8 HYPERBOLIC FUNCTIONS

DEFINITION OF HYPERBOLIC FUNCTIONS

8.1 *Hyperbolic sine* of x $= \sinh x = \dfrac{e^x - e^{-x}}{2}$

8.2 *Hyperbolic cosine* of x $= \cosh x = \dfrac{e^x + e^{-x}}{2}$

8.3 *Hyperbolic tangent* of x $= \tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

8.4 *Hyperbolic cotangent* of x $= \coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

8.5 *Hyperbolic secant* of x $= \operatorname{sech} x = \dfrac{2}{e^x + e^{-x}}$

8.6 *Hyperbolic cosecant* of x $= \operatorname{csch} x = \dfrac{2}{e^x - e^{-x}}$

RELATIONSHIPS AMONG HYPERBOLIC FUNCTIONS

8.7 $$\tanh x = \frac{\sinh x}{\cosh x}$$

8.8 $$\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}$$

8.9 $$\operatorname{sech} x = \frac{1}{\cosh x}$$

8.10 $$\operatorname{csch} x = \frac{1}{\sinh x}$$

8.11 $$\cosh^2 x - \sinh^2 x = 1$$

8.12 $$\operatorname{sech}^2 x + \tanh^2 x = 1$$

8.13 $$\coth^2 x - \operatorname{csch}^2 x = 1$$

FUNCTIONS OF NEGATIVE ARGUMENTS

8.14 $\sinh(-x) = -\sinh x$ **8.15** $\cosh(-x) = \cosh x$ **8.16** $\tanh(-x) = -\tanh x$

8.17 $\operatorname{csch}(-x) = -\operatorname{csch} x$ **8.18** $\operatorname{sech}(-x) = \operatorname{sech} x$ **8.19** $\coth(-x) = -\coth x$

ADDITION FORMULAS

8.20 $$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

8.21 $$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

8.22 $$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

8.23 $$\coth(x \pm y) = \frac{\coth x \coth y \pm 1}{\coth y \pm \coth x}$$

DOUBLE ANGLE FORMULAS

8.24 $$\sinh 2x = 2 \sinh x \cosh x$$

8.25 $$\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$$

8.26 $$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$$

HALF ANGLE FORMULAS

8.27 $$\sinh\frac{x}{2} = \pm\sqrt{\frac{\cosh x - 1}{2}} \qquad [+ \text{ if } x > 0, \ - \text{ if } x < 0]$$

8.28 $$\cosh\frac{x}{2} = \sqrt{\frac{\cosh x + 1}{2}}$$

8.29 $$\tanh\frac{x}{2} = \pm\sqrt{\frac{\cosh x - 1}{\cosh x + 1}} \qquad [+ \text{ if } x > 0, \ - \text{ if } x < 0]$$
$$= \frac{\sinh x}{\cosh x + 1} = \frac{\cosh x - 1}{\sinh x}$$

