DIFFERENTIAL EQUATIONS

SPECIAL FORMULAS

Certain types of differential equations occur sufficiently often to justify the use of formulas for the corresponding particular solutions. The following set of tables I to XIV covers all first, second, and nth order ordinary linear differential equations with constant coefficients for which the right members are of the form $P(x)e^{rx}\sin sx$ or $P(x)e^{rx}\cos sx$, where r and s are constants and $P(x)$, is a polynomial of degree n.

When the right member of a reducible linear partial differential equation with constant coefficients is not zero, particular solutions for certain types of right members are contained in tables XV to XXI. In these tables both F and P are used to denote polynomials, and it is assumed that no denominator is zero. In any formula the roles of x and y may be reversed throughout, changing a formula in which x dominates to one in which y dominates. Tables XIX, XX, XXI are applicable whether the equations are reducible or not. The symbol $\binom{m}{n}$ stands for $\dfrac{m!}{(m-n)!n!}$ and is the $n{+}1$ st coefficient in the expansion of $(\mathrm{a}+\mathrm{b})^m$. Also $0!=1$ by definition.

The tables as herewith given are those contained in the text *Differential Equations* by Ginn and Company (1955) and are published with their kind permission and that of the author, Professor Frederick H. Steen.

Solution of Linear Differential Equations with Constant Coefficients

Any linear differential equation with constant coefficients may be written in the form

$$p(D)y = R(x)$$

where D is the differential operation

$$Dy = \frac{dy}{dx}$$

$p(D)$ is a polynomial in D,
y is the dependent variable,
x is the independent variable,
$R(x)$ is an arbitrary function of x.

A power of D represents repeated differentiation, that is

$$D^n y = \frac{d^n y}{dx^n}$$

For such an equation, the general solution may be written in the form

$$y = y_c + y_p$$

where y_p is any particular solution, and y_c is called the *complementary function*. This complementary function is defined as the general solution of the *homogeneous equation*, which is the original differential equation with the right side replaced by zero, i.e.

$$p(D)y = 0$$

The complementary function y_c may be determined as follows:

1. Factor the polynomial $p(D)$ into real and complex linear factors, just as if D were a variable instead of an operator.
2. For each nonrepeated linear factor of the form $(D-a)$, where a is real, write down a term of the form

$$ce^{ax}$$

where c is an arbitrary constant.

3. For each repeated real linear factor of the form $(D-a)^n$, write down n terms of the form

$$c_1e^{ax} + c_2xe^{ax} + c_3x^2e^{ax} + \cdots + c_nx^{n-1}e^{ax}$$

where the c_i's are arbitrary constants.

4. For each non-repeated conjugate complex pair of factors of the form $(D-a+ib)(D-a-ib)$, write down 2 terms of the form

$$c_1e^{ax}\cos bx + c_2e^{ax}\sin bx$$

5. For each repeated conjugate complex pair of factors of the form $(D-a+ib)^n(D-a-ib)^n$, write down $2n$ terms of the form

$$c_1e^{ax}\cos bx + c_2e^{ax}\sin bx + c_3xe^{ax}\cos bx + c_4xe^{ax}\sin bx$$
$$+ \cdots + c_{2n-1}x^{n-1}e^{ax}\cos bx + c_{2n}x^{n-1}e^{ax}\sin bx$$

6. The sum of all the terms thus written down is the complementary function y_c.

To find the particular solution y_p, use the following tables, as shown in the examples. For cases not shown in the tables, there are various methods of finding y_p. The most general method is called *variation of parameters*. The following example illustrates the method:

Find y_p for $(D^2-4)\ y = e^x$.

This example can be solved most easily by use of equation 63 in the tables following. However it is given here as an example of the method of variation of parameters.

The complementary function is

$$y_c = c_1e^{2x} + c_2e^{-2x}$$

To find y_p, replace the constants in the complementary function with unknown functions,

$$y_p = ue^{2x} + ve^{-2x}$$

We now prepare to substitute this assumed solution into the original equation. We begin by taking all the necessary derivatives:

$$y_p = ue^{2x} + ve^{-2x}$$
$$y_p' = 2ue^{2x} + 2ve^{-2x} + u'e^{2x} - v'e^{-2x}$$

For each derivative of y_p except the highest, we set the sum of all the terms containing u' and v' to 0. Thus the above equation becomes

$$u'e^{2x} + v'e^{-2x} = 0 \quad \text{and} \quad y_p' = 2ue^{2x} - 2ve^{-2x}$$

Continuing to differentiate, we have

$$y_p'' = 4ue^{2x} + 4ve^{-2x} + 2u'e^{2x} - 2v'e^{-2x}$$

When we substitute into the original equation, all the terms not containing u' or v' cancel out. This is a consequence of the method by which y_p was set up.

