22 FORMULAS from VECTOR ANALYSIS

VECTORS AND SCALARS

Various quantities in physics such as temperature, volume and speed can be specified by a real number. Such quantities are called *scalars.*

Other quantities such as force, velocity and momentum require for their specification a direction as well as magnitude. Such quantities are called *vectors.* A vector is represented by an arrow or directed line segment indicating direction. The magnitude of the vector is determined by the length of the arrow, using an appropriate unit.

NOTATION FOR VECTORS

A vector is denoted by a bold faced letter such as **A** [Fig. 22-1]. The magnitude is denoted by $|\mathbf{A}|$ or A. The tail end of the arrow is called the *initial point* while the head is called the *terminal point.*

FUNDAMENTAL DEFINITIONS

1. **Equality of vectors.** Two vectors are equal if they have the same magnitude and direction. Thus $\mathbf{A} = \mathbf{B}$ in Fig. 22-1.

2. **Multiplication of a vector by a scalar.** If m is any real number (scalar), then $m\mathbf{A}$ is a vector whose magnitude is $|m|$ times the magnitude of **A** and whose direction is the same as or opposite to **A** according as $m > 0$ or $m < 0$. If $m = 0$, then $m\mathbf{A} = \mathbf{0}$ is called the *zero* or *null vector.*

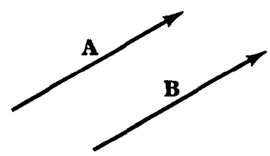

Fig. 22-1

3. **Sums of vectors.** The sum or resultant of **A** and **B** is a vector $\mathbf{C} = \mathbf{A} + \mathbf{B}$ formed by placing the initial point of **B** on the terminal point of **A** and joining the initial point of **A** to the terminal point of **B** [Fig. 22-2(*b*)]. This definition is equivalent to the parallelogram law for vector addition as indicated in Fig. 22-2(*c*). The vector $\mathbf{A} - \mathbf{B}$ is defined as $\mathbf{A} + (-\mathbf{B})$.

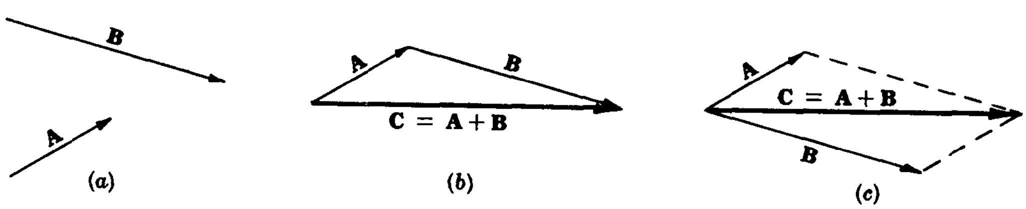

Fig. 22-2

Extensions to sums of more than two vectors are immediate. Thus Fig. 22-3 shows how to obtain the sum **E** of the vectors **A**, **B**, **C** and **D**.

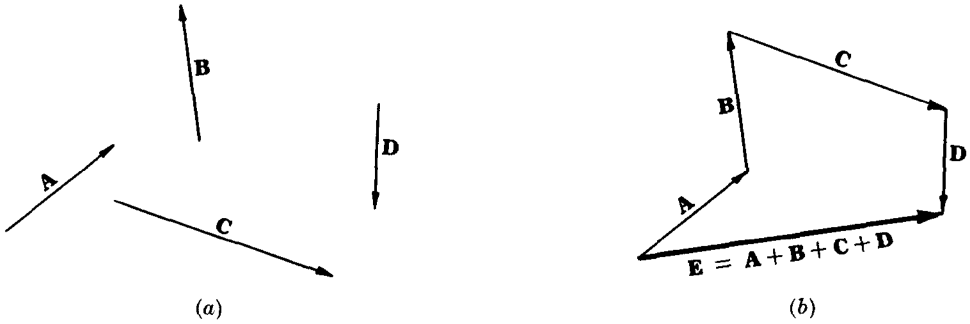

Fig. 22-3

4. **Unit vectors.** A *unit vector* is a vector with unit magnitude. If **A** is a vector, then a unit vector in the direction of **A** is $\mathbf{a} = \mathbf{A}/A$ where $A > 0$.

LAWS OF VECTOR ALGEBRA

If **A**, **B**, **C** are vectors and m, n are scalars, then

22.1 $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ Commutative law for addition

22.2 $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$ Associative law for addition

22.3 $m(n\mathbf{A}) = (mn)\mathbf{A} = n(m\mathbf{A})$ Associative law for scalar multiplication

22.4 $(m + n)\mathbf{A} = m\mathbf{A} + n\mathbf{A}$ Distributive law

22.5 $m(\mathbf{A} + \mathbf{B}) = m\mathbf{A} + m\mathbf{B}$ Distributive law

COMPONENTS OF A VECTOR

A vector **A** can be represented with initial point at the origin of a rectangular coordinate system. If **i**, **j**, **k** are unit vectors in the directions of the positive x, y, z axes, then

22.6 $$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$$

where $A_1\mathbf{i}, A_2\mathbf{j}, A_3\mathbf{k}$ are called *component vectors* of **A** in the **i**, **j**, **k** directions and A_1, A_2, A_3 are called the *components* of **A**.

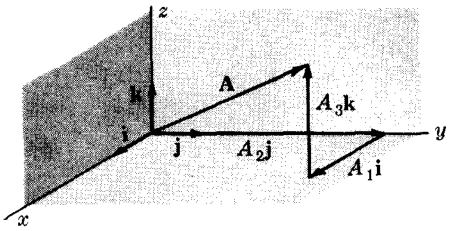

Fig. 22-4

DOT OR SCALAR PRODUCT

22.7 $$\mathbf{A} \cdot \mathbf{B} = AB \cos\theta \qquad 0 \leqq \theta \leqq \pi$$

where θ is the angle between **A** and **B**.

Fundamental results are

22.8 $$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$$ Commutative law

22.9 $$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}$$ Distributive law

22.10 $$\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$$

where $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$.

