14 INDEFINITE INTEGRALS

DEFINITION OF AN INDEFINITE INTEGRAL

If $\frac{dy}{dx} = f(x)$, then y is the function whose derivative is $f(x)$ and is called the *anti-derivative* of $f(x)$ or the *indefinite integral* of $f(x)$, denoted by $\int f(x)\,dx$. Similarly if $y = \int f(u)\,du$, then $\frac{dy}{du} = f(u)$. Since the derivative of a constant is zero, all indefinite integrals differ by an arbitrary constant.

For the definition of a definite integral, see page 94. The process of finding an integral is called *integration.*

GENERAL RULES OF INTEGRATION

In the following, u, v, w are functions of x; a, b, p, q, n any constants, restricted if indicated; $e = 2.71828\ldots$ is the natural base of logarithms; $\ln u$ denotes the natural logarithm of u where it is assumed that $u > 0$ [in general, to extend formulas to cases where $u < 0$ as well, replace $\ln u$ by $\ln|u|$]; all angles are in radians; all constants of integration are omitted but implied.

14.1 $$\int a\,dx = ax$$

14.2 $$\int af(x)\,dx = a\int f(x)\,dx$$

14.3 $$\int (u \pm v \pm w \pm \cdots)\,dx = \int u\,dx \pm \int v\,dx \pm \int w\,dx \pm \cdots$$

14.4 $$\int u\,dv = uv - \int v\,du \qquad \text{[Integration by parts]}$$

For generalized integration by parts, see 14.48.

14.5 $$\int f(ax)\,dx = \frac{1}{a}\int f(u)\,du$$

14.6 $$\int F\{f(x)\}\,dx = \int F(u)\frac{dx}{du}\,du = \int \frac{F(u)}{f'(x)}\,du \quad \text{where } u = f(x)$$

14.7 $$\int u^n\,du = \frac{u^{n+1}}{n+1}, \quad n \neq -1 \qquad \text{[For } n = -1\text{, see 14.8]}$$

14.8 $$\int \frac{du}{u} = \ln u \quad \text{if } u > 0 \text{ or } \ln(-u) \text{ if } u < 0$$
$$= \ln|u|$$

14.9 $$\int e^u\,du = e^u$$

14.10 $$\int a^u\,du = \int e^{u \ln a}\,du = \frac{e^{u \ln a}}{\ln a} = \frac{a^u}{\ln a}, \quad a > 0,\ a \neq 1$$

14.11 $$\int \sin u\,du = -\cos u$$

14.12 $$\int \cos u\,du = \sin u$$

14.13 $$\int \tan u\,du = \ln \sec u = -\ln \cos u$$

14.14 $$\int \cot u\,du = \ln \sin u$$

14.15 $$\int \sec u\,du = \ln(\sec u + \tan u) = \ln \tan\left(\frac{u}{2}+\frac{\pi}{4}\right)$$

14.16 $$\int \csc u\,du = \ln(\csc u - \cot u) = \ln \tan\frac{u}{2}$$

14.17 $$\int \sec^2 u\,du = \tan u$$

14.18 $$\int \csc^2 u\,du = -\cot u$$

14.19 $$\int \tan^2 u\,du = \tan u - u$$

14.20 $$\int \cot^2 u\,du = -\cot u - u$$

14.21 $$\int \sin^2 u\,du = \frac{u}{2} - \frac{\sin 2u}{4} = \tfrac{1}{2}(u - \sin u \cos u)$$

14.22 $$\int \cos^2 u\,du = \frac{u}{2} + \frac{\sin 2u}{4} = \tfrac{1}{2}(u + \sin u \cos u)$$

14.23 $$\int \sec u \tan u\,du = \sec u$$

14.24 $$\int \csc u \cot u\,du = -\csc u$$

14.25 $$\int \sinh u\,du = \cosh u$$

14.26 $$\int \cosh u\,du = \sinh u$$

14.27 $$\int \tanh u\,du = \ln \cosh u$$

14.28 $$\int \coth u\,du = \ln \sinh u$$

14.29 $$\int \operatorname{sech} u\,du = \sin^{-1}(\tanh u) \quad \text{or} \quad 2\tan^{-1} e^u$$

14.30 $$\int \operatorname{csch} u\,du = \ln \tanh\frac{u}{2} \quad \text{or} \quad -\coth^{-1} e^u$$

14.31 $$\int \operatorname{sech}^2 u\,du = \tanh u$$

14.32 $$\int \operatorname{csch}^2 u\,du = -\coth u$$

14.33 $$\int \tanh^2 u\,du = u - \tanh u$$

14.34 $\int \coth^2 u\,du = u - \coth u$

14.35 $\int \sinh^2 u\,du = \frac{\sinh 2u}{4} - \frac{u}{2} = \frac{1}{2}(\sinh u \cosh u - u)$

14.36 $\int \cosh^2 u\,du = \frac{\sinh 2u}{4} + \frac{u}{2} = \frac{1}{2}(\sinh u \cosh u + u)$

14.37 $\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u$

14.38 $\int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u$

14.39 $\int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}$

14.40 $\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left(\frac{u-a}{u+a}\right) = -\frac{1}{a}\coth^{-1}\frac{u}{a} \qquad u^2 > a^2$

14.41 $\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left(\frac{a+u}{a-u}\right) = \frac{1}{a}\tanh^{-1}\frac{u}{a} \qquad u^2 < a^2$

14.42 $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a}$

14.43 $\int \frac{du}{\sqrt{u^2+a^2}} = \ln(u+\sqrt{u^2+a^2}) \quad \text{or} \quad \sinh^{-1}\frac{u}{a}$

14.44 $\int \frac{du}{\sqrt{u^2-a^2}} = \ln(u+\sqrt{u^2-a^2})$

14.45 $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right|$

14.46 $\int \frac{du}{u\sqrt{u^2+a^2}} = -\frac{1}{a}\ln\left(\frac{a+\sqrt{u^2+a^2}}{u}\right)$

14.47 $\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-u^2}}{u}\right)$

14.48 $\int f^{(n)}g\,dx = f^{(n-1)}g - f^{(n-2)}g' + f^{(n-3)}g'' - \cdots (-1)^n \int fg^{(n)}\,dx$

This is called *generalized integration* by parts.

IMPORTANT TRANSFORMATIONS

Often in practice an integral can be simplified by using an appropriate transformation or substitution and formula 14.6, page 57. The following list gives some transformations and their effects.

14.49 $\int F(ax+b)\,dx = \frac{1}{a}\int F(u)\,du \qquad \text{where } u = ax+b$

14.50 $\int F(\sqrt{ax+b})\,dx = \frac{2}{a}\int u\,F(u)\,du \qquad \text{where } u = \sqrt{ax+b}$

14.51 $\int F(\sqrt[n]{ax+b})\,dx = \frac{n}{a}\int u^{n-1}F(u)\,du \qquad \text{where } u = \sqrt[n]{ax+b}$

14.52 $\int F(\sqrt{a^2-x^2})\,dx = a\int F(a\cos u)\cos u\,du \qquad \text{where } x = a\sin u$

14.53 $\int F(\sqrt{x^2+a^2})\,dx = a\int F(a\sec u)\sec^2 u\,du \qquad \text{where } x = a\tan u$

14.54 $\displaystyle\int F(\sqrt{x^2-a^2})\,dx = a\int F(a\tan u)\sec u\tan u\,du$ where $x = a\sec u$

14.55 $\displaystyle\int F(e^{ax})\,dx = \frac{1}{a}\int \frac{F(u)}{u}\,du$ where $u = e^{ax}$

14.56 $\displaystyle\int F(\ln x)\,dx = \int F(u)\,e^u\,du$ where $u = \ln x$

14.57 $\displaystyle\int F\left(\sin^{-1}\frac{x}{a}\right)dx = a\int F(u)\cos u\,du$ where $u = \sin^{-1}\frac{x}{a}$

Similar results apply for other inverse trigonometric functions.

14.58 $\displaystyle\int F(\sin x, \cos x)\,dx = 2\int F\left(\frac{2u}{1+u^2}, \frac{1-u^2}{1+u^2}\right)\frac{du}{1+u^2}$ where $u = \tan\frac{x}{2}$

SPECIAL INTEGRALS

Pages 60 through 93 provide a table of integrals classified under special types. The remarks given on page 57 apply here as well. It is assumed in all cases that division by zero is excluded.

INTEGRALS INVOLVING $ax + b$

14.59 $\displaystyle\int \frac{dx}{ax+b} = \frac{1}{a}\ln(ax+b)$

14.60 $\displaystyle\int \frac{x\,dx}{ax+b} = \frac{x}{a} - \frac{b}{a^2}\ln(ax+b)$

14.61 $\displaystyle\int \frac{x^2\,dx}{ax+b} = \frac{(ax+b)^2}{2a^3} - \frac{2b(ax+b)}{a^3} + \frac{b^2}{a^3}\ln(ax+b)$

14.62 $\displaystyle\int \frac{x^3\,dx}{ax+b} = \frac{(ax+b)^3}{3a^4} - \frac{3b(ax+b)^2}{2a^4} + \frac{3b^2(ax+b)}{a^4} - \frac{b^3}{a^4}\ln(ax+b)$

14.63 $\displaystyle\int \frac{dx}{x(ax+b)} = \frac{1}{b}\ln\left(\frac{x}{ax+b}\right)$

14.64 $\displaystyle\int \frac{dx}{x^2(ax+b)} = -\frac{1}{bx} + \frac{a}{b^2}\ln\left(\frac{ax+b}{x}\right)$

14.65 $\displaystyle\int \frac{dx}{x^3(ax+b)} = \frac{2ax-b}{2b^2x^2} + \frac{a^2}{b^3}\ln\left(\frac{x}{ax+b}\right)$

14.66 $\displaystyle\int \frac{dx}{(ax+b)^2} = \frac{-1}{a(ax+b)}$

14.67 $\displaystyle\int \frac{x\,dx}{(ax+b)^2} = \frac{b}{a^2(ax+b)} + \frac{1}{a^2}\ln(ax+b)$

14.68 $\displaystyle\int \frac{x^2\,dx}{(ax+b)^2} = \frac{ax+b}{a^3} - \frac{b^2}{a^3(ax+b)} - \frac{2b}{a^3}\ln(ax+b)$

14.69 $\displaystyle\int \frac{x^3\,dx}{(ax+b)^2} = \frac{(ax+b)^2}{2a^4} - \frac{3b(ax+b)}{a^4} + \frac{b^3}{a^4(ax+b)} + \frac{3b^2}{a^4}\ln(ax+b)$

14.70 $\displaystyle\int \frac{dx}{x(ax+b)^2} = \frac{1}{b(ax+b)} + \frac{1}{b^2}\ln\left(\frac{x}{ax+b}\right)$

14.71 $\displaystyle\int \frac{dx}{x^2(ax+b)^2} = \frac{-a}{b^2(ax+b)} - \frac{1}{b^2x} + \frac{2a}{b^3}\ln\left(\frac{ax+b}{x}\right)$

14.72 $$\int \frac{dx}{x^3(ax+b)^2} = -\frac{(ax+b)^2}{2b^4x^2} + \frac{3a(ax+b)}{b^4x} - \frac{a^3x}{b^4(ax+b)} - \frac{3a^2}{b^4}\ln\left(\frac{ax+b}{x}\right)$$

14.73 $$\int \frac{dx}{(ax+b)^3} = \frac{-1}{2(ax+b)^2}$$

14.74 $$\int \frac{x\,dx}{(ax+b)^3} = \frac{-1}{a^2(ax+b)} + \frac{b}{2a^2(ax+b)^2}$$

14.75 $$\int \frac{x^2\,dx}{(ax+b)^3} = \frac{2b}{a^3(ax+b)} - \frac{b^2}{2a^3(ax+b)^2} + \frac{1}{a^3}\ln(ax+b)$$

14.76 $$\int \frac{x^3\,dx}{(ax+b)^3} = \frac{x}{a^3} - \frac{3b^2}{a^4(ax+b)} + \frac{b^3}{2a^4(ax+b)^2} - \frac{3b}{a^4}\ln(ax+b)$$

14.77 $$\int \frac{dx}{x(ax+b)^3} = \frac{a^2x^2}{2b^3(ax+b)^2} - \frac{2ax}{b^3(ax+b)} - \frac{1}{b^3}\ln\left(\frac{ax+b}{x}\right)$$

14.78 $$\int \frac{dx}{x^2(ax+b)^3} = \frac{-a}{2b^2(ax+b)^2} - \frac{2a}{b^3(ax+b)} - \frac{1}{b^3x} + \frac{3a}{b^4}\ln\left(\frac{ax+b}{x}\right)$$

14.79 $$\int \frac{dx}{x^3(ax+b)^3} = \frac{a^4x^2}{2b^5(ax+b)^2} - \frac{4a^3x}{b^5(ax+b)} - \frac{(ax+b)^2}{2b^5x^2} - \frac{6a^2}{b^5}\ln\left(\frac{ax+b}{x}\right)$$

14.80 $$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{(n+1)a}. \qquad \text{If } n=-1, \text{ see 14.59.}$$

14.81 $$\int x(ax+b)^n\,dx = \frac{(ax+b)^{n+2}}{(n+2)a^2} - \frac{b(ax+b)^{n+1}}{(n+1)a^2}, \quad n \neq -1, -2$$

If $n = -1, -2$, see 14.60, 14.67.

14.82 $$\int x^2(ax+b)^n\,dx = \frac{(ax+b)^{n+3}}{(n+3)a^3} - \frac{2b(ax+b)^{n+2}}{(n+2)a^3} + \frac{b^2(ax+b)^{n+1}}{(n+1)a^3}$$

If $n = -1, -2, -3$, see 14.61, 14.68, 14.75.

14.83 $$\int x^m(ax+b)^n\,dx = \begin{cases} \dfrac{x^{m+1}(ax+b)^n}{m+n+1} + \dfrac{nb}{m+n+1}\displaystyle\int x^m(ax+b)^{n-1}\,dx \\ \dfrac{x^m(ax+b)^{n+1}}{(m+n+1)a} - \dfrac{mb}{(m+n+1)a}\displaystyle\int x^{m-1}(ax+b)^n\,dx \\ \dfrac{-x^{m+1}(ax+b)^{n+1}}{(n+1)b} + \dfrac{m+n+2}{(n+1)b}\displaystyle\int x^m(ax+b)^{n+1}\,dx \end{cases}$$

INTEGRALS INVOLVING $\sqrt{ax+b}$

14.84 $$\int \frac{dx}{\sqrt{ax+b}} = \frac{2\sqrt{ax+b}}{a}$$

14.85 $$\int \frac{x\,dx}{\sqrt{ax+b}} = \frac{2(ax-2b)}{3a^2}\sqrt{ax+b}$$

14.86 $$\int \frac{x^2\,dx}{\sqrt{ax+b}} = \frac{2(3a^2x^2-4abx+8b^2)}{15a^3}\sqrt{ax+b}$$

14.87 $$\int \frac{dx}{x\sqrt{ax+b}} = \begin{cases} \dfrac{1}{\sqrt{b}}\ln\left(\dfrac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right) \\ \dfrac{2}{\sqrt{-b}}\tan^{-1}\sqrt{\dfrac{ax+b}{-b}} \end{cases}$$

14.88 $$\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{ax+b}} \qquad \text{[See 14.87]}$$

14.89 $$\int \sqrt{ax+b}\,dx = \frac{2\sqrt{(ax+b)^3}}{3a}$$

14.90 $$\int x\sqrt{ax+b}\,dx = \frac{2(3ax-2b)}{15a^2}\sqrt{(ax+b)^3}$$

14.91 $$\int x^2\sqrt{ax+b}\,dx = \frac{2(15a^2x^2-12abx+8b^2)}{105a^3}\sqrt{(ax+b)^3}$$

14.92 $$\int \frac{\sqrt{ax+b}}{x}\,dx = 2\sqrt{ax+b} + b\int \frac{dx}{x\sqrt{ax+b}}$$ [See 14.87]

14.93 $$\int \frac{\sqrt{ax+b}}{x^2}\,dx = -\frac{\sqrt{ax+b}}{x} + \frac{a}{2}\int \frac{dx}{x\sqrt{ax+b}}$$ [See 14.87]

14.94 $$\int \frac{x^m}{\sqrt{ax+b}}\,dx = \frac{2x^m\sqrt{ax+b}}{(2m+1)a} - \frac{2mb}{(2m+1)a}\int \frac{x^{m-1}}{\sqrt{ax+b}}\,dx$$

14.95 $$\int \frac{dx}{x^m\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{(m-1)bx^{m-1}} - \frac{(2m-3)a}{(2m-2)b}\int \frac{dx}{x^{m-1}\sqrt{ax+b}}$$

14.96 $$\int x^m\sqrt{ax+b}\,dx = \frac{2x^m}{(2m+3)a}(ax+b)^{3/2} - \frac{2mb}{(2m+3)a}\int x^{m-1}\sqrt{ax+b}\,dx$$

14.97 $$\int \frac{\sqrt{ax+b}}{x^m}\,dx = -\frac{\sqrt{ax+b}}{(m-1)x^{m-1}} + \frac{a}{2(m-1)}\int \frac{dx}{x^{m-1}\sqrt{ax+b}}$$

14.98 $$\int \frac{\sqrt{ax+b}}{x^m}\,dx = \frac{-(ax+b)^{3/2}}{(m-1)bx^{m-1}} - \frac{(2m-5)a}{(2m-2)b}\int \frac{\sqrt{ax+b}}{x^{m-1}}\,dx$$

14.99 $$\int (ax+b)^{m/2}\,dx = \frac{2(ax+b)^{(m+2)/2}}{a(m+2)}$$

14.100 $$\int x(ax+b)^{m/2}\,dx = \frac{2(ax+b)^{(m+4)/2}}{a^2(m+4)} - \frac{2b(ax+b)^{(m+2)/2}}{a^2(m+2)}$$

14.101 $$\int x^2(ax+b)^{m/2}\,dx = \frac{2(ax+b)^{(m+6)/2}}{a^3(m+6)} - \frac{4b(ax+b)^{(m+4)/2}}{a^3(m+4)} + \frac{2b^2(ax+b)^{(m+2)/2}}{a^3(m+2)}$$

14.102 $$\int \frac{(ax+b)^{m/2}}{x}\,dx = \frac{2(ax+b)^{m/2}}{m} + b\int \frac{(ax+b)^{(m-2)/2}}{x}\,dx$$

14.103 $$\int \frac{(ax+b)^{m/2}}{x^2}\,dx = -\frac{(ax+b)^{(m+2)/2}}{bx} + \frac{ma}{2b}\int \frac{(ax+b)^{m/2}}{x}\,dx$$

14.104 $$\int \frac{dx}{x(ax+b)^{m/2}} = \frac{2}{(m-2)b(ax+b)^{(m-2)/2}} + \frac{1}{b}\int \frac{dx}{x(ax+b)^{(m-2)/2}}$$

INTEGRALS INVOLVING $ax+b$ AND $px+q$

14.105 $$\int \frac{dx}{(ax+b)(px+q)} = \frac{1}{bp-aq}\ln\left(\frac{px+q}{ax+b}\right)$$

14.106 $$\int \frac{x\,dx}{(ax+b)(px+q)} = \frac{1}{bp-aq}\left\{\frac{b}{a}\ln(ax+b) - \frac{q}{p}\ln(px+q)\right\}$$

14.107 $$\int \frac{dx}{(ax+b)^2(px+q)} = \frac{1}{bp-aq}\left\{\frac{1}{ax+b} + \frac{p}{bp-aq}\ln\left(\frac{px+q}{ax+b}\right)\right\}$$