Thus all that is necessary is to write down the terms containing u' or v' in the highest order derivative of y_p, multiply by the constant coefficient of the highest power of D in $p(D)$, and set it equal to $R(x)$. Together with the previous terms in u' and v' which were set equal to 0, this gives us as many linear equations in the first derivatives of the unknown functions as there are unknown functions. The first derivatives may then be solved for by algebra, and the unknown functions found by integration. In the present example, this becomes

$$u'e^{2x} + v'e^{-2x} = 0$$
$$2u'e^{2x} - 2v'e^{-2x} = e^x$$

We eliminate v' and u' separately, getting

$$4u'e^{2x} = e^x$$
$$4v'e^{-2x} = -e^x$$

Thus

$$u' = \tfrac{1}{4}e^{-x}$$
$$v' = -\tfrac{1}{4}e^{3x}$$

Therefore, by integrating

$$u = -\tfrac{1}{4}e^{-x}$$
$$v = -\tfrac{1}{12}e^{3x}$$

A constant of integration is not needed, since we need only one particular solution. Thus

$$y_p = ue^{2x} + ve^{-2x} = -\tfrac{1}{4}e^{-x}e^{2x} - \tfrac{1}{12}e^{3x}e^{-2x}$$
$$= -\tfrac{1}{4}e^x - \tfrac{1}{12}e^x = -\tfrac{1}{3}e^x$$

and the general solution is

$$y = y_c + y_p = c_1 e^{2x} + c_2 e^{-2x} - \tfrac{1}{3} e^x$$

The following samples illustrate the use of the tables.

Example 1. Solve $(D^2 - 4)y = \sin 3x$.

Substitution of $q = -4$, $s = 3$ in formula 24 gives

$$y_p = \frac{\sin 3x}{-9 - 4}$$

wherefore the general solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{\sin 3x}{13}$$

Example 2. Obtain a particular solution of $(D^2 - 4D + 5)y = x^2 e^{3x} \sin x$.

Applying formula 40 with $a = 2$, $b = 1$, $r = 3$, $s = 1$, $P(x) = x^2$, $s + b = 2$, $s - b = 0$, $a - r = -1$, $(a - r)^2 + (s + b)^2 = 5$, $(a - r)^2 + (s - b)^2 = 1$, we have

$$\begin{aligned} y_p &= \frac{e^{3x}\sin x}{2}\left[\left(\frac{2}{5} - \frac{0}{1}\right)x^2 + \left(\frac{2(-1)2}{25} - \frac{2(-1)0}{1}\right)2x + \left(\frac{3 \cdot 1 \cdot 2 - 2^3}{125} - \frac{3 \cdot 1 \cdot 0 - 0}{1}\right)2\right] \\ &\quad - \frac{e^{3x}\cos x}{2}\left[\left(\frac{-1}{5} - \frac{-1}{1}\right)x^2 + \left(\frac{1 - 4}{25} - \frac{1 - 0}{1}\right)2x + \left(\frac{-1 - 3(-1)4}{125} - \frac{-1 - 3(-1)0}{1}\right)2\right] \\ &= \left(\frac{1}{5}x^2 - \frac{4}{25}x - \frac{2}{125}\right)e^{3x}\sin x + \left(-\frac{2}{5}x^2 + \frac{28}{25}x - \frac{136}{125}\right)e^{3x}\cos x \end{aligned}$$

The special formulas effect a very considerable saving of time in problems of this type.

Example 3. Obtain a particular solution of $(D^2 - 4D + 5)y = x^2 e^{2x} \cos x$. (Compare with Example 2.)

Formula 40 is not applicable here since for this equation $r = a$, $s = b$, wherefore the denominator $(a - r)^2 + (s - b)^2 = 0$. We turn instead to formula 44. Substituting $a = 2$, $b = 1$, $P(x) = x^2$ and replacing sin by cos, cos by $-$sin, we obtain

$$\begin{aligned} y_p &= \frac{e^{2x}\cos x}{4}\left(x^2 - \frac{2}{4}\right) + \frac{e^{2x}\sin x}{2}\int\left(x^2 - \frac{1}{2}\right)dx \\ &= \left(\frac{x^2}{4} - \frac{1}{8}\right)e^{2x}\cos x + \left(\frac{x^3}{6} - \frac{x}{4}\right)e^{2x}\sin x \end{aligned}$$

which is the required solution.

Example 4. Find z_p for $(D_x - 3D_y)\,z = \ln(y + 3x)$.

Referring to Table XV we note that formula 69 (not 68) is applicable. This gives

$$z_p = x\ln(y + 3x)$$

It is easily seen that $-y/3\,\ln(y + 3x)$ would serve equally well.

Example 5. Solve $(D_x + 2D_y - 4)\,z = y\cos(y - 2x)$.

Since R in formula 76 contains a polynomial in x, not y, we rewrite the given equation in the form

$$(D_y + \tfrac{1}{2}\,D_x - 2)\,z = \tfrac{1}{2}\,y\cos(y - 2x)$$

Then

$$z_c = e^{2y}F\left(x - \tfrac{2}{1}y\right) = e^{2x}f(2x - y)$$

and by the formula

$$\begin{aligned} z_p &= -\frac{1}{2}\cos(y - 2x)\cdot\left(\frac{y}{2} + \frac{\frac{1}{2}}{2}\right) \\ &= -\frac{1}{8}(2y + 1)\cos(y - 2x) \end{aligned}$$

Example 6. Find z_p for $(D_x + 4D_y)^3\,z = (2x - y)^2$.

Using formula 79, we obtain

$$z_p = \frac{\iiint u^2\,du^3}{[2 + 4(-1)]^3} = \frac{u^5}{5 \cdot 4 \cdot 3 \cdot (-8)} = -\frac{(2x - y)^5}{480}$$

Example 7. Find z_p for $(D_x^3 + 5D_x^2 D_y - 7D_x + 4)z = e^{2x+3y}$.

