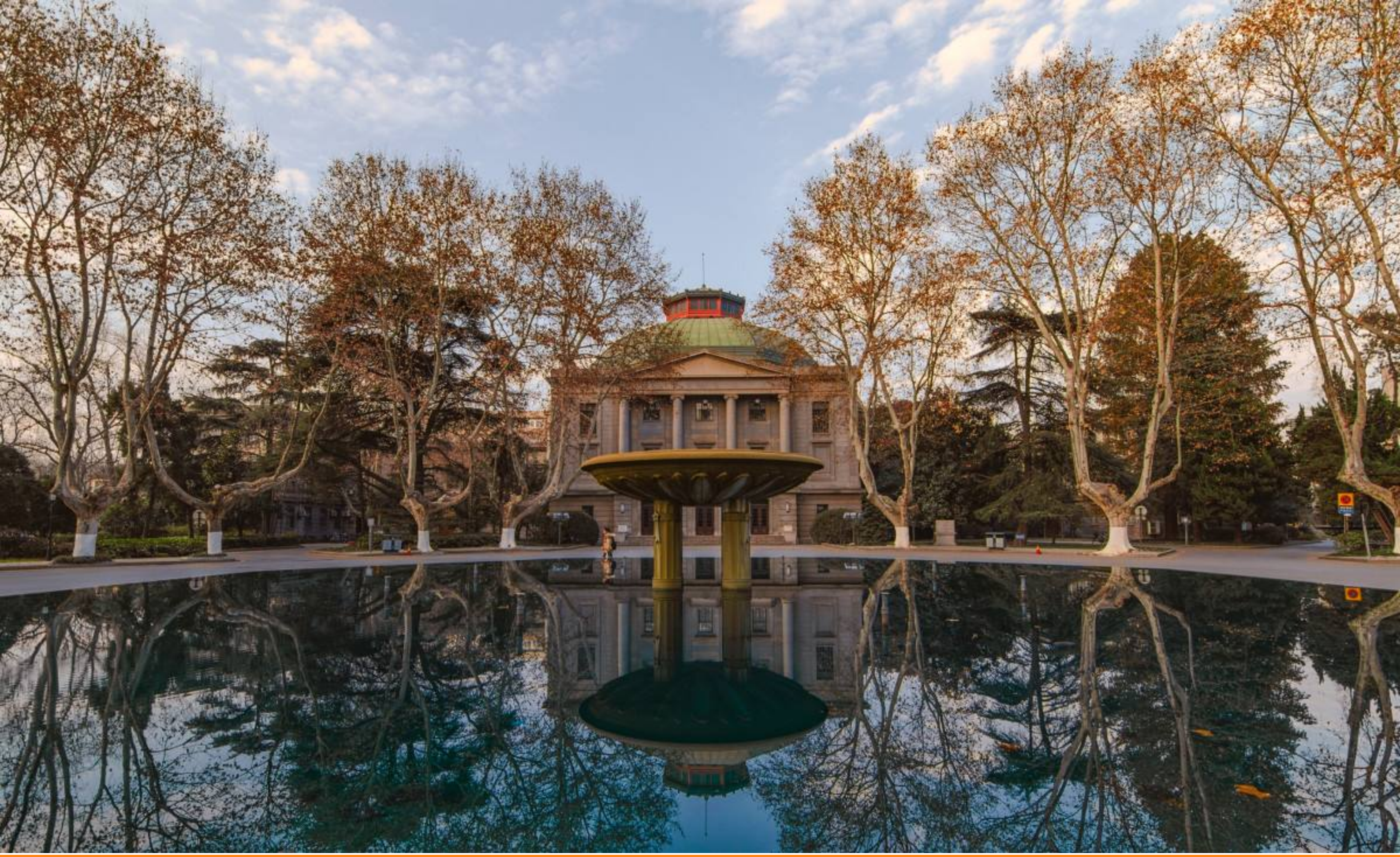

Topology I

Elementary Topology

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Rather less, but better. – Carl Friedrich Gauss

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1.1 Metric spaces

Introduction

- *Metric spaces*
- *Length spaces*
- *Gromov-Hausdorff distance*

Over the n-dimensional Euclidean space $\mathbf{R}^n$, we have a natural distance function

$$d(\boldsymbol{x}, \boldsymbol{y}) := \sqrt{\sum_{1\le i\le n}(x^i - y^i)^2}, \quad \boldsymbol{x} = (x^1, \cdots, x^n), \ \boldsymbol{y} = (y^1, \cdots, y^n),$$

which satisfies the following properties: d is always nonnegative (with zero if and only if $\boldsymbol{x} = \boldsymbol{y}$), symmetric, and $d(\boldsymbol{x}, \boldsymbol{z}) \le d(\boldsymbol{x}, \boldsymbol{y}) + d(\boldsymbol{y}, \boldsymbol{z})$.

1.1.1 Metric spaces

A **metric** on a set X is a map $d : X \times X \to \mathbf{R} \cup \{+\infty\}$ satisfying, for any $x, y, z \in X$,

(a) **(positiveness)** $d(x, y) \ge 0$ and $d(x, y) = 0$ if and only if $x = y$;

(b) **(symmetry)** $d(x, y) = d(y, x)$;

(c) **(triangle inequality)** $d(x, z) \le d(x, y) + d(y, z)$.

A **metric space** is a pair (X, d) where d is a metric on X.

Given a metric space (X, d) we define a relation $\sim$ on X by

$$x \sim y \Longleftrightarrow d(x, y) < \infty.$$

Then $\sim$ is an equivalence relation on X, and the equivalence class $[x]$ of x can be endowed with a natural metric (still denoted by d).

Exercise 1.1

For any $x \in X$, show that $([x], d)$ is a finite metric space.

A map $\boldsymbol{f} : (X, d_X) \to (Y, d_Y)$ between metric spaces is called **distance-preserving** if

$$d_Y(\boldsymbol{f}(x_1), \boldsymbol{f}(x_2)) = d_X(x_1, x_2)$$

for all $x_1, x_2 \in X$. A distance-preserving bijection is called an **isometry**.

A **semi-metric** on a set X is a map $d : X \times X \to \mathbf{R} \cup \{+\infty\}$ satisfying, for any $x, y, z \in X$,

(a) **(nonnegativity)** $d(x, y) \geq 0$;

(b) **(symmetry)** $d(x, y) = d(y, x)$;

(c) **(triangle inequality)** $d(x, z) \leq d(x, y) + d(y, z)$.

A **semi-metric space** is a pair (X, d) where d is a semi-metric on X.

Given a semi-metric space (X, d) we define a relation $\sim$ on X by

$$x \sim y \iff d(x, y) = 0.$$

Then $\sim$ is an equivalence relation on X. Define

$$\widehat{X} := X/\sim = \{[x] : x \in X\}, \quad \widehat{d}([x], [y]) := d(x, y).$$

Then $(X/d, d) := (\widehat{X}, \widehat{d})$ is a metric space.

Exercise 1.2

Show that $\widehat{d}$ is well-defined and $(X/d, d)$ is a metric space.

Exercise 1.3

Let $X = \mathbf{R}^2$ and define

$$d((x, y), (x', y')) := |(x - x') + (y - y')|.$$

Show that d is a semi-metric on X. Define $\boldsymbol{f} : \mathbf{R}^2/d \to \mathbf{R}$ by $\boldsymbol{f}([(x, y)]) := x + y$. Show that $\boldsymbol{f}$ is an isometry.

Exercise 1.4

Consider the set X of all continuous real-valued functions on $[0, 1]$. Show that

$$d(f, g) := \int_0^1 |f(x) - g(x)| dx$$

defines a metric on X. Is this still the case if continuity is weakened to integrability?

Exercise 1.5

Let (X, d_X) and (Y, d_Y) be two metric spaces. We define

$$d_{X \times Y}((x_1, y_1), (x_2, y_2)) := \sqrt{d_X(x_1, x_2) + d_Y(y_1, y_2)}.$$

Show that $(X \times Y, d_{X \times Y})$ is a metric space.

Exercise 1.6

As a subspace of $\mathbf{R}^2$, the unit circle $\mathbf{S}^1$ carries the restricted Euclidean metric from $\mathbf{R}^2$. We can define another (intrinsic) metric d_{int} by

$$d_{\text{int}}(\boldsymbol{x}, \boldsymbol{y}) := \text{the length of the shorter arc between them}.$$

Note that $d_{\text{int}}(\boldsymbol{x}, \boldsymbol{y}) \in [0, \pi]$ *for any points* $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{S}^1$.

(a) *Show that any circle arc of length less than or equal to* π*, equipped with the intrinsic metric, is isometric to a straight line segment.*

(b) *The whole circle with the intrinsic metric is not isometric to any subset of* $\mathbf{R}^2$.

Let (X, d) be a metric space. Set

$$B(x, r) := \{y \in X : d(x, y) < r\}, \quad \overline{B(x, r)} := \{y \in X : d(x, y) \leq r\}.$$

We define the **metric topology** on X by saying that $U \subseteq X$ is open if and only for any $x \in U$ there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$. Let

$$\mathscr{T} := \{\text{all open subsets in } X\} \cup \{\emptyset, X\}. \tag{1.1.1.1}$$

The set $\mathscr{T}$ satisfies the following

(1) $U \cap V \in \mathscr{T}$ for any $U, V \in \mathscr{T}$;

(2) $\cup_{i \in I} U_i \in \mathscr{T}$ for any collection $\{U_i\}_{i \in I} \subset \mathscr{T}$;

(3) $\emptyset, X \in \mathscr{T}$.

In general, we can introduce a topology on any set X. A **topology** on X is a subset $\mathscr{T}$ of 2^X satisfying

(1) $U \cap V \in \mathscr{T}$ for any $U, V \in \mathscr{T}$;

(2) $\cup_{i \in I} U_i \in \mathscr{T}$ for any collection $\{U_i\}_{i \in I} \subset \mathscr{T}$;

(3) $\emptyset, X \in \mathscr{T}$.

The pair $(X, \mathscr{T})$ is called the **topological space**, and any element in $\mathscr{T}$ is said to be **open**; a **closed set** is a subset of X whose complement in X is open. In particular, we see that any metric space must be a topological space.

Given a metric space (X, d).

- (X, d) is **complete** if any Cauchy sequence[1] in X has a limit in X.
- A **diameter** of $S \subset X$ is defined by

$$\operatorname{diam}(S) := \sup_{x, y \in S} d(x, y). \tag{1.1.1.2}$$

- (X, d) is **compact** if any sequence in X has a converging subsequence.
- $S \subset X$ is an **ϵ-net** if $S_\epsilon := \{x \in X : d(x, S) = \inf_{y \in X} d(x, y) \leq \epsilon\} = X$.
- X is **totally bounded** if for any $\epsilon > 0$ there exists a finite ϵ-net in X.

The following theorem is a basic result.

Theorem 1.1

Let (X, d) *be a metric space. Then* X *is compact if and only if* X *is complete and totally bounded.*

[1] A sequence $\{x_n\}_{n \in \mathbf{N}}$ in X is said to be **Cauchy** if for any $\epsilon > 0$ there exists an integer $n_0 \in \mathbf{N}$ such that $d(x_n, x_m) < \epsilon$ whenever $n, m \geq n_0$.

1.1.2 Length spaces

Let $(X, \mathscr{T})$ be a topological space. We say X is **Hausdorff** if for any different points $x, y \in X$, there exist open sets $U_x, U_y \subseteq X$ such that $x \in U_x$, $y \in U_y$, and $U_x \cap U_y = \emptyset$. Clearly that any metric space must be Hausdorff.

A **path** γ in X is a continuous[2] map $\gamma : I \to X$. where $I \subseteq \mathbf{R}$ is an interval (i.e., connected subset) of $\mathbf{R}$. A **length structure** on X is a pair $(\mathscr{A}, L)$, where $\mathscr{A}$ is a collection of paths (called **admissible paths**) and $L : \mathscr{A} \to \overline{\mathbf{R}_+}$ is a map on $\mathscr{A}$ (called the **length of paths**), satisfying

(a) ($\mathscr{A}$ is closed under restrictions) If $\gamma : [a,b] \to X$ in $\mathscr{A}$, then $\gamma|_{[c,d]} \in \mathscr{A}$ for any subinterval $[c,d] \subseteq [a,b]$;

(b) ($\mathscr{A}$ is closed under concatenations of paths) If $\gamma_1 : [a,c] \to X$ and $\gamma_2 : [c,d] \to X$ are admissible with $\gamma_1(c) = \gamma_2(c)$, then the obvious product $\gamma := \gamma_1 \cdot \gamma_2 : [a,d] \to X$ is also admissible;

(c) ($\mathscr{A}$ is closed under reparameterizations) If $\gamma : [a,b] \to X$ is admissible and $\varphi : [c,d] \to [a,b]$ is a homeomorphism, then $\gamma \circ \varphi : [c,d] \to x$ is also admissible;

(d) (Length of paths is additive) $L(\gamma|_{[a,b]}) = L(\gamma|_{[a,c]}) + L(\gamma|_{[c,b]})$ for any $c \in [a,b]$ and $\gamma : [a,b] \to X$ in $\mathscr{A}$;

(e) (The length of a piece of a path continuously depends on the piece) If $\gamma : [a,b] \to X$ is a admissible path with $L(\gamma) < +\infty$, then $L(\gamma, a, t) := L(\gamma_{[a,t]})$ is continuous in t;

(f) ($L(\cdot)$ is invariant under reparametrizations) $L(\gamma \circ \varphi) = L(\gamma)$ for any admissible path γ and any homeomorphism φ as in (c);

(g) (Length structures agree with the topology of X) For any $x \in X$ and open set $U_x \ni x$, one has
$$\inf_{\gamma \in \mathscr{A}} \{L(\gamma) : \gamma(a) = x,\ \gamma(b) \in X \setminus U_x\} > 0.$$

Let $(X, \mathscr{T})$ is a Hausdorff space with a length structure $(\mathscr{A}, L)$. Define
$$d_L(x,y) := \inf_{\gamma \in \mathscr{A}} \{L(\gamma) | \gamma : [a,b] \to X,\ \gamma(a) = x,\ \gamma(b) = y\}. \tag{1.1.2.1}$$
If there is no path between x and y, we set $d_L(x,y) = +\infty$. We say a length structure $(\mathscr{A}, L)$ is **complete** if for any $x, y \in X$ with $d_L(x,y) < +\infty$, there exists a path $\gamma \in \mathscr{A}$ joining x and y such that $d_L(x,y) = L(\gamma)$.

Exercise 1.7

Show that (X, d_L) is a metric space, where d_L is defined in (1.1.2.1). ♠

Note that d_L is not necessarily a finite metric.

Definition 1.1. (Length space)

*A **length space** is a metric space (X, d) such that $d = d_L$ for some length structure $(\mathscr{A}, L)$. In this case, we call d as an **intrinsic metric** on X.* ♣

[2]That is, $\gamma^{-1}(U)$ is open in I for any open set U in X.

In the following we shall construct a length structure on a given metric space (X, d). If $\gamma : [a, b] \to X$ is a path in X, for any partition $\{y_i\}_{0 \le i \le N}$ of $[a, b]$ with $a = y_0 \le y_1 \le y_2 \le \cdots \le y_N = b$, we define

$$\Sigma\left(\gamma, \{y_i\}_{0 \le i \le N}\right) := \sum_{1 \le i \le N} d\left(\gamma(y_{i-1}), \gamma(y_i)\right).$$

The **length of** γ is given by

$$L_d(\gamma) := \sup_{\{y_i\}_{0 \le i \le N}} \Sigma\left(\gamma, \{y_i\}_{0 \le i \le N}\right). \tag{1.1.2.2}$$

The path γ is said to be **rectifiable** if $L_d(\gamma) < +\infty$. The length structure $(\mathscr{A}, L)$ now is defined by

$$\mathscr{A} := \{\text{all paths parametrized by closed intervals}\}, \quad L := L_d. \tag{1.1.2.3}$$

Theorem 1.2

Let (X, d) be a metric space, for the length structure $(\mathscr{A}, L)$ defined in (1.1.2.3), we have the following properties:

(a) $L(\gamma) \ge d(\gamma(a), \gamma(b))$ *for any path* $\gamma : [a, b] \to X$*;*

(b) $L(\gamma|_{[a,c]}) + L(\gamma|_{[c,b]}) = L(\gamma)$ *for any path* $\gamma : [a, b] \to X$ *and* $a < c < b$*;*

(c) If γ *is rectifiable, then* $L(\gamma, c, d) := L(\gamma|_{[c,d]})$ *is continuous in* c *and* d*;*

(d) L *is a lower semi-continuous functional on* $C([a, b], X)$ *with respect to the pointwise convergence; that is, if* $\{\gamma_i\}_{i \in \mathbf{N}}$ *is a sequence of rectifiable paths in* X *with the same domain and* $\gamma_i(t)$ *converges to* $\gamma(t)$ *for every given* $t \in [a, b]$*, then* $L(\gamma) \le \liminf_{i \to \infty} L(\gamma_i)$.

♡

Proof. (a) For any partition $\{y_i\}_{0 \le i \le N}$ of $[a, b]$ we have

$$\Sigma\left(b\gamma, \{y_i\}_{0 \le i \le N}\right) = \sum_{1 \le i \le M} d\left(\gamma(y_{i-1}), \gamma(y_i)\right) \ge d(\gamma(y_0), \gamma(y_N)) = d(\gamma(a), \gamma(b))$$

by the triangle inequality, so that $L(\gamma) \ge d(\gamma(a), \gamma(b))$.

(b) Given partitions $\{y'_i\}_{0 \le i \le N'}$ and $\{y''_j\}_{0 \le j \le N''}$ of $[a, c]$ and $[c, b]$, respectively. Then $\{y'_i\}_{0 \le i \le N'} \cup \{y''_j\}_{0 \le j \le N''}$ forms a partition of $[a, b]$ and

$$\begin{aligned} L(\gamma) &\ge \Sigma\left(\gamma, \{y'_i\}_{0 \le i \le N'} \cup \{y''_j\}_{0 \le j \le N''}\right) \\ &= \sum_{1 \le i \le N} d\left(\gamma(y'_{i-1}), \gamma(y'_i)\right) + \sum_{1 \le j \le N''} d\left(\gamma(y''_{j-1}), \gamma(y''_j)\right) \\ &= \Sigma\left(\gamma|_{[a,c]}, \{y'_i\}_{0 \le i \le N'}\right) + \Sigma\left(\gamma|_{[c,b]}, \{y''_j\}_{0 \le j \le N''}\right). \end{aligned}$$

Hence $L(\gamma) \ge L(\gamma|_{[a,c]}) + L(\gamma|_{[c,b]})$.

Conversely, for any partition $\{y_i\}_{0 \le i \le N}$, we see that $\{y_i\}_{0 \le i \le N} \cup \{c\}$ is also a partition of $[a, b]$. Consequently, we get

$$L(\gamma|_{[a,c]}) + L(\gamma|_{[c,b]}) \ge \Sigma\left(\gamma, \{y_i\}_{0 \le i \le N} \cup \{c\}\right) \ge \Sigma\left(\gamma, \{y_i\}_{0 \le i \le N}\right)$$

and then $L(\gamma|_{[a,c]}) + L(\gamma|_{[c,b]}) \ge L(\gamma)$.

(c) We only prove that $L(\boldsymbol{\gamma}, c, d)$ is continuous in $d \in (a, b]$. For any $\epsilon > 0$, since $L(\boldsymbol{\gamma}) < +\infty$), there exists a partition $\{y_i\}_{0\le i\le N}$ of $[a, b]$ such that $0 \le L(\boldsymbol{\gamma}) - \Sigma(\boldsymbol{\gamma}, \{y_i\}_{0\le i\le N}) < \epsilon$. We may assume that $y_{j-1} < d = y_j$ for some j, otherwise we can use the partition $\{y_i\}_{0\le i\le N} \cup \{d\}$. Then

$$L(\boldsymbol{\gamma}|_{[y_{j-1},d]}) - d(\boldsymbol{\gamma}(y_{j-1}), \boldsymbol{\gamma}(d)) \le L(\boldsymbol{\gamma}) - \Sigma(\boldsymbol{\gamma}, (y_i)_{0\le i\le N}) < \epsilon$$

by (a) and (b). For any $d' \in [y_{j-1}, d]$, one has

$$\begin{aligned} L\left(\boldsymbol{\gamma}|_{[y_{j-1},d]}\right) - L\left(\boldsymbol{\gamma}|_{[y_{j-1},d']}\right) &\le \epsilon + d\left(\boldsymbol{\gamma}(y_{j-1}), \boldsymbol{\gamma}(d)\right) - L\left(\boldsymbol{\gamma}|_{[y_{j-1},d']}\right) \\ &\le \epsilon + d\left(\boldsymbol{\gamma}(y_{j-1}), \boldsymbol{\gamma}(d)\right) - d\left(\boldsymbol{\gamma}(y_{j-1}), \boldsymbol{\gamma}(d')\right) \\ &\le \epsilon + d(\boldsymbol{\gamma}(d'), \boldsymbol{\gamma}(d)). \end{aligned}$$

Since $\boldsymbol{\gamma}$ is continuous, it follows that $L(\boldsymbol{\gamma}, c, d') - L(\boldsymbol{\gamma}, c, d) = L(\boldsymbol{\gamma}, d', d) \le 2\epsilon$ when d' is very close to d. Thus $L(\boldsymbol{\gamma}, c, d)$ is continuous in d.

(d) We first assume that $L(\boldsymbol{\gamma}) < +\infty$. For any $\epsilon > 0$ there exists a partition $\{y_i\}_{0\le i\le N}$ of $[a, b]$ such that $L(\boldsymbol{\gamma}) - \Sigma(\boldsymbol{\gamma}, \{y_i\}_{0\le i\le N}) < \epsilon$. Choose $j \gg 1$ so that $d(\boldsymbol{\gamma}_j(y_i), \boldsymbol{\gamma}(y_i)) < \epsilon/N$ for each $0 \le i \le N$. Compute

$$L(\boldsymbol{\gamma}) \le \Sigma(\boldsymbol{\gamma}, \{y_i\}_{0\le i\le N}) + \epsilon \le \Sigma(\boldsymbol{\gamma}_j, \{y_i\}_{0\le i\le N}) + \epsilon + \frac{\epsilon}{N} \cdot 2 \cdot N \le L(\boldsymbol{\gamma}_j) + 3\epsilon$$

where we used

$$\begin{aligned} d\left(\boldsymbol{\gamma}(y_{i-1}), \boldsymbol{\gamma}(y_i)\right) &\le d\left(\boldsymbol{\gamma}(y_{i-1}), \boldsymbol{\gamma}_j(y_{i-1})\right) + d\left(\boldsymbol{\gamma}_j(y_{i-1}), \boldsymbol{\gamma}_j(y_i)\right) + d\left(\boldsymbol{\gamma}(y_i), \boldsymbol{\gamma}_j(y_i)\right) \\ &\le \frac{\epsilon}{N} \cdot 2 + d\left(\boldsymbol{\gamma}_j(y_{i-1}), \gamma_j(y_i)\right). \end{aligned}$$

Therefore $L(\boldsymbol{\gamma}) \le \liminf_{j\to\infty} L(\boldsymbol{\gamma}_j)$.

When $L(\boldsymbol{\gamma}) = +\infty$, we choose a partition $\{y_i\}_{0\le i\le N}$ such that $\Sigma(\boldsymbol{\gamma}, \{y_i\}_{0\le i\le N}) > 1/\epsilon$. Then

$$\frac{1}{\epsilon} < \Sigma(\boldsymbol{\gamma}, \{y_i\}_{0\le i\le N}) \le \Sigma(\boldsymbol{\gamma}_j, \{y_i\}_{0\le i\le N}) + \frac{\epsilon}{N} \cdot 2 \cdot N \le L(\boldsymbol{\gamma}_j) + 2\epsilon$$

for $j \gg 1$. Letting $j \to \infty$ yields $L(\boldsymbol{\gamma}_j) \to \infty$. □

For a metric space (X, d), we have a length structure $(\mathscr{A}, L)$, where $L := L_d$, and then the **induced intrinsic metric**

$$d^\wedge := d_L. \tag{1.1.2.4}$$

However, $d^\wedge$ may be equal to d. As a direct consequence of Theorem 1.2, $d \le d^\wedge$ always holds.

Example 1.1

Consider the subset

$$X := \bigcup_{n\in\mathbf{N}} \left[(0,1), \left(\frac{1}{n}, 0\right)\right] \cup [(0,1),(0,0)] \subset \mathbf{R}^2.$$

For the standard metric $d := d_{\mathbf{R}^2}|_X$ *we see that* $d((1/n, 0), (0, 0)) = 1/n \to 0$ *as* $n \to \infty$. *However the limit* $\lim_{n\to\infty} d^\wedge((1/n, 0), (0, 0))$ *does not exist; in fact, for any* *path γ connecting $(1/n, 0)$ with $(1/m, 0)$, we have*

$$\begin{aligned} L(\gamma) &= L_d(\gamma) \geq d\left(\left(\frac{1}{n}, 0\right), (0,0)\right) + d\left(\left(\frac{1}{m}, 0\right), (0,0)\right) \\ &= \sqrt{1 + \frac{1}{n^2}} + \sqrt{1 + \frac{1}{m^2}} > 2. \end{aligned}$$

Given a metric space (X, d), we have the following diagram:

$$\begin{tikzcd} (X, d) \arrow[r] & (\mathscr{A}, L = L_d) \arrow[d] \\ & (X, d^\wedge = d_L) \arrow[d] \\ (X, (d^\wedge)^\wedge = d_{\widehat{L}}) & (\mathscr{A}, \widehat{L} = L_{d^\wedge}) \arrow[l] \end{tikzcd}$$

by Theorem 1.2. A natural question is whether $d = (d^\wedge)^\wedge$.

Theorem 1.3

Let (X, d) be a metric space and $\widehat{d}$ the intrinsic metric induced by d.

(a) If $L_d(\gamma) < +\infty$, then $L_d(\gamma) = L_{d^\wedge}(\gamma)$.

(b) The intrinsic metric induced by $d^\wedge$ coincides with $d^\wedge$, i.e., $(d^\wedge)^\wedge = d^\wedge$.

Proof. (a) For any two points $x, y \in X$, choose a path $\gamma : [a, b] \to X$ from x to y. Then

$$L_d(\gamma) \geq d(\gamma(a), \gamma(b)) = d(x, y)$$

which implies $d(x, y) \leq d^\wedge(x, y)$ and hence $L_d(\gamma) \leq L_{d^\wedge}(\gamma)$.

Conversely, for any path $\gamma : [a, b] \to X$ and any partition $\{y_i\}_{0 \leq i \leq N}$ of the interval $[a, b]$, we have

$$\sum_{\widehat{d}} (\gamma, \{y_i\}_{0 \leq i \leq N}) \leq \sum_{1 \leq i \leq N} d^\wedge(\gamma(y_{i-1}), \gamma(y_i)) \leq \sum_{1 \leq i \leq N} L_d\left(\gamma|_{[y_{i-1}, y_i]}\right) = L_d(\gamma)$$

according to Theorem 1.2 (b). Part (b) follows from an obvious way. □

Example 1.2

Let $(\mathcal{M}, \boldsymbol{g})$ be a connected Riemannian manifold. Define

$$L(x, y) = L_d(\gamma) := \int_a^b |\dot{\gamma}(t)|_{\boldsymbol{g}} dt, \tag{1.1.2.5}$$

where

$$d(x, y) = d_{\boldsymbol{g}}(x, y) := \inf\left\{L_{\boldsymbol{g}}(\gamma) | \gamma \text{ is a piecewise } C^1\text{-path joining } x \text{ to } y\right\}. \tag{1.1.2.6}$$

Then $(\mathcal{M}, d)$ is a metric space.

1.1.3 Gromov-Hausdorff distance

In this subsection, all proofs can be found in BBI's book "A course in metric geometry".

Let (Z, d_Z) be a metric space. Given an $\epsilon > 0$, the **ϵ-neighborhood** of a subset $S \subset Z$ is given by

$$S_\epsilon := \left\{ z \in Z : d_Z(z, S) = \inf_{x \in S} d_Z(z, x) \le \epsilon \right\}. \tag{1.1.3.1}$$

We say a subset $S \subset X$ is an **ϵ-net** if $S_\epsilon = X$. The **Hausdorff distance** of $A, B \subset Z$ is defined to be the quantity

$$d^Z_{\mathbf{H}}(A, B) := \inf \{\epsilon > 0 : A \subset B_\epsilon \text{ and } B \subset A_\epsilon\}. \tag{1.1.3.2}$$

If no such ϵ, we put $d^Z_{\mathbf{H}}(A, B) = +\infty$. A basic fact is that the set of all compact subsets of Z, equipped with $d^Z_{\mathbf{H}}$, forms a metric space.

For any metric spaces (X, d_X) and (Y, d_Y), we define **Gromov-Hausdorff distance** between them by

$$\begin{aligned} & d_{\mathbf{GH}}\left((X, d_X), (Y, d_Y)\right) \\ := & \inf_{(Z, d_Z), \boldsymbol{f}, \boldsymbol{g}} \left\{ d^Z_{\mathbf{H}}\left(\boldsymbol{f}(X), \boldsymbol{g}(Y)\right) : \begin{array}{c} \boldsymbol{f} : (X, d_X) \to (Z, d_Z) \\ \boldsymbol{g} : (Y, d_Y) \to (Z, d_Z) \\ \text{are isometric embeddings} \\ \text{into a metric space } (Z, d_Z) \end{array} \right\}. \end{aligned} \tag{1.1.3.3}$$

One can show that if $(X, d_X), (Y, d_Y)$ are compact metric spaces with zero Gromov-Hausdorff distance, then (X, d_X) and (Y, d_Y) are isometric. Therefore, the set of all compact metric space, modulo the isometric equivalence, is a metric space with respect to $d_{\mathbf{GH}}$.

A **distortion** of a map $\boldsymbol{f} : (X, d_X) \to (Y, d_Y)$ between metric spaces is defined to be

$$\operatorname{dis}(\boldsymbol{f}) := \sup_{x_1, x_2 \in X} \left| d_X(x_1, x_2) - d_Y(\boldsymbol{f}(x_1), \boldsymbol{f}(x_2)) \right|. \tag{1.1.3.4}$$

Note that $\operatorname{dis}(\boldsymbol{f}) = 0$ if and only if $\boldsymbol{f}$ is distance-preserving. Such a $\boldsymbol{f}$ is called an **ϵ-isometry** if

$$\operatorname{dis}(\boldsymbol{f}) \le \epsilon \quad \text{and} \quad \boldsymbol{f}(X) \text{ is an } \epsilon\text{-net in } Y.$$

The following result characterizes the Gromov-Hausdorff distances with ϵ-isometries.

Lemma 1.1

Let (X, d_X) and (Y, d_Y) be metric spaces and $\epsilon > 0$.

(a) If $d_{\mathbf{GH}}((X, d_X), (Y, d_Y)) < \epsilon$, then there exist ϵ-isometries $\boldsymbol{f} : (X, d_X) \to (Y, d_Y)$ and $g : (Y, d_Y) \to (X, d_X)$.

(b) If $\boldsymbol{f} : (X, d_X) \to (Y, d_Y)$ is an ϵ-isometry, then $d_{\mathbf{GH}}((X, d_X), (Y, d_Y)) < 2\epsilon$. ♡

A **pointed metric space** is a triple (X, d_X, x_0), where (X, d_X) is a metric space and x_0 is a fixed base-point. A **pointed map** $\boldsymbol{f} : (X, d_X, x_0) \to (Y, d_Y, y_0)$ is a map $\boldsymbol{f} : (X, d_X) \to (Y, d_Y)$ which preserves the base-points (that is, $\boldsymbol{f}(x_0) = y_0$). An **$\epsilon$-pointed-isometry** $\boldsymbol{f}$:

$(X, d_X, x_0) \to (Y, d_Y, y_0)$ is a pointed map f satisfying

$$|d_X(x_1, x_2) - d_Y(\boldsymbol{f}(x_1), f(x_2))| < \epsilon$$

for any $x_1, x_2 \in B_X(x_0, \epsilon^{-1})$, and $B_Y(y_0, \epsilon^{-1}) \subset (\boldsymbol{f}(B_X(x_0, \epsilon^{-1})))_\epsilon$. For two pointed metric spaces (X, d_X, x_0) and (Y, d_Y, y_0) define the **pointed-Gromov-Hausdorff distance** as

$$\begin{aligned} & d_{\mathbf{GH}}^{\mathrm{pt}}((x, d_X, x_0), (Y, d_Y, y_0)) \\ := \quad & \inf \left\{ \epsilon > 0 : \begin{array}{c} \text{there exist } \epsilon\text{-pointed-isometries} \\ \boldsymbol{f} : (X, d_X, x_0) \to (Y, d_Y, y_0) \text{ and} \\ \boldsymbol{g} : (Y, d_Y, y_0) \to (X, d_X, x_0) \end{array} \right\}. \end{aligned} \tag{1.1.3.5}$$

Lemma 1.2

Given $\epsilon > 0$. If $d_{\mathbf{GH}}^{\mathrm{pt}}((X, d_X, x_0), (Y, d_Y, y_0)) < \epsilon$, then there exist 2ϵ-isometries $\boldsymbol{f} : (B_X(x_0, \epsilon^{-1}), d_X) \to (B_Y(y_0, \epsilon^{-1} + \epsilon), d_Y)$ with $\boldsymbol{f}(x_0) = y_0$ and $\boldsymbol{g} : (B_Y(y_0, \epsilon^{-1}), d_Y) \to (B_X(x_0, \epsilon^{-1} + \epsilon), d_X)$ with $\boldsymbol{g}(y_0) = x_0$. ♡

Proof. By definition, there exist 2ϵ-pointed-isometries $\boldsymbol{f} : (X, d_X, x_0) \to (Y, d_Y, y_0)$ and $\boldsymbol{g} : (Y, d_Y, y_0) \to (X, d_X, x_0)$. For any $x \in B_X(x_0, \epsilon^{-1})$ we have $|d_X(x, x_0) - d_Y(\boldsymbol{f}(x), y_0)| < \epsilon$ and then $\boldsymbol{f}(B_X(x_0, \epsilon^{-1})) \subseteq B_Y(y_0, \epsilon^{-1} + \epsilon)$. □

We say that a sequence of pointed metric spaces (X_i, d_{X_i}, x_i) **converges** to a pointed metric space (Y, d_Y, y_0) **in the sense of pointed Gromov-Hausdorff distance**, if

$$\lim_{i \to \infty} d_{\mathbf{GH}}^{\mathrm{pt}}((X_i, d_{X_i}, x_i), (Y, d_Y, y_0)) = 0.$$

A subset S in a topological space X is **precompact** if any sequence in S has a subsequence that converges to a point in X.

Theorem 1.4. (Gromov's precompactness theorem)

Let

$$\begin{aligned} \mathfrak{M} \quad &:= \quad \textit{a collection of compact metric spaces}, \\ \mathfrak{M}^{\mathrm{pt}} \quad &:= \quad \textit{a collection of pointed metric spaces}. \end{aligned}$$

(1) If $\mathfrak{M}$ is uniformly totally bounded, that is,

(i) there exists $D < \infty$ such that $\operatorname{diam}(X) \leq D$ for any $X \in \mathfrak{M}$, and

(ii) for any $\epsilon > 0$ there exists $N(\epsilon) \in \mathbf{N}$ such that any $X \in \mathfrak{M}$ contains an ϵ-net $S \subseteq X$ consisting of at most $N(\epsilon)$ points,

then $\mathfrak{M}$ is precompact in the collection of metric spaces with respect to the Gromov-Hausdorff distance.

(2) If $\mathfrak{M}^{\mathrm{pt}}$ has the property that

for any $\epsilon, \rho > 0$, there exists $N(\epsilon, \rho) \in \mathbf{N}$ such that any $(X, x_0) \in \mathfrak{M}^{\mathrm{pt}}$ contains an ϵ-net $S \subseteq B(x_0, \rho) \subseteq X$ consisting of at most $N(\epsilon, \rho)$ points, *then $\mathfrak{M}^{\mathrm{pt}}$ is precompact in the collection of pointed metric spaces with respect to the pointed Gromov-Hausdorff distance.* ♡

As a corollary of volume comparison theorem, we have

Theorem 1.5. (Gromov)

Given $K \in \mathbf{R}$, $m \geq 2$, let

$$\mathfrak{M}^{\mathrm{pt}}_{m,K} := \left\{ \begin{array}{c} \text{pointed complete Riemannian } m\text{-manifolds } (\mathcal{M}, \boldsymbol{g}, O) \\ \text{with } \mathbf{Ric}_{\boldsymbol{g}} \geq (m-1)K\boldsymbol{g} \text{ and } O \in \mathcal{M} \end{array} \right\}.$$

Then $\mathfrak{M}^{\mathrm{pt}}_{m,K}$ is precompact in the collection of pointed metric spaces with respect to pointed Gromov-Hausdorff distance. ♡

The **tangent cone** of a metric space (X, d) at $x \in X$ is given by

$$(T_x X, d_x, 0_x) := \lim_{\alpha_k \to \infty} (X, \alpha_k d, x) \tag{1.1.3.6}$$

provided this limit (in the sense of pointed Gromov-Hausdorff distance) exists for any sequence $\alpha_k \to \infty$ and is independent of $\{\alpha_k\}_{k \in \mathbf{N}}$ (up to isometry).

(1) The tangent cone is a metric space.

(2) The tangent cone of $(\mathcal{M}, \boldsymbol{g})$ at $p \in \mathcal{M}$ is isometric to $(T_x\mathcal{M}, \boldsymbol{g}(x), 0_x)$.

(3) For any $c > 0$, we have an isometry $(T_x X, c d_x, 0_x) \cong (T_x X, d_x, 0_x)$.

The **Gromov-Hausdorff asymptotic cone** of a metric space (X, d) at $x \in X$ is given by

$$(\mathbf{A}X, d_{\mathbf{A}X}, 0) := \lim_{\omega_i \to 0} (X, \omega_i d, x) \tag{1.1.3.7}$$

provided this limit exists and is independent (up to isometry) of the sequence $\{\omega_i\}_{i \in \mathbf{N}}$.

(1) For any $c > 0$, we have an isometry $(\mathbf{A}X, c d_{\mathbf{A}X}, 0) \cong (\mathbf{A}X, d_{\mathbf{A}X}, 0)$.

(2) If $(\mathcal{M}, \boldsymbol{g})$ is a complete noncompact Riemannian manifold with nonnegative sectional curvature, then the asymptotic cone exists and is isometric to a Euclidean metric cone. (For a proof of this result, read Theorem I.26 in *The Ricci Flow: Techniques and Applications. Part* III*: Geometric-Analytic Aspects*.)

The **topological cone** of a topological space X is defined by

$$\mathbf{Cone}(X) := (X \times [0, \infty))/(X \times \{0\}) = \{[(x, r)] : x \in X,\ r \in [0, \infty)\}. \tag{1.1.3.8}$$

We call the point $X \times \{0\}$ the **vertex** of $\mathbf{Cone}(X)$. It is clear that $\mathbf{Cone}(\mathbf{S}^{m-1}) \cong \mathbf{R}^m$.

We define the **Euclidean metric cone** of a metric space (X, d) with $\mathrm{diam}(X) \leq \pi$ as follows. It is a topological cone $\mathbf{Cone}(X)$ equipped with the metric $d_{\mathbf{Cone}(X)}$ defined by

$$d_{\mathbf{Cone}(X)}\left([(x_1, r_1)], [(x_2, r_2)]\right) := \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\left(d(x_1, x_2)\right)}. \tag{1.1.3.9}$$

One can show that $(\mathbf{Cone}(X), d_{\mathbf{Cone}(X)})$ is a metric space, and for any $c > 0$

$$d_{\mathbf{Cone}(X)}\left([(x_1, c r_1)], [(x_2, c r_2)]\right) = c\, d_{\mathbf{Cone}(X)}\left([(x_1, r_1)], [(x_2, r_2)]\right).$$

For Riemannian manifolds, we have the following equivalent description of Euclidean metric cones. Given a Riemannian manifold $(\mathcal{M}, \boldsymbol{g})$ with diameter $\leq \pi$, define

$$\boldsymbol{g}_{\mathbf{Cone}} := r^2\boldsymbol{g} + dr \otimes dr \tag{1.1.3.10}$$

on $\mathcal{M} \times (0, \infty)$. We can show that

$$d_{\boldsymbol{g}_{\mathbf{Cone}}} = d_{\mathbf{Cone}(\mathcal{M})} \tag{1.1.3.11}$$

on $\mathcal{M} \times (0, \infty) \subset \mathbf{Cone}(\mathcal{M})$. Indeed, for any $x_1, x_2 \in \mathcal{M}$ and $r_1, r_2 \in (0, \infty)$, one has

$$d_{\boldsymbol{g}_{\mathbf{Cone}}}((x_1, r_1), (x_2, r_2)) = d_{\mathbf{Cone}(\mathcal{M})}([(x_1, r_1)], [(x_2, r_2)]) . \tag{1.1.3.12}$$

In fact, given two points $x_1, x_2 \in \mathcal{M}$, let $\boldsymbol{\gamma} : [0, d(x_1, x_2)] \to \mathcal{M}$ be a unit speed minimal geodesic joining x_1 to x_2, where $d := d_{\boldsymbol{g}}$. Given $r_1, r_2 \in (0, \infty)$, join the two points $(x_1, r_1), (x_2, r_2) \in \mathcal{M} \times (0, \infty)$ by paths

$$\overline{\boldsymbol{\gamma}} : [0, d(x_1, x_2)] \longrightarrow \mathcal{M} \times (0, \infty), \quad u \longmapsto (\boldsymbol{\gamma}(u), r(u))$$

where $r : [0, d(x_1, x_2)] \to (0, \infty)$ satisfies $r(0) = r_1$ and $r(d(x_1, x_2)) = r_2$. The length of $\overline{\boldsymbol{\gamma}}$ with respect to $\boldsymbol{g}_{\mathbf{Cone}}$ is given by

$$L_{\boldsymbol{g}_{\mathbf{Cone}}}(\overline{\boldsymbol{\gamma}}) = \int_0^{d(x_1,x_2)} \sqrt{r(u)^2 + \dot{r}(u)^2}\, du.$$

For any variation $\eth r = s$ with $s(0) = s(d(x_1, x_2)) = 0$, we have

$$\begin{aligned}
\eth L_{\boldsymbol{g}_{\mathbf{Cone}}}(\overline{\boldsymbol{\gamma}}) &= \int_0^{d(x_1,x_2)} \left[r(u)^2 + \dot{r}(u)^2\right]^{-1/2} \left[r(u)s(u) + \dot{r}(u)\dot{s}(u)\right] du \\
&= \int_0^{d(x_1,x_2)} \left[r(u)^2 + \dot{r}(u)^2\right]^{-1/2} r(u)s(u) \\
&\quad - \int_0^{d(x_1,x_2)} \left\{ - \left[r(u)^2 + \dot{r}(u)^2\right]^{-3/2} \left[r(u)\dot{r}(u) + \dot{r}(u)\ddot{r}(u)\right] \dot{r}(u) \right. \\
&\qquad \left. + \left[r(u)^2 + \dot{r}(u)^2\right]^{-1/2} \ddot{r}(u) \right\} s(u)\, du \\
&= \int_0^{d(x_1,x_2)} \left[r(u)^2 + \dot{r}(u)^2\right]^{-3/2} \left[-r(u)\ddot{r}(u) + 2\dot{r}(u)^2 + r(u)^2\right] r(u)s(u)\, du.
\end{aligned}$$

The Euler-Lagrange equation is now of the form

$$r(u) = \frac{1}{a\cos u + b\sin u}$$

for some $a, b \in \mathbf{R}$. Let $\alpha := d(x_1, x_2) \in [0, \pi]$ and consider the Euclidean planar triangle with side-angle-side equal to r_1-α-r_2 and vertices with polar coordinates $(r_1, 0), (0, 0)$, and (r_2, α). By the law of sines, we obtain

$$a = \frac{1}{r_1}, \quad b = \frac{r_2^{-1} - r_1^{-1}\cos\alpha}{\sin\alpha} = \frac{\frac{r_1}{r_2} - \cos\alpha}{r_1 \sin\alpha}$$

and

$$\begin{aligned}
d_{\boldsymbol{g}_{\mathbf{Cone}}}((x_1, r_1), (x_2, r_2)) &= \int_0^\alpha \sqrt{r(u)^2 + \dot{r}(u)^2}\, du \\
= \sqrt{a^2 + b^2} \int_0^\alpha \frac{du}{(a\cos u + b\sin u)^2} &= \sqrt{a^2 + b^2} \frac{-\cos u}{b(b\sin u + a\cos u)}\bigg|_0^\alpha
\end{aligned}$$

$$= \sqrt{a^2+b^2}\frac{ab^2\sin\alpha}{b\sin\alpha + a\cos\alpha} = \sqrt{r_1^2+r_2^2-2r_1r_2\cos\alpha}$$
$$= d_{\mathbf{Cone}(\mathcal{M}^m)}\left([(x_1,r_1)],[(x_2,r_2)]\right).$$

1.2 Topological spaces

Introduction

- Continuous maps and bases
- Categories: an introduction
- Subspaces
- Separation axioms
- Topological manifolds

A **topological space** is a pair $(X, \mathscr{T})$, where X is a set and $\mathscr{T}$ is a set of 2^X, satisfying
(1) $U \cap V \in \mathscr{T}$ for any $U, V \in \mathscr{T}$;
(2) $\cup_{i\in I}U_i \in \mathscr{T}$ for any collection $\{U_i\}_{i\in I} \subseteq \mathscr{T}$;
(3) $\emptyset, X \in \mathscr{T}$.
An element of $\mathscr{T}$ is called an **open subset** of X (or **open set** if both X and $\mathscr{T}$ are understood), while a subset in X is called **closed** it its complement in open.

Suppose that $(X, \mathscr{T})$ is a topological space and $A \subseteq X$. Define

$$\mathscr{T}_A := \{A \cap U | I \in \mathscr{T}\}.$$

Then clearly $(A, \mathscr{T}_A)$ is also a topological space.

If $(X, \mathscr{T})$ is a topological space, then, by definition, any intersection of finitely many open subsets of X is still an open subsets of X.

We have seen in Section 1.1 that any metric space is always a topological space. Hence the Euclidean space $\mathbf{R}^m$ is a topological space, whose topology is called the **Euclidean topology**. The following are some standard subsets of $\mathbf{R}^m$:

- The **unit interval**:
$$I = [0,1] = \{x \in \mathbf{R} | 0 \le x \le 1\}.$$
- The **(open) unit ball** in $\mathbf{R}^m$:
$$\mathbf{B}^m := \left\{\boldsymbol{x} = (x^1, \cdots, x^m) \in \mathbf{R}^m \middle| |\boldsymbol{x}| = \left(\sum_{1\le i\le m}(x^i)^2\right)^{1/2} < 1\right\}.$$
When $m = 2$, we call $\mathbf{B} := \mathbf{B}^2$ the **(open) unit disk**.
- The **(closed) unit ball** in $\mathbf{R}^m$:
$$\mathbf{D}^m \equiv \overline{\mathbf{B}}^m \equiv \overline{\mathbf{B}^m} := \{\boldsymbol{x} \in \mathbf{R}^m | |\boldsymbol{x}| \le 1\}.$$

When $m = 2$, we call $\mathbf{D} := \mathbf{D}^2$ the **closed unit disk**.

- The **(unit) circle**:

$$\mathbf{S}^1 := \{\boldsymbol{x} \in \mathbf{R}^2 | |\boldsymbol{x}| = 1\} = \{z \in \mathbf{C} | |z| = 1\}.$$

- The **(unit) m-sphere**:

$$\mathbf{S}^m := \{\boldsymbol{x} \in \mathbf{R}^{m+1} | |\boldsymbol{x}| = 1\}.$$

Let $(X, \mathscr{T})$ be a topological space. Then

- X and $\emptyset$ are closed subsets of X.
- Any union of finitely many closed subsets of X is a closed subset of X.
- Any intersection of arbitrarily many closed subsets of X is a closed subset of X.

For example, let $X = \{1, 2, 3\}$ and

$$\mathscr{T} := \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}\}.$$

Then $(X, \mathscr{T})$ is a topological space. Clearly $\emptyset$, $\{3\}$, $\{2, 3\}$, and $\{1, 2, 3\}$ are all closed subsets of X. However, the subset $\{2\}$ is neither open nor closed of X. To describe $\{2\}$, introduce the following concepts.

Suppose that $(X, \mathscr{T})$ is a topological space and A is any subset of X.

- The **closure of A in X** is

$$\mathbf{Cle}(A) \equiv \overline{A} := \bigcap \{B \subset X | B \supset A \text{ and } B \text{ is closed in } X\}.$$

Then $\overline{A}$ is closed in X.

- The **interior of A** is

$$\mathbf{Int}(A) \equiv \mathring{A} := \bigcup \{C \subset X | C \subset A \text{ and } C \text{ is open in } X\}.$$

Then $\mathring{A}$ is open in X.

- The **exterior of A** is

$$\mathbf{Ext}(A) := X \setminus \overline{A}.$$

- The **boundary of A** is

$$\mathbf{Bdy}(A) \equiv \partial A := X \setminus (\mathbf{Int}(A) \cup \mathbf{Ext}(A)).$$

By definition, we obtain

$$\mathring{A} \subseteq A \subseteq \overline{A}, \quad X = \mathbf{Int}(A) \coprod \mathbf{Ext}(A) \coprod \mathbf{Bdy}(A).$$

1.2.1 Continuous maps and bases

Let $(X, \mathscr{T})$ be a topological space. A **neighborhood** of $x \in X$ is a set $N \subseteq X$ containing an open subset $U \in \mathscr{T}$ with the property $x \in U \subset N$.

(1) An neighborhood may not be open.

(2) An neighborhood can be the entire space X.

(3) The intersection of two neighborhoods of x is also a neighborhood of x.

A **neighborhood basis** at $x \in X$ is a set $\mathscr{B}_x \subset 2^X$ of neighborhoods of x with the property that every neighborhood N of x in X contains some $B \in \mathscr{B}_x$.

Exercise 1.8

Let $(X, \mathscr{T})$ be a topological space and $A \subseteq X$ be any subset. It is easy to prove the following statements:

(1) A point is in $\mathbf{Int}(A)$ *if and only if it has a neighborhood contained in A.*

(2) A point is in $\mathbf{Ext}(A)$ *if and only if it has a neighborhood contained in $X \setminus A$.*

(3) A point is in $\mathbf{Bdy}(A)$ *if and only if every neighborhood of it contains both a point of A and a point if $X \setminus A$.*

(4) A point is in $\overline{A}$ if and only if every neighborhood of it contain a point of A.

(5) $\overline{A} = A \cup \partial A = \mathring{A} \cup \partial A$.

(6) $\mathbf{Int}(A)$ *and* $\mathbf{Ext}(A)$ *are open in X, while $\overline{A}$ and ∂A are closed in X.*

(7) The following are equivalent:

- *A is open in X.*
- $A = \mathbf{Int}(A)$.
- *A contains none of its boundary points.*
- *Every point of A has a neighborhood contained in A.*

(8) The following are equivalent:

- *A is closed in X.*
- $A = \overline{A}$.
- *A contains all of its boundary points.*
- *Every point of $X \setminus A$ has a neighborhood contained in $X \setminus A$.*

(9) $\overline{X \setminus A} = X \setminus \mathring{A}$ *and* $(X \setminus A)^\circ = X \setminus \overline{A}$.

(10) Let $\mathscr{A}$ be a collection of subsets of X. Show that

$$\overline{\bigcap_{A \in \mathscr{A}}} \subseteq \bigcap_{A \in \mathscr{A}} \overline{A}, \quad \overline{\bigcup_{A \in \mathscr{A}} A} \supseteq \bigcup_{A \in \mathscr{A}} \overline{A}$$

and

$$\left(\bigcap_{A \in \mathscr{A}} A\right)^\circ \subseteq \bigcap_{A \in \mathscr{A}} \mathring{A}, \quad \left(\bigcup_{A \in \mathscr{A}} A\right)^\circ \supseteq \bigcup_{A \in \mathscr{A}} \mathring{A}.$$

Suppose that $(X, \mathscr{T})$ is a topological space and $A \subseteq X$ is a subset.

- We say that $x \in X$ is a **limit point (or accumulation point, cluster point) of A** if every neighborhood of x contains a point of A other than x (note that x itself may not be in A).
- We say that $x \in X$ is an **isolated point of A** if x has a neighborhood U in X such that $U \cap X = \{x\}$.
- A is said to be **dense in X** if $\overline{A} = X$.

Clearly ever point of A is either a limit point or an isolated point, but not both.

Exercise 1.9

Let $(X, \mathscr{T})$ be a topological space and $A \subseteq X$. Show that

- *A is closed if and only if it contains all of its limit points.*
- *A is dense if and only if every nonempty open subset of X contains a point of A.*

Theorem 1.6

Let (X, d) be a metric space. If $A \subseteq X$, then

$$\overline{A} = \{x \in X : \lim_{n\to\infty} x_n = x \text{ for some sequence } \{x_n\}_{n\in\mathbf{N}} \subset A\}. \tag{1.2.1.1}$$

Proof. If $x \in X$ with $\lim_{n\to\infty} x_n = x$ for some sequence $\{x_n\}_{n\in\mathbf{N}} \subset A$, any open set around x contains at least one point in A. Hence $x \in \overline{A}$. Conversely, for a point $x \in \overline{A}$, consider the ball $B(x, n^{-1})$. Then $x \in B(x, n^{-1}) \cap A$ for some x_n. Thus $d(x, x_n) < n^{-1}$. □ □

A **continuous map** $\boldsymbol{f} : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ between topological spaces is a map $\boldsymbol{f} : X \to Y$ such that $\boldsymbol{f}^{-1}(U) \in \mathscr{T}_X$ whenever $U \in \mathscr{T}_Y$.

We say that a map $\boldsymbol{f} : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is **continuous at $\boldsymbol{x}$** if for any neighborhood N of $\boldsymbol{f}(x)$ in Y there exists a neighborhood M of x in X with $\boldsymbol{f}(M) \subset N$. From the definition of neighborhoods, $\boldsymbol{f}$ is continuous at x if and only if for any neighborhood N of $\boldsymbol{f}(x)$ in Y the inverse image $\boldsymbol{f}^{-1}(N)$ is a neighborhood of x in X.