CROSS OR VECTOR PRODUCT

22.11 $$\mathbf{A}\times\mathbf{B} = AB\sin\theta\,\mathbf{u} \qquad 0 \leqq \theta \leqq \pi$$

where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$ and $\mathbf{u}$ is a unit vector perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A}, \mathbf{B}, \mathbf{u}$ form a *right-handed system* [i.e. a right-threaded screw rotated through an angle less than 180° from $\mathbf{A}$ to $\mathbf{B}$ will advance in the direction of $\mathbf{u}$ as in Fig. 22-5].

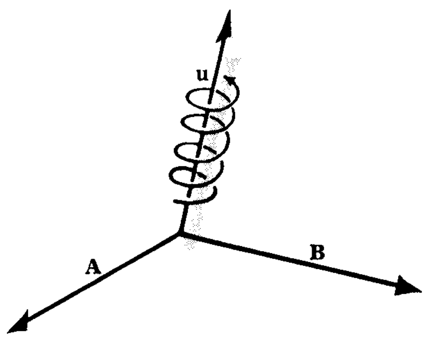

Fig. 22-5

Fundamental results are

22.12 $$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$
$$= (A_2B_3 - A_3B_2)\mathbf{i} + (A_3B_1 - A_1B_3)\mathbf{j} + (A_1B_2 - A_2B_1)\mathbf{k}$$

22.13 $$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$$

22.14 $$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}$$

22.15 $|\mathbf{A}\times\mathbf{B}|$ = area of parallelogram having sides $\mathbf{A}$ and $\mathbf{B}$

MISCELLANEOUS FORMULAS INVOLVING DOT AND CROSS PRODUCTS

22.16 $$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = A_1B_2C_3 + A_2B_3C_1 + A_3B_1C_2 - A_3B_2C_1 - A_2B_1C_3 - A_1B_3C_2$$

22.17 $|\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})|$ = volume of parallelepiped with sides $\mathbf{A}, \mathbf{B}, \mathbf{C}$

22.18 $$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

22.19 $$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{A}(\mathbf{B}\cdot\mathbf{C})$$

22.20 $$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$$

22.21 $$(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = \mathbf{C}\{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{D})\} - \mathbf{D}\{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})\}$$
$$= \mathbf{B}\{\mathbf{A}\cdot(\mathbf{C}\times\mathbf{D})\} - \mathbf{A}\{\mathbf{B}\cdot(\mathbf{C}\times\mathbf{D})\}$$

DERIVATIVES OF VECTORS

The derivative of a vector function $\mathbf{A}(u) = A_1(u)\mathbf{i} + A_2(u)\mathbf{j} + A_3(u)\mathbf{k}$ of the scalar variable u is given by

22.22 $$\frac{d\mathbf{A}}{du} = \lim_{\Delta u \to 0} \frac{\mathbf{A}(u+\Delta u) - \mathbf{A}(u)}{\Delta u} = \frac{dA_1}{du}\mathbf{i} + \frac{dA_2}{du}\mathbf{j} + \frac{dA_3}{du}\mathbf{k}$$

Partial derivatives of a vector function $\mathbf{A}(x, y, z)$ are similarly defined. We assume that all derivatives exist unless otherwise specified.

FORMULAS INVOLVING DERIVATIVES

22.23 $$\frac{d}{du}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\frac{d\mathbf{B}}{du} + \frac{d\mathbf{A}}{du}\cdot\mathbf{B}$$

22.24 $$\frac{d}{du}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times\frac{d\mathbf{B}}{du} + \frac{d\mathbf{A}}{du}\times\mathbf{B}$$

22.25 $$\frac{d}{du}\{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})\} = \frac{d\mathbf{A}}{du}\cdot(\mathbf{B}\times\mathbf{C}) + \mathbf{A}\cdot\left(\frac{d\mathbf{B}}{du}\times\mathbf{C}\right) + \mathbf{A}\cdot\left(\mathbf{B}\times\frac{d\mathbf{C}}{du}\right)$$

22.26 $$\mathbf{A}\cdot\frac{d\mathbf{A}}{du} = A\,\frac{dA}{du}$$

22.27 $$\mathbf{A}\cdot\frac{d\mathbf{A}}{du} = 0 \quad \text{if } |\mathbf{A}| \text{ is a constant}$$

THE DEL OPERATOR

The operator *del* is defined by

22.28 $$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

In the results below we assume that $U = U(x, y, z)$, $V = V(x, y, z)$, $\mathbf{A} = \mathbf{A}(x, y, z)$ and $\mathbf{B} = \mathbf{B}(x, y, z)$ have partial derivatives.

THE GRADIENT

22.29 $$\text{Gradient of } U = \text{grad } U = \nabla U = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)U = \frac{\partial U}{\partial x}\mathbf{i} + \frac{\partial U}{\partial y}\mathbf{j} + \frac{\partial U}{\partial z}\mathbf{k}$$

THE DIVERGENCE

22.30 $$\begin{aligned}\text{Divergence of } \mathbf{A} = \text{div } \mathbf{A} = \nabla\cdot\mathbf{A} &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}) \\ &= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\end{aligned}$$

THE CURL

22.31 Curl of $\mathbf{A}$ $=$ curl $\mathbf{A}$ $=$ $\nabla \times \mathbf{A}$

$$= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\mathbf{k}$$

THE LAPLACIAN

22.32 Laplacian of U $= \nabla^2 U = \nabla \cdot (\nabla U) = \dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} + \dfrac{\partial^2 U}{\partial z^2}$

22.33 Laplacian of $\mathbf{A}$ $= \nabla^2 \mathbf{A} = \dfrac{\partial^2 \mathbf{A}}{\partial x^2} + \dfrac{\partial^2 \mathbf{A}}{\partial y^2} + \dfrac{\partial^2 \mathbf{A}}{\partial z^2}$

THE BIHARMONIC OPERATOR

22.34 Biharmonic operator on U $= \nabla^4 U = \nabla^2(\nabla^2 U)$

$$= \frac{\partial^4 U}{\partial x^4} + \frac{\partial^4 U}{\partial y^4} + \frac{\partial^4 U}{\partial z^4} + 2\frac{\partial^4 U}{\partial x^2\,\partial y^2} + 2\frac{\partial^4 U}{\partial y^2\,\partial z^2} + 2\frac{\partial^4 U}{\partial x^2\,\partial z^2}$$

MISCELLANEOUS FORMULAS INVOLVING ∇

22.35 $\nabla(U+V) = \nabla U + \nabla V$

22.36 $\nabla \cdot (\mathbf{A}+\mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$

22.37 $\nabla \times (\mathbf{A}+\mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$

22.38 $\nabla \cdot (U\mathbf{A}) = (\nabla U) \cdot \mathbf{A} + U(\nabla \cdot \mathbf{A})$

22.39 $\nabla \times (U\mathbf{A}) = (\nabla U) \times \mathbf{A} + U(\nabla \times \mathbf{A})$

22.40 $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

22.41 $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - \mathbf{B}(\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B})$

22.42 $\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{B} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{B})$

22.43 $\nabla \times (\nabla U) = \mathbf{0}$, i.e. the curl of the gradient of U is zero.