By formula 87

$$z_p = \frac{e^{2x+3y}}{2^3 + 5 \cdot 2^2 \cdot 3 - 7 \cdot 2 + 4} = \frac{e^{2x+3y}}{58}$$

Example 8. Find z_p for

$$(D_x^4 + 6D_x^3 D_y + D_x D_y + D_y^2 + 9)z = \sin(3x + 4y)$$

Since every term in the left member is of even degree in the two operators D_x and D_y, formula 90 is applicable. It gives

$$z_p = \frac{\sin(3x+4y)}{(-9)^2 + 6(-9)(-12) + (-12) + (-16) + 9} = \frac{\sin(3x+4y)}{710}$$

TABLE I: $(D-a)y = R$

R	y_p
1. e^{rx}	$\dfrac{e^{rx}}{r-a}$
2. $\sin sx$*	$-\dfrac{a\sin sx + s\cos sx}{a^2+s^2} = \dfrac{1}{\sqrt{a^2+s^2}}\sin\left(sx + \tan^{-1}\dfrac{s}{a}\right)$
3. $P(x)$	$-\dfrac{1}{a}\left[P(x) + \dfrac{P'(x)}{a} + \dfrac{P''(x)}{a^2} + \cdots + \dfrac{P^{(n)}(x)}{a^n}\right]$
4. $e^{rx}\sin sx$*	Replace a by $a-r$ in formula 2 and multiply by e^{rx}.
5. $P(x)\,e^{rx}$	Replace a by $a-r$ in formula 3 and multiply by e^{rx}.
6. $P(x)\sin sx$*	$-\sin sx\left[\dfrac{a}{a^2+s^2}P(x) + \dfrac{a^2-s^2}{(a^2+s^2)^2}P'(x) + \dfrac{a^3-3as^2}{(a^2+s^2)^3}P''(x) + \cdots + \dfrac{a^k - \binom{k}{2}a^{k-2}s^2 + \binom{k}{4}a^{k-4}s^4 - \cdots}{(a^2+s^2)^k}P^{(k-1)}(x) + \cdots\right]$ $-\cos sx\left[\dfrac{s}{a^2+s^2}P(x) + \dfrac{2as}{(a^2+s^2)^2}P'(x) + \dfrac{3a^2s-s^3}{(a^2+s^2)^3}P''(x) + \cdots + \dfrac{\binom{k}{1}a^{k-1}s - \binom{k}{3}a^{k-3}s^3 + \cdots}{(a^2+s^2)^k}P^{(k-1)}(x) + \cdots\right]$
7. $P(x)e^{rx}\sin sx$*	Replace a by $a-r$ in formula 6 and multiply by e^{rx}.
8. e^{ax}	xe^{ax}
9. $e^{ax}\sin sx$*	$-\dfrac{e^{ax}\cos sx}{s}$
10. $P(x)e^{ax}$	$e^{ax}\displaystyle\int P(x)\,dx$
11. $P(x)e^{ax}\sin sx$	$\dfrac{e^{ax}\sin sx}{s}\left[\dfrac{P'(x)}{s^3} - \dfrac{P'''(x)}{s^3} + \dfrac{P^{\text{v}}(x)}{s^5} - \cdots\right] - \dfrac{e^{ax}\cos sx}{s}\left[P(x) - \dfrac{P''(x)}{s^2} + \dfrac{P^{\text{iv}}(x)}{s^4} - \cdots\right]$

*For $\cos sx$ in R replace "sin" by "cos" and "cos" by "$-$sin" in y_p.

$$D^n = \frac{d^n}{dx^n} \qquad \binom{m}{n} = \frac{m!}{(m-n)!n!} \qquad 0! = 1$$

TABLE II: $(D-a)^2y = R$

R	y_p
12. e^{rx}	$\dfrac{e^{rx}}{(r-a)^2}$
13. $\sin sx$*	$\dfrac{1}{(a^2+s^2)}[(a^2-s^2)\sin sx + 2as\cos sx] = \dfrac{1}{a^2+s^2}\sin\left(sx + \tan^{-1}\dfrac{2as}{a^2-s^2}\right)$
14. $P(x)$	$\dfrac{1}{a^2}\left[P(x) + \dfrac{2P'(x)}{a} + \dfrac{3P''(x)}{a^2} + \cdots + \dfrac{(n+1)P^{(n)}(x)}{a^n}\right]$
15. $e^{rx}\sin sx$*	Replace a by $a-r$ in formula 13 and multiply by e^{rx}.
16. $P(x)e^{rx}$	Replace a by $a-r$ in formula 14 and multiply by e^{rx}.
17. $P(x)\sin sx$*	$\sin sx\left[\dfrac{a^2-s^2}{(a^2+s^2)^2}P(x) + 2\dfrac{a^3-3as^2}{(a^2+s^2)^3}P'(x) + 3\dfrac{a^4-6a^2s^2+s^4}{(a^2+s^2)^4}P''(x) + \cdots + (k-1)\dfrac{a^k - \binom{k}{2}a^{k-2}s^2 + \binom{k}{4}a^{k-4}s^4 - \cdots}{(a^2+s^2)^k}P^{(k-2)}(x) + \cdots\right]$ $+\cos sx\left[\dfrac{2as}{(a^2+s^2)^2}P(x) + 2\dfrac{3a^2s-s^3}{(a^2+s^2)^3}P'(x) + 3\dfrac{4a^3s-4as^3}{(a^2+s^2)^4}P''(x) + \cdots + (k-1)\dfrac{\binom{k}{1}a^{k-1}s - \binom{k}{3}a^{k-3}s^3 + \cdots}{(a^2+s^2)^k}P^{(k-2)}(x) + \cdots\right]$
18. $P(x)e^{rx}\sin sx$*	Replace a by $a-r$ in formula 17 and multiply by e^{rx}.
19. e^{ax}	$\frac{1}{2}x^2e^{ax}$
20. $e^{ax}\sin sx$*	$-\dfrac{e^{ax}\sin sx}{s^2}$
21. $P(x)e^{ax}$	$e^{ax}\displaystyle\int\int P(x)\,dx\,dx$
22. $P(x)e^{ax}\sin sx$*	$-\dfrac{e^{ax}\sin sx}{s^2}\left[P(x) - \dfrac{3P''(x)}{s^2} + \dfrac{5P^{\text{iv}}(x)}{s^4} - \dfrac{7P^{\text{vi}}(x)}{s^6} + \cdots\right] - \dfrac{e^{ax}\cos sx}{s^2}\left[\dfrac{2P'(x)}{s} + \dfrac{4P'''(x)}{s^3} - \dfrac{6P^{\text{v}}(x)}{s^5} - \cdots\right]$

