Name: ______________________ **ID:** ______________________

Task 1 (13 points)

In this question we compare the convergence of a few methods we have done in class.

$$\dot{y} = \frac{y+t}{y-t}; \qquad y(0) = 1$$

(a) (2 points) Matlab can solve simple differential equations symbolically. For example,

```
>> equation = 'Dy = (y+t)/(y-t)';
>> initial = 'y(0) = 1';
>> y = dsolve(equation, initial, 't');
>> pretty(y)
>> x = linspace( 0, 1, 20);
>> z = eval(vectorize(y));
>> plot(x,z)
```

Run the above commands and write down the exact solution below.

(b) (2 points) Study and use the M-function butcher and euler to estimate the convergence rate of Euler's method for the above IVP over the interval $[0, 1]$. In addition, compute the error as the difference between your approximation and the exact solution at $t = 1$.

(c) (2 points) Repeat part (b), but this time do it for Heun's method. The M-function heun is given.

(d) (2 points) Repeat to part (b), but this time do it for Aadms-Bashforth's method. The M-function ab2 is given.

(e) (2 points) Repeat to part (b), but this time do it for the classic 4-order Runge Kutta's method. The M-function rk4 is given.

(f) (3 points) Repeat to part (b), but this time do it for the classic 4-order Taylor's method. The M-function taylor is given.

Task 2 (8 points)

Matlab has its implementation of Runge-Kutta. This question looks at the build-in functions ode23 and ode45, which implement versions of 2nd/3rd-order Runge-Kutta and 4th/5th-oder Runge Kutta, respectively.

(a) (2 points) Study and use ode23 to solve the following

$$\dot{y} = \frac{y+t}{y-t}; \qquad y(0) = 1$$

Compare and comment the convergence with the methods we considered in Task 1.

(b) (1 point) Write an M-function for the vector-valued function

$$\mathbf{\Phi}(t, \mathbf{x}) = \begin{bmatrix} 10(x_2 - x_1) \\ 28x_1 - x_2 - x_1x_3 \\ -8/3x_3 + x_1x_2 \end{bmatrix}$$

The input of your function shall be a scalar t and a vector in $\mathbb{R}^3$. The output of your function shall be a column vector.

(c) (2 points) Study and use ode45 to solve the following IVP over $[0, 20]$.

$$\dot{x} = \mathbf{\Phi}(\mathbf{x}); \qquad \mathbf{x}(0) = \begin{bmatrix} -8 \\ 8 \\ 27 \end{bmatrix}$$

(d) (3 points) Let $\mathbf{X}$ denote the matrix that contains approximations you have found in part (c) for x_1, x_2 and x_3 as its columns at various t_k values. Plot the following

```
>> plot(X(:,1), X(:,2))
>> plot(X(:,1), X(:,3))
>> plot(X(:,2), X(:,3))
>> subplot(3,1,1)
>> plot(t,X(:,1))
>> subplot(3,1,2)
>> plot(t,X(:,2))
>> subplot(3,1,3)
>> plot(t,X(:,3))
```

What do those graphs suggest? What happens if we change the initial condition $\mathbf{x}_0$?

Task 3 (9 points)

There are IVPs that defeat the explicit Adams-Bashforth and Runge-Kutta methods. In fact, for such problems, the higher order method perform even more poorly than the low order methods. These problems are call "stiff" ODEs. Consider

$$y' = \lambda(-y + \sin x); \qquad y(0) = 0$$

whose exact solution is

$$y(x) = \frac{\lambda}{1+\lambda^2} e^{-\lambda x} + \frac{\lambda^2}{1+\lambda^2} \sin x - \frac{\lambda}{1+\lambda^2} \cos x$$

(a) (4 points) Run the following commands. The M-functions stiff2ode and stiff2solution are given.

```
>> clear all
>> h = 0.1; % mesh size
>> [x,y] = meshgrid ( 0:h:2*pi, -1:h:1 );
>> px = ones ( size ( x ) );
>> py = stiff2ode ( x, y );
>> quiver ( x, y, px, py )
>> axis tight equal
>> hold on
>> x1=(0:h:2*pi);
>> y1=stiff2solution(x1);
>> plot(x1,y1,'r')
>> hold off
```

What does the direction field seem to suggest? What happens if λ increases? Describe your understanding of stiffness.

(b) (2 points) Use euler to solve the IVP when $\lambda = 10000$ for 40 steps over the interval $[0, 2\pi]$. Comment your solution and provide an explanation of it.

(c) (2 points) Use rk4 to solve the IVP when $\lambda = 10000$ for 40 steps over the interval $[0, 2\pi]$. Is it better than Euler's? What happens if we increase the number of steps?

(d) (1 point) Study and use the build-in function ode15s to solve the IVP when $\lambda = 10000$. Write down the estimated value of $y(2\pi)$.

Task 4 (10 points)
Consider the following boundary value problem.

$$\ddot{y} - 3\dot{y} + 2y = 0; \qquad y(0) = 0; \quad y(1) = 10$$

(a) (2 points) Study and use the built-in function bvp4c to solve the above BVP over $[0, 1]$. Write the estimated value of $y(1/\pi)$.

(b) (4 points) Write an M-function to solve the above BVP using the shooting method.

(c) (4 points) Write an M-function to solve the above BVP using the finite-difference.