15 DEFINITE INTEGRALS

DEFINITION OF A DEFINITE INTEGRAL

Let $f(x)$ be defined in an interval $a \leqq x \leqq b$. Divide the interval into n equal parts of length $\Delta x = (b-a)/n$. Then the definite integral of $f(x)$ between $x = a$ and $x = b$ is defined as

15.1 $$\int_a^b f(x)\,dx \quad = \quad \lim_{n\to\infty} \{f(a)\,\Delta x \;+\; f(a+\Delta x)\,\Delta x \;+\; f(a+2\Delta x)\,\Delta x \;+\; \cdots \;+\; f(a+(n-1)\,\Delta x)\,\Delta x\}$$

The limit will certainly exist if $f(x)$ is piecewise continuous.

If $f(x) = \frac{d}{dx}g(x)$, then by the fundamental theorem of the integral calculus the above definite integral can be evaluated by using the result

15.2 $$\int_a^b f(x)\,dx \quad = \quad \int_a^b \frac{d}{dx}g(x)\,dx \quad = \quad g(x)\Big|_a^b \quad = \quad g(b) - g(a)$$

If the interval is infinite or if $f(x)$ has a singularity at some point in the interval, the definite integral is called an *improper integral* and can be defined by using appropriate limiting procedures. For example,

15.3 $$\int_a^\infty f(x)\,dx \quad = \quad \lim_{b\to\infty} \int_a^b f(x)\,dx$$

15.4 $$\int_{-\infty}^\infty f(x)\,dx \quad = \quad \lim_{\substack{a\to-\infty\\ b\to\infty}} \int_a^b f(x)\,dx$$

15.5 $$\int_a^b f(x)\,dx \quad = \quad \lim_{\epsilon\to 0} \int_a^{b-\epsilon} f(x)\,dx \qquad \text{if } b \text{ is a singular point}$$

15.6 $$\int_a^b f(x)\,dx \quad = \quad \lim_{\epsilon\to 0} \int_{a+\epsilon}^{b} f(x)\,dx \qquad \text{if } a \text{ is a singular point}$$

GENERAL FORMULAS INVOLVING DEFINITE INTEGRALS

15.7 $$\int_a^b \{f(x) \pm g(x) \pm h(x) \pm \cdots\}\,dx \quad = \quad \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx \pm \int_a^b h(x)\,dx \pm \cdots$$

15.8 $$\int_a^b c\,f(x)\,dx \quad = \quad c\int_a^b f(x)\,dx \qquad \text{where } c \text{ is any constant}$$

15.9 $$\int_a^a f(x)\,dx \quad = \quad 0$$

15.10 $$\int_a^b f(x)\,dx \quad = \quad -\int_b^a f(x)\,dx$$

15.11 $$\int_a^b f(x)\,dx \quad = \quad \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

15.12 $$\int_a^b f(x)\,dx \quad = \quad (b-a)\,f(c) \qquad \text{where } c \text{ is between } a \text{ and } b$$

This is called the *mean value theorem* for definite integrals and is valid if $f(x)$ is continuous in $a \leqq x \leqq b$.

15.13 $$\int_a^b f(x)\,g(x)\,dx = f(c)\int_a^b g(x)\,dx \qquad \text{where } c \text{ is between } a \text{ and } b$$

This is a generalization of 15.12 and is valid if $f(x)$ and $g(x)$ are continuous in $a \leqq x \leqq b$ and $g(x) \geqq 0$.

LEIBNITZ'S RULE FOR DIFFERENTIATION OF INTEGRALS

15.14 $$\frac{d}{d\alpha}\int_{\phi_1(\alpha)}^{\phi_2(\alpha)} F(x,\alpha)\,dx = \int_{\phi_1(\alpha)}^{\phi_2(\alpha)} \frac{\partial F}{\partial \alpha}dx + F(\phi_2,\alpha)\frac{d\phi_1}{d\alpha} - F(\phi_1,\alpha)\frac{d\phi_2}{d\alpha}$$

APPROXIMATE FORMULAS FOR DEFINITE INTEGRALS

In the following the interval from $x = a$ to $x = b$ is subdivided into n equal parts by the points $a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b$ and we let $y_0 = f(x_0)$, $y_1 = f(x_1)$, $y_2 = f(x_2)$, $\ldots$, $y_n = f(x_n)$, $h = (b-a)/n$.