*For $\cos sx$ in R replace "sin" by "cos" by "$-$sin" in y_p.

TABLE III: $(D^2+q)y=R$

R	y_p
23. e^{rx}	$\dfrac{e^{rx}}{r^2+q}$
24. $\sin sx$*	$\dfrac{\sin sx}{-s^2+q}$
25. $P(x)$	$\dfrac{1}{q}\left[P(x)-\dfrac{P''(x)}{q}+\dfrac{P^{\text{iv}}(x)}{q^2}-\cdots+(-1)^k\dfrac{P^{(2k)}(x)}{qk}\cdots\right]$
26. $e^{rx}\sin sx$	$\dfrac{(r^2-s^2+q)e^{rx}\sin sx-2rse^{rx}\cos sx}{(r^2-s^2+q)^2+(2rs)^2}=\dfrac{e^{rx}}{\sqrt{(r^2-s^2+q)^2+(2rs)^2}}\sin\left[sx-\tan^{-1}\dfrac{2rs}{r^2-s^2+q}\right]$
27. $P(x)e^{rx}$	$\dfrac{e^{rx}}{r^2+q}\left[P(x)-\dfrac{2r}{r^2+q}P'(x)+\dfrac{3r^2-q}{(r^2+q)^2}P''(x)-\dfrac{4r^3-4qr}{(r^2+q)^3}P'''(x)+\cdots\right.$ $\left.+\cdots+(-1)^{k-1}\dfrac{\binom{k}{1}r^{k-1}-\binom{k}{3}r^{k-3}q+\binom{k}{5}r^{k-5}q^2-\cdots}{(r^2+q)^{k-1}}P^{(k-1)}(x)+\cdots\right]$
28. $P(x)\sin sx$*	$\dfrac{\sin sx}{(-s^2+q)}\left[P(x)-\dfrac{3s^2+q}{(-s^2+q)^2}P''(x)+\dfrac{5s^4+10s^2q+q^2}{(-s^2+q)^4}P^{\text{iv}}(x)+\cdots\right.$ $\left.+(-1)^k\dfrac{\binom{2k+1}{1}s^{2k}+\binom{2k+1}{3}s^{2k-2}q+\binom{2k+1}{5}s^{2k-4}q^2+\cdots}{(-s^2+q)^{2k}}P^{(2k)}(x)+\cdots\right]$ $-\dfrac{s\cos sx}{(-s^2+q)}\left[\dfrac{2P'(x)}{(-s^2+q)}-\dfrac{4s^2+4q}{(-s^2+q)^3}P'''(x)+\cdots+(-1)^{k+1}\dfrac{\binom{2k}{1}s^{2k-2}+\binom{2k}{3}s^{2k-4}q+\cdots}{(-s^2+q)^{2k-1}}P^{(2k-1)}(x)+\cdots\right]$

TABLE IV: $(D^2+b^2)y=R$

29. $\sin bx$*	$-\dfrac{x\cos bx}{2b}$
30. $P(x)\sin bx$*	$\dfrac{\sin bx}{(2b)^2}\left[P(x)-\dfrac{P''(x)}{(2b)^2}+\dfrac{P^{\text{iv}}(x)}{(2b)^4}-\cdots\right]-\dfrac{\cos bx}{2b}\displaystyle\int\left[P(x)-\dfrac{P''(x)}{(2b)^2}+\cdots\right]dx$

* For $\cos sx$ in R replace "sin" by "cos" and "cos" by "$-$sin" in y_p.

TABLE V: $(D^2+pD+q)y=R$

R	y_p
31. e^{rx}	$\dfrac{e^{rx}}{r^2+pr+q}$
32. $\sin sx$*	$\dfrac{(q-s^2)\sin sx-ps\cos sx}{(q-s^2)^2+(ps)^2}=\dfrac{1}{\sqrt{(q-s^2)^2+(ps)^2}}\sin\left(sx-\tan^{-1}\dfrac{ps}{q-s^2}\right)$
33. $P(x)$	$\dfrac{1}{q}\left[P(x)-\dfrac{p}{q}P'(x)+\dfrac{p^2-q}{q^2}P''(x)-\dfrac{p^3-2pq}{q^3}P'''(x)+\cdots+(-1)^n\dfrac{p^n-\binom{n-1}{1}p^{n-2}q+\binom{n-2}{2}p^{n-4}q^2-\cdots}{q^n}P^{(n)}(x)\right]$
34. $e^{rx}\sin sx$*	Replace p by $p+2r$, q by $q+pr+r^2$ in formula 32 and multiply by e^{rx}.
35. $P(x)e^{rx}$	Replace p by $p+2r$, q by $q+pr+r^2$ in formula 33 and multiply by e^{rx}.

