' < 0$

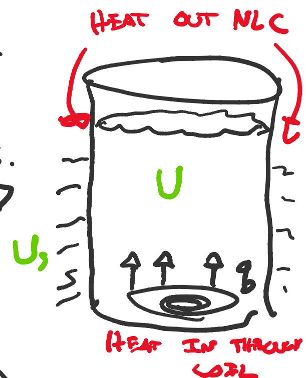
NOTE: CAN SOLVE WITH ~~SUBSTITUTION~~ + LINEAR ODE THEORY.

$$\boxed{U = U_s + (U_0 - U_s)e^{-\frac{hS}{cm}t}}$$



Note: Decrease faster when ΔU is large.

Ex: Water Heater:



$\Rightarrow$ Equations:

$$\underbrace{cmU'}_{\text{Rate of change of heat: J/s}} = -hS(U - U_s) + q$$

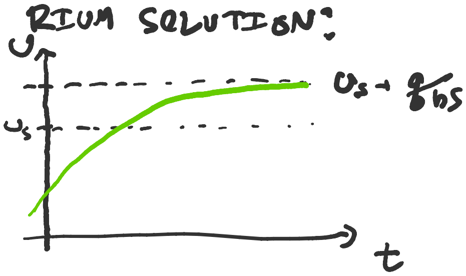
- q is const. heat in in J/s.

How does the addition of q effect solutions?

Write

$$U' = -\frac{hS}{cm}\left(U - U_s - \frac{q}{hS}\right)$$

So if U_s is constant, this just acts as a shift in equilibrium solution:



$$U = U_s + \frac{q}{hS} + \left(U_0 - U_s - \frac{q}{hS}\right)e^{-\frac{hS}{cm}t}$$

~~Substitution~~ + Linear ODE theory.

Exc: What is solution?

$$\vec{U}' = A\vec{U}.$$

$$U = U_s + \frac{q}{hs} + \left(U_0 - U_s - \frac{q}{hs}\right) e^{-\frac{hs}{c\rho V} t}$$

Fouriers Law and Heat Flux

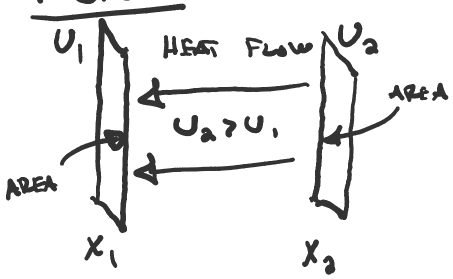
Newtons law is about heat transfer at the boundry of two materials. Heat flow within a (static) material can be modeled using Fouriers law.

Heat flux: The rate of heat heat flow (in positive x dir) per unit time per unit area

$J(x)$ $\quad \frac{W}{m^2} = \frac{J}{s\, m^2}$

Usually not J for heat flux

Picture:



Fouriers law: Heat flux increases as ΔU increases, and decreases as Δx increases.

$J(x)$ proportional to $\frac{\Delta U}{\Delta x}$ so

for $\Delta x \ll 1$

$$J(x) = -K \frac{dU}{dx} \quad \frac{J}{s \cdot m^2}$$

This is slightly nonstandard.

Why $-KU'$? For $\Delta U = U(x_2) - U(x_1)$ so if $U(x_2) > U(x_1)$, i.e. $\Delta U > 0$ then hotter at x_2 then at x_1 so heat flows from x_2 to x_1.

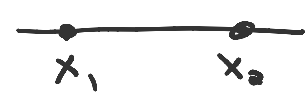
If we change x orientation

$x \mapsto -x$

then need to change

$\Delta U \mapsto x_1 - x_2$

to be "right to left"

