13 DERIVATIVES

DEFINITION OF A DERIVATIVE

If $y = f(x)$, the derivative of y or $f(x)$ with respect to x is defined as

13.1 $$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

where $h = \Delta x$. The derivative is also denoted by y', df/dx or $f'(x)$. The process of taking a derivative is called *differentiation.*

GENERAL RULES OF DIFFERENTIATION

In the following, u, v, w are functions of x; a, b, c, n are constants [restricted if indicated]; $e = 2.71828\ldots$ is the natural base of logarithms; $\ln u$ is the natural logarithm of u [i.e. the logarithm to the base e] where it is assumed that $u > 0$ and all angles are in radians.

13.2 $$\frac{d}{dx}(c) = 0$$

13.3 $$\frac{d}{dx}(cx) = c$$

13.4 $$\frac{d}{dx}(cx^n) = ncx^{n-1}$$

13.5 $$\frac{d}{dx}(u \pm v \pm w \pm \cdots) = \frac{du}{dx} \pm \frac{dv}{dx} \pm \frac{dw}{dx} \pm \cdots$$

13.6 $$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

13.7 $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

13.8 $$\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$$

13.9 $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v(du/dx) - u(dv/dx)}{v^2}$$

13.10 $$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

13.11 $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \text{(Chain rule)}$$

13.12 $$\frac{du}{dx} = \frac{1}{dx/du}$$

13.13 $$\frac{dy}{dx} = \frac{dy/du}{dx/du}$$

DERIVATIVES OF TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS

13.14 $\frac{d}{dx}\sin u = \cos u \frac{du}{dx}$

13.15 $\frac{d}{dx}\cos u = -\sin u \frac{du}{dx}$

13.16 $\frac{d}{dx}\tan u = \sec^2 u \frac{du}{dx}$

13.17 $\frac{d}{dx}\cot u = -\csc^2 u \frac{du}{dx}$

13.18 $\frac{d}{dx}\sec u = \sec u \tan u \frac{du}{dx}$

13.19 $\frac{d}{dx}\csc u = -\csc u \cot u \frac{du}{dx}$

13.20 $\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad \left[-\frac{\pi}{2} < \sin^{-1} u < \frac{\pi}{2}\right]$

13.21 $\frac{d}{dx}\cos^{-1} u = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad [0 < \cos^{-1} u < \pi]$

13.22 $\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx} \qquad \left[-\frac{\pi}{2} < \tan^{-1} u < \frac{\pi}{2}\right]$

13.23 $\frac{d}{dx}\cot^{-1} u = \frac{-1}{1+u^2}\frac{du}{dx} \qquad [0 < \cot^{-1} u < \pi]$

13.24 $\frac{d}{dx}\sec^{-1} u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} = \frac{\pm 1}{u\sqrt{u^2-1}}\frac{du}{dx} \qquad \begin{bmatrix} + \text{ if } 0 < \sec^{-1} u < \pi/2 \\ - \text{ if } \pi/2 < \sec^{-1} u < \pi \end{bmatrix}$

13.25 $\frac{d}{dx}\csc^{-1} u = \frac{-1}{|u|\sqrt{u^2-1}}\frac{du}{dx} = \frac{\mp 1}{u\sqrt{u^2-1}}\frac{du}{dx} \qquad \begin{bmatrix} - \text{ if } 0 < \csc^{-1} u < \pi/2 \\ + \text{ if } -\pi/2 < \csc^{-1} u < 0 \end{bmatrix}$

DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

13.26 $\frac{d}{dx}\log_a u = \frac{\log_a e}{u}\frac{du}{dx} \qquad a \neq 0, 1$

13.27 $\frac{d}{dx}\ln u = \frac{d}{dx}\log_e u = \frac{1}{u}\frac{du}{dx}$

13.28 $\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}$

13.29 $\frac{d}{dx}e^u = e^u \frac{du}{dx}$

13.30 $\frac{d}{dx}u^v = \frac{d}{dx}e^{v \ln u} = e^{v \ln u}\frac{d}{dx}[v \ln u] = vu^{v-1}\frac{du}{dx} + u^v \ln u \frac{dv}{dx}$

DERIVATIVES OF HYPERBOLIC AND INVERSE HYPERBOLIC FUNCTIONS

13.31 $\frac{d}{dx}\sinh u = \cosh u \frac{du}{dx}$

13.32 $\frac{d}{dx}\cosh u = \sinh u \frac{du}{dx}$

13.33 $\frac{d}{dx}\tanh u = \operatorname{sech}^2 u \frac{du}{dx}$

13.34 $\frac{d}{dx}\coth u = -\operatorname{csch}^2 u \frac{du}{dx}$

13.35 $\frac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u \tanh u \frac{du}{dx}$

