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Homework 1
MATH4570
Professor Wang

1)

1) a) Show set of all numbers $a+b\sqrt{2}$ is a field

<u>Identity for Sum</u>: $e+x = x+e = x \rightarrow e=0, x = a+b\sqrt{2}$

$$(0+0\sqrt{2}) + (a+b\sqrt{2}) = (a+b\sqrt{2}) + (0+0\sqrt{2})$$
$$= a+b\sqrt{2} = a+b\sqrt{2}$$

$\therefore$ there exists $e \in F$ s.t. $e = 0+0\sqrt{2} = 0$ ✓

<u>Associativity for Sum</u>: Let $c+d\sqrt{2}$ and $x+y\sqrt{2} \in F$

$$(a+b\sqrt{2}+c+d\sqrt{2}) + x+y\sqrt{2} = (a+b\sqrt{2}+x+y\sqrt{2}) + c+d\sqrt{2} \checkmark$$

<u>Inverse for Sum</u>: Let $a = a+b\sqrt{2}$
$a^{-1} = -a-b\sqrt{2}$

$\therefore$ $a + a^{-1} = a+b\sqrt{2} - a - b\sqrt{2} = a^{-1} + a = 0$ ✓
$\forall$ $a = a+b\sqrt{2} \in F$ the inverse is $a^{-1} = -a-b\sqrt{2}$

<u>Commutativity for Sum</u>: Let $a = a+b\sqrt{2}$, $b = c+d\sqrt{2}$

$$a+b\sqrt{2} + c+d\sqrt{2} = c+d\sqrt{2} + a+b\sqrt{2} \checkmark$$

<u>Multiplicative Identity</u>: Let $e' = (1+0\sqrt{2})$, $a = a$, $b = 0\sqrt{2}$

$$e' \times (a+b\sqrt{2}) = (a+b\sqrt{2}) \times e'$$

$$(1+0\sqrt{2})(a+b\sqrt{2}) = (a+b\sqrt{2})(1+0\sqrt{2})$$
$$\Rightarrow a+b\sqrt{2} = a+b\sqrt{2}$$

$\therefore$ $\forall$ $(a+b\sqrt{2}) \in F$, mult. id. is $1+0\sqrt{2} = 1$ ✓

Associativity of Mult.: Let $a = a+b\sqrt{2}$, $b = c+d\sqrt{2}$,
$C = x+y\sqrt{2}$

$$\checkmark\left(\overset{a}{(a+b\sqrt{2})}\overset{b}{(c+d\sqrt{2})}\right)\times\overset{c}{(x+y\sqrt{2})} = \overset{a}{(a+b\sqrt{2})}\times\left(\overset{b}{(c+d\sqrt{2})}\overset{c}{(x+y\sqrt{2})}\right)$$

Distrib. of Mult.:

$$(a+b\sqrt{2})\times\left(c+d\sqrt{2}+x+y\sqrt{2}\right) = (a+b\sqrt{2})(c+d\sqrt{2})+(a+b\sqrt{2})(x+y\sqrt{2})$$

$$= ac + ad\sqrt{2} + ax + ay\sqrt{2} + bc\sqrt{2} + 2bd + bx\sqrt{2} + 2by$$

$$\checkmark = (ac + ad\sqrt{2} + bc\sqrt{2} + 2bd) + (ax + ay\sqrt{2} + bx\sqrt{2} + 2by)$$

Commutative for Mult.:

$$(a+b\sqrt{2})(c+d\sqrt{2}) = \underline{ac} + \underline{ad\sqrt{2}} + \underline{bc\sqrt{2}} + \underline{2bd}$$ $\checkmark$

$$(c+d\sqrt{2})(a+b\sqrt{2}) = \underline{ac} + \underline{bc\sqrt{2}} + \underline{ad\sqrt{2}} + \underline{2bd}$$

Inverse for Mult.

