11 SPECIAL PLANE CURVES

LEMNISCATE

11.1 Equation in polar coordinates:

$$r^2 = a^2 \cos 2\theta$$

11.2 Equation in rectangular coordinates:

$$(x^2 + y^2)^2 = a^2(x^2 - y^2)$$

11.3 Angle between AB' or $A'B$ and x axis $= 45°$

11.4 Area of one loop $= \frac{1}{2}a^2$

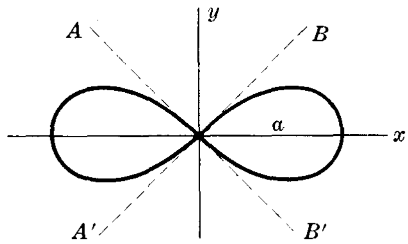

Fig. 11-1

CYCLOID

11.5 Equations in parametric form:

$$\begin{cases} x = a(\phi - \sin \phi) \\ y = a(1 - \cos \phi) \end{cases}$$

11.6 Area of one arch $= 3\pi a^2$

11.7 Arc length of one arch $= 8a$

This is a curve described by a point P on a circle of radius a rolling along x axis.

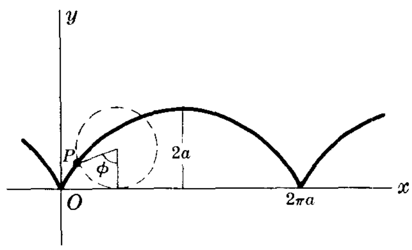

Fig. 11-2

HYPOCYCLOID WITH FOUR CUSPS

11.8 Equation in rectangular coordinates:

$$x^{2/3} + y^{2/3} = a^{2/3}$$

11.9 Equations in parametric form:

$$\begin{cases} x = a \cos^3 \theta \\ y = a \sin^3 \theta \end{cases}$$

11.10 Area bounded by curve $= \frac{3}{8}\pi a^2$

11.11 Arc length of entire curve $= 6a$

This is a curve described by a point P on a circle of radius $a/4$ as it rolls on the inside of a circle of radius a.

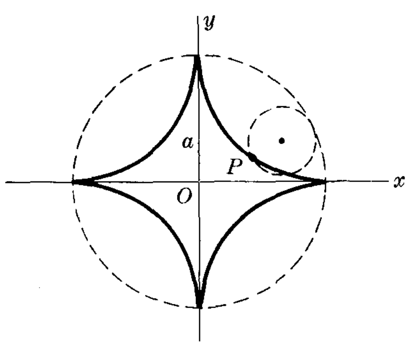

Fig. 11-3

CARDIOID

11.12 Equation: $r = a(1 + \cos\theta)$

11.13 Area bounded by curve $= \frac{3}{2}\pi a^2$

11.14 Arc length of curve $= 8a$

This is the curve described by a point P of a circle of radius a as it rolls on the outside of a fixed circle of radius a. The curve is also a special case of the limacon of Pascal [see 11.32].

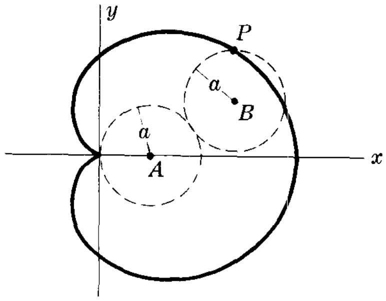

Fig. 11-4

CATENARY

11.15 Equation: $y = \frac{a}{2}(e^{x/a} + e^{-x/a}) = a \cosh\frac{x}{a}$

This is the curve in which a heavy uniform chain would hang if suspended vertically from fixed points A and B.

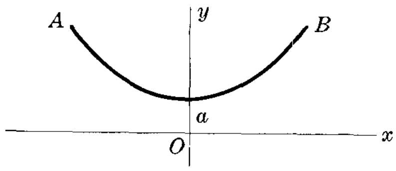

Fig. 11-5

THREE-LEAVED ROSE

11.16 Equation: $r = a \cos 3\theta$

The equation $r = a \sin 3\theta$ is a similar curve obtained by rotating the curve of Fig. 11-6 counterclockwise through 30° or $\pi/6$ radians.

In general $r = a \cos n\theta$ or $r = a \sin n\theta$ has n leaves if n is odd.