Theorem 1.7

A map $\boldsymbol{f} : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is continuous if and only if $\boldsymbol{f}$ is continuous at each point in X.

Proof. Suppose first that $\boldsymbol{f}$ is continuous. Let N be a neighborhood of $\boldsymbol{f}(x)$ in Y and choose an open set U in Y with $\boldsymbol{f}(x) \in U \subset N$. Then $x \in \boldsymbol{f}^{-1}(U) \subset \boldsymbol{f}^{-1}(N)$ and $\boldsymbol{f}^{-1}(N)$ is a neighborhood of x in X. Hence $\boldsymbol{f}$ is continuous at x.

Conversely, given an open set $U \subset Y$. For any $x \in \boldsymbol{f}^{-1}(U)$, we see that $\boldsymbol{f}^{-1}(U)$ is a neighborhood of x so that $x \in V_x \subset \boldsymbol{f}^{-1}(U)$ for some open set V_x in X. Then

$$\boldsymbol{f}^{-1}(U) = \bigcup_{x \in \boldsymbol{f}^{-1}(U)} V_x \subset \mathscr{T}_X$$

and $\boldsymbol{f}$ is continuous. □

Example 1.3

(1) *Every constant map $\boldsymbol{f} : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is continuous.*

(2) *The identity map $\mathbf{1}_X : (X, \mathscr{T}_X) \to (X, \mathscr{T}_X)$ is continuous.*

(3) *If $\boldsymbol{f} : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is continuous, so is the restriction of $\boldsymbol{f}$ to any open $U \subseteq X$.*

(4) *For continuous maps $\boldsymbol{f} : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ and $\boldsymbol{g} : (Y, \mathscr{T}_Y) \to (Z, \mathscr{T}_Z)$, the*

composite map $\boldsymbol{g}\circ\boldsymbol{f}:(X,\mathscr{T}_X)\to(Z,\mathscr{T}_Z)$ is also continuous. We also observe that for any continuous map $\boldsymbol{f}:(X,\mathscr{T}_X)\to(Y,\mathscr{T}_Y)$ we have $\boldsymbol{f}\circ\mathbf{1}_X=f$ and $\mathbf{1}_Y\circ\boldsymbol{f}=\boldsymbol{f}$. ♠

We say a map $\boldsymbol{f}:(X,\mathscr{T}_X)\to(Y,\mathscr{T}_Y)$ is a **homeomorphism** if $\boldsymbol{f}$ is bijective and $\boldsymbol{f}$, $\boldsymbol{f}^{-1}$ are continuous. In this case, we say that $(X,\mathscr{T}_X)$ and $(Y,\mathscr{T}_Y)$ are **homeomorphic** or **topologically equivalent**, and denote by $X\approx Y$.

Exercise 1.10

(1) *Let $\boldsymbol{f}:(X,\mathscr{T}_X)\to(Y,\mathscr{T}_Y)$ be a bijection between topological spaces. Show that $\boldsymbol{f}$ is homeomorphic if and only if $\boldsymbol{f}(\mathscr{T}_X)=\mathscr{T}_Y$, that is, $U\in\mathscr{T}_X$ if and only if $\boldsymbol{f}(U)\in\mathscr{T}_Y$.*
(2) *Let $\boldsymbol{f}:(X,\mathscr{T}_X)\to(Y,\mathscr{T}_Y)$ be a homeomorphism and $U\in\mathscr{T}_X$. Show that $\boldsymbol{f}(U)\in\mathscr{T}_Y$ and $\boldsymbol{f}|_U$ is a homeomorphism from $(U,\mathscr{T}_U)\to(\boldsymbol{f}(U),\mathscr{T}_{\boldsymbol{f}(U)})$.*
(3) *Any* **translation**

$$\mathbf{R}^m\longrightarrow\mathbf{R}^m,\quad \boldsymbol{x}\longmapsto\boldsymbol{x}+\boldsymbol{x}_0$$

and **dilation**

$$\mathbf{R}^m\longrightarrow\mathbf{R}^m,\quad \boldsymbol{x}\longmapsto\lambda\boldsymbol{x}$$

are homeomorphic.
(4) *The map $\boldsymbol{F}:\mathbf{B}^m\to\mathbf{R}^m$ given by*

$$\mathbf{F}(\boldsymbol{x}):=\frac{\boldsymbol{x}}{1-|\boldsymbol{x}|}.$$

(5) *Let $C:=\{(x,y,z)\in\mathbf{R}^3|\max\{|x|,|y|,|z|\}=1\}$ be the cubical surface of side 2 centered at $(0,0,0)$. Define*

$$\boldsymbol{F}:C\longrightarrow\mathbf{S}^2,\quad (x,y,z)\longmapsto\frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}.$$

Show that $\boldsymbol{F}$ is homeomorphic.
(6) *Let $X:=[0,1)\subseteq\mathbf{R}$ and $Y:=\mathbf{S}^1\subseteq\mathbf{C}$. Define*

$$e:X\longrightarrow Y,\quad e(s):=e^{2\pi\sqrt{-1}s}=\cos(2\pi s)+\sqrt{-1}\sin(2\pi s).$$

Show that e is continuous and bijective, but is not homeomorphic.
(7) *Consider the set, denoted $\mathbf{Max}(n,\mathbf{K})$, of all $n\times n$ matrices over $\mathbf{K}=\mathbf{R}$ or $\mathbf{K}=\mathbf{C}$. We can view $\mathbf{Max}(n,\mathbf{K})$ as the Euclidean space $\mathbf{K}^{n^2}$, i.e., the following mapping*

$$\Psi:\mathbf{Max}(n,\mathbf{K})\longrightarrow\mathbf{K}^{n^2},\quad A=(a_{ij})\longmapsto(a_{11},\cdots,a_{1n},\cdots,a_{n1},\cdots,a_{nn})$$

is bijective. We give $\mathbf{Max}(n,\mathbf{K})$ the topology induced by Ψ as follows. A nonempty set $\mathcal{U}$ in $\mathbf{Max}(n,\mathbf{K})$ is open if $\Psi(\mathcal{U})$ is open in $\mathbf{K}^{n^2}$. Together with the empty set, all those nonempty open set form a topoloy on $\mathbf{Max}(n,\mathbf{K})$. Under this topology, Ψ is a homeomorphism.
Let

$$\mathbf{GL}(n,\mathbf{K}):=\{A\in\mathbf{Max}(n,\mathbf{K})|\det(A)\neq 0\}$$

be the general linear group over $\mathbf{K}$. *Then the determinant map*

$$\det : \mathbf{Max}(n, \mathbf{K}) \longrightarrow \mathbf{K}, \quad A \longmapsto \det(A)$$

gives the following relation

$$\mathbf{GL}(n, \mathbf{K}) = \mathbf{Max}(n, \mathbf{K}) \setminus \det{}^{-1}(0).$$

The continuity of $\det$ *shows that* $\det^{-1}(0)$ *is closed in* $\mathbf{Max}(n, \mathbf{K})$ *and then* $\mathbf{GL}(n, \mathbf{K})$ *is open.*

- $\mathbf{GL}(1, \mathbf{C}) = \mathbf{C} \setminus \{0\} =: \mathbf{C}^*$, *while* $\mathbf{GL}(1, \mathbf{R}) = \mathbf{R} \setminus \{0\} =: \mathbf{R}^*$.
- *For any two nonzero complex number* z, w, *we can always find a continuous map* $\gamma : [0, 1] \to \mathbf{C}^*$ *such that* $\gamma(0) = z$ *and* $\gamma(1) = w$. *Thus shows that* $\mathbf{GL}(1, \mathbf{C})$ *is* **path-connected** *(A topological space* $(X, \mathscr{T})$ *is said to be path-connected if for any two points* $x, y \in X$, *there is a continuous map* $\boldsymbol{\gamma} : [0, 1] \to X$ *such that* $\boldsymbol{\gamma}(0) = x$ *and* $\boldsymbol{\gamma}(1) = y$. *Such a continuous map* $\boldsymbol{\gamma}$ *is also called a* **path***).*
- *We now show that* $\mathbf{GL}(n, \mathbf{C})$ *is also path-connected for any* $n \geq 2$. *It suffices to prove that there is a path connecting any given nonsingular matrix* $A \in \mathbf{GL}(n, \mathbf{C})$ *to the identity matrix* I_n. *According to linear algebra, we have the following decomposition*

$$A = CBC^{-1}, \quad C \in \mathbf{GL}(n, \mathbf{C}), \quad B = \begin{bmatrix} \lambda_1 & * & * & \cdots & * \\ 0 & \lambda_2 & * & \cdots & * \\ 0 & 0 & \lambda_3 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

where $\lambda_1, \cdots, \lambda_n$ *are eigenvalues of* A *so all are nonzero. Then* $A(t) := CB(t)C^{-1}$, *where*

$$B(t) = \begin{bmatrix} \lambda_1 & (1-t)* & (1-t)* & \cdots & (1-t)* \\ 0 & \lambda_2 & (1-t)* & \cdots & (1-t)* \\ 0 & 0 & \lambda_3 & \cdots & (1-t)* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

gives a path from A *to* CDC^{-1}, *with* $D := \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$. *Since* $\mathbf{GL}(1, \mathbf{C})$ *is path-connected, it follows that for each* i, *there is a path* $\lambda_i(t)$ *from* λ_i *to* 1. *Therefore* $CD(t)C^{-1}$, *with* $D(t) = \mathrm{diag}(\lambda_1(t), \cdots, \lambda_n(t))$, *gives rise to a path from* CDC^{-1} *to* I_n. *The composition of the above two paths implies a path from* A *to* I_n.
- *However,* $\mathbf{GL}(1, \mathbf{R})$ *is not path-connected.* ♠

A map $\boldsymbol{f} : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is **open** if $\boldsymbol{f}(U) \subseteq \mathscr{T}_Y$ for all $U \in \mathscr{T}_X$; $\boldsymbol{f}$ is **closed** if

$f(C)$ is closed in Y for all closed set $C \subseteq X$.

Exercise 1.11

Let $f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ be a bijective continuous map. Show that the following are equivalent:

- *f is a homeomorphism.*
- *f is open.*
- *f is closed.*

Proposition 1.1

Let $f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ be a map between topological spaces. Show that

(1) *f is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$.*

(2) *f is closed if and only if $f(\overline{A}) \supseteq \overline{f(A)}$ for all $A \subseteq X$.*

(3) *f is continuous if and only if $f^{-1}(\mathring{B}) \subseteq (f^{-1}(B))^\circ$ for all $B \subseteq Y$.*

(4) *f is open if and only if $f^{-1}(\mathring{B}) \supseteq (f^{-1}(B))^\circ$ for all $B \subseteq Y$.*

Proof. We only give proves of (1) and (2).

(1) Let f be a continuous map and A be a given subset of X. For any point $x \in \overline{A}$, we shall prove $f(x) \in \overline{f(A)}$. According to **Exercise 1.8** (4), it suffices to verify that every neighborhood N of $f(x)$ contains a point in $f(A)$. By definition, we have an open subset V of $f(x)$ such that $V \subseteq N$. Then, by continuity, $U := f^{-1}(V) \in \mathscr{T}_X$ and $x \in U$. Hence by **Exercise 1.8** (4) again, we can find a point $x' \in A \cap U$, so that $y' := f(x') \in f(A) \cap V$ and $f(x) \in \overline{f(A)}$.

Conversely, assume $f(\overline{A}) \subseteq \overline{f(A)}$ holds for all $A \subseteq X$. We first show that for any closed subset B of Y, $f^{-1}(B)$ is closed in X. Let $A := f^{-1}(B) \subseteq X$. To prove $f^{-1}(B)$ is closed in X, we verify that $\overline{A} = A$ by **Exercise 1.8** (8). If $x \in \overline{A}$, then

$$f(x) \in f(\overline{A}) \subseteq \overline{f(A)} = \overline{f(f^{-1}(B))} \subseteq \overline{B} = B$$

so that $x \in A$. Hence $\overline{A} = A$. Next we show the continuity of f. Let $V \in \mathscr{T}_Y$ and set $B := Y \setminus V$. Then

$$f^{-1}(B) = f^{-1}(Y) \setminus f^{-1}(V) = X \setminus f^{-1}(V).$$

Since $f^{-1}(B)$ is closed, it follows that $f^{-1}(V)$ is open.

(2) Let f be a closed map. Then for any $A \subseteq X$, we have that $f(\overline{A})$ is closed and then

$$f(A) \subseteq f(\overline{A}), \quad \overline{f(A)} \subseteq \overline{f(\overline{A})} = f(\overline{A}).$$

Conversely, assume that $f(\overline{A}) \supseteq \overline{f(A)}$ holds for all $A \subseteq X$. For a closed subset C of X, we get $\overline{f(C)} \subseteq f(\overline{C}) = f(C)$ so that $f(C)$ is closed in Y. □

In **Proposition 4.4**, together with **Theorem 1.7**, we actually proved that for a map $f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ between topological spaces, the following are equivalent:

- f is continuous,
- for any $x \in X$, f is continuous at x,
- for any closed subset B of Y, $f^{-1}(B)$ is closed in X,
- for any $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.

A map $f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ between topological spaces is said to be **locally homeomorphic** if every point $x \in X$ has a neighborhood $U \subseteq X$ such that $f(U) \in \mathscr{T}_Y$ and $f|_U : U \to f(U)$ is a homeomorphism.

Exercise 1.12

Show that

- *Every homeomorphism is a local homeomorphism.*
- *Every local homeomorphism is continuous and open.*
- *Every bijective local homeomorphism is a homeomorphism.*

Given a topological space $(X, \mathscr{T})$. A **basis for** $\mathscr{T}$ is a subset $\mathscr{B} \subseteq \mathscr{T}$ such that any open set U of X can be written as an union of some members of $\mathscr{B}$.

(1) Let $\mathscr{B} \subseteq \mathscr{T}$ be a basis for $\mathscr{T}$. Then clearly
   - $X = \cup_{B \in \mathscr{B}} B$, and
   - if $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$, there exists an element $B_3 \in \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

(2) Consider a subset $\mathscr{C} \subseteq \mathscr{T}$ satisfying

$$\text{for any } U \in \mathscr{T} \text{ and } x \in U \text{ there exists } C_x \in \mathscr{C} \text{ such that } x \in C_x \subseteq U.$$

Then $\mathscr{C}$ is a basis for $\mathscr{T}$. In fact, one has

$$U = \bigcup_{x \in U} C_x, \quad C_x \in \mathscr{C}.$$

Proposition 1.2

Let X be a set and suppose that $\mathscr{B}$ is a collection of subsets of X. Then $\mathscr{B}$ is a basis for some topology on X if and only if it satisfies the following two conditions:

(1) $X = \cup_{B \in \mathscr{B}} B$.

(2) *If $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$, there exists an element $B_3 \in \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.*

If so, there is a unique topology on X for which $\mathscr{B}$ is a basis, called the **topology generated by** $\mathscr{B}$.

Proof. One direction is obvious. Suppose now that $\mathscr{B}$ satisfies (1) and (2). Let $\mathscr{T}_{\mathscr{B}}$ be the collection of all unions of elements of $\mathscr{B}$.

We shall verify that $\mathscr{T}_{\mathscr{B}}$ is a topology on X.

- $X, \emptyset \in \mathscr{T}_{\mathscr{B}}$. By condition (1), $X \in \mathscr{T}_{\mathscr{B}}$, and $\emptyset \in \mathscr{T}_{\mathscr{B}}$ as the union of the empty collection of elements of $\mathscr{B}$.
- $\mathscr{T}_{\mathscr{B}}$ is closed under arbitrary unions. For any collection $\{U_i\}_{i\in I} \subseteq \mathscr{T}_{\mathscr{B}}$, write each U_i as
$$U_i = \bigcup_{j\in J_i} B_{ij}.$$
Then
$$\bigcup_{i\in I} U_i = \bigcup_{i\in I}\bigcup_{j\in J_i} B_{ij} \in \mathscr{T}_{\mathscr{B}}.$$
- $\mathscr{T}_{\mathscr{B}}$ is closed under finite intersections. It is enough to prove that whenever $U, V \in \mathscr{T}_{\mathscr{B}}$ one has $U \cap V \in \mathscr{T}_{\mathscr{B}}$. Given $x \in U \cap V$. Then there exist $B_1, B_2 \in \mathscr{B}$ such that $x \in B_1 \subseteq U$ and $x \in B_2 \subseteq V$. By condition (2), there exists $B_x \in \mathscr{B}$ such that $x \in B_x \subseteq B_1 \cap B_2 \subseteq U \cap V$. Consequently,
$$U \cap V = \bigcup_{x\in U\cap V} B_x$$
so that $U \cap V \in \mathscr{T}_{\mathscr{B}}$.

Next we show that $\mathscr{B}$ is a basis for $\mathscr{T}_{\mathscr{B}}$. By definition of $\mathscr{T}_{\mathscr{B}}$, it is clear.

If $\mathscr{T}'$ is another topology on X for which $\mathscr{B}$ is a basis, then $\mathscr{B} \subseteq \mathscr{T}$ and, for every $U \in \mathscr{T}'$, we have $U = \cup_{i\in I} B_i$, for a collection $\{B_i\}_{i\in I}$ of $\mathscr{B}$. Hence $U \in \mathscr{T}_{\mathscr{B}}$ and then $\mathscr{T}' \subseteq \mathscr{T}_{\mathscr{B}}$. Since $\mathscr{B} \subseteq \mathscr{T}'$, it follows that $\mathscr{T}_{\mathscr{B}} \subseteq \mathscr{T}'$. Therefore $\mathscr{T}' = \mathscr{T}_{\mathscr{B}}$. □

Given a topological space $(X, \mathscr{T})$. A **subbasis for** $\mathscr{T}$ is a set $\mathscr{S} \subset 2^X$ such that $\mathscr{B}$ is a basis for $\mathscr{T}$, where $\mathscr{B}$ is the collection of all finite intersections of members of $\mathscr{S}$ (so $\emptyset, X \in \mathscr{B}$).

Given any collection $\mathscr{S}$ of subsets of any set X, we can construct a topology $\mathscr{T}$ on X such that $\mathscr{S}$ is a subbasis for $\mathscr{T}$. Indeed, we define $\mathscr{T}$ to be the collection of arbitrary unions of the finite intersections of members of $\mathscr{C}$, assuming the convenience that $\emptyset$ is the union of an empty collection of sets and X is the intersection of an empty collection of sets. This topology is called the **topology generated by** $\mathscr{S}$.

A topology space $(X, \mathscr{T})$ is said to be **first countable** if for any $x \in X$ there exists a countable **neighborhood basis at** $\boldsymbol{x}$ (that is, a collection $\mathscr{B}_x$ of neighborhoods of x such that every neighborhood of contains some $B \in \mathscr{B}_x$). $(X, \mathscr{T})$ is said to be **second countable** if $\mathscr{T}$ has a countable basis $\mathscr{B}$.

(1) Any metric space is first countable. If (X, d) is a metric space, then d induces the metric topology $\mathscr{T}$. For each $x \in X$, the collection $\{B(x, r)\}_{r\in\mathbf{Q}^+}$ is a neighborhood basis at x.
(2) Euclidean spaces are second countable. Consider the countable base $\mathscr{B} = \{\mathbf{B}^m(\boldsymbol{x}, r) | \boldsymbol{x} \in \mathbf{Q}^m,\ r \in \mathbf{Q}^+\}$ of $\mathbf{R}^m$.
(3) Not every metric space is second countable, for example $X = [0, 1]$ with the trivial metric.

(4) Any second countable space is first countable.

Theorem 1.8

Let (X, d) be a metric space. X is second countable if and only if X contains a countable dense subset.

Proof. If X is second countable, then X has a countable basis $\mathscr{B} = \{B_i\}_{i\in\mathbf{N}}$. Choose $x_i \in B_i$ for each $i \in \mathbf{N}$ and let $A := \{x_i\}_{i\in\mathbf{N}}$. Then A is a countable dense set. Conversely, let $A = \{x_i\}_{i\in\mathbf{N}}$ be a countable dense set in X. If we take $\mathscr{B} = (B(x_i, r))_{i\in\mathbf{N}, r\in\mathbf{Q}_{>0}}$, then $\mathscr{B}$ is a countable basis. □

A topological space is said to be **separable** if it contains a countable dense subset.

Let $(X, \mathscr{T})$ be a topological space and (Y, d) is a metric space. We say a sequence of maps $\{\boldsymbol{f}_n\}_{n\in\mathbf{N}}$ between X and Y **converges uniformly** to a map $\boldsymbol{f} : X \to Y$ if

$$\lim_{n\to\infty}\left(\sup_{x\in X} d(\boldsymbol{f}_i(x), \boldsymbol{f}(x))\right) = 0.$$

In this case, we write it as $\boldsymbol{f}_n \rightrightarrows \boldsymbol{f}$.

Theorem 1.9

Let $\{\boldsymbol{f}_n\}_{n\in\mathbf{N}}, \boldsymbol{f}$ be maps between a topological space $(X, \mathscr{T})$ and a metric space (Y, d). If $\boldsymbol{f}_n$ are continuous and $\boldsymbol{f}_n \rightrightarrows \boldsymbol{f}$, then $\boldsymbol{f}$ is continuous.

Proof. Given $\epsilon > 0$ there is an integer $n_0 \in \mathbf{N}$ such that $d(\boldsymbol{f}(x), \boldsymbol{f}(x_0)) < \frac{\epsilon}{3}$ for any $x \in X$ and $n \geq n_0$. Given $x_0 \in X$ there exists a neighborhood N of x_0 in X such that

$$d(\boldsymbol{f}_{m_0}(x), \boldsymbol{f}_{n_0}(x_0)) < \frac{\epsilon}{3}, \quad x \in N.$$

Hence

$$d(\boldsymbol{f}(x), \boldsymbol{f}(x_0)) \leq d(\boldsymbol{f}(x), \boldsymbol{f}_{n_0}(x)) + d(\boldsymbol{f}_{n_0}(x), \boldsymbol{f}_{n_0}(x_0)) + d(\boldsymbol{f}_{n_0}(x_0), \boldsymbol{f}(x_0)) < \epsilon.$$

Thus $\boldsymbol{f}$ is continuous at x_0. By Theorem 4.1, we see that $\boldsymbol{f}$ is continuous. □

Let X be a set and $\mathbb{P}$ some condition given on subsets of X. Consider a topology $\mathscr{T}$ on X whose open sets satisfy $\mathbb{P}$.

(1) $\mathscr{T}$ is the **smallest topology** or **coarsest topology** satisfying $\mathbb{P}$, if for any topology $\mathscr{T}'$ on X whose open sets satisfying $\mathbb{P}$, then $\mathscr{T}$-open sets are $\mathscr{T}'$-open. In this case we use the notion $\mathscr{T} \prec \mathscr{T}'$.

(2) $\mathscr{T}$ is the **largest topology** or **finest topology** satisfying $\mathbb{P}$, if for any topology $\mathscr{T}'$ on X whose open sets satisfying $\mathbb{P}$, then $\mathscr{T}'$-open sets are $\mathscr{T}$-open. In this case, we use the notion $\mathscr{T}' \prec \mathscr{T}$.

Example 1.4

(1) **(Trivial topology)** *Any set X can be equipped with a topology $\mathscr{T}_{\mathbf{t}} = \{\emptyset, X\}$. This is the smallest topology on X.*

(2) **(Discrete topology)** *Any set X can also be equipped with another topology $\mathscr{T}_{\mathbf{d}} = 2^X$. This is the largest topology on X.*

(3) **(Finite complement topology)** *Given an infinite set X. The collection of subsets whose complement are finite, together with the empty set, forms a topology $\mathscr{T}_{\mathbf{fc}}$.*

(4) **(Countable complement topology)** *Given an infinite set X. The collection of subsets whose complement are countable, together with the empty set, forms a topology $\mathscr{T}_{\mathbf{cc}}$.*

(5) **(Particular point topology)** *Given an infinite set X and $x \in X$. Show that*

$$\mathscr{T}_{\mathbf{pp}} := \{U \subseteq X | U = \emptyset \text{ or } x \in U\}$$

is a topology on X.

(6) **(Excluded point topology)** *Given an infinite set X and $x \in X$. Show that*

$$\mathscr{T}_{\mathbf{ep}} := \{U \subseteq X | U = X \text{ or } x \notin U\}$$

is a topology on X.

(7) *Show that $(\mathbf{R}, \mathscr{T}_{\mathbf{pp}})$ is first countable and separable, but not second countable.*

(8) *Show that $(\mathbf{R}, \mathscr{T}_{\mathbf{ep}})$ is first countable, but not second countable or separable.*

(9) *Show that $(\mathbf{R}, \mathscr{T}_{\mathbf{fc}})$ is separable, but not first or second countable.* ♠

1.2.2 Categories: an introduction

A **category** $\mathfrak{C}$ is a triple

$$(\mathbf{Ob}(\mathfrak{C}), \mathbf{Hom}_{\mathfrak{C}}(\cdot, \cdot), \circ),$$

where $\mathbf{Ob}(\mathfrak{C})$ is a class of **objects**, $\mathbf{Hom}_{\mathfrak{C}}(X, Y)$ is a set for every pair (X, Y) of objects (its element, called **morphism**, is usually written as $f : X \to Y$; we call X is the domain of f and Y the range of f), and $\circ$ is the composition on sets (that is, given $X, Y, Z \in \mathbf{Ob}(\mathfrak{C})$, we define $\circ : \mathbf{Hom}_{\mathfrak{C}}(X, Y) \times \mathbf{Hom}_{\mathfrak{C}}(Y, Z) \to \mathbf{Hom}_{\mathfrak{C}}(X, Z)$ by $\circ(f, g) := g \circ f$), satisfying the following axioms

(1) **(Associativity)** $h \circ (g \circ f) = (h \circ g) \circ f$ for any morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

(2) **(Identity)** For any $X \in \mathbf{Ob}(\mathfrak{C})$ there exists a morphism $\mathbf{1}_X \in \mathbf{Hom}_{\mathfrak{C}}(X, X)$, called the **identity morphism** of X, such that

$$\mathbf{1}_X \circ f = f, \quad g \circ \mathbf{1}_X = g$$

for any $f \in \mathbf{Hom}_{\mathfrak{C}}(Y, X)$ and $g \in \mathbf{Hom}_{\mathfrak{C}}(X, Y)$. Note that $\mathbf{1}_X$ is unique.

$\mathfrak{C}$ is called **small** if $\mathbf{Ob}(\mathfrak{C})$ is a set.

If morphisms $f \in \mathbf{Hom}_{\mathfrak{C}}(X, Y)$ and $g \in \mathbf{Hom}_{\mathfrak{C}}(Y, X)$ satisfying $g \circ f = \mathbf{1}_X$, we say that g is a **left inverse** of f and f is a **right inverse** of g respectively. A **two-sided inverse** or an **inverse** of a morphism f is a morphism which is both a left inverse of f and a right inverse of f. A morphism $f : X \to Y$ is said to be an **equivalence** if there exists a morphism $g : Y \to X$ which is a two-sided inverse of f.

(1) If $f : X \to Y$ has a left inverse and a right inverse, then they are equal and f is an equivalence. Indeed, if $g' : Y \to X$ is a left inverse and $g'' : Y \to X$ is a right inverse, then $g' \circ f = \mathbf{1}_X$ and $f \circ g'' = \mathbf{1}_Y$ so that $g' = g' \circ \mathbf{1}_Y = g' \circ (f \circ g'') = \mathbf{1}_X \circ g'' = g''$.

(2) If $f : X \to Y$ is an equivalence, then by (1) it has a unique inverse $f^{-1} : Y \to X$ and f^{-1} is also an equivalence.

(3) We say two objects X and Y are **equivalent**, written as $X \approx Y$, if there exists an equivalence $f : X \to Y$.

(4) It is clear that $\approx$ is an equivalence relation in any set of objects of $\mathfrak{C}$.

An equivalence $f : X \to Y$ is also called an **isomorphism**. An **endomorphism** is a morphism $f : X \to X$ with same domain and range. An **automorphism** is an endomorphism which is an isomorphism. Two morphisms f and g are **parallel** if they have same domain and same range, i.e., $f, g : X \rightrightarrows Y$, or

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y.$$

A morphism $f : X \to Y$ is a **monomorphism** if for any pair of parallel morphisms $g_1, g_2 : Z \rightrightarrows X$, $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.

$$Z \underset{g_2}{\overset{g_1}{\rightrightarrows}} X \xrightarrow{f} Y$$

A morphism $f : X \to Y$ is an **epimorphism** if for any pair of parallel morphisms $g_1, g_2 : Y \rightrightarrows Z$, $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$.

$$X \xrightarrow{f} Y \underset{g_2}{\overset{g_1}{\rightrightarrows}} Z.$$

Note that $f : X \to Y$ is a monomorphism if and only if the map $f\circ : \mathbf{Hom}_{\mathfrak{C}}(Z, X) \to \mathbf{Hom}_{\mathfrak{C}}(Z, Y)$ is injective for any object $Z \in \mathbf{Ob}(\mathfrak{C})$, and $f : X \to Y$ is an epimorphism if and only if the map $\circ f : \mathbf{Hom}_{\mathfrak{C}}(Y, Z) \to \mathbf{Hom}_{\mathfrak{C}}(X, Z)$ is injective for any object $Z \in \mathbf{Ob}(\mathfrak{C})$.

If

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

are morphisms and if f and g are monomorphisms (resp. epimorphisms, resp. isomorphisms), then $g \circ f$ is a monomorphism (resp. epimorphism, resp. isomorphism).

We sometimes write $f : X \rightarrowtail Y$ or else $f : X \hookrightarrow Y$ to denote a monomorphism and $f : X \twoheadrightarrow Y$ to denote an epimorphism.

Example 1.5

(1) $\mathbf{Set}$ *is the category of sets and set maps.*

(2) $\mathbf{Group}$ *is the category of groups and group homomorphisms.*

(3) $\mathbf{AGroup}$ *is the category of Abelian groups and group homomorphisms.*

(4) $\mathbf{Ring}$ *is the category of rings and ring homomorphisms.*

(5) $\mathbf{CRing}$ *is the category of commutative rings and ring homomorphisms.*

(6) $\mathbf{Mod}_R$ *is the category of R-modules and R-module homomorphisms.*

(7) $\mathbf{Vect}(\mathbf{R})$ *is the category of real vector spaces and $\mathbf{R}$-linear maps.*

(8) $\mathbf{Vect}(\mathbf{C})$ *is the category of complex vector spaces and $\mathbf{C}$-linear maps.*

(9) $\mathbf{Top}$ *is the the category of topological spaces and continuous maps.*

(10) $\mathbf{Top}_*$ *is the category of pointed topological space and pointed continuous maps. A pointed topological space is a pair (X, x_0) where X is a topological space and $x_0 \in X$; a pointed continuous map $\boldsymbol{f} : (X, x_0) \to (Y, y_0)$ is a continuous map $\boldsymbol{f} : X \to Y$ such that $\boldsymbol{f}(x_0) = y_0$.*

The fundamental groups π_1 roughly speaking is a map from $\mathbf{Top}_*$ to $\mathbf{Group}$; namely associated to a pointed topological space (X, x_0) a group $\pi_1(X, x_0)$. We also can define higher homotopy groups $\pi_i(X, x_0)$ which turns out to be Abelian for any $i \geq 2$.

Let $\mathfrak{C}$ be a category. A **subcategory** $\mathfrak{C}'$ of $\mathfrak{C}$ is itself a category and satisfies

- any object in $\mathfrak{C}'$ is also object in $\mathfrak{C}$,
- For any $X, Y \in \mathbf{Ob}(\mathfrak{C}')$, we have $\mathbf{Hom}_{\mathfrak{C}'}(X, Y) \subseteq \mathbf{Hom}_{\mathfrak{C}}(X, Y)$, and
- $\mathbf{1}_X \in \mathbf{Hom}_{\mathfrak{C}'}(X, X)$ for any $X \in \mathbf{Ob}(\mathfrak{C}')$.

A subcategory $\mathfrak{C}'$ is called a **full subcategory** if moreover

$$\mathbf{Hom}_{\mathfrak{C}'}(X, Y) = \mathbf{Hom}_{\mathfrak{C}}(X, Y)$$

for any $X, Y \in \mathbf{Ob}(\mathfrak{C}')$.

$\mathbf{Top}$ is a subcategory, but not a full subcategory, of $\mathbf{Set}$.

The **opposite category** $\mathfrak{C}^\circ$ of a category $\mathfrak{C}$ is defined by

$$\mathbf{Ob}(\mathfrak{C}^\circ) := \mathbf{Ob}(\mathfrak{C}), \quad \mathbf{Hom}_{\mathfrak{C}^\circ}(X, Y) := \mathbf{Hom}_{\mathfrak{C}}(Y, X).$$

Exercise 1.13

Let $\mathfrak{C}$ be a category. An object $P \in \mathbf{Ob}(\mathfrak{C})$ is **initial** *if for any $Y \in \mathbf{Ob}(\mathfrak{C})$, $\mathbf{Hom}_{\mathfrak{C}}(P, Y)$ has exactly one element. An object $Q \in \mathbf{Ob}(\mathfrak{C})$ is* **final** *if for any $X \in \mathbf{Ob}(\mathfrak{C})$, $\mathbf{Hom}_{\mathfrak{C}}(X, Q)$ has exactly one element. Show that two initial (resp. final) objects are isomorphic.*

A **(covariant) functor** $\boldsymbol{F} : \mathfrak{C} \to \mathfrak{C}'$ between two categories consists of

(1) a map $\boldsymbol{F} : \mathbf{Ob}(\mathfrak{C}) \to \mathbf{Ob}(\mathfrak{C}')$, and

(2) for any two objects $X, Y \in \mathbf{Ob}(\mathfrak{C})$, a map

$$\boldsymbol{F} : \mathbf{Hom}_{\mathfrak{C}}(X, Y) \to \mathbf{Hom}_{\mathfrak{C}'}(\boldsymbol{F}(X), \boldsymbol{F}(X')).$$

These data satisfy

$$\boldsymbol{F}(\mathbf{1}_X) = \mathbf{1}_{\boldsymbol{F}(X)}, \quad \boldsymbol{F}(f \circ g) = \boldsymbol{F}(f) \circ \boldsymbol{F}(g). \tag{1.2.2.1}$$

A **contravariant functor** $\boldsymbol{G} : \mathfrak{C} \to \mathfrak{C}'$ is a covariant functor from $\mathfrak{C}^\circ$ to $\mathfrak{C}'$.

Example 1.6

(1) *Let $\mathfrak{C}$ be a category and take an object $X \in \mathbf{Ob}(\mathfrak{C})$. Define*

$$\mathbf{Hom}_{\mathfrak{C}}(X, \cdot) : \mathfrak{C} \longrightarrow \mathbf{Set}, \quad Z \longmapsto \mathbf{Hom}_{\mathfrak{C}}(X, Z), \tag{1.2.2.2}$$

$$\mathbf{Hom}_{\mathfrak{C}}(\cdot, X) : \mathfrak{C} \longrightarrow \mathbf{Set}, \quad Z \longmapsto \mathbf{Hom}_{\mathfrak{C}}(Z, X). \tag{1.2.2.3}$$

Then $\mathbf{Hom}_{\mathfrak{C}}(X, \cdot)$ is a functor, while $\mathbf{Hom}_{\mathfrak{C}}(\cdot, X)$ is a contravariant functor.

(2) *The* **forgetful functor** $\boldsymbol{F} : \mathbf{Top} \to \mathbf{Set}$.

(3) *The* **fundamental group functor** $\pi_1 : \mathbf{Top}_* \to \mathbf{Group}$, *given by* $(X, x) \mapsto \pi_1(X, x)$.

Let $\boldsymbol{F}_1, \boldsymbol{F}_2 : \mathfrak{C} \to \mathfrak{C}'$ be two functors. A **morphism** or **natural transformation** $\boldsymbol{\Theta}$ from $\boldsymbol{F}_1$ to $\boldsymbol{F}_2$ consists of

$$\boldsymbol{\Theta}(X) \in \mathbf{Hom}_{\mathfrak{C}'}(\boldsymbol{F}_1(X), \boldsymbol{F}_2(X)) \quad \text{whenever} \quad X \in \mathbf{Ob}(\mathfrak{C}).$$

These data satisfy the following commutative diagram:

$$\begin{tikzcd}
\boldsymbol{F}_1(X) \arrow[r, "\boldsymbol{\Theta}(X)"] \arrow[d, "\boldsymbol{F}_1(f)"'] & \boldsymbol{F}_2(X) \arrow[d, "\boldsymbol{F}_2(f)"] \\
\boldsymbol{F}_1(Y) \arrow[r, "\boldsymbol{\Theta}(Y)"'] & \boldsymbol{F}_2(Y)
\end{tikzcd} \qquad \boldsymbol{F}_2(f) \circ \boldsymbol{\Theta}(X) = \boldsymbol{\Theta}(Y) \circ \boldsymbol{F}_1(f) \tag{1.2.2.4}$$

for any $X, Y \in \mathbf{Ob}(\mathfrak{C})$ and any $f \in \mathbf{Hom}_{\mathfrak{C}}(X, Y)$.

Definition 1.2

For two categories $\mathfrak{C}$ and $\mathfrak{C}'$, define a new category $\mathbf{Func}(\mathfrak{C}, \mathfrak{C}')$ as follows:

$$\mathbf{Ob}(\mathbf{Func}(\mathfrak{C}, \mathfrak{C}')) := \{\textit{functors from } \mathfrak{C} \textit{ to } \mathfrak{C}'\},$$

$$\mathbf{Hom}_{\mathbf{Func}(\mathfrak{C}, \mathfrak{C}')}(\boldsymbol{F}_1, \boldsymbol{F}_2) := \{\textit{morphisms from } \boldsymbol{F}_1 \textit{ to } \boldsymbol{F}_2\}.$$

Let $\mathfrak{C}$ be a category. A functor $\boldsymbol{F} : \mathfrak{C} \to \mathbf{Set}$ is said to be **representable** if there exists an object $X \in \mathbf{Ob}(\mathfrak{C})$ such that $\boldsymbol{F}$ is isomorphic to $\mathbf{Hom}_{\mathfrak{C}}(X, \cdot)$ in the category $\mathbf{Func}(\mathfrak{C}, \mathbf{Set})$.

Exercise 1.14

(1) *Show that $\mathbf{Func}(\mathfrak{C}, \mathfrak{C}')$ is a category.*

(2) *If $\boldsymbol{F} : \mathfrak{C} \to \mathbf{Set}$ is representable for some $X \in \mathbf{Ob}(\mathfrak{C})$, then X is unique to isomorphism and is called a* **representative** *of $\boldsymbol{F}$.*

Let $\boldsymbol{F} : \mathfrak{C} \to \mathfrak{C}'$ be a functor.

(1) $\boldsymbol{F}$ is **fully faithful** if for any objects $X, Y \in \mathbf{Ob}(\mathfrak{C})$, the map

$$\mathbf{Hom}_{\mathfrak{C}}(X,Y) \longrightarrow \mathbf{Hom}_{\mathfrak{C}'}(\boldsymbol{F}(X), \boldsymbol{F}(Y))$$

is bijective.

(2) $\boldsymbol{F}$ is an **equivalence of categories** if $\boldsymbol{F}$ is fully faithful and for any object $X' \in \mathbf{Ob}(\mathfrak{C}')$ there exists $X \in \mathbf{Ob}(\mathfrak{C})$ and an isomorphism $X' \to \boldsymbol{F}(X)$.

Exercise 1.15

$\boldsymbol{F} \in \mathbf{Ob}(\mathbf{Func}(\mathfrak{C}, \mathfrak{C}'))$ *is an equivalence of categories if and only if there exists* $\boldsymbol{F}' \in \mathbf{Ob}(\mathbf{Func}(\mathfrak{C}', \mathfrak{C}))$ *and isomorphisms*

$$\boldsymbol{F} \circ \boldsymbol{F}' \to \mathbf{1}_{\mathfrak{C}'} \quad \text{in } \mathbf{Func}(\mathfrak{C}', \mathfrak{C}') \quad \text{and} \quad \boldsymbol{F}' \circ \boldsymbol{F} \to \mathbf{1}_{\mathfrak{C}} \quad \text{in } \mathbf{Func}(\mathfrak{C}, \mathfrak{C}).$$

♠

Let $\mathfrak{C}$ be a category. Define

$$\mathfrak{C}^{\vee} := \mathbf{Func}(\mathfrak{C}^{\circ}, \mathbf{Set}) \tag{1.2.2.5}$$

and

$$\mathbf{Hom}_{\mathfrak{C}} : \mathfrak{C} \longrightarrow \mathfrak{C}^{\vee}, \quad X \longmapsto \mathbf{Hom}_{\mathfrak{C}}(\cdot, X). \tag{1.2.2.6}$$

Theorem 1.10. (Yoneda's lemma)

(1) *For any* $X \in \mathbf{Ob}(\mathfrak{C})$ *and any* $\boldsymbol{F} \in \mathbf{Ob}(\mathfrak{C}^{\vee})$, *one has*

$$\mathbf{Hom}_{\mathfrak{C}^{\vee}}(\mathbf{Hom}_{\mathfrak{C}}(X), F) \quad \text{is isomorphic to} \quad \boldsymbol{F}(X) \quad \text{in } \mathbf{Set}. \tag{1.2.2.7}$$

(2) $\mathbf{Hom}_{\mathfrak{C}}$ *is a fully faithful functor.*

♡

Proof. (1) To $f \in \mathbf{Hom}_{\mathfrak{C}^{\vee}}(\mathbf{Hom}_{\mathfrak{C}}(\cdot, X), \boldsymbol{F})$, we associate $\phi(f) \in \boldsymbol{F}(X)$ as follows:

$$f(X) : \mathbf{Hom}_{\mathfrak{C}}(X,X) \longrightarrow \boldsymbol{F}(X), \quad \mathbf{1}_X \longmapsto \phi(f) := f(X)(\mathbf{1}_X).$$

Conversely, to $s \in \boldsymbol{F}(X)$, we can associate $\psi(s) \in \mathbf{Hom}_{\mathfrak{C}^{\vee}}(\mathbf{Hom}_{\mathfrak{C}}(\cdot, X), \boldsymbol{F})$ as follows: For $Y \in \mathbf{Ob}(\mathfrak{C})$,

$$\mathbf{Hom}_{\mathfrak{C}}(Y,X) \xrightarrow{\ \boldsymbol{F}\ } \mathbf{Hom}_{\mathbf{Set}}(\boldsymbol{F}(X), \boldsymbol{F}(Y)) \xrightarrow{\ s\ } \boldsymbol{F}(Y)$$

we define $\psi(s)(Y) := s \circ F$, where

$$s(f) := f(s), \quad f \in \mathbf{Hom}_{\mathbf{Set}}(\boldsymbol{F}(X), \boldsymbol{F}(Y)).$$

Then ϕ and ψ are inverse to each other.

(2) For any $X, Y \in \mathbf{Ob}(\mathfrak{C})$,

$$\mathbf{Hom}_{\mathscr{C}^{\vee}}(\mathbf{Hom}_{\mathfrak{C}}(\cdot, X), \mathbf{Hom}_{\mathfrak{C}}(\cdot, Y)) \to \mathbf{Hom}_{\mathfrak{C}}(\cdot, Y)(X) = \mathbf{Hom}_{\mathfrak{C}}(X,Y)$$

is bijective by (1). □

1.2.3 Subspaces

Let $(X, \mathscr{T})$ be a topological space and $S \subseteq X$ is a subset. $S \subseteq X$ is a **subspace** if $(S, \mathscr{T}_S)$ is a topological space, where $\mathscr{T}_S := \{U \cap S | U \in \mathscr{T}\}$ is the **relative topology** or **subspace**

topology. The inclusion map

$$\iota_S : (S, \mathscr{T}_S) \longrightarrow (X, \mathscr{T}) \tag{1.2.3.1}$$

is continuous.

Example 1.7

Consider the subspaces

$$S_1 := [0,1] \cup (2,3), \quad S_2 := \{1/n\}_{n \geq 1}$$

of $\mathbf{R}$*. Note that* $[0,1]$ *is not an open interval of* $\mathbf{R}$*, but is an open subset of* S_1*, since* $[0,1] = S_1 \cap (-1,2)$*. In* S_2*, the one-point sets* $\{1/n\}$ *are all open so the subspace topology on* S_2 *is discrete.* ♠

Let S be a subspace of $(X, \mathscr{T})$. We sat a subset $U \subseteq S$ is **relatively open** or **relatively closed in S** if U is open or closed in the subspace topology $\mathscr{T}_S$.

Proposition 1.3

Suppose that S *is a subspace of a topological space* $(X, \mathscr{T})$*.*

(1) *If* $U \subseteq S \subseteq X$*,* U *is open in* S *and* S *is open in* X*, then* U *is open in* X*. The same is true with "closed" in place of "open".*

(2) *If* U *is a subset of* S *that is either open or closed in* X*, then it is also open or closed in* S*, respectively.*

(3) *If* $U \subseteq S \subseteq X$*, then the closure of* U *in* S *is equal to* $\overline{U} \cap S$*.*

(4) *If* $U \subseteq S \subseteq X$*, then the interior of* U *in* S *contains* $\mathring{U} \cap S$*.* ♡

Proof. (1) U is open in S implies $U = S \cap V$ for some $V \in \mathscr{T}$. Because $S \in \mathscr{T}$, we must have $U \in \mathscr{T}$.

Suppose now that U is closed in S and S is closed in X. Then $X \setminus S \in \mathscr{T}$ and $S \setminus U = S \cap V$ for some open set $V \in \mathscr{T}$. Hence

$$\begin{aligned} X \setminus U &= (X \setminus S) \cup (S \setminus U) = (X \setminus S) \cup (S \cap V) \\ &= ((X \setminus S) \cup S) \cap ((X \setminus S) \cup V) = (X \setminus S) \cup V \in \mathscr{T}. \end{aligned}$$

Thus U is closed in X.

(2) Assume that U is open in X. Then $U = S \cap U$ is open in S.

(3) Write $\overline{U}^S$ as the closure of U in S. We shall prove that $\overline{U}^S = \overline{U} \cap S$. Suppose $x \in \overline{U}^S$ and N is any neighborhood of x in X. Then $N' := N \cap S$ is a neighborhood of x in S, so $N' \cap U \neq \emptyset$. Because $U \subset S$, we get $N \cap U \neq \emptyset$ and $x \in \overline{U} \cap S$. Conversely, let $x \in \overline{U} \cap S$ and N' is any neighborhood of x in S. By definition, N' contains an open subset $V' \in \mathscr{T}_S$ satisfying $V' \ni x$. Write $V' = N \cap S$ for some $N \in \mathscr{T}$. Then N itself is a neighborhood of x in X, so $N \cap U \neq \emptyset$. Consequently, $V' \cap U \neq \emptyset$ and $N' \cap U \neq \emptyset$, so $x \in \overline{U}^S$.

(4) Write $\mathbf{Int}_S(U)$ as the interior of U in S. Let $x \in \mathring{U} \cap S$. Then we can find a neighborhood $N \subseteq X$ containing x. Set $N' := N \cap S$ a neighborhood of x in S. Hence

$x \in \mathbf{Int}_S(U)$. □

In **Proposition 1.3** (4), we can find an example so that $\mathring{U} \cap S \subsetneq \mathbf{Int}_S(U)$. Indeed, consider $X = \mathbf{R}$, $S = [0,1] \cup (2,3)$ and $U = [0,1]$. Then $\mathbf{Int}_S(U) = [0,1]$ but $\mathring{U} \cap S = (0,1)$.

Theorem 1.11. (Characteristic property of the subspace topology)

(1) *Suppose that* $(X, \mathscr{T})$ *is a topological space and* $S \subseteq X$ *is a subspace. For any topological space* $(Y, \mathscr{T}_Y)$, *a map* $\boldsymbol{f} : (Y, \mathscr{T}_Y) \to (S, \mathscr{T}_S)$ *is continuous if and only if the composite map* $\boldsymbol{\iota}_S \circ \boldsymbol{f} : (Y, \mathscr{T}_Y) \to (X, \mathscr{T})$ *is continuous:*

$$\begin{tikzcd} & X \\ Y \arrow[ur, "\boldsymbol{\iota}_S \circ \boldsymbol{f}"] \arrow[r, "\boldsymbol{f}"'] & S \arrow[u, "\boldsymbol{\iota}_S"'] \end{tikzcd}$$

(2) **(Uniqueness of the subspace topology)** *Suppose that* S *is a subset of a topological space* $(X, \mathscr{T})$. *The subspace topology on* S *is the unique topology for which the characteristic property* (1) *holds.* ♡

Proof. (1) Suppose that $\boldsymbol{\iota}_S \circ \boldsymbol{f} : (Y, \mathscr{T}_Y) \to (X, \mathscr{T})$ is continuous. If $U \in \mathscr{T}_S$, then $U = V \cap S = \boldsymbol{\iota}_S^{-1}(V)$ for some $V \in \mathscr{T}$. Thus

$$\boldsymbol{f}^{-1}(U) = \boldsymbol{f}^{-1}(\boldsymbol{\iota}_S^{-1}(V)) = (\boldsymbol{\iota}_S \circ \boldsymbol{f})^{-1}(V) \in \mathscr{T}_Y.$$

Conversely, suppose that $\boldsymbol{f}$ is continuous. For any $V \in \mathscr{T}$, we have

$$(\boldsymbol{\iota}_S \circ \boldsymbol{f})^{-1}(V) = \boldsymbol{f}^{-1}(\boldsymbol{\iota}_S^{-1}(V)) = \boldsymbol{f}^{-1}(S \cap V) \in \mathscr{T}_Y.$$

Hence $\boldsymbol{\iota}_S \circ \boldsymbol{f}$ is continuous.

(2) Suppose S is equipped with some topology $\mathscr{T}'_S$ that satisfies the characteristic property (1). For convenience, let S_s denote the topological space $(S, \mathscr{T}_S)$ and $S_{s'}$ denote the topological space $(S, \mathscr{T}'_A)$. To prove $\mathscr{T}_S = \mathscr{T}'_S$ we should verify that (by **Exercise 1.10**) the identity map $\mathbf{1}_S : S \to S$ is a homeomorphism between S_s and $S_{s'}$.

Clearly that the inclusion map $\boldsymbol{\iota}_s : S_s \to (X, \mathscr{T})$ is continuous. Since $S_{s'}$ satisfies the characteristic property, it follows that the inclusion map $\boldsymbol{\iota}_{s'} : S_{s'} \to (X, \mathscr{T})$ is also continuous. Consider the following two diagrams:

$$\begin{tikzcd} & X \\ S_s \arrow[ur, "\boldsymbol{\iota}_{s'} \circ \mathbf{1}_{ss'}"] \arrow[r, "\mathbf{1}_{ss'}"'] & S_{s'} \arrow[u, "\boldsymbol{\iota}_{s'}"'] \end{tikzcd} \qquad \begin{tikzcd} & X \\ S_{s'} \arrow[ur, "\boldsymbol{\iota}_s \circ \mathbf{1}_{s's}"] \arrow[r, "\mathbf{1}_{s's}"'] & S_s \arrow[u, "\boldsymbol{\iota}_s"'] \end{tikzcd}$$

Here

$$\mathbf{1}_{ss'} : S_s \longrightarrow S_{s'}, \quad \mathbf{1}_{s's} : S_{s'} \longrightarrow S_s.$$

Because $\boldsymbol{\iota}_{s'} \circ \mathbf{1}_{ss'} = \boldsymbol{\iota}_s$ and $\iota_s \circ \mathbf{1}_{s's} = \boldsymbol{\iota}_{s'}$, we, using the characteristic property, obtain the continuities of both $\mathbf{1}_{ss'}$ and $\mathbf{1}_{s's}$. So $\mathbf{1}_S : S \to S$ is a homeomorphism between S_s and $S_{s'}$. □

Corollary 1.1

Let $f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ be a continuous map between topological spaces.

(1) **(Restricting the domain)** *The restriction of f to any subspace $S \subseteq X$ is continuous.*

(2) **(Restricting the codomain)** *If T is a subspace of Y that contains $f(X)$, then $f : X \to T$ is continuous.*

(3) **(Expanding the codomain)** *If Y is a subspace of Z, then $f : X \to Z$ is continuous.* ♡

Proof. (1) Because $f|_S = f \circ \iota_S$. (2) By **Theorem 1.11**. (3) Because the map $X \to Z$ is the composite of $f : X \to Y$ with the inclusion $Y \hookrightarrow Z$. □

Proposition 1.4

Suppose S is a subspace of a topological space $(X, \mathscr{T})$.

(1) *If $R \subseteq S$ is a subspace of S, then R is a subspace of X.*

(2) *If $\mathscr{B}$ is a basis for $\mathscr{T}$, then $\mathscr{B}_S := \{B \cap S | B \in \mathscr{B}\}$ is a basis for $\mathscr{T}_S$.*

(3) *Every subspace of a first countable space is first countable.*

(4) *Every subspace of a second countable space if second countable.* ♡

Proof. (1) It suffices to prove $(\mathscr{T}_S)_R = \mathscr{T}_R$. Let $U \in (\mathscr{T}_S)_R$. Then $U = R \cap V$ for some $V \in \mathscr{T}_S$. But V can be written as $V = S \cap W$ for some $W \in \mathscr{T}$, we obtain $U = R \cap (S \cap W) = R \cap W \in \mathscr{T}_R$. Conversely, Let $U \in \mathscr{T}_R$. Then $U = R \cap W$ for some $W \in \mathscr{T}$ and

$$U = R \cap V, \quad V := S \cap W \in \mathscr{T}_S.$$

Hence $U \in (\mathscr{T}_S)_R$.

(2) By definition of subspace topology, $\mathscr{B}_S \subseteq \mathscr{T}_S$. For any $U \in \mathscr{T}_S$, we have $U = S \cap W$ for some $W \in \mathscr{T}$. Since $\mathscr{B}$ is a basis for $\mathscr{T}$, it follows that

$$W = \bigcup_{i \in I} B_i, \quad B_i \in \mathscr{B}.$$

Therefore $U = \cup_{i \in I} B_i \cap S$ with $B_i \cap S \in \mathscr{B}_S$. Thus $\mathscr{B}_S$ is a basis for $\mathscr{T}_S$.