14.108 $$\int \frac{x\,dx}{(ax+b)^2(px+q)} = \frac{1}{bp-aq}\left\{\frac{q}{bp-aq}\ln\left(\frac{ax+b}{px+q}\right) - \frac{b}{a(ax+b)}\right\}$$

14.109 $$\int \frac{x^2\,dx}{(ax+b)^2(px+q)} = \frac{b^2}{(bp-aq)a^2(ax+b)} + \frac{1}{(bp-aq)^2}\left\{\frac{q^2}{p}\ln(px+q) + \frac{b(bp-2aq)}{a^2}\ln(ax+b)\right\}$$

14.110 $$\int \frac{dx}{(ax+b)^m(px+q)^n} = \frac{-1}{(n-1)(bp-aq)}\left\{\frac{1}{(ax+b)^{m-1}(px+q)^{n-1}} + a(m+n-2)\int\frac{dx}{(ax+b)^m(px+q)^{n-1}}\right\}$$

14.111 $$\int \frac{ax+b}{px+q}\,dx = \frac{ax}{p} + \frac{bp-aq}{p^2}\ln(px+q)$$

14.112 $$\int \frac{(ax+b)^m}{(px+q)^n}\,dx = \begin{cases} \dfrac{-1}{(n-1)(bp-aq)}\left\{\dfrac{(ax+b)^{m+1}}{(px+q)^{n-1}} + (n-m-2)a\displaystyle\int\frac{(ax+b)^m}{(px+q)^{n-1}}\,dx\right\} \\ \dfrac{-1}{(n-m-1)p}\left\{\dfrac{(ax+b)^m}{(px+q)^{n-1}} + m(bp-aq)\displaystyle\int\frac{(ax+b)^{m-1}}{(px+q)^n}\,dx\right\} \\ \dfrac{-1}{(n-1)p}\left\{\dfrac{(ax+b)^m}{(px+q)^{n-1}} - ma\displaystyle\int\frac{(ax+b)^{m-1}}{(px+q)^{n-1}}\,dx\right\} \end{cases}$$

INTEGRALS INVOLVING $\sqrt{ax+b}$ AND $px+q$

14.113 $$\int \frac{px+q}{\sqrt{ax+b}}\,dx = \frac{2(apx+3aq-2bp)}{3a^2}\sqrt{ax+b}$$

14.114 $$\int \frac{dx}{(px+q)\sqrt{ax+b}} = \begin{cases} \dfrac{1}{\sqrt{bp-aq}\,\sqrt{p}}\ln\left(\dfrac{\sqrt{p(ax+b)}-\sqrt{bp-aq}}{\sqrt{p(ax+b)}+\sqrt{bp-aq}}\right) \\ \dfrac{2}{\sqrt{aq-bp}\,\sqrt{p}}\tan^{-1}\sqrt{\dfrac{p(ax+b)}{aq-bp}} \end{cases}$$

14.115 $$\int \frac{\sqrt{ax+b}}{px+q}\,dx = \begin{cases} \dfrac{2\sqrt{ax+b}}{p} + \dfrac{\sqrt{bp-aq}}{p\sqrt{p}}\ln\left(\dfrac{\sqrt{p(ax+b)}-\sqrt{bp-aq}}{\sqrt{p(ax+b)}+\sqrt{bp-aq}}\right) \\ \dfrac{2\sqrt{ax+b}}{p} - \dfrac{2\sqrt{aq-bp}}{p\sqrt{p}}\tan^{-1}\sqrt{\dfrac{p(ax+b)}{aq-bp}} \end{cases}$$

14.116 $$\int (px+q)^n\sqrt{ax+b}\,dx = \frac{2(px+q)^{n+1}\sqrt{ax+b}}{(2n+3)p} + \frac{bp-aq}{(2n+3)p}\int\frac{(px+q)^n}{\sqrt{ax+b}}\,dx$$

14.117 $$\int \frac{dx}{(px+q)^n\sqrt{ax+b}} = \frac{\sqrt{ax+b}}{(n-1)(aq-bp)(px+q)^{n-1}} + \frac{(2n-3)a}{2(n-1)(aq-bp)}\int\frac{dx}{(px+q)^{n-1}\sqrt{ax+b}}$$

14.118 $$\int \frac{(px+q)^n}{\sqrt{ax+b}}\,dx = \frac{2(px+q)^n\sqrt{ax+b}}{(2n+1)a} + \frac{2n(aq-bp)}{(2n+1)a}\int\frac{(px+q)^{n-1}\,dx}{\sqrt{ax+b}}$$

14.119 $$\int \frac{\sqrt{ax+b}}{(px+q)^n}\,dx = \frac{-\sqrt{ax+b}}{(n-1)p(px+q)^{n-1}} + \frac{a}{2(n-1)p}\int\frac{dx}{(px+q)^{n-1}\sqrt{ax+b}}$$

INTEGRALS INVOLVING $\sqrt{ax+b}$ AND $\sqrt{px+q}$

14.120 $$\int \frac{dx}{\sqrt{(ax+b)(px+q)}} = \begin{cases} \dfrac{2}{\sqrt{ap}}\ln\left(\sqrt{a(px+q)}+\sqrt{p(ax+b)}\right) \\ \dfrac{2}{\sqrt{-ap}}\tan^{-1}\sqrt{\dfrac{-p(ax+b)}{a(px+q)}} \end{cases}$$

14.121 $$\int \frac{x\,dx}{\sqrt{(ax+b)(px+q)}} = \frac{\sqrt{(ax+b)(px+q)}}{ap} - \frac{bp+aq}{2ap}\int\frac{dx}{\sqrt{(ax+b)(px+q)}}$$

14.122 $\displaystyle\int \sqrt{(ax+b)(px+q)}\,dx = \frac{2apx+bp+aq}{4ap}\sqrt{(ax+b)(px+q)} - \frac{(bp-aq)^2}{8ap}\int \frac{dx}{\sqrt{(ax+b)(px+q)}}$

14.123 $\displaystyle\int \sqrt{\frac{px+q}{ax+b}}\,dx = \frac{\sqrt{(ax+b)(px+q)}}{a} + \frac{aq-bp}{2a}\int \frac{dx}{\sqrt{(ax+b)(px+q)}}$

14.124 $\displaystyle\int \frac{dx}{(px+q)\sqrt{(ax+b)(px+q)}} = \frac{2\sqrt{ax+b}}{(aq-bp)\sqrt{px+q}}$

INTEGRALS INVOLVING $x^2 + a^2$

14.125 $\displaystyle\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

14.126 $\displaystyle\int \frac{x\,dx}{x^2+a^2} = \frac{1}{2}\ln(x^2+a^2)$

14.127 $\displaystyle\int \frac{x^2\,dx}{x^2+a^2} = x - a\tan^{-1}\frac{x}{a}$

14.128 $\displaystyle\int \frac{x^3\,dx}{x^2+a^2} = \frac{x^2}{2} - \frac{a^2}{2}\ln(x^2+a^2)$

14.129 $\displaystyle\int \frac{dx}{x(x^2+a^2)} = \frac{1}{2a^2}\ln\left(\frac{x^2}{x^2+a^2}\right)$

14.130 $\displaystyle\int \frac{dx}{x^2(x^2+a^2)} = -\frac{1}{a^2x} - \frac{1}{a^3}\tan^{-1}\frac{x}{a}$

14.131 $\displaystyle\int \frac{dx}{x^3(x^2+a^2)} = -\frac{1}{2a^2x^2} - \frac{1}{2a^4}\ln\left(\frac{x^2}{x^2+a^2}\right)$

14.132 $\displaystyle\int \frac{dx}{(x^2+a^2)^2} = \frac{x}{2a^2(x^2+a^2)} + \frac{1}{2a^3}\tan^{-1}\frac{x}{a}$

14.133 $\displaystyle\int \frac{x\,dx}{(x^2+a^2)^2} = \frac{-1}{2(x^2+a^2)}$

14.134 $\displaystyle\int \frac{x^2\,dx}{(x^2+a^2)^2} = \frac{-x}{2(x^2+a^2)} + \frac{1}{2a}\tan^{-1}\frac{x}{a}$

14.135 $\displaystyle\int \frac{x^3\,dx}{(x^2+a^2)^2} = \frac{a^2}{2(x^2+a^2)} + \frac{1}{2}\ln(x^2+a^2)$

14.136 $\displaystyle\int \frac{dx}{x(x^2+a^2)^2} = \frac{1}{2a^2(x^2+a^2)} + \frac{1}{2a^4}\ln\left(\frac{x^2}{x^2+a^2}\right)$

14.137 $\displaystyle\int \frac{dx}{x^2(x^2+a^2)^2} = -\frac{1}{a^4x} - \frac{x}{2a^4(x^2+a^2)} - \frac{3}{2a^5}\tan^{-1}\frac{x}{a}$

14.138 $\displaystyle\int \frac{dx}{x^3(x^2+a^2)^2} = -\frac{1}{2a^4x^2} - \frac{1}{2a^4(x^2+a^2)} - \frac{1}{a^6}\ln\left(\frac{x^2}{x^2+a^2}\right)$

14.139 $\displaystyle\int \frac{dx}{(x^2+a^2)^n} = \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{2n-3}{(2n-2)a^2}\int \frac{dx}{(x^2+a^2)^{n-1}}$

14.140 $\displaystyle\int \frac{x\,dx}{(x^2+a^2)^n} = \frac{-1}{2(n-1)(x^2+a^2)^{n-1}}$

14.141 $\displaystyle\int \frac{dx}{x(x^2+a^2)^n} = \frac{1}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{1}{a^2}\int \frac{dx}{x(x^2+a^2)^{n-1}}$

14.142 $\displaystyle\int \frac{x^m\,dx}{(x^2+a^2)^n} = \int \frac{x^{m-2}\,dx}{(x^2+a^2)^{n-1}} - a^2\int \frac{x^{m-2}\,dx}{(x^2+a^2)^n}$

14.143 $\displaystyle\int \frac{dx}{x^m(x^2+a^2)^n} = \frac{1}{a^2}\int \frac{dx}{x^m(x^2+a^2)^{n-1}} - \frac{1}{a^2}\int \frac{dx}{x^{m-2}(x^2+a^2)^n}$

INTEGRALS INVOLVING $x^2 - a^2$, $x^2 > a^2$

14.144 $$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln\left(\frac{x-a}{x+a}\right) \quad \text{or} \quad -\frac{1}{a}\coth^{-1}\frac{x}{a}$$

14.145 $$\int \frac{x\,dx}{x^2 - a^2} = \frac{1}{2}\ln(x^2 - a^2)$$

14.146 $$\int \frac{x^2\,dx}{x^2 - a^2} = x + \frac{a}{2}\ln\left(\frac{x-a}{x+a}\right)$$

14.147 $$\int \frac{x^3\,dx}{x^2 - a^2} = \frac{x^2}{2} + \frac{a^2}{2}\ln(x^2 - a^2)$$

14.148 $$\int \frac{dx}{x(x^2 - a^2)} = \frac{1}{2a^2}\ln\left(\frac{x^2 - a^2}{x^2}\right)$$

14.149 $$\int \frac{dx}{x^2(x^2 - a^2)} = \frac{1}{a^2 x} + \frac{1}{2a^3}\ln\left(\frac{x-a}{x+a}\right)$$

14.150 $$\int \frac{dx}{x^3(x^2 - a^2)} = \frac{1}{2a^2x^2} - \frac{1}{2a^4}\ln\left(\frac{x^2}{x^2 - a^2}\right)$$

14.151 $$\int \frac{dx}{(x^2 - a^2)^2} = \frac{-x}{2a^2(x^2 - a^2)} - \frac{1}{4a^3}\ln\left(\frac{x-a}{x+a}\right)$$

14.152 $$\int \frac{x\,dx}{(x^2 - a^2)^2} = \frac{-1}{2(x^2 - a^2)}$$

14.153 $$\int \frac{x^2\,dx}{(x^2 - a^2)^2} = \frac{-x}{2(x^2 - a^2)} + \frac{1}{4a}\ln\left(\frac{x-a}{x+a}\right)$$

14.154 $$\int \frac{x^3\,dx}{(x^2 - a^2)^2} = \frac{-a^2}{2(x^2 - a^2)} + \frac{1}{2}\ln(x^2 - a^2)$$

14.155 $$\int \frac{dx}{x(x^2 - a^2)^2} = \frac{-1}{2a^2(x^2 - a^2)} + \frac{1}{2a^4}\ln\left(\frac{x^2}{x^2 - a^2}\right)$$

14.156 $$\int \frac{dx}{x^2(x^2 - a^2)^2} = -\frac{1}{a^4 x} - \frac{x}{2a^4(x^2 - a^2)} - \frac{3}{4a^5}\ln\left(\frac{x-a}{x+a}\right)$$

14.157 $$\int \frac{dx}{x^3(x^2 - a^2)^2} = -\frac{1}{2a^4x^2} - \frac{1}{2a^4(x^2 - a^2)} + \frac{1}{a^6}\ln\left(\frac{x^2}{x^2 - a^2}\right)$$

14.158 $$\int \frac{dx}{(x^2 - a^2)^n} = \frac{-x}{2(n-1)a^2(x^2 - a^2)^{n-1}} - \frac{2n-3}{(2n-2)a^2}\int \frac{dx}{(x^2 - a^2)^{n-1}}$$

14.159 $$\int \frac{x\,dx}{(x^2 - a^2)^n} = \frac{-1}{2(n-1)(x^2 - a^2)^{n-1}}$$

14.160 $$\int \frac{dx}{x(x^2 - a^2)^n} = \frac{-1}{2(n-1)a^2(x^2 - a^2)^{n-1}} - \frac{1}{a^2}\int \frac{dx}{x(x^2 - a^2)^{n-1}}$$

14.161 $$\int \frac{x^m\,dx}{(x^2 - a^2)^n} = \int \frac{x^{m-2}\,dx}{(x^2 - a^2)^{n-1}} + a^2\int \frac{x^{m-2}\,dx}{(x^2 - a^2)^n}$$

14.162 $$\int \frac{dx}{x^m(x^2 - a^2)^n} = \frac{1}{a^2}\int \frac{dx}{x^{m-2}(x^2 - a^2)^n} - \frac{1}{a^2}\int \frac{dx}{x^m(x^2 - a^2)^{n-1}}$$

INTEGRALS INVOLVING $a^2 - x^2, \quad x^2 < a^2$

14.163 $$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln\left(\frac{a+x}{a-x}\right) \quad \text{or} \quad \frac{1}{a}\tanh^{-1}\frac{x}{a}$$

14.164 $$\int \frac{x\,dx}{a^2 - x^2} = -\frac{1}{2}\ln(a^2 - x^2)$$

14.165 $$\int \frac{x^2\,dx}{a^2 - x^2} = -x + \frac{a}{2}\ln\left(\frac{a+x}{a-x}\right)$$

14.166 $$\int \frac{x^3\,dx}{a^2 - x^2} = -\frac{x^2}{2} - \frac{a^2}{2}\ln(a^2 - x^2)$$

14.167 $$\int \frac{dx}{x(a^2 - x^2)} = \frac{1}{2a^2}\ln\left(\frac{x^2}{a^2 - x^2}\right)$$

14.168 $$\int \frac{dx}{x^2(a^2 - x^2)} = -\frac{1}{a^2 x} + \frac{1}{2a^3}\ln\left(\frac{a+x}{a-x}\right)$$

14.169 $$\int \frac{dx}{x^3(a^2 - x^2)} = -\frac{1}{2a^2x^2} + \frac{1}{2a^4}\ln\left(\frac{x^2}{a^2 - x^2}\right)$$

14.170 $$\int \frac{dx}{(a^2 - x^2)^2} = \frac{x}{2a^2(a^2 - x^2)} + \frac{1}{4a^3}\ln\left(\frac{a+x}{a-x}\right)$$

14.171 $$\int \frac{x\,dx}{(a^2 - x^2)^2} = \frac{1}{2(a^2 - x^2)}$$

14.172 $$\int \frac{x^2\,dx}{(a^2 - x^2)^2} = \frac{x}{2(a^2 - x^2)} - \frac{1}{4a}\ln\left(\frac{a+x}{a-x}\right)$$

14.173 $$\int \frac{x^3\,dx}{(a^2 - x^2)^2} = \frac{a^2}{2(a^2 - x^2)} + \frac{1}{2}\ln(a^2 - x^2)$$

14.174 $$\int \frac{dx}{x(a^2 - x^2)^2} = \frac{1}{2a^2(a^2 - x^2)} + \frac{1}{2a^4}\ln\left(\frac{x^2}{a^2 - x^2}\right)$$

14.175 $$\int \frac{dx}{x^2(a^2 - x^2)^2} = \frac{-1}{a^4 x} + \frac{x}{2a^4(a^2 - x^2)} + \frac{3}{4a^5}\ln\left(\frac{a+x}{a-x}\right)$$

14.176 $$\int \frac{dx}{x^3(a^2 - x^2)^2} = \frac{-1}{2a^4x^2} + \frac{1}{2a^4(a^2 - x^2)} + \frac{1}{a^6}\ln\left(\frac{x^2}{a^2 - x^2}\right)$$

14.177 $$\int \frac{dx}{(a^2 - x^2)^n} = \frac{x}{2(n-1)a^2(a^2 - x^2)^{n-1}} + \frac{2n-3}{(2n-2)a^2}\int \frac{dx}{(a^2 - x^2)^{n-1}}$$

14.178 $$\int \frac{x\,dx}{(a^2 - x^2)^n} = \frac{1}{2(n-1)(a^2 - x^2)^{n-1}}$$

14.179 $$\int \frac{dx}{x(a^2 - x^2)^n} = \frac{1}{2(n-1)a^2(a^2 - x^2)^{n-1}} + \frac{1}{a^2}\int \frac{dx}{x(a^2 - x^2)^{n-1}}$$

14.180 $$\int \frac{x^m\,dx}{(a^2 - x^2)^n} = a^2\int \frac{x^{m-2}dx}{(a^2 - x^2)^n} - \int \frac{x^{m-2}dx}{(a^2 - x^2)^{n-1}}$$

14.181 $$\int \frac{dx}{x^m(a^2 - x^2)^n} = \frac{1}{a^2}\int \frac{dx}{x^m(a^2 - x^2)^{n-1}} + \frac{1}{a^2}\int \frac{dx}{x^{m-2}(a^2 - x^2)^n}$$

INTEGRALS INVOLVING $\sqrt{x^2+a^2}$

14.182 $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x+\sqrt{x^2+a^2}\right) \quad \text{or} \quad \sinh^{-1}\frac{x}{a}$

14.183 $\displaystyle\int \frac{x\,dx}{\sqrt{x^2+a^2}} = \sqrt{x^2+a^2}$

14.184 $\displaystyle\int \frac{x^2\,dx}{\sqrt{x^2+a^2}} = \frac{x\sqrt{x^2+a^2}}{2} - \frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right)$

14.185 $\displaystyle\int \frac{x^3\,dx}{\sqrt{x^2+a^2}} = \frac{(x^2+a^2)^{3/2}}{3} - a^2\sqrt{x^2+a^2}$

14.186 $\displaystyle\int \frac{dx}{x\sqrt{x^2+a^2}} = -\frac{1}{a}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$

14.187 $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{a^2x}$

14.188 $\displaystyle\int \frac{dx}{x^3\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{2a^2x^2} + \frac{1}{2a^3}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$

14.189 $\displaystyle\int \sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right)$

