16 THE GAMMA FUNCTION

DEFINITION OF THE GAMMA FUNCTION $\Gamma(n)$ FOR $n > 0$

16.1
$$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}\,dt \qquad n > 0$$

RECURSION FORMULA

16.2
$$\Gamma(n+1) = n\,\Gamma(n)$$

16.3
$$\Gamma(n+1) = n! \quad \text{if } n = 0, 1, 2, \ldots \text{ where } 0! = 1$$

THE GAMMA FUNCTION FOR $n < 0$

For $n < 0$ the gamma function can be defined by using 16.2, i.e.

16.4
$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

GRAPH OF THE GAMMA FUNCTION

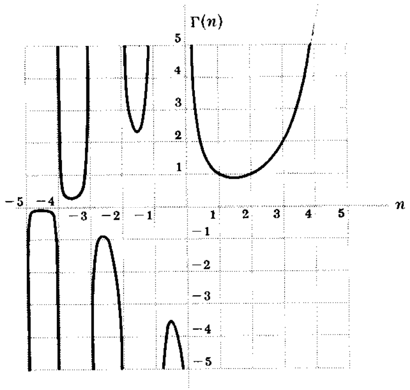

Fig. 16-1

SPECIAL VALUES FOR THE GAMMA FUNCTION

16.5
$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

16.6
$$\Gamma(m+\tfrac{1}{2}) = \frac{1\cdot 3\cdot 5\cdots(2m-1)}{2^m}\sqrt{\pi} \qquad m = 1, 2, 3, \ldots \; = \frac{(2m)!\sqrt{\pi}}{4^m m!}$$

16.7
$$\Gamma(-m+\tfrac{1}{2}) = \frac{(-1)^m 2^m\sqrt{\pi}}{1\cdot 3\cdot 5\cdots(2m-1)} \qquad m = 1, 2, 3, \ldots$$

RELATIONSHIPS AMONG GAMMA FUNCTIONS

16.8 $$\Gamma(p)\,\Gamma(1-p) \;=\; \frac{\pi}{\sin p\pi}$$

16.9 $$2^{2x-1}\,\Gamma(x)\,\Gamma(x+\tfrac{1}{2}) \;=\; \sqrt{\pi}\,\Gamma(2x)$$

This is called the *duplication formula.*

16.10 $$\Gamma(x)\,\Gamma\left(x+\frac{1}{m}\right)\Gamma\left(x+\frac{2}{m}\right)\cdots\Gamma\left(x+\frac{m-1}{m}\right) \;=\; m^{1/2-mx}\,(2\pi)^{(m-1)/2}\,\Gamma(mx)$$

For $m = 2$ this reduces to 16.9.

OTHER DEFINITIONS OF THE GAMMA FUNCTION

16.11 $$\Gamma(x+1) \;=\; \lim_{k\to\infty} \frac{1\cdot 2\cdot 3\cdots k}{(x+1)(x+2)\cdots(x+k)}\,k^x$$

16.12 $$\frac{1}{\Gamma(x)} \;=\; xe^{\gamma x}\prod_{m=1}^{\infty}\left\{\left(1+\frac{x}{m}\right)e^{-x/m}\right\}$$

This is an infinite product representation for the gamma function where γ is Euler's constant.

DERIVATIVES OF THE GAMMA FUNCTION

16.13 $$\Gamma'(1) \;=\; \int_0^\infty e^{-x}\ln x\,dx \;=\; -\gamma$$

16.14 $$\frac{\Gamma'(x)}{\Gamma(x)} \;=\; -\gamma + \left(\frac{1}{1}-\frac{1}{x}\right) + \left(\frac{1}{2}-\frac{1}{x+1}\right) + \cdots + \left(\frac{1}{n}-\frac{1}{x+n-1}\right) + \cdots$$

ASYMPTOTIC EXPANSIONS FOR THE GAMMA FUNCTION

16.15 $$\Gamma(x+1) \;=\; \sqrt{2\pi x}\,x^x e^{-x}\left\{1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51{,}840x^3} + \cdots\right\}$$

This is called *Stirling's asymptotic series.*

If we let $x = n$ a positive integer in 16.15, then a useful approximation for $n!$ where n is large [e.g. $n > 10$] is given by *Stirling's formula*

16.16 $$n! \;\sim\; \sqrt{2\pi n}\,n^n e^{-n}$$

where $\sim$ is used to indicate that the ratio of the terms on each side approaches 1 as $n \to \infty$.

MISCELLANEOUS RESULTS

16.17 $$|\Gamma(ix)|^2 \;=\; \frac{\pi}{x\sinh \pi x}$$

17 THE BETA FUNCTION

DEFINITION OF THE BETA FUNCTION $B(m, n)$

17.1 $$B(m,n) \;=\; \int_0^1 t^{m-1}(1-t)^{n-1}\,dt \qquad m>0,\; n>0$$

RELATIONSHIP OF BETA FUNCTION TO GAMMA FUNCTION

17.2 $$B(m,n) \;=\; \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Extensions of $B(m,n)$ to $m<0$, $n<0$ is provided by using 16.4, page 101.

SOME IMPORTANT RESULTS

17.3 $$B(m,n) \;=\; B(n,m)$$

17.4 $$B(m,n) \;=\; 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$$

17.5 $$B(m,n) \;=\; \int_0^{\infty} \frac{t^{m-1}}{(1+t)^{m+n}}\,dt$$

17.6 $$B(m,n) \;=\; r^n(r+1)^m \int_0^1 \frac{t^{m-1}(1-t)^{n-1}}{(r+t)^{m+n}}\,dt$$