(3) Assume that S is a subspace of a first countable topological space $(X, \mathscr{T})$. For any $x \in S$, we can find a countable neighborhood basis $\mathscr{B}_x = \{B_i\}_{i \in \mathbf{N}}$ at x in X. Then $\mathscr{B}_{S,x} := \{B_i \cap S\}_{i \in \mathbf{N}}$ is a countable neighborhood basis at x in S, so $(S, \mathscr{T}_S)$ is first countable.

(4) The proof is similar to (3). □

Let $(X, \mathscr{T})$ be a topological space and S be closed in X. Clearly that if A is closed in S, then it is also closed in X.

Theorem 1.12. (Gluing lemma)

Let $(X, \mathscr{T})$ be a topological space and $X = A \cup B$, where A, B are closed in X.

(1) *If $f : A \to Y$ and $g : B \to Y$ are continuous and $f|_{A \cap B} = g|_{A \cap B}$, then f, g*

combine to give a continuous function $\boldsymbol{h} : X \to Y$ *given by*

$$\boldsymbol{h}(x) = \begin{cases} \boldsymbol{f}(x), & x \in A, \\ \boldsymbol{g}(x), & x \in B. \end{cases}$$

(2) *If* $\boldsymbol{f} : X \to Y$ *is a map, then* $\boldsymbol{f}$ *is continuous provided* $\boldsymbol{f}|_A, \boldsymbol{f}|_B$ *are continuous.* ♡

Proof. Evidently (2) directly follows from (1). To prove (1), we use criterion of continuity below **Proposition 4.4**, that is, $\boldsymbol{f} : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is continuous if and only if $\boldsymbol{f}^{-1}(B)$ is closed in X for any closed subset B of Y.

For any closed subset C of Y, one has

$$\boldsymbol{h}^{-1}(C) = \boldsymbol{f}^{-1}(C) \cup \boldsymbol{g}^{-1}(C).$$

Since $\boldsymbol{f}$ and $\boldsymbol{g}$ are continuous, we have that $\boldsymbol{f}^{-1}(C)$ and $\boldsymbol{g}^{-1}(C)$ are closed in A and B respectively. Hence $\boldsymbol{h}^{-1}(C)$ is closed in X. □

1.2.4 Separation axioms

Let $(X, \mathscr{T})$ be a topological space.

(1) $\boldsymbol{T_0}$**-space**: for any $x \neq y \in X$, there exists $U \in \mathscr{T}$ containing x but not y, or $V \in \mathscr{T}$ containing y but not x.

(2) $\boldsymbol{T_1}$**-space**: for any $x \neq y \in X$, there exist $U, V \in \mathscr{T}$ such that U contains x but not y and V contains y but not x.

(3) $\boldsymbol{T_2}$**-space** or **Hausdorff space**: for any $x \neq y \in X$, there exist $U, V \in \mathscr{T}$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

(4) $\boldsymbol{T_3}$**-space** or **regular space**: it is T_1-space, and for any $x \in X$, any closed set $F \subseteq X$ with $x \notin F$, there exist $U, V \in \mathscr{T}$ such that $x \in U$, $F \subseteq V$, and $U \cap V = \emptyset$.

(5) $\boldsymbol{T_4}$**-space** or **normal space**: it is T_1-space, and for any closed sets $F, G \subseteq X$ with $F \cap G = \emptyset$, there exist $U, V \in \mathscr{T}$ such that $F \subseteq U$, $G \subseteq V$, and $U \cap V = \emptyset$.

By definition, it is clear that

$$\{\text{normal spaces}\} \subsetneq \{\text{regular spaces}\} \subsetneq \{\text{Hausdorff spaces}\}.$$

Indeed, we shall prove that any Hausdorff space must be T_1. By **Theorem 1.13** below, we shall prove that any one-point set $\{x\}$ in a Hausdorff space $(X, \mathscr{T})$ is closed. For any $y \in X$ and $y \neq x$, we can find disjoint open subsets $U, V \in \mathscr{T}$ such that $y \in U$ and $x \in V$ respectively. But $x \notin U$, we have $y \notin \overline{\{x\}}$. Thus $\overline{\{x\}} = \{x\}$.

We now give examples on strict inclusions.

(1) $\mathbf{R}_K$ is Hausdorff but not regular. Recall that $\mathbf{R}_K$ denotes the real line with the topology generated by the basis consisting of all open intervals (a, b) and all subsets of the form $(a, b) \setminus K$, where $K = \{1/n\}_{n \in \mathbf{N}}$. Observe that K is closed in $\mathbf{R}_K$ and does not contain 0. Suppose that there exist disjoint open sets U and V containing 0 and K, respectively. We

shall find a contradiction. Choose a basis element, which must be of the form $(a, b) \setminus K$, containing 0 and lying in U. Take n sufficiently large so that $1/n \in (a, b)$ and then choose a basis element, which must be of the form (c, d), about $1/n$ contained in B. Choose z satisfying $\max(c, 1/(n+1)) < z < 1/n$; then $z \in U \cap V$! Therefore $\mathbf{R}_K$ is not regular.

(2) $\mathbf{R}_\ell$ is regular. Recall that $\mathbf{R}_\ell$ denotes the real line with the topology generated by the basis consisting of all half-open intervals of the form $[a, b)$ with $a < b$. Clearly one-point sets are closed in $\mathbf{R}_\ell$ so that, according to **Theorem 1.13** below, $\mathbf{R}_\ell$ is T_1. To prove normality, we suppose that A and B are disjoint closed subsets in $\mathbf{R}_\ell$. For $a \in A$ choose a basis element $[a, x_a)$ not intersecting B and similarly for any $b \in B$ choose a basis element $[b, x_b)$ not intersecting A. Then open subsets

$$U := \bigcup_{a\in A} [a, x_a), \quad V := \bigcup_{b\in B} [b, x_b)$$

are disjoint open subsets about A and B respectively.

(3) The Sorgenfrey plane $\mathbf{R}_\ell^2 := \mathbf{R}_\ell \times \mathbf{R}_\ell$ is regular but not normal.

Theorem 1.13

(1) *A topological space is T_1 if and only if one-point sets are closed sets.*

(2) *Any subspaces of a T_i-space is T_i, where $i = 2, 3$.*

(3) *A T_2-space is T_3 if and only if for any $x \in X$ and any open subset U of x, there exists an open subset V of x such that $\overline{V} \subseteq U$.*

(4) *A T_2-space is T_4 if and only if for any open set U and closed set A with $A \subseteq U$, there exists an open set V such that $A \subseteq V \subseteq \overline{V} \subseteq U$.*

(5) *Any metric space is normal.* ♡

Proof. (1) Let $(X, \mathscr{T})$ be a T_1-space. Given any point $x \in X$. For any $y \in X \setminus \{x\}$ we have an open set $U_y \in \mathscr{T}$ such that $y \in U_y$ but $x \notin U_y$. Then $X \setminus \{x\} = \cup_{y\in X\setminus\{x\}} U_y$ is open and $\{x\}$ is closed in X.

Conversely, assume that a topological space $(X, \mathscr{T})$ satisfies the properties that any one-point set is closed. Consider any two distinct points $x, y \in X$. We have that $U_y := X \setminus \{x\}, V_x := X \setminus \{y\}$ are open. Therefore $U_y \in \mathscr{T}$ is an open set such that $y \in U_y$ but $x \notin U_y$. The similar result holds for V_x. Hence $(X, \mathscr{T})$ is T_1.

(2) Assume first that $(X, \mathscr{T})$ is T_2 and S is a subspace of $(X, \mathscr{T})$. For any $x \neq y$ in S, we can find two disjoint open subsets $U, V \in \mathscr{T}$ of x, y respectively, satisfying $U \cap V = \emptyset$. Then $(U \cap S) \cap (V \cap S) = \emptyset$ so $(S, \mathscr{T}_S)$ is Hausdorff.

Next assume that $(X, \mathscr{T})$ is T_3 and S is subspace of $(X, \mathscr{T})$. Clearly $(S, \mathscr{T}_S)$ is T_1 space. Let $x \in S$ and B be a closed subset of S disjoint from x. Then

$$\overline{B} \cap S = \overline{B}^S = B$$

by **Proposition 1.3** (3), so $x \notin \overline{B}$. Using the regularity of X, there exist disjoint open subsets $U \ni x$ and $V \supseteq \overline{B}$. Then $U \cap S$ and $V \cap S$ are disjoint open subsets in Y containing x and B

respectively. Therefore $(S, \mathscr{T}_S)$ is regular.

(3) Let $(X, \mathscr{T})$ be a Hausdorff space. Suppose first that $(X, \mathscr{T})$ is regular. Take any $x \in X$ and any open subset U of x, and let $B := X \setminus U$. By regularity, there exist disjoint open subsets $V \ni x$ and $W \supset B$. We claim that $\overline{V} \cap B = \emptyset$. Otherwise, we can find $y \in B \cap \overline{V}$ so that W is a neighborhood of y but disjoint from V, contradicting $y \in \overline{V}$. Hence $\overline{V} \subseteq U$.

Conversely, suppose that $x \in X$ and a closed subset B not containing x. Let $U := X \setminus B$/ By hypothesis, there exists an open subset V of x such that $\overline{V} \subset U$. The open subsets V and $X \setminus \overline{V}$ are disjoint open subsets containing x and B respectively.

(4) The proof is essentially similar to that of (3).

(5) Let (X, d) be a metric space with the induced metric topology $\mathscr{T}$, and A, B be disjoint closed subsets of X. For any $a \in A$, there exists $\epsilon_a > 0$ so that $B(a, \epsilon_a) \cap B = \emptyset$. Similarly, for any $b \in B$ there exists $\epsilon_b > 0$ so that $B(b, \epsilon_b) \cap A = \emptyset$. Define

$$U := \bigcup_{a\in A} B(a, \epsilon_a/2), \quad V := \bigcup_{b\in B} B(b, \epsilon_b/2).$$

Then U and V both are open and contain A and B respectively. We claim that $U \cap V = \emptyset$. Otherwise, for $z \in U \cap V$, we have $z \in B(a, \epsilon_a/2) \cap B(b, \epsilon_b/2)$ for some $a \in A$ and $b \in B$. Then

$$d(a, b) \le d(a, z) + d(z, b) < \frac{\epsilon_a + \epsilon_b}{2}.$$

If $\epsilon_a \le \epsilon_b$, then $d(a, b) < \epsilon_b$ so that $a \in B(b, \epsilon_b)$. If $\epsilon_b \le \epsilon_a$, then $b \in B(a, \epsilon_a$. Both give a contradiction. □

Exercise 1.16

Prove (4) *in* **Theorem 1.13**. ♠

Remark that a subspace of a normal space need not be normal. For example, consider the product space (for its definition see Section 1.3) $\mathbf{R}^J$, where J is uncountable. It can be proved that $\mathbf{R}^J$ is regular but not normal. Moreover $\mathbf{R}^J$ is homeomorphic to the subspace $(0, 1)^J$ of $[0, 1]^J$, which is normal.

Theorem 1.14

Any regular space with a countable basis is normal. ♡

Proof. Let $(X, \mathscr{T})$ be a regular space with a countable basis $\mathscr{B}$. Let A and B be disjoint closed subsets of X. For any $x \in A$ there exists an open subset U_x not intersecting B. By **Theorem 1.13** (3), we can find an open subset V_x of x such that $\overline{V_x} \subseteq U_x$. Choose an element of $\mathscr{B}$ containing x and contained in V_x; therefore we obtain a countable open subsets $\{U_n\}_{n\in\mathbf{N}} \subseteq \mathscr{B}$ such that

$$A \subseteq \bigcup_{n\in\mathbf{N}} U_n =: U, \quad \overline{U_n} \cap B = \emptyset.$$

Similarly, we can find another countable open subsets $\{V_n\}_{n\in\mathbf{N}}$ such that

$$B \subseteq \bigcup_{n\in\mathbf{N}} V_n =: V, \quad \overline{V_n} \cap A = \emptyset.$$

Then U and V are open subsets containing A and B respectively, but they may have nonempty intersection. Given $n \in \mathbf{N}$, set

$$U'_n := U_n \setminus \bigcup_{1\le i\le n} \overline{V_i}, \quad V'_n := V_n \setminus \bigcup_{1\le i\le n} \overline{U_i}.$$

Then U'_n and V'_n are open, and

$$A \subseteq \bigcup_{n\in\mathbf{N}} U'_n, \quad B \subseteq \bigcup_{n\in\mathbf{N}} V'_n.$$

Finally, define

$$U' := \bigcup_{n\in\mathbf{N}} U'_n, \quad V' := \bigcup_{n\in\mathbf{N}} V'_n.$$

Then $U' \cap V' = \emptyset$. Indeed, if $x \in U' \cap V'$, then $x \in U'_j \cap V'_k$ for some $j, k \in \mathbf{N}$. Assume $j \le k$. Then $x \in U_j$ but $x \notin \cup_{1\le i\le j}\overline{V_i}$; since $j \le k$, $x \notin \cup_{1\le i\le k}\overline{U_k}$ implies $x \notin \overline{U_j}$. Hence $U' \cap V' = \emptyset$. $\square$

Consider a metric space (X, d), and A, B are disjoint closed subsets of X. Define a map

$$f : X \longrightarrow [0,1], \quad x \longmapsto \frac{d(x,A)}{d(x,A) + d(x,B)}$$

where $d(x,A) := \inf\{d(x,a)|a \in A\}$. Clearly that

$$\boldsymbol{f}|_A \equiv 0, \quad \boldsymbol{f}|_B \equiv 1.$$

We claim that f is continuous. Consider a given point $x \in X$ and $\{x_n\}_{n\in\mathbf{N}} \subset X$ with $d(x_n, x) \to 0$ as $n \to +\infty$. We may assume that $x \notin A \cup B$, since A and B are closed. In this case, $d(x_n, A) > 0$ and $d(x_n, B) > 0$ for sufficiently large n. Compute

$$\begin{aligned} f(x_n) - f(x) &= \frac{d(x_n,A)}{d(x_n,A) + d(x_n,B)} - \frac{d(x,A)}{d(x,A)+d(x,B)} \\ &= \frac{d(x_n,A)d(x,B) - d(x,A)d(x_n,B)}{[d(x_n,A) + d(x_n,B)][d(x,A)+d(x,B)]} \\ &= \frac{[d(x_n,A) - d(x,A)]d(x,B) + d(x,A)[d(x,B) - d(x_n,B)]}{[d(x_n,A)+d(x_n,B)][d(x,A)+d(x,B)]}. \end{aligned}$$

From the triangle inequalities

$$|d(x_n,A) - d(x,A)| \le d(x_n,x), \quad |d(x_n,B) - d(x,B)| \le d(x_n,x),$$

we conclude that

$$|f(x_n) - f(x)| \le \frac{d(x_n,x)}{d(x_n,A) + d(x_n,B)}$$

which tends to 0 as $n \to \infty$. Thus f is continuous.

According to **Theorem 1.13** (5), any metric spaces are normal. We ask whether the above construction holds for normal spaces.

Theorem 1.15. (Urysohn's lemma)

Let $(X, \mathscr{T})$ be a normal space and A, B be disjoint closed subsets of X. For any closed interval $[a, b] \subseteq \mathbf{R}$, there exists a continuous map $f : X \to [a, b]$ such that

$$f|_A \equiv a \quad \textit{and} \quad f|_B \equiv b.$$

♡

Proof. Since the map $[0, 1] \to [a, b]$, $t \mapsto (1 - t)a + tb$, is continuous, we may assume that $[a, b] = [0, 1]$.

Step 1. Let $P := [0, 1] \cap \mathbf{Q}$. We shall define for each $p \in P$ an open set $U_p \subseteq X$, in such a way that $\overline{U_p} \subseteq U_q$ whenever $p < q$. Moreover

$$A \subseteq U_0 \subseteq \overline{U_0} \subseteq U_1, \quad U_1 := X \setminus B.$$

Step 2. We define

$$U_p := \begin{cases} \emptyset, & \text{if } p < 0 \text{ and } p \in \mathbf{Q}, \\ X, & \text{if } p > 1 \text{ and } p \in \mathbf{Q}. \end{cases}$$

Clearly for any $p, q \in \mathbf{Q}$ with $p < q$, we still have $\overline{U_p} \subseteq U_q$.

Step 3. Given $x \in X$, define

$$\mathbf{Q}(x) := \{p \in \mathbf{Q} | U_p \ni x\} \subseteq \mathbf{Q}.$$

Then $\mathbf{Q} \cap (1, +\infty) \subset \mathbf{Q}(x)$. Thus $\mathbf{Q}(x)$ is bounded below and its greatest lower bound lies in $[0, 1]$. Define

$$f(x) := \inf \mathbf{Q}(x) = \inf\{p \in \mathbf{Q} | x \in U_p\}, \quad x \in X,$$

which is well-defined.

Step 4. If $x \in A$, then $x \in U_p$ for all $p \in \mathbf{Q} \cap [0, +\infty)$, so that $f(x) = 0$. If $x \in B$, then $x \in U_p$ for all $p \in \mathbf{Q} \cap (1, +\infty)$, so that $f(x) = 1$.

Step 5. We claim

$$x \in \overline{U_r} \Longrightarrow f(x) \leq r \quad \text{and} \quad x \notin U_r \Longrightarrow f(x) \geq r.$$

Indeed, if $x \in \overline{U_r}$, then $x \in U_s$ for all rational numbers $s > r$; therefore $f(x) \leq r$. If $x \notin U_r$, then $x \notin U_s$ for any rational numbers $s < r$; so $f(x) \geq r$.

Step 6. f is continuous. For any $x_0 \in X$ and any open interval $(c, d) \subseteq \mathbf{R}$ containing $f(x_0)$, we want to find an open subset $U \ni x_0$ such that $f(U) \subset (c, d)$. Choose $p, q \in \mathbf{Q}$ such that

$$c < p < f(x_0) < q < d.$$

By **Step 5**, $f(x_0) < q$ implies $x_0 \in U_q$, and $f(x_0) > p$ implies $x_0 \notin \overline{U_p}$. Hence $x_0 \in U := U_q \setminus \overline{U_p}$ which is open in X. To verify $f(U) \subset (c, d)$, take $x \in U$. Then $x \in U_q \subset \overline{U_q}$ so that $f(x) \leq q$ by **Step 5**. From $x \notin \overline{U_p}$ we get $f(x) \geq p$ again by **Step 5**. Thus $p \leq f(x) \leq q$ and $f(U) \subset (c, d)$.

Return back to **Step 1**. Because P is countable, we can arrange the elements of P in an infinite sequence, denote by $\mathscr{P}$, so that 1 and 0 are the first two elements of the sequence. Define

$$U_1 := X \setminus B \supseteq A.$$

By normality (see **Theorem 1.13** (4)), there exists an open subset U_0 such that $A \subset U_0 \subseteq \overline{U_0} \subseteq U_1$.

Let P_n $(n \geq 2)$ denote the set of the first n rational numbers in $\mathscr{P}$. Assume that U_p is defined for all $p \in P_n$ satisfying the condition

$$p < q \implies \overline{U_p} \subset U_q. \tag{1.2.4.1}$$

Let r denote the next rational number in $\mathscr{P}$. Consider

$$P_{n+1} := P_n \cup \{r\}.$$

It is a finite subset of $[0, 1]$, and we can find $p, q \in P_{n+1}$ such that $p < r < q$ in the usual order relation and in such a way that there are no other elements of P_{n+1} in $[p, r]$ and $[r, q]$. Such p and q are called immediate predecessor and immediate successor respectively. Since p, q actually lie in P_n, we have $\overline{U_p} \subset U_q$. Applying the normality of X to the pair $(\overline{U_p}, X \setminus U_q)$ of disjoints closed subsets, there exists an open subset U_r such that $\overline{U_p} \subseteq U_r \subseteq \overline{U_r} \subseteq U_q$. We claim that (1.2.4.1) holds for every pair of elements of P_{n+1}. If both of them lie in P_n, (1.2.4.1) holds trivially. If one is r and another is $s \in P_n$, then either $s \leq p$ so that $\overline{U_s} \subset \overline{U_p} \subseteq U_r$, or $s \geq q$ so that $\overline{U_r} \subset \overline{U_q} \subseteq U_s$. Hence (1.2.4.1) also holds in P_{n+1}.

By induction on n, we have defined the open subset U_p for each $p \in P$. □

If $\mathscr{P}$ in **Step 1** is the following sequence

$$\mathscr{P} = \left\{1, 0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \cdots\right\},$$

then

$$P_3 = P_2 \cup \{1/2\} \implies 0 < \frac{1}{2} < 1 \implies U_0 \subseteq U_{\frac{1}{2}} \subseteq U_1,$$

$$P_4 = P_3 \cup \{1/3\} \implies 0 < \frac{1}{3} < \frac{1}{2} \implies U_0 \subseteq U_{\frac{1}{3}} \subseteq U_{\frac{1}{2}},$$

$$P_5 = P_4 \cup \{2/3\} \implies \frac{1}{2} < \frac{2}{3} < 1 \implies U_{\frac{1}{2}} \subseteq U_{\frac{2}{3}} \subseteq U_1,$$

$$P_6 = P_5 \cup \{1/4\} \implies 0 < \frac{1}{4} < \frac{1}{3} \implies U_0 \subseteq U_{\frac{1}{4}} \subseteq U_{\frac{1}{3}},$$

$$P_7 = P_6 \cup \{3/4\} \implies \frac{2}{3} < \frac{3}{4} < 1 \implies U_{\frac{2}{3}} \subseteq U_{\frac{3}{4}} \subseteq U_1,$$

$$P_8 = P_7 \cup \{1/5\} \implies 0 < \frac{1}{5} < \frac{1}{4} \implies U_0 \subseteq U_{\frac{1}{5}} \subseteq U_{\frac{1}{4}}.$$

Definition 1.3

Let A, B be two subsets of a topological space $(X, \mathscr{T})$. We say that A and B **can be separated by a continuous function**, *if there exists a continuous function $f : X \to [0,1]$ such that $f|_A \equiv 0$ and $f|_B \equiv 1$.* ♣

Observe that the following are equivalent for a topological space $(X, \mathscr{T})$:

(1) $(X, \mathscr{T})$ is normal.

(2) Every pair of disjoint closed subsets in X can be separated by disjoint open subsets.

(3) Every pair of disjoint closed subsets can be separated by a continuous function.

Let $(X, \mathscr{T})$ be a topological space and $f : X \to \mathbf{R}$ be a continuous function. The **support of f** is

$$\mathbf{supp}(f) := \overline{\{x \in X | f(x) \neq 0\}}. \tag{1.2.4.2}$$

If A is a closed subset of X and U is an open subset containing A, a continuous function $f : X \to [0,1]$ such that $f|_A \equiv 1$ and $\mathbf{supp}(f) \subseteq U$ is called a **bump function for A supported in U**.

Corollary 1.2. (Existence of bump functions)

Let $(X, \mathscr{T})$ be a normal space. If A is a closed subset of X and U is an open subset containing A, there exists a bump function for A supported in U. ♡

Proof. By normality, we can find an open subset V such that $A \subseteq V \subseteq \overline{V} \subseteq U$. Apply **Theorem 1.15** to the pair $(A, B := X \setminus V)$. □

Theorem 1.16. (Tietze's extension theorem)

Let $(X, \mathscr{T})$ be a normal space and $A \subseteq X$ be closed subset.

(1) *Any continuous function of A into $[a, b] \subset \mathbf{R}$ may be extended to a continuous map of X into $[a, b]$.*

(2) *Any continuous function of A into $\mathbf{R}$ may be extended to a continuous map of X into $\mathbf{R}$.* ♡

Proof. Given a continuous function $f : A \to [-r, r]$, we construct a continuous function $g : X \to \mathbf{R}$ such that

$$|g(x)| \leq \frac{1}{3}r, \quad x \in X \quad \text{and} \quad |g(a) - f(a)| \leq \frac{2}{3}r, \quad a \in A.$$

Define

$$I_1 := [-r, -r/3], \quad I_2 := [-r/3, r/3], \quad I_3 := [r/3, r], \quad B := f^{-1}(I_1), \quad C := f^{-1}(I_3).$$

Since f is continuous, it follows that B, C are closed disjoint subsets in A and then B, C are closed disjoint subset of X, by **Proposition 1.3** (1). According to **Theorem 1.15**, there exists a continuous function $g : X \to [-r/3, r/3]$ such that

$$g|_B \equiv -\frac{1}{3}r, \quad g|_C = \frac{1}{3}r.$$

Then $|g(x)| \le r/3$ for all $x \in X$. Given $a \in A$. If $a \in B$, then $|g(a)-f(a)| = |-r/3-f(a)| \le 2r/3$; if $a \in C$, then $|g(a) - f(a)| = |r/3 - f(a)| \le 2r/3$; if $a \notin B \cup C$, then $f(a), g(a) \in I_2$ and $|g(a) - f(a)| \le 2r/3$.

(1) Since $[a, b]$ is homeomorphic to $[-1, 1]$, we may assume $[a, b] = [-1, 1]$. Let $f : A \to [-1, 1]$ be a continuous map. Then there exists a continuous function $g_1 : X \to \mathbf{R}$ such that

$$|g_1(x)| \le \frac{1}{3}, \quad x \in X \quad \text{and} \quad |g_1(a) - f(a)| \le \frac{2}{3}, \quad a \in A.$$

The function $f - g_1$ maps A into $[-2/3, 2/3]$. We can find a continuous function $g_2 : X \to \mathbf{R}$ such that

$$|g_2(x)| \le \frac{1}{3}\left(\frac{2}{3}\right), \quad x \in X \quad \text{and} \quad |f(a) - g_1(a) - g_2(a)| \le \left(\frac{2}{3}\right)^2, \quad a \in A.$$

In general, for any $n \ge 2$, we can construct a continuous function bg_n defined on X such that

$$|g_n(x)| \le \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}, \quad x \in X \quad \text{and} \quad \left| f(a) - \sum_{1 \le i \le n} g_i(a) \right| \le \left(\frac{2}{3}\right)^n, \quad a \in A.$$

Define

$$g(x) := \sum_{n \ge 1} g_n(x), \quad x \in X.$$

Since

$$\sum_{1 \le n \le N} |g_n(x)| \le \frac{1}{3} \sum_{1 \le n \le N} \left(\frac{2}{3}\right)^{n-1} = 1,$$

it follow that the above series is absolutely and uniformly convergent and then g is continuous on X. Clearly that $g(a) = f(a)$ for all $a \in A$, and g maps X into $[-1, 1]$.

(2) We may, without loss of generality, assume that $f : A \to (-1, 1) \cong \mathbf{R}$ is a continuous function. By (1), f can be extended to a continuous map $g : X \to [-1, 1]$. Define

$$D := g^{-1}(\{-1\}) \cup g^{-1}(\{1\})$$

which is closed in X. from $g|_A = f(A) \subseteq (-1, 1)$, we see that $A \cap D = \emptyset$. By **Theorem 1.15**, there exists a continuous function $\phi : X \to [0, 1]$ such that

$$\phi|_D = 0, \quad \phi|_A = 1.$$

Define

$$h(x) := \phi(x)g(x).$$

Then g is continuous on X and is an extension of f:

$$h(a) = \phi(a)g(a) = g(a) = f(a), \quad a \in A.$$

For any $x \in X$, if $x \in D$ then $h(x) = 0 \cdot g(x) = 0$; if $x \notin D$ then $h(x) = \phi(x)g(x) \in (-1, 1)$. Thus h is a continuous from X into $(-1, 1) \cong \mathbf{R}$. □

1.2.5 Topological manifolds

Let $(X, \mathscr{T})$ be a topological space.

(1) A **cover** of X is a collection $\mathscr{U}$ of subsets of X such that any point $x \in X$ is contained in some $U \in \mathscr{U}$. If each of the sets in $\mathscr{U}$ is open (resp. closed), then we say $\mathscr{U}$ is an **open cover** (resp. a **closed cover**).

(2) Given a cover $\mathscr{U}$, a **subcover** of $\mathscr{U}$ is a subcollection $\mathscr{U}' \subseteq \mathscr{U}$ that is still a cover of X.

Theorem 1.17

Any open cover of a second countable topological space has a countable subcover. ♡

Proof. Let $\mathscr{B}$ be a countable basis for X and let $\mathscr{U}$ be any open cover of X. Define

$$\mathscr{B}' := \{B \in \mathscr{B} : B \subset U \text{ for some } U \in \mathscr{U}\} \neq \emptyset.$$

It is clear that $\mathscr{B}'$ is countable. For any $B \in \mathscr{B}'$ let $U_B \in \mathscr{U}$ be such that $U_B \supset B$ as above. Let

$$\mathscr{U}' := \{U_B \in \mathscr{U} : B \in \mathscr{B}'\}.$$

This is a countable subcover of X. □ □

A topological space $(X, \mathscr{T})$ is said to be a **Lindelöf space** if every open cover of X has a countable subcover.

Exercise 1.17

Recall **Example 1.4***. Show that $(\mathbf{R}, \mathscr{T}_{\mathbf{pp}})$ is not Lindelöf, $(\mathbf{R}, \mathscr{T}_{\mathbf{ep}})$ is Lindelöf, and $(\mathbf{R}, \mathscr{T}_{\mathbf{fc}})$ is Lindelöf.* ♠

A topological space $\mathcal{M} = (\mathcal{M}, \mathscr{T})$ is **local Euclidean of dimension m** if for any $p \in \mathcal{M}$ there is an open set $\mathcal{U} \subset \mathcal{M}$ containing p, called **coordinate domain**, that is homeomorphic to an open subset U of $\mathbf{R}^m$. The homeomorphism is denoted by $\boldsymbol{\varphi} : \mathcal{U} \to \boldsymbol{\varphi}(\mathcal{U}) = U$, and is called a **coordinate map** on $\mathcal{U}$. The pair $(\mathcal{U}, \boldsymbol{\varphi})$ is called a **coordinate chart** for $\mathcal{M}$. Write

$$\boldsymbol{\varphi}(q) := (x^1(q), \cdots, x^m(q)), \quad q \in \mathcal{U}. \tag{1.2.5.1}$$

We see that x^i can be viewed as a function on $\mathcal{U}$ and $\boldsymbol{\varphi} = (x^1, \cdots, x^m)$.

Equivalently, $\mathcal{M}$ is local Euclidean of dimension m if and only if any point $p \in \mathcal{M}$ has an open set $\mathcal{U} \subset \mathcal{M}$ of p that is homeomorphic to an open ball in $\mathbf{R}^m$. In this case we usually say $\mathcal{U}$ is a **coordinate ball** in $\mathcal{M}$. We may also assume that $\varphi(p) = 0 \in \mathbf{R}^m$.

An **m-dimensional topological manifold** is a second countable Hausdorff space $\mathcal{M}$ that is local Euclidean of dimension m. Write $\dim(\mathcal{M}) = m$ and call the **dimension** of $\mathcal{M}$.

A typical example of m-dimensional topological manifold is $\mathbf{R}^m$.

Proposition 1.5

Every open subset of an m-dimensional topological manifold is still an m-dimensional topological manifold. ♡

Proof. Let $\mathcal{M}$ be an m-dimensional topological manifold and $\mathcal{V}$ be an open subset of $\mathcal{M}$.

(1) $\mathcal{V}$ is locally Euclidean of dimension m. For any $p \in \mathcal{V} \subseteq \mathcal{M}$, we have an open subset U in $\mathcal{M}$ containing p, that is homeomorphic to an open subset U of $\mathbf{R}^m$. Then $\mathcal{U} \cap \mathcal{V}$ still contains p, is open, and is homeomorphic to an open subset of $\mathbf{R}^m$.

(2) $\mathcal{V}$ is second countable. By Proposition 1.4 (4).

(3) $\mathcal{V}$ is Hausdorff. By Theorem 1.13 (2). □

The **(closed) m-dimensional upper half-space** $\mathbf{H}^m \subseteq \mathbf{R}^m$ is defined by

$$\mathbf{H}^m := \{(x^1, \cdots, x^m) \in \mathbf{R}^m | x^m \ge 0\}. \tag{1.2.5.2}$$

The boundary and interior of $\mathbf{H}^m$ are

$$\partial\mathbf{H}^m = \{(x^1, \cdots, x^m) \in \mathbf{H}^m | x^m = 0\}, \quad \mathbf{Int}(\mathbf{H}^m) = \{(x^1, \cdots, x^m) \in \mathbf{H}^m | x^m > 0\}.$$

When $m = 0$, we have $\mathbf{H}^0 = \mathbf{R}^0 = \{0\}$, so $\mathbf{Int}(\mathbf{H}^0) = \mathbf{H}^0$ and $\partial\mathbf{H}^0 = \emptyset$.

When $m = 2$, we obtain the **upper half-plane**

$$\mathbf{H} \equiv \mathbf{H}^2 := \{\tau \in \mathbf{C} | \mathrm{Im}(\tau) > 0\}. \tag{1.2.5.3}$$

Define the **modular group**

$$\mathbf{SL}_2(\mathbf{Z}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \mathbf{Z},\ ad - bc = 1 \right\} \tag{1.2.5.4}$$

and the associated map

$$\mathbf{SL}_2(\mathbf{Z}) \times \overline{\mathbf{C}} \longrightarrow \overline{\mathbf{C}}, \quad \left(\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tau \right) \longmapsto \gamma(\tau) := \frac{a\tau + b}{c\tau + d}. \tag{1.2.5.5}$$

Here $\overline{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$ can be considered as the Riemann sphere $\mathbf{S}^2$.

Exercise 1.18

(1) *Let Γ be the subgroup of $\mathbf{SL}_2(\mathbf{Z})$ generated by*

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Prove $\Gamma = \mathbf{SL}_2(\mathbf{Z})$.

(2) *For any* $\gamma, \gamma' \in \mathbf{SL}_2(\mathbf{Z})$ *and any* $\tau \in \mathbf{H}$*, one has*

$$(\gamma \cdot \gamma')(z) = \gamma(\gamma'(z)).$$

(3) *Prove*

$$\mathbf{Im}(\gamma(\tau)) = \frac{\mathbf{Im}(\tau)}{|c\tau + d|^2}, \quad \frac{d}{d\tau}\gamma(\tau) = \frac{1}{(c\tau + d)^2}, \quad \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{SL}_2(\mathbf{Z}).$$

Since $\mathbf{H}^m \subseteq \mathbf{R}^m$ we can give it the subspace topology of $\mathbf{R}^m$. An **m-dimensional topological manifold with boundary** is a second countable Hausdorff space $\mathcal{M}$ in which every point $p \in \mathcal{M}$ has an open subset $\mathcal{U}$ containing p, which is homeomorphic, under the map φ, either to an open subset of $\mathbf{R}^m$ or to an open subset of $\mathbf{H}^m$. The set $\mathcal{U}$ is called a **coordinate domain** of p, φ is called a **coordinate map**, and the pair $(\mathcal{U}, \varphi)$ is called a **coordinate chart for** $\mathcal{M}$.

We say $(\mathcal{U}, \varphi)$ is an **interior chart** if $\varphi(\mathcal{U})$ is an open subset of $\mathbf{R}^m$, and a **boundary chart** if $\varphi(\mathcal{U})$ is an open subset of $\mathbf{H}^m$ with $\varphi(\mathcal{U}) \cap \partial\mathbf{H}^m \neq \emptyset$.

Let $\mathcal{M}$ be an m-dimensional topological manifold with boundary. A point $p \in \mathcal{M}$ is called an **interior point of** $\mathcal{M}$ if it is in the domain of an interior chart, and it is called a **boundary point of** $\mathcal{M}$ if it is in the domain of a boundary chart that takes p to $\partial\mathbf{H}^m$. The **boundary of** $\mathcal{M}$, denoted by $\partial\mathcal{M}$, is the set of all its boundary points, and its **interior**, denoted by $\mathbf{Int}(\mathcal{M})$ or $\mathring{\mathcal{M}}$, is the set of all its interior points.

Proposition 1.6

If $\mathcal{M}$ is an m-dimensional topological manifold with boundary, then $\mathbf{Int}(\mathcal{M})$ *is an open subset of $\mathcal{M}$, which is itself an m-dimensional topological manifold.*

Exercise 1.19

Prove **Proposition 1.6**.

When $\mathcal{M}$ is $\mathbf{H}^m$, an m-dimensional topological manifold with boundary, we have $\mathbf{H}^m = \mathbf{Int}(\mathbf{H}^m) \coprod \partial\mathbf{H}^m$, a disjoint union. Actually this fact holds for any m-dimensional topological manifold with boundary $\mathcal{M}$, i.e., $\mathcal{M} = \mathbf{Int}(\mathcal{M}) \coprod \partial\mathcal{M}$.

Proposition 1.7

If $\mathcal{M}$ is a nonempty m-dimensional topological manifold with boundary, then $\partial\mathcal{M}$ is closed in $\mathcal{M}$, and $\mathcal{M}$ is an m-dimensional topological manifold if and only if $\partial\mathcal{M} = \emptyset$.

Proof. According to **Proposition 1.6** and the above fact, $\partial\mathcal{M} = \mathcal{M} \setminus \mathbf{Int}(\mathcal{M})$ is closed. If $\mathcal{M}$ is an m-dimensional topological manifold, then $\mathcal{M} = \mathbf{Int}(\mathcal{M})$ so $\partial\mathcal{M} = \emptyset$. If $\partial\mathcal{M} = \emptyset$, then $\mathcal{M} = \mathbf{Int}(\mathcal{M})$ is an m-dimensional topological manifold by **Proposition 1.6**. $\square$

Let $\mathscr{U} := \{U_i\}_{1\leq i\leq n}$ be an open cover of a topological space $(X, \mathscr{T})$. A **partition of unity subordinate to** $\mathscr{U}$ is a finite family of continuous functions $\phi_i : X \to \mathbf{R}$, with the following properties:

- $0 \leq \phi_i(x) \leq 1$ for all i and all $x \in X$;
- $\mathrm{Supp}(\phi_i) \subseteq U_i$;
- $\sum_{1\leq i\leq n} \phi_i(x) = x$ for all $x \in X$.

Theorem 1.18. (Existence of finite partitions of unity)

If $\mathscr{U} = \{U_i\}_{1\leq i\leq n}$ is a finite open cover of a normal space $(X, \mathscr{T})$, then there exists a partition of unity subordinate to $\mathscr{U}$. ♡

Proof. Basic idea is to construct two finite open covers $\mathscr{W} = \{W_i\}_{1\leq i\leq n}$ and $\mathscr{V} = \{V_i\}_{1\leq i\leq n}$ of X so that

$$\overline{W_i} \subseteq V_i \subseteq \overline{V_i} \subseteq U_i.$$

Then, by **Theorem 1.15** we can find $\{\phi_i\}_{1\leq i\leq n}$.

Step 1: We can shrink $\mathscr{U} = \{U_i\}_{1\leq i\leq n}$ to an open cover $\mathscr{V} = \{V_i\}_{1\leq i\leq n}$ of X such that $\overline{V_i} \subseteq U_i$ for each i.

Let $A := X \setminus \cup_{2\leq i\leq n} U_i$. Then A is closed in X and is contained in U_1. By normality, **Theorem 1.13** (4), there exists an open subset V_1 of X such that

$$A \subseteq V_1 \subseteq \overline{V_1} \subseteq U_1.$$

So $\{V_1, U_2, \cdots, U_n\}$ is still an open cover of X. In general, given open subsets $V_1, \cdots, V_{k-1}$ such that $\{V_1, \cdots, V_{k-1}, U_k, U_{k+1}, \cdots, U_n\}$ is an open cover of X, let

$$A := X \setminus \left(\left(\bigcup_{1\leq i\leq k-1} V_i \right) \cup \left(\bigcup_{k+1\leq j\leq n} U_j \right) \right).$$

Then A is closed in X and is contained in U_k. By normality again, there exists an open subset V_k such that $A \subseteq V_k \subseteq \overline{V_k} \subseteq U_k$. Then $\{V_1, \cdots, V_{k-1}, V_k, U_{k+1}, \cdots, U_n\}$ is an open cover of X. After finite steps, we prove **Step 1**.

Step 2: Complete the proof. Using **Step 1** again, we can choose an open cover $\mathcal{W} = \{W_i\}_{1\leq i\leq n}$ of X such that

$$\overline{W_i} \subseteq V_i \subseteq \overline{V_i} \subseteq U_i.$$

By **Theorem 1.15**, for each i, there exists a continuous function $\psi_i : X \to [0, 1]$ such that $\psi_i(\overline{W_i}) = 1$ and $\psi_i(X \setminus V_i) = 0$. Because $\psi_i^{-1}(\mathbf{R} \setminus \{0\}) \subseteq V_i$, we have $\mathbf{Supp}(\psi_i) \subseteq \overline{V_i} \subseteq U_i$. Since $\mathscr{W}$ covers X, the finite sum $\sum_{1\leq i\leq n} \psi_i(x) > 0$ for all $x \in X$. Define

$$\phi_i(x) := \frac{\psi_i(x)}{\sum_{1\leq j\leq n} \psi_j(x)}, \quad x \in X.$$

Then $\{\phi_i\}_{1\le i\le n}$ is a partition of unity subordinate to $\mathscr{U}$. $\square$

1.3 Constructions

Introduction

- *Product spaces*
- *Quotient spaces*
- *The de Rham cohomology groups on* $\mathbf{R}^n$

We in this section discuss two important constructions of new topological spaces from old one. At first we introduce topological embeddings.

An injective continuous map that is a homeomorphism onto its image (in the subspace topology) is called a **topological embedding**.

Proposition 1.8

(1) *A continuous injective map that is either open or closed is a topological embeddings.*

(2) *A surjective topological embedding is a homeomorphism.*

Proof. (1) Let $\boldsymbol{f} : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ be a continuous injective map. Then $\boldsymbol{f}$ defines a bijective map $\boldsymbol{f} : X \to \boldsymbol{f}(X)$ that is continuous. To prove $\boldsymbol{f} : (X, \mathscr{T}_X) \to (\boldsymbol{f}(X), \mathscr{T}_{\boldsymbol{f}(X)})$ is homeomorphic, we need to check that it is open or closed by **Exercise 1.11**.

If $\boldsymbol{f} : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is open (resp. closed) and $A \in \mathscr{T}_X$, then $\boldsymbol{f}(A)$ is open (resp. closed) in Y. By **Proposition 1.3** (2), $\boldsymbol{f}(A)$ is also open (resp. closed) in $\boldsymbol{f}(X)$. Thus $\boldsymbol{f} : (X, \mathscr{T}_X) \to (\boldsymbol{f}(X), \mathscr{T}_{\boldsymbol{f}(X)})$ is open (resp. closed).

(2) Clearly. $\square$

If S is a subspace of a topological space $(X, \mathscr{T})$, then the inclusion map $\boldsymbol{\iota}_S : (S, \mathscr{T}_S) \to (X, \mathscr{T})$ is a topological embedding.

Example 1.8. (Examples of topological embeddings)

(1) *Let* $\boldsymbol{F} : \mathbf{R} \to \mathbf{R}^2$ *be the injective continuous map*

$$\boldsymbol{F}(s) := (s, s^2).$$

Its image is the parabola $P := \{(x, y) \in \mathbf{R}^2 | y = x^2\}$. *Hence* $\boldsymbol{F}$ *is an injective continuous map. Denote* $\pi : \mathbf{R}^2 \to \mathbf{R}$ *by* $\pi(x, y) := x$. *Then* $\pi|_P$ *is continuous and gives the inverse of* $\boldsymbol{F} : \mathbf{R} \to P$. *Therefore* $\boldsymbol{F}$ *is a topological embedding.*

(2) *Let* $\boldsymbol{f} : [0, 1) \to \mathbf{R}^2$ *be the map* $\boldsymbol{f}(s) := (\cos(2\pi s), \sin(2\pi s))$. *It is clear that* $\boldsymbol{f}$ *is not a homeomorphisms onto its image in the subspace topology. However the restriction of* f *to any interval* $[0, b)$ *for* $b \in (0, 1)$ *is an topological embedding.*

(3) *Let* $U \subseteq \mathbf{R}^n$ *be an open subset and* : $\mathbf{U} \to \mathbf{R}^\mathbf{k}$ *be any continuous map. The* **graph of** $\boldsymbol{f}$ *is the subset* $\Gamma_{\boldsymbol{f}} \subseteq \mathbf{R}^{n+k}$ *defined by*

$$\Gamma_{\boldsymbol{f}} := \{(\boldsymbol{x}, \boldsymbol{y}) \in U \times \mathbf{R}^k | \boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})\},$$

with the subspace topology inherited from $\mathbf{R}^{n+k}$. *Define two maps*

$$\boldsymbol{\Phi}_{\boldsymbol{f}} : U \longrightarrow \mathbf{R}^{n+k}, \quad \boldsymbol{x} \longmapsto (\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x}))$$

and

$$\boldsymbol{\pi} : \mathbf{R}^{n+k} \longrightarrow \mathbf{R}^n, \quad (x^1, \cdots, x^n, x^{n+1}, \cdots, x^{n+k}) \longmapsto (x^1, \cdots, x^n).$$

Then $\boldsymbol{\Phi}_{\boldsymbol{f}}$ *is a continuous bijection from* U *onto* $\Gamma_{\boldsymbol{f}}$ *and* $\boldsymbol{\pi}|_{\Gamma_{\boldsymbol{f}}}$ *is a continuous inverse for* $\boldsymbol{\Phi}_{\boldsymbol{f}}$. *Hence* $\boldsymbol{\Phi}_{\boldsymbol{f}}$ *is a topological embedding and* $\Gamma_{\boldsymbol{f}}$ *is homeomorphic to* U, *so that* $\Gamma_{\boldsymbol{f}}$ *is a topological manifold.*

(4) *Consider the* $\boldsymbol{n}$**-dimensional unit sphere** *or* **unit** $\boldsymbol{n}$**-sphere**

$$\mathbf{S}^n = \left\{ \boldsymbol{x} = (x^1, \cdots, x^{n+1}) \in \mathbf{R}^{n+1} | \sum_{1 \leq i \leq n+1} (x^i)^2 = 1 \right\}.$$

For example, $\mathbf{S}^0 = \{-1, 1\}$, $\mathbf{S}^1$ *is the unit circle in* $\mathbf{R}^2$, *and* $\mathbf{S}^2$ *is the spherical surface of radius* 1 *in* $\mathbf{R}^3$. *For each* $i \in \{1, \cdots, n+1\}$, *define*

$$U_i^{\pm} := \{\boldsymbol{x} \in \mathbf{S}^n | \pm x^i > 0\}.$$

Then

$$\mathbf{S}^n = \bigcup_{1 \leq i \leq n+1} U_i^+ \cup \bigcup_{1 \leq i \leq n+1} U_i^-.$$

On each $U_i^{\pm}$, *we see that* $\boldsymbol{x} \in \mathbf{S}^n \cap U_i^{\pm}$ *if and only if*

$$x^i = \pm \sqrt{1 - \sum_{1 \leq j \neq i \leq n+1} (x^j)^2}.$$

Hence $\mathbf{S}^n \cap U_i^{\pm}$ *is the graph of a continuous function, is therefore locally Euclidean of dimension* n. *Thus* $\mathbf{S}^n$ *is an* n*-dimensional topological manifold.*

Using the stereographic projection we can give an explicit coordinate map. Let $N := (0, \cdots, 0, 1) \in \mathbf{S}^n$ *and consider the map*

$$\boldsymbol{\varphi} : \mathbf{S}^n \setminus \{N\} \longrightarrow \mathbf{R}^n, \quad (x^1, \cdots, x^n, x^{n+1}) \longmapsto \left(\frac{x^1}{1 - x^{n+1}}, \cdots, \frac{x^n}{1 - x^{n+1}} \right).$$

Evidently,

$$\boldsymbol{\varphi}^{-1}(u^1, \cdots, u^n) = \left(\frac{2u^1}{1 + |\boldsymbol{u}|^2}, \cdots, \frac{2u^n}{1 + |\boldsymbol{u}|^2}, \frac{|\boldsymbol{u}|^2 - 1}{1 + |\boldsymbol{u}|^2} \right), \quad \boldsymbol{u} = (u^1, \cdots, u^n).$$

Similarly, we let $S := \{0, \cdots, 0, -1\}$ *and consider the the map*

$$\boldsymbol{\psi} : \mathbf{S}^n \setminus \{S\} \longrightarrow \mathbf{R}^n, \quad (x^1, \cdots, x^n, x^{n+1}) \longmapsto \left(\frac{x^1}{1 + x^{n+1}}, \cdots, \frac{x^n}{1 + x^{n+1}} \right).$$

Evidently,

$$\boldsymbol{\psi}^{-1}(u^1,\cdots,u^n) = \left(\frac{2u^1}{1+|\boldsymbol{u}|^2},\cdots,\frac{2u^n}{1+|\boldsymbol{u}|^2},\frac{1-|\boldsymbol{u}|^2}{1+|\boldsymbol{u}|^2}\right), \quad \boldsymbol{u} = (u^1,\cdots,u^n).$$

Consequently, $\{(\mathbf{S}^n \setminus N, \boldsymbol{\varphi}), (\mathbf{S}^n \setminus S, \boldsymbol{\psi})\}$ *covers* $\mathbf{S}^n$.

(5) *Suppose that* $\mathcal{M}$ *is a topological manifold and* $\mathcal{S} \subset \mathcal{M}$. *Under the relative topology,* $\mathcal{S}$ *is itself a topological space and furthermore a topological manifold. When* $\mathcal{U}$ *is open, we say that* $\mathcal{U}$ *is an* **open submanifold** *of* $\mathcal{M}$.

Let us denote $\mathbf{GL}(n,\mathbf{R})$ *the set of all* $n \times n$ *singular real matrices. Since any* $n \times n$ *matrix can be viewed as a point in* $\mathbf{R}^{n^2}$, *the "determent function" given by*

$$\det : \mathbf{R}^{n^2} \supset \mathbf{GL}(n,\mathbf{R}) \longrightarrow \mathbf{R}^{n^2}, \quad A \longmapsto \det(A)$$

shows that $\mathbf{GL}(n,\mathbf{R}) = \mathbf{R}^{n^2} \setminus \det^{-1}(0)$ *is a topological manifold (actually it is an open submanifold).*

If $f_i : X_i \to Y_i$ are maps for $i = 1,\cdots,n$, their **product map** is

$$f_1 \times \cdots \times f_i : X_1 \times \cdots \times X_n \longrightarrow Y_1 \times \cdots \times Y_n$$

given by

$$f_1 \times \cdots \times f_n(x_1,\cdots,x_n) := (f_1(x_1),\cdots,f_n(x_n)).$$

1.3.1 Product spaces

Let $(X_i, \mathscr{T}_i)$, $1 \le i \le n$ be topological spaces. Define

$$\mathscr{B} := \{U_1 \times \cdots \times U_n | U_i \in \mathscr{T}_i, \ 1 \le i \le n\}.$$

According to **Proposition 1.2**, $\mathscr{B}$ is a basis for some topology. We call such a topology the **product topology** $\mathscr{T}_{X_1\times\cdots\times X_n}$ or $\mathscr{T}_1 \otimes \cdots \otimes \mathscr{T}_n$ on $X_1 \times \cdots \times X_n$. The topological space $(X_1 \times \cdots \times X_n, \mathscr{T}_1 \otimes \cdots \otimes \mathscr{T}_n)$ is called the **product space**. The basis subsets of the form $U_1 \times \cdots \times U_n$ are called **product open subsets**.

For example, on $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$, the product topology is generated by sets of the form $I \times J$, where I and J are open subsets of $\mathbf{R}$. Hence the product topology on $\mathbf{R}^2$ is generated by open rectangle, so that it is the same as the metric topology induced by the Euclidean distance function.

Consider the above product space $(X_1 \times \cdots \times X_n, \mathscr{T}_1 \otimes \cdots \otimes \mathscr{T}_n)$. The canonical projection

$$\boldsymbol{\pi}_i : X_1 \times \cdots \times X_n \longrightarrow X_i \tag{1.3.1.1}$$

is actually continuous. Indeed, for any $U_i \in \mathscr{T}_i$ we have

$$\boldsymbol{\pi}^{-1}(U_i) = X_1 \times \cdots \times X_{i-1} \times U_i \times X_{i+1} \times \cdots \times X_n \in \mathscr{T}_{X_1\times\cdots\times X_n}$$

so that $\boldsymbol{\pi}_i$ is continuous.

Theorem 1.19

(1) **(Characteristic property of product topologies)** *Let $X_1 \times \cdots \times X_n$ be a product space. For any topological space B, a map $\boldsymbol{f} : B \to X_1 \times \cdots \times X_n$ is continuous if and only if each of its component functions $\boldsymbol{f}_i := \boldsymbol{\pi}_i \circ \boldsymbol{f}$ is continuous:*

$$\begin{tikzcd} & X_1 \times \cdots \times X_n \arrow[d, "\boldsymbol{\pi}_i"] \\ B \arrow[ur, "\boldsymbol{f}"] \arrow[r, "\boldsymbol{f}_i"'] & X_i \end{tikzcd}$$

(2) **(Uniqueness of the product topology)** *Let $X_1, \cdots, X_n$ be topological spaces. The product topology on $X_1 \times \cdots \times X_n$ is the unique topology that satisfies the characteristic property* (1).

(3) *Let $X_1, \cdots, X_n$ be topological spaces.*

(a) The projection maps $\pi_i : X_1 \times \cdots \times X_n \to X_i$ are all continuous and open.

(b) The product topology is "associative" in the sense that the three product topologies $X_1 \times X_2 \times X_3$, $(X_1 \times X_2) \times X_3$, and $X_1 \times (X_2 \times X_3)$ on the set $X_1 \times X_2 \times X_3$ are all equal.

(c) For any i and points $x_j \in X_j$, $j \neq i$, the map $\boldsymbol{f}_i : X_i \to X_1 \times \cdots \times X_n$ given by

$$\boldsymbol{f}_i(x) := (x_1, \cdots, x_{i-1}, x, x_{i+1}, \cdots, x_n)$$

is a topological embedding of X_i into the product space.

(d) If for each i, $\mathscr{B}_i$ is a basis for the topology of X_i, then the set $\{B_1 \times \cdots \times B_n : B_i \in \mathscr{B}_i\}$ is a basis for the product topology on $X_1 \times \cdots \times X_n$.

(e) If A_i is the subspace of X_i for $i = 1, \cdots, n$, the product topology and the subspace topology on $A_1 \times \cdots \times A_n \subseteq X_1 \times \cdots \times X_n$ are equal.