$$\frac{1}{a}+\left(\frac{b}{a^2-2b^2}\right)\left(\frac{2b}{a}\right)$$

$$(a+b\sqrt{2})^{-1} := x+y\sqrt{2}$$

$$(a+b\sqrt{2})*(x+y\sqrt{2}) = 1$$

$$(ax+2by)+(ay+bx)\sqrt{2} = 1$$

$\therefore$ Need to solve $ax+2by=1$ and $ay+bx=0$

$$-\frac{b}{a}\cdot\frac{1}{a-2b^2/a}$$

Using $\begin{bmatrix} a & 2b & 1 \\ b & a & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2b/a & 1/a \\ b & a & 0 \end{bmatrix}$

$$\rightarrow \begin{bmatrix} 1 & 2b/a & 1/a \\ 0 & a-\frac{2b^2}{a} & -b/a \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2b/a & 1/a \\ 0 & 1 & \frac{-b}{a^2-2b^2} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & a/a^2-2b^2 \\ 0 & 1 & -\frac{b}{a^2-2b^2} \end{bmatrix} \therefore \quad x = \frac{a}{a^2-2b^2} \quad y = \frac{-b}{a^2-2b^2} \quad \therefore (a+b\sqrt{2})^{-1} = \left(\frac{a}{a^2-2b^2} - \frac{b}{a^2-2b^2}\sqrt{2}\right) \checkmark$$

b) Show set of all numbers $a+b\sqrt{-1}$ is field
where $a+b\sqrt{-1} = a+bi$

<u>Id. for Sum:</u> $(0+0i)+(a+bi) = (a+bi)+(0+0i)$
$a+bi = a+bi$
$\therefore\ \exists\ e \in F$ s.t. $e = (0+0i) = 0$ ✓

<u>Ass. for Sum:</u> $((a+bi)+(c+di))+(x+yi) = (a+bi)+((c+di)+(x+yi))$
$a+bi+c+di+x+yi = a+bi+c+di+x+yi$ ✓

<u>Inverse Sum:</u> Let $a = a+bi$, $a^{-1} = -a-bi$
$a+a^{-1} = a+bi-a-bi = 0$ ✓

<u>Commn. Sum:</u> $(a+bi)+(c+di) = (c+di)+(a+bi)$
$a+bi+c+di = c+di+a+bi$ ✓

<u>Mult. Id.:</u> $(1+0i)(a+bi) = (a+bi)(1+0i)$
$= (1)(a+bi) = (a+bi)(1)$
$\therefore\ \forall\ (a+bi)\in F,\ (1+0i) = 1$ ✓

<u>Asso. Mult.:</u> $((a+bi)(c+di))(x+yi) = (a+bi)((c+di)(x+yi))$
$= (a+bi)(c+di)(x+yi) = (a+bi)(c+di)(x+yi)$ ✓

<u>Distn. Mult.:</u> $(a+bi)((c+di)+(x+yi)) = ((a+bi)(c+di))+((a+bi)(x+yi))$

$= ac+adi+ax+ayi+bci-bd+bxi-by$
$= (ac+adi+bci-bd)+(ax+ayi+bxi+by)$ ✓

<u>Commn. Mult.:</u> $(a+bi)(c+di) = (c+di)(a+bi)$
$ac+adi+bci-bd = ac+bci+adi-bd$ ✓

Inverse Product:

Let $a = a+bi$
$a^{-1} = -a-bi = x+yi$

$= (a+bi)(x+yi) = 1$
$= (ax + ayi + bxi - by) = 1$
$= (ax-by) + (ay+bx)i = 1$

$\therefore$ Need to solve $ax-by=1$ and $ay+bx=0$

$a+\frac{b^2}{a}$

$$A = \begin{bmatrix} a & -b & 1 \\ b & a & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -b/a & 1/a \\ b & a & 0 \end{bmatrix}$$

$\frac{2a}{a}$

$$= \begin{bmatrix} 1 & -b/a & 1/a \\ 0 & a+\frac{b^2}{a} & -b/a \end{bmatrix} = \begin{bmatrix} 1 & -b/a & 1/a \\ 0 & 1 & -\frac{b}{a^2+b^2} \end{bmatrix}$$