MULTIPLE ANGLE FORMULAS

8.30 $$\sinh 3x = 3\sinh x + 4\sinh^3 x$$

8.31 $$\cosh 3x = 4\cosh^3 x - 3\cosh x$$

8.32 $$\tanh 3x = \frac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$$

8.33 $$\sinh 4x = 8\sinh^3 x \cosh x + 4\sinh x \cosh x$$

8.34 $$\cosh 4x = 8\cosh^4 x - 8\cosh^2 x + 1$$

8.35 $$\tanh 4x = \frac{4\tanh x + 4\tanh^3 x}{1 + 6\tanh^2 x + \tanh^4 x}$$

POWERS OF HYPERBOLIC FUNCTIONS

8.36 $\sinh^2 x = \frac{1}{2}\cosh 2x - \frac{1}{2}$

8.37 $\cosh^2 x = \frac{1}{2}\cosh 2x + \frac{1}{2}$

8.38 $\sinh^3 x = \frac{1}{4}\sinh 3x - \frac{3}{4}\sinh x$

8.39 $\cosh^3 x = \frac{1}{4}\cosh 3x + \frac{3}{4}\cosh x$

8.40 $\sinh^4 x = \frac{3}{8} - \frac{1}{2}\cosh 2x + \frac{1}{8}\cosh 4x$

8.41 $\cosh^4 x = \frac{3}{8} + \frac{1}{2}\cosh 2x + \frac{1}{8}\cosh 4x$

SUM, DIFFERENCE AND PRODUCT OF HYPERBOLIC FUNCTIONS

8.42 $\sinh x + \sinh y = 2\sinh\frac{1}{2}(x+y)\cosh\frac{1}{2}(x-y)$

8.43 $\sinh x - \sinh y = 2\cosh\frac{1}{2}(x+y)\sinh\frac{1}{2}(x-y)$

8.44 $\cosh x + \cosh y = 2\cosh\frac{1}{2}(x+y)\cosh\frac{1}{2}(x-y)$

8.45 $\cosh x - \cosh y = 2\sinh\frac{1}{2}(x+y)\sinh\frac{1}{2}(x-y)$

8.46 $\sinh x\sinh y = \frac{1}{2}\{\cosh(x+y) - \cosh(x-y)\}$

8.47 $\cosh x\cosh y = \frac{1}{2}\{\cosh(x+y) + \cosh(x-y)\}$

8.48 $\sinh x\cosh y = \frac{1}{2}\{\sinh(x+y) + \sinh(x-y)\}$

EXPRESSION OF HYPERBOLIC FUNCTIONS IN TERMS OF OTHERS

In the following we assume $x > 0$. If $x < 0$ use the appropriate sign as indicated by formulas 8.14 to 8.19.

	$\sinh x = u$	$\cosh x = u$	$\tanh x = u$	$\coth x = u$	$\operatorname{sech} x = u$	$\operatorname{csch} x = u$
$\sinh x$	u	$\sqrt{u^2-1}$	$u/\sqrt{1-u^2}$	$1/\sqrt{u^2-1}$	$\sqrt{1-u^2}/u$	$1/u$
$\cosh x$	$\sqrt{1+u^2}$	u	$1/\sqrt{1-u^2}$	$u/\sqrt{u^2-1}$	$1/u$	$\sqrt{1+u^2}/u$
$\tanh x$	$u/\sqrt{1+u^2}$	$\sqrt{u^2-1}/u$	u	$1/u$	$\sqrt{1-u^2}$	$1/\sqrt{1+u^2}$
$\coth x$	$\sqrt{u^2+1}/u$	$u/\sqrt{u^2-1}$	$1/u$	u	$1/\sqrt{1-u^2}$	$\sqrt{1+u^2}$
$\operatorname{sech} x$	$1/\sqrt{1+u^2}$	$1/u$	$\sqrt{1-u^2}$	$\sqrt{u^2-1}/u$	u	$u/\sqrt{1+u^2}$
$\operatorname{csch} x$	$1/u$	$1/\sqrt{u^2-1}$	$\sqrt{1-u^2}/u$	$\sqrt{u^2-1}$	$u/\sqrt{1-u^2}$	u