TABLE VI: $(D-b)(D-a)y=R$

36. $P(x)\sin sx$*	$\dfrac{\sin sx}{b-a}\left[\left(\dfrac{a}{a^2+s^2}-\dfrac{b}{b^2+s^2}\right)P(x)+\left(\dfrac{a^2-s^2}{(a^2+s^2)^2}-\dfrac{b^2-s^2}{(b^2+s^2)^2}\right)P'(x)+\left(\dfrac{a^3-3as^2}{(a^2+s^2)^3}-\dfrac{b^3-3bs^2}{(b^2+s^2)^3}\right)P''(x)+\cdots\right]$ $+\dfrac{\cos sx}{b-a}\left[\left(\dfrac{s}{a^2+s^2}-\dfrac{s}{b^2+s^2}\right)P(x)+\left(\dfrac{2as}{(a^2+s^2)^2}-\dfrac{2bs}{(b^2+s^2)^2}\right).P'(x)\right.$ $\left.+\left(\dfrac{3a^2s-s^2}{(a^2+s^2)^3}-\dfrac{3b^2s-s^3}{(b^2+s^2)^3}\right)P''(x)+\cdots\right]^{\dagger}$
37. $P(x)e^{rx}\sin sx$*	Replace a by $a-r$, b by b - r in formula 36 and multiply by e^{rx}.
38. $P(x)e^{ax}$	$\dfrac{e^{ax}}{a-b}\left[\displaystyle\int P(x)\,dx+\dfrac{P(x)}{(b-a)}+\dfrac{P'(x)}{(b-a)^2}+\dfrac{P''(x)}{(b-a)^3}+\cdots+\dfrac{P^{(n)}(x)}{(b-a)^{n+1}}\right]$

*For $\cos sx$ in R replace "sin" by "cos" and "cos" by "$-$sin" in y_p.

† For additional terms, compare with formula 6.

TABLE VII: $(D^2 - 2aD + a^2 + b^2)y = R$

R	y_p
39. $P(x)\sin sx^*$	$\frac{\sin sx}{2b}\left[\left(\frac{s+b}{a^2+(s+b)^2}-\frac{s-b}{a^2+(s-b)^2}\right)P(x)+\left(\frac{2a(s+b)}{[a^2+(s+b)^2]^2}-\frac{2a(s-b)}{[a^2+(s-b)^2]^2}\right)P'(x)+\left(\frac{3a^2(s+b)-(s+b)^3}{[a^2+(s+b)^2]^3}-\frac{3a^2(s-b)-(s-b)^3}{[a^2+(s-b)^2]^3}\right)P''(x)+\cdots\right]$ $-\frac{\cos sx}{2b}\left[\left(\frac{a}{a^2+(s+b)^2}-\frac{a}{a^2+(s-b)^2}\right)P(x)+\left(\frac{a^2-(s+b)^2}{[a^2+(s+b)^2]^2}-\frac{a^2-(s-b)^2}{[a^2+(s-b)^2]^2}\right)P'(x)+\left(\frac{a^2-3a(s+b)^2}{[a^2+(s+b)^2]^3}-\frac{a^3-3a(s-b)^2}{[a^2+(s-b)^2]^3}\right)P''(x)+\cdots\right]^\dagger$
40. $P(x)e^{rx}\sin sx^*$	Replace a by $a-r$ in formula 39 and multiply by e^{rx}.
41. $P(x)e^{ax}$	$\frac{e^{ax}}{b^2}\left[P(x)-\frac{P''(x)}{b^2}+\frac{P^{\text{iv}}(x)}{b^4}-\cdots\right]$
42. $e^{ax}\sin sx^*$	$\frac{e^{ax}\sin sx}{-s^2+b^2}$
43. $e^{ax}\sin bx^*$	$-\frac{xe^{ax}\cos bx}{2b}$
44. $P(x)e^{ax}\sin bx^*$	$\frac{e^{ax}\sin bx}{(2b)^2}\left[P(x)-\frac{P''(x)}{(2b)^2}+\frac{P^{\text{iv}}(x)}{(2b)^4}-\cdots\right]-\frac{e^{ax}\cos bx}{2b}\int\left[P(x)-\frac{P''(x)}{(2b)^2}+\frac{P^{\text{iv}}(x)}{(2b)^4}-\cdots\right]dx$

*For cos sx in R replace "sin' by "cos" and "cos" by "−sin" in y_p.

†For additional terms, compare with formula 6.

TABLE VIII: $f(D)y = [D^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0]y = R$

R	y_p
45. e^{rx}	$\frac{e^{rx}}{f(r)}$
46. $\sin sx$*	$\frac{[a_0-a_2s^2+a_4s^4-\cdots]\sin sx-[a_1s-a_3s^3+a_5s^5+\cdots]\cos sx}{[a_0-a_2s^2+a_4s^4-\cdots]^2+[a_1s-a_3s^3+a_5s^5-\cdots]^2}$

TABLE IX: $f(D^2)y = R$

47. $\sin sx$*	$\frac{\sin sx}{f(-s^2)}=\frac{\sin sx}{a_0-a_2s^2+\cdots\pm s^{2n}}$