$-\frac{b}{a} \cdot \frac{1}{a+b^2/a}$

$$= \begin{bmatrix} 1 & 0 & a/a^2+b^2 \\ 0 & 1 & -b/a^2+b^2 \end{bmatrix} = rref(A)$$

$x = a/a^2+b^2, \quad y = -\frac{b}{a^2+b^2}$

$(a+bi)^{-1} = x+yi = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i$ ✓

$\boxed{\therefore\ a+bi \text{ is a field on mult. \& add.}}$

2)

2) Given that set F is all $n \times n$ matrices $\mathbb{R}^{n \times n}$ and is a ring, we need to prove that it is not a field if $n > 1$

Prove: Inverse for product is not true

$(\forall a \neq 0 \in F, \exists x \in F$ s.t. $ax = e$

Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$B^{-1} = \frac{1}{|B|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ where $|B|$ is determinant of B.

B^{-1} does not exist when $|B| = 0$ because $1/0 = \infty$. Therefore, any $n \times n$ matrix when $n > 1$ cannot be a field because not all the matrices have inverses (if their determinant $= 0$

3)

$$\mathbb{Z}_3 = \{[0], [1], [2]\}$$

3)

+	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

×	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

4)

b) Show set of all numbers $a + b\sqrt{-1}$ is field

where $a + b\sqrt{-1} = a + bi$

Id. for Sum: $(0 + 0i) + (a + bi) = (a + bi) + (0 + 0i)$

$a + bi = a + bi$

$\therefore \exists e \in F$ s.t. $e = (0 + 0i) = 0$ ✓

Ass. for Sum: $((a+bi) + (c+di)) + (x+yi) = (a+bi) + ((c+di) + (x+yi))$

$a + bi + c + di + x + yi = a + bi + c + di + x + yi$ ✓

Inverse Sum: Let $a = a + bi$, $a^{-1} = -a - bi$

$a + a^{-1} = a + bi - a - bi = 0$ ✓

Commu. Sum: $(a+bi) + (c+di) = (c+di) + (a+bi)$

$a + bi + c + di = c + di + a + bi$ ✓

Mult. Id.: $(1 + 0i)(a + bi) = (a+bi)(1+0i)$

$= (1)(a+bi) = (a+bi)(1)$

$\therefore \forall (a+bi) \in F$, $(1+0i) = 1$ ✓

Asso. Mult.: $((a+bi)(c+di))(x+yi) = (a+bi)((c+di)(x+yi))$

$= (a+bi)(c+di)(x+yi) = (a+bi)(c+di)(x+yi)$ ✓

Distr. Mult.: $(a+bi)((c+di) + (x+yi)) = ((a+bi)(c+di)) + ((a+bi)(x+yi))$

$= ac + adi + ax + ayi + bci - bd + bxi - by$

$= (ac + adi + bci - bd) + (ax + ayi + bxi - by)$ ✓

Commu. Mult.: $(a+bi)(c+di) = (c+di)(a+bi)$

$ac + adi + bci - bd = ac + bci + adi - bd$ ✓

Inverse Product:

Let $a = a+bi$
$a^{-1} = -a-bi = x+yi$

$= (a+bi)(x+yi) = 1$
$= (ax + ayi + bxi - by) = 1$
$= (ax-by) + (ay+bx)i = 1$

$\therefore$ Need to solve $ax-by=1$ and $ay+bx=0$

$a - \frac{b^2}{a}$

$A = \begin{bmatrix} a & -b & 1 \\ b & a & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -b/a & 1/a \\ b & a & 0 \end{bmatrix}$

$\frac{2a}{a}$

$= \begin{bmatrix} 1 & -b/a & 1/a \\ 0 & a+\frac{b^2}{a} & -b/a \end{bmatrix} = \begin{bmatrix} 1 & -b/a & 1/a \\ 0 & 1 & -\frac{b}{a^2+b^2} \end{bmatrix}$