GRAPHS OF HYPERBOLIC FUNCTIONS

8.49 $y = \sinh x$

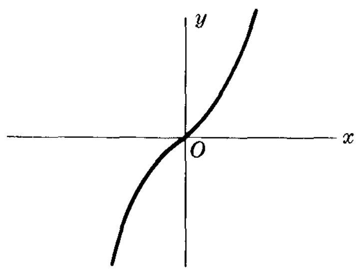

Fig. 8-1

8.50 $y = \cosh x$

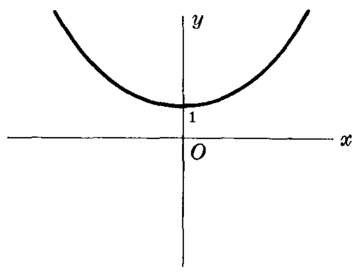

Fig. 8-2

8.51 $y = \tanh x$

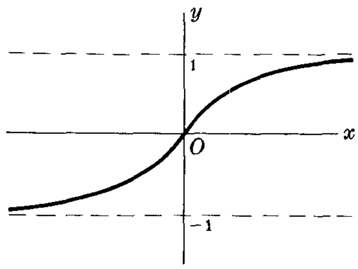

Fig. 8-3

8.52 $y = \coth x$

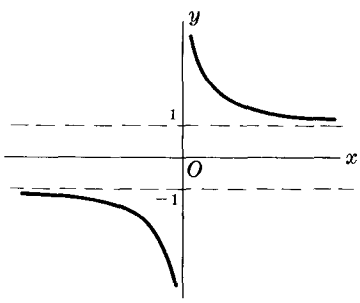

Fig. 8-4

8.53 $y = \operatorname{sech} x$

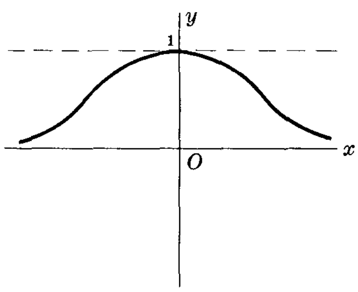

Fig. 8-5

8.54 $y = \operatorname{csch} x$

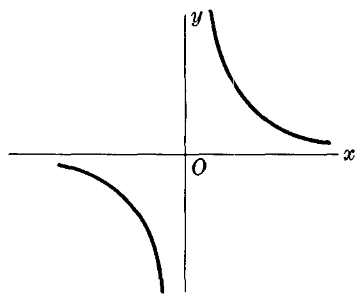

Fig. 8-6

INVERSE HYPERBOLIC FUNCTIONS

If $x = \sinh y$, then $y = \sinh^{-1} x$ is called the *inverse hyperbolic sine* of x. Similarly we define the other inverse hyperbolic functions. The inverse hyperbolic functions are multiple-valued and as in the case of inverse trigonometric functions [see page 17] we restrict ourselves to principal values for which they can be considered as single-valued.

The following list shows the principal values [unless otherwise indicated] of the inverse hyperbolic functions expressed in terms of logarithmic functions which are taken as real valued.

8.55 $$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}) \qquad -\infty < x < \infty$$

8.56 $$\cosh^{-1} x = \ln(x + \sqrt{x^2-1}) \qquad x \geqq 1 \quad [\cosh^{-1} x > 0 \text{ is principal value}]$$

8.57 $$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad -1 < x < 1$$

8.58 $$\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) \qquad x > 1 \text{ or } x < -1$$

8.59 $$\operatorname{sech}^{-1} x = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right) \qquad 0 < x \leqq 1 \quad [\operatorname{sech}^{-1} x > 0 \text{ is principal value}]$$

8.60 $$\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right) \qquad x \neq 0$$

