12 FORMULAS from SOLID ANALYTIC GEOMETRY

DISTANCE d BETWEEN TWO POINTS $P_1(x_1, y_1, z_1)$ AND $P_2(x_2, y_2, z_2)$

12.1 $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

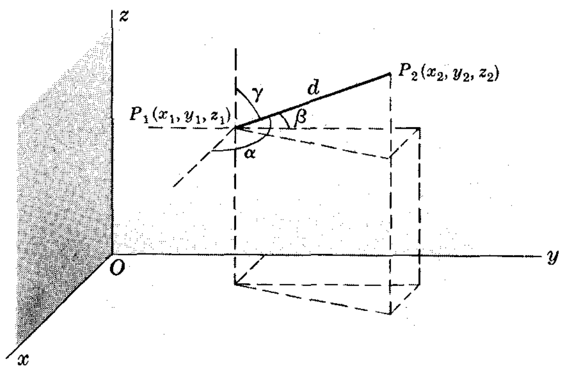

Fig. 12-1

DIRECTION COSINES OF LINE JOINING POINTS $P_1(x_1, y_1, z_1)$ AND $P_2(x_2, y_2, z_2)$

12.2 $$l = \cos\alpha = \frac{x_2 - x_1}{d}, \quad m = \cos\beta = \frac{y_2 - y_1}{d}, \quad n = \cos\gamma = \frac{z_2 - z_1}{d}$$

where α, β, γ are the angles which line P_1P_2 makes with the positive x, y, z axes respectively and d is given by 12.1 [see Fig. 12-1].

RELATIONSHIP BETWEEN DIRECTION COSINES

12.3 $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \quad \text{or} \quad l^2 + m^2 + n^2 = 1$$

DIRECTION NUMBERS

Numbers L, M, N which are proportional to the direction cosines l, m, n are called *direction numbers.* The relationship between them is given by

12.4 $$l = \frac{L}{\sqrt{L^2 + M^2 + N^2}}, \quad m = \frac{M}{\sqrt{L^2 + M^2 + N^2}}, \quad n = \frac{N}{\sqrt{L^2 + M^2 + N^2}}$$

EQUATIONS OF LINE JOINING $P_1(x_1, y_1, z_1)$ AND $P_2(x_2, y_2, z_2)$ IN STANDARD FORM

12.5 $$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \quad \text{or} \quad \frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

These are also valid if l, m, n are replaced by L, M, N respectively.

EQUATIONS OF LINE JOINING $P_1(x_1, y_1, z_1)$ AND $P_2(x_2, y_2, z_2)$ IN PARAMETRIC FORM

12.6 $$x = x_1 + lt, \quad y = y_1 + mt, \quad z = z_1 + nt$$

These are also valid if l, m, n are replaced by L, M, N respectively.

ANGLE ϕ BETWEEN TWO LINES WITH DIRECTION COSINES l_1, m_1, n_1 AND l_2, m_2, n_2

12.7 $$\cos\phi = l_1 l_2 + m_1 m_2 + n_1 n_2$$

GENERAL EQUATION OF A PLANE

12.8 $$Ax + By + Cz + D = 0 \qquad [A, B, C, D \text{ are constants}]$$

EQUATION OF PLANE PASSING THROUGH POINTS (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)

12.9 $$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

or

12.10 $$\begin{vmatrix} y_2 - y_1 & z_2 - z_1 \\ y_3 - y_1 & z_3 - z_1 \end{vmatrix}(x - x_1) + \begin{vmatrix} z_2 - z_1 & x_2 - x_1 \\ z_3 - z_1 & x_3 - x_1 \end{vmatrix}(y - y_1) + \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}(z - z_1) = 0$$

EQUATION OF PLANE IN INTERCEPT FORM

12.11 $$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

where a, b, c are the intercepts on the x, y, z axes respectively.

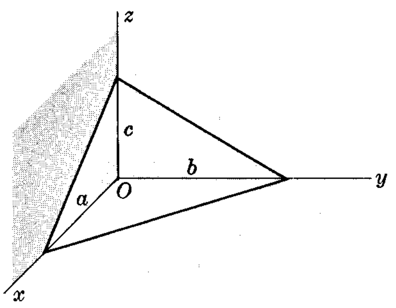

Fig. 12-2

EQUATIONS OF LINE THROUGH (x_0, y_0, z_0) AND PERPENDICULAR TO PLANE $Ax + By + Cz + D = 0$

12.12 $$\frac{x - x_0}{A} = \frac{y - y_0}{B} = \frac{z - z_0}{C} \quad \text{or} \quad x = x_0 + At, \quad y = y_0 + Bt, \quad z = z_0 + Ct$$

Note that the direction numbers for a line perpendicular to the plane $Ax + By + Cz + D = 0$ are A, B, C.