(f) If each X_i is Hausdorff so is $X_1 \times \cdots \times X_n$.

(g) If each X_i is second countable so is $X_1 \times \cdots \times X_n$.

(4) *A product of continuous maps is continuous, and a product of homeomorphisms is a homeomorphism.*

(5) *If $\mathcal{M}_1, \cdots, \mathcal{M}_n$ are topological manifolds of dimensions $m_1, \cdots, m_n$, respectively, the product spade $\mathcal{M}_1 \times \cdots \times \mathcal{M}_n$ is a topological manifold of dimension $m_1 + \cdots + m_n$.* ♡

Proof. (1) Suppose first that all $\boldsymbol{f}_i$ are continuous. To prove $\boldsymbol{f}$ is continuous, it suffices to prove that $\boldsymbol{f}^{-1}(U_1 \times \cdots \times U_n)$ in open in B. According tyo

$$\boldsymbol{f}^{-1}(U_1 \times \cdots \times U_n) = \boldsymbol{f}_1^{-1}(U_1) \cap \cdots \cap \boldsymbol{f}_n^{-1}(U_n),$$

we see that $\boldsymbol{f}$ is continuous.

Conversely, if $\boldsymbol{f}$ is continuous, then for any $U_i \in \mathscr{T}_i$ we have

$$\boldsymbol{f}_i^{-1}(U_i) = \boldsymbol{f}^{-1}(X_1 \times \cdots X_{i-1} \times U \times X_{i+1} \times \cdots \times X_n)$$

that is open in B. Hence $\boldsymbol{f}_i$ is continuous.

$(2) - (5)$: as an exercise. □

Exercise 1.20

Prove $(2) - (5)$ *in* **Theorem 1.19**.

From **Theorem 1.19**, we have the **-torus**

$$\mathbf{T}^m := \underbrace{\mathbf{S}^1 \times \cdots \times \mathbf{S}^1}_{m} \tag{1.3.1.2}$$

which is an m-dimensional topological manifold. When $m = 2$, we call $\mathbf{T}^2$ the **torus**.

For any indexed family $(X_\alpha)_{\alpha \in J}$ of sets, define its Cartesian product by

$$\prod_{\alpha \in J} X_\alpha := \left\{ \boldsymbol{x} : J \to \bigcup_{\alpha \in J} X_\alpha \middle| x_\alpha := \boldsymbol{x}(\alpha) \in X_\alpha \right\}.$$

When $X_\alpha = X$ for all $\alpha \in J$, we write $X^J := \prod_{\alpha \in A} X_\alpha$.

Given a collection $(X_\alpha, \mathscr{T}_\alpha)_{\alpha \in J}$ of topological spaces, define the topology $\mathscr{T}$ to be the topology generated by the basis

$$\left\{ \prod_{\alpha \in J} U_\alpha \middle| U_\alpha \in \mathscr{T}_\alpha \text{ and } U_\alpha = X_\alpha \text{ for all but a finite number of } \alpha \right\}.$$

We call $\mathscr{T}$ the **Tychonoff topology** or **product topology** on $\prod_{\alpha \in J} X_\alpha$. The **box topology** is generalized by the basis

$$\left\{ \prod_{\alpha \in J} U_\alpha \middle| U_\alpha \in \mathscr{T}_\alpha \text{ for all } \alpha \in A \right\}.$$

Most theorems about finite products will also hold for arbitrary products if we use the product topology. Whenever we consider the product $\prod_{\alpha \in J} X_\alpha$ we shall assume it is given by the product topology.

Theorem 1.20

(1) *Suppose that the topology* $\mathscr{T}_\alpha$ *on each* X_α *is generated by a basis* $\mathscr{B}_\alpha$, $\alpha \in J$. *Then the basis for the box topology on* $\prod_{\alpha \in J} X_\alpha$ *is*

$$\left\{ \prod_{\alpha \in J} B_\alpha \middle| B_\alpha \in \mathscr{B}_\alpha \right\},$$

and the basis for the product topology on $\prod_{\alpha \in J} X_\alpha$ *is*

$$\left\{ \prod_{\alpha \in J} B_\alpha \middle| \begin{array}{l} B_\alpha \in \mathscr{B}_\alpha \textit{ for finitely many indices } \alpha \textit{ and} \\ B_\alpha = X_\alpha \textit{ for all the remaining indices} \end{array} \right\}.$$

(2) *Let* $\boldsymbol{\pi}_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$ *be the natural projection, that is,* $\boldsymbol{\pi}_\beta(\boldsymbol{x}) = x_\beta$. *Define*

$$\mathscr{S} := \bigcup_{\beta \in J} \mathscr{S}_\beta, \quad \mathscr{S}_\beta := \left\{ \boldsymbol{\pi}_\beta^{-1}(U_\beta) | U_\beta \in \mathscr{T}_\beta \right\}.$$

Then $\mathscr{S}$ *is a subbasis for the product topology.*

(3) *Let* A_α, $\alpha \in J$, *be subspace of* X_α.

(3.1) $\prod_{\alpha\in J} A_\alpha$ *is a subspace of* $\prod_{\alpha\in J} X_\alpha$ *if both products are given by the box product, or if both products are given by the product product.*

(3.2) *If* $\prod_{\alpha\in J} X_\alpha$ *is given either the box or the product topology, then*

$$\prod_{\alpha\in J}\overline{A_\alpha}=\overline{\prod_{\alpha\in J}A_\alpha}.$$

(4) *If each* X_α*,* $\alpha\in J$*, is Hausdorff, then* $\prod_{\alpha\in J} X_\alpha$ *is Hausdorff in both the box and product topologies.*

(5) **(Characteristic property of infinite product spaces)** *Let* $(X_\alpha,\mathscr{T}_\alpha)_{\alpha\in J}$ *be an indexed family of topological spaces. For any topological space* $(B,\mathscr{T}_B)$*, a map* $\boldsymbol{f}: B\to\prod_{\alpha\in J} X_\alpha$ *(with the product topology) ia continuous if and only if its component functions* $\boldsymbol{f}_\alpha=\boldsymbol{\pi}_\alpha\circ\boldsymbol{f}$ *is continuous. The product topology is the unique topology on* $\prod_{\alpha\in J} X_\alpha$ *that satisfies this property.* ♡

Proof. (1) is clear.

(2) Let $\mathscr{B}_{\mathscr{S}}$ be the basis on $\prod_{\alpha\in A} X_\alpha$ consisting of finite intersections of elements of $\mathscr{S}$. Then for any $B\in\mathscr{B}_{\mathscr{S}}$ we have

$$B=\bigcap_{1\le i\le n}\pi_{\beta_i}^{-1}(U_{\beta_i})$$

where $\beta_1,\cdots,\beta_n$ are distinct and $U_{\beta_i}\in\mathscr{T}_{\beta_i}$. Defining $U_\alpha=X_\alpha$ if $\alpha\neq\beta_1,\cdots,\beta_n$, we obtain $B=\prod_{\alpha\in J}U_\alpha$ which is a basis element for the product topology.

(3) – (4): as an exercise.

(5) Consider the diagram

$$\begin{tikzcd} B \arrow[r, "\boldsymbol{f}"] \arrow[dr, "\boldsymbol{f}_\beta"'] & \prod_{\alpha\in J} X_\alpha \arrow[d, "\boldsymbol{\pi}_\beta"] \\ & X_\beta \end{tikzcd}$$

Assume first that $\boldsymbol{f}$ is continuous. For any $U_\beta\in\mathscr{T}_\beta$, $\boldsymbol{\pi}_\beta^{-1}(U_\beta)$ is a subbasis element for the product topology on $\prod_{\alpha\in J} X_\alpha$, we see that $\boldsymbol{\pi}_\beta$ is continuous and then $\boldsymbol{f}_\beta=\boldsymbol{\pi}_\beta\circ\boldsymbol{f}$ is also continuous.

Conversely, assume that each $\boldsymbol{f}_\alpha$ is continuous. For any subbasis element $\boldsymbol{\pi}_\beta^{-1}(U_\beta)$ for the product topology on $\prod_{\alpha\in J} X_\alpha$, where $U_\beta\in\mathscr{T}_\beta$, one has

$$\boldsymbol{f}^{-1}(\boldsymbol{\pi}_\beta^{-1}(U_\beta))=\boldsymbol{f}_\beta^{-1}(U_\beta)\in\mathscr{T}_B.$$

Hence $\boldsymbol{f}$ is continuous. The uniqueness is similar to that of finite product topology. □

Exercise 1.21

Prove **Theorem 1.20** (3) – (4). ♠

Example 1.9

This example shows that **Theorem 1.20** (5) *may not hold if the product topology is replaced by the box topology. Consider* $\mathbf{R}^\omega := \prod_{n\geq 1} X_n$ *with* $X_n \equiv \mathbf{R}$*, and define*

$$\begin{tikzcd} \mathbf{R} \arrow[r, "\boldsymbol{f}"] \arrow[dr, "\boldsymbol{f}_n"'] & \mathbf{R}^\omega \arrow[d, "\boldsymbol{\pi}_n"] \\ & \mathbf{R} = X_n \end{tikzcd}$$

where $\boldsymbol{f}(t) := (t, t, t, \cdots)$ *and* $f_n(t) := t$*. If* $\mathbf{R}^\omega$ *is given the box topology, then* $\boldsymbol{f}$ *is not continuous. For example, let*

$$B := \prod_{n\geq 1}\left(-\frac{1}{n}, \frac{1}{n}\right)$$

be an open subset of $\mathbf{R}^\omega$ *relative to the box topology. We prove that* $\boldsymbol{f}^{-1}(B)$ *is not open in* $\mathbf{R}$*. Otherwise,* $\boldsymbol{f}^{-1}(B)$ *contains some interval* $(-\delta, \delta)$ *about* $0 \in \mathbf{R}$*. So* $\boldsymbol{f}((-\delta, \delta)) \subseteq B$ *and*

$$\boldsymbol{f}_n((-\delta, \delta)) = (-\delta, \delta) \subseteq \left(-\frac{1}{n}, \frac{1}{n}\right)$$

for all $n \geq 1$*, a contradiction!*

Theorem 1.21

A topological space $(X, \mathscr{T})$ *is Hausdorff if and only if* $\Delta_X := \{(x, y) \in X \times X : x = y\} \subset X \times X$ *is closed in* $(X \times X, \mathscr{T}_{X\times X})$*.*

Proof. Suppose first that Δ_X is closed in $X \times X$ with respect to the product topology. Then $X \times X \setminus \Delta_X$ is open. If $x \neq y$ in X, then $(x, y) \in U \times V \subset \mathscr{T}_{X\times X} \cap (X \times X \setminus \Delta_X)$. Hence X is Hausdorff.

For any $(x, y) \in X \times X \setminus \Delta_X$, we have $x \in U \in \mathscr{T}_X$ and $y \in V \in \mathscr{T}_Y$ for some open sets U, V with $U \cap V = \emptyset$. Hence $(x, y) \in U \times V$. If $U \times V \subset \Delta_X$, then $z \in U \cap V$, a contradiction. Therefore $U \times V \subset X \times X \setminus \Delta_X$ and Δ_X is closed. $\square$

1.3.2 Quotient spaces

Let $(X, \mathscr{T}_X)$ be a topological space, Y be a set, and $\boldsymbol{q} : X \to Y$ a surjective map.

(1) Define a topology $\mathscr{T}_Y$ on Y by

$$\mathscr{T}_Y := \{V \subseteq Y : \boldsymbol{q}^{-1}(V) \in \mathscr{T}_X\}. \tag{1.3.2.1}$$

This is called the **quotient topology** induced by $\boldsymbol{q}$. In this topology, $\boldsymbol{q} : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is continuous (called a **quotient map**).

(2) $(Y, \mathscr{T}_Y)$ is a topological space and $\mathscr{T}_Y$ is the largest topology on Y which makes $\boldsymbol{q}$ continuous.

Let $(X, \mathscr{T}_X)$ be a topological space and $\sim$ be an equivalence relation on X.

(1) Let $Y := X/\sim := \{[x] : x \in X\}$ be the set of equivalence classes, and define

$$\boldsymbol{\pi} : X \longrightarrow Y, \quad x \longmapsto [x],$$

be the natural projection.

(2) $X/\sim$ together with the topology determined by $\boldsymbol{\pi}$ is called the **quotient space** of X by the given equivalence relation. The quotient map $\boldsymbol{\pi}$ is called the **projection**.

More generally, a surjective map $\pi : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ between topological spaces is called a **quotient map** if the following condition holds:

$$V \in \mathscr{T}_Y \Longleftrightarrow \pi^{-1}(V) \in \mathscr{T}_X.$$

It is clear that a quotient map is continuous.

Example 1.10

(1) **(Quotient spaces may not be Hausdorff)** *Let* $X = \mathbf{R}$ *and* $x \sim y$ *if and only* $x - y \in \mathbf{Q}$. *Then* $X/\sim$ *is noncountable. If* $\mathscr{T}_X := \mathscr{T}_{\mathbf{R}}$, *then* $\mathscr{T}_{X/\sim}$ *is the trivial topology on* $X/\sim$.

(2) *Let* $\mathbf{P}^n := (\mathbf{R}^{n+1}\setminus\{\mathbf{0}\})/\sim$ *be the quotient space of* $\mathbf{R}^{n+1}\setminus\{\mathbf{0}\}$ *where the equivalence relation is defined by*

$$\boldsymbol{x} \sim \boldsymbol{y} \Longleftrightarrow \boldsymbol{y} = \lambda\boldsymbol{x} \text{ for some } \lambda \neq 0.$$

Equivalently, $\mathbf{P}^n$ *is the set of* 1*-dimensional linear subspaces in* $\mathbf{R}^{n+1}$. *Denote the quotient map by*

$$\boldsymbol{\pi} : \mathbf{R}^{n+1} \setminus \{\mathbf{0}\} \longrightarrow \mathbf{P}^n, \quad \boldsymbol{x} \longmapsto [\boldsymbol{x}] = [x^1, \cdots, x^{n+1}]. \tag{1.3.2.2}$$

If $\mathcal{U}_i$ *is given by*

$$\mathcal{U}_i := \left\{[\boldsymbol{x}] \in \mathbf{P}^n : x^i \neq 0\right\}, \quad 1 \le i \le n+1,$$

then $(\mathcal{U}_i)_{1\le i\le n+1}$ *is an open cover of* $\mathbf{P}^n$ *with respect to the quotient topology induced by* $\boldsymbol{\pi}$. *Define*

$$\boldsymbol{\varphi}_i([\boldsymbol{x}]) := \left(\frac{x^1}{x^i}, \cdots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \cdots, \frac{x^{n+1}}{x^i}\right);$$

then $\boldsymbol{\varphi}_i$ *is homeomorphic and hence* $\mathbf{P}^n$ *is locally Euclidean.* **Exercise 1.22** *below implies that* $\mathbf{P}^n$ *is second countable. The Hausdorff property of* $\mathbf{P}^n$ *will be proved in Section 1.5.*

(3) *Let* $(X, \mathscr{T})$ *be any topological space and* A *be any subset of* X. *Define a relation* $\sim$ *on* X *by*

$$x_1 \sim x_2 \quad \Longleftrightarrow \quad (x_1, x_2) \in A \times A \text{ or } (X \setminus A) \times (X \setminus A).$$

Clearly that $\sim$ *is an equivalence relation on* X. *The quotient space* $X \setminus A := X\setminus \sim$ *is said to be obtained by* **collapsing A to a point**.

For example, $\overline{\mathbf{B}^n} \setminus \mathbf{S}^{n-1}$ *is homeomorphic to* $\mathbf{S}^{n-1}$.

(4) *If* $(X, \mathscr{T})$ *is any topological space, the quotient*

$$\mathbf{C}(X) := (X \times [0,1])/(X \times \{0\}) \tag{1.3.2.3}$$

obtained from $X \times [0,1]$ by collapsing one end to a point is called the **cone on X**. *For example, $\mathbf{C}(\mathbf{S}^n)$ is homeomorphic to $\mathbf{B}^n$.*

(5) *Let $(X_i, \mathscr{T}_i)_{1\leq i\leq n}$ be a finite family of nonempty topological spaces. For each $i \in \{1, \cdots, n\}$, let $x_i \in X_i$ be a fixed point. The* **wedge product** $\vee_{1\leq i\leq n} X_i$ *of $X_1, \cdots, X_n$ is the quotient space obtained from the disjoint union $\coprod_{1\leq i\leq n} X_i$ by collapsing the set $\{x_1, \cdots, x_n\}$ to a point.*

For example, the wedge sum $\mathbf{R} \vee \mathbf{R}$ is homeomorphic to the union of the x-axis and the y-axis in the plane, and the wedge sum $\mathbf{S}^1 \vee \mathbf{S}^1$ is homeomorphic to the figure-eight space consisting of the union of the two circles of radius 1 centered at $(0,1)$ and $(0,-1)$ in the plane.

(6) *Quotient maps may not be open or closed. For example, consider*

$$\pi_1 : \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R}, \quad (x,y) \longmapsto x$$

and $A := \{(x,y) \in \mathbf{R} \times \mathbf{R} | x \geq 0 \text{ or } y = 0\}$. Let $\boldsymbol{q} : A \to \mathbf{R}$ be obtained by restricting π_1 on A. Then $\boldsymbol{q}$is a quotient map but is neither open or closed.

(7) *Consider $X := [0,1] \cup [2,3] \subseteq \mathbf{R}$ with the subspace topology, $Y := [0,2] \subseteq \mathbf{R}$ with the subspace topology. Define a map*

$$p : X \longrightarrow Y, \quad x \longmapsto \begin{cases} x, & x \in [0,1], \\ x-1, & x \in [2,3]. \end{cases}$$

Then p is surjective, continuous, closed, quotient, but is not open. For $A := [0,1) \cup [2,3] \subseteq X$, define $q : A \to Y$ to be the restriction of p on A; then q is continuous, surjective, but is not quotient.

(8) *Consider the projection $\pi_1 : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, $(x,y) \mapsto x$. Then π_1 is continuous, surjective, open, but is not closed (consider $C := \{(x,y) \in \mathbf{R} \times \mathbf{R} | xy = 1\}$). Let $A := C \cup \{(0,0)\}$ and $q := \pi_1|_A$; then q is continuous, surjective, but is not a quotient map.*

(9) *Define*

$$\boldsymbol{q} : \mathbf{R} \longrightarrow A := \{a,b,c\}, \quad x \longmapsto \begin{cases} a, & x > 0, \\ b, & x < 0, \\ c, & x = 0. \end{cases}$$

Then the quotient topology on A induced by $\boldsymbol{q}$ is $\mathscr{T}_A = \{\{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \emptyset\}$. ♠

If $\boldsymbol{q} : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is a quotient map, a subset $U \subseteq X$ is said to be **saturated** (with respect to $\boldsymbol{q}$) if $U = \boldsymbol{q}^{-1}(V)$ for some subset $V \subseteq Y$.

(1) U is saturated if and only if $U = \boldsymbol{q}^{-1}(\boldsymbol{q}(U))$.

(2) A subset $\boldsymbol{q}^{-1}(y) \subseteq X$ for $y \in Y$ is called a **fiber** of $\boldsymbol{q}$; a saturated set is one that is a union of fibers.

Theorem 1.22

(1) **(Characteristic property of quotient topologies)** *Let $\boldsymbol{q} : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ be a quotient map. For any topological space $(B, \mathscr{T}_B)$, a map $\boldsymbol{f} : Y \to B$ is continuous if and only if the composite map $\boldsymbol{f} \circ \boldsymbol{q}$ is continuous:*

$$\begin{tikzcd} X \arrow[d, "\boldsymbol{q}"'] \arrow[dr, "\boldsymbol{f}\circ\boldsymbol{q}"] & \\ Y \arrow[r, "\boldsymbol{f}"'] & B \end{tikzcd}$$

(2) **(Characterization of quotient maps)** *Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be topological spaces, and let $\boldsymbol{q} : X \to Y$ be any surjective map. Then $\boldsymbol{q}$ is a quotient map if and only if the characteristic property holds.*

(3) **(Passing to the quotient)** *Suppose that $\boldsymbol{q} : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is a quotient map. $(B, \mathscr{T}_B)$ is a topological space, and $\boldsymbol{f} : (X, \mathscr{T}_X) \to (B, \mathscr{T}_B)$ is any continuous map that is constant on the fibers of $\boldsymbol{q}$ (i.e., if $\boldsymbol{q}(x) = \boldsymbol{q}(x')$, then $\boldsymbol{f}(x) = \boldsymbol{f}(x')$). Then there exists a unique continuous map $\widetilde{\boldsymbol{f}} : (Y, \mathscr{T}_Y) \to (B, \mathscr{T}_B)$ such that $\boldsymbol{f} = \widetilde{\boldsymbol{f}} \circ \boldsymbol{q}$:*

$$\begin{tikzcd} X \arrow[d, "\boldsymbol{q}"'] \arrow[dr, "\boldsymbol{f}"] & \\ Y \arrow[r, "\widetilde{\boldsymbol{f}}"'] & B \end{tikzcd}$$

In this situation, we say that the map f **passes to the quotient** *or* **descends to the quotients**.

(4) **(Uniqueness of quotient spaces)** *Suppose $\boldsymbol{q}_1 : (X, \mathscr{T}_X) \to (Y_1, \mathscr{T}_{Y_1})$ and $\boldsymbol{q}_2 : (X, \mathscr{T}_X) \to (Y_2, \mathscr{T}_{Y_2})$ are quotient maps that make the some identifications (i.e., $\boldsymbol{q}_1(x) = \boldsymbol{q}_1(x')$ if and only if $\boldsymbol{q}_2(x) = \boldsymbol{q}_2(x')$). Then there is a unique homeomorphism $\boldsymbol{\varphi} : (Y_1, \mathscr{T}_{Y_1}) \to (Y_2, \mathscr{T}_{Y_2})$ such that $\boldsymbol{\varphi} \circ \boldsymbol{q}_1 = \boldsymbol{q}_2$.*

(5) **(Composition property of quotient maps)** *Suppose $\boldsymbol{q}_1 : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_{Y_1})$ and $\boldsymbol{q}_2 : (Y, \mathscr{T}_Y) \to (Z, \mathscr{T}_Z)$ are quotient maps. Then their composition $\boldsymbol{q}_2 \circ \boldsymbol{q}_1 : (X, \mathscr{T}_X) \to (Z, \mathscr{T}_Z)$ is also a quotient map.*

(6) *A continuous surjective map $\boldsymbol{q} : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is a quotient map if and only if it takes saturated open sets to open sets, or saturated closed sets to closed set.*

(7) *Suppose that $\boldsymbol{q} : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is a quotient map. The restriction of $\boldsymbol{q}$ to any saturated open or closed set is a quotient map.*

(8) *If $\boldsymbol{q} : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is a surjective continuous map that is also an open or closed map, then it is a quotient map.* ♡

Proof. (1) For any $U \in \mathscr{T}_B$, $\boldsymbol{f}^{-1}(U)$ is open in Y if and only if $\boldsymbol{q}^{-1}(\boldsymbol{f}^{-1}(U)) = (\boldsymbol{f} \circ \boldsymbol{q})^{-1}(U)$ is open in X.

(2): as an exercise.

(3) For any $y \in Y$, we choose some $x \in X$ with $\boldsymbol{q}(x) = y$ (because $\boldsymbol{q}$ is surjective). Define

$$\widetilde{\boldsymbol{f}}(y) := \boldsymbol{f}(x).$$

Since $\boldsymbol{f}$ is constant on the fibers of $\boldsymbol{q}$, it follows that $\widetilde{\boldsymbol{f}}$ is well-defined. The continuity of $\widetilde{\boldsymbol{f}}$ follows from (1).

If there exists another continuous map $\widetilde{\boldsymbol{f}}' : (Y, \mathscr{T}_Y) \to (B, \mathscr{T}_B)$ such that $\boldsymbol{f} = \widetilde{\boldsymbol{f}}' \circ \boldsymbol{q}$. Then

$$\widetilde{\boldsymbol{f}}'(y) = \widetilde{\boldsymbol{f}}'(\boldsymbol{q}(x)) = \boldsymbol{f}(x) = (\widetilde{\boldsymbol{f}} \circ \boldsymbol{q})(x) = \widetilde{\boldsymbol{f}}(y).$$

(4) Consider three diagrams

$$\begin{tikzcd} X \arrow[d, "\boldsymbol{q}_1"'] \arrow[dr, "\boldsymbol{q}_2"] & \\ Y_1 \arrow[r, dashed, "\exists \widetilde{\boldsymbol{q}}_2"'] & Y_2 \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d, "\boldsymbol{q}_2"'] \arrow[dr, "\boldsymbol{q}_1"] & \\ Y_2 \arrow[r, dashed, "\exists \widetilde{\boldsymbol{q}}_1"'] & Y_1 \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d, "\boldsymbol{q}_1"'] \arrow[dr, "\boldsymbol{q}_1"] & \\ Y_1 \arrow[r, "\mathbf{1}_{Y_1}", "\widetilde{\boldsymbol{q}}_1 \circ \widetilde{\boldsymbol{q}}_2"'] & Y_1 \end{tikzcd}$$

By (3), there exist $\widetilde{\boldsymbol{q}}_2$ and $\widetilde{q}_1$. Since

$$\widetilde{\boldsymbol{q}}_1 \circ \widetilde{\boldsymbol{q}}_2 \circ \boldsymbol{q}_1 = \widetilde{\boldsymbol{q}}_1 \circ \boldsymbol{q}_2 = \boldsymbol{q}_1,$$

we obtain from (2) that $\widetilde{\boldsymbol{q}}_1 \circ \widetilde{\boldsymbol{q}}_2 = \mathbf{1}_{Y_1}$. Similarly we have $\widetilde{\boldsymbol{q}}_2 \circ \widetilde{\boldsymbol{q}}_1 = \mathbf{1}_{Y_2}$.

(5) – (6): as an exercise.

(7) Let A be a saturated open set and let $\boldsymbol{\pi} = \boldsymbol{q}|_A : A \to \boldsymbol{q}(A)$. Given $V \subseteq \boldsymbol{q}(A)$ and assume $\boldsymbol{\pi}^{-1}(V)$ is open in A, we shall show that V is open in $\boldsymbol{q}(A)$. We first claim

$$\boldsymbol{\pi}^{-1}(V) = \boldsymbol{q}^{-1}(V).$$

Indeed, if $V \subseteq \boldsymbol{q}(A)$ and A is saturated, then $\boldsymbol{q}^{-1}(V) \subseteq A$ (why?) and hence $\boldsymbol{\pi}^{-1}(V) = \boldsymbol{q}^{-1}(V)$ (why?)

Now $\boldsymbol{\pi}^{-1}(V)$ is open in A and A is open in X, so that $\boldsymbol{\pi}^{-1}(V)$ is open in X. From $\boldsymbol{q}^{-1}(V) = \boldsymbol{\pi}^{-1}(V)$, we see that $\boldsymbol{q}^{-1}(V)$ is open in X. Since $\boldsymbol{q}$ is a quotient map, it follows that V is open in Y and in particular open in $\boldsymbol{q}(A)$.

(8) Assume that $\boldsymbol{q}$ is open. Given $V \subseteq Y$ and assume that $\boldsymbol{q}^{-1}(V)$ is open in X, we shall show that V is open in Y. Because $\boldsymbol{q}$ is surjective, we get $\boldsymbol{q}(\boldsymbol{q}^{-1}(V)) = V$. Hence $V = \boldsymbol{q}(\boldsymbol{q}^{-1}(V))$ is open in Y (because $\boldsymbol{q}$ is open). □

Exercise 1.22

Prove **Theorem 1.22** *(2), (5), and (6).* ♠

Note that the product of two quotient maps need not be a quotient map. If $\boldsymbol{q} : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is a quotient map and $\mathscr{T}_X$ is Hausdorff, then $\mathscr{T}_Y$ need not be Hausdorff.

Lemma 1.3

Suppose $\pi : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is a quotient map and X is second countable. If Y is locally Euclidean, it is second countable. ♡

Proof. Since Y is locally Euclidean, it follows that Y has an open cover $\mathscr{U} = (\mathcal{U}_\alpha)_{\alpha \in A}$ with

$\mathcal{U}_\alpha$ is homeomorphic to some open ball. Then $\pi^{-1}(\mathscr{U}) := \{\pi^{-1}(\mathcal{U}_\alpha) : \mathcal{U}_\alpha \in \mathscr{U}\}$ is an open cover of X. The second countability of X shows that (by **Theorem 1.17**) there exists a countable subcover of $\pi^{-1}(\mathscr{U})$. Let $\mathscr{U}' \subseteq \mathscr{U}$ denote a subcover such that $\pi^{-1}(\mathscr{U}')$ is a countable subcover of X and $\mathscr{U}'$ is a countable cover of Y by Euclidean balls. Each such ball has a countable basis, and the union of all these bases is a countable basis for Y. □

1.3.3 The de Rham cohomology groups on $\mathbf{R}^n$

Let U be an open set in $\mathbf{R}^n$ and $\Omega^p(U)$ denote the space of smooth p-forms on U generated by dx^I, where

$$dx^I := dx^{i_1} \wedge \cdots \wedge dx^{i_p}, \quad I = (1 \le i_1 < \cdots < i_p \le n). \tag{1.3.3.1}$$

over $C^\infty(U)$. Any element of $\Omega^p(U)$ can be written as

$$\omega = \sum_{|I|=p} \omega_I dx^I.$$

Then

$$\Omega^\bullet(U) := \bigoplus_{0 \le p \le n} \Omega^p(U) \tag{1.3.3.2}$$

is a graded algebra. Define a differential operator $\boldsymbol{d}$ (**exterior differentiation**) by

$$\boldsymbol{d} : \Omega^p(U) \longrightarrow \Omega^{p+1}(U) \tag{1.3.3.3}$$

as follows:

(i) if $f \in \Omega^0(U)$, then

$$\boldsymbol{d}f = \sum_{1 \le i \le n} \frac{\partial f}{\partial x^i} dx^i = df;$$

(ii) if $\omega = \sum_{|I|=p} \omega_I dx^I \in \Omega^p(U)$, then

$$\boldsymbol{d}\omega := \sum_{|I|=p} \boldsymbol{d}\omega_I \wedge dx^I = \sum_{|I|=p} d\omega_I \wedge dx^I.$$

It is easy to show that $\boldsymbol{d}^2 = \boldsymbol{d} \circ \boldsymbol{d} = 0$ and

$$\boldsymbol{d}(\omega \wedge \eta) = \boldsymbol{d}\omega \wedge \eta + (-1)^{\deg(\omega)} \omega \wedge \boldsymbol{d}\eta, \tag{1.3.3.4}$$

where $\deg(\omega) = p$ if $\omega \in \Omega^p(U)$.

The pair $(\Omega^\bullet(U), \boldsymbol{d})$ is called the **de Rham cohomological complex** on U. We denote by $Z^p(U)$ and $B^p(U)$ the set of **closed p-forms** and **exact p-forms** respectively. Note that exact forms are always closed, however the closed forms may not be exact[3]. The **p-th de Rham cohomology group** of U is

$$H^p_{\mathbf{DR}}(U) := \frac{Z^p(U)}{B^p(U)}, \quad 0 \le p \le n. \tag{1.3.3.5}$$

[3]Consider $U = \mathbf{R}^2 \setminus \{(0,0)\}$ and $\omega = (xdy - ydx)/(x^2 + y^2)$

Set

$$H^{\bullet}_{\mathbf{DR}}(U) := \bigoplus_{0\le p\le n} H^p_{\mathbf{DR}}(U). \tag{1.3.3.6}$$

For any $[\omega], [\eta] \in H^{\bullet}_{\mathbf{DR}}(U)$ we can define

$$[\omega]\cdot[\eta] := [\omega\wedge\eta] \tag{1.3.3.7}$$

which is independent of the choice of representatives.

If $\boldsymbol{F}: U \to V$ is a smooth map from an open subset $U \subset \mathbf{R}^n$ to another open subset $V \subset \mathbf{R}^m$, and if $\omega \in \Omega^p(V)$, the **pull-back** $\boldsymbol{F}^*\omega$ is the smooth p-form on U obtained by making the substitution $\boldsymbol{y} = \boldsymbol{F}(\boldsymbol{x}) = (F^1(\boldsymbol{x}), \cdots, F^m(\boldsymbol{x})$ in ω

$$(\boldsymbol{F}^*\omega)(\boldsymbol{x}) = \sum_{|I|=p} \omega_I(\boldsymbol{F}(\boldsymbol{x}))dF^{i_1}\wedge\cdots\wedge dF^{i_p}. \tag{1.3.3.8}$$

If we have a second smooth map $\boldsymbol{G}: V \to W$ and if η is a smooth form on W, then $\boldsymbol{F}^*(\boldsymbol{G}^*\eta)$ is obtained by substituting $\boldsymbol{z} = \boldsymbol{G}(\boldsymbol{y})$ and $\boldsymbol{y} = \boldsymbol{F}(\boldsymbol{x})$, thus

$$\boldsymbol{F}^*(\boldsymbol{G}^*\eta) = (\boldsymbol{G}\circ\boldsymbol{F})^*\eta. \tag{1.3.3.9}$$

Moreover, we have

$$\boldsymbol{d}(\boldsymbol{F}^*\omega) = \boldsymbol{F}^*(\boldsymbol{d}\omega) \tag{1.3.3.10}$$

for any smooth form ω on V. Consequently, the smooth map $\boldsymbol{F}: U \to V$ induces a morphism on cohomology groups:

$$\boldsymbol{F}^*: H^p_{\mathbf{DR}}(V) \longrightarrow H^p_{\mathbf{DR}}(U), \quad [\omega] \longmapsto [\boldsymbol{F}^*\omega]. \tag{1.3.3.11}$$

Let ω be a smooth form on $[0,1]\times U$. For $(t,\boldsymbol{x}) \in [0,1]\times U$ we write

$$\omega(t,\boldsymbol{x}) = \sum_{|I|=p} \omega_I(t,\boldsymbol{x})dx^I + \sum_{|J|=p-1} \widetilde{\omega}_J(t,\boldsymbol{x})dt\wedge dx^J. \tag{1.3.3.12}$$

Define

$$\mathbf{K}: \Omega^p([0,1]\times U) \longrightarrow \Omega^{p-1}(U), \quad \omega \longmapsto \sum_{|J|=p-1}\left(\int_0^1 \widetilde{\omega}_J(t,\boldsymbol{x})\,dt\right)dx^J. \tag{1.3.3.13}$$

We then have the following diagram

$$\begin{tikzcd} \Omega^p([0,1]\times U) \arrow[r, "\boldsymbol{d}"] \arrow[d, "\mathbf{K}"'] & \Omega^{p+1}([0,1]\times U) \arrow[d, "\mathbf{K}"] \\ \Omega^{p-1}(U) \arrow[r, "\boldsymbol{d}"'] & \Omega^p(U) \end{tikzcd}$$

Observe that

$$(\mathbf{K}\boldsymbol{d}+\boldsymbol{d}\mathbf{K})\,\omega = \sum_{|I|=p}\left(\int_0^1 \frac{\partial\omega_I}{\partial t}(t,\boldsymbol{x})\,dt\right)dx^I = \sum_{|I|=p}[\omega_I(1,\boldsymbol{x})-\omega_I(0,\boldsymbol{x})]\,dx^I \tag{1.3.3.14}$$

where ω is given in (1.3.3.12). Thus

$$(\mathbf{K}\boldsymbol{d}+\boldsymbol{d}\mathbf{K})\,\omega = \iota_1^*\omega - \iota_0^*\omega \tag{1.3.3.15}$$

where $\iota_t: U \to [0,1]\times U$ is the injection $\boldsymbol{x} \mapsto (t,\boldsymbol{x})$.

Theorem 1.23. (Poincaré's lemma) *Let $U \subseteq \mathbf{R}^n$ be a star-shaped open set with respect to $\mathbf{0}$ (that is, $\mathbf{0} \in U$ and any line segment $\overline{\mathbf{0}\boldsymbol{x}}$ is entirely contained in U). If $\omega \in Z^p(U)$ and $p \geq 1$, then $\omega = \boldsymbol{d}\eta$ for some $\eta \in \Omega^{p-1}(U)$. Consequently, $H^p_{\mathbf{DR}}(U) = \{0\}$ for any star-shaped open set (with respect to some point in U) and $p \geq 1$, and $H^0_{\mathbf{DR}}(U) = \mathbf{R}$ for any star-shaped open set.* ♡

Proof. Let $\boldsymbol{H} : [0,1] \times U \to U$ be defined by $\boldsymbol{H}(t, \boldsymbol{x}) := t\boldsymbol{x}$ and set

$$\boldsymbol{F}(\boldsymbol{x}) := \boldsymbol{x}, \quad \boldsymbol{G}(\boldsymbol{x}) := \mathbf{0}, \quad \boldsymbol{x} \in \Omega.$$

Then

$$\boldsymbol{F} = \boldsymbol{H} \circ \iota_1, \quad \boldsymbol{G} = \boldsymbol{H} \circ \iota_0,$$

so that

$$\boldsymbol{d}\left(\mathbf{K}(\boldsymbol{H}^*\omega)\right) = \iota *_1 \boldsymbol{H}^*\omega - \iota_0^*\boldsymbol{H}^*\omega = \boldsymbol{F}^*\omega - \boldsymbol{G}^*\omega = \begin{cases} \omega - \omega(\mathbf{0}), & p = 0, \\ \omega, & p \geq 1. \end{cases}$$

When $p \geq 1$, we get $\eta = \mathbf{K}(\boldsymbol{H}^*\omega)$. □

A direct sum of vector spaces

$$C^\bullet := \bigoplus_{p \in \mathbf{Z}} C^p$$

is called a **differential complex** if there are homomorphisms

$$\cdots \longrightarrow C^{p-1} \xrightarrow{\boldsymbol{d}^{p-1}} C^p \xrightarrow{\boldsymbol{d}^p} C^{p+1} \xrightarrow{\boldsymbol{d}^{p+1}} \cdots$$

such that $\boldsymbol{d}^{p+1} \circ \boldsymbol{d}^p = 0$ for any $p \in \mathbf{Z}$. $\boldsymbol{d}$ is the **differential operator** of the complex $C^\bullet$. The **cohomology** of $C^\bullet$ is the direct sum of vector spaces

$$H^\bullet(C^\bullet) := \bigoplus_{p \in \mathbf{Z}} H^p(C^\bullet) \tag{1.3.3.16}$$

where

$$H^p(C^\bullet) := \frac{\operatorname{Ker}(\boldsymbol{d}^p) \cap C^p}{\operatorname{Im}(\boldsymbol{d}^{p-1}) \cap C^p}.$$

A map $\boldsymbol{f}^\bullet : A^\bullet \to B^\bullet$ between two differential complexes is a **chain map** if it commutes with the differential operators of $A^\bullet$ and $B^\bullet$: $\boldsymbol{f}^\bullet \circ \boldsymbol{d}^\bullet_{A^\bullet} = \boldsymbol{d}^\bullet_{B^\bullet} \circ \boldsymbol{f}^\bullet$. That is,

$$\begin{tikzcd}
\cdots \arrow[r] & A^{p-1} \arrow[r, "\boldsymbol{d}^{p-1}_{A^\bullet}"] \arrow[d, "\boldsymbol{f}^{p-1}"] & A^p \arrow[r, "\boldsymbol{d}^p_{A^\bullet}"] \arrow[d, "\boldsymbol{f}^p"] & A^{p+1} \arrow[r, "\boldsymbol{d}^{p+1}_{A^\bullet}"] \arrow[d, "\boldsymbol{f}^{p+1}"] & \cdots \\
\cdots \arrow[r] & B^{p-1} \arrow[r, "\boldsymbol{d}^{p-1}_{B^\bullet}"] & B^p \arrow[r, "\boldsymbol{d}^p_{B^\bullet}"] & B^{p+1} \arrow[r, "\boldsymbol{d}^{p+1}_{B^\bullet}"] & \cdots
\end{tikzcd}$$

for any $p \in \mathbf{Z}$,

$$\boldsymbol{f}^p \circ \boldsymbol{d}^{p-1}_{A^\bullet} = \boldsymbol{d}^{p-1}_{B^\bullet} \circ \boldsymbol{f}^{p-1}.$$

A sequence of vector spaces

$$\cdots \longrightarrow V^{i-1} \xrightarrow{\boldsymbol{f}^{i-1}} V^i \xrightarrow{\boldsymbol{f}^i} V^{i+1} \longrightarrow \cdots$$

is said to be **exact** if for all $i \in \mathbf{Z}$, $\mathrm{Ker}(\boldsymbol{f}^i) = \mathrm{Im}(\boldsymbol{f}^{i-1})$. An exact sequence of vector spaces $0 \to U \to V \to W \to 0$ is called a **short exact sequence**.

Given a short exact sequence of differential complexes

$$0 \longrightarrow A^\bullet \xrightarrow{\boldsymbol{f}^\bullet} B^\bullet \xrightarrow{\boldsymbol{g}^\bullet} C^\bullet \longrightarrow 0$$

in which the maps $\boldsymbol{f}^\bullet$ and $\boldsymbol{g}^\bullet$ are chain maps, there is a long exact sequence of cohomology groups:

$$\cdots \longrightarrow H^p(A^\bullet) \xrightarrow{H^p(\boldsymbol{f}^\bullet)} H^p(B^\bullet) \xrightarrow{H^p(\boldsymbol{g}^\bullet)} H^p(C^\bullet) \xrightarrow{\boldsymbol{\delta}^p} H^{p+1}(A^\bullet) \longrightarrow$$

Indeed, the map $H^p(\boldsymbol{f}^\bullet) : H^p(A^\bullet) \to H^p(B^\bullet)$ is defined by

$$H^p(\boldsymbol{f}^\bullet)([a]) := [\boldsymbol{f}^p(a)].$$

If $a \in \mathrm{Ker}(\boldsymbol{d}^p_{A^\bullet})$, then

$$0 = \boldsymbol{f}^{p+1} \circ \boldsymbol{d}^p_{A^\bullet}(a) = \boldsymbol{d}^p_{B^\bullet} \circ \boldsymbol{f}^p(a)$$

which implies $\boldsymbol{f}^p(a) \in \mathrm{Ker}(\boldsymbol{d}^p_{B^\bullet})$. If $[a_1] = [a_2] \in H^p(A^\bullet)$, then

$$a_2 = a_1 + \boldsymbol{d}^{p-1}_{A^\bullet}(a)$$

for some $a \in A^{p-1}$. Then

$$\boldsymbol{f}^p(a_2) = \boldsymbol{f}^p(a_1) + \boldsymbol{f}^p \circ \boldsymbol{d}^{p-1}_{A^\bullet}(a) = \boldsymbol{f}^p(a_1) + \boldsymbol{d}^{p-1}_{B^\bullet} \circ \boldsymbol{f}^{p-1}(a).$$

Thus $[\boldsymbol{f}^p(a_1)] = [\boldsymbol{f}^p(a_2)]$.

Similarly, we can define $H^p(\boldsymbol{g}^\bullet) : H^p(B^\bullet) \to H^p(C^\bullet)$.

Next we define $\boldsymbol{\delta}^p : H^p(C^\bullet) \to H^{p+1}(A^\bullet)$ as follows.

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & A^{p+1} \ni a \arrow[u, "\boldsymbol{d}^{p+1}_{A^\bullet}"] \arrow[r, "\boldsymbol{f}^{p+1}"] & B^{p+1} \ni \boldsymbol{d}^p_{B^\bullet}(b) \arrow[u, "\boldsymbol{d}^{p+1}_{B^\bullet}"] \arrow[r, "\boldsymbol{g}^{p+1}"] & C^{p+1} \arrow[u, "\boldsymbol{d}^{p+1}_{C^\bullet}"] \arrow[r] & 0 \\
0 \arrow[r] & A^p \arrow[u, "\boldsymbol{d}^p_{A^\bullet}"] \arrow[r, "\boldsymbol{f}^p"] & B^p \ni b \arrow[u, "\boldsymbol{d}^p_{B^\bullet}"] \arrow[r, "g^p"] & C^p \ni c \arrow[u, "\boldsymbol{d}^p_{C^\bullet}"] \arrow[r] & 0 \\
 & \vdots \arrow[u, "\boldsymbol{d}^{p-1}_{A^\bullet}"] & \vdots \arrow[u, "\boldsymbol{d}^{p-1}_{B^\bullet}"] & \vdots \arrow[u, "\boldsymbol{d}^{p-1}_{C^\bullet}"] &
\end{tikzcd}$$

For any $c \in \mathrm{Ker}(\boldsymbol{d}^p_{C^\bullet})$, since $\boldsymbol{g}^p$ is surjective, it follows that $\boldsymbol{g}^p(b) = c$ for some $b \in B^p$. Because

$$0 = \boldsymbol{d}^p_{C^\bullet} \circ \boldsymbol{g}^p(b) = \boldsymbol{g}^{p+1} \circ \boldsymbol{d}^p_{B^\bullet}(b),$$

we get $\boldsymbol{d}^p_{B^\bullet}(b) \in \mathrm{Ker}(\boldsymbol{g}^{p+1}) = \mathrm{Im}(\boldsymbol{f}^{p+1})$. Thus $\boldsymbol{f}^{p+1}(a) = \boldsymbol{d}^p_{B^\bullet}(b)$ for some $a \in A^{p+1}$. From

$$0 = \boldsymbol{d}^{p+1}_{B^\bullet} \circ \boldsymbol{d}^p_{B^\bullet}(b) = \boldsymbol{d}^{p+1}_{B^\bullet} \circ \boldsymbol{f}^{p+1}(a) = \boldsymbol{f}^{p+2} \circ \boldsymbol{d}^{p+1}_{A^\bullet}(a)$$

and the injectivity of $\boldsymbol{f}^{p+2}$, we see that $\boldsymbol{d}^{p+1}_{A^\bullet}(a) = 0$ and then a is closed. Define

$$\boldsymbol{\delta}^p([c]) := [a].$$

It is clear that the map $\boldsymbol{\delta}^p$ is independent of the choice of a, b, c.

Proposition 1.9

The map δ^p is independent of the choice of a, b, c. ♡

Proof. Assume $[c] = [c']$. Then $c' = c + \boldsymbol{d}_{C^\bullet}^{p-1}(\widetilde{c})$ for some $\widetilde{c} \in C^{p-1}$. For c', we can also find $b' \in B^p$ and $a' \in A^{p+1}$ such that

$$\boldsymbol{g}^p(b') = c', \quad \boldsymbol{d}_{A^\bullet}^{p+1}(a') = 0, \quad \boldsymbol{f}^{p+1}(a') = \boldsymbol{d}_{B^\bullet}^p(b').$$

Since $\boldsymbol{g}^p(b' - b) = c' - c = \boldsymbol{d}_{C^\bullet}^{p-1}(\widetilde{c})$, it follows that

$$0 = \boldsymbol{d}_{C^\bullet}^p \circ \boldsymbol{g}^p(b' - b) = \boldsymbol{g}^{p+1} \circ \boldsymbol{d}_{B^\bullet}^p(b' - b)$$

and then $\boldsymbol{d}_{B^\bullet}^p(b' - b) \in \mathbf{Ker}(\boldsymbol{g}^{p+1}) = \mathbf{Im}(\boldsymbol{f}^{p+1})$. Hence $\boldsymbol{f}^{p+1}(\widetilde{a}) = \boldsymbol{d}_{B^\bullet}^p(b' - b)$ for some $\widetilde{a} \in A^{p+1}$. Moreover

$$0 = \boldsymbol{d}_{B^\bullet}^{p+1} \circ \boldsymbol{f}^{p+1}(\widetilde{a}) = \boldsymbol{f}^{p+2} \circ \boldsymbol{d}_{A^\bullet}^{p+1}(\widetilde{a}) \quad \Longrightarrow \quad \boldsymbol{d}_{A^\bullet}^{p+1}(\widetilde{a}) = 0.$$

From $\boldsymbol{f}^{p+1}(\widetilde{a}) = \boldsymbol{f}^{p+1}(a' - a)$ and the injectivity of $\boldsymbol{f}^{p+1}$, we obtain $a' = a + \widetilde{a}$ for some closed vector $\widetilde{a}$. Therefore $[a'] = [a]$. □

1.4 Topological groups and matrix Lie groups

Introduction

- *Subgroups and left/right translations*
- *Matrix Lie groups*
- *Group actions*

A **topological group** is a group G endowed with a topology such that

$$\boldsymbol{\mu} : G \times G \longrightarrow G, \quad (g_1, g_2) \longmapsto g_1 g_2 \quad \text{and} \quad \boldsymbol{\iota} : G \longrightarrow G, \quad g \longmapsto g^{-1}$$

are both continuous. If the topology is discrete, G is called a **discrete group**.

For example, $\mathbf{R}$, together with additive group structure and Euclidean topology, is a topological group; $\mathbf{R}^* := \mathbf{R} \setminus \{0\}$, together with multiplication and the relative topology, is a topological group; the **general linear group** $\mathbf{GL}(n, \mathbf{R}) := \{n \times n \text{ real nonsingular matrices}\}$, together with matrix multiplication and the relative topology inherited from $\mathbf{R}^{n^2}$, is a topologiocal group.

1.4.1 Subgroups and left/right translations

Let G be a topological group. A **subgroup** H of G is a subset which is also a subgroup in the algebraic sense. Under the relative topology, H is also a topological group.

If A, B are subsets of G we let

$$AB := \{ab : a \in A, \ b \in B\}, \quad A^{-1} := \{a^{-1} : a \in A\}.$$

We say a subset A of G is **symmetric** if $A = A^{-1}$.

Proposition 1.10

(1) *Any finite product of topological groups is a topological group.*

(2) *Let G be a topological group with unity element e.*

(a) *The symmetric neighborhoods of e form a neighborhood basis at e.*

(b) *If $g \in G$ and U is any neighborhood of g, then there is a symmetric neighborhood V of e such that $VgV^{-1} \subseteq U$.*

(c) *If U is any neighborhood of e and n is any positive integer, then there exists a symmetric neighborhood V of e such that $V^n \subseteq U$.*

(3) *Let G be a topological group. If H is any subgroup of G, then $\overline{H}$ is also a subgroup of G. If H is a normal subgroup then so is $\overline{H}$.* ♡

Proof. Exercise. □

By **Proposition 1.10**, $\mathbf{R}^n, \mathbf{S}^1, \mathbf{T}^n$ are topological groups. Since the **orthogonal group**

$$\mathbf{O}(n) = \{A \in \mathbf{GL}(m, \mathbf{R}) : AA^T = I\}$$

is a subgroup of $\mathbf{GL}(m, \mathbf{R})$, $\mathbf{O}(m)$ is a topological group.

Let G be a topological group and $g \in G$.

(1) The **left translation** is defined by

$$\boldsymbol{L}_g : G \longrightarrow G, \quad g' \longmapsto gg'. \tag{1.4.1.1}$$

Then $\boldsymbol{L}_g^{-1} = \boldsymbol{L}_{g^{-1}}$ and $\boldsymbol{L}_{g_1} \circ \boldsymbol{L}_{g_2} = \boldsymbol{L}_{g_1 g_2}$.

(2) The **right translation** is defined by

$$\boldsymbol{R}_g : G \longrightarrow G, \quad g' \longmapsto g'g^{-1}. \tag{1.4.1.2}$$

Then $\boldsymbol{R}_g^{-1} = \boldsymbol{R}_{g^{-1}}$ and $\boldsymbol{R}_{g_1} \circ \boldsymbol{R}_{g_2} = \boldsymbol{R}_{g_1 g_2}$.

(3) The **adjoint homomorphism** is defined by

$$\mathbf{ad}_g : G \longrightarrow G, \quad g' \longmapsto gg'g^{-1}. \tag{1.4.1.3}$$

Then $\mathbf{ad}_g = \boldsymbol{L}_g \circ \boldsymbol{R}_g = \boldsymbol{R}_g \circ \boldsymbol{L}_g$.

1.4.2 Group actions

Let X be a topological space and G a topological group.

(1) The **left action of G on X** is a continuous map

$$G \times X \longrightarrow X, \quad (g, x) \longmapsto g \cdot x \tag{1.4.2.1}$$

with the properties that

$$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x, \quad 1 \cdot x = x$$

for any $g_1, g_2 \in G$ and $x \in X$.

(2) The **right action of G on X** is a continuous map

$$X \times G \longrightarrow X, \quad (x, g) \longmapsto x \cdot g \tag{1.4.2.2}$$

with the properties that

$$(x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2), \quad x \cdot 1 = x$$

for any $g_1, g_2 \in G$ and $x \in X$.

Observe that $\boldsymbol{L}_g$ induces a left action of G on itself by setting

$$g \cdot x := \boldsymbol{L}_g x.$$

Similarly, $\boldsymbol{R}_g$ induces a right action of G on itself by setting

$$x \cdot g := \boldsymbol{R}_{g^{-1}} x.$$

We now always assume that $G \circlearrowleft X$ is a left action. For any $x \in X$, we define the **orbit of $\boldsymbol{x}$** by

$$G \cdot x := \{g \cdot x : g \in G\} \subseteq X. \tag{1.4.2.3}$$

If $G \cdot x \cap G \cdot y \neq \emptyset$, then $g \cdot g = y$ for some $g \in G$ and $G \cdot x = G \cdot y$. We can define an equivalence relation $\sim$ on X by

$$x \sim y \Longleftrightarrow g \cdot x = y \text{ for some } g \in G. \tag{1.4.2.4}$$

Then the equivalence class $[x]$ of x is precisely the orbit $G \cdot x$ of x. Define the **orbit space** by

$$G \setminus X := X / \sim = \{[x] : x \in X\} = \bigcup_{x \in X} G \cdot x. \tag{1.4.2.5}$$

Example 1.11

(1) *The left action* $\mathbf{GL}(n, \mathbf{R}) \times \mathbf{R}^n \to \mathbf{R}^n$ *given by* $(A, \boldsymbol{x}) \mapsto A\boldsymbol{x}$ *has orbits that are* $\{\mathbf{0}\}$ *and* $\mathbf{R}^n \setminus \{\mathbf{0}\}$.

(2) *The left action* $\mathbf{O}(n) \times \mathbf{R}^n \to \mathbf{R}^n$ *given by* $(A, \boldsymbol{x}) \mapsto A\boldsymbol{x}$ *has orbits that are* $\{\mathbf{0}\}$ *and spheres at* $\mathbf{0}$.