RELATIONS BETWEEN INVERSE HYPERBOLIC FUNCTIONS

8.61 $$\operatorname{csch}^{-1} x = \sinh^{-1}(1/x)$$

8.62 $$\operatorname{sech}^{-1} x = \cosh^{-1}(1/x)$$

8.63 $$\coth^{-1} x = \tanh^{-1}(1/x)$$

8.64 $$\sinh^{-1}(-x) = -\sinh^{-1} x$$

8.65 $$\tanh^{-1}(-x) = -\tanh^{-1} x$$

8.66 $$\coth^{-1}(-x) = -\coth^{-1} x$$

8.67 $$\operatorname{csch}^{-1}(-x) = -\operatorname{csch}^{-1} x$$

GRAPHS OF INVERSE HYPERBOLIC FUNCTIONS

8.68 $y = \sinh^{-1} x$

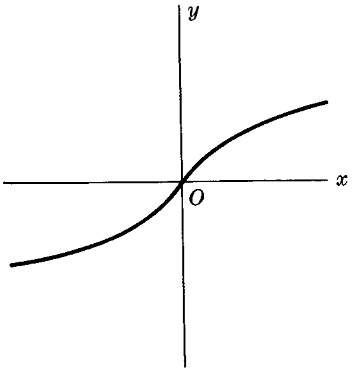

Fig. 8-7

8.69 $y = \cosh^{-1} x$

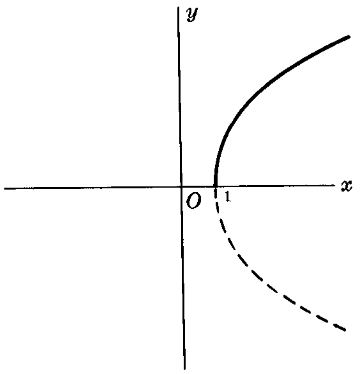

Fig. 8-8

8.70 $y = \tanh^{-1} x$

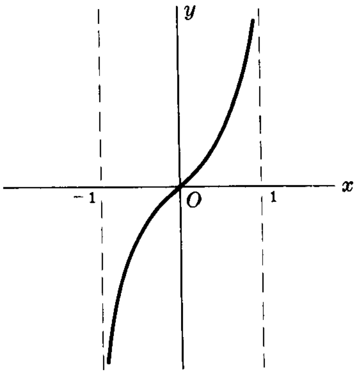

Fig. 8-9

8.71 $y = \coth^{-1} x$

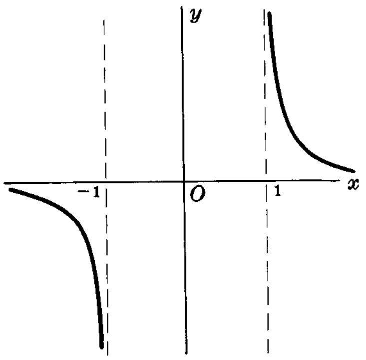

Fig. 8-10

8.72 $y = \operatorname{sech}^{-1} x$

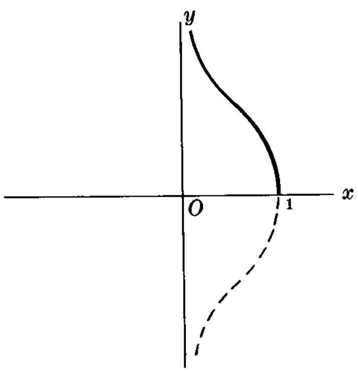

Fig. 8-11

8.73 $y = \operatorname{csch}^{-1} x$

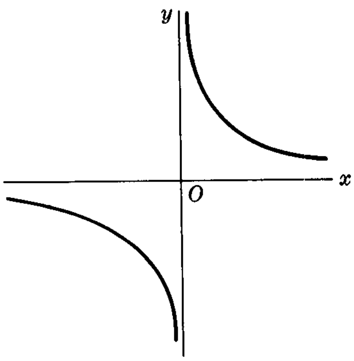

Fig. 8-12

RELATIONSHIP BETWEEN HYPERBOLIC AND TRIGONOMETRIC FUNCTIONS

8.74 $\sin(ix) = i\sinh x$

8.75 $\cos(ix) = \cosh x$

8.76 $\tan(ix) = i\tanh x$

8.77 $\csc(ix) = -i\,\mathrm{csch}\,x$

8.78 $\sec(ix) = \mathrm{sech}\,x$

8.79 $\cot(ix) = -i\coth x$

8.80 $\sinh(ix) = i\sin x$

8.81 $\cosh(ix) = \cos x$

8.82 $\tanh(ix) = i\tan x$

8.83 $\mathrm{csch}(ix) = -i\csc x$

8.84 $\mathrm{sech}(ix) = \sec x$

8.85 $\coth(ix) = -i\cot x$

PERIODICITY OF HYPERBOLIC FUNCTIONS

In the following k is any integer.

8.86 $\sinh(x + 2k\pi i) = \sinh x$

8.87 $\cosh(x + 2k\pi i) = \cosh x$

8.88 $\tanh(x + k\pi i) = \tanh x$

8.89 $\mathrm{csch}(x + 2k\pi i) = \mathrm{csch}\,x$

8.90 $\mathrm{sech}(x + 2k\pi i) = \mathrm{sech}\,x$

8.91 $\coth(x + k\pi i) = \coth x$

RELATIONSHIP BETWEEN INVERSE HYPERBOLIC AND INVERSE TRIGONOMETRIC FUNCTIONS

8.92 $\sin^{-1}(ix) = i\sinh^{-1}x$

8.93 $\sinh^{-1}(ix) = i\sin^{-1}x$

8.94 $\cos^{-1}x = \pm i\cosh^{-1}x$

8.95 $\cosh^{-1}x = \pm i\cos^{-1}x$

8.96 $\tan^{-1}(ix) = i\tanh^{-1}x$

8.97 $\tanh^{-1}(ix) = i\tan^{-1}x$

8.98 $\cot^{-1}(ix) = -i\coth^{-1}x$

8.99 $\coth^{-1}(ix) = -i\cot^{-1}x$

8.100 $\sec^{-1}x = \pm i\,\mathrm{sech}^{-1}x$

8.101 $\mathrm{sech}^{-1}x = \pm i\sec^{-1}x$

8.102 $\csc^{-1}(ix) = -i\,\mathrm{csch}^{-1}x$

8.103 $\mathrm{csch}^{-1}(ix) = -i\csc^{-1}x$