14.190 $\displaystyle\int x\sqrt{x^2+a^2}\,dx = \frac{(x^2+a^2)^{3/2}}{3}$

14.191 $\displaystyle\int x^2\sqrt{x^2+a^2}\,dx = \frac{x(x^2+a^2)^{3/2}}{4} - \frac{a^2x\sqrt{x^2+a^2}}{8} - \frac{a^4}{8}\ln\left(x+\sqrt{x^2+a^2}\right)$

14.192 $\displaystyle\int x^3\sqrt{x^2+a^2}\,dx = \frac{(x^2+a^2)^{5/2}}{5} - \frac{a^2(x^2+a^2)^{3/2}}{3}$

14.193 $\displaystyle\int \frac{\sqrt{x^2+a^2}}{x}\,dx = \sqrt{x^2+a^2} - a\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$

14.194 $\displaystyle\int \frac{\sqrt{x^2+a^2}}{x^2}\,dx = -\frac{\sqrt{x^2+a^2}}{x} + \ln\left(x+\sqrt{x^2+a^2}\right)$

14.195 $\displaystyle\int \frac{\sqrt{x^2+a^2}}{x^3}\,dx = -\frac{\sqrt{x^2+a^2}}{2x^2} - \frac{1}{2a}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$

14.196 $\displaystyle\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}}$

14.197 $\displaystyle\int \frac{x\,dx}{(x^2+a^2)^{3/2}} = \frac{-1}{\sqrt{x^2+a^2}}$

14.198 $\displaystyle\int \frac{x^2\,dx}{(x^2+a^2)^{3/2}} = \frac{-x}{\sqrt{x^2+a^2}} + \ln\left(x+\sqrt{x^2+a^2}\right)$

14.199 $\displaystyle\int \frac{x^3\,dx}{(x^2+a^2)^{3/2}} = \sqrt{x^2+a^2} + \frac{a^2}{\sqrt{x^2+a^2}}$

14.200 $\displaystyle\int \frac{dx}{x(x^2+a^2)^{3/2}} = \frac{1}{a^2\sqrt{x^2+a^2}} - \frac{1}{a^3}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$

14.201 $\displaystyle\int \frac{dx}{x^2(x^2+a^2)^{3/2}} = -\frac{\sqrt{x^2+a^2}}{a^4x} - \frac{x}{a^4\sqrt{x^2+a^2}}$

14.202 $\displaystyle\int \frac{dx}{x^3(x^2+a^2)^{3/2}} = \frac{-1}{2a^2x^2\sqrt{x^2+a^2}} - \frac{3}{2a^4\sqrt{x^2+a^2}} + \frac{3}{2a^5}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$

14.203 $$\int (x^2+a^2)^{3/2}\,dx = \frac{x(x^2+a^2)^{3/2}}{4} + \frac{3a^2x\sqrt{x^2+a^2}}{8} + \frac{3}{8}a^4 \ln\left(x+\sqrt{x^2+a^2}\right)$$

14.204 $$\int x(x^2+a^2)^{3/2}\,dx = \frac{(x^2+a^2)^{5/2}}{5}$$

14.205 $$\int x^2(x^2+a^2)^{3/2}\,dx = \frac{x(x^2+a^2)^{5/2}}{6} - \frac{a^2x(x^2+a^2)^{3/2}}{24} - \frac{a^4x\sqrt{x^2+a^2}}{16} - \frac{a^6}{16}\ln\left(x+\sqrt{x^2+a^2}\right)$$

14.206 $$\int x^3(x^2+a^2)^{3/2}\,dx = \frac{(x^2+a^2)^{7/2}}{7} - \frac{a^2(x^2+a^2)^{5/2}}{5}$$

14.207 $$\int \frac{(x^2+a^2)^{3/2}}{x}\,dx = \frac{(x^2+a^2)^{3/2}}{3} + a^2\sqrt{x^2+a^2} - a^3\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$$

14.208 $$\int \frac{(x^2+a^2)^{3/2}}{x^2}\,dx = -\frac{(x^2+a^2)^{3/2}}{x} + \frac{3x\sqrt{x^2+a^2}}{2} + \frac{3}{2}a^2\ln\left(x+\sqrt{x^2+a^2}\right)$$

14.209 $$\int \frac{(x^2+a^2)^{3/2}}{x^3}\,dx = -\frac{(x^2+a^2)^{3/2}}{2x^2} + \frac{3}{2}\sqrt{x^2+a^2} - \frac{3}{2}a\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$$

INTEGRALS INVOLVING $\sqrt{x^2-a^2}$

14.210 $$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left(x+\sqrt{x^2-a^2}\right), \qquad \int \frac{x\,dx}{\sqrt{x^2-a^2}} = \sqrt{x^2-a^2}$$

14.211 $$\int \frac{x^2\,dx}{\sqrt{x^2-a^2}} = \frac{x\sqrt{x^2-a^2}}{2} + \frac{a^2}{2}\ln\left(x+\sqrt{x^2-a^2}\right)$$

14.212 $$\int \frac{x^3\,dx}{\sqrt{x^2-a^2}} = \frac{(x^2-a^2)^{3/2}}{3} + a^2\sqrt{x^2-a^2}$$

14.213 $$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right|$$

14.214 $$\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2x}$$

14.215 $$\int \frac{dx}{x^3\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{2a^2x^2} + \frac{1}{2a^3}\sec^{-1}\left|\frac{x}{a}\right|$$

14.216 $$\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\ln\left(x+\sqrt{x^2-a^2}\right)$$

14.217 $$\int x\sqrt{x^2-a^2}\,dx = \frac{(x^2-a^2)^{3/2}}{3}$$

14.218 $$\int x^2\sqrt{x^2-a^2}\,dx = \frac{x(x^2-a^2)^{3/2}}{4} + \frac{a^2x\sqrt{x^2-a^2}}{8} - \frac{a^4}{8}\ln\left(x+\sqrt{x^2-a^2}\right)$$

14.219 $$\int x^3\sqrt{x^2-a^2}\,dx = \frac{(x^2-a^2)^{5/2}}{5} + \frac{a^2(x^2-a^2)^{3/2}}{3}$$

14.220 $$\int \frac{\sqrt{x^2-a^2}}{x}\,dx = \sqrt{x^2-a^2} - a\sec^{-1}\left|\frac{x}{a}\right|$$

14.221 $$\int \frac{\sqrt{x^2-a^2}}{x^2}\,dx = -\frac{\sqrt{x^2-a^2}}{x} + \ln\left(x+\sqrt{x^2-a^2}\right)$$

14.222 $$\int \frac{\sqrt{x^2-a^2}}{x^3}\,dx = -\frac{\sqrt{x^2-a^2}}{2x^2} + \frac{1}{2a}\sec^{-1}\left|\frac{x}{a}\right|$$

14.223 $$\int \frac{dx}{(x^2-a^2)^{3/2}} = -\frac{x}{a^2\sqrt{x^2-a^2}}$$

14.224 $$\int \frac{x\,dx}{(x^2-a^2)^{3/2}} = \frac{-1}{\sqrt{x^2-a^2}}$$

14.225 $$\int \frac{x^2\,dx}{(x^2-a^2)^{3/2}} = -\frac{x}{\sqrt{x^2-a^2}} + \ln\left(x+\sqrt{x^2-a^2}\right)$$

14.226 $$\int \frac{x^3\,dx}{(x^2-a^2)^{3/2}} = \sqrt{x^2-a^2} - \frac{a^2}{\sqrt{x^2-a^2}}$$

14.227 $$\int \frac{dx}{x(x^2-a^2)^{3/2}} = \frac{-1}{a^2\sqrt{x^2-a^2}} - \frac{1}{a^3}\sec^{-1}\left|\frac{x}{a}\right|$$

14.228 $$\int \frac{dx}{x^2(x^2-a^2)^{3/2}} = -\frac{\sqrt{x^2-a^2}}{a^4x} - \frac{x}{a^4\sqrt{x^2-a^2}}$$

14.229 $$\int \frac{dx}{x^3(x^2-a^2)^{3/2}} = \frac{1}{2a^2x^2\sqrt{x^2-a^2}} - \frac{3}{2a^4\sqrt{x^2-a^2}} - \frac{3}{2a^5}\sec^{-1}\left|\frac{x}{a}\right|$$

14.230 $$\int (x^2-a^2)^{3/2}\,dx = \frac{x(x^2-a^2)^{3/2}}{4} - \frac{3a^2x\sqrt{x^2-a^2}}{8} + \frac{3}{8}a^4\ln\left(x+\sqrt{x^2-a^2}\right)$$

14.231 $$\int x(x^2-a^2)^{3/2}\,dx = \frac{(x^2-a^2)^{5/2}}{5}$$

14.232 $$\int x^2(x^2-a^2)^{3/2}\,dx = \frac{x(x^2-a^2)^{5/2}}{6} + \frac{a^2x(x^2-a^2)^{3/2}}{24} - \frac{a^4x\sqrt{x^2-a^2}}{16} + \frac{a^6}{16}\ln\left(x+\sqrt{x^2-a^2}\right)$$

14.233 $$\int x^3(x^2-a^2)^{3/2}\,dx = \frac{(x^2-a^2)^{7/2}}{7} + \frac{a^2(x^2-a^2)^{5/2}}{5}$$

14.234 $$\int \frac{(x^2-a^2)^{3/2}}{x}\,dx = \frac{(x^2-a^2)^{3/2}}{3} - a^2\sqrt{x^2-a^2} + a^3\sec^{-1}\left|\frac{x}{a}\right|$$

14.235 $$\int \frac{(x^2-a^2)^{3/2}}{x^2}\,dx = -\frac{(x^2-a^2)^{3/2}}{x} + \frac{3x\sqrt{x^2-a^2}}{2} - \frac{3}{2}a^2\ln\left(x+\sqrt{x^2-a^2}\right)$$

14.236 $$\int \frac{(x^2-a^2)^{3/2}}{x^3}\,dx = -\frac{(x^2-a^2)^{3/2}}{2x^2} + \frac{3\sqrt{x^2-a^2}}{2} - \frac{3}{2}a\sec^{-1}\left|\frac{x}{a}\right|$$

INTEGRALS INVOLVING $\sqrt{a^2-x^2}$

14.237 $$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

14.238 $$\int \frac{x\,dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}$$

14.239 $$\int \frac{x^2\,dx}{\sqrt{a^2-x^2}} = -\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

14.240 $$\int \frac{x^3\,dx}{\sqrt{a^2-x^2}} = \frac{(a^2-x^2)^{3/2}}{3} - a^2\sqrt{a^2-x^2}$$

14.241 $$\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$$

14.242 $$\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x}$$

14.243 $$\int \frac{dx}{x^3\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{2a^2x^2} - \frac{1}{2a^3}\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$$

14.244 $$\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

14.245 $$\int x\sqrt{a^2-x^2}\,dx = -\frac{(a^2-x^2)^{3/2}}{3}$$

14.246 $$\int x^2\sqrt{a^2-x^2}\,dx = -\frac{x(a^2-x^2)^{3/2}}{4} + \frac{a^2x\sqrt{a^2-x^2}}{8} + \frac{a^4}{8}\sin^{-1}\frac{x}{a}$$

14.247 $$\int x^3\sqrt{a^2-x^2}\,dx = \frac{(a^2-x^2)^{5/2}}{5} - \frac{a^2(a^2-x^2)^{3/2}}{3}$$

14.248 $$\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \sqrt{a^2-x^2} - a\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$$

14.249 $$\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx = -\frac{\sqrt{a^2-x^2}}{x} - \sin^{-1}\frac{x}{a}$$

14.250 $$\int \frac{\sqrt{a^2-x^2}}{x^3}\,dx = -\frac{\sqrt{a^2-x^2}}{2x^2} + \frac{1}{2a}\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$$

14.251 $$\int \frac{dx}{(a^2-x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2-x^2}}$$

14.252 $$\int \frac{x\,dx}{(a^2-x^2)^{3/2}} = \frac{1}{\sqrt{a^2-x^2}}$$

14.253 $$\int \frac{x^2\,dx}{(a^2-x^2)^{3/2}} = \frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{x}{a}$$

14.254 $$\int \frac{x^3\,dx}{(a^2-x^2)^{3/2}} = \sqrt{a^2-x^2} + \frac{a^2}{\sqrt{a^2-x^2}}$$

14.255 $$\int \frac{dx}{x(a^2-x^2)^{3/2}} = \frac{1}{a^2\sqrt{a^2-x^2}} - \frac{1}{a^3}\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$$

14.256 $$\int \frac{dx}{x^2(a^2-x^2)^{3/2}} = -\frac{\sqrt{a^2-x^2}}{a^4x} + \frac{x}{a^4\sqrt{a^2-x^2}}$$

14.257 $$\int \frac{dx}{x^3(a^2-x^2)^{3/2}} = \frac{-1}{2a^2x^2\sqrt{a^2-x^2}} + \frac{3}{2a^4\sqrt{a^2-x^2}} - \frac{3}{2a^5}\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$$

14.258 $$\int (a^2-x^2)^{3/2}\,dx = \frac{x(a^2-x^2)^{3/2}}{4} + \frac{3a^2x\sqrt{a^2-x^2}}{8} + \frac{3}{8}a^4\sin^{-1}\frac{x}{a}$$

14.259 $$\int x(a^2-x^2)^{3/2}\,dx = -\frac{(a^2-x^2)^{5/2}}{5}$$

14.260 $$\int x^2(a^2-x^2)^{3/2}\,dx = -\frac{x(a^2-x^2)^{5/2}}{6} + \frac{a^2x(a^2-x^2)^{3/2}}{24} + \frac{a^4x\sqrt{a^2-x^2}}{16} + \frac{a^6}{16}\sin^{-1}\frac{x}{a}$$

14.261 $$\int x^3(a^2-x^2)^{3/2}\,dx = \frac{(a^2-x^2)^{7/2}}{7} - \frac{a^2(a^2-x^2)^{5/2}}{5}$$

14.262 $$\int \frac{(a^2-x^2)^{3/2}}{x}\,dx = \frac{(a^2-x^2)^{3/2}}{3} + a^2\sqrt{a^2-x^2} - a^3\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$$

14.263 $$\int \frac{(a^2-x^2)^{3/2}}{x^2}\,dx = -\frac{(a^2-x^2)^{3/2}}{x} - \frac{3x\sqrt{a^2-x^2}}{2} - \frac{3}{2}a^2\sin^{-1}\frac{x}{a}$$

14.264 $$\int \frac{(a^2-x^2)^{3/2}}{x^3}\,dx = -\frac{(a^2-x^2)^{3/2}}{2x^2} - \frac{3\sqrt{a^2-x^2}}{2} + \frac{3}{2}a\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$$

INTEGRALS INVOLVING $ax^2 + bx + c$

14.265 $$\int \frac{dx}{ax^2+bx+c} = \begin{cases} \dfrac{2}{\sqrt{4ac-b^2}} \tan^{-1} \dfrac{2ax+b}{\sqrt{4ac-b^2}} \\[2ex] \dfrac{1}{\sqrt{b^2-4ac}} \ln\left(\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right) \end{cases}$$

If $b^2 = 4ac$, $ax^2+bx+c = a(x+b/2a)^2$ and the results on pages 60-61 can be used. If $b = 0$ use results on page 64. If a or $c = 0$ use results on pages 60-61.

14.266 $$\int \frac{x\,dx}{ax^2+bx+c} = \frac{1}{2a}\ln(ax^2+bx+c) - \frac{b}{2a}\int \frac{dx}{ax^2+bx+c}$$

14.267 $$\int \frac{x^2\,dx}{ax^2+bx+c} = \frac{x}{a} - \frac{b}{2a^2}\ln(ax^2+bx+c) + \frac{b^2-2ac}{2a^2}\int \frac{dx}{ax^2+bx+c}$$

14.268 $$\int \frac{x^m\,dx}{ax^2+bx+c} = \frac{x^{m-1}}{(m-1)a} - \frac{c}{a}\int \frac{x^{m-2}\,dx}{ax^2+bx+c} - \frac{b}{a}\int \frac{x^{m-1}\,dx}{ax^2+bx+c}$$

14.269 $$\int \frac{dx}{x(ax^2+bx+c)} = \frac{1}{2c}\ln\left(\frac{x^2}{ax^2+bx+c}\right) - \frac{b}{2c}\int \frac{dx}{ax^2+bx+c}$$

14.270 $$\int \frac{dx}{x^2(ax^2+bx+c)} = \frac{b}{2c^2}\ln\left(\frac{ax^2+bx+c}{x^2}\right) - \frac{1}{cx} + \frac{b^2-2ac}{2c^2}\int \frac{dx}{ax^2+bx+c}$$

14.271 $$\int \frac{dx}{x^n(ax^2+bx+c)} = -\frac{1}{(n-1)cx^{n-1}} - \frac{b}{c}\int \frac{dx}{x^{n-1}(ax^2+bx+c)} - \frac{a}{c}\int \frac{dx}{x^{n-2}(ax^2+bx+c)}$$

14.272 $$\int \frac{dx}{(ax^2+bx+c)^2} = \frac{2ax+b}{(4ac-b^2)(ax^2+bx+c)} + \frac{2a}{4ac-b^2}\int \frac{dx}{ax^2+bx+c}$$

14.273 $$\int \frac{x\,dx}{(ax^2+bx+c)^2} = -\frac{bx+2c}{(4ac-b^2)(ax^2+bx+c)} - \frac{b}{4ac-b^2}\int \frac{dx}{ax^2+bx+c}$$

14.274 $$\int \frac{x^2\,dx}{(ax^2+bx+c)^2} = \frac{(b^2-2ac)x+bc}{a(4ac-b^2)(ax^2+bx+c)} + \frac{2c}{4ac-b^2}\int \frac{dx}{ax^2+bx+c}$$

14.275 $$\int \frac{x^m\,dx}{(ax^2+bx+c)^n} = -\frac{x^{m-1}}{(2n-m-1)a(ax^2+bx+c)^{n-1}} + \frac{(m-1)c}{(2n-m-1)a}\int \frac{x^{m-2}\,dx}{(ax^2+bx+c)^n}$$
$$- \frac{(n-m)b}{(2n-m-1)a}\int \frac{x^{m-1}\,dx}{(ax^2+bx+c)^n}$$

14.276 $$\int \frac{x^{2n-1}\,dx}{(ax^2+bx+c)^n} = \frac{1}{a}\int \frac{x^{2n-3}\,dx}{(ax^2+bx+c)^{n-1}} - \frac{c}{a}\int \frac{x^{2n-3}\,dx}{(ax^2+bx+c)^n} - \frac{b}{a}\int \frac{x^{2n-2}\,dx}{(ax^2+bx+c)^n}$$

14.277 $$\int \frac{dx}{x(ax^2+bx+c)^2} = \frac{1}{2c(ax^2+bx+c)} - \frac{b}{2c}\int \frac{dx}{(ax^2+bx+c)^2} + \frac{1}{c}\int \frac{dx}{x(ax^2+bx+c)}$$

14.278 $$\int \frac{dx}{x^2(ax^2+bx+c)^2} = -\frac{1}{cx(ax^2+bx+c)} - \frac{3a}{c}\int \frac{dx}{(ax^2+bx+c)^2} - \frac{2b}{c}\int \frac{dx}{x(ax^2+bx+c)^2}$$

14.279 $$\int \frac{dx}{x^m(ax^2+bx+c)^n} = -\frac{1}{(m-1)cx^{m-1}(ax^2+bx+c)^{n-1}} - \frac{(m+2n-3)a}{(m-1)c}\int \frac{dx}{x^{m-2}(ax^2+bx+c)^n}$$
$$- \frac{(m+n-2)b}{(m-1)c}\int \frac{dx}{x^{m-1}(ax^2+bx+c)^n}$$

INTEGRALS INVOLVING $\sqrt{ax^2+bx+c}$

In the following results if $b^2 = 4ac$, $\sqrt{ax^2+bx+c} = \sqrt{a}\,(x + b/2a)$ and the results on pages 60-61 can be used. If $b = 0$ use the results on pages 67-70. If $a = 0$ or $c = 0$ use the results on pages 61-62.