$-\frac{b}{a} \cdot \frac{1}{a+b^2/a}$

$= \begin{bmatrix} 1 & 0 & a/a^2+b^2 \\ 0 & 1 & -b/a^2+b^2 \end{bmatrix} = rref(A)$

$x = a/a^2+b^2, \quad y = \frac{-b}{a^2+b^2}$

$(a+bi)^{-1} = x+yi = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i$ ✓

$\boxed{\therefore a+bi \text{ is a field on mult. \& add.}}$

5,6,7,8,9,10)

5) $\boxed{B \text{ is in rref}}$

6) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$\boxed{A+B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad AB = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}}$$

7) $A = \begin{bmatrix} 6 & -1 & 1 \\ t & 0 & 1 \\ 0 & 1 & t \end{bmatrix}$

When $|A| = 0$, A^{-1} DNE,

$\therefore \ |A| = 0 = 6(0-1) + (t^2) + (t-0)$

$|A| = 0 = -6 + t^2 + t$

$t^2 + t - 6 = 0$

$(t+3)(t-2) = 0$

$t = -3, 2$

$\boxed{\text{When } t=-3 \text{ or } t=2, \ A \text{ does not have an inverse}}$

8) a) $= \left[\begin{array}{cc|c} 1 & h & 4 \\ 3 & 6 & 8 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & h & 4 \\ 0 & 6-3h & -4 \end{array}\right] \rightarrow R_2/_{6-3h}$

$= \left[\begin{array}{cc|c} 1 & h & 4 \\ 0 & 1 & \frac{-4}{6-3h} \end{array}\right] \xrightarrow{R_1 - hR_2} \left[\begin{array}{cc|c} 1 & 0 & \frac{24-8h}{6-3h} \\ 0 & 1 & -4/6-3h \end{array}\right]$

$\boxed{\text{All values of } h \text{ except } 2 \text{ and } -2/9}$

$4 - \frac{-4h}{6-3h}$

$24 - 12h + 4h$

$\frac{24-8h}{6-3h}$

b) $\left[\begin{array}{cc|c} -4 & 12 & h \\ 2 & -6 & -3 \end{array}\right] \xrightarrow[R_2/2]{R_1/-4} \left[\begin{array}{cc|c} 1 & -3 & -h/4 \\ 1 & -3 & -3/2 \end{array}\right]$

$\xrightarrow{R_2 - R_1} \left[\begin{array}{cc|c} 1 & -3 & -h/4 \\ 0 & -6 & \frac{-6-h}{4} \end{array}\right]$

$\frac{-6}{4} - \frac{h}{4}$

$\frac{-6-h}{4} \cdot -\frac{1}{6}$

$\left[\begin{array}{cc|c} 1 & -3 & -h/4 \\ 0 & 1 & \frac{6+h}{24} \end{array}\right]$

$= \left[\begin{array}{cc|c} 1 & 0 & -6-3h/8 \\ 0 & 1 & 6+h/24 \end{array}\right]$

$\frac{6+h}{\cancel{24}_8} \cdot -\cancel{3}' = \frac{-6-h}{\cancel{8}} + \frac{\cancel{-h}}{8}$

$\boxed{h \text{ can't be } -2/3 \text{ or } 18}$

9) a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\boxed{\text{4 soln}}$

b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & * \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}$ $\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\boxed{\text{11 solutions}}$

c) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\boxed{\text{1 solution}}$

10) $\boxed{\begin{array}{ll} a = \mathbb{R} & d = \mathbb{R} \\ c = 0 & b = \mathbb{R} \\ e = 0 & \end{array}}$

12)

In [6]:
```python
from sympy import Matrix, pprint
import numpy as np
import galois

GF7 = galois.GF(7)
GF7

A = GF7([[3, 1, 4, 1],
    [5, 2, 6, 5],
    [0, 5, 2, 1]]);

B = GF7([1,5,1])

GF7.row_reduce(A)
```