Rectangular formula

15.15 $$\int_a^b f(x)\,dx \approx h(y_0 + y_1 + y_2 + \cdots + y_{n-1})$$

Trapezoidal formula

15.16 $$\int_a^b f(x)\,dx \approx \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n)$$

Simpson's formula (or parabolic formula) for n even

15.17 $$\int_a^b f(x)\,dx \approx \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$$

DEFINITE INTEGRALS INVOLVING RATIONAL OR IRRATIONAL EXPRESSIONS

15.18 $$\int_0^\infty \frac{dx}{x^2 + a^2} = \frac{\pi}{2a}$$

15.19 $$\int_0^\infty \frac{x^{p-1}\,dx}{1 + x} = \frac{\pi}{\sin p\pi}, \qquad 0 < p < 1$$

15.20 $$\int_0^\infty \frac{x^m\,dx}{x^n + a^n} = \frac{\pi a^{m+1-n}}{n \sin[(m+1)\pi/n]}, \qquad 0 < m+1 < n$$

15.21 $$\int_0^\infty \frac{x^m\,dx}{1 + 2x\cos\beta + x^2} = \frac{\pi}{\sin m\pi}\,\frac{\sin m\beta}{\sin\beta}$$

15.22 $$\int_0^a \frac{dx}{\sqrt{a^2 - x^2}} = \frac{\pi}{2}$$

15.23 $$\int_0^a \sqrt{a^2 - x^2}\,dx = \frac{\pi a^2}{4}$$

15.24 $$\int_0^a x^m(a^n - x^n)^p\,dx = \frac{a^{m+1+np}\,\Gamma[(m+1)/n]\,\Gamma(p+1)}{n\Gamma[(m+1)/n + p + 1]}$$

15.25 $$\int_0^\infty \frac{x^m\,dx}{(x^n + a^n)^r} = \frac{(-1)^{r-1}\pi a^{m+1-nr}\,\Gamma[(m+1)/n]}{n\sin[(m+1)\pi/n](r-1)!\,\Gamma[(m+1)/n - r + 1]}, \qquad 0 < m+1 < nr$$

DEFINITE INTEGRALS INVOLVING TRIGONOMETRIC FUNCTIONS

All letters are considered positive unless otherwise indicated.

15.26 $$\int_0^\pi \sin mx \sin nx\, dx = \begin{cases} 0 & m, n \text{ integers and } m \neq n \\ \pi/2 & m, n \text{ integers and } m = n \end{cases}$$

15.27 $$\int_0^\pi \cos mx \cos nx\, dx = \begin{cases} 0 & m, n \text{ integers and } m \neq n \\ \pi/2 & m, n \text{ integers and } m = n \end{cases}$$

15.28 $$\int_0^\pi \sin mx \cos nx\, dx = \begin{cases} 0 & m, n \text{ integers and } m + n \text{ odd} \\ 2m/(m^2 - n^2) & m, n \text{ integers and } m + n \text{ even} \end{cases}$$

15.29 $$\int_0^{\pi/2} \sin^2 x\, dx = \int_0^{\pi/2} \cos^2 x\, dx = \frac{\pi}{4}$$

15.30 $$\int_0^{\pi/2} \sin^{2m} x\, dx = \int_0^{\pi/2} \cos^{2m} x\, dx = \frac{1 \cdot 3 \cdot 5 \cdots 2m-1}{2 \cdot 4 \cdot 6 \cdots 2m} \frac{\pi}{2}, \qquad m = 1, 2, \ldots$$

15.31 $$\int_0^{\pi/2} \sin^{2m+1} x\, dx = \int_0^{\pi/2} \cos^{2m+1} x\, dx = \frac{2 \cdot 4 \cdot 6 \cdots 2m}{1 \cdot 3 \cdot 5 \cdots 2m+1}, \qquad m = 1, 2, \ldots$$