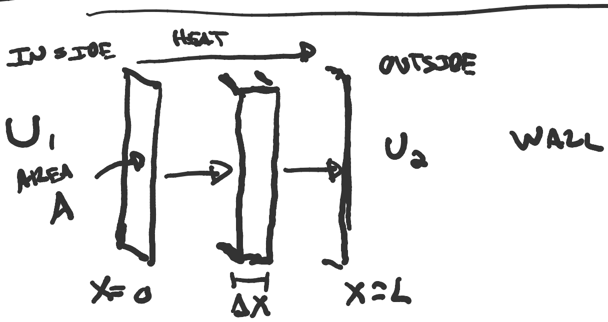


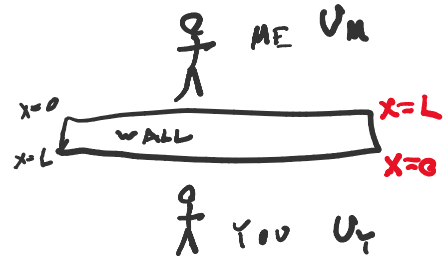
x_1 so heat flows from
x_2 to x_1.

Now: $J(x)$ is deriv. (change) of heat flow as we move in x dir. So if $\Delta U > 0$, change in heat is against dir. of increasing x. Hence sign.

Def: Thermal Equilibrium occurs when no temps are changing.

Ex: Heat Conduction Through a Wall





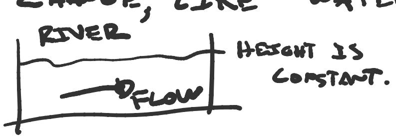
I chose x to increase from me to you, but you could chose x to increase from you to me.

Want to use Fouriers law to understand how heat conducts through volume.

In section of width Δx

$$\left\{\begin{matrix}\text{Rate of change}\\ \text{of heat in section}\end{matrix}\right\} = \left\{\begin{matrix}\text{Heat conducted}\\ \text{at } x\end{matrix}\right\} - \left\{\begin{matrix}\text{Heat cond.}\\ \text{at } x+\Delta x\end{matrix}\right\}$$

At thermal eq, heat content in section is constant. Heat may be flowing through but no net change, like water in river



Here:

$$0 = J(x)A - J(x+\Delta x)A$$

$$0 = J(x)A - J(x+\Delta x)A$$

SO CAN WRITE DIFF EQ:

$$0 = \lim_{\Delta x \to 0} \frac{J(x)A - J(x+\Delta x)A}{\Delta x}$$

$$= -A\frac{dJ}{dx}$$

$$= -A\left(\frac{d}{dx}\left(-k\frac{dU}{dx}\right)\right)$$

$$= A \cdot k \frac{d^2U}{dx^2}$$

SECOND ORDER DIFF EQ.

WHAT IS SOLUTION TO

$$\frac{d^2U}{dx^2} = 0 \quad , \quad \text{WITH} \quad U(0) = U_1, \; U(L) = U_2,$$

INTEGRATE TWICE:

$$U = ax + b$$ ← GENERAL SOLUTION TO DIFFEQ.

WITH INITIAL CONDITIONS

$$U(0) = b = U_1$$

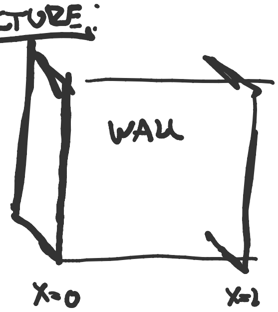
$$U(L) = aL + U_1 = U_2$$

SO

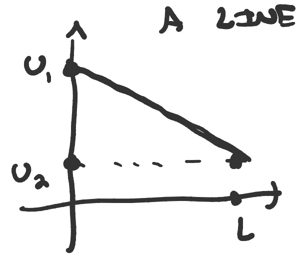
$$a = \frac{U_2 - U_1}{L}$$

$$U = \frac{U_2 - U_1}{L} x + U_1$$

PICTURE:

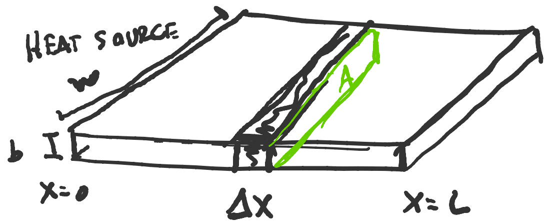


SO TEMP IS A LINE.