22.44 $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, i.e. the divergence of the curl of $\mathbf{A}$ is zero.

22.45 $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

INTEGRALS INVOLVING VECTORS

If $\mathbf{A}(u) = \dfrac{d}{du}\mathbf{B}(u)$, then the *indefinite integral* of $\mathbf{A}(u)$ is

22.46 $$\int \mathbf{A}(u)\,du \;=\; \mathbf{B}(u) + \mathbf{c} \qquad \mathbf{c} = \text{constant vector}$$

The *definite integral* of $\mathbf{A}(u)$ from $u = a$ to $u = b$ in this case is given by

22.47 $$\int_a^b \mathbf{A}(u)\,du \;=\; \mathbf{B}(b) - \mathbf{B}(a)$$

The definite integral can be defined as on page 94.

LINE INTEGRALS

Consider a space curve C joining two points $P_1(a_1, a_2, a_3)$ and $P_2(b_1, b_2, b_3)$ as in Fig. 22-6. Divide the curve into n parts by points of subdivision $(x_1, y_1, z_1), \ldots, (x_{n-1}, y_{n-1}, z_{n-1})$. Then the *line integral* of a vector $\mathbf{A}(x, y, z)$ along C is defined as

22.48 $$\int_C \mathbf{A}\cdot d\mathbf{r} \;=\; \int_{P_1}^{P_2} \mathbf{A}\cdot d\mathbf{r} \;=\; \lim_{n\to\infty} \sum_{p=1}^{n} \mathbf{A}(x_p, y_p, z_p)\cdot \Delta\mathbf{r}_p$$

where $\Delta\mathbf{r}_p = \Delta x_p\,\mathbf{i} + \Delta y_p\,\mathbf{j} + \Delta z_p\,\mathbf{k}$, $\Delta x_p = x_{p+1} - x_p$, $\Delta y_p = y_{p+1} - y_p$, $\Delta z_p = z_{p+1} - z_p$ and where it is assumed that as $n \to \infty$ the largest of the magnitudes $|\Delta\mathbf{r}_p|$ approaches zero. The result 22.48 is a generalization of the ordinary definite integral [page 94].

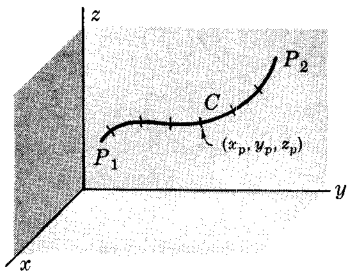

Fig. 22-6

The line integral 22.48 can also be written

22.49 $$\int_C \mathbf{A}\cdot d\mathbf{r} \;=\; \int_C (A_1\,dx + A_2\,dy + A_3\,dz)$$

using $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$.

PROPERTIES OF LINE INTEGRALS

22.50 $$\int_{P_1}^{P_2} \mathbf{A}\cdot d\mathbf{r} \;=\; -\int_{P_2}^{P_1} \mathbf{A}\cdot d\mathbf{r}$$

22.51 $$\int_{P_1}^{P_2} \mathbf{A}\cdot d\mathbf{r} \;=\; \int_{P_1}^{P_3} \mathbf{A}\cdot d\mathbf{r} + \int_{P_3}^{P_2} \mathbf{A}\cdot d\mathbf{r}$$

INDEPENDENCE OF THE PATH

In general a line integral has a value which depends on the particular path C joining points P_1 and P_2 in a region $\mathcal{R}$. However, in case $\mathbf{A} = \nabla\phi$ or $\nabla \times \mathbf{A} = \mathbf{0}$ where ϕ and its partial derivatives are continuous in $\mathcal{R}$, the line integral $\int_C \mathbf{A}\cdot d\mathbf{r}$ is independent of the path. In such case

22.52 $$\int_C \mathbf{A}\cdot d\mathbf{r} \;=\; \int_{P_1}^{P_2} \mathbf{A}\cdot d\mathbf{r} \;=\; \phi(P_2) - \phi(P_1)$$

where $\phi(P_1)$ and $\phi(P_2)$ denote the values of ϕ at P_1 and P_2 respectively. In particular if C is a closed curve,

22.53 $$\int_C \mathbf{A}\cdot d\mathbf{r} \;=\; \oint_C \mathbf{A}\cdot d\mathbf{r} \;=\; 0$$

where the circle on the integral sign is used to emphasize that C is closed.

MULTIPLE INTEGRALS

Let $F(x,y)$ be a function defined in a region $\mathcal{R}$ of the xy plane as in Fig. 22-7. Subdivide the region into n parts by lines parallel to the x and y axes as indicated. Let $\Delta A_p = \Delta x_p\,\Delta y_p$ denote an area of one of these parts. Then the integral of $F(x,y)$ over $\mathcal{R}$ is defined as

22.54 $$\int_{\mathcal{R}} F(x,y)\,dA \;=\; \lim_{n\to\infty}\sum_{p=1}^{n} F(x_p,y_p)\,\Delta A_p$$

provided this limit exists.