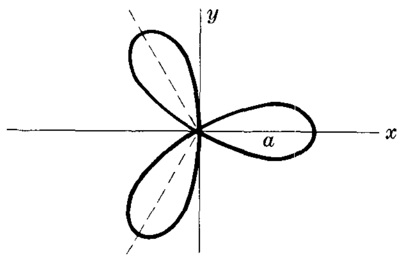

Fig. 11-6

FOUR-LEAVED ROSE

11.17 Equation: $r = a \cos 2\theta$

The equation $r = a \sin 2\theta$ is a similar curve obtained by rotating the curve of Fig. 11-7 counterclockwise through 45° or $\pi/4$ radians.

In general $r = a \cos n\theta$ or $r = a \sin n\theta$ has $2n$ leaves if n is even.

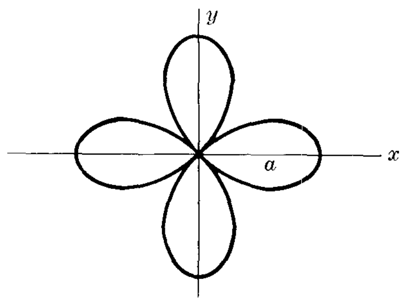

Fig. 11-7

EPICYCLOID

11.18 Parametric equations:

$$\begin{cases} x = (a+b)\cos\theta - b\cos\left(\dfrac{a+b}{b}\right)\theta \\ y = (a+b)\sin\theta - b\sin\left(\dfrac{a+b}{b}\right)\theta \end{cases}$$

This is the curve described by a point P on a circle of radius b as it rolls on the outside of a circle of radius a.

The cardioid [Fig. 11-4] is a special case of an epicycloid.

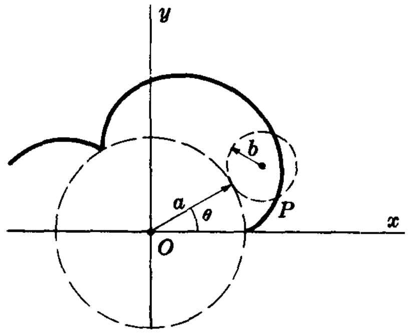

Fig. 11-8

GENERAL HYPOCYCLOID

11.19 Parametric equations:

$$\begin{cases} x = (a-b)\cos\phi + b\cos\left(\dfrac{a-b}{b}\right)\phi \\ y = (a-b)\sin\phi - b\sin\left(\dfrac{a-b}{b}\right)\phi \end{cases}$$

This is the curve described by a point P on a circle of radius b as it rolls on the inside of a circle of radius a.

If $b = a/4$, the curve is that of Fig. 11-3.

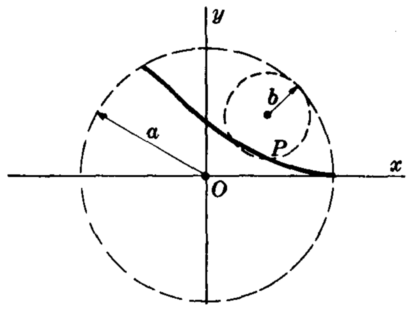

Fig. 11-9

TROCHOID

11.20 Parametric equations: $\begin{cases} x = a\phi - b\sin\phi \\ y = a - b\cos\phi \end{cases}$

This is the curve described by a point P at distance b from the center of a circle of radius a as the circle rolls on the x axis.

If $b < a$, the curve is as shown in Fig. 11-10 and is called a *curtate cycloid.*

If $b > a$, the curve is as shown in Fig. 11-11 and is called a *prolate cycloid.*

If $b = a$, the curve is the cycloid of Fig. 11-2.

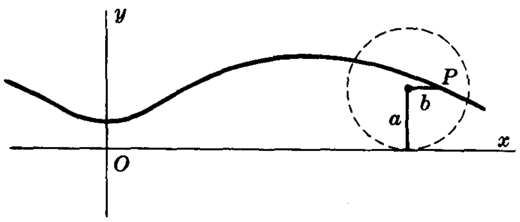

Fig. 11-10

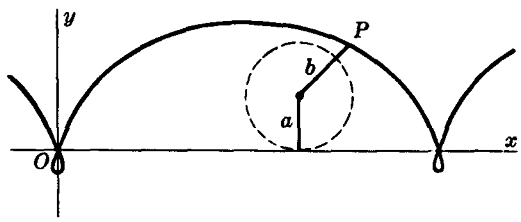

Fig. 11-11

TRACTRIX

11.21 Parametric equations: $\begin{cases} x = a(\ln \cot \frac{1}{2}\phi - \cos \phi) \\ y = a \sin \phi \end{cases}$

This is the curve described by endpoint P of a taut string PQ of length a as the other end Q is moved along the x axis.

