5 TRIGONOMETRIC FUNCTIONS

DEFINITION OF TRIGONOMETRIC FUNCTIONS FOR A RIGHT TRIANGLE

Triangle ABC has a right angle (90°) at C and sides of length a, b, c. The trigonometric functions of angle A are defined as follows.

5.1 $\quad \textit{sine} \text{ of } A = \sin A = \dfrac{a}{c} = \dfrac{\text{opposite}}{\text{hypotenuse}}$

5.2 $\quad \textit{cosine} \text{ of } A = \cos A = \dfrac{b}{c} = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

5.3 $\quad \textit{tangent} \text{ of } A = \tan A = \dfrac{a}{b} = \dfrac{\text{opposite}}{\text{adjacent}}$

5.4 $\quad \textit{cotangent} \text{ of } A = \cot A = \dfrac{b}{a} = \dfrac{\text{adjacent}}{\text{opposite}}$

5.5 $\quad \textit{secant} \text{ of } A = \sec A = \dfrac{c}{b} = \dfrac{\text{hypotenuse}}{\text{adjacent}}$

5.6 $\quad \textit{cosecant} \text{ of } A = \csc A = \dfrac{c}{a} = \dfrac{\text{hypotenuse}}{\text{opposite}}$

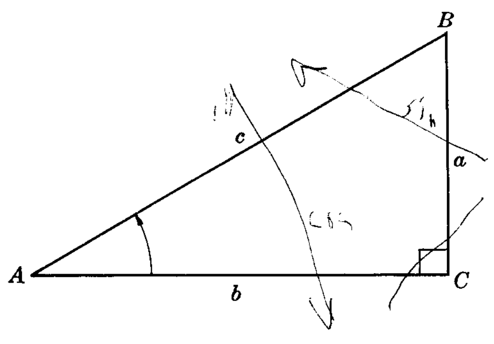

Fig. 5-1

EXTENSIONS TO ANGLES WHICH MAY BE GREATER THAN 90°

Consider an xy coordinate system [see Fig. 5-2 and 5-3 below]. A point P in the xy plane has coordinates (x, y) where x is considered as positive along OX and negative along OX' while y is positive along OY and negative along OY'. The distance from origin O to point P is positive and denoted by $r = \sqrt{x^2 + y^2}$. The angle A described *counterclockwise* from OX is considered *positive*. If it is described *clockwise* from OX it is considered *negative*. We call $X'OX$ and $Y'OY$ the x and y axis respectively.

The various quadrants are denoted by I, II, III and IV called the first, second, third and fourth quadrants respectively. In Fig. 5-2, for example, angle A is in the second quadrant while in Fig. 5-3 angle A is in the third quadrant.

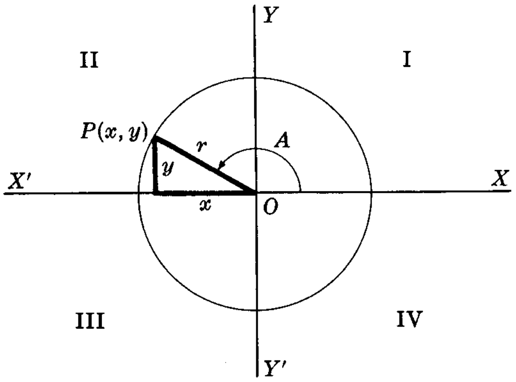

Fig. 5-2

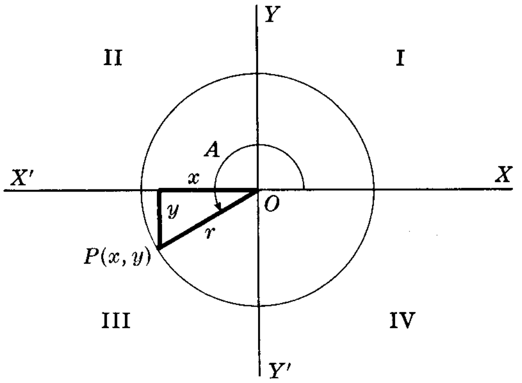

Fig. 5-3

For an angle A in any quadrant the trigonometric functions of A are defined as follows.