15.32 $$\int_0^{\pi/2} \sin^{2p-1} x \cos^{2q-1} x\, dx = \frac{\Gamma(p)\,\Gamma(q)}{2\,\Gamma(p+q)}$$

15.33 $$\int_0^\infty \frac{\sin px}{x}\, dx = \begin{cases} \pi/2 & p > 0 \\ 0 & p = 0 \\ -\pi/2 & p < 0 \end{cases}$$

15.34 $$\int_0^\infty \frac{\sin px \cos qx}{x}\, dx = \begin{cases} 0 & p > q > 0 \\ \pi/2 & 0 < p < q \\ \pi/4 & p = q > 0 \end{cases}$$

15.35 $$\int_0^\infty \frac{\sin px \sin qx}{x^2}\, dx = \begin{cases} \pi p/2 & 0 < p \leqq q \\ \pi q/2 & p \geqq q > 0 \end{cases}$$

15.36 $$\int_0^\infty \frac{\sin^2 px}{x^2}\, dx = \frac{\pi p}{2}$$

15.37 $$\int_0^\infty \frac{1 - \cos px}{x^2}\, dx = \frac{\pi p}{2}$$

15.38 $$\int_0^\infty \frac{\cos px - \cos qx}{x}\, dx = \ln \frac{q}{p}$$

15.39 $$\int_0^\infty \frac{\cos px - \cos qx}{x^2}\, dx = \frac{\pi(q-p)}{2}$$

15.40 $$\int_0^\infty \frac{\cos mx}{x^2 + a^2}\, dx = \frac{\pi}{2a} e^{-ma}$$

15.41 $$\int_0^\infty \frac{x \sin mx}{x^2 + a^2}\, dx = \frac{\pi}{2} e^{-ma}$$

15.42 $$\int_0^\infty \frac{\sin mx}{x(x^2 + a^2)}\, dx = \frac{\pi}{2a^2}(1 - e^{-ma})$$

15.43 $$\int_0^{2\pi} \frac{dx}{a + b \sin x} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

15.44 $$\int_0^{2\pi} \frac{dx}{a + b \cos x} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

15.45 $$\int_0^{\pi/2} \frac{dx}{a + b \cos x} = \frac{\cos^{-1}(b/a)}{\sqrt{a^2 - b^2}}$$

15.46 $\displaystyle\int_0^{2\pi} \frac{dx}{(a + b\sin x)^2} = \int_0^{2\pi} \frac{dx}{(a + b\cos x)^2} = \frac{2\pi a}{(a^2 - b^2)^{3/2}}$

15.47 $\displaystyle\int_0^{2\pi} \frac{dx}{1 - 2a\cos x + a^2} = \frac{2\pi}{1 - a^2}, \quad 0 < a < 1$

15.48 $\displaystyle\int_0^{\pi} \frac{x\sin x\,dx}{1 - 2a\cos x + a^2} = \begin{cases} (\pi/a)\ln(1 + a) & |a| < 1 \\ \pi\ln(1 + 1/a) & |a| > 1 \end{cases}$

15.49 $\displaystyle\int_0^{\pi} \frac{\cos mx\,dx}{1 - 2a\cos x + a^2} = \frac{\pi a^m}{1 - a^2}, \quad a^2 < 1, \quad m = 0, 1, 2, \ldots$

15.50 $\displaystyle\int_0^{\infty} \sin ax^2\,dx = \int_0^{\infty} \cos ax^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}$

15.51 $\displaystyle\int_0^{\infty} \sin ax^n\,dx = \frac{1}{na^{1/n}}\Gamma(1/n)\sin\frac{\pi}{2n}, \quad n > 1$

15.52 $\displaystyle\int_0^{\infty} \cos ax^n\,dx = \frac{1}{na^{1/n}}\Gamma(1/n)\cos\frac{\pi}{2n}, \quad n > 1$

15.53 $\displaystyle\int_0^{\infty} \frac{\sin x}{\sqrt{x}}\,dx = \int_0^{\infty} \frac{\cos x}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2}}$