(3) *The left action* $\mathbf{R}^* \times (\mathbf{R}^n \setminus \{\mathbf{0}\}) \to \mathbf{R}^n \setminus \{\mathbf{0}\}$ *given by* $(r, \boldsymbol{x}) \mapsto r\boldsymbol{x}$ *has orbits that are lines through* $\mathbf{0}$.

♠

Theorem 1.24

Let G be a topological group acted from left on a topological space X. Then

(i) $\boldsymbol{\pi} : X \to G \setminus X$ *is open, and*

(ii) $G \setminus X$ *is Hausdorff if and only if* $\{(x, y) \in X \times X : y = g \cdot x \text{ for some } g \in G\}$ *is closed in* $X \times X$.

♡

Proof. (i) Let $U \subset X$ be open and consider $\boldsymbol{\pi}^{-1}(\boldsymbol{\pi}(U))$. To prove $\boldsymbol{\pi}$ is open, we shall verify $\boldsymbol{\pi}^{-1}(\boldsymbol{\pi}(U))$ is open in X. For any $x \in \boldsymbol{\pi}^{-1}(\boldsymbol{\pi}(U))$, set $U_x := \{g \cdot x : g \in G\}$. The continuity implies that U_x is a neighborhood of x and $U_x \subset \boldsymbol{\pi}^{-1}(\boldsymbol{\pi}(U))$.

(ii) $\Delta_{G\backslash X} = \{([x],[x]) : [x] \in G \setminus X\}$ implies that

$$(\boldsymbol{\pi} \times \boldsymbol{\pi})^{-1}\Delta_{G\backslash X} = \{(x,y) \in X \times X : g \cdot x = y \text{ for some } g \in G\}.$$

Since $\boldsymbol{\pi} \times \boldsymbol{\pi}$ is open, surjective, and continuous, it follows that $\Delta_{G\backslash X}$ is closed if and only if $(\boldsymbol{\pi} \times \boldsymbol{\pi})^{-1}(\Delta_{G\backslash X})$ is closed. □

In general, we can show that if $\boldsymbol{f} : X \to Y$ is open, surjective, and continuous, then a subset $S \subseteq X$ is closed if and only if $\boldsymbol{f}^{-1}(S) \subseteq X$ is closed. Indeed, if S is closed, then

$$\boldsymbol{f}^{-1}(Y \setminus S) = \boldsymbol{f}^{-1}(Y) \setminus \boldsymbol{f}^{-1}(S) = X \setminus \boldsymbol{f}^{-1}(S)$$

is open in X. If $\boldsymbol{f}^{-1}(S)$ is closed, then $\boldsymbol{f}(X \setminus \boldsymbol{f}^{-1}(S)) = Y \setminus S$ is open in Y.

1.4.3 Matrix Lie groups

Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. The **general linear map over $\mathbf{K}$** is the group

$$\mathbf{GL}(n,\mathbf{K}) := \{n \times n \text{ invertible matrices with } \mathbf{K}\text{-entries}\} \subseteq \mathbf{K}^{n^2}. \tag{1.4.3.1}$$

Then $\mathbf{GL}(n,\mathbf{K})$ is a topological group. Define

$$\mathbf{Max}(n,\mathbf{K}) := \{n \times n \text{ matrices with } \mathbf{K}\text{-entries}\} \cong \mathbf{K}^{n^2}. \tag{1.4.3.2}$$

We say a sequence $\{A_m\}_{m\geq 1}$ of complex matrices in $\mathbf{Max}(n,\mathbf{C})$ **converges** to a matrix $A \in \mathbf{Max}(n,\mathbf{C})$ if $(A_m)_{ij} \to A_{ij}$ as $m \to \infty$ for all $1 \leq i,j \leq n$.

Definition 1.4

A **matrix Lie group** *is any subgroup G of $\mathbf{GL}(n,\mathbf{C})$ with the following property: if $\{A_m\}_{m\geq 1}$ is any sequence of matrices in G and A_m converges to some matrix A, then either $A \in G$ or A is not invertible.* ♣

Equivalently $G \subseteq \mathbf{GL}(n,\mathbf{C})$ is a matrix Lie group if and only if it is a closed subgroup.

Example 1.12. (Counterexamples of matrix Lie groups)

(1) *The set of all $n \times n$ invertible matrices all of whose entries are rational.*

(2) *Let a be irrational and consider*

$$G := \left\{ \begin{bmatrix} e^{\sqrt{-1}t} & 0 \\ 0 & e^{\sqrt{-1}ta} \end{bmatrix} \middle| t \in \mathbf{R} \right\}.$$

Clearly $-I \notin G$. However we can find a sequence of matrices in G which converges to $-I$, so G is not a matrix Lie group. Actually

$$\overline{G} = \left\{ \begin{bmatrix} e^{\sqrt{-1}t} & 0 \\ 0 & e^{\sqrt{-1}s} \end{bmatrix} \middle| s,t \in \mathbf{R} \right\} \cong \mathbf{S}^1 \times \mathbf{S}^1 = \mathbf{T}^2.$$ ♠

We introduce some examples of matrix Lie groups.

(1) **The general linear groups** $\mathbf{GL}(n, \mathbf{R})$ and $\mathbf{GL}(n, \mathbf{C})$. Moreover $\mathbf{GL}(n, \mathbf{R})$ is a subgroup of $\mathbf{GL}(n, \mathbf{C})$.

(2) **The special linear groups** $\mathbf{SL}(n, \mathbf{K})$:

$$\mathbf{SL}(n, \mathbf{C}) := \{A \in \mathbf{GL}(n, \mathbf{K}) | \det A = 1\}. \tag{1.4.3.3}$$

(3) **The orthogonal and special orthogonal groups**. We say $A \in \mathbf{Max}(n, \mathbf{R})$ is **orthogonal** if

$$\sum_{1 \le k \le n} A_{ki} A_{kj} = \delta_{ij}, \quad 1 \le i, j \le n.$$

The usual inner product on $\mathbf{R}^n$ is

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \sum_{1 \le k \le n} x^k y^k, \quad \boldsymbol{x} = (x^1, \cdots, x^n), \ \boldsymbol{y} = (y^1, \cdots, y^n) \in \mathbf{R}^n.$$

Then A is orthogonal is equivalent to $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle A\boldsymbol{x}, A\boldsymbol{y} \rangle$ for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n$. Hence A is orthogonal if and only if $A^\mathsf{T} A = I$, where A^T denotes the transpose of A, i.e., $(A^\mathsf{T})_{ij} = A_{ji}$.

If A is orthogonal, then $1 = \det(A^\mathsf{T} A) = (\det A)^2$ and $\det A = \pm 1$. Let

$$\mathbf{O}(n) := \{A \in \mathbf{GL}(n, \mathbf{R}) | A \text{ orthogonal}\} \tag{1.4.3.4}$$

and

$$\mathbf{SO}(n) := \{A \in \mathbf{O}(n) | \det A = 1\}. \tag{1.4.3.5}$$

For $n = 2$, $\mathbf{SO}(2)$ is the group of rotations on the plane, i.e.,

$$\mathbf{SO}(2) = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \middle| \theta \in \mathbf{R} \right\}. \tag{1.4.3.6}$$

(4) **The unitary and special unitary groups.** We say $A \in \mathbf{Max}(n, \mathbf{C})$ is **unitary** if

$$\sum_{1 \le k \le n} \overline{A_{ki}} A_{kj} = \delta_{ij}, \quad 1 \le i, j \le n.$$

The usual inner product on $\mathbf{C}^n$ is

$$\langle \boldsymbol{z}, \boldsymbol{w} \rangle := \sum_{1 \le k \le n} \overline{z^k} w^k, \quad \boldsymbol{z} = (z^1, \cdots, z^n), \ \boldsymbol{w} = (w^1, \cdots, w^n) \in \mathbf{C}^n.$$

Then A is unitary is equivalent to $\langle \boldsymbol{z}, \boldsymbol{w} \rangle = \langle A\boldsymbol{z}, A\boldsymbol{w} \rangle$ for any $\boldsymbol{z}, \boldsymbol{w} \in \mathbf{C}^n$. Hence A is unitary if and only if $A^* A = I$, where A^* denotes the adjoint of A, i.e., $(A^*_{ij} = \overline{A_{ji}}$.

If A is unitary, then $1 = \det(A^* A) = |\det A|^2$ and $\det A \in \mathbf{S}^1$. Let

$$\mathbf{U}(n) := \{A \in \mathbf{L}(n, \mathbf{C}) | A \text{ unitary}\} \tag{1.4.3.7}$$

and

$$\mathbf{SU}(n) := \{A \in \mathbf{U}(n) | \det A = 1\}. \tag{1.4.3.8}$$

(5) **The complex orthogonal groups**. Instead the inner product on $\mathbf{C}^n$, we consider the bilinear form $(\cdot, \cdot)$ on $\mathbf{C}^n$,

$$(\boldsymbol{z}, \boldsymbol{w}) := \sum_{1 \le k \le n} z^k w^k, \quad \boldsymbol{z} = (z^1, \cdots, z^n), \ \boldsymbol{w} = (w^1, \cdots, w^n) \in \mathbf{C}^n.$$

We say $A \in \mathbf{Max}(n, \mathbf{C})$ is **(complex) orthogonal** of $(\boldsymbol{z}, \boldsymbol{w}) = (A\boldsymbol{z}, A\boldsymbol{w})$ for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbf{C}^n$. If A is orthogonal, then $1 = \det A \cdot \det A$ and $\det A = \pm 1$. Let

$$\mathbf{O}(n, \mathbf{C}) := \{A \in \mathbf{Max}(n, \mathbf{C}) | A \text{ orthogonal}\} \tag{1.4.3.9}$$

and

$$\mathbf{SO}(n, \mathbf{C}) := \{A \in \mathbf{O}(n, \mathbf{C}) | \det A = 1\}. \tag{1.4.3.10}$$

(6) **The generalized orthogonal and Lorentz groups**. For any positive integers n and k, define a symmetric bilinear form $[\cdot, \cdot]_{n,k}$ on $\mathbf{R}^{n+k}$:

$$[\boldsymbol{x}, \boldsymbol{y}]_{n,k} := \sum_{1 \le i \le n} x^i y^i - \sum_{n+1 \le j \le n+k} x^j y^j. \tag{1.4.3.11}$$

Let

$$\mathbf{O}(n, k) := \{A \in \mathbf{Max}(n+k, \mathbf{R}) | [Ax, Ay]_{n,k} = [x, y]_{n,k}\}. \tag{1.4.3.12}$$

For $A \in \mathbf{Max}(n+k, \mathbf{R})$, let

$$A^{(i)} := \begin{bmatrix} A_{1,i} \\ \vdots \\ A_{n+k,i} \end{bmatrix}, \qquad 1 \le i \le n+k.$$

Then

$$A \in \mathbf{O}(n,k) \Longleftrightarrow \begin{cases} [A^{(i)}, A^{(j)}]_{n,k} = 0, & i \ne j, \\ [A^{(i)}, A^{(i)}]_{n,k} = 1, & 1 \le i \le n \\ [A^{(i)}, A^{(i)}]_{n,k} = -1, & n+1 \le i \le n+k. \end{cases}$$

Let

$$g := \begin{bmatrix} I_n & 0 \\ 0 & -I_k \end{bmatrix}$$

Then

$$A\mathbf{O}(n, k) \quad \Longleftrightarrow \quad A^{\mathsf{T}} g A = g. \tag{1.4.3.13}$$

Hence $\det A = \pm 1$.

When $(n, k) = (3, 1)$, we obtain the Lorentz group $\mathbf{O}(3, 1)$.

(7) **The symplectic groups**. Define the following skew-symmetric bilinear form B on $\mathbf{R}^{2n}$:

$$B[x, y] := \sum_{1 \le k \le n} (x^k y^{n+k} - x^{n+k} y^k). \tag{1.4.3.14}$$

Let us define the **real symplectic group** by

$$\mathbf{Sp}(n, \mathbf{R}) := \{A \in \mathbf{Max}(2n, \mathbf{R}) | B[A\boldsymbol{x}, A\boldsymbol{y}] = B[\boldsymbol{x}, \boldsymbol{y}], \ \forall\, \boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^{2n}\}. \tag{1.4.3.15}$$

Clearly $\mathbf{Sp}(1, \mathbf{R}) = \mathbf{SL}(2, \mathbf{R})$. If define

$$J := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

then $B[x, y] = \langle x, Jy\rangle$ and

$$A \in \mathbf{Sp}(n, \mathbf{R}) \quad \Longrightarrow \quad A^{\mathsf{T}}JA = J. \tag{1.4.3.16}$$

Hence $\det A = \pm 1$ for all $A \in \mathbf{Sp}(n, \mathbf{R})$. Actually, we can prove that $\det A = 1$ for all $A \in \mathbf{Sp}(n, \mathbf{R})$.

Similarly we can define the **complex symplectic group**

$$\mathbf{Sp}(n, \mathbf{C}) := \{A \in \mathbf{Max}(2n, \mathbf{C}) | B[A\boldsymbol{x}, A\boldsymbol{y}] = B[\boldsymbol{x}, \boldsymbol{y}], \ \forall\, \boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^{2n}\}. \tag{1.4.3.17}$$

Then

$$A \in \mathbf{Sp}(n, \mathbf{C}) \quad \Longleftrightarrow \quad A^{\mathsf{T}}JA = J. \tag{1.4.3.18}$$

Hence $\det A = \pm 1$ for all $A \in \mathbf{Sp}(n, \mathbf{C})$. Actually, we can prove that $\det A = 1$ for all $A \in \mathbf{Sp}(n, \mathbf{C})$. Clearly $\mathbf{Sp}(1, \mathbf{C}) = \mathbf{SL}(2, \mathbf{C})$.

The **compact symplectic group**[4] is defined to be

$$\mathbf{Sp}(n) := \mathbf{Sp}(n, \mathbf{C}) \cap \mathbf{U}(2n). \tag{1.4.3.19}$$

Clearly $\mathbf{Sp}(1) = \mathbf{SU}(2)$.

The special and general linear groups, the orthogonal and unitary groups, and the symplectic groups make up the **classical groups**.

(8) The **Heisenberg group** H is

$$H := \left\{ A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \in \mathbf{Max}(3, \mathbf{R}) \middle| a, b, c \in \mathbf{R} \right\} \subseteq \mathbf{SL}(3, \mathbf{R}). \tag{1.4.3.20}$$

(9) $\mathbf{R}^* = \{a \in \mathbf{R} | a \neq 0\} \cong \mathbf{GL}(1, \mathbf{R})$, $\mathbf{C}^* = \{a \in \mathbf{C} | a \neq 0\} \cong \mathbf{GL}(1, \mathbf{C})$, and $\mathbf{S}^1 = \{z \in \mathbf{C}^* || z| = 1\} \cong \mathbf{U}(1)$. The additive group $\mathbf{R}$ is isomorphic to $\mathbf{GL}^+(1, \mathbf{R})$ the group of 1×1 matrices with positive determinant, via the map$x \mapsto [e^x]$. Moreover, the additive group $\mathbf{R}^n$ is isomorphic to the group of diagonl real matrices with positive diagonal entries, i.e.,

$$\mathbf{R}^n \cong \mathbf{GL}^+(1, \mathbf{R}) \times \cdots \times \mathbf{GL}^+(1, \mathbf{R}) \ (n \text{ times}).$$

(10) **The Euclidean and Poincaré groups**. Let

$$\mathbf{E}(n) := \{\text{distance-preserving bijections on } \mathbf{R}^n\} \tag{1.4.3.21}$$

that is, $\boldsymbol{f} \in \mathbf{E}(n)$ if and only if $\boldsymbol{f} : \mathbf{R}^n \to \mathbf{R}^n$ is bijective and $|\boldsymbol{x} - \boldsymbol{y}| = |\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})|$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n$. Note that $\boldsymbol{f}$ need not to be linear. Then $\mathbf{O}(n)$ is a subgroup of $\mathbf{E}(n)$ and is the group of all linear distance-preserving maps on $\mathbf{R}^n$.

For $\boldsymbol{x} \in \mathbf{R}^n$ define the **translation by** $\boldsymbol{x}$ by

$$\boldsymbol{T}_{\boldsymbol{x}} : \mathbf{R}^n \longrightarrow \mathbf{R}^n, \quad \boldsymbol{y} \longmapsto \boldsymbol{x} + \boldsymbol{y}.$$

A basic fact about $\mathbf{R}(n)$ is any $\boldsymbol{T} \in \mathbf{E}(n)$ can be written uniquely as an orthogonal linear transformation followed by a translation, i.e., $\boldsymbol{T} = \boldsymbol{T}_{\boldsymbol{x}} \circ \boldsymbol{R}$ for some $\boldsymbol{x} \in \mathbf{R}^n$ and

[4] Some books used $\mathbf{Sp}(n)$ for our $\mathbf{Sp}(n, \mathbf{C})$ and $\mathbf{USp}(n)$ for our $\mathbf{Sp}(n)$.

$\boldsymbol{R} \in \mathbf{O}(n)$. Write

$$\{\boldsymbol{x}, \boldsymbol{R}\} := \boldsymbol{T}_x \circ \boldsymbol{R}.$$

Then for $\boldsymbol{y} \in \mathbf{R}^n$, one has $\{\boldsymbol{x}, \boldsymbol{R}\}\boldsymbol{y} = \boldsymbol{R}\boldsymbol{y} + \boldsymbol{x}$ and

$$\{\boldsymbol{x}_1, \boldsymbol{R}_1\}\{x_2, \boldsymbol{R}_2\}\boldsymbol{y} = \boldsymbol{R}_1(\boldsymbol{R}_2\boldsymbol{y} + \boldsymbol{x}_2) + \boldsymbol{x}_1 = \boldsymbol{R}_1\boldsymbol{R}_2\boldsymbol{y} + (\boldsymbol{x}_1 + \boldsymbol{R}_1\boldsymbol{x}_2).$$

The product on $\mathbf{E}(n)$ is

$$\{\boldsymbol{x}_1, \boldsymbol{R}_1\}\{\boldsymbol{x}_2, \boldsymbol{R}_2\} = \{\boldsymbol{x}_1 + \boldsymbol{R}_1\boldsymbol{x}_2, \boldsymbol{R}1 \circ \boldsymbol{R}_2\} \tag{1.4.3.22}$$

and the inverse of $\{\boldsymbol{x}, \boldsymbol{R}\}$ is

$$\{\boldsymbol{x}, \boldsymbol{R}\}^{-1} = \{-\boldsymbol{R}^{-1}\boldsymbol{x}, \boldsymbol{R}^{-1}\}.$$

Because translations are not linear, we conclude that $\mathbf{E}(n)$is not a subgroup of $\mathbf{GL}(n, \mathbf{R})$. However, $\mathbf{E}(n)$ is isomorphic to a subgroup of $\mathbf{GL}(n+1, \mathbf{R})$ via the following map

$$\{\boldsymbol{x}, \boldsymbol{R}\} \longmapsto \begin{bmatrix} \boldsymbol{R} & \boldsymbol{x}^{\mathsf{T}} \\ \mathbf{0} & 1 \end{bmatrix} \in \mathbf{GL}(n+1, \mathbf{R}).$$

The Poincaré group is defined to be

$$\mathbf{P}(n, 1) := \{\boldsymbol{T} = \boldsymbol{T}_{\boldsymbol{x}} \circ \boldsymbol{A} | \boldsymbol{x} \in \mathbf{R}^{n+1},\ \boldsymbol{A} \in \mathbf{O}(n, 1)\}. \tag{1.4.3.23}$$

This is the group of affine transformations of $\mathbf{R}^{n+1}$ which preserves the Lorentz distance

$$d_{\mathbf{L}}(\boldsymbol{x}, \boldsymbol{y}) := \sum_{1 \le i \le n} (x^i - y^i)^2 - (x^{n+1} - y^{n+1})^2, \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^{n+1}.$$

Here an affine transformation is one of the form $\boldsymbol{x} \mapsto \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$, where $\boldsymbol{A}$ is a linear transformation and $\boldsymbol{b}$ is a constant. Similarly, we can prove that $\mathbf{P}(n, 1)$ is isomorphic to the group of $(n+2) \times (n+2)$ matrices of the form

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{x}^{\mathsf{T}} \\ \mathbf{0} & 1 \end{bmatrix} \qquad \boldsymbol{A} \in \mathbf{O}(n, 1),\ \boldsymbol{x} \in \mathbf{R}^{n+1}$$

Exercise 1.23

(1) *Show that* $\mathbf{O}(n)$ *is a group, a subgroup of* $\mathbf{GL}(n, \mathbf{C})$*, and is a matrix Lie group.*

(2) *Show that* $\mathbf{SO}(n)$ *is a group, a subgroup of* $\mathbf{O}(n)$*, and is a matrix Lie group.*

(3) *Deduce (1.4.3.6) from (1.4.3.5).*

(4) *Show that* $\mathbf{U}(n)$ *is a group, a subgroup of* $\mathbf{GL}(n, \mathbf{C})$*, and is a matrix Lie group.*

(5) *Show that* $\mathbf{SU}(n)$ *is a group, a subgroup of* $\mathbf{U}(n)$*, and is a matrix Lie group.*

(6) *Show that* $\mathbf{O}(n, k)$ *is a group, a subgroup of* $\mathbf{GL}(n+k, \mathbf{R})$*, and a matrix Lie group.*

(7) *Determine elements in* $\mathbf{O}(1, 1)$.

(8) *Verify (1.4.3.13).*

(9) *Show that* $\mathbf{Sp}(n, \mathbf{R})$ *is a group, a subgroup of* $\mathbf{GL}(2n, \mathbf{R})$*, and a matrix Lie group.*

(10) *Show that the Heisenberg* H *is a group, a subgroup of* $\mathbf{GL}(3, \mathbf{R})$*, and a matrix Lie group.*

Actually, every $A \in \mathbf{SU}(2)$ can be expressed in the form

$$A = \begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix}, \qquad \alpha, \beta \in \mathbf{C}, \ |\alpha|^2 + |\beta|^2 = 1.$$

Hence we can view $\mathbf{SU}(2)$ as the three-dimensional sphere $\mathbf{S}^3$ inside $\mathbf{C}^2 = \mathbf{R}^4$.

Definition 1.5

A matrix Lie group G is said to be **compact** *if the following conditions holds:*

(1) *if A_m is any sequence of matrices in G and A_m converges to a matrix A, then $A \in G$, and*

(2) *there exists a constant C such that for all $A \in G$, $||A||_\infty = \max_{1 \le i,j \le n} |A_{ij}| \le C$.* ♣

If we view $\mathbf{Max}(n, \mathbf{C})$ as $\mathbf{C}^{n^2}$, then G is compact in the sense of **Definition 1.5** if and only if G is a closed and bounded subset of $\mathbf{C}^{n^2}$.

Example 1.13

(1) *Compact matrix Lie groups:* $\mathbf{O}(n)$, $\mathbf{SO}(n)$, $\mathbf{U}(n)$, $\mathbf{SU}(n)$, $\mathbf{Sp}(n)$.

(2) *Noncompact matrix Lie groups:* $\mathbf{GL}(n, \mathbf{R})$, $\mathbf{GL}(n, \mathbf{C})$, $\mathbf{SL}(n, \mathbf{R})$, $\mathbf{SL}(n, \mathbf{C})$, $\mathbf{O}(n, \mathbf{C})$, $\mathbf{SO}(n, \mathbf{C})$, $\mathbf{O}(n, k)$, $\mathbf{SO}(n, k)$, *the Heisenberg group H*, $\mathbf{Sp}(n, \mathbf{R})$, $\mathbf{Sp}(n, \mathbf{C})$, $\mathbf{E}(n)$, $\mathbf{P}(n, 1)$, $\mathbf{R}$, $\mathbf{R}^*$, $\mathbf{C}$, $\mathbf{C}^*$. ♠

Exercise 1.24

Verify **Example 1.13**. ♠

Definition 1.6

A matrix Lie group G is **path-connected** *if given any two matrices A and B in G, there exists a continuous path $A(t)$ in G, $0 \le t \le 1$, such that $A(0) = A$ and $A(1) = B$.* ♣

Since any matrix Lie group G can be viewed as a subset of $\mathbf{C}^{n^2}$ that is simply-connected, it follows that any matrix Lie group is path-connected if and only if it is path-connected.

A matrix Lie group which is not path-connected can be decomposed uniquely as a union of **path-connected components**, such that two elements in the same path-connected component can be joined by a continuous path, but two elements in the different path-connected components cannot.

Proposition 1.11

(1) *If G is a matrix Lie group, then the path-connected component of G containing the identity is a subgroup of G.*

(2) *For any $n \ge 1$, the group $\mathbf{GL}(n, \mathbf{C})$ is path-connected.*

(3) *For any $n \ge 1$, the group $\mathbf{SL}(n, \mathbf{C})$ is path-connected.*

(4) *For any $n \ge 1$, the groups $\mathbf{U}(n)$ and $\mathbf{SU}(n)$ are path-connected.* ♡

Proof. (1) Let H be the path-connected component of G containing the identity e. For $A, B \in H \subseteq G$, there exists a continuous paths $A(t), B(t)$, such that

$$A(0) = B(0) = e, \quad A(1) = A, \quad B(1) = B.$$

Then $(A(t)B(t)$ is a continuous path connecting e and AB. Hence the product AB is still in H. Similarly the inverse of A is again in H. Hence H is a subgroup.

(2) When $n = 1$, $\mathbf{GL}(1, \mathbf{C}) \cong \mathbf{C}^*$ that is obviously path-connected.

When $n \geq 2$, we claim that any element $A \in \mathbf{GL}(n, \mathbf{C})$ can be connected to the identity I by a continuous path lying in $\mathbf{GL}(n, \mathbf{C})$. From linear algebra, we know that any matrix is similar to an upper triangular matrix. That is, for any $A \in \mathbf{Max}(n, \mathbf{C})$ there exists $C \in \mathbf{GL}(n, \mathbf{C})$ such that

$$A = CBC^{-1}, \quad B = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & \ddots & * & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

If $A \in \mathbf{GL}(n, \mathbf{C})$, then all λ_i are nonzero. Set

$$B(t) = \begin{bmatrix} \lambda_1 & (1-t)* & \cdots & (1-t)* \\ 0 & \ddots & (1-t)* & \vdots \\ \vdots & \ddots & \ddots & (1-t)* \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} \quad (0 \leq t \leq 1), \quad D = \mathrm{diag}(\lambda_1, \cdots, \lambda_n).$$

Then $A(t) := CB(t)C^{-1}$ is a continuous path in $\mathbf{GL}(n, \mathbf{C})$ from A to CDC^{-1} and $A(t) \in \mathbf{GL}(n, \mathbf{C})$. Next, for each $i \in \{1, \cdots, n\}$, choose a continuous path $\lambda_i(t) \in \mathbf{C}^*$, $1 \leq t \leq 2$, such that $\lambda_i(1) = \lambda_i$ and $\lambda_i(2) = 1$. Then $\widetilde{A}(t)$ is a continuous path in $\mathbf{GL}(n, \mathbf{C})$ from CDC^{-1} to I_n, where

$$\widetilde{A}(t) = C \cdot \mathrm{diag}(\lambda_1(t), \cdots, \lambda_n(t)) \cdot C^{-1}, \quad 1 \leq t \leq 2.$$

Hence we find a continuous path in $\mathbf{GL}(n, \mathbf{C})$ from A to I_n.

(3) In the proof of (2), define $\lambda_i(t)$, $1 \leq i \leq n-1$ as before and

$$\lambda_n(t) = \frac{1}{\lambda_1(t) \cdots \lambda_{n-1}(t)}.$$

(4) From linear algebra, any unitary matrix $U \in \mathbf{U}(n)$ can be written as

$$U = U_1 \cdot \mathrm{diag}(e^{\sqrt{-1}\theta_1}, \cdots, e^{\sqrt{-1}\theta_n}) \cdot U_1^{-1}, \quad U_1 \in \mathbf{U}(n), \ \theta_1, \cdots, \theta_n \in \mathbf{R}.$$

If we define

$$U(t) = U_1 \cdot \mathrm{diag}(e^{\sqrt{-1}(1-t)\theta_1}, \cdots, e^{\sqrt{-1}(1-t)\theta_n}) \cdot U_1^{-1}, \quad 0 \leq t \leq 1,$$

then $U(t) \in \mathbf{U}(n)$ and connects U to I_n. $\square$

Group	Connected?	Components
$\mathbf{GL}(n, \mathbf{C})$	Y	1
$\mathbf{SL}(n, \mathbf{C})$	Y	1
$\mathbf{GL}(n, \mathbf{R})$	N	2
$\mathbf{SL}(n, \mathbf{R})$	Y	1
$\mathbf{O}(n)$	N	2
$\mathbf{SO}(n)$	Y	1
$\mathbf{U}(n)$	Y	1
$\mathbf{SU}(n)$	Y	1
$\mathbf{O}(n, 1)$	N	4
$\mathbf{SO}(n, 1)$	N	2
Heisenberg	Y	1
$\mathbf{E}(n)$	N	2
$P(n, 1)$	N	4

Definition 1.7

A matrix Lie group G is **simply-connected** *if it is path-connected and any loop in G can be shrunk continuously to a point in G. Precisely, assume that G is path-connected. Then G is simply-connected if for any given continuous path $A(t)$, $0 \leq t \leq 1$, lying in G with $A(0) = A(1)$, there exists a continuous function $A(s,t)$, $0 \leq s, t \leq 1$, such that*

$$A(s,0) = A(s,1) \ (0 \leq s \leq 1), \quad A(0,t) = A(t), \quad A(1,t) = A(1,0) \ (0 \leq t \leq 1).$$

♣

Define $A_s(t) := A(s,t)$. Then $A(s,0) = A(s,1)$ means that A_s is a loop; $A(0,t) = A(t)$ means $A_0 = A$; $A(1,t) = A(1,0)$ means A_1 is the point $A(0) = A(1)$.

Proposition 1.12

The group $\mathbf{SU}(2)$ is simply-connected. ♡

Proof. Any $A \in \mathbf{SU}(2)$ can be written as

$$A = \begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix}, \quad \alpha, \beta \in \mathbf{C} \text{ with } |\alpha|^2 + |\beta|^2 = 1.$$

Define

$$\mathbf{SU}(2) \longrightarrow \mathbf{S}^3 \subseteq \mathbf{C}^2 \cong \mathbf{R}^4, \quad \begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix} \longmapsto (\alpha, \beta).$$

The above map is homeomorphic so that $\mathbf{SU}(2)$ is simply-connected. □

Group	Simply-connected?	Fundamental group
$\mathbf{SO}(2)$	N	$\mathbf{Z}$
$\mathbf{SO}(n)$ $(n \geq 3)$	N	$\mathbf{Z}/2\mathbf{Z}$
$\mathbf{U}(n)$	X	$\mathbf{Z}$
$\mathbf{SU}(n)$	Y	$\{1\}$
$\mathbf{Sp}(n)$	Y	$\{1\}$
$\mathbf{GL}^+(n, \mathbf{R})$	N	$\mathbf{Z}$ $(n = 2)$, $\mathbf{Z}/2\mathbf{Z}$ $(n \geq 3)$
$\mathbf{GL}(n, \mathbf{C})$	N	$\mathbf{Z}$
$\mathbf{SL}(n, \mathbf{R})$ $(n \geq 2)$	N	$\mathbf{Z}$ $(n = 2)$, $\mathbf{Z}/2\mathbf{Z}$ $(n \geq 3)$
$\mathbf{SL}(n, \mathbf{C})$	Y	$\{1\}$
$\mathbf{SO}(n, \mathbf{C})$	N	$\mathbf{Z}$ $(n = 2)$, $\mathbf{Z}/2\mathbf{Z}$ $(n \geq 3)$
$\mathbf{Sp}(n, \mathbf{R})$	N	$\mathbf{Z}$
$\mathbf{Sp}(n, \mathbf{C})$	Y	$\{1\}$

Let $X \in \mathbf{Max}(n, \mathbf{K})$. Define formally

$$e^X \equiv \exp X := \sum_{m \geq 0} \frac{X^m}{m!}. \tag{1.4.3.24}$$

Regarding $\mathbf{Max}(n, \mathbf{C})$ as $\mathbf{C}^{n^2}$, we can define the **Hilbert-Schmidt norm** by

$$||X|| := \left(\sum_{1 \leq i,j \leq n} |X_{ij}|^2 \right)^{1/2}. \tag{1.4.3.25}$$

Then we can pullback all analysis on $\mathbf{C}^{n^2}$ to $\mathbf{Max}(n, \mathbf{C})$. A sequence $\{X_m\}_{m \geq 1}$ in $\mathbf{Max}(n, \mathbf{C})$ **converges** to X, if $||X_m - X|| \to 0$ as $m \to \infty$. A sequence $\{X_m\}_{m \geq 1}$ in $\mathbf{Max}(n, \mathbf{C})$ is a **Cauchy sequence** if $||X_m - X_\ell|| \to 0$ as $m, \ell \to \infty$. Therefore, any Cauchy sequence in $\mathbf{Max}(n, \mathbf{C})$ converges.

Proposition 1.13

For any $X \in \mathbf{Max}(n, \mathbf{K})$, the series (1.4.3.24) converges. Moreover e^X is continuous in X. ♡

Proof. Since

$$\sum_{0 \leq m \leq M} \left|\left| \frac{X^m}{m!} \right|\right| \leq \sum_{0 \leq m \leq M} \frac{||X||^m}{m!} \leq e^{||X||},$$

it follows that the series $\sum_{m \geq 0} ||X^m/m!||$ is convergent. Hence $\{\sum_{0 \leq i \leq m} X^i/i!\}_{m \geq 0}$ is a Cauchy sequence and then is a convergent sequence. Thus e^X is well-defined. Because the series (1.4.3.24) uniformly converges on $\{X \in \mathbf{Max}(n, \mathbf{C}) | ||X|| \leq R\}$ and each $X^m/m!$ is continuous, we can conclude that e^X is continuous. □

Consider the following 2×2 matrices

$$X = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} b & c \\ 0 & b \end{bmatrix}$$

Clearly

$$X^2 = \begin{bmatrix} -a^2 & 0 \\ 0 & -a^2 \end{bmatrix}, \quad X^3 = \begin{bmatrix} 0 & a^3 \\ -a^3 & 0 \end{bmatrix}, \quad X^4 = \begin{bmatrix} a^4 & 0 \\ 0 & a^4 \end{bmatrix}, \quad \cdots.$$

Then

$$e^X = \begin{bmatrix} 1 - \frac{a^2}{2!} + \frac{a^4}{4!} + \cdots & -\left(a - \frac{a^3}{3!} + \cdots\right) \\ a - \frac{a^3}{3!} + \cdots & 1 - \frac{a^2}{2!} + \frac{a^4}{4!} + \cdots \end{bmatrix} = \begin{bmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{bmatrix}$$

On the other hand

$$XY \neq YX, \quad e^Y = \begin{bmatrix} e^b & e^b c \\ 0 & e^b \end{bmatrix}, \quad e^X e^Y \neq e^Y e^X.$$

To compute e^X for more general matrices X, we first prove the following

Proposition 1.14

Let $X, Y \in \mathbf{Max}(n, \mathbf{C})$. Then

(1) $e^0 = I_n$.

(2) $(e^X)^* = e^{X^*}$.

(3) e^X *is invertible and* $(e^X)^{-1} = e^{-X}$.

(4) $e^{(\alpha+\beta)X} = e^{\alpha X} e^{\beta X}$ *for all* $\alpha, \beta \in \mathbf{C}$.

(5) *If* $XY = YX$, *then* $e^{X+Y} = e^X e^Y = e^Y e^X$.

(6) *If* C *is invertible, then* $e^{CXC^{-1}} = Ce^X C^{-1}$.

(7) $||e^X|| \leq e^{||X||}$.

(8) e^{tX} *is a smooth curve in* $\mathbf{Max}(n, \mathbf{C})$ *and*

$$\frac{d}{dt} e^{tX} = Xe^{tX} = e^{tX} X, \quad \left.\frac{d}{dt}\right|_{t=0} e^{tX} = X.$$

♡

Proof. $(1) - (2)$ and $(6) - (8)$ are clearly. Also $(3) - (4)$ follow from (5). Since $XY = YX$, it follows that for any $m \geq 1$ we have

$$(X+Y)^m = \sum_{0 \leq k \leq m} \binom{m}{k} X^k Y^{m-k}.$$

Hence

$$\begin{aligned} e^X e^Y &= \left(\sum_{m\geq 0} \frac{X^m}{m!}\right)\left(\sum_{m \geq 0} \frac{Y^m}{m!}\right) = \sum_{m \geq 0} \left(\sum_{0 \leq k \leq m} \frac{X^k}{k!} \frac{Y^{m-k}}{(m-k)!}\right) \\ &= \sum_{m \geq 0} \frac{1}{m!} \left(\sum_{0 \leq k \leq m} \binom{m}{k} X^k Y^{m-k}\right) = \sum_{m \geq 0} \frac{1}{m!} (X+Y)^m = e^{X+Y}. \end{aligned}$$

Now we have $e^X e^Y = e^{X+Y} = e^{Y+X} = e^Y e^X$. □

Exercise 1.25

Prove $(1)-(2)$ *and* $(6)-(8)$ *in* **Proposition 1.14**.

Return back to the above matrix

$$X=\begin{bmatrix}0 & -a\\ a & 0\end{bmatrix}$$

Because the eigenvectors of X are

$$\begin{bmatrix}1\\ \sqrt{-1}\end{bmatrix},\quad \begin{bmatrix}\sqrt{-1}\\ 1\end{bmatrix}$$

with eigenvalues $-\sqrt{-1}a$ and $\sqrt{-1}a$, respectively, we get

$$X=CDC^{-1},\quad C:=\begin{bmatrix}1 & \sqrt{-1}\\ \sqrt{-1} & 1\end{bmatrix},\quad D=\begin{bmatrix}-\sqrt{-1}a & 0\\ 0 & \sqrt{-1}a\end{bmatrix}$$

From **Proposition 1.14** (6), we have

$$\begin{aligned}e^X=Ce^DC^{-1}&=\begin{bmatrix}1 & \sqrt{-1}\\ \sqrt{-1} & 1\end{bmatrix}\begin{bmatrix}e^{-\sqrt{-1}a} & 0\\ 0 & e^{\sqrt{-1}a}\end{bmatrix}\begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{-1}}{2}\\ -\frac{\sqrt{-1}}{2} & \frac{1}{2}\end{bmatrix}\\ &=\begin{bmatrix}\cos a & -\sin a\\ \sin a & \cos a\end{bmatrix}\end{aligned}$$

For any $X,Y\in\mathbf{Max}(n,\mathbf{C})$, we know from Proposition 1.14 (8) that e^{tX} is smooth and then

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\left(e^{tX}Ye^{-tX}\right) &= \frac{d}{dt}\Big|_{t=0}\left(I_n+tX+t^2\frac{X^2}{2}+\cdots\right)Y\left(I_n-tX+t^2\frac{X^2}{2}+\cdots\right)\\ &= \frac{d}{dt}\Big|_{t=0}\left(Y+t(XY-YX)+t^2(\cdots)\right) = XY-YX.\end{aligned}$$

Definition 1.8

For any $X\in\mathbf{Max}(n,\mathbf{K})$ *define*

$$\mathbf{ad}_X:\mathbf{Max}(n,\mathbf{K})\longrightarrow\mathbf{Max}(n,\mathbf{K}),\quad Y\longmapsto[X,Y]:=XY-YX. \tag{1.4.3.26}$$

We also call $[X,Y]$ *the* **Lie bracket** *of* X *and* Y.

If $g(z)=\sum_{m\ge0}a_m(z-a)^m$ is a formal series, then define

$$g(\mathbf{ad}_X):=\sum_{m\ge0}a_m\left(\mathbf{ad}_X-a\mathbf{1}_{\mathbf{Max}(n,\mathbf{K})}\right)^m:\mathbf{Max}(n,\mathbf{K})\longrightarrow\mathbf{Max}(n,\mathbf{K}).$$

In particular, we obtain the operator $e^{\mathbf{ad}_X}$ for each $X\in\mathbf{Max}(n,\mathbf{K})$. Moreover we can prove

$$e^XYe^{-X}=e^{\mathbf{ad}_X}(Y). \tag{1.4.3.27}$$

Indeed,

$$e^{\mathbf{ad}_X}(Y)=\left(\sum_{m\ge0}\frac{(\mathbf{ad}_X)^m}{m!}\right)Y,\quad e^XYe^{-X}=\sum_{m\ge0}\sum_{0\le k\le m}\frac{X^kY}{k!}\frac{(-X)^{m-k}}{(m-k)!}.$$

To prove (1.4.3.27) we need to verify

$$(\mathbf{ad}_X)^m Y = \sum_{0\le k\le m}\binom{m}{k}X^kY(-X)^{m-k}.$$

When $m=1$, it is trivial. Assume that $(\mathbf{ad}_X)^mY = \sum_{0\le k\le m}\binom{m}{k}X^kY(-X)^{m-k}$. Then

$$\begin{aligned}(\mathbf{ad}_X)^{m+1}Y &= \mathbf{ad}_X\left((\mathbf{ad}_X)^mY\right) = \left[X, \sum_{0\le k\le m}\binom{m}{k}X^kY(-X)^{m-k}\right]\\ &= \sum_{0\le k\le m}\binom{m}{k}\left(X^{k+1}Y(-X)^{m-k} - X^kY(-X)^{m+1-k}\right)\\ &= \sum_{1\le k\le m+1}\binom{m}{k-1}X^kY(-X)^{m+1-k} + \sum_{0\le k\le m}\binom{m}{k}X^kY(-X)^{m+1-k}\\ &= \sum_{0\le k\le m+1}\binom{m+1}{k}X^kY(-X)^{m+1-k}.\end{aligned}$$

A **finite-dimensional K-Lie algebra** is a finite-dimensional $\mathbf{K}$-vector space $\mathfrak{g}$, together with a map $[\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}$ into $\mathfrak{g}$, with the following properties

(1) $[\cdot,\cdot]$ is bilinear,

(2) $[X,Y]=-[Y,X]$ for all $X,Y\in\mathfrak{g}$,

(3) **(Jacobi identity)** $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$ for all $X,Y,Z\in\mathfrak{g}$.

It is clear that $\mathbf{Max}(n,\mathbf{K})$ together with $[\cdot,\cdot]$ defined in (1.4.3.26) is a finite-dimensional $\mathbf{K}$-Lie algebra.

Theorem 1.25

Every invertible $n\times n$ matrix can be expressed as e^X for some $X\in\mathbf{Max}(n,\mathbf{C})$. ♡

Recall that the function

$$\ln z = \sum_{m\ge 1}(-1)^{m+1}\frac{(z-1)^m}{m}$$

is defined and holomorphic in $\{z\in\mathbf{C}||z-1|<1\}$. In particular, when $|u|<\ln 2$, $\ln(e^u)=u$, when $|z-1|<1$, $e^{\ln z}=z$.

Definition 1.9

For any $A\in\mathbf{Max}(n,\mathbf{C})$, define

$$\ln A := \sum_{m\ge 1}(-1)^{m+1}\frac{(A-I_n)^m}{m}. \tag{1.4.3.28}$$

♣

As in the proof of **Proposition 1.13**, we can prove that the series (1.4.3.28) is defined and continuous in $\{A\in\mathbf{Max}(n,\mathbf{C})|||A-I_n||<1\}$. Moreover, when $||X||<\ln 2$, $\ln e^X = X$, when $||A-I_n||<1$, $e^{\ln A}=A$.

To extend **Proposition 1.14** (5) to the case that $[X, Y] \neq 0$, we need the following definition:

$$g(z) = \frac{\ln z}{1 - \frac{1}{z}} = \frac{z \ln z}{z - 1} = 1 + \sum_{m \geq 1} \frac{(-1)^{m+1}}{m(m+1)}(z-1)^m, \quad |z - 1| < 1. \tag{1.4.3.29}$$

Exercise 1.26

Verify (1.4.3.29).

For any $X, Y \in \mathbf{Max}(n, \mathbf{C})$ with sufficiently small $||X||$ and $||Y||$, we have

$$\begin{aligned} e^X e^Y &= I_n + (X+Y) + \left(\frac{1}{2}X^2 + XY + \frac{1}{2}Y^2\right) \\ &\quad + \left(\frac{1}{6}X^3 + \frac{1}{2}X^2Y + \frac{1}{2}XY^2 + \frac{1}{6}Y^3\right) + \cdots \end{aligned}$$

Then

$$e^X e^Y - I_n = (X+Y) + \left(\frac{1}{2}X^2 + XY + \frac{1}{2}Y^2\right) + \cdots$$

and

$$\begin{aligned} (e^X e^Y - I_n)^2 &= (X+Y)^2 + \Big(X^3 + Y^3 + XYX + YXY \\ &\quad + \frac{1}{2}Y^2X + \frac{1}{2}YX^2 + \frac{3}{2}X^2Y + \frac{2}{3}XY^2\Big) + \cdots \end{aligned}$$

Therefore

$$\begin{aligned} \ln(e^X e^Y) &= \sum_{m \geq 1} \frac{(-1)^{m+1}}{m}(e^X e^Y - 1)^m = (e^X e^Y - I_n) - \frac{1}{2}(e^X e^Y - I_n)^2 + \cdots \\ &= X + Y + \frac{1}{2}X^2 + XY + \frac{1}{2}Y^2 - \frac{1}{2}(X^2 + XY + YX + Y^2) + \cdots \\ &= X + Y + \frac{XY - YX}{2} + \cdots = X + Y + \frac{1}{2}[X, Y] + \cdots. \end{aligned}$$

The higher order terms are given by the famous Baker-Campbell-Hausdorff formula.

Theorem 1.26. (Baker-Campbell-Hausdorff)

For all $X, Y \in \mathbf{Max}(n, \mathbf{C})$ with $||X||$ and $||Y||$ sufficiently small, we have

$$\ln(e^X e^Y) = X + \int_0^1 g(e^{\mathbf{ad}_X} \circ e^{t\mathbf{ad}_Y})(Y)\,dt. \tag{1.4.3.30}$$

The above formula gives the next terms in the series of $\ln(e^X e^Y)$.

Exercise 1.27

Show by (1.4.3.30) that

$$\ln(e^X e^Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \cdots \tag{1.4.3.31}$$

for sufficiently small X and Y.

We will give a proof of **Theorem 1.26** in a special case that X and Y commute with $[X, Y]$ (we now do not assume further that $||X||$ and $||Y||$ are sufficiently small).

Theorem 1.27

If $X, Y \in \mathbf{Max}(n, \mathbf{C})$ *satisfies*

$$[X, [X, Y]] = [Y, [X, Y]] = 0, \tag{1.4.3.32}$$

then

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]}. \tag{1.4.3.33}$$

Proof. For any $t \in \mathbf{R}$ we shall prove

$$e^{tX} e^{tY} = e^{tX+tY+\frac{t^2}{2}[X,Y]}$$

which is equivalent, under the assumption (1.4.3.32),

$$e^{t(X+Y)} = e^{tX} e^{tY} e^{-\frac{t^2}{2}[X,Y]}.$$

Set

$$A(t) := e^{t(X+Y)}, \quad B(t) := e^{tX} e^{tY} e^{-\frac{t^2}{2}[X,Y]}.$$

Clearly that

$$\frac{d}{dt}A(t) = A(t)(X+Y)$$

by **Proposition 1.14** (8). For $B(t)$ one has

$$\begin{aligned} \frac{d}{dt}B(t) &= \frac{d}{dt}\left(e^{tX} e^{tY} e^{-\frac{t^2}{2}[X,Y]}\right) \\ &= \left(\frac{d}{dt}e^{tX}\right) e^{tY} e^{-\frac{t^2}{2}[X,Y]} + e^{tX}\left(\frac{d}{dt}e^{tY}\right) e^{-\frac{t^2}{2}[X,Y]} + e^{tX} e^{tY}\left(\frac{d}{dt}e^{-\frac{t^2}{2}[X,Y]}\right) \\ &= e^{tX} X e^{tY} e^{-\frac{t^2}{2}[X,Y]} + e^{tX} e^{tY} Y e^{-\frac{t^2}{2}[X,Y]} + e^{tX} e^{tY} e^{-\frac{t^2}{2}[X,Y]}(-t[X,Y]). \end{aligned}$$

Since Y commutes with $[X, Y]$, it follows that

$$e^{tX} e^{tY} Y e^{-\frac{t^2}{2}[X,Y]} = e^{tX} e^{tY} e^{-\frac{t^2}{2}[X,Y]} Y.$$

Using (1.4.3.27) and $[Y, [X, Y]] = 0$ yields

$$X e^{tY} = e^{tY} e^{-tY} X e^{tY} = e^{tY} e^{-\mathbf{ad}_Y}(X) = X - t[Y, X] = X + t[X, Y].$$

Consequently,

$$\begin{aligned} \frac{d}{dt}B(t) &= e^{tX}(X + t[X,Y]) e^{-\frac{t^2}{2}[X,Y]} + e^{tX} e^{tY} e^{-\frac{t^2}{2}[X,Y]} Y + e^{tX} e^{tY} e^{-\frac{t^2}{2}[X,Y]}(-t[X,Y]) \\ &= e^{tX} e^{tY} e^{-\frac{t^2}{2}[X,Y]}(X + t[X,Y] + Y - t[X,Y]) = B(t)(X+Y). \end{aligned}$$

Because both $A(t)$ and $B(t)$ satisfy the same differential equation, and $A(0) = B(0) = I_n$, we can conclude that $A(t) \equiv B(t)$. □

Exercise 1.28

(1) *Show that for any two elements* X, Y *of the Heisenberg group* H, *we always have (1.4.3.32).*

(2) *Show by direct computation that for any* $X, Y \in \mathfrak{h}$, *we have* (1.4.3.33), *where*

$$\mathfrak{h} := \left\{ \begin{bmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{bmatrix} \middle| \alpha, \beta, \gamma \in \mathbf{R} \right\}. \tag{1.4.3.34}$$

Actually, $\mathfrak{h}$ defined in (1.4.3.34) is the Lie algebra of the Heisenberg group H in the sense of **Definition 1.10**.

Definition 1.10

Let G be a matrix Lie group. The **Lie algebra of G**, *denoted $\mathfrak{g}$, is the set of all matrices X such that e^{tX} is in G for all $t \in \mathbf{R}$.*

If we write $A(t) := e^{tX}$, then $A(0) = I_n$, $A(t)$ is continuous in t, and $A(t+s) = A(t)A(s)$ for all $t, s \in \mathbf{R}$. Such a function $A : \mathbf{R} \to \mathbf{GKL}(n, \mathbf{C})$ is called a **one-parameter subgroup** of $\mathbf{GL}(n, \mathbf{C})$. Conversely, we can prove that for a given one-parameter subgroup $A(t)$ of $\mathbf{GL}(n, \mathbf{C})$ there exists a unique $X \in \mathbf{Max}(n, \mathbf{C})$ such that $A(t) = e^{tX}$.

Theorem 1.28

For any $X \in \mathbf{Max}(n, \mathbf{C})$ we have

$$\det(e^X) = e^{\mathrm{tr}(X)}. \tag{1.4.3.35}$$

Proof. From linear algebra, we know that X can be uniquely written as $X = S + N$, where $S, N \in \mathbf{Mat}(n, \mathbf{C})$, $SN = NS$, S is diagonalizable and N is nilpotent. From $SN = NS$, we obtain

$$e^X = e^{S+N} = e^S e^N, \quad \det(e^X) = \det(e^S)\det(e^N).$$

Write $S = C\mathrm{diag}(\lambda_1, \cdots, \lambda_n)C^{-1}$ for some $C \in \mathbf{GL}(n, \mathbf{C})$. Then

$$e^S = C\mathrm{diag}(e^{\lambda_1}, \cdots, e^{\lambda_n})C^{-1}$$

and

$$\det(e^S) = \prod_{1 \le i \le n} e^{\lambda_i} = e^{\sum_{1 \le i \le n} \lambda_i} = e^{\mathrm{tr}(S)}.$$

Since N is nilpotent, from linear algebra, we can find some $D \in \mathbf{GL}(n, \mathbf{C})$ such that

$$N = D \begin{bmatrix} 0 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} D^{-1}$$

Hence

$$e^N = D \begin{bmatrix} 1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} D^{-1}$$

So $\det(e^N) = 1$ and $\mathrm{tr}(N) = 0$. Consequently $\det(e^X) = e^{\mathrm{tr}(S)} = e^{\mathrm{tr}(X)}$. $\square$

We now can determine Lie algebras of classical groups.

(1) $G = \mathbf{GL}(n, \mathbf{C})$: If $X \in \mathbf{Max}(n, \mathbf{C})$, we have $e^{tX} \in \mathbf{GL}(n, \mathbf{C})$. Hence the Lie algebra $\mathfrak{gl}(n, \mathbf{C})$ of $\mathbf{GL}(n, \mathbf{C})$ is $\mathbf{Max}(n, \mathbf{C})$.

(2) $G = \mathbf{GL}(n, \mathbf{R})$: The Lie algebra $\mathfrak{gl}(n, \mathbf{R})$ of $\mathbf{GL}(n, \mathbf{R})$ is $\mathbf{Max}(n, \mathbf{R})$.