5.7 $$\sin A = y/r$$

5.8 $$\cos A = x/r$$

5.9 $$\tan A = y/x$$

5.10 $$\cot A = x/y$$

5.11 $$\sec A = r/x$$

5.12 $$\csc A = r/y$$

RELATIONSHIP BETWEEN DEGREES AND RADIANS

A *radian* is that angle θ subtended at center O of a circle by an arc MN equal to the radius r.

Since 2π radians $= 360°$ we have

5.13 $$1 \text{ radian} = 180°/\pi = 57.29577\ 95130\ 8232\ldots°$$

5.14 $$1° = \pi/180 \text{ radians} = 0.01745\ 32925\ 19943\ 29576\ 92\ldots \text{radians}$$

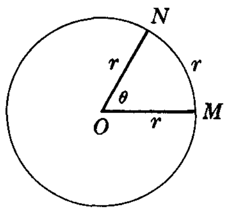

Fig. 5-4

RELATIONSHIPS AMONG TRIGONOMETRIC FUNCTIONS

5.15 $$\tan A = \frac{\sin A}{\cos A}$$

5.16 $$\cot A = \frac{1}{\tan A} = \frac{\cos A}{\sin A}$$

5.17 $$\sec A = \frac{1}{\cos A}$$

5.18 $$\csc A = \frac{1}{\sin A}$$

5.19 $$\sin^2 A + \cos^2 A = 1$$

5.20 $$\sec^2 A - \tan^2 A = 1$$

5.21 $$\csc^2 A - \cot^2 A = 1$$

SIGNS AND VARIATIONS OF TRIGONOMETRIC FUNCTIONS

Quadrant	$\sin A$	$\cos A$	$\tan A$	$\cot A$	$\sec A$	$\csc A$
I	$+$ 0 to 1	$+$ 1 to 0	$+$ 0 to ∞	$+$ ∞ to 0	$+$ 1 to ∞	$+$ ∞ to 1
II	$+$ 1 to 0	$-$ 0 to -1	$-$ $-\infty$ to 0	$-$ 0 to $-\infty$	$-$ $-\infty$ to -1	$+$ 1 to ∞
III	$-$ 0 to -1	$-$ -1 to 0	$+$ 0 to ∞	$+$ ∞ to 0	$-$ -1 to $-\infty$	$-$ $-\infty$ to -1
IV	$-$ -1 to 0	$+$ 0 to 1	$-$ $-\infty$ to 0	$-$ 0 to $-\infty$	$+$ ∞ to 1	$-$ -1 to $-\infty$