TABLE X: $(D-a)^n y = R$

48. e^{rx}	$\frac{e^{rx}}{(r-a)^n}$
49. $\sin sx^*$	$\frac{(-1)^n}{(a^2+s^2)^n}\{[a^n-\binom{n}{2}a^{n-2}s^2+\binom{n}{4}a^{n-4}s^4-\cdots]\sin sx+[\binom{n}{1}a^{n-1}s-\binom{n}{3}a^{n-3}s^3+\cdots]\cos sx\}$
50. $P(x)$	$\frac{(-1)^n}{a^n}\left[P(x)+\binom{n}{1}\frac{P'(x)}{a}+\binom{n+1}{2}\frac{P''(x)}{a^2}+\binom{n+2}{3}\frac{P'''(x)}{a^2}+\cdots\right]$
51. $e^{rx}\sin sx^*$	Replace a by $a-r$ in formula 49 and multiply by e^{rx}.
52. $e^{rx}P(x)$	Replace a by $a-r$ in formula 50 and multiply by e^{rx}.
53. $P(x)\sin sx^*$	$(-1)^n\sin sx[A_nP(x)+\binom{n}{1}A_{n+1}P'(x)+\binom{n+1}{2}A_{n+2}P''(x)+\binom{n+2}{3}A_{n+3}P'''(x)+\cdots]$ $+(-1)^n\cos sx[B_nP(x)+\binom{n}{1}B_{n+1}P'(x)+\binom{n+1}{2}B_{n+2}P''(x)+\binom{n+2}{3}B_{n+3}P'''(x)+\cdots]$ $A_1=\frac{a}{a^2+s^2},\ A_2=\frac{a^2-s^2}{(a^2+s^2)^2},\ldots,A_k=\frac{a^k-\binom{k}{2}a^{k-2}s^2+\binom{k}{4}a^{k-4}s^4-\cdots}{(a^2+s^2)^k}$ $B_1=\frac{a}{a^2+s^2},\ B_2=\frac{2as}{(a^2+s^2)^2},\ldots,B_k=\frac{\binom{k}{1}a^{k-1}s-\binom{k}{3}a^{k-3}s^3+\cdots}{(a^2+s^2)^k}$
54. $e^{rx}\sin sx^*$	Replace a by $a-r$ in formula 53 and multiply by e^{rx}.

55. $e^{ax}P(x)$ $\qquad e^{ax}\int\int\cdots\int P(x)\,dx^n$

56. $P(x)e^{ax}\sin sx^*$

$$\frac{(-1)^{(n-1)/2}e^{ax}\sin sx}{s^n}\left[\binom{n}{n-1}\frac{P'(x)}{s}-\binom{n+2}{n-1}\frac{P'''(x)}{s^3}+\binom{n+4}{n-1}\frac{P^{\mathrm{v}}(x)}{s^5}-\cdots\right]$$

$$+\frac{(-1)^{(n+1)/2}e^{ax}\cos sx}{s^n}\left[\binom{n-1}{n-1}P(x)-\binom{n+1}{n-1}\frac{P''(x)}{s^2}+\binom{n+3}{n-1}\frac{P^{\mathrm{iv}}(x)}{s^4}-\cdots\right]\quad (n\text{ odd})$$

$$\frac{(-1)^{n/2}e^{ax}\sin sx}{s^n}\left[\binom{n-1}{n-1}P(x)-\binom{n+1}{n-1}\frac{P''(x)}{s^2}+\binom{n+3}{n-1}\frac{P^{\mathrm{iv}}(x)}{s^4}-\cdots\right]$$

$$+\frac{(-1)^{n/2}e^{ax}\cos sx}{s^n}\left[\binom{n}{n-1}\frac{P'(x)}{s}-\binom{n+2}{n-1}\frac{P'''(x)}{s^3}+\binom{n+4}{n-1}\frac{P^{\mathrm{v}}(x)}{s^5}-\cdots\right]\quad (n\text{ even})$$

*For cos sx in R replace "sin" by "cos" and "cos" by "−sin" in y_p.

TABLE XI: $(D-a)^n f(D)y=R$

57. e^{ax} $\qquad \dfrac{x^n}{n!}\cdot\dfrac{e^{ax}}{f(a)}$

*For cos sx in R replace "sin" by "cos" and "cos" by "−sin" in y_p.

TABLE XII: $(D^2+q)^n y=R$

R	y_p
58. e^{rx}	$e^{rx}/(r^2+q)^n$
59. $\sin sx^*$	$\sin sx/(q-s^2)^n$
60. $P(x)$	$\dfrac{1}{q^n}\left[P(x)-\binom{n}{1}\dfrac{P''(x)}{q^2}+\binom{n+1}{2}\dfrac{P^{\mathrm{iv}}(x)}{q^2}-\binom{n+2}{3}\dfrac{P^{\mathrm{vi}}(x)}{q^3}+\cdots\right]$
61. $e^{rx}\sin sx^*$	$\dfrac{e^{rx}}{(A^2+B^2)^n}\{[A^n-\binom{n}{2}A^{n-2}B^2+\binom{n}{4}A^{n-4}B^4-\cdots]\sin sx-[\binom{n}{1}A^{n-1}B-\binom{n}{3}A^{n-3}B^3+\cdots]\cos sx\}$ $A=r^2-s^2+q,\quad B=2rs$

TABLE XIII: $(D^2+b^2)^n y=R$

62. $\sin bx^*$ $\qquad (-1)^{(n+1)/2}\dfrac{x^n\cos bx}{n!(2b)^n}\quad (n\text{ odd}),\qquad (-1)^{n/2}\dfrac{x^n\sin bx}{n!(2b)^n}\quad (n\text{ even})$