14.280 $$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{1}{\sqrt{a}} \ln\left(2\sqrt{a}\sqrt{ax^2+bx+c} + 2ax + b\right) \\ -\dfrac{1}{\sqrt{-a}} \sin^{-1}\left(\dfrac{2ax+b}{\sqrt{b^2-4ac}}\right) \quad \text{or} \quad \dfrac{1}{\sqrt{a}} \sinh^{-1}\left(\dfrac{2ax+b}{\sqrt{4ac-b^2}}\right) \end{cases}$$

14.281 $$\int \frac{x\,dx}{\sqrt{ax^2+bx+c}} = \frac{\sqrt{ax^2+bx+c}}{a} - \frac{b}{2a}\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

14.282 $$\int \frac{x^2\,dx}{\sqrt{ax^2+bx+c}} = \frac{2ax-3b}{4a^2}\sqrt{ax^2+bx+c} + \frac{3b^2-4ac}{8a^2}\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

14.283 $$\int \frac{dx}{x\sqrt{ax^2+bx+c}} = \begin{cases} -\dfrac{1}{\sqrt{c}} \ln\left(\dfrac{2\sqrt{c}\sqrt{ax^2+bx+c} + bx + 2c}{x}\right) \\ \dfrac{1}{\sqrt{-c}} \sin^{-1}\left(\dfrac{bx+2c}{|x|\sqrt{b^2-4ac}}\right) \quad \text{or} \quad -\dfrac{1}{\sqrt{c}} \sinh^{-1}\left(\dfrac{bx+2c}{|x|\sqrt{4ac-b^2}}\right) \end{cases}$$

14.284 $$\int \frac{dx}{x^2\sqrt{ax^2+bx+c}} = -\frac{\sqrt{ax^2+bx+c}}{cx} - \frac{b}{2c}\int \frac{dx}{x\sqrt{ax^2+bx+c}}$$

14.285 $$\int \sqrt{ax^2+bx+c}\,dx = \frac{(2ax+b)\sqrt{ax^2+bx+c}}{4a} + \frac{4ac-b^2}{8a}\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

14.286 $$\int x\sqrt{ax^2+bx+c}\,dx = \frac{(ax^2+bx+c)^{3/2}}{3a} - \frac{b(2ax+b)}{8a^2}\sqrt{ax^2+bx+c} - \frac{b(4ac-b^2)}{16a^2}\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

14.287 $$\int x^2\sqrt{ax^2+bx+c}\,dx = \frac{6ax-5b}{24a^2}(ax^2+bx+c)^{3/2} + \frac{5b^2-4ac}{16a^2}\int \sqrt{ax^2+bx+c}\,dx$$

14.288 $$\int \frac{\sqrt{ax^2+bx+c}}{x}\,dx = \sqrt{ax^2+bx+c} + \frac{b}{2}\int \frac{dx}{\sqrt{ax^2+bx+c}} + c\int \frac{dx}{x\sqrt{ax^2+bx+c}}$$

14.289 $$\int \frac{\sqrt{ax^2+bx+c}}{x^2}\,dx = -\frac{\sqrt{ax^2+bx+c}}{x} + a\int \frac{dx}{\sqrt{ax^2+bx+c}} + \frac{b}{2}\int \frac{dx}{x\sqrt{ax^2+bx+c}}$$

14.290 $$\int \frac{dx}{(ax^2+bx+c)^{3/2}} = \frac{2(2ax+b)}{(4ac-b^2)\sqrt{ax^2+bx+c}}$$

14.291 $$\int \frac{x\,dx}{(ax^2+bx+c)^{3/2}} = \frac{2(bx+2c)}{(b^2-4ac)\sqrt{ax^2+bx+c}}$$

14.292 $$\int \frac{x^2\,dx}{(ax^2+bx+c)^{3/2}} = \frac{(2b^2-4ac)x + 2bc}{a(4ac-b^2)\sqrt{ax^2+bx+c}} + \frac{1}{a}\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

14.293 $$\int \frac{dx}{x(ax^2+bx+c)^{3/2}} = \frac{1}{c\sqrt{ax^2+bx+c}} + \frac{1}{c}\int \frac{dx}{x\sqrt{ax^2+bx+c}} - \frac{b}{2c}\int \frac{dx}{(ax^2+bx+c)^{3/2}}$$

14.294 $$\int \frac{dx}{x^2(ax^2+bx+c)^{3/2}} = -\frac{ax^2+2bx+c}{c^2x\sqrt{ax^2+bx+c}} + \frac{b^2-2ac}{2c^2}\int \frac{dx}{(ax^2+bx+c)^{3/2}} - \frac{3b}{2c^2}\int \frac{dx}{x\sqrt{ax^2+bx+c}}$$

14.295 $$\int (ax^2+bx+c)^{n+1/2}\,dx = \frac{(2ax+b)(ax^2+bx+c)^{n+1/2}}{4a(n+1)} + \frac{(2n+1)(4ac-b^2)}{8a(n+1)}\int (ax^2+bx+c)^{n-1/2}\,dx$$

14.296 $$\int x(ax^2+bx+c)^{n+1/2}\,dx = \frac{(ax^2+bx+c)^{n+3/2}}{a(2n+3)} - \frac{b}{2a}\int (ax^2+bx+c)^{n+1/2}\,dx$$

14.297 $$\int \frac{dx}{(ax^2+bx+c)^{n+1/2}} = \frac{2(2ax+b)}{(2n-1)(4ac-b^2)(ax^2+bx+c)^{n-1/2}} + \frac{8a(n-1)}{(2n-1)(4ac-b^2)}\int \frac{dx}{(ax^2+bx+c)^{n-1/2}}$$

14.298 $$\int \frac{dx}{x(ax^2+bx+c)^{n+1/2}} = \frac{1}{(2n-1)c(ax^2+bx+c)^{n-1/2}} + \frac{1}{c}\int \frac{dx}{x(ax^2+bx+c)^{n-1/2}} - \frac{b}{2c}\int \frac{dx}{(ax^2+bx+c)^{n+1/2}}$$

INTEGRALS INVOLVING $x^3 + a^3$

Note that for formulas involving $x^3 - a^3$ replace a by $-a$.

14.299 $$\int \frac{dx}{x^3+a^3} = \frac{1}{6a^2}\ln\frac{(x+a)^2}{x^2-ax+a^2} + \frac{1}{a^2\sqrt{3}}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

14.300 $$\int \frac{x\,dx}{x^3+a^3} = \frac{1}{6a}\ln\frac{x^2-ax+a^2}{(x+a)^2} + \frac{1}{a\sqrt{3}}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

14.301 $$\int \frac{x^2\,dx}{x^3+a^3} = \frac{1}{3}\ln(x^3+a^3)$$

14.302 $$\int \frac{dx}{x(x^3+a^3)} = \frac{1}{3a^3}\ln\left(\frac{x^3}{x^3+a^3}\right)$$

14.303 $$\int \frac{dx}{x^2(x^3+a^3)} = -\frac{1}{a^3x} - \frac{1}{6a^4}\ln\frac{x^2-ax+a^2}{(x+a)^2} - \frac{1}{a^4\sqrt{3}}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

14.304 $$\int \frac{dx}{(x^3+a^3)^2} = \frac{x}{3a^3(x^3+a^3)} + \frac{1}{9a^5}\ln\frac{(x+a)^2}{x^2-ax+a^2} + \frac{2}{3a^5\sqrt{3}}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

14.305 $$\int \frac{x\,dx}{(x^3+a^3)^2} = \frac{x^2}{3a^3(x^3+a^3)} + \frac{1}{18a^4}\ln\frac{x^2-ax+a^2}{(x+a)^2} + \frac{1}{3a^4\sqrt{3}}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

14.306 $$\int \frac{x^2\,dx}{(x^3+a^3)^2} = -\frac{1}{3(x^3+a^3)}$$

14.307 $$\int \frac{dx}{x(x^3+a^3)^2} = \frac{1}{3a^3(x^3+a^3)} + \frac{1}{3a^6}\ln\left(\frac{x^3}{x^3+a^3}\right)$$

14.308 $$\int \frac{dx}{x^2(x^3+a^3)^2} = -\frac{1}{a^6x} - \frac{x^2}{3a^6(x^3+a^3)} - \frac{4}{3a^6}\int \frac{x\,dx}{x^3+a^3}$$ [See 14.300]

14.309 $$\int \frac{x^m\,dx}{x^3+a^3} = \frac{x^{m-2}}{m-2} - a^3\int \frac{x^{m-3}\,dx}{x^3+a^3}$$

14.310 $$\int \frac{dx}{x^n(x^3+a^3)} = \frac{-1}{a^3(n-1)x^{n-1}} - \frac{1}{a^3}\int \frac{dx}{x^{n-3}(x^3+a^3)}$$

INTEGRALS INVOLVING $x^4 \pm a^4$

14.311 $$\int \frac{dx}{x^4+a^4} = \frac{1}{4a^3\sqrt{2}}\ln\left(\frac{x^2+ax\sqrt{2}+a^2}{x^2-ax\sqrt{2}+a^2}\right) - \frac{1}{2a^3\sqrt{2}}\tan^{-1}\frac{ax\sqrt{2}}{x^2-a^2}$$

14.312 $$\int \frac{x\,dx}{x^4+a^4} = \frac{1}{2a^2}\tan^{-1}\frac{x^2}{a^2}$$

14.313 $$\int \frac{x^2\,dx}{x^4+a^4} = \frac{1}{4a\sqrt{2}}\ln\left(\frac{x^2-ax\sqrt{2}+a^2}{x^2+ax\sqrt{2}+a^2}\right) - \frac{1}{2a\sqrt{2}}\tan^{-1}\frac{ax\sqrt{2}}{x^2-a^2}$$

14.314 $$\int \frac{x^3\,dx}{x^4+a^4} = \frac{1}{4}\ln(x^4+a^4)$$

14.315 $$\int \frac{dx}{x(x^4+a^4)} = \frac{1}{4a^4}\ln\left(\frac{x^4}{x^4+a^4}\right)$$

14.316 $$\int \frac{dx}{x^2(x^4+a^4)} = -\frac{1}{a^4x} - \frac{1}{4a^5\sqrt{2}}\ln\left(\frac{x^2-ax\sqrt{2}+a^2}{x^2+ax\sqrt{2}+a^2}\right) + \frac{1}{2a^5\sqrt{2}}\tan^{-1}\frac{ax\sqrt{2}}{x^2-a^2}$$

14.317 $$\int \frac{dx}{x^3(x^4+a^4)} = -\frac{1}{2a^4x^2} - \frac{1}{2a^6}\tan^{-1}\frac{x^2}{a^2}$$

14.318 $$\int \frac{dx}{x^4-a^4} = \frac{1}{4a^3}\ln\left(\frac{x-a}{x+a}\right) - \frac{1}{2a^3}\tan^{-1}\frac{x}{a}$$

14.319 $$\int \frac{x\,dx}{x^4-a^4} = \frac{1}{4a^2}\ln\left(\frac{x^2-a^2}{x^2+a^2}\right)$$

14.320 $$\int \frac{x^2\,dx}{x^4-a^4} = \frac{1}{4a}\ln\left(\frac{x-a}{x+a}\right) + \frac{1}{2a}\tan^{-1}\frac{x}{a}$$

14.321 $$\int \frac{x^3\,dx}{x^4-a^4} = \frac{1}{4}\ln(x^4-a^4)$$

14.322 $$\int \frac{dx}{x(x^4-a^4)} = \frac{1}{4a^4}\ln\left(\frac{x^4-a^4}{x^4}\right)$$

14.323 $$\int \frac{dx}{x^2(x^4-a^4)} = \frac{1}{a^4x} + \frac{1}{4a^5}\ln\left(\frac{x-a}{x+a}\right) + \frac{1}{2a^5}\tan^{-1}\frac{x}{a}$$

14.324 $$\int \frac{dx}{x^3(x^4-a^4)} = \frac{1}{2a^4x^2} + \frac{1}{4a^6}\ln\left(\frac{x^2-a^2}{x^2+a^2}\right)$$

INTEGRALS INVOLVING $x^n \pm a^n$

14.325 $$\int \frac{dx}{x(x^n+a^n)} = \frac{1}{na^n}\ln\frac{x^n}{x^n+a^n}$$

14.326 $$\int \frac{x^{n-1}\,dx}{x^n+a^n} = \frac{1}{n}\ln(x^n+a^n)$$

14.327 $$\int \frac{x^m\,dx}{(x^n+a^n)^r} = \int \frac{x^{m-n}\,dx}{(x^n+a^n)^{r-1}} - a^n\int \frac{x^{m-n}\,dx}{(x^n+a^n)^r}$$

14.328 $$\int \frac{dx}{x^m(x^n+a^n)^r} = \frac{1}{a^n}\int \frac{dx}{x^m(x^n+a^n)^{r-1}} - \frac{1}{a^n}\int \frac{dx}{x^{m-n}(x^n+a^n)^r}$$

14.329 $$\int \frac{dx}{x\sqrt{x^n+a^n}} = \frac{1}{n\sqrt{a^n}}\ln\left(\frac{\sqrt{x^n+a^n}-\sqrt{a^n}}{\sqrt{x^n+a^n}+\sqrt{a^n}}\right)$$

14.330 $$\int \frac{dx}{x(x^n-a^n)} = \frac{1}{na^n}\ln\left(\frac{x^n-a^n}{x^n}\right)$$

14.331 $$\int \frac{x^{n-1}\,dx}{x^n-a^n} = \frac{1}{n}\ln(x^n-a^n)$$

14.332 $$\int \frac{x^m\,dx}{(x^n-a^n)^r} = a^n\int \frac{x^{m-n}\,dx}{(x^n-a^n)^r} + \int \frac{x^{m-n}\,dx}{(x^n-a^n)^{r-1}}$$

14.333 $$\int \frac{dx}{x^m(x^n-a^n)^r} = \frac{1}{a^n}\int \frac{dx}{x^{m-n}(x^n-a^n)^r} - \frac{1}{a^n}\int \frac{dx}{x^m(x^n-a^n)^{r-1}}$$

14.334 $$\int \frac{dx}{x\sqrt{x^n-a^n}} = \frac{2}{n\sqrt{a^n}}\cos^{-1}\sqrt{\frac{a^n}{x^n}}$$

14.335 $$\int \frac{x^{p-1}\,dx}{x^{2m}+a^{2m}} = \frac{1}{ma^{2m-p}} \sum_{k=1}^{m} \sin\frac{(2k-1)p\pi}{2m} \tan^{-1}\left(\frac{x + a\cos[(2k-1)\pi/2m]}{a\sin[(2k-1)\pi/2m]}\right)$$
$$- \frac{1}{2ma^{2m-p}} \sum_{k=1}^{m} \cos\frac{(2k-1)p\pi}{2m} \ln\left(x^2 + 2ax\cos\frac{(2k-1)\pi}{2m} + a^2\right)$$

where $0 < p \leqq 2m$.

14.336 $$\int \frac{x^{p-1}\,dx}{x^{2m}-a^{2m}} = \frac{1}{2ma^{2m-p}} \sum_{k=1}^{m-1} \cos\frac{kp\pi}{m} \ln\left(x^2 - 2ax\cos\frac{k\pi}{m} + a^2\right)$$
$$- \frac{1}{ma^{2m-p}} \sum_{k=1}^{m-1} \sin\frac{kp\pi}{m} \tan^{-1}\left(\frac{x - a\cos(k\pi/m)}{a\sin(k\pi/m)}\right)$$
$$+ \frac{1}{2ma^{2m-p}}\{\ln(x-a) + (-1)^p \ln(x+a)\}$$

where $0 < p \leqq 2m$.

14.337 $$\int \frac{x^{p-1}\,dx}{x^{2m+1}+a^{2m+1}}$$
$$= \frac{2(-1)^{p-1}}{(2m+1)a^{2m-p+1}} \sum_{k=1}^{m} \sin\frac{2kp\pi}{2m+1} \tan^{-1}\left(\frac{x + a\cos[2k\pi/(2m+1)]}{a\sin[2k\pi/(2m+1)]}\right)$$
$$- \frac{(-1)^{p-1}}{(2m+1)a^{2m-p+1}} \sum_{k=1}^{m} \cos\frac{2kp\pi}{2m+1} \ln\left(x^2 + 2ax\cos\frac{2k\pi}{2m+1} + a^2\right)$$
$$+ \frac{(-1)^{p-1}\ln(x+a)}{(2m+1)a^{2m-p+1}}$$

where $0 < p \leqq 2m+1$.

14.338 $$\int \frac{x^{p-1}\,dx}{x^{2m+1}-a^{2m+1}}$$
$$= \frac{-2}{(2m+1)a^{2m-p+1}} \sum_{k=1}^{m} \sin\frac{2kp\pi}{2m+1} \tan^{-1}\left(\frac{x - a\cos[2k\pi/(2m+1)]}{a\sin[2k\pi/(2m+1)]}\right)$$
$$+ \frac{1}{(2m+1)a^{2m-p+1}} \sum_{k=1}^{m} \cos\frac{2kp\pi}{2m+1} \ln\left(x^2 - 2ax\cos\frac{2k\pi}{2m+1} + a^2\right)$$
$$+ \frac{\ln(x-a)}{(2m+1)a^{2m-p+1}}$$

where $0 < p \leqq 2m+1$.

INTEGRALS INVOLVING sin ax

14.339 $$\int \sin ax\,dx = -\frac{\cos ax}{a}$$

14.340 $$\int x\sin ax\,dx = \frac{\sin ax}{a^2} - \frac{x\cos ax}{a}$$

14.341 $$\int x^2\sin ax\,dx = \frac{2x}{a^2}\sin ax + \left(\frac{2}{a^3} - \frac{x^2}{a}\right)\cos ax$$

14.342 $$\int x^3\sin ax\,dx = \left(\frac{3x^2}{a^2} - \frac{6}{a^4}\right)\sin ax + \left(\frac{6x}{a^3} - \frac{x^3}{a}\right)\cos ax$$

14.343 $$\int \frac{\sin ax}{x}\,dx = ax - \frac{(ax)^3}{3\cdot 3!} + \frac{(ax)^5}{5\cdot 5!} - \cdots$$

14.344 $$\int \frac{\sin ax}{x^2}\,dx = -\frac{\sin ax}{x} + a\int \frac{\cos ax}{x}\,dx$$ [see 14.373]

14.345 $$\int \frac{dx}{\sin ax} = \frac{1}{a}\ln(\csc ax - \cot ax) = \frac{1}{a}\ln\tan\frac{ax}{2}$$

14.346 $$\int \frac{x\,dx}{\sin ax} = \frac{1}{a^2}\left\{ax + \frac{(ax)^3}{18} + \frac{7(ax)^5}{1800} + \cdots + \frac{2(2^{2n-1}-1)B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$$

14.347 $$\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

14.348 $$\int x\sin^2 ax\,dx = \frac{x^2}{4} - \frac{x\sin 2ax}{4a} - \frac{\cos 2ax}{8a^2}$$

14.349 $$\int \sin^3 ax\,dx = -\frac{\cos ax}{a} + \frac{\cos^3 ax}{3a}$$

14.350 $$\int \sin^4 ax\,dx = \frac{3x}{8} - \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$$

14.351 $$\int \frac{dx}{\sin^2 ax} = -\frac{1}{a}\cot ax$$

14.352 $$\int \frac{dx}{\sin^3 ax} = -\frac{\cos ax}{2a\sin^2 ax} + \frac{1}{2a}\ln\tan\frac{ax}{2}$$

14.353 $$\int \sin px\sin qx\,dx = \frac{\sin(p-q)x}{2(p-q)} - \frac{\sin(p+q)x}{2(p+q)}$$ [If $p = \pm q$, see 14.368.]