(3) $G = \mathbf{SL}(n, \mathbf{C})$: If $e^{tX} \in \mathbf{SL}(n, \mathbf{C})$, then from (1.4.3.35) we have

$$1 = \det(e^{tX}) = e^{\mathrm{tr}(tX)} = e^{t\cdot\mathrm{tr}(X)} \quad \Longleftrightarrow \quad \mathrm{tr}(X) = 0.$$

Hence the Lie algebra $\mathfrak{sl}(n, \mathbf{C})$ of $\mathbf{SL}(n, \mathbf{C})$ is given by

$$\mathfrak{sl}(n, \mathbf{C}) = \{X \in \mathbf{Max}(n, \mathbf{C}) | \mathrm{tr}(X) = 0\}. \tag{1.4.3.36}$$

(4) $G = \mathbf{SL}(n, \mathbf{R})$: The Lie algebra $\mathfrak{sl}(n, \mathbf{R})$ of $\mathbf{SL}(n, \mathbf{R})$ is given by

$$\mathfrak{sl}(n, \mathbf{R}) = \{X \in \mathbf{Max}(n, \mathbf{R}) | \mathrm{tr}(X) = 0\}. \tag{1.4.3.37}$$

(5) $G = \mathbf{U}(n)$: $e^{tX} \in \mathbf{U}(n)$ if and only if $(e^{tX})^* = (e^{tX})^{-1} = e^{-tX}$, or $e^{tX^*} = e^{-tX}$. Thus the Lie algebra $\mathfrak{u}(n)$ of $\mathbf{U}(n)$ is

$$\mathfrak{u}(n) = \{X \in \mathbf{Max}(n, \mathbf{C}) | X^* = -X\}. \tag{1.4.3.38}$$

(6) $G = \mathbf{SU}(n)$: The Lie algebra $\mathfrak{su}(n)$ of $\mathbf{SU}(n)$ is

$$\mathfrak{su}(n) = \{X \in \mathbf{Max}(n, \mathbf{C}) | X^* = -X, \ \mathrm{tr}(X) = 0\}. \tag{1.4.3.39}$$

(7) $G = \mathbf{O}(n)$: $e^{tX} \in \mathbf{O}(n)$ if and only if $(e^{tX})^\intercal = (e^{tX})^{-1}$ or $e^{tX^\intercal} = e^{-tX}$. Thus the Lie algebra $\mathfrak{o}(n)$ of $\mathbf{O}(n)$ is

$$\mathfrak{o}(n) = \{X \in \mathbf{Max}(n, \mathbf{R}) | X^\intercal = -X\}. \tag{1.4.3.40}$$

If $X \in \mathfrak{o}(n)$, then $\mathrm{tr}(X) = 0$ and $\det(e^X) = 0$.

(8) $G = \mathbf{SO}(n)$: The Lie algebra $\mathfrak{so}(n)$ of $\mathbf{SO}(n)$ is the same as $\mathfrak{o}(n)$.

(9) $G = \mathbf{O}(n, k)$: The Lie algebra of $\mathbf{O}(n, k)$ is given by

$$\mathfrak{o}(n, k) = \left\{ X \in \mathbf{Max}(n + k, \mathbf{R}) \middle| gX^\intercal g = -X, \ g = \begin{bmatrix} I_n & 0 \\ 0 & -I_k \end{bmatrix} \right\}. \tag{1.4.3.41}$$

(10) $G = \mathbf{SO}(n, k)$: the Lie algebra $\mathfrak{so}(n, k)$ is the same as $\mathfrak{o}(n, k)$.

(11) $G = \mathbf{Sp}(n, \mathbf{C})$: The Lie algebra of $\mathbf{Sp}(n, \mathbf{C})$ is given by

$$\mathfrak{sp}(n, \mathbf{C}) = \left\{ \begin{bmatrix} A & B \\ C & -A^\intercal \end{bmatrix} \middle| A \in \mathbf{Max}(n, \mathbf{C}) \text{ and } B, C \text{ symmetric} \right\}. \tag{1.4.3.42}$$

(12) $G = \mathbf{Sp}(n, \mathbf{R})$: The Lie algebra of $\mathbf{Sp}(n, \mathbf{R})$ is given by

$$\mathfrak{sp}(n, \mathbf{R}) = \left\{ X \in \mathbf{Max}(2n, \mathbf{R}) | JX^{\mathsf{T}}J = X,\ J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \right\}. \tag{1.4.3.43}$$

(13) $G = \mathbf{Sp}(n)$: The Lie algebra of $\mathbf{Sp}(n)$ is given by $\mathfrak{sp}(n) = \mathfrak{sp}(n, \mathbf{C}) \cap \mathfrak{u}(2n)$.

(14) $G =$ Heisenberg group H: The Lie algebra of H is given by (1.4.3.34).

(15) $G = \mathbf{E}(n)$: The Lie algebra of $\mathbf{E}(n)$ is given by

$$\mathfrak{e}(n) = \left\{ \begin{bmatrix} Y & \boldsymbol{y}^{\mathsf{T}} \\ \mathbf{0} & 1 \end{bmatrix} \middle| Y \in \mathbf{Max}(n, \mathbf{R}),\ Y^{\mathsf{T}} = -Y \right\}. \tag{1.4.3.44}$$

(16) $G = \mathbf{P}(n, 1)$: The Lie algebra of $\mathbf{P}(n, 1)$ is given by

$$\mathfrak{p}(n, 1) = \left\{ \begin{bmatrix} Y & \boldsymbol{y}^{\mathsf{T}} \\ \mathbf{0} & 0 \end{bmatrix} \middle| Y \in \mathfrak{so}(n, 1) \right\}. \tag{1.4.3.45}$$

1.5 Connectedness and compactness

Introduction

- *Connectedness*
- *Path-connectedness*
- *Compactness*
- *Paracompact spaces*

1.5.1 Connectedness

Let $(X, \mathscr{T})$ be a topological space.

(1) A **separation** of X is a pair (U, V) such that

$$\emptyset \neq U, V \subseteq \mathscr{T}, \quad U \cap V = \emptyset, \quad X = U \cup V.$$

(2) X is called **disconnected** if there exists a separation of X. If there does not, X is called **connected**. Thus, a topological space X is called connected if it is not the disjoint of two nonempty open subsets.

(3) A subset A of X is called **clopen** if it is both open and closed in X. It is clear that X is connected if and only if its only clopen subsets are X and $\emptyset$. If A is a nonempty proper subset of X that is clopen, then the pair $(U = A, V = X \setminus A)$ is a separation of X. Conversely, if X is connected, then for any subset $A \subseteq X$ that is clopen in X, we conclude that $(U = A, V = X \setminus A)$ is not a separation.

(4) A **discrete valued map** is a continuous map from X to a discrete space D.

(5) X is connected if and only if every discrete valued map on X is constant. If X is connected and $f : X \to D$ is a discrete valued map and if $y \in D$ is in the range of f, then $f^{-1}(y)$ is clopen and nonempty and so must equal X, and so f is constant with only value y.

Conversely, if X is not connected then $X = U \cup V$ for some disjoint clopen sets U and V. Then the map $f : X \to \{0, 1\}$ which is 0 on U and is 1 on V is a nonconstant discrete value map.

(6) A subset A of X is called **connected**, if A is a connected with respect to the subspace topology.

(7) Define a relation $\sim$ on X by

$$x \sim y \iff x \text{ and } y \text{ belong to a connected subset of } X.$$

Then $\sim$ is an equivalence relation on X, and the equivalence class $[x]$ is called a **connected component** or **component** of X.

(8) Define a relation $\sim$ on X by

$$x \sim y \iff f(x) = f(y) \text{ for every discrete valued map } f \text{ on } X.$$

Then $\sim$ is an equivalence relation on X, and the equivalence class $[x]$ is called a **quasi-component** of X.

Proposition 1.15

(1) If $f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is continuous and X is connected, then $f(X)$ is connected.

(2) If $(Y_i)_{i\in I}$ is a collection of connected subsets in a topological space X and if no two of the Y_i are disjoint, then $\cup_{i\in I} Y_i$ is connected.

(3) Components of a topological space X are connected and closed. Each connected subset is contained in a component. Thus the components are "maximal connected subsets". Components are either equal or disjoint, and fill out X.

(4) Quasi-components of a topological space X are closed. Each connected subsets is contained in a quasi-component. In particular, each component is contained in a quasi-component. Quasi-components are either equal or disjoint, and fill out X.

(5) **(Intermediate value theorem)** *If $f : X \to \mathbf{R}$ is continuous and $x, y \in X$, then for any M between $f(x)$ and $f(y)$ we have $f(z) = M$ for some $z \in X$.*

(6) If $X_1, \cdots, X_k$ are connected, then $X_1 \times \cdots \times X_k$ is connected.

(7) If $\sim$ is an equivalence relation on a topological space X, then $X/\sim$ is connected.

(8) Suppose that $(X, \mathscr{T})$ is a topological space and U, V are disjoint open subsets of X. If S is a connected subset of X contained in $U \cup V$, then either $S \subseteq U$ or $S \subseteq V$.

(9) If $(X, \mathscr{T})$ is a topological space that contains a dense connected subset S, then X is connected.

(10) Let $(X, \mathscr{T})$ be a topological space. The union of a collection of connected subspaces of X that have a point in common is connected.

(11) Suppose that $(X, \mathscr{T})$ is a topological space and $A \subseteq X$ is connected. If a subset B of X satisfies $A \subseteq B \subseteq \overline{A}$, then B is connected. In particular, $\overline{A}$ is connected. ♡

Proof. (1) Let $d : f(X) \to D$ be a discrete valued map. Then $d \circ f : X \to D$ is a discrete valued map on X and hence must be constant. This implies that d is constant on $f(X)$ and then $f(X)$ is connected.

(2) Let $d : \cup_{i\in I}Y_i \to D$ be a discrete valued map. Let p, q be any two points in $\cup_{i\in I}Y_i$. Suppose $x \in Y_i$, $y \in Y_j$, and $z \in Y_i \cap Y_j$. Because d is constant on each Y_i, we conclude that $d(x) = d(z) = d(y)$. Hence d is constant.

(3) By definition, the component of X containing x is the union of all connected subsets containing p, hence it is connected by (2). Since the closure of a connected set is connected[5], it follows that components of X is closed.

(4) If $x \in X$ then the quasi-component containing p is the set

$$\{y \in X : d(y) = d(x) \text{ for all discrete valued maps } d\} = \bigcap_{d \text{ discrete value maps}} d^{-1}(d(x))$$

which is an intersection of closed subsets and hence is closed.

(5) This follows from the fact that $f(X)$ is an interval (a nonempty subset of $\mathbf{R}$ is connected if and only if it is an interval).

(6) By induction, we need only to consider a product of two topological spaces. Let X, Y be connected and $d : X \times Y \to D$ a discrete valued map. Given a point $(x_0, y_0) \in X \times Y$ and define maps

$$\iota_{x_0} : Y \longrightarrow X \times Y, \quad y \longmapsto (x_0, y) \quad \iota_{y_0} : X \longrightarrow X \times Y, \quad x \longmapsto (x, y_0).$$

Then we get maps $d \circ \iota_{x_0} : Y \to D$ and $d \circ \iota_{y_0} : X \to D$. Since X and Y are connected, it follows that $d\circ_{x_0} = c_{x_0}$ and $d \circ \iota_{y_0} = c_{y_0}$ for some constants c_{x_0}, c_{y_0} (may depending on (x_0, y_0)). In particular, $c_{x_0} = c_{y_0} = d(x_0, y_0) =: c_{x_0,y_0}$. Thus $d(x_0, y) = d(x, y_0) = c_{x_0,y_0}$ for any $x \in X, y \in Y$. We now show that the constant c_{x_0,y_0} is independent of the choice of (x_0, y_0). Indeed, for another pair (x_1, y_1), we also have $d(x_1, y) = d(x, y_1) = c_{x_1,y_1}$ for any $x \in X, y \in Y$. Then

$$c_{x_1,y_1} = d(x_0, y_1) = c_{x_0,y_0}.$$

Thus $d(x_0, y_0) = c$ for some constant c that is independent of the choice of (x_0, y_0). Since (x_0, y_0) was arbitrary, we get d is constant and then $X \times Y$ is connected.

(7) This follows from the fact that $X \to X/\sim$ is surjective and (1).

(8) Since U, V are open in X, it follows that $A \cap U$ and $A \cap V$ are open in A. The connectedness of A implies that $A \cap U = \emptyset$ or $A \cap V = \emptyset$. Thus $A \subseteq V$ or $A \subseteq U$.

(9) Let $A \subseteq X$ be a dense connected subset of X. Assume that (U, V) is a separation of X. By (8), we have $A \subseteq U$ or $A \subseteq V$. If $A \subseteq U$, then $X = \overline{A} \subseteq \overline{U} = U$ (since U is both open and closed). But this implies $V = \emptyset$. Hence X is connected.

(10) Let $\{A_\alpha\}_{\alpha\in I}$ be a collection of connected subspaces of X and $p \in \cap_{\alpha\in I}A_\alpha$. Set $Y := \cup_{\alpha\in I}A_\alpha$ and suppose (C, D) is a separation of Y. Then $p \in C$ or $p \in D$. Assume

[5]Suppose that A is a connected subset. If U and V ar disjoint open subsets of X that separates $\overline{A}$, then A is connected in one of them, say $A \subseteq U$. Since $U \cap V = \emptyset$, it follows that $\overline{A} \subseteq U$, which means that $\overline{A} \cap V = \emptyset$, a contradiction.

that $p \in C$. Since A_α is connected, from (8), we must have $A_\alpha \subseteq C$ for all $\alpha \in I$. Thus $Y = \cup_{\alpha \in I} A_\alpha \subseteq C$. This contradiction shows that Y is connected.

(11) Consider a separation (C, D) of B. According to (8), $A \subseteq C$ or $A \subseteq D$. Suppose that $A \subseteq C$. Then $\overline{A} \subseteq \overline{C}$. Since $\overline{C} \cap D = C \cap D = \emptyset$, it follows that $B \cap D = \emptyset$. This contradicts the fact that $\emptyset \neq D \subseteq B$. □

Example 1.14

$\mathbf{R}^\omega$ *is not connected in the box topology, but it is connected in the product topology. Indeed, we have*

$$\mathbf{R}^\omega = A \cup B$$

where $A \cap B = \emptyset$, *and*

$$A := \{\textit{bounded sequences of } \mathbf{R}\} \neq \emptyset, \quad B := \{\textit{unbounded sequences of } \mathbf{R}\} \neq \emptyset.$$

We claim that both A *and* B *are clopen in the box topology. For all* $\boldsymbol{a} = (a^i)_{i \geq 1} \in \mathbf{R}^\omega$ *consider the open set*

$$U_{\boldsymbol{a}} := \prod_{i \geq 1} (a^i - 1, a^i + 1).$$

Then $U_{\boldsymbol{a}} \subseteq A$ *if* $\boldsymbol{a} \in A$ *and* $U_{\boldsymbol{a}} \in B$ *if* $\boldsymbol{a} \in B$.

To prove that $\mathbf{R}^\omega$ *is connected in the product topology we shall use the fact that* $\mathbf{R}$ *is connected (see* **Theorem** 1.29*). Define*

$$\begin{aligned} \widetilde{\mathbf{R}}^n &:= \left\{ \boldsymbol{x} = (x^i)_{i \geq 1} \in \mathbf{R}^\omega \middle| x^i = 0 \textit{ for } i > n \right\} \cong \mathbf{R}^n, \\ \mathbf{R}^\infty &:= \bigcup_{n \geq 1} \widetilde{\mathbf{R}}^n \end{aligned}$$

From **Proposition** 1.15 *(6),* $\mathbf{R}^n$ *is connected. Since* $\widetilde{\mathbf{R}}^n$ *have the common point* $\mathbf{0}$*, it follows from* **Proposition** 1.15 *(10) that* $\mathbf{R}^\infty$ *is also connected. We now claim that*

$$\overline{\mathbf{R}^\infty} = \mathbf{R}^\omega$$

which, by **Proposition** 1.15 *(11), implies the connectedness of* $\mathbf{R}^\omega$ *in the product topology. For* $\boldsymbol{a} = (a^i)_{i \geq 1} \in \mathbf{R}^\omega$*, let* $U := \prod_{i \geq 1} U_i$ *be a basis element for the product topology that contains* $\boldsymbol{a}$*. Then there exists an integer* $N \in \mathbf{N}$ *such that* $U_i = \mathbf{R}$ *for* $i > N$*. Hence* $\boldsymbol{x} = (x^1, \cdots, x^n, 0, \cdots, 0) \in \mathbf{R}^\infty \cap U$*. Thus* $U \cap \mathbf{R}^\infty \neq \emptyset$ *and* $\boldsymbol{a} \in \overline{\mathbf{R}^\infty}$. ♠

A subset $I \subseteq \mathbf{R}$ is said to be an **interval** if it contains more than one element, and whenever $a, b \in I$, every c such that $a < c < b$ is also in I. An interval must be one of the nine types of sets

$$[a, b], \quad (a, b), \quad [a, b), \quad (a, b], \quad (-\infty, b], \quad (-\infty, b), \quad [a, +\infty), \quad (a, +\infty), \quad (-\infty, +\infty).$$

Theorem 1.29

$\mathbf{R}$ is connected. Moreover a nonempty subset of $\mathbf{R}$ is connected if and only if it is a singleton or an interval.

Proof. We assume that $I \subseteq \mathbf{R}$ contains at least two points.

Assume first that I is not an interval. By definition, there exists $a, b, c \in \mathbf{R}$ so that $a < c < b$, $a, b \in I$ but $c \notin I$. Then he sets $(-\infty, c) \cap I$ and $(c, +\infty) \cap I$ form a separation for I. Thus I is not connected.

Now we assume that I is an interval. It is not connected, there are open subsets $U, V \subseteq \mathbf{R}$ such that $(U \cap I, V \cap I)$ forms a separation for I. Choose $a \in U \cap I$ and $b \in V \cap I$. We may without loss of generality that $a < b$. Since I is a interval, it follows that $[a, b] \subseteq I$. Since U and V are open, there exists $\epsilon > 0$ such that $[a, a + \epsilon) \subseteq U \cap I$ and $(b - \epsilon, b] \subseteq V \cap I$. Let $c; = \sup(U \cap [a, b])$. Then $a + \epsilon \leq c`b - \epsilon$ and $c \in (a, b) \subseteq I \subseteq U \cup V$. If $c \in U$, then $(c - \epsilon', c + \epsilon') \subseteq U \cap (a, b)$ for some $\epsilon' > 0$; if $c \in V$, then $(c - \epsilon'', c + \epsilon'') \subseteq V$ for some $\epsilon'' > 0$. Both cases contradict with the definition of c. Therefore I is connected. □

But in general the components of X need not to be open in X. For example, consider $\mathbf{Q} \subseteq \mathbf{R}$. Then each component of $\mathbf{Q}$ consists of a single point so that this component is not open.

1.5.2 Path-connectedness

Let $(X, \mathscr{T})$ be a topological space and $x, y \in X$. A **path in X from x to y** is a continuous map $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.

(1) X is **path-connected** if for any two points x, y in X there is a path in X from x to y.

(2) Define a relation $\sim$ on X by

$$x \sim y \iff \text{there exists a path in } X \text{ from } x \text{ to } y.$$

Then $\sim$ is an equivalence relation on X, and the equivalence class $[x]$ is called a **path-component** of X.

(3) X is called **locally connected** if for each $x \in X$ and each neighborhood N of x, there is a connected neighborhood V of x with $V \subseteq N$. Equivalently, X is locally connected if it admits a basis of connected open subsets.

(4) X is called **locally path-connected** if it admits a basis of path-connected open subsets. Equivalently, X is locally path-connected if for each $x \in X$ and each neighborhood N of x, there is a path-connected neighborhood V of x with $V \subseteq N$.

Note 1.1

(1) *The following sets*

$$\mathbf{R}^n, \quad \mathbf{R}^n \setminus \{\mathbf{0}\}\ (n \geq 2), \quad \mathbf{B}^n\ (n \geq 1), \quad \mathbf{S}^n\ (n \geq 1), \quad \mathbf{T}^n, \quad \textit{convex sets}$$

are all path-connected.

(2) *The* **topologist's sin curve**

$$X=\left\{(x,y)\in\mathbf{R}^2\middle|x=0,\ -1\le y\le 1\right\}\cup\left\{(x,y)\in\mathbf{R}^2\middle|y=\sin\frac{1}{x},\ 0<x\le 1\right\}$$

is connected, but not path-connected.

Proof. Indeed, Write $S:=\{(x,\sin(1/x))|x\in(0,1]\}$. Then S is connected since $S=(1\times f)((0,1])$ for $f(x):=\sin(1/x)$ (by **Proposition 1.15** (1)), $\overline{S}$ is also connected (by **Proposition 1.15** (11)), and $\overline{S}=X$.

We now show that $\overline{S}$ is not path-connected. Suppose there exists a path $\boldsymbol{f}:[a,c]\to\overline{S}$ beginning at $(0,0)$ and ending at a point of S. Set

$$I:=\{t\in[a,c]|\boldsymbol{f}(t)\in\{0\}\times[-1,1]\}.$$

Then I is closed in $[a,c]$ and $b:=\sup I$ exists. Moreover $\boldsymbol{f}:[b,c]\to\overline{S}$ is a path that maps b into $\{0\}\times[-1,1]$ and maps $(b,c]$ to S. Write $\boldsymbol{f}(t)=(x(t),y(t)),t\in[b,c]$. Then $x(b)=0$, $x(t)>0$ for all $t\in(b,c]$, and $y(t)=\sin(1/x(t))$ for $t\in(b,c]$. Given $n\in\mathbf{N}$, there exists $u\in(0,x(b+\frac{1}{n}))$ such that $\sin\frac{1}{u}=(-1)^n$. Then there exists $t_n\in(b,b+\frac{1}{n})$ such that $x(t_n)=u$. Hence $y(t_n)=\sin(\frac{1}{u})=(-1)^n$. But $t_n\to b$, on the other hand we have $y(t_n)\to y(b)\in[-1,1]$. This contradiction indicates X is not path-connected. □

(3) *If $U\subseteq\mathbf{R}^2$ is an open and connected subspace, then U is path-connected.*

Proof. Given $\boldsymbol{x}_0\in U$, define

$$\widetilde{U}:=\{\boldsymbol{x}\in U|\exists\text{ path in }U\text{ joining }\boldsymbol{x}_0\text{ to }\boldsymbol{x}\}\subseteq U.$$

Since $\boldsymbol{x}_0\in\widetilde{U}$, we claim that $\widetilde{U}$ is both open and closed in U so that $\widetilde{U}=U$. For $\boldsymbol{x}\in\widetilde{U}$, there exists a path $\boldsymbol{f}\in[0,1]\to U$ such that $\boldsymbol{f}(0)=\boldsymbol{x}_0$ and $\boldsymbol{f}(1)=\boldsymbol{x}$. Since U is open, it follows that there exists an open ball $\mathbf{B}^2(\boldsymbol{x},\epsilon)\subseteq U$ and then $\mathbf{B}^2(\boldsymbol{x},\epsilon)\subseteq\widetilde{U}$. To prove the closedness of $\widetilde{U}$, we take a sequence $\{\boldsymbol{x}_n\}_{n\ge1}$ in $\widetilde{U}$ such that $\boldsymbol{x}_n\to\boldsymbol{x}\in U$. For any $\epsilon>0$, there exists $N\in\mathbf{N}$ such that $\boldsymbol{x}_n\in\mathbf{B}^2(\boldsymbol{x},\epsilon)$ for all $n\ge N$. Since $\boldsymbol{x}_n\in\widetilde{U}$, we can find a path connecting $\boldsymbol{x}_0$ to $\boldsymbol{x}_n$. So we can find a path joining $\boldsymbol{x}_0$ to $\boldsymbol{x}$. Thus $\boldsymbol{x}\in\widetilde{U}$. □

(4) *We have the following relations:*

$$\begin{tikzcd}
\mathbf{PC} & \mathbf{C} \arrow[l, "+\mathbf{LPC}"'] & \mathbf{PC} \arrow[l] \\
 & \mathbf{LC} & \mathbf{LPC} \arrow[l] \arrow[u, "+\mathbf{C}"']
\end{tikzcd}$$

where $\mathbf{C}$ = *connected,* $\mathbf{PC}$ = *path-connected,* $\mathbf{LC}$ = *locally connected, and* $\mathbf{LPC}$ = *locally path-connected.* ♣

Proposition 1.16

Let $(X, \mathscr{T})$ be a topological space.

(1) If X is path-connected, then X is connected.

(2) Each path-component is contained in a single component, and each components is a disjoint union of path components.

(3) If $A \subset X$ is path-connected, then A is contained in a single path-component.

(4) If X is locally connected, then any open subset of X is locally connected and each component of X is open in X.

(5) If X is locally path-connected, then X is locally connected, any open subset of X is locally path-connected, each path-component of X is open, and its path-components are the same as its components.

(6) If X is locally path-connected, then X is connected if and only if it is path-connected.

(7) If $f : X \to Y$ is continuous and X is path-connected, then $f(X)$ is path-connected.

(8) Any finite product of path-connected spaces is path-connected.

(9) If $\sim$ is an equivalence relation on a path-connected space X, then $X/\sim$ is path-connected. ♡

Proof. (1) Suppose that X is path-connected but not connected. Choose a separation (U, V) of X, choose $p \in U$ and $q \in V$, and choose a path γ from p to q in X. Then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint open subsets of $[0, 1]$ that cover $[0, 1]$; moreover $0 \in f^{-1}(U)$ and $1 \in f^{-1}(V)$, so that $(f^{-1}(U), f^{-1}(V))$ is a separation of $[0, 1]$, which is a contradiction.

(2) and (3) can be proved as **Proposition 1.10** (3).

(4) The first statement is obvious. Let A be a component of X. If $x \in A$, then x has a connected neighborhood U by local connectedness. Then $U \subset A$. Thus A is open.

(5) The first two statements are obvious. Let B be a path-component and A the single component containing A. By (2), A can be written as a disjoint union of path-components, each of which is open in X and thus in A. If $B \neq A$, then $(B, A \setminus B)$ is a separation of A, which is a contradiction since A is connected.

(6) If X is connected, then X has only one component and hence one path-component; thus X is path-connected. Conversely, if X is path-connected, by (5), X has only one component so that X is connected. □

Exercise 1.29

Prove **Proposition 1.16** *(7)-(9).* ♠

Consider the topologist's sin curve $X = \overline{S}$ in **Remark 1.1**. It has one component, two path components S and $V := \{0\} \times [-1, 1]$, S is open in $\overline{S}$ but not closed, and V is closed but not open in $\overline{S}$. Let

$$\widehat{S} := \overline{S} \setminus (\{0\} \times (\mathbf{Q} \cap [-1, 1])).$$

Then $\widehat{S}$ has still one component but uncountably many path components.

Example 1.15

(1) *Each interval in* $\mathbf{R}$ *is both connected and locally connected.*

(2) $[-1,0)\cup(0,1]\subseteq\mathbf{R}$ *is not connected, but is locally connected.*

(3) *The topologist's sine curve* X *is connected but not locally connected. Otherwise,* X *must be path-connected!*

(4) $\mathbf{Q}$ *is neither connected nor locally connected.*

There are no relations between connectedness and local connecteness.

Theorem 1.30

Any topological manifold is locally path-connected. Consequently, by **Proposition 1.16**, *for any topological manifold* $\mathcal{M}$, $\mathcal{M}$ *is connected if and only if it is path-connected.*

Proof. By definition, for any m-dimensional topological manifold $\mathcal{M}$ we have $\mathcal{M}=\cup_{i\in\mathbf{N}}\mathcal{U}_i$ where $\varphi_i:\mathcal{U}_i\to\mathbf{B}_i\subset\mathbf{R}^m$ is homeomorphic for each i. So $\mathcal{M}$ is locally path-connected. □

1.5.3 Compactness

Let $(X,\mathscr{T})$ be a topological space.

(1) X is **compact** if any open cover of X has a finite subcover.

(2) A collection $\mathscr{U}$ of sets has the **finite intersection property** if the intersection of any finite subcollection is nonempty.

A topological space $(X,\mathscr{T})$ is compact if and only if for every collection of closed subsets of X which has the finite intersection property, the intersection of the entire collection is nonempty.

Proof. Given a collection $\mathscr{U}$ of subsets of X, let $\mathscr{C}:=\{X\setminus U|U\in\mathscr{U}\}$ be the collection of their complements.

(a) $\mathscr{U}$ is a collection of open subsets if and only if $\mathscr{C}$ is a collection of closed subsets.

(b) $\{U_i\}_{1\le i\le n}\subseteq\mathcal{U}$ covers X if and only if $\cap_{1\le i\le n}C_i=\emptyset$ where $C_i:=X\setminus U_i$.

(c) $\mathscr{U}$ covers X if and only if $\cap_{C\in\mathscr{C}}C=\emptyset$.

Then

$$X \text{ is compact} \iff \left\{\text{given any collection } \mathscr{U} \text{ of open subsets of } X \right.$$

$$\left.\text{that covers } X,\ \exists \text{ finite subcollection of } \mathscr{A} \text{ covers } X\right\} \iff$$

$$\left\{\text{given any collection } \mathcal{U} \text{ of open subsets of } X,\ \text{ if no finite subcollection of}\right.$$

$$\left.\mathscr{U} \text{ covers } X,\ \text{then } \mathscr{U} \text{ does not cover } X\right\} \iff \text{the desired statement}$$

by the above statements $(a) - (c)$. □

(3) X is **locally compact** if for any $x \in X$, there is a compact subset of X containing a neighborhood of x.

(4) A subset $S \subseteq X$ is **precompact** or **relatively compact** if $\overline{S} \subseteq X$ is compact.

(5) A continuous map $f : X \to Y$ between topological spaces is **proper** if $f^{-1}(K) \subseteq X$ is compact whenever $K \subseteq X$ is compact.

Recall that a subset $S \subseteq \mathbf{R}^n$ is compact if it is closed and bounded.

Example 1.16

(1) $\mathbf{R}$ *is not compact, since the open cover* $\mathscr{U} = \{(n, n+2)\}_{n\in\mathbf{Z}}$ *of* $\mathbf{R}$ *contains no finite subcollection that cover* $\mathbf{R}$*.*

(2) $X = \{0\} \cup \{1/n\}_{n\geq 1}$ *is compact. Given an open cover* $\mathscr{U}$ *of* X*, there exists an open subset* $U \in \mathscr{U}$ *containing* 0*. Take* $N \in \mathbf{N}$ *such that* $1/n \in U$ *for all* $n > N$*. For each* $i \in \{1, \cdots, N\}$*, we choose* $U_i \in \mathscr{U}$ *containing* $1/i$*. Then* $U, U_1, \cdots, U_N$ *form a finite subcollection of* $\mathscr{U}$ *that contains* X*.*

(3) *If* X *is finite, then* X *is compact.*

(4) $(0, 1]$ *is not compact, because the open cover* $\mathscr{U} = \{(1/n, 1]\}_{n\geq 1}$ *contains no finite subcollection covering* $(0, 1]$*.* ♠

Let Y be a subset of a topological space $(X, \mathscr{T})$. Then Y is compact (in the subspace topology) if and only if every cover of Y by open subsets of X contains a finite subcollection covering Y. Let $\mathscr{U} = \{U_\alpha\}_{\alpha\in I}$ be a cover of Y by subsets open in X. Then $\mathscr{U}_Y := \{U_\alpha \cap Y | \alpha \in I\}$ is a cover of Y by subsets open in Y. Since Y is compact, there exists a finite subcollection $\{A_{\alpha_i} \cap Y\}_{1\leq i\leq n}$ covers Y. Thus $\{U_{\alpha_i}\}_{1\leq i\leq n}$ is a subcollection of $\mathscr{A}$ that covers Y. Conversely, let $\mathscr{U}' = \{U'_\alpha | \alpha \in I'\}$ be a cover of Y by subsets open in Y. For each $\alpha \in I'$, $U'_\alpha = U_\alpha \cap Y$ for some open subset U_α in X. Then $\mathscr{U} = \{U_\alpha\}_{\alpha\in I'}$ is a cover of Y by subsets open in X. By hypothesis, there is a finite subcollection $\{U_{\alpha_i}\}_{1\leq i\leq n}$ covers Y. Then $\{U'_{\alpha_i}\}_{1\leq i\leq n}$ is a subcollection of $\mathscr{U}'$ that covers Y.

Proposition 1.17

(1) If $f : X \to Y$ *is a continuous map between topological spaces and* X *is compact, then* $f(X)$ *is compact.*

(2) **(Extreme value theorem)** *If* $f : X \to \mathbf{R}$ *is a continuous map and* X *is compact, then* f *is bounded and* f *attains its minimum and maximum in* X*.*

(3) If X *is compact and* $U \subset X$ *is closed, then* U *is compact.*

(4) If X *is Hausdorff and* $U \subset X$ *is compact, then* U *is closed.*

(5) If X *is Hausdorff,* $K, L \subset X$ *are compact, and* $K \cap L = \emptyset$*, then* $K \subset U$ *and* $L \subset V$ *for some disjoint open subsets* $U, V \subset X$*.*

(6) The finite product of compact spaces is compact.

(7) If X is compact and $\sim$ is any equivalence relation, then $X/\sim$ is compact.

(8) **(Closed map lemma)** *Let $f: X \to Y$ be a continuous map from a compact space X to a Hausdorff space Y. Then f is closed. If f is moreover surjective (injective, bijective), then f is s quotient map (topological embedding, homeomorphism).*

(9) If X is Hausdorff, then the following are equivalent:

(a) X is locally compact,

(b) any $x \in X$ has a precompact neighborhood,

(c) X has a basis of precompact open sets.

(10) Any open or closed subset of a locally compact Hausdorff space are locally compact Hausdorff.

(11) If $f: X \to Y$ is a proper map between locally compact Hausdorff spaces, then f is a closed map. ♡

Proof. (1) Let $\mathscr{U}$ be an open cover of $f(X)$. For each $U \in \mathscr{U}$, $f^{-1}(U)$ is an open subset of X. Since $\mathscr{U}$ coves $f(X)$, it follows that $f^{-1}(\mathscr{U}) := \{f^{-1}(U) : U \in \mathscr{U}\}$ is an open cover of X. By compactness of X, there exists $N \in \mathbf{N}$ such that $(f^{-1}(U_i))_{1\leq i\leq N}$ cover X. Then $(U_i)_{1\leq i\leq N}$ covers $f(X)$.

(2) By (1), $f(X)$ is a compact subset of $\mathbf{R}$ so that $f(X)$ is closed and bounded. In particular, it contains its maximum and maximum in X.

(3) Let $\mathscr{U}$ be a cover of U by open subsets of X. Then $\mathscr{U} \cup \{X \setminus U\}$ is an open cover of X, which has a finite subcover $(U_1, \cdots, U_N, X \setminus A\}$. Therefore A can be covered by $(U_i)_{1\leq i\leq N}$ and A is compact.

(5) If the statement holds for L being a one-point set, then the same result holds for general L. For each $y \in L$ there exist disjoint open subsets $U_y, V_y \subset X$ such that $K \subset U_y$ and $y \in V_y$. By compactness of L, finitely many of these, say $(V_{y_i})_{1\leq i\leq N}$, cover L. Then setting $U := \cap_{1\leq i\leq N} U_{y_i}$ and $V := \cup_{1\leq i\leq N} V_{y_i}$ proves the result.

We now prove the result for the case $L = \{y\}$. For each $x \in K$, there exists disjoint open sets U_x containing x and V_x containing y by the Hausdorff property. The collection $\{U_x : x \in K\}$ is an open cover of K, so it has a finite subcover, say $\{U_{x_1}, \cdots, U_{x_N}\}$. Let $U := \cup_{1\leq i\leq N} U_{x_i}$ and $V := \cap_{1\leq i\leq N} V_{x_i}$. Then U and V are disjoint open sets with $K \subset U$ and $\{y\} \subset V$.

(4) For any $y \in X \setminus U$, by (5), we have $U \subset U$ and $y \in V$ for some disjoint open subsets U and V. Thus $X \setminus U$ is open.

(6) Without loss of generality, we may consider the product $X \times Y$ of compact spaces. Let $\mathscr{U}$ be an open cover of $X \times Y$. Choose $x \in X$; the slice $\{x\} \times Y$ is homeomorphic to Y, so finitely many of the sets of $\mathscr{U}$ cover it, say $(U_i)_{1\leq i\leq N}$. Since product open sets are a basis for the product topology, for each $y \in Y$ there is a product open set $V \times W \subset X \times Y$ such that $(x, y) \in V \times W \subset \cup_{1\leq i\leq N} U_i$. Finitely many of these product sets cover $\{x\} \times Y$,

say $(V_i \times W_i)_{1\le i\le M}$. If we set $Z_x := \cap_{1\le i\le M} V_i$, then the whole strip $Z_x \times Y$ is contained in $\cup_{1\le i\le N} U_i$. Because $(Z_x)_{x\in X}$ is an open cover of X, by compactness of X, we have a finite subcover, say $(Z_{x_i})_{1\le i\le K}$, of X. Since finitely many sets of $\mathscr{U}$ cover each strip $Z_{x_i} \times Y$, and finitely many such strips cover $X \times Y$.

(7) This follows from the fact that a quotient of a compact space is the image of a compact space by a continuous map.

(8) HW.

(9) $(c) \Rightarrow (b) \Rightarrow (a)$ are obvious. We now prove $(a) \Rightarrow (c)$. It suffices to show that every $x \in X$ has a neighborhood of precompact open sets. Being locally compact, X has a compact subset K containing a neighborhood U of x. Set

$$\mathscr{U}_x := \{\text{neighborhoods of } x \text{ contained in } U\}$$

so that $\mathscr{U}_x$ is a neighborhood of x. Since X is Hausdorff and K is compact, by (4), K must be closed. If $V \in \mathscr{U}_x$, then $V \subset U \subset K$ implies $\overline{V} \subset \overline{K} = K$. Thus $\overline{V}$ is compact by (3), and V is precompact.

(10) We need only to verify the local compactness. If U is open in X, then, **Theorem 1.32** yields that for any $x \in Y$ there exists a precompact neighborhood V_x of x such that $\overline{V_x} \subset X$. Thus U is locally compact.

Suppose that K is closed in X. By (9), any $x \in K$ has a precompact neighborhood U_x in X. Since $\overline{U_x \cap Z} = \overline{U_x} \cap Z$ is closed in $\overline{U_x}$ that is compact in X, it follows that $\overline{U_x \cap Z}$ is compact in X by (3). Hence $U_x \cap Z$ is a precompact neighborhood of x in Z, and Z is locally compact.

(11) For any closed set $K \subset X$, we shall show that $f(K)$ is closed in Y. Let $y \in \partial(f(K)) \subset Y$. There exists a precompact neighborhood U of y so that $y \in \partial(f(K) \cap \overline{U})$. The properness of f implies that $f^{-1}(\overline{U})$ is compact in X and then $K \cap f^{-1}(\overline{U})$ is closed in $f^{-1}(\overline{U})$. Hence $K \cap f^{-1}(\overline{U})$ is compact in X. The continuity of f implies that $f(K) \cap \overline{U} = f(K \cap f^{-1}(\overline{U}))$ is compact in Y, and then $f(K \cap f^{-1}(\overline{U}))$ is closed in Y. Therefore $y \in f(K) \cap \overline{U} \subset f(K)$. Thus $f(K)$ is closed in Y. □

Let (X, d) be a metric space and $S \subseteq X$. Define

$$\operatorname{diam}(S) := \sup_{x,y\in S} d(x, y) \tag{1.5.3.1}$$

the **diameter of** S. If $\mathscr{U}$ is an open cover of (X, d), a positive number $\delta > 0$ is called a **Lebesgue number for** $\mathscr{U}$ if any set whose diameter less than δ is contained in one of $U \in \mathscr{U}$.

Theorem 1.31. (Lebesgue number lemma)

Any open cover of a compact metric space has a Lebesgue number. ♡

Proof. Let $\mathscr{U}$ be an open cover of a compact metric space (X, d). For any $x \in X$, there is an open subset $U \in \mathscr{U}$ with $x \in U$. The openness of U implies that $B_{2r(x)}(x) \subseteq U$ for some $r(x) > 0$. Since $\{B_{r(x)}(x)\}_{x\in X}$ forms an open cover of X, it follows that $B_{r(x_1)}(x_1), \cdots, B_{r(x_n)}(x_n)$ cover X for some $n \in \mathbf{N}$. Define

$$\delta := \min_{1\le i\le n} r(x_i) > 0.$$

If $S \subseteq X$ with $\operatorname{diam}(S) < \delta$, we claim that $S \subseteq B_{2r(x_i)}(x_i)$ for some $i \in \{1, \cdots, n\}$. Let $y \in S$ and choose $y \in B_{r(x_i)}(x_i)$ for some i. Therefore for any $z \in S$,

$$d(z, x_i) \le d(z, y) + d(y, x_i) < \delta + d(y, x_i) < \delta + r(x_i) < 2r(x_i).$$

Thus δ is a Lebesgue number for $\mathscr{U}$. □

Theorem 1.32. (Shrinking lemma)

Let X be a locally compact Hausdorff space and $x \in X$. If U is a neighborhood of x, then there exists a precompact neighborhood V of x such that $\overline{V} \subset U$. ♡

Proof. By **Proposition 1.17** (9), there is a precompact neighborhood W of x so that $\overline{W} \setminus U$ is closed in $\overline{W}$. Then $\overline{W} \setminus U$ is compact by **Proposition 1.17** (3). Part (5) in **Proposition 1.17** implies that there exist open sets $Y, Y' \subseteq X$ such that $x \in Y$ and $\overline{W} \setminus U \subseteq Y'$. Let $V := Y \cap W$ be a neighborhood of x. Since $\overline{V} \subseteq \overline{W}$, it follows that $\overline{V}$ is compact by (3). Because $\overline{V} \subseteq \overline{Y}$, we get $\overline{V} \subseteq \overline{W} \setminus Y' \subseteq U$. □

Example 1.17

(1) $\mathbf{R}^n$ *is locally compact Hausdorff so that any open or closed subset of* $\mathbf{R}^n$ *is locally compact Hausdorff.*

(2) *Any topological manifold is locally compact Hausdorff.*

(3) $\mathbf{B}^n$ *and* $\mathbf{S}^{n-1}$ *are compact.*

(4) *The set* $A = \{(x, 1/x) | 0 < x \le 1\}$ *is closed in* $\mathbf{R}^2$*, but is not compact.*

(5) *The set* $A = \{(x, \sin(1/x)) | 0 < x \le 1\}$ *is bounded in* $\mathbf{R}^2$*, but is not compact.* ♠

A point x of a topological space $(X, \mathscr{T})$ is said to be an **isolated point** of X if $\{x\}$ is open in X.

Theorem 1.33

If $(X, \mathscr{T})$ is a nonempty compact Hausdorff space without any isolated points, then X is uncountable. ♡

Proof. (1) Given any nonempty open subset $U \subseteq X$ and any $x \in X$, there exists a nonempty open subset $V \subseteq U$ such that $x \notin \overline{V}$. Choose $y \in U$ different from x (when $x \notin U$, we choose any point in U because $U \neq \emptyset$; when $x \in U$, we can also choose such a y because x is not an isolated point of X). Since X is Hausdorff, there exist disjoint open subsets W_1 and W_2 about x and y respectively. Then we can set $V := W_2 \cap U$.

(2) Given a map $f : \mathbf{Z}_+ \to X$. We show that f is not surjective (and hence X is uncountable). Let $x_n := f(n)$. From (1), we can find a nonempty open subset $V_1 \subseteq X$ such that $x_1 \notin \overline{V_1}$. In general, given $V_{n-1} \neq \emptyset$, there exists a nonempty open subset $V_n \subseteq V_{n-1}$ such that $x_n \notin \overline{V_n}$. Now we have a family $\{\overline{V_n}\}_{n\geq 1}$ of nonempty closed subsets of X with the property that $\overline{V_n} \supseteq \overline{V_{n+1}}$. Because X is compact, we can find $x \in \cap_{n\geq 1}\overline{V_n}$. Consequently, for any $n \geq 1$, $x \neq x_n$, so that f is not surjective. □

Corollary 1.3

Any nonempty closed interval in $\mathbf{R}$ *is uncountable.* ♡

Exercise 1.30

Verify **Corollary 1.3**. ♠

1.5.4 Paracompact spaces

Let $(X, \mathscr{T})$ be a topological space.

(1) A collection $\mathscr{U} \subseteq 2^X$ is **locally finite** if any point $x \in X$ has a neighborhood that intersects at most finitely many of the sets in $\mathscr{U}$.

(2) Given a cover $\mathscr{U}$ of X, a cover $\mathscr{V}$ is called a **refinement of** $\mathscr{U}$ (written as $\mathscr{V} \prec \mathscr{U}$) if any $V \in \mathscr{V}$ is contained in some $U \in \mathscr{U}$. It is an **open refinement** if each $V \in \mathscr{V}$ is open in X.

(3) X is **paracompact** if any open cover of X has a locally finite open refinement.

Example 1.18

(1) $\mathscr{U} = \{(n, n+2)\}_{n\in\mathbf{Z}}$ *is locally finite in* $\mathbf{R}$.

(2) $\mathscr{U} = \{(0, 1/n)\}_{n\geq 1}$ *or* $\{(1/(n+1), 1/n)\}_{n\geq 1}$ *is locally finite in* $(0,1)$ *but not in* $\mathbf{R}$. ♠

Proposition 1.18

(1) Let X be a topological space and $\mathscr{U} \subset 2^X$.

(a) $\mathscr{U}$ is locally finite if and only if $\overline{\mathscr{U}} := \{\overline{U} : U \in \mathscr{U}\}$ is locally finite.

(b) If $\mathscr{U}$ is locally finite, then

$$\overline{\bigcup_{U\in\mathscr{U}} U} = \bigcup_{U\in\mathscr{U}} \overline{U}.$$

(2) Any second countable, locally compact Hausdorff space X is paracompact. In fact, each open cover of X has a countable, locally finite refinement consisting of open subsets with compact closure (i.e., there exists a countable, locally finite, precompact refinement).

(3) Any paracompact Hausdorff space is normal.

(4) Any closed subspace of a paracompact space is paracompact. ♡

Proof. (1) (a) Suppose that $\mathscr{U}$ is locally finite. For any $x \in X$, there exists a neighborhood W

of x such that W intersects $U_1, \cdots, U_m \in \mathscr{U}$ for some $m \in \mathbf{N}$. If $W \cap \overline{U} \neq \emptyset$, then $W \cap U \neq \emptyset$ so that $U \in \{U_1, \cdots, U_m\}$. Thus W intersects $\overline{U}$ for finitely many $\overline{U} \in \overline{\mathscr{U}}$. The converse is obvious.

(b) Observe that $\cup_{U\in\mathscr{U}}\overline{U} \subseteq \overline{\cup_{U\in\mathscr{U}}U}$. Let $x \notin \cup_{U\in\mathscr{U}}\overline{U}$. There exists some $U \in \mathscr{U}$ such that $x \in U$ and intersects $\overline{U_1}, \cdots, \overline{U_k} \in \overline{\mathscr{U}}$ by (1). Then $U \setminus \cup_{1\leq i\leq k}\overline{U_k}$ contains x and intersects none of the sets in $\mathscr{U}$; hence $x \notin \overline{\cup_{U\in\mathscr{U}}U}$.

(2) Let X be a second countable locally compact Hausdorff space. We claim that there exists a sequence $(G_i)_{i\in\mathbf{N}}$ of open sets such that $\overline{G_i}$ is compact, $\overline{G_i} \subseteq G_{i+1}$, and $X = \cup_{i\in\mathbf{N}}G_i$.

In fact, local compactness implies that X has a basis of precompact open sets by **Proposition 1.17**. The second countable assumption implies that there exists a countable basis $(U_i)_{i\in\mathbf{N}}$ of precompact open sets by **Theorem 1.17**. Let $G_1 := U_1$. Assume $G_k := U_1 \cup \cdots \cup U_{j_k}$. Let j_{k+1} be the smallest positive integer greater than j_k such that $\overline{G_k} \subseteq \cup_{1\leq i\leq j_{k+1}}U_i$. Define $G_{k+1} := \cup_{1\leq i\leq j_{k+1}}U_i$. Then $(G_i)_{i\in\mathbf{N}}$ is countable, $\overline{G_i}$ is compact, $\overline{G_i} \subseteq G_{i+1}$, and

$$X = \bigcup_{i\in\mathbf{N}} U_i \subseteq \bigcup_{i\in\mathbf{N}} G_i \subseteq \bigcup_{i\in\mathbf{N}} \overline{G_i} \subseteq X.$$

Let $\mathscr{U} = (U_\alpha)_{\alpha\in I}$ be an arbitrary open cover of X. Since $\overline{G_i}\setminus G_{i-1} \subseteq G_{i+1}\setminus\overline{G_{i-2}}$ $(i \geq 3)$ is compact, it follows that we can choose a finite subcover of the open cover $(U_\alpha \cap (G_{i+1}\setminus\overline{G_{i-2}}))_{\alpha\in I}$ of $\overline{G_i} \setminus G_{i-1}$ and choose a finite subcover of the open cover $(U_\alpha \cap G_3)_{\alpha\in I}$ of $\overline{G_2}$. Indeed, let $W_i := \overline{G_i} \setminus G_{i-1}$; the local compactness of X implies that there exists a finite subcover $(U_{ij})_{1\leq j\leq m_i}$ of the open cover $(U_\alpha \cap (G_{i+1} \setminus \overline{G_{i-2}})_{\alpha\in I}$ of W_i. Set $V_{ij} := U_{ij} \cap (G_{i+1} \setminus \overline{G_{i-2}})$ for $1 \leq j \leq m_i$. Similarly, we can define V_{ij} for $i = 1, 2$. Hence $(V_{ij})_{1\leq j\leq m_i}$ is an open subcover of W_i and

$$\bigcup_{i\in\mathbf{N}} \bigcup_{1\leq j\leq m_i} V_{ij} = \bigcup_{i\in\mathbf{N}} W_i = X.$$

If $\mathscr{V} := (V_{ij})_{i\in\mathbf{N},1\leq j\leq m_i}$, then $\mathscr{V} \prec \mathscr{U}$ and $W_i \cap \cup_{1\leq j\leq m_i} V_{kj} = \emptyset$ for $k \neq i-2, i-1, i, i+1, i+1$. Thus $\mathscr{V}$ is locally finite.

(3) We first prove that X is regular. Let $a \in X$ and B be closed subset of X disjoint from a. The Husdorff property implies that for any $b \in B$ there is an open subset $U_b \ni b$ such that $\overline{U_b} \cap \{a\} = \emptyset$. Cover X by $\{U_b\}_{b\in B}$ and $X \setminus B$, and take a locally finite open refinement $\mathscr{C}$ that covers X. Set

$$\mathscr{D} := \{C \in \mathscr{C} | C \cap B \neq \emptyset\}.$$

Then $\mathscr{D}$ covers B. For any $D \in \mathscr{D}$, $\overline{D} \cap \{a\} = \emptyset$. Let

$$V := \bigcup_{D\in\mathscr{D}} D$$

that is open in X and contains B. Since $\mathscr{D}$ is locally finite, it follows that $\overline{V} = \cup_{D\in\mathscr{D}}\overline{D}$. Hence $\overline{V} \cap \{a\} = \emptyset$ so that X is regular (because V and $X \setminus \overline{V}$ are disjoint open subsets of B and a respectively.

Let A and B be two disjoint closed subsets of X. For any $a \in A$, there exist open subsets

U_a of a and V_a containing B, respectively, such that $\overline{U_a} \cap V_a = \emptyset$. As above, we can find two disjoint open subsets V and $X \setminus V$ of B and A respectively.

(4) Let Y be a closed subspace of the paracompact space X and let $\mathscr{U}$ be an open cover of Y. For any $U \in \mathscr{U}$, there exists an open subset U' of X such that $U' \cap Y = U$. Cover X by the open subsets U', together with $X \setminus Y$. We then take a locally finite open refinement $\mathscr{V}$ of $\mathscr{U}$ that still covers X. The collection $\mathcal{C} := \{V \cap Y | V \in \mathscr{V}\}$ is a locally finite open refinement of $\mathscr{U}$. □

Note 1.2

(1) *A paracompact subspace of a Hausdorff space X need not be closed in X. For example, $(0,1)$ is paracompact, but it is not closed in $\mathbf{R}$.*

(2) **(A. H. Stone)** *Every metric space is paracompact.*

(3) *Every regular Lindelöf space is paracompact.*

(4) *The product of two paracompact spaces need not be paracompact. For example, $\mathbf{R}_\ell$ is regular and Lindelöf, it follows from (3) that $\mathbf{R}_\ell$ is paracompact, but $\mathbf{R}_\ell^2$, that is Hausdorff and not normal, is not paracompact.*

(5) $\mathbf{R}^\omega$ *is paracompact in te product topology.*

(6) $\mathbf{R}^J$ *is not paracompact of J is unvountable.* ♣

Let $\mathscr{U} = \{U_\alpha\}_{\alpha \in J}$ be an index open cover of a topological space X. An indexed family of continuous functions $\phi_\alpha : X \to [0,1]$ is a **partition of unity on X dominated by** $\mathscr{U}$, if

(1) $\mathbf{supp}(\phi_\alpha) \subseteq U_\alpha$ for all $\alpha \in J$,

(2) $\{\mathbf{supp}(\phi_\alpha)\}_{\alpha \in J}$ is locally finite, and

(3) $\sum_{\alpha \in J} \phi_\alpha(x) \equiv 1$ for all $x \in X$.