In such case the integral can also be written as

22.55 $$\int_{x=a}^{b}\int_{y=f_1(x)}^{f_2(x)} F(x,y)\,dy\,dx \;=\; \int_{x=a}^{b}\left\{\int_{y=f_1(x)}^{f_2(x)} F(x,y)\,dy\right\}dx$$

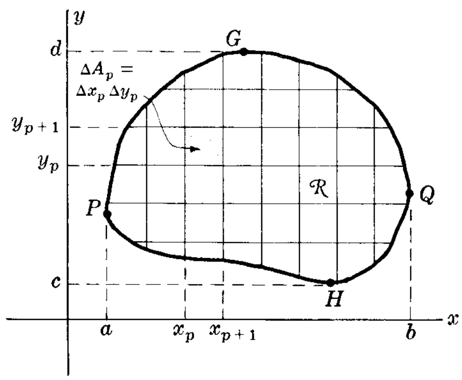

Fig. 22-7

where $y = f_1(x)$ and $y = f_2(x)$ are the equations of curves PHQ and PGQ respectively and a and b are the x coordinates of points P and Q. The result can also be written as

22.56 $$\int_{y=c}^{d}\int_{x=g_1(y)}^{g_2(y)} F(x,y)\,dx\,dy \;=\; \int_{y=c}^{d}\left\{\int_{x=g_1(y)}^{g_2(y)} F(x,y)\,dx\right\}dy$$

where $x = g_1(y)$, $x = g_2(y)$ are the equations of curves HPG and HQG respectively and c and d are the y coordinates of H and G.

These are called *double integrals* or *area integrals*. The ideas can be similarly extended to *triple* or *volume integrals* or to higher *multiple integrals*.

SURFACE INTEGRALS

Subdivide the surface S [see Fig. 22-8] into n elements of area ΔS_p, $p = 1,2,\ldots,n$. Let $\mathbf{A}(x_p,y_p,z_p) = \mathbf{A}_p$ where (x_p,y_p,z_p) is a point P in ΔS_p. Let $\mathbf{N}_p$ be a unit normal to ΔS_p at P. Then the surface integral of the normal component of $\mathbf{A}$ over S is defined as

22.57 $$\int_S \mathbf{A}\cdot\mathbf{N}\,dS \;=\; \lim_{n\to\infty}\sum_{p=1}^{n} \mathbf{A}_p\cdot\mathbf{N}_p\,\Delta S_p$$

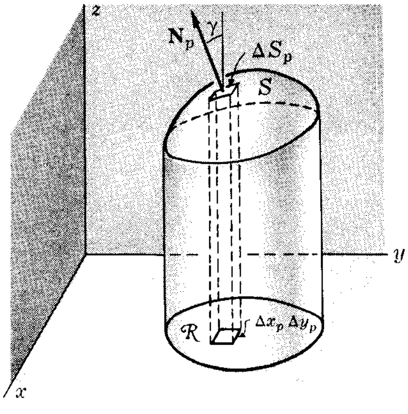

Fig. 22-8

RELATION BETWEEN SURFACE AND DOUBLE INTEGRALS

If $\mathcal{R}$ is the projection of S on the xy plane, then [see Fig. 22-8]

22.58
$$\int_S \mathbf{A}\cdot\mathbf{N}\,dS = \iint_{\mathcal{R}} \mathbf{A}\cdot\mathbf{N}\,\frac{dx\,dy}{|\mathbf{N}\cdot\mathbf{k}|}$$

THE DIVERGENCE THEOREM

Let S be a closed surface bounding a region of volume V; then if $\mathbf{N}$ is the positive (outward drawn) normal and $d\mathbf{S} = \mathbf{N}\,dS$, we have [see Fig. 22-9]

22.59
$$\int_V \nabla\cdot\mathbf{A}\,dV = \int_S \mathbf{A}\cdot d\mathbf{S}$$

The result is also called *Gauss' theorem* or *Green's theorem.*

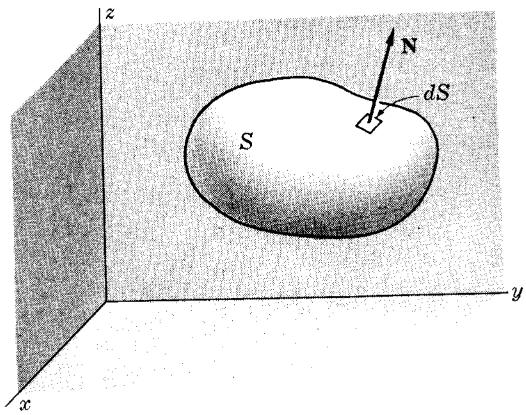

Fig. 22-9

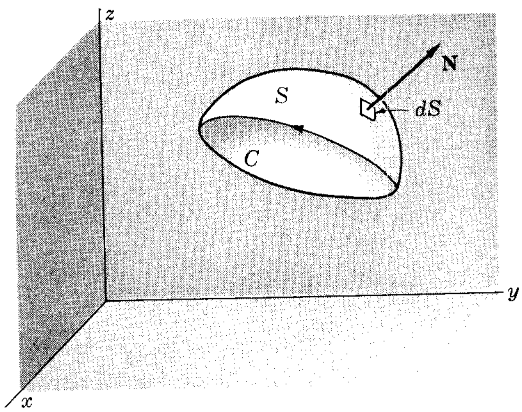

Fig. 22-10

STOKE'S THEOREM

Let S be an open two-sided surface bounded by a closed non-intersecting curve C [simple closed curve] as in Fig. 22-10. Then

22.60
$$\oint_C \mathbf{A}\cdot d\mathbf{r} = \int_S (\nabla\times\mathbf{A})\cdot d\mathbf{S}$$

where the circle on the integral is used to emphasize that C is closed.

GREEN'S THEOREM IN THE PLANE

22.61
$$\oint_C (P\,dx + Q\,dy) = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

where R is the area bounded by the closed curve C. This result is a special case of the divergence theorem or Stoke's theorem.

GREEN'S FIRST IDENTITY

22.62 $$\int_V \{\phi\,\nabla^2\psi + (\nabla\phi)\cdot(\nabla\psi)\}\,dV = \int (\phi\,\nabla\psi)\cdot d\mathbf{S}$$

where ϕ and ψ are scalar functions.