14.354 $$\int \frac{dx}{1-\sin ax} = \frac{1}{a}\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)$$

14.355 $$\int \frac{x\,dx}{1-\sin ax} = \frac{x}{a}\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right) + \frac{2}{a^2}\ln\sin\left(\frac{\pi}{4}-\frac{ax}{2}\right)$$

14.356 $$\int \frac{dx}{1+\sin ax} = -\frac{1}{a}\tan\left(\frac{\pi}{4}-\frac{ax}{2}\right)$$

14.357 $$\int \frac{x\,dx}{1+\sin ax} = -\frac{x}{a}\tan\left(\frac{\pi}{4}-\frac{ax}{2}\right) + \frac{2}{a^2}\ln\sin\left(\frac{\pi}{4}+\frac{ax}{2}\right)$$

14.358 $$\int \frac{dx}{(1-\sin ax)^2} = \frac{1}{2a}\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right) + \frac{1}{6a}\tan^3\left(\frac{\pi}{4}+\frac{ax}{2}\right)$$

14.359 $$\int \frac{dx}{(1+\sin ax)^2} = -\frac{1}{2a}\tan\left(\frac{\pi}{4}-\frac{ax}{2}\right) - \frac{1}{6a}\tan^3\left(\frac{\pi}{4}-\frac{ax}{2}\right)$$

14.360 $$\int \frac{dx}{p+q\sin ax} = \begin{cases} \dfrac{2}{a\sqrt{p^2-q^2}}\tan^{-1}\dfrac{p\tan\frac{1}{2}ax+q}{\sqrt{p^2-q^2}} \\[2ex] \dfrac{1}{a\sqrt{q^2-p^2}}\ln\left(\dfrac{p\tan\frac{1}{2}ax+q-\sqrt{q^2-p^2}}{p\tan\frac{1}{2}ax+q+\sqrt{q^2-p^2}}\right) \end{cases}$$

If $p = \pm q$ see 14.354 and 14.356.

14.361 $$\int \frac{dx}{(p+q\sin ax)^2} = \frac{q\cos ax}{a(p^2-q^2)(p+q\sin ax)} + \frac{p}{p^2-q^2}\int \frac{dx}{p+q\sin ax}$$

If $p = \pm q$ see 14.358 and 14.359.

14.362 $$\int \frac{dx}{p^2+q^2\sin^2 ax} = \frac{1}{ap\sqrt{p^2+q^2}}\tan^{-1}\frac{\sqrt{p^2+q^2}\tan ax}{p}$$

14.363 $$\int \frac{dx}{p^2-q^2\sin^2 ax} = \begin{cases} \dfrac{1}{ap\sqrt{p^2-q^2}}\tan^{-1}\dfrac{\sqrt{p^2-q^2}\tan ax}{p} \\[2ex] \dfrac{1}{2ap\sqrt{q^2-p^2}}\ln\left(\dfrac{\sqrt{q^2-p^2}\tan ax+p}{\sqrt{q^2-p^2}\tan ax-p}\right) \end{cases}$$

14.364 $$\int x^m\sin ax\,dx = -\frac{x^m\cos ax}{a} + \frac{mx^{m-1}\sin ax}{a^2} - \frac{m(m-1)}{a^2}\int x^{m-2}\sin ax\,dx$$

14.365 $$\int \frac{\sin ax}{x^n}\,dx = -\frac{\sin ax}{(n-1)x^{n-1}} + \frac{a}{n-1}\int \frac{\cos ax}{x^{n-1}}\,dx$$ [see 14.395]

14.366 $$\int \sin^n ax\,dx = -\frac{\sin^{n-1} ax\cos ax}{an} + \frac{n-1}{n}\int \sin^{n-2} ax\,dx$$

14.367 $$\int \frac{dx}{\sin^n ax} = \frac{-\cos ax}{a(n-1)\sin^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} ax}$$

14.368 $$\int \frac{x\,dx}{\sin^n ax} = \frac{-x\cos ax}{a(n-1)\sin^{n-1} ax} - \frac{1}{a^2(n-1)(n-2)\sin^{n-2} ax} + \frac{n-2}{n-1}\int \frac{x\,dx}{\sin^{n-2} ax}$$

INTEGRALS INVOLVING $\cos ax$

14.369 $$\int \cos ax\,dx = \frac{\sin ax}{a}$$

14.370 $$\int x\cos ax\,dx = \frac{\cos ax}{a^2} + \frac{x\sin ax}{a}$$

14.371 $$\int x^2\cos ax\,dx = \frac{2x}{a^2}\cos ax + \left(\frac{x^2}{a} - \frac{2}{a^3}\right)\sin ax$$

14.372 $$\int x^3\cos ax\,dx = \left(\frac{3x^2}{a^2} - \frac{6}{a^4}\right)\cos ax + \left(\frac{x^3}{a} - \frac{6x}{a^3}\right)\sin ax$$

14.373 $$\int \frac{\cos ax}{x}\,dx = \ln x - \frac{(ax)^2}{2\cdot 2!} + \frac{(ax)^4}{4\cdot 4!} - \frac{(ax)^6}{6\cdot 6!} + \cdots$$

14.374 $$\int \frac{\cos ax}{x^2}\,dx = -\frac{\cos ax}{x} - a\int \frac{\sin ax}{x}\,dx$$ [See 14.343]

14.375 $$\int \frac{dx}{\cos ax} = \frac{1}{a}\ln(\sec ax + \tan ax) = \frac{1}{a}\ln\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

14.376 $$\int \frac{x\,dx}{\cos ax} = \frac{1}{a^2}\left\{\frac{(ax)^2}{2} + \frac{(ax)^4}{8} + \frac{5(ax)^6}{144} + \cdots + \frac{E_n(ax)^{2n+2}}{(2n+2)(2n)!} + \cdots\right\}$$

14.377 $$\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

14.378 $$\int x\cos^2 ax\,dx = \frac{x^2}{4} + \frac{x\sin 2ax}{4a} + \frac{\cos 2ax}{8a^2}$$

14.379 $$\int \cos^3 ax\,dx = \frac{\sin ax}{a} - \frac{\sin^3 ax}{3a}$$

14.380 $$\int \cos^4 ax\,dx = \frac{3x}{8} + \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$$

14.381 $$\int \frac{dx}{\cos^2 ax} = \frac{\tan ax}{a}$$

14.382 $$\int \frac{dx}{\cos^3 ax} = \frac{\sin ax}{2a\cos^2 ax} + \frac{1}{2a}\ln\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

14.383 $$\int \cos ax\cos px\,dx = \frac{\sin(a-p)x}{2(a-p)} + \frac{\sin(a+p)x}{2(a+p)}$$ [If $a = \pm p$, see 14.377.]

14.384 $$\int \frac{dx}{1-\cos ax} = -\frac{1}{a}\cot\frac{ax}{2}$$

14.385 $$\int \frac{x\,dx}{1-\cos ax} = -\frac{x}{a}\cot\frac{ax}{2} + \frac{2}{a^2}\ln\sin\frac{ax}{2}$$

14.386 $$\int \frac{dx}{1+\cos ax} = \frac{1}{a}\tan\frac{ax}{2}$$

14.387 $$\int \frac{x\,dx}{1+\cos ax} = \frac{x}{a}\tan\frac{ax}{2} + \frac{2}{a^2}\ln\cos\frac{ax}{2}$$

14.388 $$\int \frac{dx}{(1-\cos ax)^2} = -\frac{1}{2a}\cot\frac{ax}{2} - \frac{1}{6a}\cot^3\frac{ax}{2}$$

14.389 $$\int \frac{dx}{(1+\cos ax)^2} = \frac{1}{2a}\tan\frac{ax}{2} + \frac{1}{6a}\tan^3\frac{ax}{2}$$

14.390 $$\int \frac{dx}{p+q\cos ax} = \begin{cases} \dfrac{2}{a\sqrt{p^2-q^2}}\tan^{-1}\sqrt{(p-q)/(p+q)}\,\tan\tfrac{1}{2}ax \\ \dfrac{1}{a\sqrt{q^2-p^2}}\ln\left(\dfrac{\tan\tfrac{1}{2}ax+\sqrt{(q+p)/(q-p)}}{\tan\tfrac{1}{2}ax-\sqrt{(q+p)/(q-p)}}\right) \end{cases}$$ [If $p = \pm q$ see 14.384 and 14.386.]

14.391 $$\int \frac{dx}{(p+q\cos ax)^2} = \frac{q\sin ax}{a(q^2-p^2)(p+q\cos ax)} - \frac{p}{q^2-p^2}\int \frac{dx}{p+q\cos ax}$$ [If $p = \pm q$ see 14.388 and 14.389.]

14.392 $$\int \frac{dx}{p^2+q^2\cos^2 ax} = \frac{1}{ap\sqrt{p^2+q^2}}\tan^{-1}\frac{p\tan ax}{\sqrt{p^2+q^2}}$$

14.393 $$\int \frac{dx}{p^2-q^2\cos^2 ax} = \begin{cases} \dfrac{1}{ap\sqrt{p^2-q^2}}\tan^{-1}\dfrac{p\tan ax}{\sqrt{p^2-q^2}} \\ \dfrac{1}{2ap\sqrt{q^2-p^2}}\ln\left(\dfrac{p\tan ax-\sqrt{q^2-p^2}}{p\tan ax+\sqrt{q^2-p^2}}\right) \end{cases}$$

14.394 $$\int x^m\cos ax\,dx = \frac{x^m\sin ax}{a} + \frac{mx^{m-1}}{a^2}\cos ax - \frac{m(m-1)}{a^2}\int x^{m-2}\cos ax\,dx$$

14.395 $$\int \frac{\cos ax}{x^n}\,dx = -\frac{\cos ax}{(n-1)x^{n-1}} - \frac{a}{n-1}\int \frac{\sin ax}{x^{n-1}}\,dx$$ [See 14.365]

14.396 $$\int \cos^n ax\,dx = \frac{\sin ax\cos^{n-1}ax}{an} + \frac{n-1}{n}\int \cos^{n-2}ax\,dx$$

14.397 $$\int \frac{dx}{\cos^n ax} = \frac{\sin ax}{a(n-1)\cos^{n-1}ax} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2}ax}$$

14.398 $$\int \frac{x\,dx}{\cos^n ax} = \frac{x\sin ax}{a(n-1)\cos^{n-1}ax} - \frac{1}{a^2(n-1)(n-2)\cos^{n-2}ax} + \frac{n-2}{n-1}\int \frac{x\,dx}{\cos^{n-2}ax}$$

INTEGRALS INVOLVING $\sin ax$ AND $\cos ax$

14.399 $$\int \sin ax\cos ax\,dx = \frac{\sin^2 ax}{2a}$$

14.400 $$\int \sin px\cos qx\,dx = -\frac{\cos(p-q)x}{2(p-q)} - \frac{\cos(p+q)x}{2(p+q)}$$

14.401 $$\int \sin^n ax\cos ax\,dx = \frac{\sin^{n+1}ax}{(n+1)a}$$ [If $n = -1$, see 14.440.]

14.402 $$\int \cos^n ax\sin ax\,dx = -\frac{\cos^{n+1}ax}{(n+1)a}$$ [If $n = -1$, see 14.429.]

14.403 $$\int \sin^2 ax\cos^2 ax\,dx = \frac{x}{8} - \frac{\sin 4ax}{32a}$$

14.404 $$\int \frac{dx}{\sin ax\cos ax} = \frac{1}{a}\ln\tan ax$$

14.405 $$\int \frac{dx}{\sin^2 ax\cos ax} = \frac{1}{a}\ln\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right) - \frac{1}{a\sin ax}$$

14.406 $$\int \frac{dx}{\sin ax\cos^2 ax} = \frac{1}{a}\ln\tan\frac{ax}{2} + \frac{1}{a\cos ax}$$

14.407 $$\int \frac{dx}{\sin^2 ax\cos^2 ax} = -\frac{2\cot 2ax}{a}$$

14.408 $$\int \frac{\sin^2 ax}{\cos ax}\,dx = -\frac{\sin ax}{a} + \frac{1}{a}\ln\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right)$$

14.409 $$\int \frac{\cos^2 ax}{\sin ax}\,dx = \frac{\cos ax}{a} + \frac{1}{a}\ln\tan\frac{ax}{2}$$

14.410 $$\int \frac{dx}{\cos ax(1 \pm \sin ax)} = \mp\frac{1}{2a(1 \pm \sin ax)} + \frac{1}{2a}\ln\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right)$$

14.411 $$\int \frac{dx}{\sin ax(1 \pm \cos ax)} = \pm\frac{1}{2a(1 \pm \cos ax)} + \frac{1}{2a}\ln\tan\frac{ax}{2}$$

14.412 $$\int \frac{dx}{\sin ax \pm \cos ax} = \frac{1}{a\sqrt{2}}\ln\tan\left(\frac{ax}{2} \pm \frac{\pi}{8}\right)$$

14.413 $$\int \frac{\sin ax\,dx}{\sin ax \pm \cos ax} = \frac{x}{2} \mp \frac{1}{2a}\ln(\sin ax \pm \cos ax)$$

14.414 $$\int \frac{\cos ax\,dx}{\sin ax \pm \cos ax} = \pm\frac{x}{2} + \frac{1}{2a}\ln(\sin ax \pm \cos ax)$$

14.415 $$\int \frac{\sin ax\,dx}{p + q\cos ax} = -\frac{1}{aq}\ln(p + q\cos ax)$$

14.416 $$\int \frac{\cos ax\,dx}{p + q\sin ax} = \frac{1}{aq}\ln(p + q\sin ax)$$

14.417 $$\int \frac{\sin ax\,dx}{(p + q\cos ax)^n} = \frac{1}{aq(n-1)(p+q\cos ax)^{n-1}}$$

14.418 $$\int \frac{\cos ax\,dx}{(p + q\sin ax)^n} = \frac{-1}{aq(n-1)(p+q\sin ax)^{n-1}}$$

14.419 $$\int \frac{dx}{p\sin ax + q\cos ax} = \frac{1}{a\sqrt{p^2+q^2}}\ln\tan\left(\frac{ax + \tan^{-1}(q/p)}{2}\right)$$

14.420 $$\int \frac{dx}{p\sin ax + q\cos ax + r} = \begin{cases} \dfrac{2}{a\sqrt{r^2-p^2-q^2}}\tan^{-1}\left(\dfrac{p+(r-q)\tan(ax/2)}{\sqrt{r^2-p^2-q^2}}\right) \\[2ex] \dfrac{1}{a\sqrt{p^2+q^2-r^2}}\ln\left(\dfrac{p-\sqrt{p^2+q^2-r^2}+(r-q)\tan(ax/2)}{p+\sqrt{p^2+q^2-r^2}+(r-q)\tan(ax/2)}\right) \end{cases}$$

If $r = q$ see 14.421. If $r^2 = p^2 + q^2$ see 14.422.

14.421 $$\int \frac{dx}{p\sin ax + q(1+\cos ax)} = \frac{1}{ap}\ln\left(q + p\tan\frac{ax}{2}\right)$$

14.422 $$\int \frac{dx}{p\sin ax + q\cos ax \pm \sqrt{p^2+q^2}} = \frac{-1}{a\sqrt{p^2+q^2}}\tan\left(\frac{\pi}{4} \mp \frac{ax + \tan^{-1}(q/p)}{2}\right)$$

14.423 $$\int \frac{dx}{p^2\sin^2 ax + q^2\cos^2 ax} = \frac{1}{apq}\tan^{-1}\left(\frac{p\tan ax}{q}\right)$$

14.424 $$\int \frac{dx}{p^2\sin^2 ax - q^2\cos^2 ax} = \frac{1}{2apq}\ln\left(\frac{p\tan ax - q}{p\tan ax + q}\right)$$

14.425 $$\int \sin^m ax\cos^n ax\,dx = \begin{cases} -\dfrac{\sin^{m-1} ax\cos^{n+1} ax}{a(m+n)} + \dfrac{m-1}{m+n}\displaystyle\int \sin^{m-2} ax\cos^n ax\,dx \\[2ex] \dfrac{\sin^{m+1} ax\cos^{n-1} ax}{a(m+n)} + \dfrac{n-1}{m+n}\displaystyle\int \sin^m ax\cos^{n-2} ax\,dx \end{cases}$$

14.426 $$\int \frac{\sin^m ax}{\cos^n ax}\,dx = \begin{cases} \dfrac{\sin^{m-1} ax}{a(n-1)\cos^{n-1} ax} - \dfrac{m-1}{n-1}\displaystyle\int \frac{\sin^{m-2} ax}{\cos^{n-2} ax}\,dx \\ \dfrac{\sin^{m+1} ax}{a(n-1)\cos^{n-1} ax} - \dfrac{m-n+2}{n-1}\displaystyle\int \frac{\sin^m ax}{\cos^{n-2} ax}\,dx \\ \dfrac{-\sin^{m-1} ax}{a(m-n)\cos^{n-1} ax} + \dfrac{m-1}{m-n}\displaystyle\int \frac{\sin^{m-2} ax}{\cos^n ax}\,dx \end{cases}$$

14.427 $$\int \frac{\cos^m ax}{\sin^n ax}\,dx = \begin{cases} \dfrac{-\cos^{m-1} ax}{a(n-1)\sin^{n-1} ax} - \dfrac{m-1}{n-1}\displaystyle\int \frac{\cos^{m-2} ax}{\sin^{n-2} ax}\,dx \\ \dfrac{-\cos^{m+1} ax}{a(n-1)\sin^{n-1} ax} - \dfrac{m-n+2}{n-1}\displaystyle\int \frac{\cos^m ax}{\sin^{n-2} ax}\,dx \\ \dfrac{\cos^{m-1} ax}{a(m-n)\sin^{n-1} ax} + \dfrac{m-1}{m-n}\displaystyle\int \frac{\cos^{m-2} ax}{\sin^n ax}\,dx \end{cases}$$

14.428 $$\int \frac{dx}{\sin^m ax \cos^n ax} = \begin{cases} \dfrac{1}{a(n-1)\sin^{m-1} ax \cos^{n-1} ax} + \dfrac{m+n-2}{n-1}\displaystyle\int \frac{dx}{\sin^m ax \cos^{n-2} ax} \\ \dfrac{-1}{a(m-1)\sin^{m-1} ax \cos^{n-1} ax} + \dfrac{m+n-2}{m-1}\displaystyle\int \frac{dx}{\sin^{m-2} ax \cos^n ax} \end{cases}$$