EXACT VALUES FOR TRIGONOMETRIC FUNCTIONS OF VARIOUS ANGLES

Angle A in degrees	Angle A in radians	$\sin A$	$\cos A$	$\tan A$	$\cot A$	$\sec A$	$\csc A$
0°	0	0	1	0	∞	1	∞
15°	$\pi/12$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$2-\sqrt{3}$	$2+\sqrt{3}$	$\sqrt{6}-\sqrt{2}$	$\sqrt{6}+\sqrt{2}$
30°	$\pi/6$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	2
45°	$\pi/4$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\pi/3$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	2	$\frac{2}{3}\sqrt{3}$
75°	$5\pi/12$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$2+\sqrt{3}$	$2-\sqrt{3}$	$\sqrt{6}+\sqrt{2}$	$\sqrt{6}-\sqrt{2}$
90°	$\pi/2$	1	0	$\pm\infty$	0	$\pm\infty$	1
105°	$7\pi/12$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$-\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$-(2+\sqrt{3})$	$-(2-\sqrt{3})$	$-(\sqrt{6}+\sqrt{2})$	$\sqrt{6}-\sqrt{2}$
120°	$2\pi/3$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	-2	$\frac{2}{3}\sqrt{3}$
135°	$3\pi/4$	$\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
150°	$5\pi/6$	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	2
165°	$11\pi/12$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$-\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$-(2-\sqrt{3})$	$-(2+\sqrt{3})$	$-(\sqrt{6}-\sqrt{2})$	$\sqrt{6}+\sqrt{2}$
180°	π	0	-1	0	$\mp\infty$	-1	$\pm\infty$
195°	$13\pi/12$	$-\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$-\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$2-\sqrt{3}$	$2+\sqrt{3}$	$-(\sqrt{6}-\sqrt{2})$	$-(\sqrt{6}+\sqrt{2})$
210°	$7\pi/6$	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	-2
225°	$5\pi/4$	$-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
240°	$4\pi/3$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	-2	$-\frac{2}{3}\sqrt{3}$
255°	$17\pi/12$	$-\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$-\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$2+\sqrt{3}$	$2-\sqrt{3}$	$-(\sqrt{6}+\sqrt{2})$	$-(\sqrt{6}-\sqrt{2})$
270°	$3\pi/2$	-1	0	$\pm\infty$	0	$\mp\infty$	-1
285°	$19\pi/12$	$-\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$-(2+\sqrt{3})$	$-(2-\sqrt{3})$	$\sqrt{6}+\sqrt{2}$	$-(\sqrt{6}-\sqrt{2})$
300°	$5\pi/3$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	2	$-\frac{2}{3}\sqrt{3}$
315°	$7\pi/4$	$-\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
330°	$11\pi/6$	$-\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	-2
345°	$23\pi/12$	$-\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$-(2-\sqrt{3})$	$-(2+\sqrt{3})$	$\sqrt{6}-\sqrt{2}$	$-(\sqrt{6}+\sqrt{2})$
360°	2π	0	1	0	$\mp\infty$	1	$\mp\infty$

For tables involving other angles see pages 206-211 and 212-215.

GRAPHS OF TRIGONOMETRIC FUNCTIONS

In each graph x is in radians.