TABLE XIV: $(D^n-q)y=R$

63. e^{rx} $\qquad e^{rx}/(r^n-q)$

64. $P(x)$ $\qquad -\dfrac{1}{q}\left[P(x)\dfrac{P^{(n)}(x)}{q}+\dfrac{P^{(2n)}(x)}{q^2}+\cdots\right]$

65. $\sin sx^*$ $\qquad -\dfrac{q\sin sx+(-1)^{(n-1)/2}s^n\cos sx}{q^2+s^{2n}}\quad (n\text{ odd}),\qquad \dfrac{\sin sx}{(-s^2)^{n/2}-q}\quad (n\text{ even})$

66. $e^{rx}\sin sx^*$ $\qquad \dfrac{Ae^{rx}\sin sx-Be^{rx}\cos sx}{A^2+B^2}=\dfrac{e^{rx}}{\sqrt{A^2+B^2}}\sin\left(sx-\tan^{-1}\dfrac{B}{A}\right)$

$A=[r^n-\binom{n}{2}r^{n-2}s^2+\binom{n}{4}r^{n-4}s^4-\cdots]-q,\quad B=[\binom{n}{1}r^{n-1}s-\binom{n}{3}r^{n-3}s^3+\cdots]$

*For cos sx in R replace "sin" by "cos" and "cos" by "− sin" in y_p.

TABLE XV: $(D_x+mD_y)z=R$

R	z_p
67. e^{ax+by}	$\dfrac{e^{ax+by}}{a+mb}$
68. $f(ax+by)$	$\dfrac{\int f(u)\,du}{a+mb},\quad u=ax+by$
69. $f(y-mx)$	$xf(y-mx)$
70. $\phi(x,y)f(y-mx)$	$f(y-mx)\int\phi(x,a+mx)\,dx\quad (a=y-mx$ after integration$)$

TABLE XVI: $(D_x + mD_y - k)z = R$

71. e^{ax+by} $\qquad \dfrac{e^{ax+by}}{a+mb-k}$

72. $\sin(ax+by)^*$ $\qquad -\dfrac{(a+bm)\cos(ax+by)+k\sin(ax+by)}{(a+bm)^2+k^2}$

73. $e^{\alpha x+\beta y}\sin(ax+by)^*$ $\qquad$ Replace k in 72 by $k-\alpha-m\beta$ and multiply by $e^{\alpha x+\beta y}$

74. $e^{xk}f(ax+by)$ $\qquad \dfrac{e^{kx}\int f(u)\,du}{a+mb}, \quad u = ax+by$

75. $f(y-mx)$ $\qquad -\dfrac{f(y-mx)}{k}$

76. $p(x)f(y-mx)$ $\qquad -\dfrac{1}{k}f(y-mx)\left[p(x)+\dfrac{P'(x)}{k}+\dfrac{P''(x)}{k^2}+\cdots+\dfrac{P^{(n)}(x)}{k^n}\right]$

77. $e^{kx}f(y-mx)$ $\qquad xe^{kx}f(y-mx)$

*For $\cos(ax+by)$ replace "sin" by "cos" and "cos" by "−sin" in z_p.

$D_x = \dfrac{\partial}{\partial x}; \quad D_y = \dfrac{\partial}{\partial y}; \quad D_{x^k}D_{y^r} = \dfrac{\partial^{k+r}}{\partial_{x^k}\partial_{y^r}}$

TABLE XVII: $(D_z + mD_y)^n z = R$

R $\qquad z_p$

78. e^{ax+by} $\qquad \dfrac{e^{ax+by}}{(a+mb)^n}$

79. $f(ax+by)$ $\qquad \dfrac{\int\int\cdots\int f(u)\,du^n}{(a+mb)^n}, \quad u = ax+by$

80. $f(y-mx)$ $\qquad \dfrac{x^n}{n!}f(y-mx)$

81. $\phi(x,y)f(y+mx)$ $\qquad f(y-mx)\displaystyle\int\int\cdots\int \phi(x, a+mx)\,dx^n$ $(a = y-mx$ after integration)

TABLE XVIII: $(D_x + mD_y - k)^n z = R$

82. e^{ax+by} $\qquad \dfrac{e^{ax+by}}{(a+mb-k)^n}$

83. $f(y-mx)$ $\qquad \dfrac{(-1)^n f(y-mx)}{k^n}$

84. $P(x)f(y-mx)$ $\qquad \dfrac{(-1)^n}{k^n}f(y-mx)\left[P(x)+\binom{n}{1}\dfrac{P'(x)}{k}+\binom{n+1}{2}\dfrac{P''(x)}{k^2}+\binom{n+2}{3}\dfrac{P'''(x)}{k^3}+\cdots\right]$

85. $e^{kz}f(ax+by)$ $\qquad \dfrac{e^{kx}\int\int\cdots\int f(u)\,du^n}{(a+mb)^n}, \quad u = ax+by$

86. $e^{kx}f(y-mx)$ $\qquad \dfrac{x^n}{n!}e^{kx}f(y-mx)$

TABLE XIX: $\left[D_x^n + a_1 D_x^{n-1}D_y + a_2 D_x^{n-2}D_y^2 + \cdots + a^n D_y^n\right]z = R$

87. e^{ax+by} $\qquad \dfrac{e^{ax+by}}{a+a_1a^{n-1}b+a_2a^{n-2}b^2+\cdots+a_nb^n}$

88. $f(ax+by)$ $\qquad \dfrac{\int\int\cdots\int f(u)du^n}{a^n+a_1a^{n-1}b+a_2a^{n-2}b^2+\cdots+a^nb^n}, \ (u = ax+by)$

TABLE XX: $F(D_x, D_y)z = R$

89. e^{ax+by} $\qquad \dfrac{e^{ax+by}}{F(a,b)}$

TABLE XXI: $F\left(D_x^2, D_xD_y, D_y^2\right)z = R$

90. $\sin(ax+by)^*$ $\qquad \dfrac{\sin(ax+by)}{F(-a^2, -ab, -b^2)}$

*For $\cos(ax+by)$replace "sin " by "cos", and "cos" by "−sin" in z_p.