INTEGRALS INVOLVING $\tan ax$

14.429 $$\int \tan ax\,dx = -\frac{1}{a}\ln\cos ax = \frac{1}{a}\ln\sec ax$$

14.430 $$\int \tan^2 ax\,dx = \frac{\tan ax}{a} - x$$

14.431 $$\int \tan^3 ax\,dx = \frac{\tan^2 ax}{2a} + \frac{1}{a}\ln\cos ax$$

14.432 $$\int \tan^n ax \sec^2 ax\,dx = \frac{\tan^{n+1} ax}{(n+1)a}$$

14.433 $$\int \frac{\sec^2 ax}{\tan ax}\,dx = \frac{1}{a}\ln\tan ax$$

14.434 $$\int \frac{dx}{\tan ax} = \frac{1}{a}\ln\sin ax$$

14.435 $$\int x\tan ax\,dx = \frac{1}{a^2}\left\{\frac{(ax)^3}{3} + \frac{(ax)^5}{15} + \frac{2(ax)^7}{105} + \cdots + \frac{2^{2n}(2^{2n}-1)B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$$

14.436 $$\int \frac{\tan ax}{x}\,dx = ax + \frac{(ax)^3}{9} + \frac{2(ax)^5}{75} + \cdots + \frac{2^{2n}(2^{2n}-1)B_n(ax)^{2n-1}}{(2n-1)(2n)!} + \cdots$$

14.437 $$\int x\tan^2 ax\,dx = \frac{x\tan ax}{a} + \frac{1}{a^2}\ln\cos ax - \frac{x^2}{2}$$

14.438 $$\int \frac{dx}{p+q\tan ax} = \frac{px}{p^2+q^2} + \frac{q}{a(p^2+q^2)}\ln(q\sin ax + p\cos ax)$$

14.439 $$\int \tan^n ax\,dx = \frac{\tan^{n-1} ax}{(n-1)a} - \int \tan^{n-2} ax\,dx$$

INTEGRALS INVOLVING $\cot ax$

14.440 $\displaystyle\int \cot ax\,dx = \frac{1}{a}\ln\sin ax$

14.441 $\displaystyle\int \cot^2 ax\,dx = -\frac{\cot ax}{a} - x$

14.442 $\displaystyle\int \cot^3 ax\,dx = -\frac{\cot^2 ax}{2a} - \frac{1}{a}\ln\sin ax$

14.443 $\displaystyle\int \cot^n ax\csc^2 ax\,dx = -\frac{\cot^{n+1} ax}{(n+1)a}$

14.444 $\displaystyle\int \frac{\csc^2 ax}{\cot ax}\,dx = -\frac{1}{a}\ln\cot ax$

14.445 $\displaystyle\int \frac{dx}{\cot ax} = -\frac{1}{a}\ln\cos ax$

14.446 $\displaystyle\int x\cot ax\,dx = \frac{1}{a^2}\left\{ax - \frac{(ax)^3}{9} - \frac{(ax)^5}{225} - \cdots - \frac{2^{2n}B_n(ax)^{2n+1}}{(2n+1)!} - \cdots\right\}$

14.447 $\displaystyle\int \frac{\cot ax}{x}\,dx = -\frac{1}{ax} - \frac{ax}{3} - \frac{(ax)^3}{135} - \cdots - \frac{2^{2n}B_n(ax)^{2n-1}}{(2n-1)(2n)!} - \cdots$

14.448 $\displaystyle\int x\cot^2 ax\,dx = -\frac{x\cot ax}{a} + \frac{1}{a^2}\ln\sin ax - \frac{x^2}{2}$

14.449 $\displaystyle\int \frac{dx}{p+q\cot ax} = \frac{px}{p^2+q^2} - \frac{q}{a(p^2+q^2)}\ln(p\sin ax + q\cos ax)$

14.450 $\displaystyle\int \cot^n ax\,dx = -\frac{\cot^{n-1} ax}{(n-1)a} - \int \cot^{n-2} ax\,dx$

INTEGRALS INVOLVING $\sec ax$

14.451 $\displaystyle\int \sec ax\,dx = \frac{1}{a}\ln(\sec ax + \tan ax) = \frac{1}{a}\ln\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right)$

14.452 $\displaystyle\int \sec^2 ax\,dx = \frac{\tan ax}{a}$

14.453 $\displaystyle\int \sec^3 ax\,dx = \frac{\sec ax\tan ax}{2a} + \frac{1}{2a}\ln(\sec ax + \tan ax)$

14.454 $\displaystyle\int \sec^n ax\tan ax\,dx = \frac{\sec^n ax}{na}$

14.455 $\displaystyle\int \frac{dx}{\sec ax} = \frac{\sin ax}{a}$

14.456 $\displaystyle\int x\sec ax\,dx = \frac{1}{a^2}\left\{\frac{(ax)^2}{2} + \frac{(ax)^4}{8} + \frac{5(ax)^6}{144} + \cdots + \frac{E_n(ax)^{2n+2}}{(2n+2)(2n)!} + \cdots\right\}$

14.457 $\displaystyle\int \frac{\sec ax}{x}\,dx = \ln x + \frac{(ax)^2}{4} + \frac{5(ax)^4}{96} + \frac{61(ax)^6}{4320} + \cdots + \frac{E_n(ax)^{2n}}{2n(2n)!} + \cdots$

14.458 $\displaystyle\int x\sec^2 ax\,dx = \frac{x}{a}\tan ax + \frac{1}{a^2}\ln\cos ax$

14.459 $$\int \frac{dx}{q + p \sec ax} = \frac{x}{q} - \frac{p}{q}\int \frac{dx}{p + q \cos ax}$$

14.460 $$\int \sec^n ax\, dx = \frac{\sec^{n-2} ax \tan ax}{a(n-1)} + \frac{n-2}{n-1}\int \sec^{n-2} ax\, dx$$

INTEGRALS INVOLVING $\csc ax$

14.461 $$\int \csc ax\, dx = \frac{1}{a}\ln(\csc ax - \cot ax) = \frac{1}{a}\ln \tan \frac{ax}{2}$$

14.462 $$\int \csc^2 ax\, dx = -\frac{\cot ax}{a}$$

14.463 $$\int \csc^3 ax\, dx = -\frac{\csc ax \cot ax}{2a} + \frac{1}{2a}\ln \tan \frac{ax}{2}$$

14.464 $$\int \csc^n ax \cot ax\, dx = -\frac{\csc^n ax}{na}$$

14.465 $$\int \frac{dx}{\csc ax} = -\frac{\cos ax}{a}$$

14.466 $$\int x \csc ax\, dx = \frac{1}{a^2}\left\{ax + \frac{(ax)^3}{18} + \frac{7(ax)^5}{1800} + \cdots + \frac{2(2^{2n-1}-1)B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$$

14.467 $$\int \frac{\csc ax}{x}\, dx = -\frac{1}{ax} + \frac{ax}{6} + \frac{7(ax)^3}{1080} + \cdots + \frac{2(2^{2n-1}-1)B_n(ax)^{2n-1}}{(2n-1)(2n)!} + \cdots$$

14.468 $$\int x \csc^2 ax\, dx = -\frac{x \cot ax}{a} + \frac{1}{a^2}\ln \sin ax$$

14.469 $$\int \frac{dx}{q + p \csc ax} = \frac{x}{q} - \frac{p}{q}\int \frac{dx}{p + q \sin ax}$$ [See 14.360]

14.470 $$\int \csc^n ax\, dx = -\frac{\csc^{n-2} ax \cot ax}{a(n-1)} + \frac{n-2}{n-1}\int \csc^{n-2} ax\, dx$$

INTEGRALS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

14.471 $$\int \sin^{-1}\frac{x}{a}\, dx = x \sin^{-1}\frac{x}{a} + \sqrt{a^2 - x^2}$$

14.472 $$\int x \sin^{-1}\frac{x}{a}\, dx = \left(\frac{x^2}{2} - \frac{a^2}{4}\right)\sin^{-1}\frac{x}{a} + \frac{x\sqrt{a^2 - x^2}}{4}$$

14.473 $$\int x^2 \sin^{-1}\frac{x}{a}\, dx = \frac{x^3}{3}\sin^{-1}\frac{x}{a} + \frac{(x^2 + 2a^2)\sqrt{a^2 - x^2}}{9}$$

14.474 $$\int \frac{\sin^{-1}(x/a)}{x}\, dx = \frac{x}{a} + \frac{(x/a)^3}{2 \cdot 3 \cdot 3} + \frac{1 \cdot 3 (x/a)^5}{2 \cdot 4 \cdot 5 \cdot 5} + \frac{1 \cdot 3 \cdot 5 (x/a)^7}{2 \cdot 4 \cdot 6 \cdot 7 \cdot 7} + \cdots$$

14.475 $$\int \frac{\sin^{-1}(x/a)}{x^2}\, dx = -\frac{\sin^{-1}(x/a)}{x} - \frac{1}{a}\ln\left(\frac{a + \sqrt{a^2 - x^2}}{x}\right)$$

14.476 $$\int \left(\sin^{-1}\frac{x}{a}\right)^2 dx = x\left(\sin^{-1}\frac{x}{a}\right)^2 - 2x + 2\sqrt{a^2 - x^2}\sin^{-1}\frac{x}{a}$$

14.477 $$\int \cos^{-1}\frac{x}{a}\,dx = x\cos^{-1}\frac{x}{a} - \sqrt{a^2-x^2}$$

14.478 $$\int x\cos^{-1}\frac{x}{a}\,dx = \left(\frac{x^2}{2}-\frac{a^2}{4}\right)\cos^{-1}\frac{x}{a} - \frac{x\sqrt{a^2-x^2}}{4}$$

14.479 $$\int x^2\cos^{-1}\frac{x}{a}\,dx = \frac{x^3}{3}\cos^{-1}\frac{x}{a} - \frac{(x^2+2a^2)\sqrt{a^2-x^2}}{9}$$

14.480 $$\int \frac{\cos^{-1}(x/a)}{x}\,dx = \frac{\pi}{2}\ln x - \int\frac{\sin^{-1}(x/a)}{x}\,dx \qquad \text{[See 14.474]}$$

14.481 $$\int \frac{\cos^{-1}(x/a)}{x^2}\,dx = -\frac{\cos^{-1}(x/a)}{x} + \frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$$

14.482 $$\int\left(\cos^{-1}\frac{x}{a}\right)^2 dx = x\left(\cos^{-1}\frac{x}{a}\right)^2 - 2x - 2\sqrt{a^2-x^2}\cos^{-1}\frac{x}{a}$$

14.483 $$\int \tan^{-1}\frac{x}{a}\,dx = x\tan^{-1}\frac{x}{a} - \frac{a}{2}\ln(x^2+a^2)$$

14.484 $$\int x\tan^{-1}\frac{x}{a}\,dx = \tfrac{1}{2}(x^2+a^2)\tan^{-1}\frac{x}{a} - \frac{ax}{2}$$

14.485 $$\int x^2\tan^{-1}\frac{x}{a}\,dx = \frac{x^3}{3}\tan^{-1}\frac{x}{a} - \frac{ax^2}{6} + \frac{a^3}{6}\ln(x^2+a^2)$$

14.486 $$\int \frac{\tan^{-1}(x/a)}{x}\,dx = \frac{x}{a} - \frac{(x/a)^3}{3^2} + \frac{(x/a)^5}{5^2} - \frac{(x/a)^7}{7^2} + \cdots$$

14.487 $$\int \frac{\tan^{-1}(x/a)}{x^2}\,dx = -\frac{1}{x}\tan^{-1}\frac{x}{a} - \frac{1}{2a}\ln\left(\frac{x^2+a^2}{x^2}\right)$$

14.488 $$\int \cot^{-1}\frac{x}{a}\,dx = x\cot^{-1}\frac{x}{a} + \frac{a}{2}\ln(x^2+a^2)$$

14.489 $$\int x\cot^{-1}\frac{x}{a}\,dx = \tfrac{1}{2}(x^2+a^2)\cot^{-1}\frac{x}{a} + \frac{ax}{2}$$

14.490 $$\int x^2\cot^{-1}\frac{x}{a}\,dx = \frac{x^3}{3}\cot^{-1}\frac{x}{a} + \frac{ax^2}{6} - \frac{a^3}{6}\ln(x^2+a^2)$$

14.491 $$\int \frac{\cot^{-1}(x/a)}{x}\,dx = \frac{\pi}{2}\ln x - \int\frac{\tan^{-1}(x/a)}{x}\,dx \qquad \text{[See 14.486]}$$

14.492 $$\int \frac{\cot^{-1}(x/a)}{x^2}\,dx = -\frac{\cot^{-1}(x/a)}{x} + \frac{1}{2a}\ln\left(\frac{x^2+a^2}{x^2}\right)$$

14.493 $$\int \sec^{-1}\frac{x}{a}\,dx = \begin{cases} x\sec^{-1}\dfrac{x}{a} - a\ln(x+\sqrt{x^2-a^2}) & 0<\sec^{-1}\dfrac{x}{a}<\dfrac{\pi}{2} \\[2ex] x\sec^{-1}\dfrac{x}{a} + a\ln(x+\sqrt{x^2-a^2}) & \dfrac{\pi}{2}<\sec^{-1}\dfrac{x}{a}<\pi \end{cases}$$

14.494 $$\int x\sec^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^2}{2}\sec^{-1}\dfrac{x}{a} - \dfrac{a\sqrt{x^2-a^2}}{2} & 0<\sec^{-1}\dfrac{x}{a}<\dfrac{\pi}{2} \\[2ex] \dfrac{x^2}{2}\sec^{-1}\dfrac{x}{a} + \dfrac{a\sqrt{x^2-a^2}}{2} & \dfrac{\pi}{2}<\sec^{-1}\dfrac{x}{a}<\pi \end{cases}$$

14.495 $$\int x^2\sec^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^3}{3}\sec^{-1}\dfrac{x}{a} - \dfrac{ax\sqrt{x^2-a^2}}{6} - \dfrac{a^3}{6}\ln(x+\sqrt{x^2-a^2}) & 0<\sec^{-1}\dfrac{x}{a}<\dfrac{\pi}{2} \\[2ex] \dfrac{x^3}{3}\sec^{-1}\dfrac{x}{a} + \dfrac{ax\sqrt{x^2-a^2}}{6} + \dfrac{a^3}{6}\ln(x+\sqrt{x^2-a^2}) & \dfrac{\pi}{2}<\sec^{-1}\dfrac{x}{a}<\pi \end{cases}$$

14.496 $$\int \frac{\sec^{-1}(x/a)}{x}\,dx = \frac{\pi}{2}\ln x + \frac{a}{x} + \frac{(a/x)^3}{2\cdot 3\cdot 3} + \frac{1\cdot 3(a/x)^5}{2\cdot 4\cdot 5\cdot 5} + \frac{1\cdot 3\cdot 5(a/x)^7}{2\cdot 4\cdot 6\cdot 7\cdot 7} + \cdots$$

14.497 $$\int \frac{\sec^{-1}(x/a)}{x^2}\,dx = \begin{cases} -\dfrac{\sec^{-1}(x/a)}{x} + \dfrac{\sqrt{x^2-a^2}}{ax} & 0 < \sec^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\ -\dfrac{\sec^{-1}(x/a)}{x} - \dfrac{\sqrt{x^2-a^2}}{ax} & \dfrac{\pi}{2} < \sec^{-1}\dfrac{x}{a} < \pi \end{cases}$$

14.498 $$\int \csc^{-1}\frac{x}{a}\,dx = \begin{cases} x\csc^{-1}\dfrac{x}{a} + a\ln(x+\sqrt{x^2-a^2}) & 0 < \csc^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\ x\csc^{-1}\dfrac{x}{a} - a\ln(x+\sqrt{x^2-a^2}) & -\dfrac{\pi}{2} < \csc^{-1}\dfrac{x}{a} < 0 \end{cases}$$

14.499 $$\int x\csc^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^2}{2}\csc^{-1}\dfrac{x}{a} + \dfrac{a\sqrt{x^2-a^2}}{2} & 0 < \csc^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\ \dfrac{x^2}{2}\csc^{-1}\dfrac{x}{a} - \dfrac{a\sqrt{x^2-a^2}}{2} & -\dfrac{\pi}{2} < \csc^{-1}\dfrac{x}{a} < 0 \end{cases}$$

14.500 $$\int x^2\csc^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^3}{3}\csc^{-1}\dfrac{x}{a} + \dfrac{ax\sqrt{x^2-a^2}}{6} + \dfrac{a^3}{6}\ln(x+\sqrt{x^2-a^2}) & 0 < \csc^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\ \dfrac{x^3}{3}\csc^{-1}\dfrac{x}{a} - \dfrac{ax\sqrt{x^2-a^2}}{6} - \dfrac{a^3}{6}\ln(x+\sqrt{x^2-a^2}) & -\dfrac{\pi}{2} < \csc^{-1}\dfrac{x}{a} < 0 \end{cases}$$

14.501 $$\int \frac{\csc^{-1}(x/a)}{x}\,dx = -\left(\frac{a}{x} + \frac{(a/x)^3}{2\cdot 3\cdot 3} + \frac{1\cdot 3(a/x)^5}{2\cdot 4\cdot 5\cdot 5} + \frac{1\cdot 3\cdot 5(a/x)^7}{2\cdot 4\cdot 6\cdot 7\cdot 7} + \cdots\right)$$

14.502 $$\int \frac{\csc^{-1}(x/a)}{x^2}\,dx = \begin{cases} -\dfrac{\csc^{-1}(x/a)}{x} - \dfrac{\sqrt{x^2-a^2}}{ax} & 0 < \csc^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\ -\dfrac{\csc^{-1}(x/a)}{x} + \dfrac{\sqrt{x^2-a^2}}{ax} & -\dfrac{\pi}{2} < \csc^{-1}\dfrac{x}{a} < 0 \end{cases}$$

14.503 $$\int x^m\sin^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\sin^{-1}\frac{x}{a} - \frac{1}{m+1}\int \frac{x^{m+1}}{\sqrt{a^2-x^2}}\,dx$$

14.504 $$\int x^m\cos^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\cos^{-1}\frac{x}{a} + \frac{1}{m+1}\int \frac{x^{m+1}}{\sqrt{a^2-x^2}}\,dx$$

14.505 $$\int x^m\tan^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\tan^{-1}\frac{x}{a} - \frac{a}{m+1}\int \frac{x^{m+1}}{x^2+a^2}\,dx$$

14.506 $$\int x^m\cot^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\cot^{-1}\frac{x}{a} + \frac{a}{m+1}\int \frac{x^{m+1}}{x^2+a^2}\,dx$$

14.507 $$\int x^m\sec^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^{m+1}\sec^{-1}(x/a)}{m+1} - \dfrac{a}{m+1}\displaystyle\int \frac{x^m\,dx}{\sqrt{x^2-a^2}} & 0 < \sec^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\ \dfrac{x^{m+1}\sec^{-1}(x/a)}{m+1} + \dfrac{a}{m+1}\displaystyle\int \frac{x^m\,dx}{\sqrt{x^2-a^2}} & \dfrac{\pi}{2} < \sec^{-1}\dfrac{x}{a} < \pi \end{cases}$$

14.508 $$\int x^m\csc^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^{m+1}\csc^{-1}(x/a)}{m+1} + \dfrac{a}{m+1}\displaystyle\int \frac{x^m\,dx}{\sqrt{x^2-a^2}} & 0 < \csc^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\ \dfrac{x^{m+1}\csc^{-1}(x/a)}{m+1} - \dfrac{a}{m+1}\displaystyle\int \frac{x^m\,dx}{\sqrt{x^2-a^2}} & -\dfrac{\pi}{2} < \csc^{-1}\dfrac{x}{a} < 0 \end{cases}$$

INTEGRALS INVOLVING e^{ax}

14.509 $$\int e^{ax}\,dx = \frac{e^{ax}}{a}$$

14.510 $$\int xe^{ax}\,dx = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right)$$

14.511 $$\int x^2e^{ax}\,dx = \frac{e^{ax}}{a}\left(x^2 - \frac{2x}{a} + \frac{2}{a^2}\right)$$

