18 BASIC DIFFERENTIAL EQUATIONS and SOLUTIONS

DIFFERENTIAL EQUATION	SOLUTION
18.1 Separation of variables $f_1(x)\,g_1(y)\,dx + f_2(x)\,g_2(y)\,dy = 0$	$\int \frac{f_1(x)}{f_2(x)}\,dx + \int \frac{g_2(y)}{g_1(y)}\,dy = c$
18.2 Linear first order equation $\frac{dy}{dx} + P(x)y = Q(x)$	$y\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$
18.3 Bernoulli's equation $\frac{dy}{dx} + P(x)y = Q(x)y^n$	$v\,e^{(1-n)\int P\,dx} = (1-n)\int Q\,e^{(1-n)\int P\,dx}\,dx + c$ where $v = y^{1-n}$. If $n = 1$, the solution is $\ln y = \int (Q - P)\,dx + c$
18.4 Exact equation $M(x,y)\,dx + N(x,y)\,dy = 0$ where $\partial M/\partial y = \partial N/\partial x$.	$\int M\,\partial x + \int \left(N - \frac{\partial}{\partial y}\int M\,\partial x\right) dy = c$ where ∂x indicates that the integration is to be performed with respect to x keeping y constant.
18.5 Homogeneous equation $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$	$\ln x = \int \frac{dv}{F(v) - v} + c$ where $v = y/x$. If $F(v) = v$, the solution is $y = cx$.

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18.6 $y\,F(xy)\,dx + x\,G(xy)\,dy = 0$	$\ln x = \int \frac{G(v)\,dv}{v\{G(v)-F(v)\}} + c$ where $v = xy$. If $G(v) = F(v)$, the solution is $xy = c$.
18.7 Linear, homogeneous second order equation $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$ a, b are real constants.	Let m_1, m_2 be the roots of $m^2 + am + b = 0$. Then there are 3 cases. **Case 1.** m_1, m_2 real and distinct: $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ **Case 2.** m_1, m_2 real and equal: $y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$ **Case 3.** $m_1 = p + qi$, $m_2 = p - qi$: $y = e^{px}(c_1 \cos qx + c_2 \sin qx)$ where $p = -a/2$, $q = \sqrt{b - a^2/4}$.
18.8 Linear, nonhomogeneous second order equation $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = R(x)$ a, b are real constants.	There are 3 cases corresponding to those of entry 18.7 above. **Case 1.** $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \frac{e^{m_1 x}}{m_1 - m_2}\int e^{-m_1 x} R(x)\,dx + \frac{e^{m_2 x}}{m_2 - m_1}\int e^{-m_2 x} R(x)\,dx$ **Case 2.** $y = c_1 e^{m_1 x} + c_2 x e^{m_1 x} + x e^{m_1 x}\int e^{-m_1 x} R(x)\,dx - e^{m_1 x}\int x e^{-m_1 x} R(x)\,dx$ **Case 3.** $y = e^{px}(c_1 \cos qx + c_2 \sin qx) + \frac{e^{px}\sin qx}{q}\int e^{-px} R(x) \cos qx\,dx - \frac{e^{px}\cos qx}{q}\int e^{-px} R(x) \sin qx\,dx$
18.9 Euler or Cauchy equation $x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = S(x)$	Putting $x = e^t$, the equation becomes $\frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = S(e^t)$ and can then be solved as in entries 18.7 and 18.8 above.

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18.10 Bessel's equation $$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2x^2 - n^2)y = 0$$	$$y = c_1J_n(\lambda x) + c_2Y_n(x)$$ See pages 136-137.
18.11 Transformed Bessel's equation $$x^2\frac{d^2y}{dx^2} + (2p+1)x\frac{dy}{dx} + (\alpha^2x^{2r} + \beta^2)y = 0$$	$$y = x^{-p}\left\{c_1J_{q/r}\left(\frac{\alpha}{r}x^r\right) + c_2Y_{q/r}\left(\frac{\alpha}{r}x^r\right)\right\}$$ where $q = \sqrt{p^2-\beta^2}$.
18.12 Legendre's equation $$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$	$$y = c_1P_n(x) + c_2Q_n(x)$$ See pages 146-148.