DIFFERENTIAL EQUATIONS

Differential equation	Method of solution
Separation of variables $f_1(x)g_1(y)\,dx + f_2(x)g_2(y)\,dy = 0$	$\int \frac{f_1(x)}{f_2(x)}dx + \int \frac{g_2(y)}{g_1(y)}dy = c$
Exact equation $M(x,y)\,dx + N(x,y)\,dy = 0$ where $\partial M/\partial y = \partial N/\partial x$	$\int M\partial x + \int \left(n - \frac{\partial}{\partial y}\int M\partial x\right)dy = c$ where ∂x indicates that the integration is to be performed with respect to x keeping y constant.
Linear first order equation $\frac{dy}{dx} + P(x)y = Q(x)$	$ye^{\int P\,dx} = \int Qe^{\int P\,dx}dx + c$
Bernoulli's equation $\frac{dy}{dx} + P(x)y = Q(x)y^n$	$ve^{(1-n)\int P dx} = (1-n)\int Qe^{(1-n)\int P dx}dx + c$ where $v = y^{1-n}$. If $n = 1$, the solution is $\ln y = \int (Q - P)\,dx + c$
Homogeneous equation $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$	$\ln x = \int \frac{dv}{F(v) - v} + c$ where $v = y/x$. If $F(v) = v$, the solution is $y = cx$
Reducible to homogeneous $(a_1x + b_1y + c_1)\,dx + (a_2x + b_2y + c_2)$ $dy = 0$ $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$	Set $u = a_1x + b_1y + c_1$ $v = a_2x + b_2y + c_2$ Eliminate x and y and the equation becomes homogenous
Reducible to separable $(a_1x + b_1y + c_1)\,dx + (a_2x + b_2y + c_2)$ $dy = 0$ $\frac{a_1}{a_2} = \frac{b_1}{b_2}$	Set $u = a_1x + b_1y$ Eliminate x or y and equation becomes separable

$yF(xy)\,dx + x\,G(xy)\,dy = 0$	$\ln x = \int \frac{G(v)\,dv}{v\{G(v) - F(v)\}} + c$ where $v = xy$. If $G(v) = F(v)$, the solution is $xy = c$.
Linear, homogeneous second order equation $\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ b, c are real constants	Let m_1, m_2 be the roots of $m^2 + bm + c = 0$. Then there are 3 cases: Case 1. m_1, m_2 real and distinct: $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ Case 2. m_1, m_2 real and equal: $y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$ Case 3. $m_1 = p + qi,\ m_2 = p - qi$: $y = e^{px}(c_1 \cos qx + c_2 \sin qx)$ where $p = -b/2,\ q = \sqrt{4c - b^2}/2$
Linear, nonhomogeneous second order equation $\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = R(x)$ b, c are real constants	There are 3 cases corresponding to those immediately above: Case 1. $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \frac{e^{m_1 x}}{m_1 - m_2}\int e^{-m_1 x} R(x)\,dx + \frac{e^{m_2 x}}{m_2 - m_1}\int e^{-m_2 x} R(x)\,dx$ Case 2. $y = c_1 e^{m_1 x} + c_2 x e^{m_1 x} + x e^{m_1 x}\int e^{-m_1 x} R(x)\,dx - e^{m_1 x}\int x e^{-m_1 x} R(x)\,dx$ Case 3. $y = e^{px}(c_1 \cos qx + c_2 \sin qx) + \frac{e^{px}\sin qx}{q}\int e^{-px} R(x)\cos qx\,dx - \frac{e^{px}\cos qx}{q}\int e^{-px} R(x)\sin qx\,dx$

Euler or Cauchy equation $x^2 \dfrac{d^2y}{dx} + bx\dfrac{dy}{dx} + cy = S(x)$	Putting $x = e^t$, the equation becomes $\dfrac{d^2y}{dt^2} + (b-1)\dfrac{dy}{dt} + cy = S(e^t)$ and can then be solved as a linear second order equation.
Bessel's equation $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (\lambda^2 x^2 - n^2)y = 0$	$y = c_1 J_n(\lambda x) + c_2 Y_n(\lambda x)$
Transformed Bessel's equation $x^2 \dfrac{d^2y}{dx^2} + (2p+1)x\dfrac{dy}{dx} + (\alpha^2 x^{2r} + \beta^2)y = 0$	$y = x^{-p}\left\{c_1 J_{q/r}\left(\dfrac{\alpha}{r}x^r\right) + c_2 Y_{q/r}\left(\dfrac{\alpha}{r}x^r\right)\right\}$ where $q = \sqrt{p^2 - \beta^2}$.
Legendre's equation $(1-x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + n(n+1)y = 0$	$y = c_1 P_n(x) + c_2 Q_n(x)$