DISTANCE FROM POINT (x_0, y_0, z_0) TO PLANE $Ax + By + Cz + D = 0$

12.13 $$\frac{Ax_0 + By_0 + Cz_0 + D}{\pm\sqrt{A^2 + B^2 + C^2}}$$

where the sign is chosen so that the distance is nonnegative.

NORMAL FORM FOR EQUATION OF PLANE

12.14 $$x \cos\alpha + y \cos\beta + z \cos\gamma = p$$

where p = perpendicular distance from O to plane at P and α, β, γ are angles between OP and positive x, y, z axes.

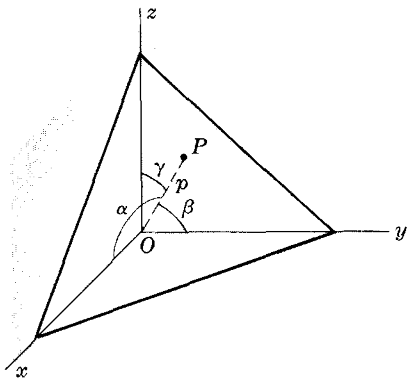

Fig. 12-3

TRANSFORMATION OF COORDINATES INVOLVING PURE TRANSLATION

12.15 $$\begin{cases} x = x' + x_0 \\ y = y' + y_0 \\ z = z' + z_0 \end{cases} \quad \text{or} \quad \begin{cases} x' = x - x_0 \\ y' = y - y_0 \\ z' = z - z_0 \end{cases}$$

where (x, y, z) are old coordinates [i.e. coordinates relative to xyz system], (x', y', z') are new coordinates [relative to $x'y'z'$ system] and (x_0, y_0, z_0) are the coordinates of the new origin O' relative to the old xyz coordinate system.

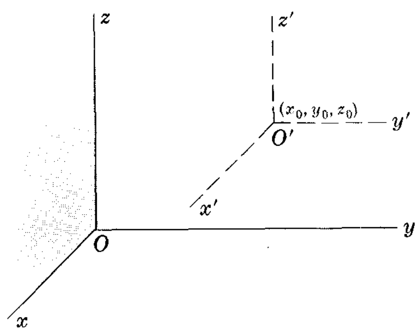

Fig. 12-4

TRANSFORMATION OF COORDINATES INVOLVING PURE ROTATION

12.16
$$\begin{cases} x = l_1x' + l_2y' + l_3z' \\ y = m_1x' + m_2y' + m_3z' \\ z = n_1x' + n_2y' + n_3z' \end{cases}$$

or
$$\begin{cases} x' = l_1x + m_1y + n_1z \\ y' = l_2x + m_2y + n_2z \\ z' = l_3x + m_3y + n_3z \end{cases}$$

where the origins of the xyz and $x'y'z'$ systems are the same and l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 are the direction cosines of the x', y', z' axes relative to the x, y, z axes respectively.

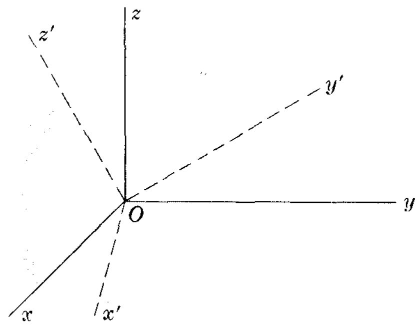

Fig. 12-5

TRANSFORMATION OF COORDINATES INVOLVING TRANSLATION AND ROTATION

12.17
$$\begin{cases} x = l_1x' + l_2y' + l_3z' + x_0 \\ y = m_1x' + m_2y' + m_3z' + y_0 \\ z = n_1x' + n_2y' + n_3z' + z_0 \end{cases}$$

or
$$\begin{cases} x' = l_1(x - x_0) + m_1(y - y_0) + n_1(z - z_0) \\ y' = l_2(x - x_0) + m_2(y - y_0) + n_2(z - z_0) \\ z' = l_3(x - x_0) + m_3(y - y_0) + n_3(z - z_0) \end{cases}$$

where the origin O' of the $x'y'z'$ system has coordinates (x_0, y_0, z_0) relative to the xyz system and l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 are the direction cosines of the x', y', z' axes relative to the x, y, z axes respectively.