Lemma 1.4. (Shrinking lemma)

Suppose that$(X, \mathscr{T})$ is a paracompact Hausdorff space. If $\mathscr{U} = \{U_\alpha\}_{\alpha \in J}$ is an indexed family of open cover of X, then $\mathscr{U}$ admits a locally finite open refinement $\mathscr{V} = \{V_\alpha\}_{\alpha \in J}$ indexed by the same index set J, such that $\overline{V_\alpha} \subseteq U_\alpha$ for each $\alpha \in J$. ♡

Proof. Let $\mathscr{A} := \{A \in \mathscr{T} | \overline{A} \subseteq U_\alpha$ for some $\alpha \in J\}$. According to **Proposition 1.18**, X is normal so that for any $x \in X$ we can find a open subset A_x such that $\overline{A_x} \subseteq U_\alpha$ for some $\alpha \in J$. Thus $\mathscr{A}$ covers X. The paracompactness implies that $\mathscr{A}$ has a locally finite open refinement $\mathscr{B} = \{B_\beta\}_{\beta \in K}$. We then find a map $f : K \to J$ such that for any $\beta \in K$ one has $\overline{B_\beta} \subseteq \overline{A} \subseteq U_{f(\beta)}$. Define, for each $\alpha \in J$,

$$V_\alpha := \begin{cases} \cup_{B_\beta \in \mathscr{B}_\alpha} B_\beta, & \mathscr{B}_\alpha := \{B_\beta \in \mathscr{B} | f(\beta) = \alpha\} \neq \emptyset, \\ \emptyset, & \mathscr{B}_\alpha = \emptyset. \end{cases}$$

For $B_\beta \in \mathscr{B}_\alpha$, we have $\overline{B_\beta} \subseteq U_{f(\beta)} = U_\alpha$. Since $\mathscr{B}_\alpha$ is locally finite, by **Proposition 1.18**,

$$\overline{V_\alpha} = \bigcup_{B_\beta \in \mathscr{B}_\alpha} \overline{B_\beta} \subseteq U_\alpha.$$

Finally we check the local finiteness of $\mathscr{V} = \{V_\alpha\}_{\alpha \in J}$. Given $x \in X$. There is a neighborhood W of x that intersects B_β for only finitely many values of $\beta = \beta_1, \cdots, \beta_N$. Then W can intersects V_α only if $\alpha \in \{f(\beta_1), \cdots, f(\beta_N)\}$. □

Theorem 1.34

Let $(X, \mathscr{T})$ be a paracompact Hausdorff space. If $\mathscr{U} = \{U_\alpha\}_{\alpha \in J}$ is an indexed open cover of X, then there is a partition of unity on X dominated by $\mathscr{U}$. ♡

Proof. By **Lemma 1.4**, we can find two locally finite indexed families of open sets $\mathscr{W} = \{W_\alpha\}_{\alpha \in J}$ and $\mathscr{V} = \{V_\alpha\}_{\alpha \in J}$ covering X such that

$$\overline{W_\alpha} \subseteq V_\alpha, \quad \overline{V_\alpha} \subseteq U_\alpha.$$

The normality, coming from **Proposition 1.18**, implies that there is a continuous function $\psi_\alpha : X \to [0, 1]$ such that

$$\psi_\alpha(\overline{W_\alpha}) = 1, \quad \psi_\alpha(X \setminus V_\alpha) = 0,$$

by **Theorem 1.15**. Hence $\mathbf{supp}(\psi_\alpha) \subseteq \overline{V_\alpha} \subseteq U_\alpha$. Because $\{\mathscr{W}$ covers X, for any $x \in X$ we can find some ψ_α such that $\psi_\alpha(x) > 0$. Define

$$\Psi(x) := \sum_{\alpha \in J} \psi_\alpha(x) \geq 0$$

which is strictly positive for each $x \in X$. Since for each $x \in X$ there is a neighborhood W_x of x that intersects $\mathbf{supp}(\psi_\alpha)$ for only finitely many values of α, it follows that the above infinite sum at $x$is actually finite. Hence Ψ is well-defined and continuous on W_x and then on X. Finally set $\phi_\alpha(x) := \psi_\alpha(x)/\Psi(x)$. □

Chapter 2 Analysis on topological groups

Introduction

- *Haar measures*
- *Iwasawa's decomposition*
- *Harish, Mellin and spherical transforms*

2.1 Haar measures

Introduction

- *Examples of Haar measures*
- *Convolution*
- *Modular functions on $\mathcal{G}$*

From now on let $\mathcal{G}$ be a **local compact group**, which in this note means a locally compact and Hausdorff topological group.

A collection $\mathscr{S}$ of subsets of $\mathcal{G}$ is called a **σ-algebra** if

(1) $\emptyset, X \in \mathscr{S}$,

(2) $A \in \mathscr{S}$ implies $X \setminus A \in \mathscr{S}$, and

(3) $\{A_n\}_{n\geq 1} \subseteq \mathscr{S}$ implies $\cup_{n\geq 1} A_n \in \mathscr{S}$.

Clearly that the topology $\mathscr{T}$ for X is a σ-algebra. Define

$$\mathscr{B}_{\mathcal{G}} := \bigcap\{\mathscr{S} | \mathscr{S} \text{ is a } \sigma\text{-algebra of } X \text{ and contains } \mathscr{T}\}.$$

Exercise 2.1

Show that $\mathscr{B}_{\mathcal{G}}$ is also a σ-algebra and is the smallest σ-algebra containing the open subsets of $\mathcal{G}$.

Elements in $\mathscr{B}_{\mathcal{G}}$ are called **Borel subsets in** $\mathcal{G}$.

For any locally compact group $\mathcal{G}$, Haar's theorem gives, up to a positive multiplicative constant, a unique countably additive, nontrivial measure μ on the Borel sets of $\mathcal{G}$ satisfying the following properties:

(1) the measure μ is left-invariant: $\mu(g\mathcal{S}) = \mu(\mathcal{S})$ for every $g \in \mathcal{G}$ and all Borel sets $\mathcal{S} \subseteq \mathcal{G}$;

(2) the measure μ is finite on every compact set: $\mu(\mathcal{K}) < \infty$ for all compact $\mathcal{K} \subseteq \mathcal{G}$;

(3) the measure μ is outer regular on Borel sets $\mathcal{S} \subseteq \mathcal{G}$: $\mu(\mathcal{S}) = \inf\{\mu(\mathcal{U}) : \mathcal{S} \subseteq \mathcal{U}, \mathcal{U} \text{ open}\}$;

(4) the measure μ is inner regular on open sets $\mathcal{U} \subseteq \mathcal{G}$: $\mu(\mathcal{U}) = \sup\{\mu(\mathcal{K}) : \mathcal{K} \subseteq \mathcal{U}, \mathcal{K} \text{ compact}\}$.

Such a measure on $\mathcal{G}$ is called a **(left) Haar measure**. If μ is a Haar measure and $g_1 \in \mathcal{G}$, then $\mathscr{B}_{\mathcal{G}} \ni \mathcal{S} \mapsto \mu(\mathcal{S}g_1)$ is left-invariant. Thus, by uniqueness of the Haar measure, there exists a function Δ from the group $\mathcal{G}$ to $\mathbb{R}^+$, called the **modular function**, such that

$$\mu(\mathcal{S}g_1) = \Delta(g_1)\mu(\mathcal{S})$$

for every Borel set $\mathcal{S} \subseteq \mathcal{G}$.

We say $\mathcal{G}$ is **unimodular** if the modular function Δ is identically 1, or equivalently, if the Haar measure is both left and right-invariant. Examples of unimodular groups are Abelian groups, compact groups, discrete groups, semisimple Lie groups and connected nilpotent Lie groups.

An example of a non-unimodular group is the group of affine transformations

$$\{x \mapsto ax + b : a \in \mathbf{R}^*,\ b \in \mathbf{R}\} = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \middle| a \in \mathbf{R}^*,\ b \in \mathbf{R} \right\}$$

on the real line. This example shows that a solvable Lie group need not be unimodular. In this group a left Hasr measure is given by $da \wedge db/a^2$, and a right Haar measure by $da \wedge db/|a|$.

Let $\mathcal{G}$ be a locally compact group with Haar measure μ. For $0 < p < +\infty$, we define $\widehat{L}^p(\mathcal{G}, \mu)$ the set of all complex-valued μ-measurable functions on $\mathcal{G}$ whose modulus to the p-th power is integrable. $\widehat{L}^\infty(\mathcal{G}, \mu)$ is the set of all complex-valued μ-measurable functions f on $\mathcal{G}$ such that for some $M > 0$ the set $\{x \in \mathcal{G} || f(x)| > M\}$ has μ-measure zero. We say two functions in $L^p(\mathcal{G}, \mu)$ are considered equal (or equivalent) if they are equal μ-almost everywhere. The equivalence class of $\widehat{L}^p(\mathcal{G}, \mu)$ is denoted $L^p(\mathcal{G}, \mu)$. Similarly we can define $L^\infty(\mathcal{G}, \mu)$.

We simply write $L^p(\mathcal{G})$ for $L^p(\mathcal{G}, \mu)$, when $p > 0$ or $p = \infty$. For $0 < p < +\infty$ define

$$||f||_{L^p(\mathcal{G})} := \left(\int_{\mathcal{G}} |f(x)|^p d\mu(x) \right)^{1/p}$$

and for $p = \infty$ define

$$||f||_{L^\infty(\mathcal{G})} := \inf \{M > 0 | \mu(\{x \in \mathcal{G} || f(x)| > M\}) = 0\}.$$

The left invariance of μ is equivalent to the fact that for all $t \in \mathcal{G}$ and all nonnegative measurable functions f on $\mathcal{G}$ we have

$$\int_{\mathcal{G}} f(tx)\, d\mu(x) = \int_{\mathcal{G}} f(x)\, d\mu(x).$$

2.1.1 Examples of Haar measures

Let $\mathcal{G} := \mathbf{R}^* = \mathbf{R} \setminus \{0\}$ with group law the usual multiplication. Then $\mathcal{G}$ is locally compact and the measure

$$\mu := \frac{dx}{|x|}$$

is invariant under multiplicative translation, because of the following identity

$$\int_{-\infty}^{+\infty} f(tx)\frac{dx}{|x|} = \int_{-\infty}^{+\infty} f(x)\frac{dx}{|x|}$$

for all $f \in L^1(\mathbf{R}^*, \mu)$ and all $t \in \mathbf{R}^*$. Then

$$\mu(A) = \int_A \frac{dx}{x} = \int_{-\infty}^{+\infty} \chi_A(x)\frac{dx}{x}.$$

Example 2.1

(1) *A Haar measure on the multiplicative group* $\mathbf{R}^+$ *is* dx/x. *That is*

$$\mu(A) = \int_{-\infty}^{+\infty} \chi_A(x)\frac{dx}{x}.$$

(2) *The* **Heisenberg group** $\mathfrak{H}^n$ *is the set* $\mathbf{C}^n \times \mathbf{R}$ *with the group operation*

$$(\boldsymbol{z}, t)(\boldsymbol{w}, s) := \left(\boldsymbol{z} + \boldsymbol{w}, t + s + 2\mathbf{Im}\sum_{1\le i\le n} z^i\overline{w^i}\right).$$

The identity in $\mathfrak{H}^n$ *is* $(\mathbf{0}, 0) \in \mathbf{C}^n \times \mathbf{R}$ *and* $(\boldsymbol{z}, t)^{-1} = (-\boldsymbol{z}, -t)$. $\mathfrak{H}^n$ *is topologically identified with* $\mathbf{C}^n \times \mathbf{R}$ *and hence is a locally compact group. Both the left and right Haar measure on* $\mathfrak{H}^n$ *is Lebesgue measure* $d\boldsymbol{z} \wedge dt$.

(3) *Let* $\mathcal{G} := \mathbf{R}^2 \setminus \{(0, y)|y \in \mathbf{R}\}$ *with group operation*

$$(x, y)(z, w) := (xz, xw + y) \quad \Longleftrightarrow \quad \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}\begin{bmatrix} z & w \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} xz & xw + y \\ 0 & 1 \end{bmatrix}$$

For $A \subseteq \mathscr{B}_{\mathcal{G}}$ *define*

$$\mu(A) := \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \chi_A(x, y)\frac{dxdy}{x^2}.$$

When $\boldsymbol{t} = (t_1, t_2) \in \mathcal{G}$ *and* $(x, y) \in A$ *we have*

$$\boldsymbol{t}(x, y) = (t_1 x, t_1 y + t_2)$$

and then

$$\begin{aligned}\mu(\boldsymbol{t}A) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \chi_{\boldsymbol{t}A}(x, y)\frac{dxdy}{x^2} &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \chi_A(x', y')\frac{d(x'/t)d((y' - t_2)/t_1)}{(x'/t_1)^2} \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \chi_A(x', y')\frac{dx'dy'}{x'^2} \quad = \quad \mu(A).\end{aligned}$$

Thus μ *is a left Haar measure on* $\mathcal{G}$. *Similarly we can prove that*

$$\nu(A) := \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \chi_A(x, y)\frac{dxdy}{|x|}$$

is a right Haar measure on $\mathcal{G}$. ♠

2.1.2 Convolution

Let $f, g \in L^1(\mathcal{G})$. Define the **convolution** $f * g$ by

$$(f * g)(x) := \int_{\mathcal{G}} f(y)g(y^{-1}x)\,d\mu(y). \tag{2.1.2.1}$$

Since

$$\left|\int_{\mathcal{G}}\int_{\mathcal{G}} f(y)g(y^{-1}x)d\mu(y)d\mu(x)\right| \le \int_{\mathcal{G}}\int_{\mathcal{G}} |f(y)||g(y^{-1}x)|d\mu(x)d\mu(y)$$
$$= \int_{\mathcal{G}}|f(y)|\int_{\mathcal{G}}|g(y^{-1}x)|d\mu(x)d\mu(y) = \int_{\mathcal{G}}|f(y)|\int_{\mathcal{G}}|g(x)|d\mu(x)d\mu(y)$$
$$= ||f||_{L^1(\mathcal{G})}||g||_{L^1(\mathcal{G})} < +\infty,$$

it follows that $f * g$ is well-defined and

$$||f * g||_{L^1(\mathcal{G})} \le ||f||_{L^1(\mathcal{G})}||g||_{L^1(\mathcal{G})}. \tag{2.1.2.2}$$

Letting $z = x^{-1}y$ in (2.1.2.1) yields

$$(f * g)(x) = \int_{\mathcal{G}} f(xz)g(z^{-1})d\mu(z). \tag{2.1.2.3}$$

Exercise 2.2

(1) *For all $f, g, h \in L^1(\mathcal{G})$, we have*

$$f * (g * h) = (f * g) * h$$

and

$$f * (g + h) = f * g + f * h, \quad (f + g) * h = f * h + g * h.$$

(2) *Verify (2.1.2.4).*

For $f, g \in L^1(\mathcal{G})$ and $t \in \mathcal{G}$, define

$${}^t f(x) := f(tx), \quad f^t(x) := f(xt).$$

Then we have

$${}^t f * g = {}^t(f * g), \quad f * g^t = (f * g)^t. \tag{2.1.2.4}$$

Theorem 2.1. (Minkowski's inequality)

Let $1 \le p \le +\infty$. For $f \in L^p(\mathcal{G})$ and $g \in L^1(\mathcal{G})$ we have $g * f$ exists μ-a.e. and satisfies

$$||g * f||_{L^p(\mathcal{G})} \le ||g||_{L^1(\mathcal{G})}||f||_{L^p(\mathcal{G})}. \tag{2.1.2.5}$$

Proof. The case $p = 1$ was proved in (2.1.2.2), and the case $p = \infty$ is obvious. Now we assume $1 < p < +\infty$. Write $p' = \frac{p}{p-1}$, and compute

$$(|g| * |f|)(x) = \int_{\mathcal{G}} |f(y^{-1}x)||g(y)|d\mu(x) = \int_{\mathcal{G}} |f(y^{-1}x)||g(y)|^{\frac{1}{p}}|g(y)|^{\frac{1}{p'}}d\mu(x)$$
$$\le \left(\int_{\mathcal{G}} |f(y^{-1}x)|^p|g(y)|d\mu(y)\right)^{1/p}\left(\int_{\mathcal{G}}|g(y)|d\mu(y)\right)^{1/p'}$$

so that

$$|||g| * |f|||_{L^p(\mathcal{G})} \le \left(||g||_{L^1(\mathcal{G})}^{p-1}\int_{\mathcal{G}}\int_{\mathcal{G}} |f(y^{-1}x)|^p|g(y)|d\mu(y)d\mu(x)\right)^{1/p}$$

$$= \left(||g||_{L^1(\mathcal{G})}^{p-1} \int_{\mathcal{G}} \int_{\mathcal{G}} |f(x)|^p d\mu(x)|g(y)|d\mu(y)\right)^{1/p} = ||f||_{L^p(\mathcal{G})}||g||_{L^1(\mathcal{G})}.$$

Now the inequality (2.1.2.5) follows from $|g * f| \leq |g| * |f|$. □

If f is a function on $\mathcal{G}$, we define

$$\widetilde{f}(x) := f(x^{-1}), \quad x \in \mathcal{G}. \tag{2.1.2.6}$$

Then (2.1.2.3) can be written as

$$(f * g)(x) = \int_{\mathcal{G}} f(xz)\widetilde{g}(z)d\mu(z).$$

Theorem 2.2. (Young's inequality)

Let $1 \leq p, q, r \leq +\infty$ satisfy

$$\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}.$$

Then for all $f \in L^p(\mathcal{G})$ and all $g \in L^r(\mathcal{G})$ satisfying $||g||_{L^r(\mathcal{G})} = ||\widetilde{g}||_{L^r(\mathcal{G})}$ we have $f * g$ exists μ-a.e. and satisfies

$$||f * g||_{L^q(\mathcal{G})} \leq ||g||_{L^r(\mathcal{G})}||f||_{L^p(\mathcal{G})}. \tag{2.1.2.7}$$

♡

Proof. If $r = +\infty$, then $p = 1$ and $q = +\infty$ in which case the inequality (2.1.2.7) is trivial. Now we assume that $r < +\infty$. From $\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}$ we have

$$\frac{1}{r'} + \frac{1}{p'} + \frac{1}{q} = 1, \quad \frac{1}{r} + \frac{1}{r'} = 1 = \frac{1}{p} + \frac{1}{p'}.$$

By Hölder's inequality we get

$$\begin{aligned}
|(|f| * |g|)(x)| \quad &\leq \int_{\mathcal{G}} |f(y)||g(y^{-1}x)|d\mu(x) \\
= \int_{\mathcal{G}} |f(y)|^{p/r'} &\left(|f(y)|^{p/q}|g(y^{-1}x)|^{r/q}\right) |g(y^{-1}x)|^{r/p'} d\mu(y) \\
\leq ||f||_{L^p(\mathcal{G})}^{p/r'} &\left(\int_{\mathcal{G}} |f(y)|^p|g(y^{-1}x)|^r d\mu(y)\right)^{1/q} \left(\int_{\mathcal{G}} |g(y^{-1}x)|^r d\mu(y)\right)^{1/p'} \\
= ||f||_{L^p(\mathcal{G})}^{p/r'} &\left(\int_{\mathcal{G}} |f(y)|^p|g(y^{-1}x)|^r d\mu(y)\right)^{1/q} \left(\int_{\mathcal{G}} |\widetilde{g}(x^{-1}y)|^r d\mu(y)\right)^{1/p'} \\
= &\left(\int_{\mathcal{G}} |f(y)|^p|g(y^{-1}x)|^r d\mu(y)\right)^{1/q} ||||_{L^p(\mathcal{G})}^{p/r'}||\widetilde{g}||_{L^r(\mathcal{G})}^{r/p'}
\end{aligned}$$

so that

$$\begin{aligned}
|||f| * |g|||_{L^q(\mathcal{G})} \quad &\leq \quad ||f||_{L^p(\mathcal{G})}^{p/r'}||\widetilde{g}||_{L^r(\mathcal{G})}^{r/p'} \left(\int_{\mathcal{G}} \int_{\mathcal{G}} |f(y)|^p|g(y^{-1}x)|^r d\mu(x)d\mu(y)\right)^{1/q} \\
= ||f||_{L^p(\mathcal{G})}^{p/r'}&||\widetilde{g}||_{L^p(\mathcal{G})}^{r/p'}||f||_{L^p(\mathcal{G})}^{p/q}||g||_{L^r(\mathcal{G})}^{r/q} \quad = \quad ||g||_{L^r(\mathcal{G})}||f||_{L^p(\mathcal{G})}
\end{aligned}$$

using again the assumption $||g||_{L^r(\mathcal{G})} = ||\widetilde{g}||_{L^r(\mathcal{G})}$. □

Exercise 2.3. (Hardy)

Use **Theorem 2.1** *to prove the following inequalities.*

(1) *For* $1 < p < +\infty$,

$$\left(\int_0^{+\infty}\left(\frac{1}{x}\int_0^x |f(t)|dt\right)^p dx\right)^{1/p} \le \frac{p}{p-1}||f||_{L^p((0,+\infty))}$$

and

$$\left(\int_0^{+\infty}\left(\int_x^{+\infty} |f(t)|dt\right)^p dx\right)^{1/p} \le p\left(\int_0^{+\infty}|f(t)|^p t^p dt\right)^{1/p}.$$

(2) *For* $0 < b < +\infty$ *and* $1 \le p < +\infty$,

$$\left(\int_0^{+\infty}\left(\int_0^x |f(t)|dt\right)^p x^{-b-1}dx\right)^{1/p} \le \frac{p}{b}\left(\int_0^{+\infty}|f(t)|^p t^{p-b-1} dt\right)^{1/p}$$

and

$$\left(\int_0^{+\infty}\left(\int_x^{+\infty} |f(t)|dt\right)^p x^{b-1}dx\right)^{1/p} \le \frac{p}{b}\left(\int_0^{+\infty}|f(t)|^p t^{p+b-1} dt\right)^{1/p}.$$

♠

2.1.3 Modular functions on $\mathcal{G}$

Let $\mathcal{G}$ be a locally compact group with Haar measure $\mu := dg$. Recall that the modular function $\Delta \equiv \Delta_{\mathcal{G}}$ on $\mathcal{G}$ is the function (actually continuous homomorphism into $\mathbb{R}^+$) such that for all $f \in C_c(\mathcal{G})$,

$$\int_{\mathcal{G}} f(gg_1)\,dg = \Delta(g_1)\int_{\mathcal{G}} f(g)\,dg. \tag{2.1.3.1}$$

Proposition 2.1

Let $\mathcal{P}$ be a locally compact group with two closed subgroups $\mathcal{A}, \mathcal{U}$ such that $\mathcal{A}$ normalizes $\mathcal{U}$, and such that the product

$$\mathcal{A}\times\mathcal{U} \longrightarrow \mathcal{A}\mathcal{U} = \mathcal{P} \tag{2.1.3.2}$$

is a topological isomorphism. Then for $f \in C_c(\mathcal{P})$ the functional

$$f \longmapsto \int_{\mathcal{U}}\int_{\mathcal{A}} f(au)\,da\,du = \int_{\mathcal{A}}\int_{\mathcal{U}} f(au)\,du\,da \tag{2.1.3.3}$$

is a Haar (left-invariant) functional on $\mathcal{P}$.

♡

Proof. Left invariant by $\mathcal{A}$ is obvious. Let $u_1 \in \mathcal{U}$. Then

$$\begin{aligned}
\int_{\mathcal{A}}\int_{\mathcal{U}} f(u_1 a u)\,du\,da &= \int_{\mathcal{A}}\int_{\mathcal{U}} f(aa^{-1}u_1 a u)\,du\,da \\
&= \int_{\mathcal{A}}\int_{\mathcal{U}} (f\circ a)(u_1^a u)\,du\,da \quad \text{where } u_1^a := a^{-1}u_1 a \in \mathcal{U} \\
&= \int_{\mathcal{A}}\int_{\mathcal{U}} f(au)\,du\,da
\end{aligned}$$

by the Haar measure property. □

Proposition 2.2

Notation is as in **Proposition 2.1**. *There exists a unique continuous homomorphism $\delta : \mathcal{A} \to \mathbb{R}^+$ such that for $f \in C_c(\mathcal{U})$,*

$$\int_{\mathcal{U}} f(a^{-1}ua)\,du = \delta(a) \int_{\mathcal{U}} f(u)\,du, \tag{2.1.3.4}$$

or in other words, for $f \in C_c(\mathcal{P})$,

$$\int_{\mathcal{U}} f(ua)\,du = \delta(a) \int_{\mathcal{U}} f(au)\,du. \tag{2.1.3.5}$$

If $\mathcal{U}$ is unimodular, then δ is the modular function on $\mathcal{P}$, that is,

$$\Delta_{\mathcal{P}}(p) = \Delta_{\mathcal{P}}(au) = \delta(a). \tag{2.1.3.6}$$

♡

Proof. For the first statement, we note that the map $u \mapsto u^a = a^{-1}ua$ is a topological group automorphism of $\mathcal{U}$, which preserves Haar measure up to a constant factor. For the second statement, we first note that the functional

$$f \longmapsto \int_{\mathcal{U}} \int_{\mathcal{A}} f(au)\,dadu$$

is right-invariant under $\mathcal{U}$. Let $a_1 \in \mathcal{A}$. Then

$$\begin{aligned} \int_{\mathcal{U}} \int_{\mathcal{A}} f(aua_1)\,dadu &= \int_{\mathcal{U}} \int_{\mathcal{A}} f(aa_1a^{-1}ua_1)\,dadu = \int_{\mathcal{U}} \int_{\mathcal{A}} f(aa_1u)\,dadu \\ &= \int_{\mathcal{U}} \int_{\mathcal{A}} f(aa_1a^{-1}au)\,dadu = \delta(a_1) \int_{\mathcal{U}} \int_{\mathcal{A}} f(au)\,dadu, \end{aligned}$$

which proves the formula (2.1.3.6). □

Proposition 2.3

Let $\mathcal{G}$ be a locally compact group with two closed subgroups $\mathcal{P}, \mathcal{K}$ such that

$$\mathcal{P} \times \mathcal{K} \longrightarrow \mathcal{P}\mathcal{K} = \mathcal{G} \tag{2.1.3.7}$$

is a topological isomorphism (not group isomorphism). Assume that $\mathcal{G}, \mathcal{K}$ are unimodular. Let dg, dp, dk be given Haar measures on $\mathcal{G}, \mathcal{P}, \mathcal{K}$ respectively. Then there is a constant c such that for all $f \in C_c(\mathcal{G})$,

$$\int_{\mathcal{G}} f(g)\,dg = c \int_{\mathcal{P}} \int_{\mathcal{K}} f(pk)\,dpdk. \tag{2.1.3.8}$$

If in addition $\mathcal{P} = \mathcal{A}\mathcal{U}$ as in **Proposition 2.1**, *with $\mathcal{U}$ unimodular, so we have the product decomposition*

$$\mathcal{U} \times \mathcal{A} \times \mathcal{K} \longrightarrow \mathcal{G}, \tag{2.1.3.9}$$

then

$$\int_{\mathcal{G}} f(g)\,dg = \int_{\mathcal{U}} \int_{\mathcal{A}} \int_{\mathcal{K}} f(uak)\delta(a)^{-1}dudadk. \tag{2.1.3.10}$$

♡

Proof. The first assertion comes from a standard fact of homogeneous spaces, that is, there exists a left-invariant Haar measure on $\mathcal{G}/\mathcal{K} = \mathcal{P}$. The second integral formula comes from **Proposition 2.2**. □

2.2 Iwasawa's decomposition

Introduction

- *Iwasawa's decomposition*
- *$\mathcal{K}$-bi-invariant functions*
- *Characters*
- *Group Fubini theorem*

The **Iwasawa decomposition** for $\mathcal{G} := \mathbf{SL}_2(\mathbb{C}) = \mathbf{SL}(2, \mathbf{C})$ is $\mathcal{G} = \mathcal{U}\mathcal{A}\mathcal{K}$, where

- $\mathcal{U}$ is the subgroup of upper triangular unipotent matrices;
- $\mathcal{A}$ is the subgroup of diagonal matrices with positive diagonal components;
- $\mathcal{K}$ is the unitary subgroup.

2.2.1 Iwasawa's decomposition

Thus $\mathcal{U}$ is the group of 2×2 matrices of the form

$$u(x) := \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \quad \text{with } x \in \mathbb{C}. \tag{2.2.1.1}$$

The elements of $\mathcal{A}$ are of the form

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \quad \text{with } a_1, a_2 > 0 \text{ and } a_1 a_2 = 1. \tag{2.2.1.2}$$

We also define the diagonal group $\mathbf{D}$ consisting of all elements

$$z = \begin{bmatrix} z_1 & 0 \\ 0 & z_1^{-1} \end{bmatrix} \quad \text{with } 0 \neq z_1 \in \mathbb{C}. \tag{2.2.1.3}$$

The elements k of $\mathcal{K}$ are the matrices satisfying (besides determinant 1)

$$k^{-1} = k^*, \quad \text{where } M^* := {}^t\overline{M} \text{ for all matrix } M. \tag{2.2.1.4}$$

Theorem 2.3. (Iwasawa's decomposition)

Every element $g \in \mathcal{G}$ has a unique expression as a product

$$g = uak \quad \text{with } u \in \mathcal{U},\ a \in \mathcal{A},\ k \in \mathcal{K}. \tag{2.2.1.5}$$

Furthermore, let $\mathrm{Pos}_2 := \mathrm{Pos}_2(\mathbb{C})$ *be the space of positive definite 2×2 Hermitian matrices. Then the map $g \mapsto gg^*$ is a bijection $\mathcal{G}/\mathcal{K} \approx \mathrm{SPos}_2$, so of $\mathcal{U}\mathcal{A} \approx \mathrm{SPos}_2$.* ♡

Proof. Let $g \in \mathcal{G}$ and e_1, e_2 be the standard (vertical) unit vectors of $\mathbb{C}^2$. Let $g^{(i)} := ge_i$. We orthonormalize $g^{(1)}, g^{(2)}$ by the standard Gram-Schmidt process, which means we can find an upper triangular matrix $B = (b_{ij})$ $(b_{21} = 0)$ such that if we let

$$e_1' := b_{11} g^{(1)}, \quad e_2' := b_{12} g^{(1)} + b_{22} g^{(2)},$$

then e_1', e_2' are orthonormal unit vectors, and we can choose both b_{11} and b_{22} greater than 0 (This corresponds to dividing by the norm). Thus $B \in \mathcal{A}\mathcal{U} = \mathcal{U}\mathcal{A}$.

Let k be the matrix such that $ke_i = e_i'$ for $i = 1, 2$. Then k is unitary and $gB = k$. Since $\det g = 1$ and the diagonal components of B are positive, it follows that $\det k = 1$, so $\mathcal{G} = \mathcal{KAU} = \mathcal{UAK}$. To show uniqueness, we use the map $g \mapsto gg^*$. Note that $gg^* = \mathbf{1}$ (identity) if and only if $g \in \mathcal{K}$. Suppose that

$$u_1ak_1(u_1ak_1)^* = u_1a^2u_1^* = u_2b^2u_2^* = u_2bk_2(u_2bk_2)^*$$

with $u_1, u_2 \in \mathcal{U}$, $a, b \in \mathcal{A}$, and $k_1, k_2 \in \mathcal{K}$. Putting $u := u_2^{-1}u_1$, we obtain $ua^2 = b^2(u^*)^{-1}$. Since u, u^* are triangular in opposite directions, they must be diagonal, and finally $a^2 = b^2$, so $a = b$ because the diagonal elements are positive. □

Corollary 2.1

Every element $g \in \mathcal{G}$ has a decomposition, called **polar**,

$$g = k_1bk_2 \quad \text{with } k_1, k_2 \in \mathcal{K} \text{ and } b \in \mathcal{A}. \tag{2.2.1.6}$$

The element b is uniquely determined up to permutation of diagonal components. Let $\mathcal{A}^+$ be the set of $a = \operatorname{diag}(a_1, a_2) \in A$ such that $a_1 \geq 1$. Then $\mathcal{G} = \mathcal{KA}^+\mathcal{K}$, and the decomposition (2.2.1.6) determines b uniquely in A^+. ♡

Proof. Since gg^* can be written as $gg^* = k_1b^2k_1^{-1}$ with $b \in \mathcal{A}$ and $k_1 \in \mathcal{K}$, where b^2 is the matrix of eigenvalues of gg^*. By the bijection in **Theorem 2.2.1**, there exists $k_2 \in \mathcal{K}$ such that $g = k_1bk_2$. □

Given $g = uak \in \mathcal{G}$ with its Iwasawa decomposition, we let

$$g_{\mathcal{A}} := a \quad \text{or also} \quad a_g := a \tag{2.2.1.7}$$

denotes the $\mathcal{A}$-projection, and similarly for $g_{\mathcal{U}}$ and $g_{\mathcal{K}}$.

2.2.2 Characters

A **character** of $\mathcal{A}$ is a continuous homomorphism $\boldsymbol{\chi} : \mathcal{A} \to \mathbb{C}^\times$ into the multiplication group of nonzero complex numbers. Additively, let $\mathfrak{a} := \operatorname{Lie}(\mathcal{A})$ be the $\mathbb{R}$-vector space of diagonal matrices $H = \operatorname{diag}(h_1, h_2)$ with trace zero, so $\mathcal{A} = \exp \mathfrak{a}$. Let $\mathfrak{a}^\vee$ be the dual space. Let $\boldsymbol{\alpha}$ be the (additive) character on $\mathfrak{a}$ defined by $\boldsymbol{\alpha}(H) := h_1 - h_2$. If $a = \exp H = \operatorname{diag}(a_1, a_2) \in \mathcal{A} = \exp \mathfrak{a}$, then we define the **Iwasawa coordinate** $\boldsymbol{y} : \mathcal{A} \to \mathbb{R}^+$ by

$$\boldsymbol{y}(a) \equiv \boldsymbol{y}_a \equiv \boldsymbol{\chi}_{\boldsymbol{\alpha}}(a) \equiv a^{\boldsymbol{\alpha}} := \frac{a_1}{a_2} = a_1^2. \tag{2.2.2.1}$$

We note that $\boldsymbol{\alpha}$ is a basis of $a^\vee$. Furthermore, $\boldsymbol{y}$ gives a group isomorphism of $\mathcal{A}$ with the positive multiplicative group. For complex s, we get

$$\boldsymbol{y}_a^s = a^{s\boldsymbol{\alpha}} = \chi_{s\boldsymbol{\alpha}}(a), \tag{2.2.2.2}$$

with a complex (multiplicative) character $\chi_{s\boldsymbol{\alpha}}$. We often write y instead of $\boldsymbol{y}_a$ and

$$a = a_y \quad \text{or} \quad a(y) = \begin{bmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{bmatrix}. \tag{2.2.2.3}$$

Let $\mathbf{D}$ = diagonal subgroup of $\mathcal{G}$ with elements

$$z = \begin{bmatrix} z_1 & 0 \\ 0 & z_1^{-1} \end{bmatrix} \tag{2.2.2.4}$$

and $\mathbf{T}$ = torus subgroup = subgroup of $\mathbf{D}$ consisting of elements

$$\epsilon = \begin{bmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_1^{-1} \end{bmatrix} \quad \text{with } |\epsilon_1| = 1. \tag{2.2.2.5}$$

Then we have a direct product decomposition

$$\mathbf{D} = \mathcal{A}\mathbf{T} \quad \text{with } z = a_z\epsilon \quad \text{and} \quad \mathbf{T} = \mathbf{D} \cap \mathcal{K}. \tag{2.2.2.6}$$

Indeed, writing $z_1 = |z_1|\epsilon_1$,

$$z = \begin{bmatrix} z_1 & 0 \\ 0 & z_1^{-1} \end{bmatrix} = \begin{bmatrix} |z_1| & 0 \\ 0 & |z_1|^{-1} \end{bmatrix} \begin{bmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_1^{-1} \end{bmatrix} =: a_z\epsilon.$$

The character $\boldsymbol{\alpha}$ is actually the restriction of a character on $\mathbf{D}$, namely

$$z^{\boldsymbol{\alpha}} = \frac{z_1}{z_2} = z_1^2. \tag{2.2.2.7}$$

The kernel of $\boldsymbol{\alpha}$ in $\mathbf{D}$ is the group $\{\pm\mathbf{1}\}$. We define $z \in \mathbf{D}$ to be **regular** if its diagonal components are distinct, in other words if $z^{\boldsymbol{\alpha}} \neq 1$. More generally, an arbitrary square matrix is called **regular** if it can be conjugated to a regular diagonal matrix.

2.2.3 $\mathcal{K}$-bi-invariant functions

Let f be a function on $\mathcal{G}$, which we write $f \in \mathbf{Fu}(\mathcal{G})$. We say that f is **$\mathcal{K}$-bi-invariant** if

$$f(k_1gk_2) = f(g), \quad \text{for all } k_1, k_2 \in \mathcal{K} \text{ and } g \in \mathcal{G}. \tag{2.2.3.1}$$

By the polar decomposition, if b is the $\mathcal{A}$-polar component of g, then $f(g) = f(b)$ and f is determined by its values on $\mathcal{A}$. Let

$$w := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{2.2.3.2}$$

be called the **Weyl element**, and let the group of order 2 that it generates $\bmod \pm\mathbf{1}$ be called the **Weyl group** $\mathcal{W} \subseteq \mathcal{K}$. Then w acts on $\mathbf{D}$ and $\mathcal{A}$ by conjugation, and for $z \in \mathbf{D}$,

$$z^w := wzw^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} z_1 & 0 \\ 0 & z_1^{-1} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} z_1^{-1} & 0 \\ 0 & z_1 \end{bmatrix} = z^{-1}. \tag{2.2.3.3}$$

Proposition 2.4

A $\mathcal{K}$-bi-invariant function is necessarily even. ♡

Proof. By the polar decomposition (2.2.1.6), $g = k_1 b k_2$, where $b \in \mathcal{A}$, so if φ is $\mathcal{K}$-bi-invariant, then

$$\varphi(g) = \varphi(b) = \varphi(wbw^{-1}) = \varphi(b^w) = \varphi(b^{-1})$$

because of $b = wb^{-1}$ and (2.2.3.3). Thus the restriction map from $\mathcal{G}$ to $\mathcal{A}$ induces bijections

$$\mathbf{Fu}(\mathcal{K} \setminus \mathcal{G}/\mathcal{K}) \rightleftarrows \mathbf{Fu}_{\text{even}}(\mathcal{A}) \quad \text{and} \quad C(\mathcal{K} \setminus \mathcal{G}/\mathcal{K}) \rightleftarrows C_{\text{even}}(\mathcal{A}). \tag{2.2.3.4}$$

Since the Iwasawa coordinate y gives an isomorphism of $\mathcal{A}$ with $\mathbb{R}^+$, the function $\nu := \ln y$ gives an isomorphism of $\mathcal{A}$ with $\mathbb{R}$. We obtain a linear isomorphism

$$\mathbf{Fu}(\mathcal{K} \setminus \mathcal{G}/\mathcal{K}) \rightleftarrows \mathbf{Fu}_{\text{even}}(\mathbb{R}). \tag{2.2.3.5}$$

Given a $\mathcal{K}$-bi-invariant function $f \in \mathbf{Fu}(\mathcal{K} \setminus \mathcal{G}/\mathcal{K})$, the corresponding even function on $\mathbb{R}$ will be denoted by f_+, so that we have $f(g) = f(b) = f_+(\nu)$. $\square$

We denote the **conjugation action** of $\mathcal{G}$ on itself by $\mathbf{c}$, so

$$\mathbf{c}(g)g' := gg'g^{-1}. \tag{2.2.3.6}$$

This action induces what we also call the conjugation action on various functors.

Let $\mathfrak{u}$ be the Lie algebra of $\mathcal{U}$, by definition the algebra of strictly upper triangular matrices, so of the form

$$\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}, \quad x \in \mathbb{C}. \tag{2.2.3.7}$$

Then $\mathfrak{u}$ is 1-dimensional as a complex vector space, with $\mathbb{C}$-basis

$$E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{2.2.3.8}$$

For $\mathfrak{u}$ as a vector space over $\mathbb{R}$, a natural basis consists of E_{12} and $\mathbf{i}E_{12}$. We have

$$\mathcal{U} = \mathbf{1} + \mathfrak{n} = \exp \mathfrak{n}, \tag{2.2.3.9}$$

where $\exp$ is the exponential given by the usual power series.

Let $a = \operatorname{diag}(a_1, a_1^{-1}) \in \mathcal{A}$ as above. Then E_{12} and $\mathbf{i}E_{12}$ are eigenvectors for the conjugation action by elements of $\mathcal{A}$. In fact[1],

$$a(E_{12})a^{-1} = a^{\boldsymbol{\alpha}}(E_{12}), \quad a(\mathbf{i}E_{12})a^{-1} = a^{\boldsymbol{\alpha}}(\mathbf{i}E_{12}). \tag{2.2.3.10}$$

Thus $\mathfrak{u} = \mathfrak{u}_{\boldsymbol{\alpha}}$ is an eigenspace, with eigencharacter $\boldsymbol{\alpha}$, which has multiplicity 2 viewing $\mathfrak{u}$ as a vector space of dimension 2 over $\mathbb{R}$. Note that ${}^t\mathfrak{u}$ is a $(-\boldsymbol{\alpha})$-eigenspace. We call $\boldsymbol{\alpha}, -\boldsymbol{\alpha}$ the

[1] For example,

$$a(E_{12})a^{-1} = \begin{bmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_1^{-1} & 0 \\ 0 & a_1 \end{bmatrix} = \begin{bmatrix} 0 & a_1^2 \\ 0 & 0 \end{bmatrix} = a_1^2 E_{12} = a^{\boldsymbol{\alpha}} E_{12}.$$

regular characters. The trace of the representation on $\mathfrak{u}$ is $2\boldsymbol{\alpha}$, and the half-trace is $\boldsymbol{\rho} := \boldsymbol{\alpha} = \boldsymbol{\rho}_{\mathcal{G}}$ (this is notation that becomes significant in the higher-dimensional case). More generally, for $z \in \mathbf{D}$,

$$zu(x)z^{-1} = \begin{bmatrix} 1 & z_1^2 x \\ 0 & 1 \end{bmatrix} = u(z^{\boldsymbol{\alpha}} x). \tag{2.2.3.11}$$

We let $\boldsymbol{\delta} = \boldsymbol{\delta}_{\mathcal{G}}$ be the **Iwasawa character** on $\mathcal{A}$, namely

$$\boldsymbol{\delta}_{\mathcal{G}}(a) := a^{2\boldsymbol{\alpha}} = \boldsymbol{y}_a^2 = a_1^4. \tag{2.2.3.12}$$

The exponent is the trace of the representation, viewed as being on a 2-dimensional $\mathbb{R}$-space. The absolute Jacobian of conjugation on $\mathfrak{u}$ viewed as an $\mathbb{R}$-space is

$$|\boldsymbol{\delta}(z)| = a_z^{2\boldsymbol{\alpha}}. \tag{2.2.3.13}$$

2.2.4 Group Fubini theorem

In the concrete application to $\mathcal{G} = \mathbf{SL}_2(\mathbb{C})$ and its Iwasawa decomposition $\mathcal{G} = \mathcal{U}\mathcal{A}\mathcal{K} = \mathcal{A}\mathcal{U}\mathcal{K}$, we have

$$\mathcal{P} = \mathcal{G}/\mathcal{K} = \mathcal{U}\mathcal{A} = \mathcal{A}\mathcal{U}, \tag{2.2.4.1}$$

and $\mathcal{U}$ is normal in $\mathcal{P}$. All four of $\mathcal{G}, \mathcal{K}, \mathcal{U}, \mathcal{A}$ are unimodular, so all the hypotheses in the previous propositions are satisfied. The character $a \mapsto \boldsymbol{\delta}(a)$ in **Proposition 2.2** is the Iwasawa character, $\boldsymbol{\delta}(a) = a^{2\boldsymbol{\alpha}}$. We normalize the Haar measure by taking $c = 1$. Thus we define the **Iwasawa measure** to be the Haar measure such that for $f \in C_c(\mathcal{G})$ we have

$$\int_{\mathcal{G}} f(g)\, dg := \int_{\mathcal{K}} \int_{\mathcal{U}} \int_{\mathcal{A}} f(auk)\, dadudk = \int_{\mathcal{K}} \int_{\mathcal{A}} \int_{\mathcal{U}} f(uak) a^{-2\boldsymbol{\alpha}} dudadk, \tag{2.2.4.2}$$

where

$$\begin{aligned} da &:= \frac{dy}{y} \quad \text{with } y = a^{\boldsymbol{\alpha}}, \\ du &:= \text{Euclidean measure } dx = dx^1 dx^2 \quad \text{if } x = x^1 + \mathbf{i}x^2 \text{ with } x^1, x^2 \in \mathbb{R}, \\ dk &:= \text{Haar measure on } \mathcal{K} \text{ giving } \mathcal{K} \text{ total measure } 1. \end{aligned}$$

Unless otherwise specified, all integral formulas are with respect to the Iwasawa measure.

The Iwasawa measure is canonically determined by the Iwasawa decomposition. We may use the notation

$$dg \equiv d_{\mathbf{IW}} g \equiv d\mu_{\mathbf{IW}}(g) \tag{2.2.4.3}$$

for the Iwasawa measure, when comparing it with other measures. Thus the basic Iwasawa measure integration formula can be written

$$\int_{\mathcal{G}} f(g)\, d\mu_{\mathbf{IW}}(g) = \int_{\mathcal{K}} \int_{\mathcal{A}} \int_{\mathcal{U}} f(uak) a^{-2\boldsymbol{\alpha}} dudadk, \quad (\mathbf{INT1}). \tag{2.2.4.4}$$

The above normalization of the Haar measure fits perfectly the normalization that comes from the quaternionic model for $\mathcal{G}/\mathcal{K} \cong \mathbb{H}^3$, i.e., the set of quaternions $z = x^1 + \mathbf{i}x^2 + \mathbf{j}y$, and the standard normalization of the measure on this homogeneous space for the action of $\mathbf{SL}_2(\mathbb{C})$ given by $(az+b)/(cz+d)$ with complex a, b, c, d. This measure is $dx^1 dx^2 dy/y^3$.

For z diagonal regular in $\mathcal{G}$, and $u \in \mathcal{U}$, the **commutator** is $[z, u] = zu(uz)^{-1} = zuz^{-1}u^{-1}$. Define

$$J_{\mathcal{G},\mathbf{com}}(z) \equiv J_{\mathbf{com}}(z) := |1 - z^{\boldsymbol{\alpha}}|^{m(\boldsymbol{\alpha})} \quad \text{with } m(\boldsymbol{\alpha}) = 2. \tag{2.2.4.5}$$

The **exponent** is the multiplicity $m(\boldsymbol{\alpha})$ of $\boldsymbol{\alpha}$. Then

$$\int_{\mathcal{U}} f(z^{-1}uzu^{-1})\,du = \frac{1}{J_{\mathbf{com}}(z^{-1})} \int_{\mathcal{U}} f(u)\,du, \quad (\mathbf{INT2}). \tag{2.2.4.6}$$

Indeed, matrix multiplication shows that

$$[z^{-1}, u] = z^{-1}uzu^{-1} = \begin{bmatrix} z_1^{-1} & 0 \\ 0 & z_1 \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 & 0 \\ 0 & z_1^{-1} \end{bmatrix} \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & (z_1^{-2} - 1)x \\ 0 & 1 \end{bmatrix}$$

whence

$$\begin{aligned} \int_{\mathcal{U}} f(z^{-1}uzu^{-1})\,du &= \int_{\mathbb{C}} f[u((z^{-\boldsymbol{\alpha}} - 1)x)]\frac{\mathbf{i}}{2} dx \wedge d\bar{x} \\ = \left[(z^{-\boldsymbol{\alpha}} - 1)\overline{(z^{-\boldsymbol{\alpha}} - 1)}\right]^{-1} \int_{\mathbb{C}} f(u(x))\frac{\mathbf{i}}{2} dx \wedge d\bar{x} &= \frac{1}{J_{\mathbf{com}}(z^{-1})} \int_{\mathcal{U}} f(u)\,du. \end{aligned}$$

From the product decomposition $\mathbf{D} = \mathcal{A}\mathbf{T}$, we give the compact group $\mathbf{T}$ the total Haar measure 1. Then we give $\mathbf{D}$ the product measure of $da = dy/y$ and this normalized measure on $\mathbf{T}$, which we write dw. Then $\mathbf{D} \setminus \mathcal{G}$ has the **homogeneous space measure**, which makes the **group Fubini theorem** valid, that is, $\mu_{\mathbf{D}\setminus\mathcal{G}}$ such that

$$\int_{\mathcal{G}} f(g)\,d\mu_{\mathbf{IW}}(g) = \int_{\mathbf{D}\setminus\mathcal{G}} \int_{\mathbf{D}} f(w\dot{g})\,dw\,d\mu_{\mathbf{D}\setminus\mathcal{G}}(\dot{g}). \tag{2.2.4.7}$$

Under the correspondence $\mathbf{D} \setminus \mathcal{G} \leftrightarrow \mathcal{U}(\mathbf{T} \setminus \mathcal{K})$, for right $\mathcal{K}$-invariant functions, we have with respect to a decomposition $g = wuk$ ($w \in \mathbf{D}$, $u \in \mathcal{U}$, $k \in \mathcal{K}$),

$$d\mu_{\mathbf{D}\setminus\mathcal{G}}(g) = du. \tag{2.2.4.8}$$

Theorem 2.4

Let $\varphi \in C_c(\mathcal{K} \setminus \mathcal{G}/\mathcal{K})$ and let z be a regular diagonal element. Then

$$g \mapsto \varphi(g^{-1}zg) \tag{2.2.4.9}$$

has compact support on $\mathbf{D} \setminus \mathcal{G}$, and we have

$$\int_{\mathbf{D}\setminus\mathcal{G}} \varphi(g^{-1}zg)\,d\mu_{\mathbf{D}\setminus\mathcal{G}}(g) = \frac{1}{J_{\mathbf{com}}(z^{-1})} \int_{\mathcal{U}} \varphi(a_z u)\,du. \tag{2.2.4.10}$$

♡

Proof. For (2.2.4.10), using $g^{-1}zg = k^{-1}(u^{-1}w^{-1}zwu)k$, one has

$$\begin{aligned} \int_{\mathbf{D}\setminus\mathcal{G}} \varphi(g^{-1}zg)\,dg &= \int_{\mathcal{U}} \varphi(u^{-1}zu)\,du = \int_{\mathcal{U}} \varphi(zz^{-1}u^{-1}zu)\,du \\ = \int_{\mathcal{U}} (\varphi \circ z)(z^{-1}u^{-1}zu)\,du &= \int_{\mathcal{U}} (\varphi \circ z)(z^{-1}uzu^{-1})\,du = \frac{1}{J_{\mathbf{com}}(z^{-1})} \int_{\mathcal{U}} \varphi(zu)\,du \end{aligned}$$

by (2.2.4.6). Since, writing $z_1 = |z_1|\epsilon$,

$$\begin{bmatrix} \epsilon^{-1} & 0 \\ 0 & \epsilon \end{bmatrix} \begin{bmatrix} z_1 & z_1 x \\ 0 & z_1^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} |z_1| & |z_1|x \\ 0 & |z_1|^{-1} \end{bmatrix} = a_z u,$$

we see that $\varphi(zu) = \varphi(a_z u)$. □

2.3 Harish, Mellin and spherical transforms

Introduction

- *The Harish transform and the orbital integral*
- *The Mellin and spherical transforms*
- *Computation of the orbital integral*
- *Gaussians on $\mathcal{G}$ and their spherical transform*
- *The polar Haar meaure and inversion*
- *Point-pair invariants, the polar heights, and the polar distance*

The Iwasawa decomposition leads into the Harish transform and Mellin transform, whose composite is the spherical transform, of Harish-Chandra, relating analytic (Fourier-type) expansion on $\mathcal{G}/\mathcal{K}$ to those on $\mathcal{A}$.

2.3.1 The Harish transform and the orbital integral

We factorize $J_{\mathbf{com}}(z)$ for diagonal z as follows. For z as given by (2.2.2.4), we have $z^{\boldsymbol{\alpha}/2} = z_1$ and $z^{-\boldsymbol{\alpha}/2} = z_1^{-1}$, so that $|z^{\boldsymbol{\alpha}/2} - z^{-\boldsymbol{\alpha}/2}| = |z_1 - z_1^{-1}| = |z_1^{-1} - z_1| = |(z^{-1})^{\boldsymbol{\alpha}/2} - (z^{-1})^{-\boldsymbol{\alpha}/2}|$. Put

$$|\mathbf{D}|(z^{-1}) = |\mathbf{D}|(z) := \left| z^{\frac{\boldsymbol{\alpha}}{2}} - z^{-\frac{\boldsymbol{\alpha}}{2}} \right|^{m(\boldsymbol{\alpha})} = \left| z^{\frac{\boldsymbol{\alpha}}{2}} - z^{-\frac{\boldsymbol{\alpha}}{2}} \right|^2 = \left| z_1 - z_1^{-1} \right|^2. \tag{2.3.1.1}$$

Then

$$J_{\mathbf{com}}(z^{-1}) = |\mathbf{D}|(z)\boldsymbol{\delta}(a_z)^{-1/2} = |\mathbf{D}|(z) z^{-\boldsymbol{\alpha}}. \tag{2.3.1.2}$$

Note that we can take $z^{\boldsymbol{\alpha}/2} = z_1$ and $z^{-\boldsymbol{\alpha}/2} = z_1^{-1}$. We define the **orbital integral** by

$$\mathbf{Orb}_{\mathbf{D}\backslash\mathcal{G}}(\varphi, z) := \int_{\mathbf{D}\backslash\mathcal{G}} \varphi(g^{-1} z g)\, d\mu_{\mathbf{D}\backslash\mathcal{G}}(g) \tag{2.3.1.3}$$

with $\mu_{\mathbf{D}\backslash\mathcal{G}}$ as in (2.2.4.8). We define the **Harish transform** of $\varphi \in C_c(\mathcal{K} \setminus \mathcal{G}/\mathcal{K})$ by

$$(\mathbf{H}\varphi)(z) := \boldsymbol{\delta}(a_z)^{1/2} \int_{\mathcal{U}} \varphi(a_z u)\, du = |\mathbf{D}|(z) \mathbf{Orb}_{\mathbf{D}\backslash\mathcal{G}}(\varphi, z). \tag{2.3.1.4}$$

The equality on the right comes from **Theorem 2.4** and (2.3.1.2). The first formula is applicable to all diagonal z, but the second, with the orbital integral, is applicable only to regular z. As usual in measure-integration theory, the equality extends to functions for which the integrals are absolutely convergent, by continuity. This will be important for us, and certain convergence propertied will be dealt with at the appropriate time and place.

Theorem 2.5

For $\varphi \in C_c(\mathcal{K} \setminus \mathcal{G}/\mathcal{K})$, the Harish transform $\mathbf{H}\varphi$ is invariant under the Weyl group, i.e.,

$$(\mathbf{H}\varphi)(a) = (\mathbf{H}\varphi)(a^{-1}). \tag{2.3.1.5}$$

Proof. $\mathbf{H}\varphi$ is invariant if and only if $(\mathbf{H}\varphi)(wzw^{-1}) = (\mathbf{H}\varphi)(z)$, or equivalently, if and only if $(\mathbf{H}\varphi)(z^{-1}) = (\mathbf{H}\varphi)(z)$ by (2.2.3.3). Using (2.2.2.6) and $\mathbf{T} \subseteq \mathcal{K}$ and the fact that φ is $\mathcal{K}$-bi-invariant, it suffices to verify that $(\mathbf{H}\varphi)(a) = (\mathbf{H}\varphi)(a^{-1})$.

