27. 北緯 ϕ の地点で、水平でなめらかな平面上で質量 m の質点を速さ $v(=|\vec{v}|)$ で平面上に打ち出す。平面上を運動する質点の奇跡を求めよ。

$$m\ddot{\vec{r}} = \vec{f} - 2m\vec{\omega} \times \dot{\vec{r}} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) \tag{1}$$

$$\vec{\omega} = \begin{pmatrix} -\omega\cos\phi \\ 0 \\ \omega\sin\phi \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} m\ddot{x} \\ m\ddot{y} \\ 0 \end{pmatrix} = -2m \begin{pmatrix} -\omega\cos\phi \\ 0 \\ \omega\sin\phi \end{pmatrix} \times \begin{pmatrix} \dot{x} \\ \dot{y} \\ 0 \end{pmatrix} = \begin{pmatrix} (2m\omega\sin\phi)\,\dot{y} \\ (-2m\omega\sin\phi)\,\dot{x} \\ 0 \end{pmatrix} \tag{3}$$

$$\therefore \begin{cases} \ddot{x} = (2\omega\sin\phi)\,\dot{y} \\ \ddot{y} = -(2\omega\sin\phi)\,\dot{x} \end{cases} \tag{4}$$

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= \begin{pmatrix} 0 & 2\omega\sin\phi \\ -2\omega\sin\phi & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \\ A &= \begin{pmatrix} 0 & 2\omega\sin\phi \\ -2\omega\sin\phi & 0 \end{pmatrix} \end{aligned} \tag{5}$$

$$\Phi_A(t) = \begin{vmatrix} t & -2\omega\sin\phi \\ 2\omega\sin\phi & t \end{vmatrix} = t^2 + 4\omega^2\sin^2\phi = 0 \tag{6}$$

$$\lambda_{\pm} = \pm(2\omega\sin\phi)\,i \tag{7}$$

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}i \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}i \end{pmatrix} \begin{pmatrix} 0 & 2\omega\sin\phi \\ -2\omega\sin\phi & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}i & -\frac{1}{\sqrt{2}}i \end{pmatrix} \\ &= \begin{pmatrix} (2\omega\sin\phi)\,i & 0 \\ 0 & -(2\omega\sin\phi)\,i \end{pmatrix} \end{aligned} \tag{8}$$

$$\begin{aligned} e^{tA} &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}i & -\frac{1}{\sqrt{2}}i \end{pmatrix} \begin{pmatrix} e^{i(2\omega\sin\phi)t} & 0 \\ 0 & e^{-(2\omega\sin\phi)t} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}i \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}i \end{pmatrix} \\ &= \begin{pmatrix} \cos\{(2\omega\sin\phi)\,t\} & \sin\{(2\omega\sin\phi)\,t\} \\ -\sin\{(2\omega\sin\phi)\,t\} & \cos\{(2\omega\sin\phi)\,t\} \end{pmatrix} \end{aligned} \tag{9}$$

$$P_1 = \frac{1}{\lambda_1 - \lambda_2}\left(A - \lambda_2 I\right) = \frac{1}{2i\left(2\omega\sin\phi\right)}\left\{A + i\left(2\omega\sin\phi\right)I\right\} \tag{10}$$

$$P_2 = \frac{1}{\lambda_2 - \lambda_1}(A - \lambda_1 I) = \frac{-1}{2i\left(2\omega\sin\phi\right)}\left\{A - i\left(2\omega\sin\phi\right)I\right\} \tag{11}$$

$$\begin{aligned}
e^{tA} &= e^{\lambda_1 t}P_1 + e^{\lambda_2 t}P_2 \\
&= e^{i(2\omega\sin\phi)t}\left\{\frac{A + i\left(2\omega\sin\phi\right)I}{2i\left(2\omega\sin\phi\right)}\right\} - e^{-i(2\omega\sin\phi)t}\left\{\frac{A - i\left(2\omega\sin\phi\right)I}{2i\left(2\omega\sin\phi\right)}\right\} \\
&= \left\{\frac{e^{i(2\omega\sin\phi)t} - e^{-i(2\omega\sin\phi)t}}{\left(2\omega\sin\phi\right)2i}\right\}A + \left\{\frac{e^{i(2\omega\sin\phi)t} + e^{-i(2\omega\sin\phi)t}}{2}\right\}I \\
&= \frac{\sin\left\{\left(2\omega\sin\phi\right)t\right\}}{\left(2\omega\sin\phi\right)}A + \cos\left\{\left(2\omega\sin\phi\right)t\right\}I \\
&= \begin{pmatrix} \cos\left\{\left(2\omega\sin\phi\right)t\right\} & \sin\left\{\left(2\omega\sin\phi\right)t\right\} \\ -\sin\left\{\left(2\omega\sin\phi\right)t\right\} & \cos\left\{\left(2\omega\sin\phi\right)t\right\} \end{pmatrix}
\end{aligned} \tag{12}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = e^{tA}\ \vec{v}_0 = e^{tA}\begin{pmatrix} v_{x_0} \\ v_{y_0} \end{pmatrix} \tag{13}$$

$$\therefore \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} v_{x_0}\cos\left\{\left(2\omega\sin\phi\right)t\right\} + v_{y_0}\sin\left\{\left(2\omega\sin\phi\right)t\right\} \\ v_{y_0}\cos\left\{\left(2\omega\sin\phi\right)t\right\} - v_{x_0}\sin\left\{\left(2\omega\sin\phi\right)t\right\} \end{pmatrix} \tag{14}$$

$$\begin{aligned}
x &= x_0 + v_{x_0}\frac{\sin\left\{\left(2\omega\sin\phi\right)t\right\}}{2\omega\sin\phi} - v_{y_0}\frac{\cos\left\{\left(2\omega\sin\phi\right)t\right\}}{2\omega\sin\phi} \\
y &= y_0 + v_{y_0}\frac{\sin\left\{\left(2\omega\sin\phi\right)t\right\}}{2\omega\sin\phi} + v_{x_0}\frac{\cos\left\{\left(2\omega\sin\phi\right)t\right\}}{2\omega\sin\phi}
\end{aligned}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 + v_{x_0}\frac{\sin\left\{(2\omega\sin\phi)t\right\}}{2\omega\sin\phi} - v_{y_0}\frac{\cos\left\{(2\omega\sin\phi)t\right\}}{2\omega\sin\phi} \\ y_0 + v_{y_0}\frac{\sin\left\{(2\omega\sin\phi)t\right\}}{2\omega\sin\phi} + v_{x_0}\frac{\cos\left\{(2\omega\sin\phi)t\right\}}{2\omega\sin\phi} \end{pmatrix} \tag{15}$$

$$m\ddot{\vec{r}} = \begin{pmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}\ddot{\vec{R}} - 2m\vec{\omega}\times\dot{\vec{r}} - m\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)$$