15.54 $\displaystyle\int_0^{\infty} \frac{\sin x}{x^p}\,dx = \frac{\pi}{2\Gamma(p)\sin(p\pi/2)}, \quad 0 < p < 1$

15.55 $\displaystyle\int_0^{\infty} \frac{\cos x}{x^p}\,dx = \frac{\pi}{2\Gamma(p)\cos(p\pi/2)}, \quad 0 < p < 1$

15.56 $\displaystyle\int_0^{\infty} \sin ax^2\cos 2bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}\left(\cos\frac{b^2}{a} - \sin\frac{b^2}{a}\right)$

15.57 $\displaystyle\int_0^{\infty} \cos ax^2\cos 2bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}\left(\cos\frac{b^2}{a} + \sin\frac{b^2}{a}\right)$

15.58 $\displaystyle\int_0^{\infty} \frac{\sin^3 x}{x^3}\,dx = \frac{3\pi}{8}$

15.59 $\displaystyle\int_0^{\infty} \frac{\sin^4 x}{x^4}\,dx = \frac{\pi}{3}$

15.60 $\displaystyle\int_0^{\infty} \frac{\tan x}{x}\,dx = \frac{\pi}{2}$

15.61 $\displaystyle\int_0^{\pi/2} \frac{dx}{1 + \tan^m x} = \frac{\pi}{4}$

15.62 $\displaystyle\int_0^{\pi/2} \frac{x}{\sin x}\,dx = 2\left\{\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots\right\}$

15.63 $\displaystyle\int_0^{1} \frac{\tan^{-1} x}{x}\,dx = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots$ ✓ catalan

15.64 $\displaystyle\int_0^{1} \frac{\sin^{-1} x}{x}\,dx = \frac{\pi}{2}\ln 2$

15.65 $\displaystyle\int_0^{1} \frac{1 - \cos x}{x}\,dx - \int_1^{\infty} \frac{\cos x}{x}\,dx = \gamma$

15.66 $\displaystyle\int_0^{\infty} \left(\frac{1}{1 + x^2} - \cos x\right)\frac{dx}{x} = \gamma$

15.67 $\displaystyle\int_0^{\infty} \frac{\tan^{-1} px - \tan^{-1} qx}{x}\,dx = \frac{\pi}{2}\ln\frac{p}{q}$

DEFINITE INTEGRALS INVOLVING EXPONENTIAL FUNCTIONS

15.68 $$\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$$

15.69 $$\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$$

15.70 $$\int_0^\infty \frac{e^{-ax}\sin bx}{x}\,dx = \tan^{-1}\frac{b}{a}$$

15.71 $$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx = \ln\frac{b}{a}$$

15.72 $$\int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

15.73 $$\int_0^\infty e^{-ax^2}\cos bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-b^2/4a}$$

15.74 $$\int_0^\infty e^{-(ax^2+bx+c)}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{(b^2-4ac)/4a}\operatorname{erfc}\frac{b}{2\sqrt{a}}$$

where $\operatorname{erfc}(p) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_p^\infty e^{-x^2}\,dx$

15.75 $$\int_{-\infty}^\infty e^{-(ax^2+bx+c)}\,dx = \sqrt{\frac{\pi}{a}}e^{(b^2-4ac)/4a}$$

15.76 $$\int_0^\infty x^n e^{-ax}\,dx = \frac{\Gamma(n+1)}{a^{n+1}}$$

15.77 $$\int_0^\infty x^m e^{-ax^2}\,dx = \frac{\Gamma[(m+1)/2]}{2a^{(m+1)/2}}$$

15.78 $$\int_0^\infty e^{-(ax^2+b/x^2)}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-2\sqrt{ab}}$$

15.79 $$\int_0^\infty \frac{x\,dx}{e^x-1} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots = \frac{\pi^2}{6}$$

15.80 $$\int_0^\infty \frac{x^{n-1}}{e^x-1}\,dx = \Gamma(n)\left(\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\cdots\right)$$

For even n this can be summed in terms of Bernoulli numbers [see pages 108-109 and 114-115].