5.22 $y = \sin x$

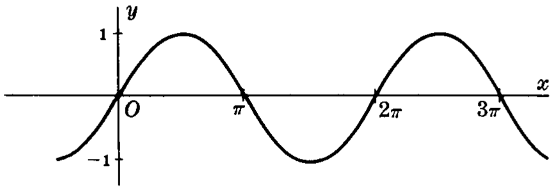

Fig. 5-5

5.23 $y = \cos x$

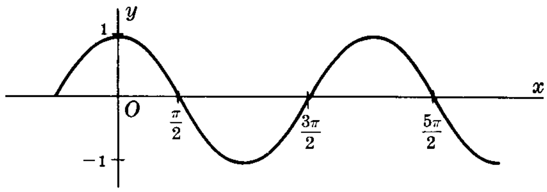

Fig. 5-6

5.24 $y = \tan x$

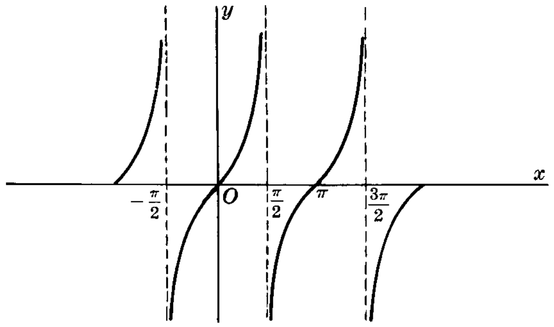

Fig. 5-7

5.25 $y = \cot x$

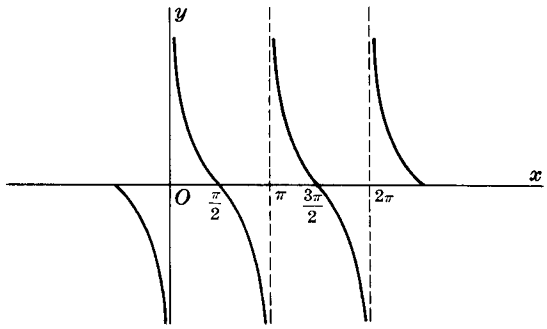

Fig. 5-8

5.26 $y = \sec x$

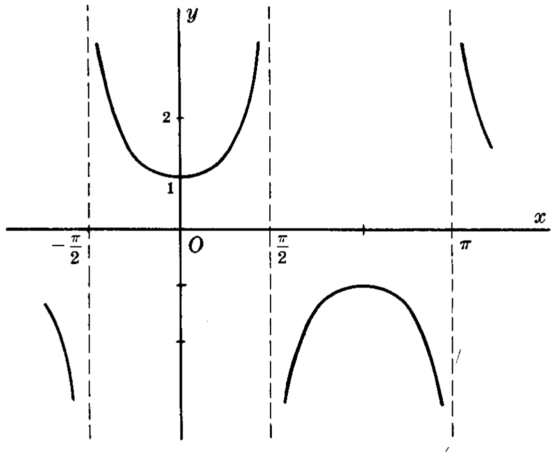

Fig. 5-9

5.27 $y = \csc x$

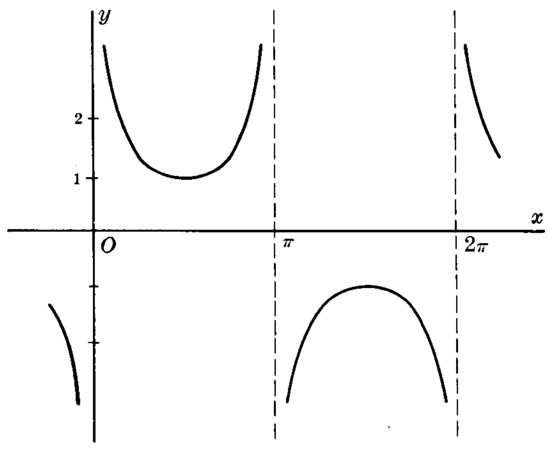

Fig. 5-10

FUNCTIONS OF NEGATIVE ANGLES

5.28 $\sin(-A) = -\sin A$ **5.29** $\cos(-A) = \cos A$ **5.30** $\tan(-A) = -\tan A$

5.31 $\csc(-A) = -\csc A$ **5.32** $\sec(-A) = \sec A$ **5.33** $\cot(-A) = -\cot A$

ADDITION FORMULAS

5.34 $$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

5.35 $$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

5.36 $$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

5.37 $$\cot(A \pm B) = \frac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$$

FUNCTIONS OF ANGLES IN ALL QUADRANTS IN TERMS OF THOSE IN QUADRANT I

	$-A$	$90^\circ \pm A$ $\frac{\pi}{2} \pm A$	$180^\circ \pm A$ $\pi \pm A$	$270^\circ \pm A$ $\frac{3\pi}{2} \pm A$	$k(360^\circ) \pm A$ $2k\pi \pm A$ k = integer
sin	$-\sin A$	$\cos A$	$\mp \sin A$	$-\cos A$	$\pm \sin A$
cos	$\cos A$	$\mp \sin A$	$-\cos A$	$\pm \sin A$	$\cos A$
tan	$-\tan A$	$\mp \cot A$	$\pm \tan A$	$\mp \cot A$	$\pm \tan A$
csc	$-\csc A$	$\sec A$	$\mp \csc A$	$-\sec A$	$\pm \csc A$
sec	$\sec A$	$\mp \csc A$	$-\sec A$	$\pm \csc A$	$\sec A$
cot	$-\cot A$	$\mp \tan A$	$\pm \cot A$	$\mp \tan A$	$\pm \cot A$

RELATIONSHIPS AMONG FUNCTIONS OF ANGLES IN QUADRANT I

	$\sin A = u$	$\cos A = u$	$\tan A = u$	$\cot A = u$	$\sec A = u$	$\csc A = u$
$\sin A$	u	$\sqrt{1-u^2}$	$u/\sqrt{1+u^2}$	$1/\sqrt{1+u^2}$	$\sqrt{u^2-1}/u$	$1/u$
$\cos A$	$\sqrt{1-u^2}$	u	$1/\sqrt{1+u^2}$	$u/\sqrt{1+u^2}$	$1/u$	$\sqrt{u^2-1}/u$
$\tan A$	$u/\sqrt{1-u^2}$	$\sqrt{1-u^2}/u$	u	$1/u$	$\sqrt{u^2-1}$	$1/\sqrt{u^2-1}$
$\cot A$	$\sqrt{1-u^2}/u$	$u/\sqrt{1-u^2}$	$1/u$	u	$1/\sqrt{u^2-1}$	$\sqrt{u^2-1}$
$\sec A$	$1/\sqrt{1-u^2}$	$1/u$	$\sqrt{1+u^2}$	$\sqrt{1+u^2}/u$	u	$u/\sqrt{u^2-1}$
$\csc A$	$1/u$	$1/\sqrt{1-u^2}$	$\sqrt{1+u^2}/u$	$\sqrt{1+u^2}$	$u/\sqrt{u^2-1}$	$\sqrt{1+u^2}$