GREEN'S SECOND IDENTITY

22.63 $$\int_V (\phi\,\nabla^2\psi - \psi\,\nabla^2\phi)\,dV = \int_S (\phi\,\nabla\psi - \psi\nabla\phi)\cdot d\mathbf{S}$$

MISCELLANEOUS INTEGRAL THEOREMS

22.64 $$\int_V \nabla\times\mathbf{A}\,dV = \int_S d\mathbf{S}\times\mathbf{A}$$

22.65 $$\int_C \phi\,d\mathbf{r} = \int_S d\mathbf{S}\times\nabla\phi$$

CURVILINEAR COORDINATES

A point P in space [see Fig. 22-11] can be located by rectangular coordinates (x, y, z) or curvilinear coordinates (u_1, u_2, u_3) where the transformation equations from one set of coordinates to the other are given by

22.66 $$x = x(u_1, u_2, u_3)$$
$$y = y(u_1, u_2, u_3)$$
$$z = z(u_1, u_2, u_3)$$

If u_2 and u_3 are constant, then as u_1 varies, the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ of P describes a curve called the u_1 coordinate curve. Similarly we define the u_2 and u_3 coordinate curves through P. The vectors $\partial\mathbf{r}/\partial u_1, \partial\mathbf{r}/\partial u_2, \partial\mathbf{r}/\partial u_3$ represent tangent vectors to the u_1, u_2, u_3 coordinate curves. Letting $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be unit tangent vectors to these curves, we have

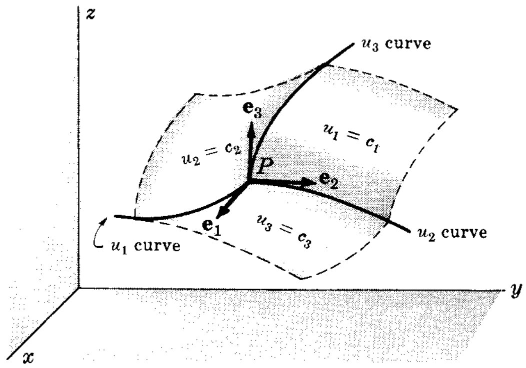

Fig. 22-11

22.67 $$\frac{\partial\mathbf{r}}{\partial u_1} = h_1\mathbf{e}_1, \quad \frac{\partial\mathbf{r}}{\partial u_2} = h_2\mathbf{e}_2, \quad \frac{\partial\mathbf{r}}{\partial u_3} = h_3\mathbf{e}_3$$

where

22.68 $$h_1 = \left|\frac{\partial\mathbf{r}}{\partial u_1}\right|, \quad h_2 = \left|\frac{\partial\mathbf{r}}{\partial u_2}\right|, \quad h_3 = \left|\frac{\partial\mathbf{r}}{\partial u_3}\right|$$

are called *scale factors*. If $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are mutually perpendicular, the curvilinear coordinate system is called *orthogonal*.

FORMULAS INVOLVING ORTHOGONAL CURVILINEAR COORDINATES

22.69 $$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1}du_1 + \frac{\partial \mathbf{r}}{\partial u_2}du_2 + \frac{\partial \mathbf{r}}{\partial u_3}du_3 = h_1\,du_1\,\mathbf{e}_1 + h_2\,du_2\,\mathbf{e}_2 + h_3\,du_3\,\mathbf{e}_3$$

22.70 $$ds^2 = d\mathbf{r}\cdot d\mathbf{r} = h_1^2\,du_1^2 + h_2^2\,du_2^2 + h_3^2\,du_3^2$$

where ds is the element of arc length.

If dV is the element of volume, then

22.71 $$dV = |(h_1\mathbf{e}_1\,du_1)\cdot(h_2\mathbf{e}_2\,du_2)\times(h_3\mathbf{e}_3\,du_3)| = h_1h_2h_3\,du_1\,du_2\,du_3$$
$$= \left|\frac{\partial \mathbf{r}}{\partial u_1}\cdot\frac{\partial \mathbf{r}}{\partial u_2}\times\frac{\partial \mathbf{r}}{\partial u_3}\right| du_1\,du_2\,du_3 = \left|\frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)}\right| du_1\,du_2\,du_3$$

where

22.72 $$\frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)} = \begin{vmatrix} \partial x/\partial u_1 & \partial x/\partial u_2 & \partial x/\partial u_3 \\ \partial y/\partial u_1 & \partial y/\partial u_2 & \partial y/\partial u_3 \\ \partial z/\partial u_1 & \partial z/\partial u_2 & \partial z/\partial u_3 \end{vmatrix}$$

is called the *Jacobian* of the transformation.

TRANSFORMATION OF MULTIPLE INTEGRALS

The result 22.72 can be used to transform multiple integrals from rectangular to curvilinear coordinates. For example, we have

22.73 $$\iiint_{\mathcal{R}} F(x,y,z)\,dx\,dy\,dz = \iiint_{\mathcal{R}'} G(u_1,u_2,u_3)\left|\frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)}\right| du_1\,du_2\,du_3$$

where $\mathcal{R}'$ is the region into which $\mathcal{R}$ is mapped by the transformation and $G(u_1,u_2,u_3)$ is the value of $F(x,y,z)$ corresponding to the transformation.

GRADIENT, DIVERGENCE, CURL AND LAPLACIAN

In the following, Φ is a scalar function and $\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3$ a vector function of orthogonal curvilinear coordinates u_1, u_2, u_3.

22.74 Gradient of Φ = grad Φ = $\nabla\Phi$ = $\dfrac{1}{h_1}\dfrac{\partial\Phi}{\partial u_1} + \dfrac{1}{h_2}\dfrac{\partial\Phi}{\partial u_2} + \dfrac{1}{h_3}\dfrac{\partial\Phi}{\partial u_3}$

22.75 Divergence of $\mathbf{A}$ = div $\mathbf{A}$ = $\nabla\cdot\mathbf{A}$ = $\dfrac{1}{h_1h_2h_3}\left[\dfrac{\partial}{\partial u_1}(h_2h_3A_1) + \dfrac{\partial}{\partial u_2}(h_3h_1A_2) + \dfrac{\partial}{\partial u_3}(h_1h_2A_3)\right]$