Out[6]:
```
GF([[1, 0, 0, 4],
    [0, 1, 0, 3],
    [0, 0, 1, 0]], order=7)
```

12) a) $A = \begin{bmatrix} 3 & 1 & 4 \\ 5 & 2 & 6 \\ 0 & 5 & 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$

$\text{rref}(A|\vec{b})$ over $\mathbb{Z}_7 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$

b) $A\vec{x} = \vec{b} \bmod 7 \longrightarrow \vec{x} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$

13)

```
In [60]: from sympy.interactive.printing import init_printing
         from sympy.matrices import Matrix, eye, zeros, ones, diag, GramSchmidt
         from sympy import *
         from sympy import shape
         import sympy as sym

         M = Matrix([[3, 11, 19, -2], [7, 23, 39, 10], [-4, -3, -2, 6]])
         display(M)
```

$$\begin{bmatrix} 3 & 11 & 19 & -2 \\ 7 & 23 & 39 & 10 \\ -4 & -3 & -2 & 6 \end{bmatrix}$$

```
In [62]: M.rref()
```

Out[62]:

$$\left(\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, (0, 1, 3)\right)$$

13) $\boxed{x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$

14)

14)

```
In [63]: from sympy.interactive.printing import init_printing
         from sympy.matrices import Matrix, eye, zeros, ones, diag, GramSchmidt
         from sympy import *
         from sympy import shape
         import sympy as sym

         M = Matrix([[3, 6, 9, 5, 25, 53], [7, 14, 21, 9, 53, 105], [-4, -8, -12, 5, -10, 11]])
         display(M)
```

$$\begin{bmatrix} 3 & 6 & 9 & 5 & 25 & 53 \\ 7 & 14 & 21 & 9 & 53 & 105 \\ -4 & -8 & -12 & 5 & -10 & 11 \end{bmatrix}$$

```
In [64]: M.rref()
```

Out[64]:

$$\left(\begin{bmatrix} 1 & 2 & 3 & 0 & 5 & 6 \\ 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ (0, 3) \right)$$

14) ~~$x_1+2x_2+3x_3$~~

$$\boxed{x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_2\begin{bmatrix}2\\0\\0\end{bmatrix} + x_3\begin{bmatrix}3\\0\\0\end{bmatrix} + x_4\begin{bmatrix}0\\1\\0\end{bmatrix} + x_5\begin{bmatrix}5\\2\end{bmatrix} = \begin{bmatrix}6\\7\\0\end{bmatrix}}$$

15)

In [65]:
```
om sympy.interactive.printing import init_printing
om sympy.matrices import Matrix, eye, zeros, ones, diag, GramSchmidt
om sympy import *
om sympy import shape
port sympy as sym

= Matrix([[2, 4, 3, 5, 6, 37], [4, 8, 7, 5, 2, 74], [-2, -4, 3, 4, -5, 20], [1, 2, 2, -1, 2, 26], [5, -10, 4, 6,
splay(M)
```

$$\begin{bmatrix} 2 & 4 & 3 & 5 & 6 & 37 \\ 4 & 8 & 7 & 5 & 2 & 74 \\ -2 & -4 & 3 & 4 & -5 & 20 \\ 1 & 2 & 2 & -1 & 2 & 26 \\ 5 & -10 & 4 & 6 & 4 & 24 \end{bmatrix}$$

In [66]:
```
M.rref()
```

Out[66]:

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -\frac{8221}{4340} \\ 0 & 1 & 0 & 0 & 0 & \frac{8591}{8680} \\ 0 & 0 & 1 & 0 & 0 & \frac{4695}{434} \\ 0 & 0 & 0 & 1 & 0 & -\frac{459}{434} \\ 0 & 0 & 0 & 0 & 1 & \frac{699}{434} \end{bmatrix}, (0, 1, 2, 3, 4) \right)$$

15) $$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -8221/4340 \\ 8591/8680 \\ 4695/434 \\ -459/434 \\ 699/434 \end{bmatrix}$$

16,17,18)

16) a) $ABC = I = 1$, $\therefore$ $(AB)C = 1 \rightarrow C = (AB)^{-1}$

$(BC)A = 1 \rightarrow A = (BC)^{-1}$

$(AC)B = 1 \rightarrow B = (AC)^{-1}$

b) Yes, using the above proof.