For extensions to other quadrants use appropriate signs as given in the preceding table.

DOUBLE ANGLE FORMULAS

5.38 $$\sin 2A = 2 \sin A \cos A$$

5.39 $$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$$

5.40 $$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

HALF ANGLE FORMULAS

5.41 $$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}} \quad \begin{bmatrix} + \text{ if } A/2 \text{ is in quadrant I or II} \\ - \text{ if } A/2 \text{ is in quadrant III or IV} \end{bmatrix}$$

5.42 $$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}} \quad \begin{bmatrix} + \text{ if } A/2 \text{ is in quadrant I or IV} \\ - \text{ if } A/2 \text{ is in quadrant II or III} \end{bmatrix}$$

5.43 $$\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}} \quad \begin{bmatrix} + \text{ if } A/2 \text{ is in quadrant I or III} \\ - \text{ if } A/2 \text{ is in quadrant II or IV} \end{bmatrix}$$

$$= \frac{\sin A}{1+\cos A} = \frac{1-\cos A}{\sin A} = \csc A - \cot A$$

MULTIPLE ANGLE FORMULAS

5.44 $$\sin 3A = 3\sin A - 4\sin^3 A$$

5.45 $$\cos 3A = 4\cos^3 A - 3\cos A$$

5.46 $$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

5.47 $$\sin 4A = 4\sin A\cos A - 8\sin^3 A\cos A$$

5.48 $$\cos 4A = 8\cos^4 A - 8\cos^2 A + 1$$

5.49 $$\tan 4A = \frac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A}$$

5.50 $$\sin 5A = 5\sin A - 20\sin^3 A + 16\sin^5 A$$

5.51 $$\cos 5A = 16\cos^5 A - 20\cos^3 A + 5\cos A$$

5.52 $$\tan 5A = \frac{\tan^5 A - 10\tan^3 A + 5\tan A}{1 - 10\tan^2 A + 5\tan^4 A}$$

See also formulas 5.68 and 5.69.

POWERS OF TRIGONOMETRIC FUNCTIONS

5.53 $$\sin^2 A = \tfrac{1}{2} - \tfrac{1}{2}\cos 2A$$

5.54 $$\cos^2 A = \tfrac{1}{2} + \tfrac{1}{2}\cos 2A$$

5.55 $$\sin^3 A = \tfrac{3}{4}\sin A - \tfrac{1}{4}\sin 3A$$

5.56 $$\cos^3 A = \tfrac{3}{4}\cos A + \tfrac{1}{4}\cos 3A$$

5.57 $$\sin^4 A = \tfrac{3}{8} - \tfrac{1}{2}\cos 2A + \tfrac{1}{8}\cos 4A$$

5.58 $$\cos^4 A = \tfrac{3}{8} + \tfrac{1}{2}\cos 2A + \tfrac{1}{8}\cos 4A$$

5.59 $$\sin^5 A = \tfrac{5}{8}\sin A - \tfrac{5}{16}\sin 3A + \tfrac{1}{16}\sin 5A$$

5.60 $$\cos^5 A = \tfrac{5}{8}\cos A + \tfrac{5}{16}\cos 3A + \tfrac{1}{16}\cos 5A$$

See also formulas 5.70 through 5.73.