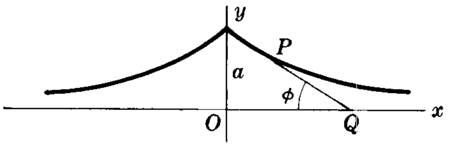

Fig. 11-12

WITCH OF AGNESI

11.22 Equation in rectangular coordinates: $y = \dfrac{8a^3}{x^2 + 4a^2}$

11.23 Parametric equations: $\begin{cases} x = 2a \cot \theta \\ y = a(1 - \cos 2\theta) \end{cases}$

In Fig. 11-13 the variable line OA intersects $y = 2a$ and the circle of radius a with center $(0, a)$ at A and B respectively. Any point P on the "witch" is located by constructing lines parallel to the x and y axes through B and A respectively and determining the point P of intersection.

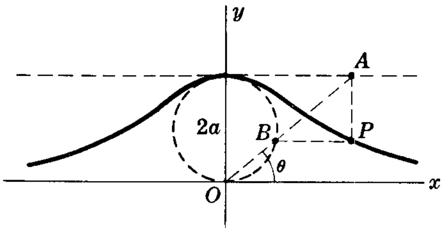

Fig. 11-13

FOLIUM OF DESCARTES

11.24 Equation in rectangular coordinates:

$$x^3 + y^3 = 3axy$$

11.25 Parametric equations:

$$\begin{cases} x = \dfrac{3at}{1+t^3} \\ y = \dfrac{3at^2}{1+t^3} \end{cases}$$

11.26 Area of loop $= \dfrac{3}{2}a^2$

11.27 Equation of asymptote: $x + y + a = 0$

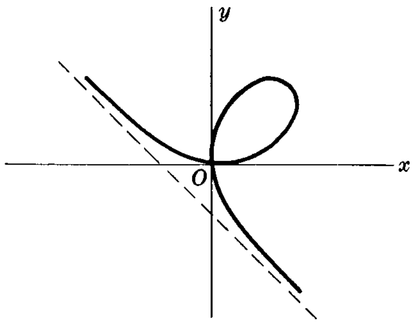

Fig. 11-14

INVOLUTE OF A CIRCLE

11.28 Parametric equations:

$$\begin{cases} x = a(\cos \phi + \phi \sin \phi) \\ y = a(\sin \phi - \phi \cos \phi) \end{cases}$$

This is the curve described by the endpoint P of a string as it unwinds from a circle of radius a while held taut.

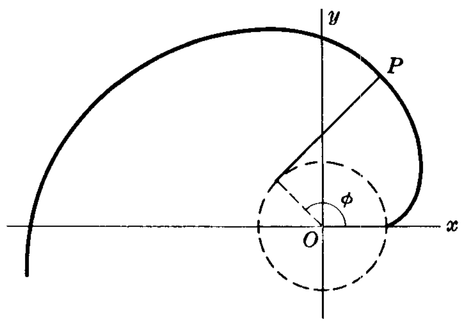

Fig. 11-15

EVOLUTE OF AN ELLIPSE

11.29 Equation in rectangular coordinates:

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$$

11.30 Parametric equations:

$$\begin{cases} ax = (a^2 - b^2)\cos^3\theta \\ by = (a^2 - b^2)\sin^3\theta \end{cases}$$

This curve is the envelope of the normals to the ellipse $x^2/a^2 + y^2/b^2 = 1$ shown dashed in Fig. 11-16.

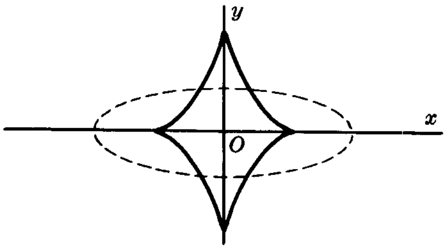

Fig. 11-16

OVALS OF CASSINI

11.31 Polar equation: $r^4 + a^4 - 2a^2r^2\cos 2\theta = b^4$

This is the curve described by a point P such that the product of its distances from two fixed points [distance $2a$ apart] is a constant b^2.