14.512 $$\int x^ne^{ax}\,dx = \frac{x^ne^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$$
$$= \frac{e^{ax}}{a}\left(x^n - \frac{nx^{n-1}}{a} + \frac{n(n-1)x^{n-2}}{a^2} - \cdots \frac{(-1)^n n!}{a^n}\right) \quad \text{if } n = \text{positive integer}$$

14.513 $$\int \frac{e^{ax}}{x}\,dx = \ln x + \frac{ax}{1\cdot 1!} + \frac{(ax)^2}{2\cdot 2!} + \frac{(ax)^3}{3\cdot 3!} + \cdots$$

14.514 $$\int \frac{e^{ax}}{x^n}\,dx = \frac{-e^{ax}}{(n-1)x^{n-1}} + \frac{a}{n-1}\int \frac{e^{ax}}{x^{n-1}}\,dx$$

14.515 $$\int \frac{dx}{p + qe^{ax}} = \frac{x}{p} - \frac{1}{ap}\ln(p + qe^{ax})$$

14.516 $$\int \frac{dx}{(p + qe^{ax})^2} = \frac{x}{p^2} + \frac{1}{ap(p + qe^{ax})} - \frac{1}{ap^2}\ln(p + qe^{ax})$$

14.517 $$\int \frac{dx}{pe^{ax} + qe^{-ax}} = \begin{cases} \dfrac{1}{a\sqrt{pq}}\tan^{-1}\left(\sqrt{\dfrac{p}{q}}\,e^{ax}\right) \\ \dfrac{1}{2a\sqrt{-pq}}\ln\left(\dfrac{e^{ax} - \sqrt{-q/p}}{e^{ax} + \sqrt{-q/p}}\right) \end{cases}$$

14.518 $$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2 + b^2}$$

14.519 $$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2 + b^2}$$

14.520 $$\int xe^{ax}\sin bx\,dx = \frac{xe^{ax}(a\sin bx - b\cos bx)}{a^2 + b^2} - \frac{e^{ax}\{(a^2 - b^2)\sin bx - 2ab\cos bx\}}{(a^2 + b^2)^2}$$

14.521 $$\int xe^{ax}\cos bx\,dx = \frac{xe^{ax}(a\cos bx + b\sin bx)}{a^2 + b^2} - \frac{e^{ax}\{(a^2 - b^2)\cos bx + 2ab\sin bx\}}{(a^2 + b^2)^2}$$

14.522 $$\int e^{ax}\ln x\,dx = \frac{e^{ax}\ln x}{a} - \frac{1}{a}\int \frac{e^{ax}}{x}\,dx$$

14.523 $$\int e^{ax}\sin^n bx\,dx = \frac{e^{ax}\sin^{n-1}bx}{a^2 + n^2b^2}(a\sin bx - nb\cos bx) + \frac{n(n-1)b^2}{a^2 + n^2b^2}\int e^{ax}\sin^{n-2}bx\,dx$$

14.524 $$\int e^{ax}\cos^n bx\,dx = \frac{e^{ax}\cos^{n-1}bx}{a^2 + n^2b^2}(a\cos bx + nb\sin bx) + \frac{n(n-1)b^2}{a^2 + n^2b^2}\int e^{ax}\cos^{n-2}bx\,dx$$

INTEGRALS INVOLVING $\ln x$

14.525 $\int \ln x\,dx = x\ln x - x$

14.526 $\int x\ln x\,dx = \frac{x^2}{2}(\ln x - \tfrac{1}{2})$

14.527 $\int x^m \ln x\,dx = \frac{x^{m+1}}{m+1}\left(\ln x - \frac{1}{m+1}\right)$ [If $m=-1$ see 14.528.]

14.528 $\int \frac{\ln x}{x}\,dx = \frac{1}{2}\ln^2 x$

14.529 $\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x}$

14.530 $\int \ln^2 x\,dx = x\ln^2 x - 2x\ln x + 2x$

14.531 $\int \frac{\ln^n x\,dx}{x} = \frac{\ln^{n+1} x}{n+1}$ [If $n=-1$ see 14.532.]

14.532 $\int \frac{dx}{x\ln x} = \ln(\ln x)$

14.533 $\int \frac{dx}{\ln x} = \ln(\ln x) + \ln x + \frac{\ln^2 x}{2\cdot 2!} + \frac{\ln^3 x}{3\cdot 3!} + \cdots$

14.534 $\int \frac{x^m\,dx}{\ln x} = \ln(\ln x) + (m+1)\ln x + \frac{(m+1)^2\ln^2 x}{2\cdot 2!} + \frac{(m+1)^3\ln^3 x}{3\cdot 3!} + \cdots$

14.535 $\int \ln^n x\,dx = x\ln^n x - n\int \ln^{n-1} x\,dx$

14.536 $\int x^m \ln^n x\,dx = \frac{x^{m+1}\ln^n x}{m+1} - \frac{n}{m+1}\int x^m \ln^{n-1} x\,dx$

If $m=-1$ see 14.531.

14.537 $\int \ln(x^2+a^2)\,dx = x\ln(x^2+a^2) - 2x + 2a\tan^{-1}\frac{x}{a}$

14.538 $\int \ln(x^2-a^2)\,dx = x\ln(x^2-a^2) - 2x + a\ln\left(\frac{x+a}{x-a}\right)$

14.539 $\int x^m \ln(x^2 \pm a^2)\,dx = \frac{x^{m+1}\ln(x^2\pm a^2)}{m+1} - \frac{2}{m+1}\int \frac{x^{m+2}}{x^2 \pm a^2}\,dx$

INTEGRALS INVOLVING $\sinh ax$

14.540 $\int \sinh ax\,dx = \frac{\cosh ax}{a}$

14.541 $\int x\sinh ax\,dx = \frac{x\cosh ax}{a} - \frac{\sinh ax}{a^2}$

14.542 $\int x^2 \sinh ax\,dx = \left(\frac{x^2}{a} + \frac{2}{a^3}\right)\cosh ax - \frac{2x}{a^2}\sinh ax$

14.543 $$\int \frac{\sinh ax}{x}\,dx = ax + \frac{(ax)^3}{3 \cdot 3!} + \frac{(ax)^5}{5 \cdot 5!} + \cdots$$

14.544 $$\int \frac{\sinh ax}{x^2}\,dx = -\frac{\sinh ax}{x} + a\int \frac{\cosh ax}{x}\,dx \qquad \text{[See 14.565]}$$

14.545 $$\int \frac{dx}{\sinh ax} = \frac{1}{a}\ln\tanh\frac{ax}{2}$$

14.546 $$\int \frac{x\,dx}{\sinh ax} = \frac{1}{a^2}\left\{ax - \frac{(ax)^3}{18} + \frac{7(ax)^5}{1800} - \cdots + \frac{2(-1)^n(2^{2n}-1)B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$$

14.547 $$\int \sinh^2 ax\,dx = \frac{\sinh ax\cosh ax}{2a} - \frac{x}{2}$$

14.548 $$\int x\sinh^2 ax\,dx = \frac{x\sinh 2ax}{4a} - \frac{\cosh 2ax}{8a^2} - \frac{x^2}{4}$$

14.549 $$\int \frac{dx}{\sinh^2 ax} = -\frac{\coth ax}{a}$$

14.550 $$\int \sinh ax\sinh px\,dx = \frac{\sinh(a+p)x}{2(a+p)} - \frac{\sinh(a-p)x}{2(a-p)}$$

For $a = \pm p$ see 14.547.

14.551 $$\int \sinh ax\sin px\,dx = \frac{a\cosh ax\sin px - p\sinh ax\cos px}{a^2+p^2}$$

14.552 $$\int \sinh ax\cos px\,dx = \frac{a\cosh ax\cos px + p\sinh ax\sin px}{a^2+p^2}$$

14.553 $$\int \frac{dx}{p+q\sinh ax} = \frac{1}{a\sqrt{p^2+q^2}}\ln\left(\frac{qe^{ax}+p-\sqrt{p^2+q^2}}{qe^{ax}+p+\sqrt{p^2+q^2}}\right)$$

14.554 $$\int \frac{dx}{(p+q\sinh ax)^2} = \frac{-q\cosh ax}{a(p^2+q^2)(p+q\sinh ax)} + \frac{p}{p^2+q^2}\int \frac{dx}{p+q\sinh ax}$$

14.555 $$\int \frac{dx}{p^2+q^2\sinh^2 ax} = \begin{cases} \dfrac{1}{ap\sqrt{q^2-p^2}}\tan^{-1}\dfrac{\sqrt{q^2-p^2}\tanh ax}{p} \\[2ex] \dfrac{1}{2ap\sqrt{p^2-q^2}}\ln\left(\dfrac{p+\sqrt{p^2-q^2}\tanh ax}{p-\sqrt{p^2-q^2}\tanh ax}\right) \end{cases}$$

14.556 $$\int \frac{dx}{p^2-q^2\sinh^2 ax} = \frac{1}{2ap\sqrt{p^2+q^2}}\ln\left(\frac{p+\sqrt{p^2+q^2}\tanh ax}{p-\sqrt{p^2+q^2}\tanh ax}\right)$$

14.557 $$\int x^m\sinh ax\,dx = \frac{x^m\cosh ax}{a} - \frac{m}{a}\int x^{m-1}\cosh ax\,dx \qquad \text{[See 14.585]}$$

14.558 $$\int \sinh^n ax\,dx = \frac{\sinh^{n-1}ax\cosh ax}{an} - \frac{n-1}{n}\int \sinh^{n-2}ax\,dx$$

14.559 $$\int \frac{\sinh ax}{x^n}\,dx = \frac{-\sinh ax}{(n-1)x^{n-1}} + \frac{a}{n-1}\int \frac{\cosh ax}{x^{n-1}}\,dx \qquad \text{[See 14.587]}$$

14.560 $$\int \frac{dx}{\sinh^n ax} = \frac{-\cosh ax}{a(n-1)\sinh^{n-1}ax} - \frac{n-2}{n-1}\int \frac{dx}{\sinh^{n-2}ax}$$

14.561 $$\int \frac{x\,dx}{\sinh^n ax} = \frac{-x\cosh ax}{a(n-1)\sinh^{n-1}ax} - \frac{1}{a^2(n-1)(n-2)\sinh^{n-2}ax} - \frac{n-2}{n-1}\int \frac{x\,dx}{\sinh^{n-2}ax}$$

INTEGRALS INVOLVING cosh ax

14.562 $$\int \cosh ax\,dx = \frac{\sinh ax}{a}$$

14.563 $$\int x\cosh ax\,dx = \frac{x\sinh ax}{a} - \frac{\cosh ax}{a^2}$$

14.564 $$\int x^2\cosh ax\,dx = -\frac{2x\cosh ax}{a^2} + \left(\frac{x^2}{a} + \frac{2}{a^3}\right)\sinh ax$$

14.565 $$\int \frac{\cosh ax}{x}\,dx = \ln x + \frac{(ax)^2}{2\cdot 2!} + \frac{(ax)^4}{4\cdot 4!} + \frac{(ax)^6}{6\cdot 6!} + \cdots$$

14.566 $$\int \frac{\cosh ax}{x^2}\,dx = -\frac{\cosh ax}{x} + a\int \frac{\sinh ax}{x}\,dx$$ [See 14.543]

14.567 $$\int \frac{dx}{\cosh ax} = \frac{2}{a}\tan^{-1} e^{ax}$$

14.568 $$\int \frac{x\,dx}{\cosh ax} = \frac{1}{a^2}\left\{\frac{(ax)^2}{2} - \frac{(ax)^4}{8} + \frac{5(ax)^6}{144} + \cdots + \frac{(-1)^n E_n (ax)^{2n+2}}{(2n+2)(2n)!} + \cdots\right\}$$

14.569 $$\int \cosh^2 ax\,dx = \frac{x}{2} + \frac{\sinh ax\cosh ax}{2a}$$

14.570 $$\int x\cosh^2 ax\,dx = \frac{x^2}{4} + \frac{x\sinh 2ax}{4a} - \frac{\cosh 2ax}{8a^2}$$

14.571 $$\int \frac{dx}{\cosh^2 ax} = \frac{\tanh ax}{a}$$

14.572 $$\int \cosh ax\cosh px\,dx = \frac{\sinh(a-p)x}{2(a-p)} + \frac{\sinh(a+p)x}{2(a+p)}$$

14.573 $$\int \cosh ax\sin px\,dx = \frac{a\sinh ax\sin px - p\cosh ax\cos px}{a^2+p^2}$$

14.574 $$\int \cosh ax\cos px\,dx = \frac{a\sinh ax\cos px + p\cosh ax\sin px}{a^2+p^2}$$

14.575 $$\int \frac{dx}{\cosh ax + 1} = \frac{1}{a}\tanh\frac{ax}{2}$$

14.576 $$\int \frac{dx}{\cosh ax - 1} = -\frac{1}{a}\coth\frac{ax}{2}$$

14.577 $$\int \frac{x\,dx}{\cosh ax + 1} = \frac{x}{a}\tanh\frac{ax}{2} - \frac{2}{a^2}\ln\cosh\frac{ax}{2}$$

14.578 $$\int \frac{x\,dx}{\cosh ax - 1} = -\frac{x}{a}\coth\frac{ax}{2} + \frac{2}{a^2}\ln\sinh\frac{ax}{2}$$

14.579 $$\int \frac{dx}{(\cosh ax + 1)^2} = \frac{1}{2a}\tanh\frac{ax}{2} - \frac{1}{6a}\tanh^3\frac{ax}{2}$$

14.580 $$\int \frac{dx}{(\cosh ax - 1)^2} = \frac{1}{2a}\coth\frac{ax}{2} - \frac{1}{6a}\coth^3\frac{ax}{2}$$

14.581 $$\int \frac{dx}{p + q\cosh ax} = \begin{cases} \dfrac{2}{a\sqrt{q^2-p^2}}\tan^{-1}\dfrac{qe^{ax}+p}{\sqrt{q^2-p^2}} \\[2ex] \dfrac{1}{a\sqrt{p^2-q^2}}\ln\left(\dfrac{qe^{ax}+p-\sqrt{p^2-q^2}}{qe^{ax}+p+\sqrt{p^2-q^2}}\right) \end{cases}$$

14.582 $$\int \frac{dx}{(p + q\cosh ax)^2} = \frac{q\sinh ax}{a(q^2-p^2)(p+q\cosh ax)} - \frac{p}{q^2-p^2}\int \frac{dx}{p+q\cosh ax}$$

14.583 $$\int \frac{dx}{p^2 - q^2 \cosh^2 ax} = \begin{cases} \dfrac{1}{2ap\sqrt{p^2-q^2}} \ln \left(\dfrac{p \tanh ax + \sqrt{p^2-q^2}}{p \tanh ax - \sqrt{p^2-q^2}} \right) \\ \dfrac{-1}{ap\sqrt{q^2-p^2}} \tan^{-1} \dfrac{p \tanh ax}{\sqrt{q^2-p^2}} \end{cases}$$

14.584 $$\int \frac{dx}{p^2 + q^2 \cosh^2 ax} = \begin{cases} \dfrac{1}{2ap\sqrt{p^2+q^2}} \ln \left(\dfrac{p \tanh ax + \sqrt{p^2+q^2}}{p \tanh ax - \sqrt{p^2+q^2}} \right) \\ \dfrac{1}{ap\sqrt{p^2+q^2}} \tan^{-1} \dfrac{p \tanh ax}{\sqrt{p^2+q^2}} \end{cases}$$

14.585 $$\int x^m \cosh ax\, dx = \frac{x^m \sinh ax}{a} - \frac{m}{a} \int x^{m-1} \sinh ax\, dx$$ [See 14.557]

14.586 $$\int \cosh^n ax\, dx = \frac{\cosh^{n-1} ax \sinh ax}{an} + \frac{n-1}{n} \int \cosh^{n-2} ax\, dx$$

14.587 $$\int \frac{\cosh ax}{x^n}\, dx = \frac{-\cosh ax}{(n-1)x^{n-1}} + \frac{a}{n-1} \int \frac{\sinh ax}{x^{n-1}}\, dx$$ [See 14.559]

14.588 $$\int \frac{dx}{\cosh^n ax} = \frac{\sinh ax}{a(n-1)\cosh^{n-1} ax} + \frac{n-2}{n-1} \int \frac{dx}{\cosh^{n-2} ax}$$

14.589 $$\int \frac{x\, dx}{\cosh^n ax} = \frac{x \sinh ax}{a(n-1)\cosh^{n-1} ax} + \frac{1}{(n-1)(n-2)a^2 \cosh^{n-2} ax} + \frac{n-2}{n-1} \int \frac{x\, dx}{\cosh^{n-2} ax}$$

INTEGRALS INVOLVING sinh ax AND cosh ax

14.590 $$\int \sinh ax \cosh ax\, dx = \frac{\sinh^2 ax}{2a}$$

14.591 $$\int \sinh px \cosh qx\, dx = \frac{\cosh (p+q)x}{2(p+q)} + \frac{\cosh (p-q)x}{2(p-q)}$$

14.592 $$\int \sinh^n ax \cosh ax\, dx = \frac{\sinh^{n+1} ax}{(n+1)a}$$ [If $n = -1$, see 14.615.]

14.593 $$\int \cosh^n ax \sinh ax\, dx = \frac{\cosh^{n+1} ax}{(n+1)a}$$ [If $n = -1$, see 14.604.]