SUM, DIFFERENCE AND PRODUCT OF TRIGONOMETRIC FUNCTIONS

5.61 $$\sin A + \sin B = 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)$$

5.62 $$\sin A - \sin B = 2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)$$

5.63 $$\cos A + \cos B = 2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)$$

5.64 $$\cos A - \cos B = 2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(B-A)$$

5.65 $$\sin A\sin B = \tfrac{1}{2}\{\cos(A-B) - \cos(A+B)\}$$

5.66 $$\cos A\cos B = \tfrac{1}{2}\{\cos(A-B) + \cos(A+B)\}$$

5.67 $$\sin A\cos B = \tfrac{1}{2}\{\sin(A-B) + \sin(A+B)\}$$

GENERAL FORMULAS

5.68 $$\sin nA = \sin A\left\{(2\cos A)^{n-1} - \binom{n-2}{1}(2\cos A)^{n-3} + \binom{n-3}{2}(2\cos A)^{n-5} - \cdots\right\}$$

5.69 $$\cos nA = \frac{1}{2}\left\{(2\cos A)^n - \frac{n}{1}(2\cos A)^{n-2} + \frac{n}{2}\binom{n-3}{1}(2\cos A)^{n-4} - \frac{n}{3}\binom{n-4}{2}(2\cos A)^{n-6} + \cdots\right\}$$

5.70 $$\sin^{2n-1} A = \frac{(-1)^{n-1}}{2^{2n-2}}\left\{\sin(2n-1)A - \binom{2n-1}{1}\sin(2n-3)A + \cdots (-1)^{n-1}\binom{2n-1}{n-1}\sin A\right\}$$

5.71 $$\cos^{2n-1} A = \frac{1}{2^{2n-2}}\left\{\cos(2n-1)A + \binom{2n-1}{1}\cos(2n-3)A + \cdots + \binom{2n-1}{n-1}\cos A\right\}$$

5.72 $$\sin^{2n} A = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{(-1)^n}{2^{2n-1}}\left\{\cos 2nA - \binom{2n}{1}\cos(2n-2)A + \cdots (-1)^{n-1}\binom{2n}{n-1}\cos 2A\right\}$$

5.73 $$\cos^{2n} A = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n-1}}\left\{\cos 2nA + \binom{2n}{1}\cos(2n-2)A + \cdots + \binom{2n}{n-1}\cos 2A\right\}$$

INVERSE TRIGONOMETRIC FUNCTIONS

If $x = \sin y$ then $y = \sin^{-1} x$, i.e. *the angle whose sine is* x or *inverse sine of* x, is a many-valued function of x which is a collection of single-valued functions called *branches*. Similarly the other inverse trigonometric functions are multiple-valued.

For many purposes a particular branch is required. This is called the *principal branch* and the values for this branch are called *principal values*.

PRINCIPAL VALUES FOR INVERSE TRIGONOMETRIC FUNCTIONS

Principal values for $x \geqq 0$	Principal values for $x < 0$
$0 \leqq \sin^{-1} x \leqq \pi/2$	$-\pi/2 \leqq \sin^{-1} x < 0$
$0 \leqq \cos^{-1} x \leqq \pi/2$	$\pi/2 < \cos^{-1} x \leqq \pi$
$0 \leqq \tan^{-1} x < \pi/2$	$-\pi/2 < \tan^{-1} x < 0$
$0 < \cot^{-1} x \leqq \pi/2$	$\pi/2 < \cot^{-1} x < \pi$
$0 \leqq \sec^{-1} x < \pi/2$	$\pi/2 < \sec^{-1} x \leqq \pi$
$0 < \csc^{-1} x \leqq \pi/2$	$-\pi/2 \leqq \csc^{-1} x < 0$

RELATIONS BETWEEN INVERSE TRIGONOMETRIC FUNCTIONS

In all cases it is assumed that principal values are used.

5.74 $\sin^{-1} x + \cos^{-1} x = \pi/2$

5.75 $\tan^{-1} x + \cot^{-1} x = \pi/2$

5.76 $\sec^{-1} x + \csc^{-1} x = \pi/2$

5.77 $\csc^{-1} x = \sin^{-1}(1/x)$

5.78 $\sec^{-1} x = \cos^{-1}(1/x)$

5.79 $\cot^{-1} x = \tan^{-1}(1/x)$

5.80 $\sin^{-1}(-x) = -\sin^{-1} x$

5.81 $\cos^{-1}(-x) = \pi - \cos^{-1} x$

5.82 $\tan^{-1}(-x) = -\tan^{-1} x$

5.83 $\cot^{-1}(-x) = \pi - \cot^{-1} x$

5.84 $\sec^{-1}(-x) = \pi - \sec^{-1} x$

5.85 $\csc^{-1}(-x) = -\csc^{-1} x$

GRAPHS OF INVERSE TRIGONOMETRIC FUNCTIONS

In each graph y is in radians. Solid portions of curves correspond to principal values.