If AB invertible, $\exists\, C$ s.t. $C(AB) = I$

$I = C(AB) = (CA)B \Rightarrow B$ is invertible $\rightarrow (CA) = B^{-1}$

17) $(AB)^2 = A^2B^2$

$\Leftrightarrow ABAB = AABB$

$\Leftrightarrow BA = AB \rightarrow$ since $BA \neq AB$, $(AB)^2 \neq A^2B^2$

18) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ $\quad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\boxed{A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$$

19)

19) a) Symmetric:

$2\times2 \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ $\quad 3\times3 \rightarrow \begin{bmatrix} 0 & 5 & 19 \\ 5 & 6 & 3 \\ 19 & 3 & 4 \end{bmatrix}$

$4\times4 \rightarrow \begin{bmatrix} 1 & 5 & 7 & -8 \\ 5 & 9 & 10 & 13 \\ 7 & 10 & 21 & 3 \\ -8 & 13 & 3 & 0 \end{bmatrix}$

Skew-Symmetric:

$2\times2 \rightarrow \begin{bmatrix} 0 & 8 \\ -8 & 0 \end{bmatrix}$ $\quad 3\times3 \rightarrow \begin{bmatrix} 0 & -1 & -4 \\ 1 & 0 & -5 \\ 4 & 5 & 0 \end{bmatrix}$

$4\times4 \rightarrow \begin{bmatrix} 0 & -1 & -3 & -5 \\ 1 & 0 & 2 & 11 \\ 3 & -2 & 0 & -13 \\ 5 & -11 & 13 & 0 \end{bmatrix}$

b) It contans all zeroes

c) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

19) An $n \times n$ matrix A is symmetric provided $A^T = A$ and skew-symmetric provided $A^T = -A$.

d) Prove that $\forall$ $n \times n$ matrix A, $A + A^T$, AA^T and A^TA are symmetric and $A - A^T$ is skew-symmetric.

$A + A^T$ is symmetric $\Leftrightarrow A + A^T = (A + A^T)^T$

Given matrices $A, B \rightarrow (A+B)^T = A^T + B^T$ and $(A^T)^T = A$

therefore, $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$ ✓

AA^T is symmetric $\Leftrightarrow AA^T = (AA^T)^T$

Given matrices $A, B \rightarrow (AB)^T = B^TA^T$ and $(A^T)^T = A$

therefore, $(AA^T)^T = (A^T)^T \times A^T = A \times A^T = A \times A^T$ ✓

A^TA is symmetric $\Leftrightarrow A^TA = (A^TA)^T$

Given $A, B \rightarrow (AB)^T = A^TB^T$ and $(A^T)^T = A$

$(A^TA)^T = A^T \times (A^T)^T = A^T \times A = A^TA$ ✓

$A - A^T$ is skew-symmetric $\Leftrightarrow (A - A^T)^T = -(A - A^T)$

Given $A, B \rightarrow (A+B)^T = A^T + B^T$ and $(A^T)^T = A$

$(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$ ✓

$(rA)^T = rA^T$ $\forall$ scalar $r \rightarrow -(A - A^T) = A^T - A$ ✓

e) Prove that any $n \times n$ matrix can be written as the sum of sym. and skew-sym matrices.