22.76 Curl of $\mathbf{A}$ = curl $\mathbf{A}$ = $\nabla\times\mathbf{A}$ = $\dfrac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$

$$= \frac{1}{h_2h_3}\left[\frac{\partial}{\partial u_2}(h_3A_3) - \frac{\partial}{\partial u_3}(h_2A_2)\right]\mathbf{e}_1 + \frac{1}{h_1h_3}\left[\frac{\partial}{\partial u_3}(h_1A_1) - \frac{\partial}{\partial u_1}(h_3A_3)\right]\mathbf{e}_2$$
$$+ \frac{1}{h_1h_2}\left[\frac{\partial}{\partial u_1}(h_2A_2) - \frac{\partial}{\partial u_2}(h_1A_1)\right]\mathbf{e}_3$$

22.77 Laplacian of Φ = $\nabla^2\Phi$ = $\dfrac{1}{h_1h_2h_3}\left[\dfrac{\partial}{\partial u_1}\left(\dfrac{h_2h_3}{h_1}\dfrac{\partial\Phi}{\partial u_1}\right) + \dfrac{\partial}{\partial u_2}\left(\dfrac{h_3h_1}{h_2}\dfrac{\partial\Phi}{\partial u_2}\right) + \dfrac{\partial}{\partial u_3}\left(\dfrac{h_1h_2}{h_3}\dfrac{\partial\Phi}{\partial u_3}\right)\right]$

Note that the biharmonic operator $\nabla^4\Phi = \nabla^2(\nabla^2\Phi)$ can be obtained from 22.77.

SPECIAL ORTHOGONAL COORDINATE SYSTEMS

Cylindrical Coordinates (r, θ, z) [See Fig. 22-12]

22.78 $$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

22.79 $$h_1^2 = 1, \quad h_2^2 = r^2, \quad h_3^2 = 1$$

22.80 $$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2}$$

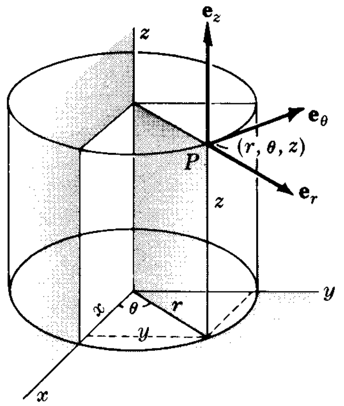

Fig. 22-12. Cylindrical coordinates.

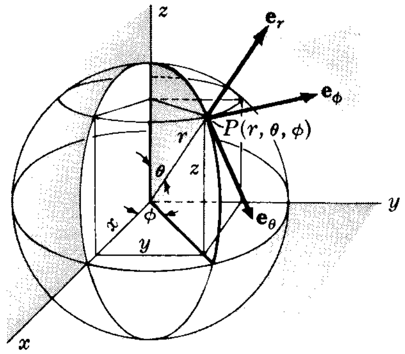

Fig. 22-13. Spherical coordinates.

Spherical Coordinates (r, θ, ϕ) [See Fig. 22-13]

22.81 $$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

22.82 $$h_1^2 = 1, \quad h_2^2 = r^2, \quad h_3^2 = r^2\sin^2\theta$$

22.83 $$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

Parabolic Cylindrical Coordinates (u, v, z)

22.84 $$x = \tfrac{1}{2}(u^2 - v^2), \quad y = uv, \quad z = z$$

22.85 $$h_1^2 = h_2^2 = u^2 + v^2, \quad h_3^2 = 1$$

22.86 $$\nabla^2\Phi = \frac{1}{u^2 + v^2}\left(\frac{\partial^2\Phi}{\partial u^2} + \frac{\partial^2\Phi}{\partial v^2}\right) + \frac{\partial^2\Phi}{\partial z^2}$$

The traces of the coordinate surfaces on the xy plane are shown in Fig. 22-14. They are confocal parabolas with a common axis.

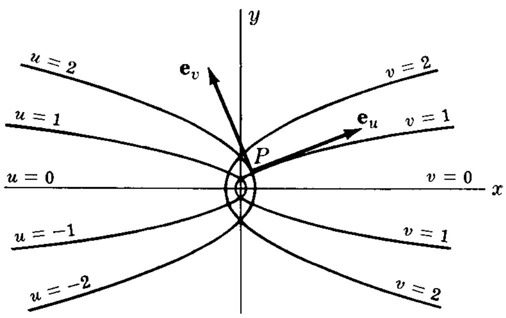

Fig. 22-14

Paraboloidal Coordinates (u, v, ϕ)

22.87 $$x = uv\cos\phi, \quad y = uv\sin\phi, \quad z = \tfrac{1}{2}(u^2 - v^2)$$

where $$u \geqq 0, \quad v \geqq 0, \quad 0 \leqq \phi < 2\pi$$

22.88 $$h_1^2 = h_2^2 = u^2 + v^2, \quad h_3^2 = u^2v^2$$

22.89 $$\nabla^2\Phi = \frac{1}{u(u^2+v^2)}\frac{\partial}{\partial u}\left(u\frac{\partial\Phi}{\partial u}\right) + \frac{1}{v(u^2+v^2)}\frac{\partial}{\partial v}\left(v\frac{\partial\Phi}{\partial v}\right) + \frac{1}{u^2v^2}\frac{\partial^2\Phi}{\partial\phi^2}$$

Two sets of coordinate surfaces are obtained by revolving the parabolas of Fig. 22-14 about the x axis which is then relabeled the z axis.

Elliptic Cylindrical Coordinates (u, v, z)

22.90 $$x = a\cosh u\cos v, \quad y = a\sinh u\sin v, \quad z = z$$

where $$u \geqq 0, \quad 0 \leqq v < 2\pi, \quad -\infty < z < \infty$$

22.91 $$h_1^2 = h_2^2 = a^2(\sinh^2 u + \sin^2 v), \quad h_3^2 = 1$$

22.92 $$\nabla^2\Phi = \frac{1}{a^2(\sinh^2 u + \sin^2 v)}\left(\frac{\partial^2\Phi}{\partial u^2} + \frac{\partial^2\Phi}{\partial v^2}\right) + \frac{\partial^2\Phi}{\partial z^2}$$

The traces of the coordinate surfaces on the xy plane are shown in Fig. 22-15. They are confocal ellipses and hyperbolas.