EX: HEAT FIN FOR HEAT DISPURSION

PICTURE:



HEAT CONDUCTS USING FOURIERS LAW, AND IS TRANSFERED TO AIR USING NEWTONS LAW. ASSUME U IS CONSTANT ALONG WIDTH WITH ONLY CHANGE IN X DIR.

- w - WIDTH OF FIN
- b - HEIGHT OF FIN
- x - PARAMETERIZES LENGTH DIR
- U_s - SURROUNDING TEMP, CONST.

AT EQUILIBRIUM:

$$\left\{\begin{matrix}\text{RATE OF CHANGE}\\ \text{OF HEAT}\\ \text{IN } \Delta x\end{matrix}\right\} = \left\{\begin{matrix}\text{CONDUCTED}\\ \text{AT}\\ x\end{matrix}\right\} - \left\{\begin{matrix}\text{CONDUCTED}\\ \text{AT}\\ x+\Delta x\end{matrix}\right\} - \left\{\begin{matrix}\text{HEAT TRANSFERED}\\ \text{TO SURROUNDING}\end{matrix}\right\}$$

EQUATION:

$$\underbrace{0}_{\text{AT EQUILIB.}} = \overbrace{J(x)A - J(x+\Delta x)A}^{\text{CONDUCTING}} - Sh\left(U(x^*) - U_s\right)$$

- x^* IS A REP. POINT IN $[x, x+\Delta x]$.
- S IS SURFACE AREA. ~~ASSUME FOR SIMPLICITY THAT $b \ll w$ SO NEGLIGABLE HEAT LOSS FROM SIDES. SO HEAT LOSS ONLY FROM TOP AND BOTTOM.~~

$$S = \underbrace{2w\cdot\Delta x}_{\text{TOP AND BOTTOM}} + \underbrace{2b\Delta x}_{\text{SIDES}}$$

- $A = bw$

SO

$$0 = bw\left(J(x) - J(x+\Delta x)\right)$$

SO

$$0 = bW\left(J(x) - J(x+\Delta x)\right)$$
$$- 2(w+b)\Delta x\, h\left(U(x^*) - U_s\right)$$

SO

$$0 = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\Big[bW\; J(x) - J(x+\Delta x)$$
$$- \Delta x\, 2(w+b)\, h\left(U(x^*) - U_s\right)\Big]$$

$$0 = -bW\frac{dJ}{dx} - 2(w+b)\,h\left(U(x) - U_s\right)$$

SO

$$0 = -bW\left(-k\frac{d^2U}{dx^2}\right) - 2(w+b)\,h\left(U - U_s\right)$$

REWRITE AS

$$\frac{d^2U}{dx^2} = \beta\left(U - U_s\right)$$

WHERE $\beta = \dfrac{2(w+b)b}{bW}$.

THIS IS SECOND ORDER, LINEAR, CONSTANT COEFF. EQUATION.

HOW DO WE SOLVE?

<u>HOMOGENIOUS CASE</u>:

$$ax'' + bx' + cx = 0 \qquad (*)$$

WE CAN SOLVE BY TURNING INTO A PAIR OF FIRST ORDER EQUATIONS.

IDEA: WRITE

$$x' = y$$

THEN $(*)$ BECOMES

$$ay' + by + cx = 0$$

$\vec{x}' = A\vec{x}$ $\begin{bmatrix} 0 & 1 \\ & -b/a \end{bmatrix}$

$$ay' + by + cx = 0$$

OR

$$\left.\begin{aligned} y' &= -\frac{c}{a}x - \frac{b}{a}y \\ x' &= y \end{aligned}\right\}$$ FIRST ORDER, LINEAR, HOMO. SYSTEM

$= \vec{x}' = A\vec{x}$

$A = \begin{bmatrix} 0 & 1 \\ -c/a & -b/a \end{bmatrix}$

SO SOLUTIONS ARE OF THE FORM

$$x = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

TO FIND r_i JUST FEED $e^{rt} = x$ INTO $(*)$:

$$a r^2 e^{rt} + b r e^{rt} + c e^{rt} = 0$$

$$\Rightarrow a r^2 + b r + c = 0$$

SO r_1 AND r_2 ARE GIVEN BY FACTORING.