5.86 $y = \sin^{-1} x$

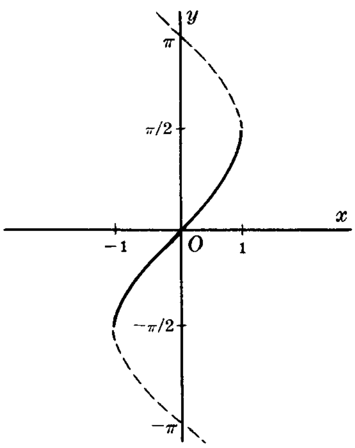

Fig. 5-11

5.87 $y = \cos^{-1} x$

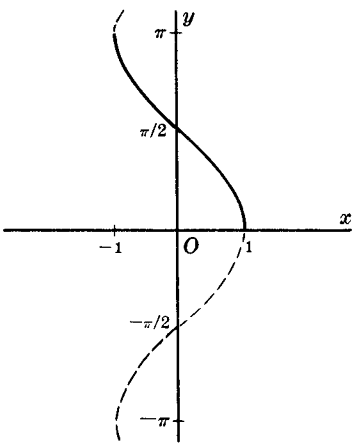

Fig. 5-12

5.88 $y = \tan^{-1} x$

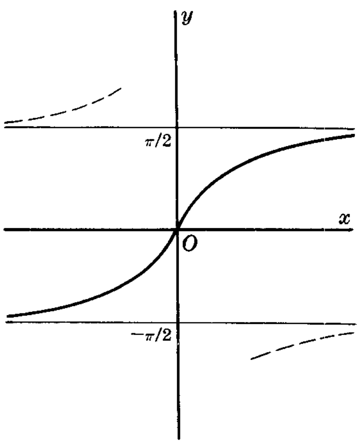

Fig. 5-13

5.89 $y = \cot^{-1} x$

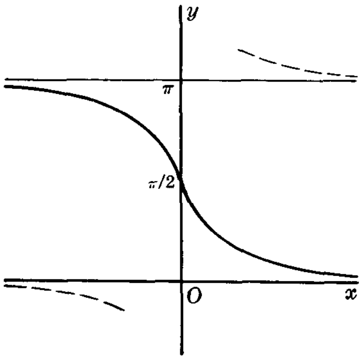

Fig. 5-14

5.90 $y = \sec^{-1} x$

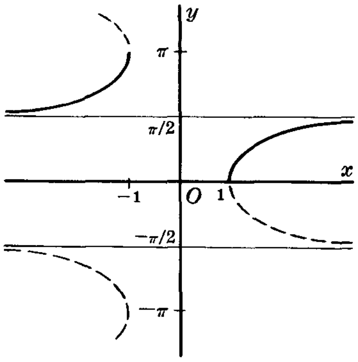

Fig. 5-15

5.91 $y = \csc^{-1} x$

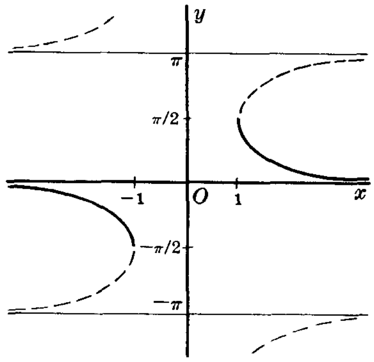

Fig. 5-16

RELATIONSHIPS BETWEEN SIDES AND ANGLES OF A PLANE TRIANGLE

The following results hold for any plane triangle ABC with sides a, b, c and angles A, B, C.