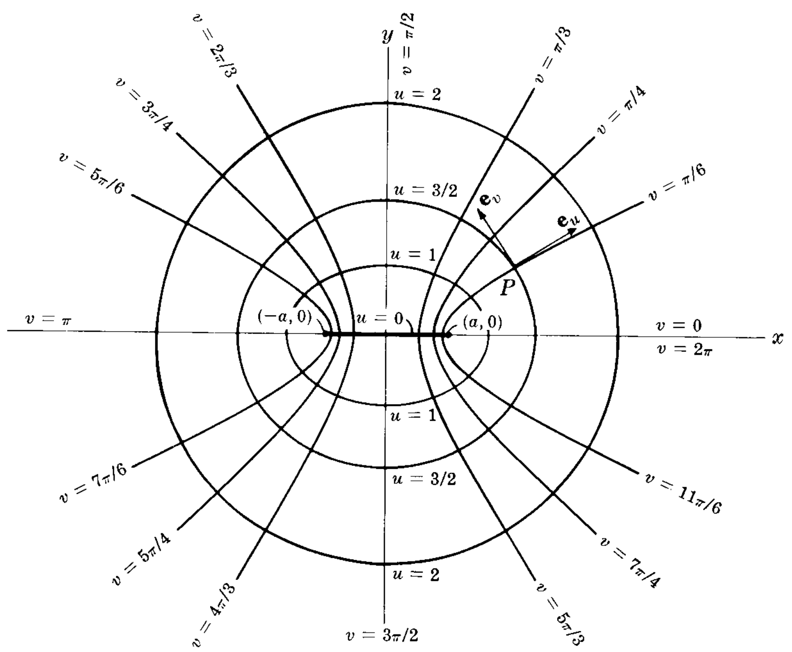

Fig. 22-15. Elliptic cylindrical coordinates.

Prolate Spheroidal Coordinates (ξ, η, ϕ)

22.93 $$x = a \sinh\xi \sin\eta \cos\phi, \quad y = a \sinh\xi \sin\eta \sin\phi, \quad z = a \cosh\xi \cos\eta$$

where $$\xi \geqq 0, \quad 0 \leqq \eta \leqq \pi, \quad 0 \leqq \phi < 2\pi$$

22.94 $$h_1^2 = h_2^2 = a^2(\sinh^2\xi + \sin^2\eta), \quad h_3^2 = a^2 \sinh^2\xi \sin^2\eta$$

22.95 $$\nabla^2\Phi = \frac{1}{a^2(\sinh^2\xi + \sin^2\eta)\sinh\xi}\frac{\partial}{\partial\xi}\left(\sinh\xi \frac{\partial\Phi}{\partial\xi}\right) + \frac{1}{a^2(\sinh^2\xi + \sin^2\eta)\sin\eta}\frac{\partial}{\partial\eta}\left(\sin\eta\frac{\partial\Phi}{\partial\eta}\right) + \frac{1}{a^2\sinh^2\xi\sin^2\eta}\frac{\partial^2\Phi}{\partial\phi^2}$$

Two sets of coordinate surfaces are obtained by revolving the curves of Fig. 22-15 about the x axis which is relabeled the z axis. The third set of coordinate surfaces consists of planes passing through this axis.

Oblate Spheroidal Coordinates (ξ, η, ϕ)

22.96 $$x = a \cosh\xi \cos\eta \cos\phi, \quad y = a \cosh\xi \cos\eta \sin\phi, \quad z = a \sinh\xi \sin\eta$$

where $$\xi \geqq 0, \quad -\pi/2 \leqq \eta \leqq \pi/2, \quad 0 \leqq \phi < 2\pi$$

22.97 $$h_1^2 = h_2^2 = a^2(\sinh^2\xi + \sin^2\eta), \quad h_3^2 = a^2 \cosh^2\xi \cos^2\eta$$

22.98 $$\nabla^2\Phi = \frac{1}{a^2(\sinh^2\xi + \sin^2\eta)\cosh\xi}\frac{\partial}{\partial\xi}\left(\cosh\xi \frac{\partial\Phi}{\partial\xi}\right) + \frac{1}{a^2(\sinh^2\xi + \sin^2\eta)\cos\eta}\frac{\partial}{\partial\eta}\left(\cos\eta\frac{\partial\Phi}{\partial\eta}\right) + \frac{1}{a^2\cosh^2\xi\cos^2\eta}\frac{\partial^2\Phi}{\partial\phi^2}$$

Two sets of coordinate surfaces are obtained by revolving the curves of Fig. 22-15 about the y axis which is relabeled the z axis. The third set of coordinate surfaces are planes passing through this axis.

Bipolar Coordinates (u, v, z)

22.99 $$x = \frac{a \sinh v}{\cosh v - \cos u}, \quad y = \frac{a \sin u}{\cosh v - \cos u}, \quad z = z$$

where $$0 \leqq u < 2\pi, \quad -\infty < v < \infty, \quad -\infty < z < \infty$$

or

22.100 $$x^2 + (y - a\cot u)^2 = a^2 \csc^2 u, \quad (x - a\coth v)^2 + y^2 = a^2 \operatorname{csch}^2 v, \quad z = z$$

22.101 $$h_1^2 = h_2^2 = \frac{a^2}{(\cosh v - \cos u)^2}, \quad h_3^2 = 1$$

22.102 $$\nabla^2\Phi = \frac{(\cosh v - \cos u)^2}{a^2}\left(\frac{\partial^2\Phi}{\partial u^2} + \frac{\partial^2\Phi}{\partial v^2}\right) + \frac{\partial^2\Phi}{\partial z^2}$$

The traces of the coordinate surfaces on the xy plane are shown in Fig. 22-16 below.