Let $A_1 = A + A^T \rightarrow$ symmetric and
$A_2 = A - A^T \rightarrow$ skew-symetric

Prove $A = A_1 + A_2$

$$A_1 + A_2 = A + A^T + A - A^T$$

$$= (A + A) + A^T + (-A^T)$$

Since $A + (-A) = 0 \rightarrow = A + A = 2A$

20)

20) a) $F(x) = x^2 \rightarrow$ bijective

b) $F(x) = x^3/(x^2+1) \rightarrow$ injective

c) $F(x) = x(x-1)(x-2)$ – surjective

d) $F(x) = e^x + 2 \rightarrow$ injective

~~21, 22, 23~~

$$L(\cdot) = \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$$

21)

21) $$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_2 - 1/4 R_1}$$

$$\begin{bmatrix} 1 & & & \\ 1/4 & 1 & & \\ & \frac{4}{15} & 1 & \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 15/4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_3 - \frac{4}{15} R_2}$$

$$\begin{bmatrix} 1 & & & \\ 1/4 & 1 & & \\ & 4/15 & 1 & \\ & & 15/56 & 1 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 15/4 & 1 & 0 \\ 0 & 0 & 56/15 & 1 \\ 0 & 0 & 0 & 209/56 \end{bmatrix}}_{U} \xrightarrow{R_4 - \frac{15}{56} R_3}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 0 & 4/15 & 1 & 0 \\ 0 & 0 & 15/56 & 1 \end{bmatrix}$$

24) Let I_n be nxn identity matrix.

Let $\vec{u}$ be unit vector in $\mathbb{R}^n$.

Define $H_n = I_n - 2\vec{u}\vec{u}^T$

Unit vector $\vec{u} = \|\vec{u}\| = 1 = \vec{u}^T\vec{u} = \vec{u}\cdot\vec{u} = 1$

a) H_n is symmetric $\Leftrightarrow (I_n - 2\vec{u}\vec{u}^T)^T = I_n - 2\vec{u}\vec{u}^T$

$\therefore (I_n - 2\vec{u}\vec{u}^T)^T = I_n^T - (2\vec{u}\vec{u}^T)^T$

$\therefore I_n^T = I_n$ so $= I_n - 2\vec{u}^T \cdot (\vec{u}^T)^T$

$= I_n - 2\vec{u}^T\vec{u} = H_n$ ✓

b) H_n is an orthogonal matrix $\Leftrightarrow H_n^T H_n = I_n$

$\therefore (I_n - 2\vec{u}\vec{u}^T)^T \times (I_n - 2\vec{u}\vec{u}^T)$

Given proof in part a, $= (I_n - 2\vec{u}\vec{u}^T)(I_n - 2\vec{u}\vec{u}^T)$

$= I_n^2 - (I_n)(2\vec{u}\vec{u}^T) - (I_n)(2\vec{u}\vec{u}^T) + 4\vec{u}^2\vec{u}^{2T}$

$= I_n - 4\vec{u}\vec{u}^T + 4(\vec{u}^T\vec{u})(\vec{u}^T\vec{u})$

$\therefore$ because $\vec{u}\vec{u}^T = 1$, $= I_n$

$\therefore H_n^T H_n = I_n$ ✓

c) What is H_n^2?

From part b, $H_n^2 = (I_n - 2\vec{u}\vec{u}^T)(I_n - 2\vec{u}\vec{u}^T)$
$= \boxed{I_n}$

d) What is $H_n \vec{u}$?

$H_n \vec{u} = (I_n - 2\vec{u}\vec{u}^T)\vec{u} = I_n \vec{u} - 2\vec{u}(\vec{u}\vec{u}^T)$

$= \vec{u} - 2\vec{u} = \boxed{-\vec{u}}$

e) $H_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\left(\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\right)$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & 2/3 \end{bmatrix} = \boxed{\begin{bmatrix} -1/3 & -2/3 & -2/3 \\ -2/3 & -1/3 & -2/3 \\ -2/3 & 2/3 & -1/3 \end{bmatrix}}$

$H_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - 2\left(\frac{1}{\sqrt{4}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{4}}\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}\right)$

$= I_4 - \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

$$\boxed{H_4 = \begin{bmatrix} 1/2 & -1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & -1/2 \\ -1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}}$$