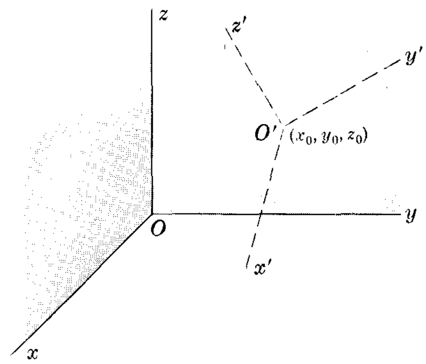

Fig. 12-6

CYLINDRICAL COORDINATES (r, θ, z)

A point P can be located by cylindrical coordinates (r, θ, z) [see Fig. 12-7] as well as rectangular coordinates (x, y, z).

The transformation between these coordinates is

12.18
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \quad \text{or} \quad \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}(y/x) \\ z = z \end{cases}$$

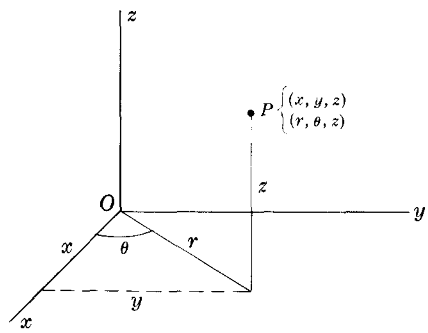

Fig. 12-7

SPHERICAL COORDINATES (r, θ, ϕ)

A point P can be located by spherical coordinates (r, θ, ϕ) [see Fig. 12-8] as well as rectangular coordinates (x, y, z).

The transformation between those coordinates is

12.19
$$\begin{cases} x = r \sin\theta \cos\phi \\ y = r \sin\theta \sin\phi \\ z = r\cos\theta \end{cases}$$

or
$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \phi = \tan^{-1}(y/x) \\ \theta = \cos^{-1}(z/\sqrt{x^2+y^2+z^2}) \end{cases}$$

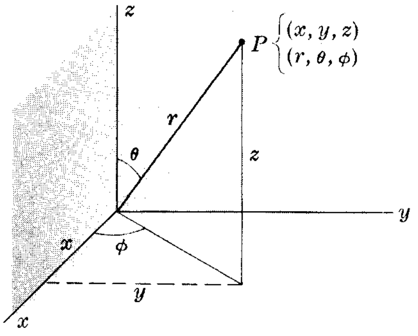

Fig. 12-8

EQUATION OF SPHERE IN RECTANGULAR COORDINATES

12.20
$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = R^2$$

where the sphere has center (x_0, y_0, z_0) and radius R.

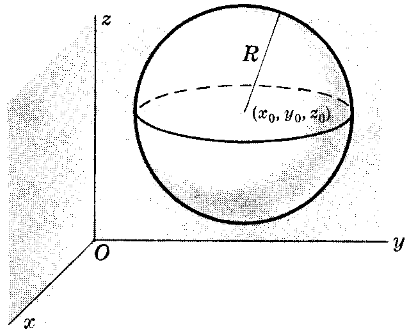

Fig. 12-9

EQUATION OF SPHERE IN CYLINDRICAL COORDINATES

12.21
$$r^2 - 2r_0 r\cos(\theta-\theta_0) + r_0^2 + (z-z_0)^2 = R^2$$

where the sphere has center (r_0, θ_0, z_0) in cylindrical coordinates and radius R.

If the center is at the origin the equation is

12.22
$$r^2 + z^2 = R^2$$

EQUATION OF SPHERE IN SPHERICAL COORDINATES

12.23
$$r^2 + r_0^2 - 2r_0 r \sin\theta \sin\theta_0 \cos(\phi-\phi_0) = R^2$$

where the sphere has center (r_0, θ_0, ϕ_0) in spherical coordinates and radius R.