13.36 $\frac{d}{dx}\operatorname{csch} u = -\operatorname{csch} u \coth u \frac{du}{dx}$

13.37 $$\frac{d}{dx}\sinh^{-1}u = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx}$$

13.38 $$\frac{d}{dx}\cosh^{-1}u = \frac{\pm 1}{\sqrt{u^2-1}}\frac{du}{dx} \qquad \begin{bmatrix} + \text{ if } \cosh^{-1}u > 0,\ u > 1 \\ - \text{ if } \cosh^{-1}u < 0,\ u > 1 \end{bmatrix}$$

13.39 $$\frac{d}{dx}\tanh^{-1}u = \frac{1}{1-u^2}\frac{du}{dx} \qquad [-1 < u < 1]$$

13.40 $$\frac{d}{dx}\coth^{-1}u = \frac{1}{1-u^2}\frac{du}{dx} \qquad [u > 1 \text{ or } u < -1]$$

13.41 $$\frac{d}{dx}\operatorname{sech}^{-1}u = \frac{\mp 1}{u\sqrt{1-u^2}}\frac{du}{dx} \qquad \begin{bmatrix} - \text{ if } \operatorname{sech}^{-1}u > 0,\ 0 < u < 1 \\ + \text{ if } \operatorname{sech}^{-1}u < 0,\ 0 < u < 1 \end{bmatrix}$$

13.42 $$\frac{d}{dx}\operatorname{csch}^{-1}u = \frac{-1}{|u|\sqrt{1+u^2}}\frac{du}{dx} = \frac{\mp 1}{u\sqrt{1+u^2}}\frac{du}{dx} \qquad [- \text{ if } u > 0,\ + \text{ if } u < 0]$$

HIGHER DERIVATIVES

The second, third and higher derivatives are defined as follows.

13.43 Second derivative $$= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x) = y''$$

13.44 Third derivative $$= \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = f'''(x) = y'''$$

13.45 nth derivative $$= \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n} = f^{(n)}(x) = y^{(n)}$$

LEIBNITZ'S RULE FOR HIGHER DERIVATIVES OF PRODUCTS

Let D^p stand for the operator $\frac{d^p}{dx^p}$ so that $D^pu = \frac{d^pu}{dx^p}$ = the pth derivative of u. Then

13.46 $$D^n(uv) = uD^nv + \binom{n}{1}(Du)(D^{n-1}v) + \binom{n}{2}(D^2u)(D^{n-2}v) + \cdots + vD^nu$$

where $\binom{n}{1}, \binom{n}{2}, \ldots$ are the binomial coefficients [page 3].

As special cases we have

13.47 $$\frac{d^2}{dx^2}(uv) = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$$

13.48 $$\frac{d^3}{dx^3}(uv) = u\frac{d^3v}{dx^3} + 3\frac{du}{dx}\frac{d^2v}{dx^2} + 3\frac{d^2u}{dx^2}\frac{dv}{dx} + v\frac{d^3u}{dx^3}$$

DIFFERENTIALS

Let $y = f(x)$ and $\Delta y = f(x+\Delta x) - f(x)$. Then

13.49 $$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x) + \epsilon = \frac{dy}{dx} + \epsilon$$

where $\epsilon \to 0$ as $\Delta x \to 0$. Thus

13.50 $$\Delta y = f'(x)\,\Delta x + \epsilon\,\Delta x$$

If we call $\Delta x = dx$ the differential of x, then we define the differential of y to be

13.51 $$dy = f'(x)\,dx$$

RULES FOR DIFFERENTIALS

The rules for differentials are exactly analogous to those for derivatives. As examples we observe that

13.52 $$d(u \pm v \pm w \pm \cdots) = du \pm dv \pm dw \pm \cdots$$

13.53 $$d(uv) = u\,dv + v\,du$$

13.54 $$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$$

13.55 $$d(u^n) = nu^{n-1}\,du$$

13.56 $$d(\sin u) = \cos u\,du$$

13.57 $$d(\cos u) = -\sin u\,du$$

PARTIAL DERIVATIVES

Let $f(x,y)$ be a function of the two variables x and y. Then we define the partial derivative of $f(x,y)$ with respect to x, keeping y constant, to be

13.58 $$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

Similarly the partial derivative of $f(x,y)$ with respect to y, keeping x constant, is defined to be

13.59 $$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

Partial derivatives of higher order can be defined as follows.

13.60 $$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

13.61 $$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \qquad \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

The results in 13.61 will be equal if the function and its partial derivatives are continuous, i.e. in such case the order of differentiation makes no difference.

The differential of $f(x,y)$ is defined as

13.62 $$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

where $dx = \Delta x$ and $dy = \Delta y$.

Extension to functions of more than two variables are exactly analogous.