The curve is as in Fig. 11-17 or Fig. 11-18 according as $b < a$ or $b > a$ respectively.

If $b = a$, the curve is a *lemniscate* [Fig. 11-1].

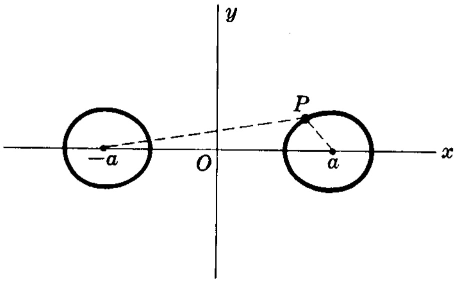

Fig. 11-17

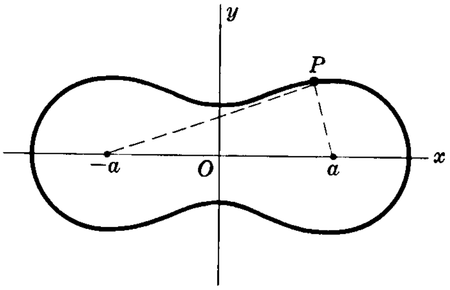

Fig. 11-18

LIMACON OF PASCAL

11.32 Polar equation: $r = b + a\cos\theta$

Let OQ be a line joining origin O to any point Q on a circle of diameter a passing through O. Then the curve is the locus of all points P such that $PQ = b$.

The curve is as in Fig. 11-19 or Fig. 11-20 according as $b > a$ or $b < a$ respectively. If $b = a$, the curve is a *cardioid* [Fig. 11-4].

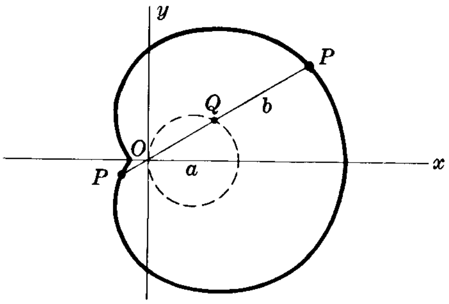

Fig. 11-19

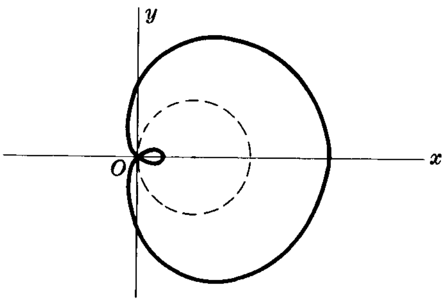

Fig. 11-20

CISSOID OF DIOCLES

11.33 Equation in rectangular coordinates:

$$y^2 = \frac{x^3}{2a - x}$$

11.34 Parametric equations:

$$\begin{cases} x = 2a \sin^2 \theta \\ y = \dfrac{2a \sin^3 \theta}{\cos \theta} \end{cases}$$

This is the curve described by a point P such that the distance OP = distance RS. It is used in the problem of *duplication of a cube*, i.e. finding the side of a cube which has twice the volume of a given cube.

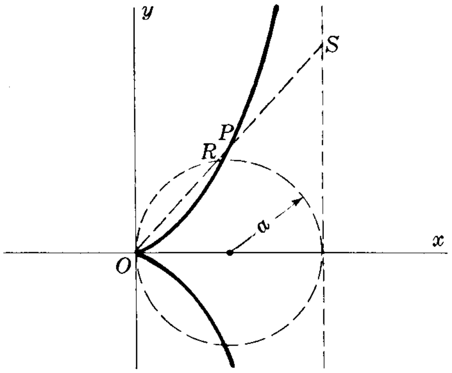

Fig. 11-21

SPIRAL OF ARCHIMEDES

11.35 Polar equation: $r = a\theta$

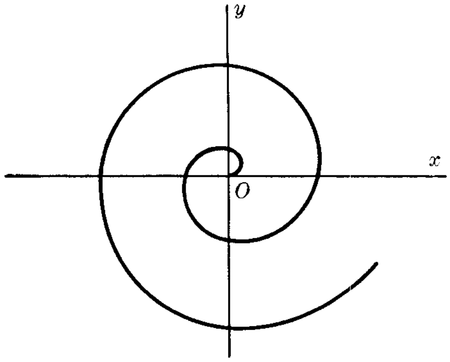

Fig. 11-22