If the center is at the origin the equation is

12.24
$$r = R$$

EQUATION OF ELLIPSOID WITH CENTER (x_0, y_0, z_0) AND SEMI-AXES a, b, c

12.25 $$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} + \frac{(z-z_0)^2}{c^2} = 1$$

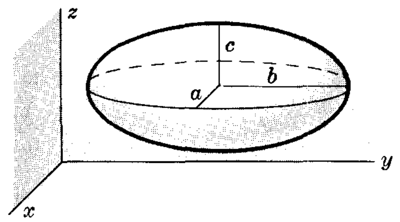

Fig. 12-10

ELLIPTIC CYLINDER WITH AXIS AS z AXIS

12.26 $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a, b are semi-axes of elliptic cross section.

If $b = a$ it becomes a circular cylinder of radius a.

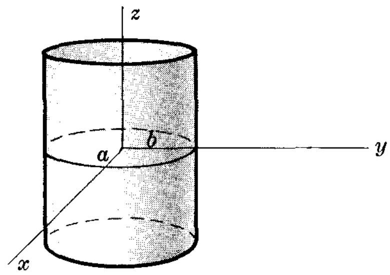

Fig. 12-11

ELLIPTIC CONE WITH AXIS AS z AXIS

12.27 $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

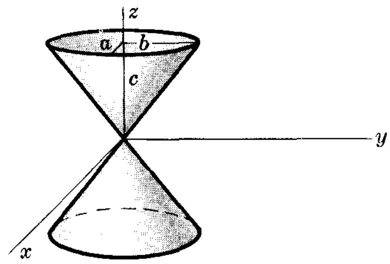

Fig. 12-12

HYPERBOLOID OF ONE SHEET

12.28 $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

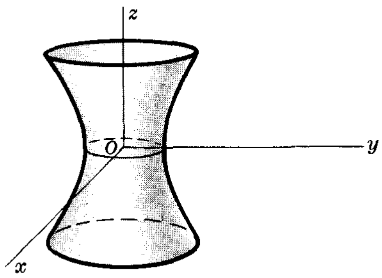

Fig. 12-13

HYPERBOLOID OF TWO SHEETS

12.29 $$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Note orientation of axes in Fig. 12-14.

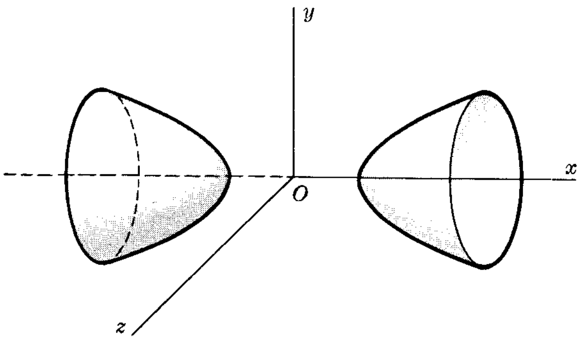

Fig. 12-14

ELLIPTIC PARABOLOID

12.30 $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

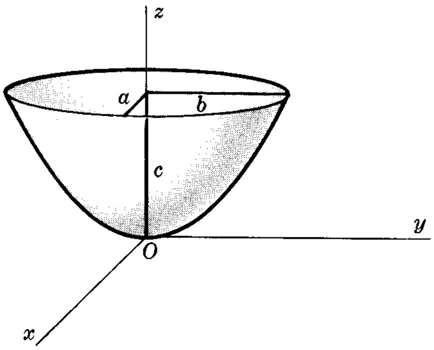

Fig. 12-15

HYPERBOLIC PARABOLOID

12.31 $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$$

Note orientation of axes in Fig. 12-16.

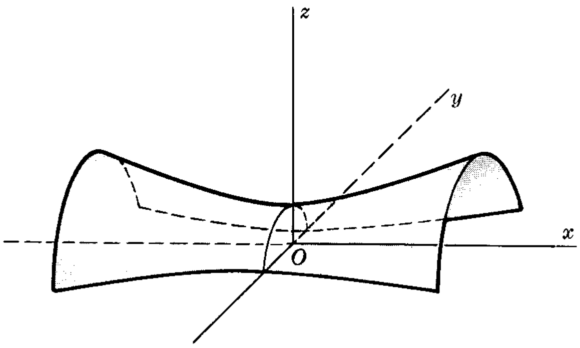

Fig. 12-16