14.594 $$\int \sinh^2 ax \cosh^2 ax\, dx = \frac{\sinh 4ax}{32a} - \frac{x}{8}$$

14.595 $$\int \frac{dx}{\sinh ax \cosh ax} = \frac{1}{a} \ln \tanh ax$$

14.596 $$\int \frac{dx}{\sinh^2 ax \cosh ax} = -\frac{1}{a} \tan^{-1} \sinh ax - \frac{\operatorname{csch} ax}{a}$$

14.597 $$\int \frac{dx}{\sinh ax \cosh^2 ax} = \frac{\operatorname{sech} ax}{a} + \frac{1}{a} \ln \tanh \frac{ax}{2}$$

14.598 $$\int \frac{dx}{\sinh^2 ax \cosh^2 ax} = -\frac{2 \coth 2ax}{a}$$

14.599 $$\int \frac{\sinh^2 ax}{\cosh ax}\, dx = \frac{\sinh ax}{a} - \frac{1}{a} \tan^{-1} \sinh ax$$

14.600 $$\int \frac{\cosh^2 ax}{\sinh ax}\, dx = \frac{\cosh ax}{a} + \frac{1}{a} \ln \tanh \frac{ax}{2}$$

14.601 $$\int \frac{dx}{\cosh ax\,(1 + \sinh ax)} = \frac{1}{2a} \ln \left(\frac{1 + \sinh ax}{\cosh ax} \right) + \frac{1}{a} \tan^{-1} e^{ax}$$

14.602 $$\int \frac{dx}{\sinh ax\,(\cosh ax+1)} = \frac{1}{2a}\ln\tanh\frac{ax}{2} + \frac{1}{2a(\cosh ax+1)}$$

14.603 $$\int \frac{dx}{\sinh ax\,(\cosh ax-1)} = -\frac{1}{2a}\ln\tanh\frac{ax}{2} - \frac{1}{2a(\cosh ax-1)}$$

INTEGRALS INVOLVING tanh ax

14.604 $$\int \tanh ax\,dx = \frac{1}{a}\ln\cosh ax$$

14.605 $$\int \tanh^2 ax\,dx = x - \frac{\tanh ax}{a}$$

14.606 $$\int \tanh^3 ax\,dx = \frac{1}{a}\ln\cosh ax - \frac{\tanh^2 ax}{2a}$$

14.607 $$\int \tanh^n ax\,\operatorname{sech}^2 ax\,dx = \frac{\tanh^{n+1} ax}{(n+1)a}$$

14.608 $$\int \frac{\operatorname{sech}^2 ax}{\tanh ax}\,dx = \frac{1}{a}\ln\tanh ax$$

14.609 $$\int \frac{dx}{\tanh ax} = \frac{1}{a}\ln\sinh ax$$

14.610 $$\int x\tanh ax\,dx = \frac{1}{a^2}\left\{\frac{(ax)^3}{3} - \frac{(ax)^5}{15} + \frac{2(ax)^7}{105} - \cdots \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$$

14.611 $$\int x\tanh^2 ax\,dx = \frac{x^2}{2} - \frac{x\tanh ax}{a} + \frac{1}{a^2}\ln\cosh ax$$

14.612 $$\int \frac{\tanh ax}{x}\,dx = ax - \frac{(ax)^3}{9} + \frac{2(ax)^5}{75} - \cdots \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_n(ax)^{2n-1}}{(2n-1)(2n)!} + \cdots$$

14.613 $$\int \frac{dx}{p+q\tanh ax} = \frac{px}{p^2-q^2} - \frac{q}{a(p^2-q^2)}\ln(q\sinh ax + p\cosh ax)$$

14.614 $$\int \tanh^n ax\,dx = \frac{-\tanh^{n-1} ax}{a(n-1)} + \int \tanh^{n-2} ax\,dx$$

INTEGRALS INVOLVING coth ax

14.615 $$\int \coth ax\,dx = \frac{1}{a}\ln\sinh ax$$

14.616 $$\int \coth^2 ax\,dx = x - \frac{\coth ax}{a}$$

14.617 $$\int \coth^3 ax\,dx = \frac{1}{a}\ln\sinh ax - \frac{\coth^2 ax}{2a}$$

14.618 $$\int \coth^n ax\,\operatorname{csch}^2 ax\,dx = -\frac{\coth^{n+1} ax}{(n+1)a}$$

14.619 $$\int \frac{\operatorname{csch}^2 ax}{\coth ax}\,dx = -\frac{1}{a}\ln\coth ax$$

14.620 $$\int \frac{dx}{\coth ax} = \frac{1}{a}\ln\cosh ax$$

14.621 $$\int x \coth ax\,dx = \frac{1}{a^2}\left\{ax + \frac{(ax)^3}{9} - \frac{(ax)^5}{225} + \cdots \frac{(-1)^{n-1}2^{2n}B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$$

14.622 $$\int x \coth^2 ax\,dx = \frac{x^2}{2} - \frac{x\coth ax}{a} + \frac{1}{a^2}\ln\sinh ax$$

14.623 $$\int \frac{\coth ax}{x}\,dx = -\frac{1}{ax} + \frac{ax}{3} - \frac{(ax)^3}{135} + \cdots \frac{(-1)^n 2^{2n}B_n(ax)^{2n-1}}{(2n-1)(2n)!} + \cdots$$

14.624 $$\int \frac{dx}{p + q\coth ax} = \frac{px}{p^2-q^2} - \frac{q}{a(p^2-q^2)}\ln(p\sinh ax + q\cosh ax)$$

14.625 $$\int \coth^n ax\,dx = -\frac{\coth^{n-1}ax}{a(n-1)} + \int \coth^{n-2}ax\,dx$$

INTEGRALS INVOLVING sech ax

14.626 $$\int \operatorname{sech} ax\,dx = \frac{2}{a}\tan^{-1}e^{ax}$$

14.627 $$\int \operatorname{sech}^2 ax\,dx = \frac{\tanh ax}{a}$$

14.628 $$\int \operatorname{sech}^3 ax\,dx = \frac{\operatorname{sech} ax\tanh ax}{2a} + \frac{1}{2a}\tan^{-1}\sinh ax$$

14.629 $$\int \operatorname{sech}^n ax\tanh ax\,dx = -\frac{\operatorname{sech}^n ax}{na}$$

14.630 $$\int \frac{dx}{\operatorname{sech} ax} = \frac{\sinh ax}{a}$$

14.631 $$\int x\operatorname{sech} ax\,dx = \frac{1}{a^2}\left\{\frac{(ax)^2}{2} - \frac{(ax)^4}{8} + \frac{5(ax)^6}{144} + \cdots \frac{(-1)^n E_n(ax)^{2n+2}}{(2n+2)(2n)!} + \cdots\right\}$$

14.632 $$\int x\operatorname{sech}^2 ax\,dx = \frac{x\tanh ax}{a} - \frac{1}{a^2}\ln\cosh ax$$

14.633 $$\int \frac{\operatorname{sech} ax}{x}\,dx = \ln x - \frac{(ax)^2}{4} + \frac{5(ax)^4}{96} - \frac{61(ax)^6}{4320} + \cdots \frac{(-1)^n E_n(ax)^{2n}}{2n(2n)!} + \cdots$$

14.634 $$\int \frac{dx}{q + p\operatorname{sech} ax} = \frac{x}{q} - \frac{p}{q}\int \frac{dx}{p + q\cosh ax}$$ [See 14.581]

14.635 $$\int \operatorname{sech}^n ax\,dx = \frac{\operatorname{sech}^{n-2}ax\tanh ax}{a(n-1)} + \frac{n-2}{n-1}\int \operatorname{sech}^{n-2}ax\,dx$$

INTEGRALS INVOLVING csch ax

14.636 $$\int \operatorname{csch} ax\,dx = \frac{1}{a}\ln\tanh\frac{ax}{2}$$

14.637 $$\int \operatorname{csch}^2 ax\,dx = -\frac{\coth ax}{a}$$

14.638 $$\int \operatorname{csch}^3 ax\,dx = -\frac{\operatorname{csch} ax\coth ax}{2a} - \frac{1}{2a}\ln\tanh\frac{ax}{2}$$

14.639 $$\int \operatorname{csch}^n ax\coth ax\,dx = -\frac{\operatorname{csch}^n ax}{na}$$

14.640 $$\int \frac{dx}{\operatorname{csch} ax} = \frac{1}{a} \cosh ax$$

14.641 $$\int x \operatorname{csch} ax\, dx = \frac{1}{a^2}\left\{ax - \frac{(ax)^3}{18} + \frac{7(ax)^5}{1800} + \cdots + \frac{2(-1)^n(2^{2n-1}-1)B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$$

14.642 $$\int x \operatorname{csch}^2 ax\, dx = -\frac{x \coth ax}{a} + \frac{1}{a^2} \ln \sinh ax$$

14.643 $$\int \frac{\operatorname{csch} ax}{x}\, dx = -\frac{1}{ax} - \frac{ax}{6} + \frac{7(ax)^3}{1080} + \cdots \frac{(-1)^n 2(2^{2n-1}-1)B_n(ax)^{2n-1}}{(2n-1)(2n)!} + \cdots$$

14.644 $$\int \frac{dx}{q + p \operatorname{csch} ax} = \frac{x}{q} - \frac{p}{q}\int \frac{dx}{p + q \sinh ax}$$ [See 14.553]

14.645 $$\int \operatorname{csch}^n ax\, dx = \frac{-\operatorname{csch}^{n-2} ax \coth ax}{a(n-1)} - \frac{n-2}{n-1}\int \operatorname{csch}^{n-2} ax\, dx$$

INTEGRALS INVOLVING INVERSE HYPERBOLIC FUNCTIONS

14.646 $$\int \sinh^{-1}\frac{x}{a}\, dx = x \sinh^{-1}\frac{x}{a} - \sqrt{x^2+a^2}$$

14.647 $$\int x \sinh^{-1}\frac{x}{a}\, dx = \left(\frac{x^2}{2} + \frac{a^2}{4}\right)\sinh^{-1}\frac{x}{a} - \frac{x\sqrt{x^2+a^2}}{4}$$

14.648 $$\int x^2 \sinh^{-1}\frac{x}{a}\, dx = \frac{x^3}{3}\sinh^{-1}\frac{x}{a} + \frac{(2a^2-x^2)\sqrt{x^2+a^2}}{9}$$

14.649 $$\int \frac{\sinh^{-1}(x/a)}{x}\, dx = \begin{cases} \dfrac{x}{a} - \dfrac{(x/a)^3}{2\cdot3\cdot3} + \dfrac{1\cdot3(x/a)^5}{2\cdot4\cdot5\cdot5} - \dfrac{1\cdot3\cdot5(x/a)^7}{2\cdot4\cdot6\cdot7\cdot7} + \cdots & |x| < a \\ \dfrac{\ln^2(2x/a)}{2} - \dfrac{(a/x)^2}{2\cdot2\cdot2} + \dfrac{1\cdot3(a/x)^4}{2\cdot4\cdot4\cdot4} - \dfrac{1\cdot3\cdot5(a/x)^6}{2\cdot4\cdot6\cdot6\cdot6} + \cdots & x > a \\ -\dfrac{\ln^2(-2x/a)}{2} + \dfrac{(a/x)^2}{2\cdot2\cdot2} - \dfrac{1\cdot3(a/x)^4}{2\cdot4\cdot4\cdot4} + \dfrac{1\cdot3\cdot5(a/x)^6}{2\cdot4\cdot6\cdot6\cdot6} - \cdots & x < -a \end{cases}$$

14.650 $$\int \frac{\sinh^{-1}(x/a)}{x^2}\, dx = -\frac{\sinh^{-1}(x/a)}{x} - \frac{1}{a}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$$

14.651 $$\int \cosh^{-1}\frac{x}{a}\, dx = \begin{cases} x\cosh^{-1}(x/a) - \sqrt{x^2-a^2}, & \cosh^{-1}(x/a) > 0 \\ x\cosh^{-1}(x/a) + \sqrt{x^2-a^2}, & \cosh^{-1}(x/a) < 0 \end{cases}$$

14.652 $$\int x\cosh^{-1}\frac{x}{a}\, dx = \begin{cases} \frac{1}{4}(2x^2-a^2)\cosh^{-1}(x/a) - \frac{1}{4}x\sqrt{x^2-a^2}, & \cosh^{-1}(x/a) > 0 \\ \frac{1}{4}(2x^2-a^2)\cosh^{-1}(x/a) + \frac{1}{4}x\sqrt{x^2-a^2}, & \cosh^{-1}(x/a) < 0 \end{cases}$$

14.653 $$\int x^2\cosh^{-1}\frac{x}{a}\, dx = \begin{cases} \frac{1}{3}x^3\cosh^{-1}(x/a) - \frac{1}{9}(x^2+2a^2)\sqrt{x^2-a^2}, & \cosh^{-1}(x/a) > 0 \\ \frac{1}{3}x^3\cosh^{-1}(x/a) + \frac{1}{9}(x^2+2a^2)\sqrt{x^2-a^2}, & \cosh^{-1}(x/a) < 0 \end{cases}$$

14.654 $$\int \frac{\cosh^{-1}(x/a)}{x}\, dx = \pm\left[\frac{1}{2}\ln^2(2x/a) + \frac{(a/x)^2}{2\cdot2\cdot2} + \frac{1\cdot3(a/x)^4}{2\cdot4\cdot4\cdot4} + \frac{1\cdot3\cdot5(a/x)^6}{2\cdot4\cdot6\cdot6\cdot6} + \cdots\right]$$
$+$ if $\cosh^{-1}(x/a) > 0$, $-$ if $\cosh^{-1}(x/a) < 0$

14.655 $$\int \frac{\cosh^{-1}(x/a)}{x^2}\, dx = -\frac{\cosh^{-1}(x/a)}{x} \mp \frac{1}{a}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$$ [$-$ if $\cosh^{-1}(x/a) > 0$, $+$ if $\cosh^{-1}(x/a) < 0$]

14.656 $$\int \tanh^{-1}\frac{x}{a}\, dx = x\tanh^{-1}\frac{x}{a} + \frac{a}{2}\ln(a^2-x^2)$$

14.657 $$\int x\tanh^{-1}\frac{x}{a}\, dx = \frac{ax}{2} + \frac{1}{2}(x^2-a^2)\tanh^{-1}\frac{x}{a}$$

14.658 $$\int x^2\tanh^{-1}\frac{x}{a}\, dx = \frac{ax^2}{6} + \frac{x^3}{3}\tanh^{-1}\frac{x}{a} + \frac{a^3}{6}\ln(a^2-x^2)$$

14.659 $\displaystyle\int \frac{\tanh^{-1}(x/a)}{x}\,dx = \frac{x}{a} + \frac{(x/a)^3}{3^2} + \frac{(x/a)^5}{5^2} + \cdots$

14.660 $\displaystyle\int \frac{\tanh^{-1}(x/a)}{x^2}\,dx = -\frac{\tanh^{-1}(x/a)}{x} + \frac{1}{2a}\ln\left(\frac{x^2}{a^2-x^2}\right)$

14.661 $\displaystyle\int \coth^{-1}\frac{x}{a}\,dx = x\coth^{-1}x + \frac{a}{2}\ln(x^2-a^2)$

14.662 $\displaystyle\int x\coth^{-1}\frac{x}{a}\,dx = \frac{ax}{2} + \tfrac{1}{2}(x^2-a^2)\coth^{-1}\frac{x}{a}$

14.663 $\displaystyle\int x^2\coth^{-1}\frac{x}{a}\,dx = \frac{ax^2}{6} + \frac{x^3}{3}\coth^{-1}\frac{x}{a} + \frac{a^3}{6}\ln(x^2-a^2)$

14.664 $\displaystyle\int \frac{\coth^{-1}(x/a)}{x}\,dx = -\left(\frac{a}{x} + \frac{(a/x)^3}{3^2} + \frac{(a/x)^5}{5^2} + \cdots\right)$

14.665 $\displaystyle\int \frac{\coth^{-1}(x/a)}{x^2}\,dx = -\frac{\coth^{-1}(x/a)}{x} + \frac{1}{2a}\ln\left(\frac{x^2}{x^2-a^2}\right)$

14.666 $\displaystyle\int \operatorname{sech}^{-1}\frac{x}{a}\,dx = \begin{cases} x\operatorname{sech}^{-1}(x/a) + a\sin^{-1}(x/a), & \operatorname{sech}^{-1}(x/a) > 0 \\ x\operatorname{sech}^{-1}(x/a) - a\sin^{-1}(x/a), & \operatorname{sech}^{-1}(x/a) < 0 \end{cases}$

14.667 $\displaystyle\int x\operatorname{sech}^{-1}\frac{x}{a}\,dx = \begin{cases} \tfrac{1}{2}x^2\operatorname{sech}^{-1}(x/a) - \tfrac{1}{2}a\sqrt{a^2-x^2}, & \operatorname{sech}^{-1}(x/a) > 0 \\ \tfrac{1}{2}x^2\operatorname{sech}^{-1}(x/a) + \tfrac{1}{2}a\sqrt{a^2-x^2}, & \operatorname{sech}^{-1}(x/a) < 0 \end{cases}$

14.668 $\displaystyle\int \frac{\operatorname{sech}^{-1}(x/a)}{x}\,dx = \begin{cases} -\tfrac{1}{2}\ln(a/x)\ln(4a/x) - \dfrac{(x/a)^2}{2\cdot 2\cdot 2} - \dfrac{1\cdot 3(x/a)^4}{2\cdot 4\cdot 4\cdot 4} - \cdots, & \operatorname{sech}^{-1}(x/a) > 0 \\ \tfrac{1}{2}\ln(a/x)\ln(4a/x) + \dfrac{(x/a)^2}{2\cdot 2\cdot 2} + \dfrac{1\cdot 3(x/a)^4}{2\cdot 4\cdot 4\cdot 4} + \cdots, & \operatorname{sech}^{-1}(x/a) < 0 \end{cases}$

14.669 $\displaystyle\int \operatorname{csch}^{-1}\frac{x}{a}\,dx = x\operatorname{csch}^{-1}\frac{x}{a} \pm a\sinh^{-1}\frac{x}{a} \qquad [+ \text{ if } x > 0,\ - \text{ if } x < 0]$

14.670 $\displaystyle\int x\operatorname{csch}^{-1}\frac{x}{a}\,dx = \frac{x^2}{2}\operatorname{csch}^{-1}\frac{x}{a} \pm \frac{a\sqrt{x^2+a^2}}{2} \qquad [+ \text{ if } x > 0,\ - \text{ if } x < 0]$

14.671 $\displaystyle\int \frac{\operatorname{csch}^{-1}(x/a)}{x}\,dx = \begin{cases} \tfrac{1}{2}\ln(x/a)\ln(4a/x) + \dfrac{1(x/a)^2}{2\cdot 2\cdot 2} - \dfrac{1\cdot 3(x/a)^4}{2\cdot 4\cdot 4\cdot 4} + \cdots & 0 < x < a \\ \tfrac{1}{2}\ln(-x/a)\ln(-x/4a) - \dfrac{(x/a)^2}{2\cdot 2\cdot 2} + \dfrac{1\cdot 3(x/a)^4}{2\cdot 4\cdot 4\cdot 4} - \cdots & -a < x < 0 \\ -\dfrac{a}{x} + \dfrac{(a/x)^3}{2\cdot 3\cdot 3} - \dfrac{1\cdot 3(a/x)^5}{2\cdot 4\cdot 5\cdot 5} + \cdots & |x| > a \end{cases}$

14.672 $\displaystyle\int x^m\sinh^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\sinh^{-1}\frac{x}{a} - \frac{1}{m+1}\int \frac{x^{m+1}}{\sqrt{x^2+a^2}}\,dx$

14.673 $\displaystyle\int x^m\cosh^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^{m+1}}{m+1}\cosh^{-1}\dfrac{x}{a} - \dfrac{1}{m+1}\displaystyle\int \frac{x^{m+1}}{\sqrt{x^2-a^2}}\,dx & \cosh^{-1}(x/a) > 0 \\ \dfrac{x^{m+1}}{m+1}\cosh^{-1}\dfrac{x}{a} + \dfrac{1}{m+1}\displaystyle\int \frac{x^{m+1}}{\sqrt{x^2-a^2}}\,dx & \cosh^{-1}(x/a) < 0 \end{cases}$

14.674 $\displaystyle\int x^m\tanh^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\tanh^{-1}\frac{x}{a} - \frac{a}{m+1}\int \frac{x^{m+1}}{a^2-x^2}\,dx$

14.675 $\displaystyle\int x^m\coth^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\coth^{-1}\frac{x}{a} - \frac{a}{m+1}\int \frac{x^{m+1}}{a^2-x^2}\,dx$

14.676 $\displaystyle\int x^m\operatorname{sech}^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^{m+1}}{m+1}\operatorname{sech}^{-1}\dfrac{x}{a} + \dfrac{a}{m+1}\displaystyle\int \frac{x^m\,dx}{\sqrt{a^2-x^2}} & \operatorname{sech}^{-1}(x/a) > 0 \\ \dfrac{x^{m+1}}{m+1}\operatorname{sech}^{-1}\dfrac{x}{a} - \dfrac{a}{m+1}\displaystyle\int \frac{x^m\,dx}{\sqrt{a^2-x^2}} & \operatorname{sech}^{-1}(x/a) < 0 \end{cases}$

14.677 $\displaystyle\int x^m\operatorname{csch}^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\operatorname{csch}^{-1}\frac{x}{a} \pm \frac{a}{m+1}\int \frac{x^m\,dx}{\sqrt{x^2+a^2}} \qquad [+ \text{ if } x > 0,\ - \text{ if } x < 0]$