15.81 $$\int_0^\infty \frac{x\,dx}{e^x+1} = \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots = \frac{\pi^2}{12}$$

15.82 $$\int_0^\infty \frac{x^{n-1}}{e^x+1}\,dx = \Gamma(n)\left(\frac{1}{1^n}-\frac{1}{2^n}+\frac{1}{3^n}-\cdots\right)$$

For some positive integer values of n the series can be summed [see pages 108-109 and 114-115].

15.83 $$\int_0^\infty \frac{\sin mx}{e^{2\pi x}-1}\,dx = \frac{1}{4}\coth\frac{m}{2}-\frac{1}{2m}$$

15.84 $$\int_0^\infty \left(\frac{1}{1+x}-e^{-x}\right)\frac{dx}{x} = \gamma$$

15.85 $$\int_0^\infty \frac{e^{-x^2}-e^{-x}}{x}\,dx = \tfrac{1}{2}\gamma$$

15.86 $$\int_0^\infty \left(\frac{1}{e^x-1}-\frac{e^{-x}}{x}\right)dx = \gamma$$

15.87 $\displaystyle\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x \sec px}\,dx \;=\; \frac{1}{2}\ln\left(\frac{b^2+p^2}{a^2+p^2}\right)$

15.88 $\displaystyle\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x \csc px}\,dx \;=\; \tan^{-1}\frac{b}{p} - \tan^{-1}\frac{a}{p}$

15.89 $\displaystyle\int_0^\infty \frac{e^{-ax}(1-\cos x)}{x^2}\,dx \;=\; \cot^{-1} a - \frac{a}{2}\ln(a^2+1)$

DEFINITE INTEGRALS INVOLVING LOGARITHMIC FUNCTIONS

15.90 $\displaystyle\int_0^1 x^m (\ln x)^n\,dx \;=\; \frac{(-1)^n n!}{(m+1)^{n+1}} \qquad m > -1,\; n = 0, 1, 2, \ldots$

If $n \neq 0, 1, 2, \ldots$ replace $n!$ by $\Gamma(n+1)$.

15.91 $\displaystyle\int_0^1 \frac{\ln x}{1+x}\,dx \;=\; -\frac{\pi^2}{12}$

15.92 $\displaystyle\int_0^1 \frac{\ln x}{1-x}\,dx \;=\; -\frac{\pi^2}{6}$

15.93 $\displaystyle\int_0^1 \frac{\ln(1+x)}{x}\,dx \;=\; \frac{\pi^2}{12}$

15.94 $\displaystyle\int_0^1 \frac{\ln(1-x)}{x}\,dx \;=\; -\frac{\pi^2}{6}$

15.95 $\displaystyle\int_0^1 \ln x \ln(1+x)\,dx \;=\; 2 - 2\ln 2 - \frac{\pi^2}{12}$

15.96 $\displaystyle\int_0^1 \ln x \ln(1-x)\,dx \;=\; 2 - \frac{\pi^2}{6}$

15.97 $\displaystyle\int_0^\infty \frac{x^{p-1}\ln x}{1+x}\,dx \;=\; -\pi^2 \csc p\pi \cot p\pi \qquad 0 < p < 1$

15.98 $\displaystyle\int_0^1 \frac{x^m - x^n}{\ln x}\,dx \;=\; \ln\frac{m+1}{n+1}$

15.99 $\displaystyle\int_0^\infty e^{-x}\ln x\,dx \;=\; -\gamma$

15.100 $\displaystyle\int_0^\infty e^{-x^2}\ln x\,dx \;=\; -\frac{\sqrt{\pi}}{4}(\gamma + 2\ln 2)$

15.101 $\displaystyle\int_0^\infty \ln\left(\frac{e^x+1}{e^x-1}\right)dx \;=\; \frac{\pi^2}{4}$

15.102 $\displaystyle\int_0^{\pi/2} \ln\sin x\,dx \;=\; \int_0^{\pi/2} \ln\cos x\,dx \;=\; -\frac{\pi}{2}\ln 2$

