POSTECH

Mathematics I - *Final Examination* June 18, 2003

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[1] (10 points)

(a) Determine whether the series given by $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges or diverges and give the reason for your answer.

(b) Determine whether the series given by $\sum_{n=1}^{\infty}(-1)^n \frac{\tan^{-1} n}{n^2+1}$ converges absolutely, conditionally or diverges and give the reason for your answer.

[2] (10 points) Find the Taylor series expansions of the following functions about the point t=1 and then obtain the radius and interval of convergence, respectively. (Hint: Obtain the power series for the function $\frac{1}{t}$ and then manipulates.)

(a)$\frac{1}{t^2}$ (b)$\ln t$

[3] (10 points) Find the general solution of the Bernoulli differential equation $\frac{dy}{dx} + y = y^5$ by finding an appropriate integrating factor. (Hint: Put $u(t) = y^{-4}(t)$ and derive the differential equation satisfied by $u(t)$.)

[4] (10 points) Let $a$, $b$ and $c$ be positive real numbers. Show that every solution of $ay''(t)+by'(t)+cy(t) = 0$ approaches zero as t approaches infinity.

[5] (10 points) Solve the following differential equations given by

(a) $y'' - 2y' - 3y = 1 + t$,

(b) $y'' + 4y = 0,\ y(0) = 0,\ y'(0) = 0$

[6](10 points) We see that $y_1(t) = t$ is one solution of the differential equation given by $(1 + t^2)y'' - 2ty' + 2y = 0$. Use the reduction of order to find the general solution of the given differential equation.

[7] (10 points) Using the method of variation of parameter, solve the initial value problem given by $y''(t) + 25y(t) = f(t)$ with $y(0) = 0$ and $y'(0) = 0$.

[8] (15 points) Let $y_1$ and $y_2$ be two solutions of the differential equation defined as $L[y](t) = 0$ in $\alpha < t < \beta$ where $L[y](t) = y''(t)+p(t)y'(t)+100y(t)$ and $p(t)$ is continuous.

(a) Show that the mapping $L$ is linear in $y$.

(b) Show that their Wronskian $W[y_1, y_2](t)$ satisfies the first order differential equation $W'(t) + p(t)W(t) = 0$.

(c) If the solutions $y_1$ and $y_2$ have a maximum or minimum at same point $t_0 \in (\alpha, \beta)$, show that they are not linearly independent on the interval $(\alpha, \beta)$.

[9] (10 points) Construct the second-order linear homogeneous differential equation having the two linearly independent solutions $y_1(t) = t$ and $y_2(t) = 1 + t^2$ on the interval $-1 < t < 1$.

[10] (10 points) It is found experimentally that a $2Kg$ mass stretches a spring $\frac{49}{90}m$. If the mass is pulled down an additional $\frac{1}{3}m$ and released, find the amplitude, period and frequency of the resulting motion, neglecting air resistance. (Use acceleration of gravity $g = 9.8m/sec^2$.)

[11] (10 points) Find the general solution in power series of $x$ for the equation $y'' = xy$ for $-\infty < x < \infty$.

[12] (15 points)

(a) (5 points) Find the general solution of the Euler equation $x^2y'' + xy' - \frac{9}{4}y = 0$ for $x > 0$.

(b) (10 points) Find two linearly independent series solutions of $x^2y'' + xy' - (x^2 - \frac{9}{4})y = 0$ for $x > 0$.

[13] (10 points) Let $F(s)$ be the Laplace transform of $f(t)$. Find Laplace transform $F(s)$ for a given $f(t)$ and conversely, inverse Laplace transform $f(t)$ for a given $F(s)$.

(a) $f(t) = t^5$ (b) $f(t) = \cos 2t$ (c) $f(t) = \sinh 3t$

(d) $F(s) = \frac{1}{(s-2)^2}$ (e) $F(s) = \frac{s^2-8}{s^3+4s^2+8s}$

[14] (10 points) Solve the initial value problem $y''(t) + 4y(t) = f(t)$ where $f(t) = 1$ for $0 \le t < 4$ and $f(t) = 0$ for $t \ge 4$ with initial data $y(0) = 3$ and $y'(0) = -2$.

[15] (10 points) Let $y(t)$ be the solution of the initial value problem $y'' + 2y' + y = e^{-t} + 3\delta(t-3)$ with initial data $y(0) = 0$ and $y'(0) = 3$. Compute $y(4)$.