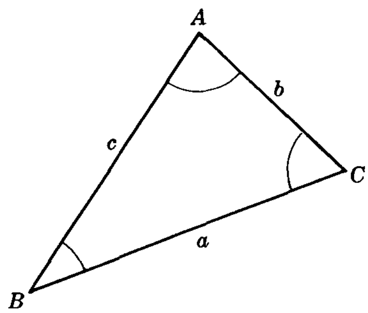

Fig. 5-17

5.92 **Law of Sines**

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

5.93 **Law of Cosines**

$$c^2 = a^2 + b^2 - 2ab \cos C$$

with similar relations involving the other sides and angles.

5.94 **Law of Tangents**

$$\frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}$$

with similar relations involving the other sides and angles.

5.95

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a+b+c)$ is the semiperimeter of the triangle. Similar relations involving angles B and C can be obtained.

See also formulas 4.5, page 5; 4.15 and 4.16, page 6.

RELATIONSHIPS BETWEEN SIDES AND ANGLES OF A SPHERICAL TRIANGLE

Spherical triangle ABC is on the surface of a sphere as shown in Fig. 5-18. Sides a, b, c [which are arcs of great circles] are measured by their angles subtended at center O of the sphere. A, B, C are the angles opposite sides a, b, c respectively. Then the following results hold.

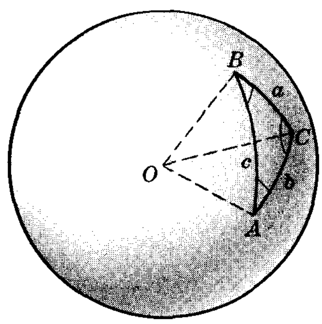

Fig. 5-18

5.96 **Law of Sines**

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

5.97 **Law of Cosines**

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$
$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

with similar results involving other sides and angles.

5.98 Law of Tangents

$$\frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)} = \frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2}(a-b)}$$

with similar results involving other sides and angles.

5.99

$$\cos \frac{A}{2} = \sqrt{\frac{\sin s \sin (s-c)}{\sin b \sin c}}$$

where $s = \frac{1}{2}(a+b+c)$. Similar results hold for other sides and angles.

5.100

$$\cos \frac{a}{2} = \sqrt{\frac{\cos (S-B) \cos (S-C)}{\sin B \sin C}}$$

where $S = \frac{1}{2}(A+B+C)$. Similar results hold for other sides and angles.

See also formula 4.44, page 10.

NAPIER'S RULES FOR RIGHT ANGLED SPHERICAL TRIANGLES

Except for right angle C, there are five parts of spherical triangle ABC which if arranged in the order as given in Fig. 5-19 would be a, b, A, c, B.

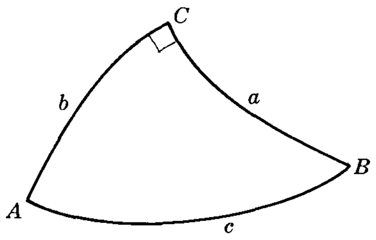

Fig. 5-19

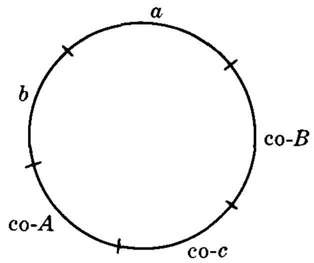

Fig. 5-20

Suppose these quantities are arranged in a circle as in Fig. 5-20 where we attach the prefix co [indicating *complement*] to hypotenuse c and angles A and B.

Any one of the parts of this circle is called a *middle part*, the two neighboring parts are called *adjacent parts* and the two remaining parts are called *opposite parts*. Then Napier's rules are

5.101 The sine of any middle part equals the product of the tangents of the adjacent parts.

5.102 The sine of any middle part equals the product of the cosines of the opposite parts.

Example: Since co-$A = 90° - A$, co-$B = 90° - B$, we have

$$\sin a = \tan b \tan (\text{co-}B) \quad \text{or} \quad \sin a = \tan b \cot B$$

$$\sin (\text{co-}A) = \cos a \cos (\text{co-}B) \quad \text{or} \quad \cos A = \cos a \sin B$$

These can of course be obtained also from the results 5.97 on page 19.