15.103 $\displaystyle\int_0^{\pi/2} (\ln\sin x)^2\,dx \;=\; \int_0^{\pi/2} (\ln\cos x)^2\,dx \;=\; \frac{\pi}{2}(\ln 2)^2 + \frac{\pi^3}{24}$

15.104 $\displaystyle\int_0^{\pi} x\ln\sin x\,dx \;=\; -\frac{\pi^2}{2}\ln 2$

15.105 $\displaystyle\int_0^{\pi/2} \sin x \ln\sin x\,dx \;=\; \ln 2 - 1$

15.106 $\displaystyle\int_0^{2\pi} \ln(a + b\sin x)\,dx \;=\; \int_0^{2\pi} \ln(a + b\cos x)\,dx \;=\; 2\pi\ln(a + \sqrt{a^2-b^2})$

15.107 $$\int_0^{\pi} \ln(a + b\cos x)\,dx = \pi \ln\left(\frac{a + \sqrt{a^2 - b^2}}{2}\right)$$

15.108 $$\int_0^{\pi} \ln(a^2 - 2ab\cos x + b^2)\,dx = \begin{cases} 2\pi \ln a, & a \geqq b > 0 \\ 2\pi \ln b, & b \geqq a > 0 \end{cases}$$

15.109 $$\int_0^{\pi/4} \ln(1 + \tan x)\,dx = \frac{\pi}{8} \ln 2$$

15.110 $$\int_0^{\pi/2} \sec x \ln\left(\frac{1 + b\cos x}{1 + a\cos x}\right) dx = \tfrac{1}{2}\{(\cos^{-1} a)^2 - (\cos^{-1} b)^2\}$$

15.111 $$\int_0^{a} \ln\left(2\sin\frac{x}{2}\right) dx = -\left(\frac{\sin a}{1^2} + \frac{\sin 2a}{2^2} + \frac{\sin 3a}{3^2} + \cdots\right)$$

See also 15.102.

DEFINITE INTEGRALS INVOLVING HYPERBOLIC FUNCTIONS

15.112 $$\int_0^{\infty} \frac{\sin ax}{\sinh bx}\,dx = \frac{\pi}{2b} \tanh \frac{a\pi}{2b}$$

15.113 $$\int_0^{\infty} \frac{\cos ax}{\cosh bx}\,dx = \frac{\pi}{2b} \operatorname{sech} \frac{a\pi}{2b}$$

15.114 $$\int_0^{\infty} \frac{x\,dx}{\sinh ax} = \frac{\pi^2}{4a^2}$$

15.115 $$\int_0^{\infty} \frac{x^n\,dx}{\sinh ax} = \frac{2^{n+1} - 1}{2^n a^{n+1}} \Gamma(n+1) \left\{\frac{1}{1^{n+1}} + \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} + \cdots\right\}$$

If n is an odd positive integer, the series can be summed [see page 108].

15.116 $$\int_0^{\infty} \frac{\sinh ax}{e^{bx} + 1}\,dx = \frac{\pi}{2b} \csc \frac{a\pi}{b} - \frac{1}{2a}$$

15.117 $$\int_0^{\infty} \frac{\sinh ax}{e^{bx} - 1}\,dx = \frac{1}{2a} - \frac{\pi}{2b} \cot \frac{a\pi}{b}$$

MISCELLANEOUS DEFINITE INTEGRALS

15.118 $$\int_0^{\infty} \frac{f(ax) - f(bx)}{x}\,dx = \{f(0) - f(\infty)\} \ln \frac{b}{a}$$

This is called *Frullani's integral.* It holds if $f'(x)$ is continuous and $\int_1^{\infty} \frac{f(x) - f(\infty)}{x}\,dx$ converges.

15.119 $$\int_0^{1} \frac{dx}{x^x} = \frac{1}{1^1} + \frac{1}{2^2} + \frac{1}{3^3} + \cdots$$

15.120 $$\int_{-a}^{a} (a + x)^{m-1}(a - x)^{n-1}\,dx = (2a)^{m+n-1} \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$