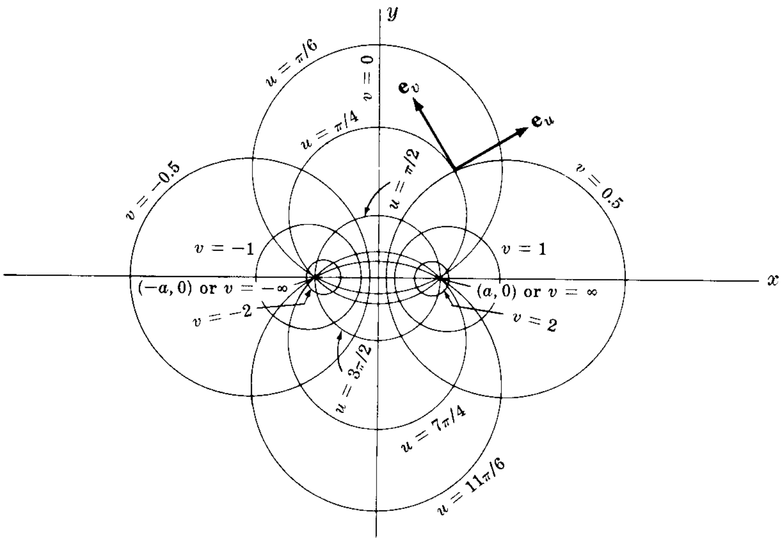

Fig. 22-16. Bipolar coordinates.

Toroidal Coordinates (u, v, ϕ)

22.103 $$x = \frac{a \sinh v \cos \phi}{\cosh v - \cos u}, \quad y = \frac{a \sinh v \sin \phi}{\cosh v - \cos u}, \quad z = \frac{a \sin u}{\cosh v - \cos u}$$

22.104 $$h_1^2 = h_2^2 = \frac{a^2}{(\cosh v - \cos u)^2}, \quad h_3^2 = \frac{a^2 \sinh^2 v}{(\cosh v - \cos u)^2}$$

22.105 $$\nabla^2\Phi = \frac{(\cosh v - \cos u)^3}{a^2} \frac{\partial}{\partial u}\left(\frac{1}{\cosh v - \cos u} \frac{\partial \Phi}{\partial u}\right) + \frac{(\cosh v - \cos u)^3}{a^2 \sinh v} \frac{\partial}{\partial v}\left(\frac{\sinh v}{\cosh v - \cos u} \frac{\partial \Phi}{\partial v}\right) + \frac{(\cosh v - \cos u)^2}{a^2 \sinh^2 v} \frac{\partial^2 \Phi}{\partial \phi^2}$$

The coordinate surfaces are obtained by revolving the curves of Fig. 22-16 about the y axis which is relabeled the z axis.

Conical Coordinates (λ, μ, ν)

22.106 $$x = \frac{\lambda \mu \nu}{ab}, \quad y = \frac{\lambda}{a}\sqrt{\frac{(\mu^2 - a^2)(\nu^2 - a^2)}{a^2 - b^2}}, \quad z = \frac{\lambda}{b}\sqrt{\frac{(\mu^2 - b^2)(\nu^2 - b^2)}{b^2 - a^2}}$$

22.107 $$h_1^2 = 1, \quad h_2^2 = \frac{\lambda^2(\mu^2 - \nu^2)}{(\mu^2 - a^2)(b^2 - \mu^2)}, \quad h_3^2 = \frac{\lambda^2(\mu^2 - \nu^2)}{(\nu^2 - a^2)(\nu^2 - b^2)}$$

Confocal Ellipsoidal Coordinates (λ, μ, ν)

22.108
$$\begin{cases} \dfrac{x^2}{a^2-\lambda} + \dfrac{y^2}{b^2-\lambda} + \dfrac{z^2}{c^2-\lambda} = 1 & \lambda < c^2 < b^2 < a^2 \\ \dfrac{x^2}{a^2-\mu} + \dfrac{y^2}{b^2-\mu} + \dfrac{z^2}{c^2-\mu} = 1 & c^2 < \mu < b^2 < a^2 \\ \dfrac{x^2}{a^2-\nu} + \dfrac{y^2}{b^2-\nu} + \dfrac{z^2}{c^2-\nu} = 1 & c^2 < b^2 < \nu < a^2 \end{cases}$$

or

22.109
$$\begin{cases} x^2 = \dfrac{(a^2-\lambda)(a^2-\mu)(a^2-\nu)}{(a^2-b^2)(a^2-c^2)} \\ y^2 = \dfrac{(b^2-\lambda)(b^2-\mu)(b^2-\nu)}{(b^2-a^2)(b^2-c^2)} \\ z^2 = \dfrac{(c^2-\lambda)(c^2-\mu)(c^2-\nu)}{(c^2-a^2)(c^2-b^2)} \end{cases}$$

22.110
$$\begin{cases} h_1^2 = \dfrac{(\mu-\lambda)(\nu-\lambda)}{4(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)} \\ h_2^2 = \dfrac{(\nu-\mu)(\lambda-\mu)}{4(a^2-\mu)(b^2-\mu)(c^2-\mu)} \\ h_3^2 = \dfrac{(\lambda-\nu)(\mu-\nu)}{4(a^2-\nu)(b^2-\nu)(c^2-\nu)} \end{cases}$$

Confocal Paraboloidal Coordinates (λ, μ, ν)

22.111
$$\begin{cases} \dfrac{x^2}{a^2-\lambda} + \dfrac{y^2}{b^2-\lambda} = z - \lambda & -\infty < \lambda < b^2 \\ \dfrac{x^2}{a^2-\mu} + \dfrac{y^2}{b^2-\mu} = z - \mu & b^2 < \mu < a^2 \\ \dfrac{x^2}{a^2-\nu} + \dfrac{y^2}{b^2-\nu} = z - \nu & a^2 < \nu < \infty \end{cases}$$

or

22.112
$$\begin{cases} x^2 = \dfrac{(a^2-\lambda)(a^2-\mu)(a^2-\nu)}{b^2-a^2} \\ y^2 = \dfrac{(b^2-\lambda)(b^2-\mu)(b^2-\nu)}{a^2-b^2} \\ z = \lambda + \mu + \nu - a^2 - b^2 \end{cases}$$

22.113
$$\begin{cases} h_1^2 = \dfrac{(\mu-\lambda)(\nu-\lambda)}{4(a^2-\lambda)(b^2-\lambda)} \\ h_2^2 = \dfrac{(\nu-\mu)(\lambda-\mu)}{4(a^2-\mu)(b^2-\mu)} \\ h_3^2 = \dfrac{(\lambda-\nu)(\mu-\nu)}{16(a^2-\nu)(b^2-\nu)} \end{cases}$$