By continuity, it suffices to prove the assertion when a is regular, so $|\mathbf{D}|(a) = |\mathbf{D}|(a^{-1})$. From the fact that $\mathcal{K}$-bi-invariant implies that φ is even, we get

$$\begin{aligned}(\mathbf{H}\varphi)(a^{-1}) &= |\mathbf{D}|(a^{-1})\mathbf{Orb}_{\mathbf{D}\backslash\mathcal{G}}(\varphi, a^{-1}) = |\mathbf{D}|(a)\int_{\mathbf{D}\backslash\mathcal{G}} \varphi(g^{-1}a^{-1}g)\, d\mu_{\mathbf{D}\backslash\mathcal{G}}(g)\\ &= |\mathbf{D}|(a)\int_{\mathbf{D}\backslash\mathcal{G}} \varphi((g^{-1}a^{-1}g)^{-1})\, d\mu_{\mathbf{D}\backslash\mathcal{G}}(g) = (\mathbf{H}\varphi)(a).\end{aligned}$$

One can argue directly on the other integral defining $\mathbf{H}$, using the Jacobian for $u \mapsto a^{-1}ua$. □

The Harish transform is therefore a linear map

$$\mathbf{H} : C_c(\mathcal{K} \setminus \mathcal{G}/\mathcal{K}) \longrightarrow C_c(\mathcal{A})^{\mathcal{W}}, \tag{2.3.1.6}$$

where the upper index $\mathcal{W}$ means the space of functions invariant under $\mathcal{W}$.

We consider next the behavior under products, notably the **convolution product** on $\mathcal{G}$, defined for a specified Haar measure by

$$(h * f)(z) := \int_{\mathcal{G}} h(zg^{-1})f(g)\, d\mu_{\mathbf{IW}} g \tag{2.3.1.7}$$

on a class of pairs of functions for which the integral is absolutely convergent. For non-commutative $\mathcal{G}$, the $\mathcal{K}$-bi-invariance (and the evenness) allows a choice of writing variables on one side or another, with inverses or not, namely for ψ, f, $\mathcal{K}$-bi-invariant such that for all z the function $z \mapsto \psi(zg)f(g)$ is in L^1, the convolution $(\psi * f)(z)$ is defined by

$$\begin{aligned}(\psi * f)(z) &:= \int_{\mathcal{G}} \psi(zg)f(g)\, d\mu_{\mathbf{IW}}(g) = \int_{\mathcal{G}} \psi(zg^{-1})f(g^{-1})\, d\mu_{\mathbf{IW}}(g)\\ &= \int_{\mathcal{G}} \psi(zg^{-1})f(g)\, d\mu_{\mathbf{IW}}(g) = \int_{\mathcal{G}} \psi(g^{-1})f(gz)\, d\mu_{\mathbf{IW}}(g) \quad (2.3.1.8)\\ &= \int_{\mathcal{G}} \psi(g)f(gz)\, d\mu_{\mathbf{IW}}(g).\end{aligned}$$

Theorem 2.6. (Gelfand)

Let $\mathcal{G}$ be a locally compact unimodular group, and let $\mathcal{K}$ be a compact subgroup. Let τ be an anti-automorphism of $\mathcal{G}$ of order 2 such that given $g \in \mathcal{G}$, there exist $k_1, k_2 \in \mathcal{K}$ satisfying ${}^{\tau}g = k_1 g k_2$. Then the convolution is commutative (whenever the convolution integral is absolutely convergent).

Proof. Since ψ, f are $\mathcal{K}$-bi-invariant and $\mathcal{G}$ is unimodular, it follows that ψ, f are τ-invariant and so is the Haar measure. Therefore

$$\begin{aligned}(\psi * f)(z) &= \int_{\mathcal{G}} \psi(zg^{-1})f(g)\,dg = \int_{\mathcal{G}} \psi({}^{\tau}z^{\tau}g^{-1})f({}^{\tau}g)\,dg \\ &= \int_{\mathcal{G}} \psi({}^{\tau}g^{-1})f({}^{\tau}z^{\tau}g)\,dg \quad (\text{by } g \mapsto gz) = \int_{\mathcal{G}} \psi(g^{-1})f(k_1 z k_2{}^{\tau}g)\,dg \\ &= \int_{\mathcal{G}} \psi({}^{\tau}g^{-1})f(k_1 z k_2 g)\,dg = \int_{\mathcal{G}} \psi(g^{-1}k_2)f(k_1 z g)\,dg \\ &= \int_{\mathcal{G}} f(zgk_2^{-1})\psi(k_2 g^{-1})\,dg = \int_{\mathcal{G}} f(z(k_2g^{-1})^{-1})\psi(k_2g^{-1})\,dg.\end{aligned}$$

Thus $\psi * f = f * \psi$. □

Theorem 2.7

For $\varphi, \psi \in C_c(\mathcal{K} \setminus \mathcal{G}/\mathcal{K})$, we have

$$\mathbf{H}(\varphi * \psi) = \mathbf{H}\varphi * \mathbf{H}\psi, \tag{2.3.1.9}$$

i.e., on $C_c(\mathcal{K}\backslash\mathcal{G}/\mathcal{K})$ the Harish transform is an algebra homomorphism for the convolution product. ♡

Proof. By continuity, we may assume that $z = a \in \mathcal{A}$. Then, in this case, $a^{\boldsymbol{\alpha}} = \boldsymbol{\delta}(a)^{1/2}$ and $a_z = a_a = a$. Hence

$$\begin{aligned}(\mathbf{H}(\varphi * \psi))(a) &= a^{\boldsymbol{\alpha}} \int_{\mathcal{U}} (\varphi * \psi)(au)\,du = a^{\boldsymbol{\alpha}} \int_{\mathcal{U}} \int_{\mathcal{G}} \varphi(aug)\psi(g^{-1})\,dgdu \\ &= a^{\boldsymbol{\alpha}} \int_{\mathcal{U}} \int_{\mathcal{G}} \varphi(ag)\psi(g^{-1}u)\,dgdu.\end{aligned}$$

Let $g = b\nu k$ be the Iwasawa decomposition with $b \in \mathcal{A}, \nu \in \mathcal{U}$, and $k \in \mathcal{K}$. Then, using (2.2.4.2), we get

$$\begin{aligned}(\mathbf{H}(\varphi * \psi))(a) &= a^{\boldsymbol{\alpha}} \int_{\mathcal{U}} \int_{\mathcal{K}} \int_{\mathcal{U}} \int_{\mathcal{A}} \varphi(ab\nu k)\psi(k^{-1}\nu^{-1}b^{-1}u)\,dbd\nu dkdu \\ &= a^{\boldsymbol{\alpha}} \int_{\mathcal{U}} \int_{\mathcal{K}} \int_{\mathcal{U}} \int_{\mathcal{A}} \varphi(ab\nu)\psi(\nu^{-1}b^{-1}u)\,dbd\nu dkdu \\ &= a^{\boldsymbol{\alpha}} \int_{\mathcal{U}} \int_{\mathcal{U}} \int_{\mathcal{A}} \varphi(ab\nu)\psi(\nu^{-1}b^{-1}u)\,dbd\nu du \\ &= a^{\boldsymbol{\alpha}} \int_{\mathcal{U}} \int_{\mathcal{U}} \int_{\mathcal{A}} \varphi(ab^{-1}\nu)\psi(\nu^{-1}bu)\,dbdud\nu \\ &= a^{\boldsymbol{\alpha}} \int_{\mathcal{U}} \int_{\mathcal{U}} \int_{\mathcal{A}} \varphi(ab^{-1}\nu)\psi(\nu^{-1}ub)\delta(b)^{-1}dbd\nu du \\ &= a^{\boldsymbol{\alpha}} \int_{\mathcal{U}} \int_{\mathcal{U}} \int_{\mathcal{A}} \varphi(ab^{-1}\nu)\psi(ub)\delta(b)^{-1}dbd\nu du \\ &= a^{\boldsymbol{\alpha}} \int_{\mathcal{U}} \int_{\mathcal{U}} \int_{\mathcal{A}} \varphi(ab^{-1}\nu)\psi(bu)\,dbd\nu du\end{aligned}$$

$$= \quad (ab^{-1})^{\boldsymbol{\alpha}} b^{\boldsymbol{\alpha}} \int_{\mathcal{U}} \int_{\mathcal{U}} \int_{\mathcal{A}} \varphi(ab^{-1}\nu)\psi(bu)\, db d\nu du$$

which is exactly the convolution $(\mathbf{H}\varphi * \mathbf{H}\psi)(a)$. □

2.3.2 The Mellin and spherical transforms

Let $\boldsymbol{\chi}$ be a character on $\mathcal{A}$ (continuous homomorphism into $\mathbb{C}^\times$). Let $F \in C_c(\mathcal{A})$. We define the **Mellin transform** of F to be the function on the group $\mathbf{ch}(\mathcal{A})$ of characters given by

$$(\mathbf{M}F)(\boldsymbol{\chi}) := \int_{\mathcal{A}} F(a)\boldsymbol{\chi}(a)\, da. \tag{2.3.2.1}$$

We use the same Haar measure da on $\mathcal{A}$ as before, with the coordinate character $\boldsymbol{y}$. Viewing $\boldsymbol{y}$ as a basis for the characters, we can write a character in the form

$$\boldsymbol{\chi}(a) := \boldsymbol{y}_a^s = a^{s\boldsymbol{\alpha}} \tag{2.3.2.2}$$

with a complex variable s. Then the Mellin transform is written with $F_{\boldsymbol{\alpha}}(y) = F(a)$:

$$(\mathbf{M}_{\boldsymbol{\alpha}}F)(s) \equiv (\mathbf{M}F_{\boldsymbol{\alpha}})(s) = \int_0^\infty F_{\boldsymbol{\alpha}}(y) y^s \frac{dy}{y}. \tag{2.3.2.3}$$

Here $y := \boldsymbol{y}_a \in [0, \infty)$. This is the orbidary Mellin transform of elementary analysis. If F does not have compact support, the integral may converge only in a restricted domain of s, usually some half-plane where the transform is analytic.

If F is even ($F(a) = F(a^{-1})$), then $\mathbf{M}F$ is also even. In other words,

$$(\mathbf{M}F)(\boldsymbol{\chi}) = (\mathbf{M}F)(\chi^{-1}). \tag{2.3.2.4}$$

For $g \in \mathcal{G}$ let $g_{\mathcal{A}}$ be its Iwasawa projection on $\mathcal{A}$. Then the function $g \mapsto \chi(g_{\mathcal{A}})$, also written $\chi(g)$, is well-defined, and is called the **Iwasawa lifting** of $\chi \in \mathbf{ch}(\mathcal{A})$ to $\mathcal{G}$. As before, let $\boldsymbol{\delta} = \boldsymbol{\delta}_{\mathcal{G}}$ be the Iwasawa character, $\boldsymbol{\delta}(a) = a^{2\boldsymbol{\alpha}}$. We define the **spherical kernel** on the product of $\mathcal{G}$ with the character group $\mathbf{ch}(\mathcal{A})$ to be the function given by

$$\Phi_\chi(g) \equiv \Phi(\chi, g) := \int_{\mathcal{K}} (\boldsymbol{\chi}\boldsymbol{\delta}^{1/2})(kg)\, dk = \int_{\mathcal{K}} (\boldsymbol{\chi}\boldsymbol{\delta}^{1/2})((kg)_{\mathcal{A}}) dk. \tag{2.3.2.5}$$

Let $\mathfrak{a} := \mathrm{Lie}(\mathcal{A})$ be the real vector space of diagonal 2×2 matrices of trace 0,

$$H = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} \quad \text{with } h_2 = -h_1. \tag{2.3.2.6}$$

Let $\boldsymbol{\zeta}$ denote a complex (additive) character of $\mathcal{A}$. Then $\boldsymbol{\chi}$ may be written as $\boldsymbol{\chi}_{\boldsymbol{\zeta}}$, with $\boldsymbol{\zeta}$ such that if $a = \exp H$, then

$$\boldsymbol{\chi}_{\boldsymbol{\zeta}}(a) := e^{\boldsymbol{\zeta}(H)} = e^{\boldsymbol{\zeta}(\ln a)}. \tag{2.3.2.7}$$

Thus the Mellin transform can also be viewed as defined on the complex dual $(\mathfrak{a}_{\mathbf{C}})^\vee$, and the spherical kernel may be written in terms of the additive variable $\boldsymbol{\xi}$, in the form

$$\Phi_{\boldsymbol{\zeta}}(g) \equiv \Phi(\boldsymbol{\zeta}, g) = \Phi(\boldsymbol{\chi}_{\boldsymbol{\zeta}}, g), \quad \boldsymbol{\zeta} \in (\mathfrak{a}_{\mathbb{C}})^\vee. \tag{2.3.2.8}$$

We let $\mathbf{S} \equiv \mathbf{S}_{\mathbf{IW}}$ be the **integral (spherical) transform** defined by this kernel,

$$(\mathbf{S}f)(\boldsymbol{\chi}) := \int_{\mathcal{G}} \Phi(\boldsymbol{\chi}, g) f(g)\, dg. \tag{2.3.2.9}$$

Theorem 2.8

Let f be measurable and $\mathcal{K}$-bi-invariant on $\mathcal{G}$. Also suppose that the repeated integral

$$\int_{\mathcal{G}}\int_{\mathcal{K}}(|\boldsymbol{\chi}|\boldsymbol{\delta}^{1/2})((kg)_{\mathcal{A}})|f(g)|dkdg$$

is absolutely convergent. Then

$$\mathbf{S}f=\mathbf{M}(\mathbf{H}f), \tag{2.3.2.10}$$

that is, $\mathbf{S}=\mathbf{M}\circ\mathbf{H}$ on $C_c(\mathcal{K}\setminus\mathcal{G}/\mathcal{K})$.

Proof. From (2.3.1.6), consider the following diagram

$$\begin{tikzcd} C_c(\mathcal{K}\setminus\mathcal{G}/\mathcal{K}) \arrow[r, "\mathbf{H}"] \arrow[dr, "\mathbf{S}"'] & C_c(\mathcal{A})^{\mathcal{W}}\subseteq C_c(\mathcal{A}) \arrow[d, "\mathbf{M}"] \\ & \mathbf{Fu}(\mathbf{ch}(\mathcal{A})) \end{tikzcd}$$

By Fubini's theorem and $\mathcal{K}$-invariance of f and dg, we have

$$\begin{aligned}(\mathbf{S}f)(\boldsymbol{\chi}) &= \int_{\mathcal{G}}\Phi(\boldsymbol{\chi},g)f(g)\,dg = \int_{\mathcal{K}}\int_{\mathcal{G}}(\boldsymbol{\chi}\boldsymbol{\delta}^{1/2})((kg)_{\mathcal{A}})f(g)\,dgdk\\ &= \int_{\mathcal{K}}\int_{\mathcal{G}}(\boldsymbol{\chi}\boldsymbol{\delta}^{1/2})(g_{\mathcal{A}})f(g)\,dgdk = \int_{\mathcal{A}}\int_{\mathcal{U}}\boldsymbol{\chi}(a)\boldsymbol{\delta}^{1/2}(a)f(au)\,dadu\\ &= \int_{\mathcal{A}}\boldsymbol{\chi}(a)(\mathbf{H}f)(a)\,da = \mathbf{M}(\mathbf{H}f)(\boldsymbol{\chi}).\end{aligned}$$

Thus $\mathbf{S}f=\mathbf{M}(\mathbf{H}f)$. □

Corollary 2.2

The spherical kernel is invariant under $\mathcal{W}$ in the variable $\boldsymbol{\chi}$, that is,

$$\Phi_{\boldsymbol{\chi}}=\Phi_{\boldsymbol{\chi}^{-1}}. \tag{2.3.2.11}$$

Proof. From (2.3.2.10), we have, for any $f\in C_c(\mathcal{K}\setminus\mathcal{G}/\mathcal{K})$,

$$\begin{aligned}(\mathbf{S}f)(\boldsymbol{\chi}^{-1}) &= \int_{\mathcal{A}}\boldsymbol{\chi}^{-1}(a)(\mathbf{H}f)(a)\,da = \int_{\mathcal{A}}\boldsymbol{\chi}(a^{-1})(\mathbf{H}f)(a)\,da\\ &= \int_{\mathcal{A}}\boldsymbol{\chi}(a)(\mathbf{H}f)(a^{-1})\,da = \int_{\mathcal{A}}\boldsymbol{\chi}(a)(\mathbf{H}f)(a)\,da = (\mathbf{S}f)(\boldsymbol{\chi}).\end{aligned}$$

Therefore $\Phi_{\boldsymbol{\chi}}=\Phi_{\boldsymbol{\chi}^{-1}}$. □

Theorem 2.9

The Mellin transform and the spherical transform are multiplicative homomorphisms, that is, for $\varphi,\psi\in C_c(\mathcal{K}\setminus\mathcal{G}/\mathcal{K})$, and $f,h\in C_c(\mathcal{A})$, we have

$$\mathbf{S}(\varphi*\psi)=\mathbf{S}(\varphi)\mathbf{S}(\psi),\quad \mathbf{M}(f*g)=\mathbf{M}(f)\mathbf{M}(h). \tag{2.3.2.12}$$

Proof. By definitions,

$$\mathbf{M}(f*h)(\boldsymbol{\chi}) = \int_{\mathcal{A}}(g*h)(a)\boldsymbol{\chi}(a)\,da = \int_{\mathcal{A}}\int_{\mathcal{A}}f(ab^{-1})h(b)\boldsymbol{\chi}(a)\,dbda.$$

$$= \int_{\mathcal{A}}\int_{\mathcal{A}} f(a)h(b)\boldsymbol{\chi}(a)\boldsymbol{\chi}(b)\,dbda \;\; = \;\; (\mathbf{M}f)(\boldsymbol{\chi})(\mathbf{M}h)(\boldsymbol{\chi}).$$

Combining this with (2.3.1.9) and (2.3.2.10) gets the result for $\mathbf{S}$. □

If $\boldsymbol{\chi} = \boldsymbol{\chi}_{s\boldsymbol{\alpha}}$ (that is, $\boldsymbol{\chi}(a) = \boldsymbol{\chi}_{s\boldsymbol{\alpha}}(a) = a_1^{2s}$), we may write

$$(\mathbf{S}_{\boldsymbol{\alpha}}f)(s) := (\mathbf{S}f)(\boldsymbol{\chi}_{s\boldsymbol{\alpha}}). \tag{2.3.2.13}$$

If $\boldsymbol{\chi} = \boldsymbol{\chi}_{\boldsymbol{\zeta}}$ with $\boldsymbol{\zeta} \in (\mathfrak{a}_{\mathbb{C}})^{\vee}$, we may write

$$(\mathbf{S}f)(\boldsymbol{\zeta}) := (\mathbf{S}f)(\boldsymbol{\chi}_{\boldsymbol{\zeta}}). \tag{2.3.2.14}$$

2.3.3 Computation of the orbital integral

Let $z \in \mathcal{G}$ be regular, that is, z is conjugate to a diagonal matrix with distinct diagonal elements (the eigenvalues)

$$z \sim \tilde{\eta} = \begin{bmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{bmatrix} \quad \text{with } 0 \neq \eta \in \mathbb{C}. \tag{2.3.3.1}$$

Without loss of generality, we may suppose that $|\eta| \geq 1$. If $|\eta| > 1$, then η is uniquely determined among the two eigenvalues η, η^{-1} of z. From the conjugation action on matrices, one sees that the centralizer in $\mathcal{G}$ of a regular diagonal element is the set of diagonal matrices. In particular, the isotropy group of $\tilde{\eta}$ under the conjugation action is the diagonal group $\mathbf{D}$. Hence the isotropy group $\mathcal{G}_z$ is conjugate to $\mathbf{D}$ by the same conjugation that diagonalizes z to $\tilde{\eta}$. We write $\eta \equiv \eta(z)$ and $\tilde{\eta} \equiv \tilde{\eta}(z)$.

Since conjugation preserves Haar measure, we have

$$\int_{\mathcal{G}_z\backslash\mathcal{G}} \varphi(g^{-1}zg)\,dg = \int_{\mathbf{D}\backslash\mathcal{G}} \varphi(g^{-1}\tilde{\eta}g)\,dg. \tag{2.3.3.2}$$

Thus the Harish transform can be computed for z in diagonal form, and we use indiscriminately identity

$$(\mathbf{H}\varphi)(z) = |\mathbf{D}|(\tilde{\eta}(z))\mathbf{Orb}_{\mathcal{G}_z\backslash\mathcal{G}}(\varphi, z) = |\mathbf{D}|(\eta)\mathbf{Orb}_{\mathbf{D}\backslash\mathcal{G}}(\varphi, \tilde{\eta}). \tag{2.3.3.3}$$

We now go on with a theorem that allows for a more explicit determination of the orbital integral, and therefore of the Harish transform. As in **Section 2.3.2**, we normalize Haar measure so that $\mathbf{T}$ as well as $\mathcal{K}$ has measure 1. On $\mathbf{D} = \mathcal{A}\mathbf{T}$ we put the product measure of $da = dy/y$ on $\mathcal{A}$ and this normalized measure on $\mathbf{T}$, which we write dw (w variable in $\mathbf{D}$). Then for $f \in L^1(\mathcal{G}/\mathcal{K})$, we have

$$\int_{\mathcal{G}} f(g)\,dg = \int_{\mathcal{U}}\int_{\mathbf{D}} f(wu)\,dwdu. \tag{2.3.3.4}$$

The dg on the left is of course the Iwasawa measure. From the modified Iwasawa decomposition

$\mathcal{G} = \mathbf{D}\mathcal{U}\mathcal{K}$, we get

$$\begin{aligned}
\mathbf{Orb}_{\mathbf{D}\backslash\mathcal{G}}(\varphi, \tilde{\eta}) &= \int_{\mathbf{D}\backslash\mathcal{G}} \varphi(g^{-1}\tilde{\eta}g)\, d\mu_{\mathbf{D}\backslash\mathcal{G}}(g) \;=\; \int_{\mathcal{U}\mathcal{K}} g(k^{-1}u^{-1}z^{-1}\tilde{\eta}zuk)\, dudk \\
&= \int_{\mathcal{U}} \varphi(u^{-1}z^{-1}\tilde{\eta}zu)\, du \;=\; \int_{\mathcal{U}} \varphi(u^{-1}\tilde{\eta}u)\, du \qquad (2.3.3.5)\\
&= \int_{\mathbb{C}} \varphi(u(-x)\tilde{\eta}u(x))\, dx \;=\; \int_{\mathbb{C}} \varphi(M_\eta(x))\, dx,
\end{aligned}$$

where

$$M_\eta(x) := u(x)^{-1}\tilde{\eta}u(x) = \begin{bmatrix} \eta & x(\eta - \eta^{-1}) \\ 0 & \eta^{-1} \end{bmatrix} \qquad (2.3.3.6)$$

The function φ is given in terms of polar coordinates, so we have to express the polar $\mathcal{A}$-coordinate of $M_\eta(x)$ in terms of x. We shall deal with the (x, y) coordinates, polar $\mathcal{A}$-coordinates, and ordinary polar coordinates in $\mathbb{C}$.

Theorem 2.10

Let the Haar measure on $\mathcal{G}$ be the Iwasawa measure as in **Subsection 2.2.4**. *Let φ be measurable on $\mathcal{G}$, $\mathcal{K}$-bi-invariant (so even), and such that the orbital integral $\mathbf{Orb}_{\mathbf{D}\backslash\mathcal{G}}(\varphi, \tilde{\eta})$ is absolutely convergent for regular $\tilde{\eta}$. For $b \in \mathcal{A}$, $y = b^{\boldsymbol{\alpha}} \geq 1$, write*

$$\varphi_{\boldsymbol{\alpha}}(y) \equiv \varphi_{\boldsymbol{\alpha}}(y(b)) := \varphi(b). \qquad (2.3.3.7)$$

Then for $|\eta| \geq 1$ and $\tilde{\eta}$ regular, we have

$$\mathbf{Ord}_{\mathbf{D}\backslash\mathcal{G}}(\varphi, \tilde{\eta}) = \frac{2\pi}{|\eta - \eta^{-1}|^2} \int_{|\eta|^2}^{\infty} \varphi_{\boldsymbol{\alpha}}(y) \frac{y - y^{-1}}{2} \frac{dy}{y}. \qquad (2.3.3.8)$$

Or in terms of the additive variable $\nu := \ln y$, using the notation

$$\varphi_{\boldsymbol{\alpha},+}(\nu) \equiv \varphi_{\boldsymbol{\alpha},+}(\ln y) := \varphi(b), \qquad (2.3.3.9)$$

the formula becomes

$$\begin{aligned}
\mathbf{Ord}_{\mathbf{D}\backslash\mathcal{G}}(\varphi, \tilde{\eta}) &= \frac{2\pi}{|\eta - \eta^{-1}|^2} \int_{|\eta|^2}^{\infty} \varphi_{\boldsymbol{\alpha},+}(\ln y) \frac{y - y^{-1}}{2} \frac{dy}{y} \\
&= \frac{2\pi}{|\eta - \eta^{-1}|^2} \int_{2\ln|\eta|}^{\infty} \varphi_{\boldsymbol{\alpha},+}(\nu) \sinh(\nu)\, d\nu. \qquad (2.3.3.10)
\end{aligned}$$

♡

Proof. To go back and forth between the variable x in $M_\eta(x)$ and the $\mathcal{A}$-variable, we use the quadratic map, namely,

$$\text{If } M_\eta(x) = k_1 b k_2 \ (k_1, k_2 \in \mathcal{K},\ b \in \mathcal{A}), \text{ then } M_\eta(x) M_\eta(x)^* = k_1 b^2 k_1^{-1}. \qquad (2.3.3.11)$$

For (2.3.3.6), we have

$$M_\eta(x) M_\eta(x)^* = \begin{bmatrix} |\eta|^2 + |x|^2(\eta - \eta^{-1})^2 & x\bar{\eta}^{-1}(\eta - \eta^{-1}) \\ \bar{x}\eta^{-1}(\bar{\eta} - \bar{\eta}^{-1}) & |\eta|^{-2} \end{bmatrix} \qquad (2.3.3.12)$$

So b^2 can be computed from the eigenvalues of $M_\eta(x)M_\eta(x)^*$, namely, the roots of the characteristic polynomial

$$T^2 - \tau(r)T + 1 = 0,$$

where $r = |x|$, and $\tau(r)$ is the trace of the matrix $M_\eta(x)M_\eta(x)^*$:

$$\tau(r) := |\eta|^2 + |\eta|^{-2} + r^2|\eta - \eta^{-1}|^2. \tag{2.3.3.13}$$

We have $r^2 = x\bar{x}$ and $dx = dx^1dx^2 = rdrd\theta$. With $b = \mathrm{diag}(b_1, b_2)$, $b_1 > 1$, we can solve for b as a function of r (and so as a function of x). Introduce the discriminant of the characteristic polynomial

$$\Delta := \tau(r)^2 - 4.$$

Then the roots of the characteristic polynomial are

$$y \equiv \boldsymbol{y}_b = \boldsymbol{y}(b) = b_1^2 = \frac{\tau(r) + \Delta^{1/2}}{2} \quad \text{and} \quad b_2^2 = \frac{\tau(r) - \Delta^{1/2}}{2}, \tag{2.3.3.14}$$

or also $\tau + \Delta^{1/2} = 2y$. Since $b_2^2 = b_1^{-2}$ we get from (2.3.3.14) the value of Δ in terms of y,

$$\Delta^{1/2} = y - y^{-1}. \tag{2.3.3.15}$$

Note that y is a variable on the positive multiplicative group. In terms of the multiplicative coordinate $y = \boldsymbol{y}(b) = b^{\boldsymbol{\alpha}}$, we write

$$\varphi(b) = \varphi_{\boldsymbol{\alpha}}(y).$$

We change variables in the integral form (2.3.3.5), $dx^1dx^2 = rdrd\theta$, to get

$$\mathbf{Orb}_{\mathbf{D}\backslash\mathcal{G}}(\varphi, \tilde{\eta}) = \int_{\mathbb{C}} \varphi(M_\eta(x))\,dx = 2\pi \int_0^\infty \varphi(b(r))\,rdr. \tag{2.3.3.16}$$

We have to write down the Jacobian between the variables y and r. By (2.3.3.13), we have

$$d\tau = |\eta - \eta^{-1}|^2 2rdr.$$

By (2.3.3.14),

$$dy = \frac{1}{2}\left(d\tau + \frac{2\tau d\tau}{2\Delta^{1/2}}\right) = \frac{\Delta^{1/2} + \tau}{2\Delta^{1/2}}d\tau = \frac{y}{\Delta^{1/2}}d\tau.$$

Hence, using (2.3.3.15),

$$\frac{dy}{y} = \frac{|\eta - \eta^{-1}|^2}{\Delta^{1/2}}2rdr \quad \text{or} \quad rdr = \frac{y - y^{-1}}{2}\frac{1}{|\eta - \eta^{-1}|^2}\frac{dy}{y}. \tag{2.3.3.17}$$

Putting (2.3.3.16) and (2.3.3.17) together yields (2.3.3.8), except determining the limits of integration.

The polar r ranges over $(0, +\infty)$. The expression (2.3.3.13) for $\tau(r)$ shows that if we put $\ell := |\eta|^2 + |\eta|^{-2}$, then

$$\tau \in (\ell, +\infty), \quad \Delta \in (\ell^2 - 4, +\infty), \quad \ell^2 - 4 = (|\eta|^2 - |\eta|^{-2})^2, \quad \ell + \sqrt{\ell^2 - 4} = 2|\eta|^2. \tag{2.3.3.18}$$

From (2.3.3.14) we get

$$y = \frac{\tau + \Delta^{1/2}}{2} \in \left(\frac{\ell + \sqrt{\ell^2 - 4}}{2}, \infty\right) = (|\eta|^2, \infty), \tag{2.3.3.19}$$

determining the limits of integration. The last formula (2.3.3.10) is obvious. □

Corollary 2.3

For $a \in \mathcal{A}^+$ (i.e., $a^{\boldsymbol{\alpha}} > 1$, so a regular), we have

$$(\mathbf{H}\varphi)(a) = 2\pi \int_{a^{\boldsymbol{\alpha}}}^{\infty} \varphi_{\boldsymbol{\alpha}}(y) \frac{y - y^{-1}}{2} \frac{dy}{y}. \tag{2.3.3.20}$$

Proof. It follows from (2.3.1.1), (2.3.3.4), and (2.3.3.9). □

2.3.4 Gaussians on $\mathcal{G}$ and their spherical transform

By a **basic $\mathcal{G}$-Gaussian** (**basic Gaussian** for short) we mean a function on $\mathcal{G}$ that is $\mathcal{K}$-bi-invariant, and with some constant $c > 0$, for all $b \in \mathcal{A}$ has the formula

$$\begin{aligned} \varphi_c(b) &= e^{-(\boldsymbol{\alpha}(\ln b))^2/2c} \frac{\boldsymbol{\alpha}(\ln b)}{\sinh(\boldsymbol{\alpha}(\ln b))} \\ &= e^{-(\ln y)^2/2c} \frac{\ln y}{(y - y^{-1})/2} \quad \text{with } y = y(b), \\ &= e^{-\nu^2/2c} \frac{\nu}{\sin h\nu} \quad \text{with } \nu = \ln y. \end{aligned} \tag{2.3.4.1}$$

We can then deal with the vector space $\mathbf{Gauss}(\mathcal{G})$ generated by the basic $\mathcal{G}$-Gaussian functions as a natural space of test functions.

Theorem 2.11

For the orbital integral, with the basic $\mathcal{G}$-Gaussian $\varphi_c \in \mathbf{Gauss}(\mathcal{G})$, and regular diagonal z, we have the value

$$\mathbf{Orb}_{\mathbf{D}\backslash\mathcal{G}}(\varphi_c, z) = \frac{2\pi c}{|z^{\boldsymbol{\alpha}/2} - z^{-\boldsymbol{\alpha}/2}|^2} e^{-(\ln|z^{\boldsymbol{\alpha}}|)^2/2c}. \tag{2.3.4.2}$$

For $a \in \mathcal{A}$ regular, $y = \boldsymbol{y}(a)$, the Harish transform is given by

$$(\mathbf{H}\varphi_c)(a) = 2\pi c e^{-(\boldsymbol{\alpha}(\ln a))^2/2c} = 2\pi c e^{-(\ln y)^2/2c}. \tag{2.3.4.3}$$

Here we use the y-variable, $\boldsymbol{\alpha}(\ln a) = \ln y$.

Proof. Letting $\tilde{\eta} = z$ (so that $\eta = z^{\boldsymbol{\alpha}/2}$) and $\varphi = \varphi_c$ in (2.3.3.10), we have

$$\begin{aligned} \mathbf{Orb}_{\mathbf{D}\backslash\mathcal{G}}(\varphi_c, z) &= \frac{2\pi c}{|\eta - \eta^{-1}|^2} \int_{2\ln\eta}^{\infty} e^{-\nu^2/2c} d\frac{\nu^2}{2c} \\ &= \frac{2\pi c}{|\eta - \eta^{-1}|^2} e^{-(2\ln\eta)^2/2c} = \frac{2\pi c}{|z^{\boldsymbol{\alpha}/2} - z^{-\boldsymbol{\alpha}/2}|^2} e^{-(\ln|z^{\boldsymbol{\alpha}}|)^2/2c}. \end{aligned}$$

The last formula follows from (2.2.2.2), (2.3.1.4), and (2.3.4.2). □

The factor 2π came originally from having taken the ordinary Euclidean measure on $\mathbb{C} \cong \mathcal{U}$, so polar coordinates for which the circle has measure 2π, which is standard, but is given priority over the counterconvention to given compact groups measure 1.

Theorem 2.12

Write a character on $\mathfrak{a} = \mathrm{Lie}(\mathcal{A})$ in the form $\boldsymbol{\chi} = s\boldsymbol{\alpha}$ with complex s. Then the spherical transform is

$$(\mathbf{S}\varphi_c)(s\boldsymbol{\alpha}) = (\mathbf{M}(\mathbf{H}\varphi_c))(s\boldsymbol{\alpha}) = (2\pi c)^{3/2} e^{s^2 c/2}. \tag{2.3.4.4}$$

On the imaginary axis $s = \mathbf{i}r$, this gives

$$(\mathbf{S}\varphi_c)(\mathbf{i}r\boldsymbol{\alpha}) = (\mathbf{M}(\mathbf{H}\varphi_c))(\mathbf{i}r\boldsymbol{\alpha}) = (2\pi c)^{3/2} e^{-r^2 c/2}. \tag{2.3.4.5}$$

♡

Proof. By **Theorem 2.11** and $\mathbf{S} = \mathbf{M} \circ \mathbf{H}$ (cf., **Theorem 2.8**) we have (cf., equations (2.3.2.3) and (2.3.2.13))

$$\frac{1}{2\pi c}(\mathbf{S}_{\boldsymbol{\alpha}}\varphi_c)(s) = \frac{1}{2\pi c}(\mathbf{S}\varphi_c)(s\boldsymbol{\alpha}) = \int_0^\infty e^{-(\ln y)^2/2c} y^s \frac{dy}{y} = \int_{-\infty}^\infty e^{-\nu^2/2c} e^{\nu s} d\nu.$$

Because of the rapid decay of Gaussians, the Mellin transform is entire in s, so it suffices to determine the integral when $s = \mathbf{i}r$ (r real) is pure imaginary. One then uses the self-duality of the function $e^{-\nu^2/2}$ with respect to $d\nu/\sqrt{2\pi}$, and a change of variables to get rid of c. □

We let the **Gauss space on** $\mathbb{R}$, $\mathbf{Gauss}(\mathbb{R})$, be the vector space generated by the Gaussian functions $r \mapsto e^{-r^2 c/2}$, with all constants $c > 0$. Sometimes, we deal with the imaginary axis $\mathbf{i}\mathbb{R}$, and $\mathbf{Gauss}(\mathbf{i}\mathbb{R})$ is the space generated by the functions having the above value at $\mathbf{i}r$.

Corollary 2.4

The spherical transform gives a linear isomorphism

$$\mathbf{Gauss}(\mathcal{G}) \longrightarrow \mathbf{Gauss}(\mathbb{R}), \quad \varphi_c \longmapsto (\mathbf{S}\varphi_c)(\cdot\boldsymbol{\alpha}). \tag{2.3.4.6}$$

♡

Corollary 2.5

With the Iwasawa measure $d\mu_{\mathbf{IW}}$, the total integral of φ_c is

$$\int_{\mathcal{G}} \varphi_c(g) d\mu_{\mathbf{IW}}(g) = (2\pi c)^{3/2} e^{c/2}. \tag{2.3.4.7}$$

♡

Proof. Taking $s = 1$ in (2.3.4.4) yields

$$(2\pi c)^{3/2} e^{c/2} = (\mathbf{S}\varphi_c)(-\boldsymbol{\alpha}) = \int_{\mathcal{G}} \Phi(-\boldsymbol{\alpha}, g)\varphi_c(g)\, dg$$

by (2.3.2.9). The spherical kernel is given by the integral

$$\Phi(-\boldsymbol{\alpha}, g) = \int_{\mathcal{K}} (-\boldsymbol{\alpha}\boldsymbol{\delta}^{1/2})((kg)_{\mathcal{A}})\, dk = \int_{\mathcal{K}} ((kg)_{\mathcal{A}})^{-\boldsymbol{\alpha}+\boldsymbol{\alpha}} dk = \int_{\mathcal{K}} dk = 1.$$

Thus, we get (2.3.4.7). □

Let $g \in \mathcal{G}$ with polar decomposition

$$g = k_1 b k_2, \quad k_1, k_2 \in \mathcal{K} \text{ and } b \in \mathcal{A}. \tag{2.3.4.8}$$

The element $b \in \mathcal{A}$ is determined only up to a permutation of its diagonal components. We call $y = \boldsymbol{y}(b) = b^{\boldsymbol{\alpha}}$ or y^{-1} a **polar coordinate** (or polar $\mathcal{A}$-coordinate) of g. Note that $(\ln y)^2$

is uniquely determine by g, and is real analytic on $\mathcal{G}$. Define the **polar height** σ to be the $\mathcal{K}$-bi-invariant function such that $\sigma \geq 0$ and

$$\sigma^2(g) = \sigma^2(b) = (\ln y)^2 = \nu^2. \tag{2.3.4.9}$$

Then the basic $\mathcal{G}$-Gaussian φ_c, given in (2.3.4.1), can be written in the form

$$\varphi_c = e^{-\sigma^2/2c} \frac{\sigma}{\sinh \sigma} = e^{-(\ln y)^2/2c} \frac{\ln y}{\sinh(\ln y)}. \tag{2.3.4.10}$$

Note that $\nu/\sinh \nu$ is a power series in $\nu^2 = (\ln y)^2$, so real analytic on $\mathcal{G}$.

For $H \in \mathfrak{a}$ given in (2.3.2.6), we define the symmetric bilinear form $\mathscr{B} : \mathfrak{a} \times \mathfrak{a} \to \mathbb{R}$ by

$$\mathscr{B}(H, H) := \operatorname{tr}(HH) = |H|^2_{\mathscr{B}}. \tag{2.3.4.11}$$

For $b = \operatorname{diag}(b_1, b_2) \in \mathcal{A}$, we have $\ln b = \operatorname{diag}(\ln b_1, \ln b_2) \in \mathfrak{a}$, and

$$|\ln b|^2_{\mathscr{B}} = (\ln b_1)^2 + (\ln b_2)^2 = 2(\ln b_1)^2 = \frac{(\ln y)^2}{2},$$

so

$$\sigma^2(b) = 2|\ln b|^2_{\mathscr{B}} = (\ln y)^2. \tag{2.3.4.12}$$

The trace form on $\mathfrak{a}$ can be extended in two natural ways to the complexification, namely, a *symmetric way* and a *Hermitian way*. The symmetric way will be relevant in the next subsection. The trace form induces a corresponding positive definite scalar product on the dual space $\mathfrak{a}^\vee$, because it induces a natural isomorphism of $\mathfrak{a}$ and $\mathfrak{a}^\vee$. Specifically, if $\lambda \in \mathfrak{a}^\vee$, then there is an element $H_\lambda \in \mathfrak{a}$ such that for all $H \in \mathfrak{a}$,

$$\mathscr{B}(H_\lambda, H) = \lambda(H). \tag{2.3.4.13}$$

Then by definition, for $\lambda, \xi \in \mathfrak{a}^\vee$, we have

$$\mathscr{B}^\vee(\xi, \lambda) := \mathscr{B}(H_\xi, H_\lambda), \tag{2.3.4.14}$$

so for example[2], $\mathscr{B}^\vee(\boldsymbol{\alpha}, \boldsymbol{\alpha}) = 2$.

We can then extend the form from $\mathfrak{a}^\vee$ to a $\mathbb{C}$-bilinear form $\mathscr{B}^\vee_{\mathbb{C}} := (\mathscr{B}^\vee)_{\mathbb{C}} = \mathscr{B}^\vee \otimes_{\mathbb{R}} \mathbb{C}$ on its complexification, i.e., to characters $\boldsymbol{\zeta} \in \mathfrak{a}^\vee_{\mathbb{C}} = (\mathfrak{a}^\vee)_{\mathbb{C}} = \mathfrak{a}^\vee \otimes_{\mathbb{R}} \mathbb{C}$ that can be written in the form $\boldsymbol{\zeta} = \boldsymbol{\xi} + \mathbf{i}\boldsymbol{\lambda}$, with $\boldsymbol{\xi}, \boldsymbol{\lambda} \in \mathfrak{a}^\vee$. We call $\boldsymbol{\xi}$ and $\boldsymbol{\lambda}$ the **real** and **imaginary parts** of $\boldsymbol{\zeta}$ respectively. Then by definition

$$\mathscr{B}^\vee_{\mathbb{C}}(\boldsymbol{\zeta}, \boldsymbol{\zeta}) := \mathscr{B}^\vee(\boldsymbol{\xi}, \boldsymbol{\xi}) + 2\mathbf{i}\mathscr{B}^\vee(\boldsymbol{\xi}, \boldsymbol{\lambda}) - \mathscr{B}^\vee(\boldsymbol{\lambda}, \boldsymbol{\lambda}). \tag{2.3.4.15}$$

In particular, for $\boldsymbol{\zeta} = s\boldsymbol{\alpha}$ with $s \in \mathbb{C}$, we have

$$\mathscr{B}^\vee_{\mathbb{C}}(\mathbf{i}\boldsymbol{\lambda}, \mathbf{i}\boldsymbol{\lambda}) = -\mathscr{B}^\vee(\boldsymbol{\lambda}, \boldsymbol{\lambda}), \quad \mathscr{B}^\vee_{\mathbb{C}}(s\boldsymbol{\alpha}, s\boldsymbol{\alpha}) = 2s^2. \tag{2.3.4.16}$$

Thus the form $\mathscr{B}^\vee_{\mathbb{C}}$ is $\mathbb{C}$-bilinear, not Hermitian. Its restriction to the imaginary axis $\mathbf{I}\mathfrak{a}^\vee$ is negative definite, corresponding to minus the original form on $\mathfrak{a}$.

[2]Let $H_{\boldsymbol{\alpha}} \in \mathfrak{a}$ be the correspondence of $\boldsymbol{\alpha} \in \mathfrak{a}^\vee$. According to $\operatorname{tr}(H_{\boldsymbol{\alpha}} H) = \boldsymbol{\alpha}(H)$, we have $H_{\boldsymbol{\alpha}} = \operatorname{diag}(1, -1)$ and then $\mathscr{B}^\vee(\boldsymbol{\alpha}, \boldsymbol{\alpha}) = \operatorname{tr}(\operatorname{diag}(1, 1)) = 2$.

2.3.5 The polar Haar measure and inversion

A $\mathcal{K}$-bi-invariant function is determined by its values on the polar $\mathcal{A}$-component b in the polar decomposition $\mathcal{G} = \mathcal{KAK}$, i.e., $g = k_1 b k_2$. We define the **polar Jacobian**

$$J_{\mathbf{P}}(b) := \left(\frac{b^{\boldsymbol{\alpha}} - b^{-\boldsymbol{\alpha}}}{2}\right)^{m(\boldsymbol{\alpha})} = \left(\frac{\boldsymbol{y}(b) - \boldsymbol{y}(b)^{-1}}{2}\right)^2 \tag{2.3.5.1}$$

with $y = \boldsymbol{y}(b) = b^{\boldsymbol{\alpha}} = b_1^2$ and $m(\boldsymbol{\alpha}) = 2$. The associated **polar measure** $d\mu_{\mathbf{P}}$ is given by

$$d\mu_{\mathbf{P}}(b) := J_{\mathbf{P}}(b)\,db = \left(\frac{e^{\nu} - e^{-\nu}}{2}\right)^2 = (\sinh\nu)^2 d\nu \tag{2.3.5.2}$$

in terms of the ν-variable, $\nu = \ln y$. It is a Jacobian computation due to Harish-Chandra in general that the functional

$$\varphi \longmapsto \int_{\mathcal{K}}\int_{\mathcal{K}}\int_{\mathcal{A}} \varphi(k_1 b k_2) J_{\mathbf{P}}(b)\,db dk_1 dk_2 \tag{2.3.5.3}$$

is a Haar functional on $\mathcal{G}$. For $\mathcal{K}$-bi-invariant functions, one does not need the two $\mathcal{K}$ integrations in the above formula. The total dk-measure of $\mathcal{K}$ is assumed to be normalized to 1. The question arises how the polar Haar measure is related to the Iwasawa measure.

Theorem 2.13

For $\varphi \in \mathbf{Gauss}(\mathcal{G})$, we have

$$\frac{1}{2\pi}\int_{\mathcal{G}} \varphi(g)\,d\mu_{\mathbf{IW}}(g) = \int_{\mathcal{A}} \varphi(b) J_{\mathbf{P}}(b)\,db = \int_{\mathcal{A}} \varphi\, d\mu_{\mathbf{P}}. \tag{2.3.5.4}$$

♡

Proof. For $\varphi_c \in \mathbf{Gauss}(\mathcal{G})$, one has

$$\frac{1}{2\pi}\int_{\mathcal{G}} \varphi_c(g)\,d\mu_{\mathbf{IW}}(g) = (2\pi)^{1/2} c^{3/2} e^{c/2}.$$

On the other hand,

$$\int_{\mathcal{A}} \varphi_c(b)\,d\mu_{\mathbf{P}}(b) = \int_{-\infty}^{\infty} e^{-\nu^2/2c}\nu\left(\frac{e^{\nu} - e^{-\nu}}{2}\right) d\nu = \int_{-\infty}^{\infty} e^{-\nu^2/2c} e^{\nu}\nu\,d\nu.$$

Using the identity

$$\frac{1}{s}\int_{-\infty}^{\infty} e^{sx} e^{-x^2/2x} x\,dx = (2\pi)^{1/2} c^{3/2} e^{s^2 c/2}, \quad s \in \mathbb{C}, \tag{2.3.5.5}$$

we get (2.3.5.4). □

Corollary 2.6

For all $f \in C_c(\mathcal{K} \setminus \mathcal{G}/\mathcal{K})$ one has

$$\frac{1}{2\pi}\int_{\mathcal{G}} f(g)\,d\mu_{\mathbf{IW}}(g) = \int_{\mathcal{A}} f(b)\,d\mu_{\mathbf{P}}(b). \tag{2.3.5.6}$$

♡

Proof. It follows from **Corollary 2.4** and the fact the $\mathbf{Gauss}(\mathbb{R})$ is dense in $C_c(\mathbb{R})$. □ □

Having two natural measures, the Iwasawa measure and the polar measure, we get two normalizations of the spherical transform, which we denote by

$$\mathbf{S}_{\mathbf{IW}} \quad \text{and} \quad \mathbf{S}_{\mathbf{P}} \quad \text{so that} \quad \mathbf{S}_{\mathbf{P}} = \frac{1}{2\pi}\mathbf{S}_{\mathbf{IW}}. \tag{2.3.5.7}$$

The $\mathbf{S}$ of proceeding sections is $\mathbf{S_{IW}}$. We may call $\mathbf{S_P}$ the **polar normalized spherical transform**, or simply the **polar spherical transform**.

We let

$$\Phi_{\mathbf{P}}(\boldsymbol{\zeta}, a) := \frac{\mathscr{B}^{\vee}(\boldsymbol{\alpha}, \boldsymbol{\alpha})}{\mathscr{B}^{\vee}_{\mathbb{C}}(\boldsymbol{\alpha}, \boldsymbol{\zeta})} \frac{a^{\boldsymbol{\zeta}} - a^{-\boldsymbol{\zeta}}}{a^{\boldsymbol{\alpha}} - a^{-\boldsymbol{\alpha}}} = \frac{2}{\mathscr{B}^{\vee}_{\mathbb{C}}(\boldsymbol{\alpha}, \boldsymbol{\zeta})} \frac{a^{\boldsymbol{\zeta}} - a^{-\boldsymbol{\zeta}}}{a^{\boldsymbol{\alpha}} - a^{-\boldsymbol{\alpha}}}. \tag{2.3.5.8}$$

Writing $\boldsymbol{\zeta} = s\boldsymbol{\alpha}$ with $s \in \mathbb{C}$, we can write this formula in the form

$$\Phi_{\mathbf{P},\boldsymbol{\alpha}}(s, a) := \Phi_{\mathbf{P}}(s\boldsymbol{\alpha}, a) = \frac{1}{s} \frac{y^s - y^{-s}}{y - y^{-1}} = \frac{1}{s} \frac{e^{s\nu} - e^{-s\nu}}{e^{\nu} - e^{-\nu}} = \frac{1}{s} \frac{\sinh(s\nu)}{\sinh \nu}. \tag{2.3.5.9}$$

Theorem 2.14

For all basic $\mathcal{G}$-Gaussians, taking the polar measure convolution $*_{\mathcal{A},\mathbf{P}}$ on $\mathcal{A}$, we have

$$\left(\Phi_{\mathbf{P}} *_{\mathcal{A},\mathbf{P}} \varphi_c\right)(s\boldsymbol{\alpha}) = (\mathbf{S_P}\varphi_c)(s\boldsymbol{\alpha}) \tag{2.3.5.10}$$

so $\Phi_{\mathbf{P}}$ is an integral kernel for $\mathbf{S_P}$ (with polar measure). ♡

Proof. By definition, (2.3.4.1) and, (2.3.5.2),

$$\begin{aligned}
\left(\Phi_{\mathbf{P}} *_{\mathcal{A},\mathbf{P}} \varphi_c\right)(s\boldsymbol{\alpha}) &= \int_{\mathcal{A}} \Phi_{\mathbf{P}}(s\boldsymbol{\alpha}, a)\varphi_c(a)\, d\mu_{\mathbf{P}}(a) \\
&= \frac{1}{s}\int_{-\infty}^{\infty} \frac{e^{s\nu} - s^{-s\nu}}{e^{\nu} - e^{-\nu}} e^{-\nu^2/2c} \frac{2\nu}{e^{\nu} - e^{-\nu}} \left(\frac{e^{\nu} - e^{-\nu}}{2}\right)^2 d\nu \\
= \frac{1}{s}\int_{-\infty}^{\infty} \left(\frac{e^{s\nu} - e^{-s\nu}}{2}\right) e^{-\nu^2/2c}\nu\, d\nu &= \frac{1}{s}\int_{-\infty}^{\infty} e^{s\nu} e^{-\nu^2/2c}\nu\, d\nu = (\mathbf{S_P}\varphi_c)(s\boldsymbol{\alpha})
\end{aligned}$$

where we used (2.3.5.5), (2.3.4.4), and (2.3.5.7). □

The spherical kernel (2.3.5.8) in polar coordinates on $\mathbf{SL}_2(\mathbb{C})$ is originally due to Gelfand and Naimark (1950).

We observe that the polar spherical kernel has a product structure. The first factor in (2.3.5.8) has a name, the **Harish-Chandra c-function**, defined for our purpose by

$$\mathbf{c}(\boldsymbol{\zeta}) := \frac{\mathscr{B}^{\vee}(\boldsymbol{\alpha}, \boldsymbol{\alpha})}{\mathscr{B}^{\vee}_{\mathbb{C}}(\boldsymbol{\alpha}, \boldsymbol{\zeta})} = \frac{1}{s} \quad \text{if } \boldsymbol{\zeta} = s\boldsymbol{\alpha}. \tag{2.3.5.11}$$

The second factor in (2.3.5.8) comes from a skew-symmetrization procedure, namely,

$$\frac{a^{\boldsymbol{\zeta}} - a^{-\boldsymbol{\zeta}}}{a^{\boldsymbol{\alpha}} - a^{-\boldsymbol{\alpha}}}.$$

We know from **Corollary 2.4** that the spherical transform is a linear isomorphism between the Gauss spaces. We want to exhibit the measure on $\mathfrak{a}^{\vee}$ with respect to which the transpose of the spherical kernel induces the inverse integral transform. As to the Haar measure on $\mathfrak{a}^{\vee}$, we have the Haar measure $da = dy/y = d\nu$ on $\mathfrak{a}^{\vee} \cong \mathbb{R}$. We let $d\mu_{\mathbf{FOU}}(\boldsymbol{\lambda})$ be the Haar measure on $\mathfrak{a}^{\vee}$ that makes Fourier inversion come out without any extra constant factor. The **Fourier**

transform F is defined by

$$(\mathbf{F}f)(\boldsymbol{\lambda}) := \int_{\mathfrak{a}} f(H)e^{-\mathbf{i}\boldsymbol{\lambda}(H)}dH, \quad \boldsymbol{\lambda} \in \mathfrak{a}^{\vee}. \tag{2.3.5.12}$$

The isomorphism of $\mathfrak{a}$ with $\mathbb{R}$ is taken via the coordinate ν. That is,

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\exp^{-1}"] & \mathfrak{a} \arrow[d, "y"'] \arrow[dr, "\nu"] & \\ & \mathbb{R}^{+} \arrow[r, "\ln"] & \mathbb{R} \end{tikzcd} \qquad \begin{tikzcd} a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \arrow[r] & H = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} \arrow[d] \arrow[dr, "\nu"] & \\ & a_1/a_2 \arrow[r] & h_1 - h_2 \end{tikzcd}$$

Thus if $\boldsymbol{\lambda} = r\boldsymbol{\alpha}$ on $\mathfrak{a}$ with $\lambda \in \mathbb{R}$, then

$$(\mathbf{F}_{\boldsymbol{\alpha},+}f)(r) \equiv (\mathbf{F}f)(\boldsymbol{\lambda}) = \int_{\mathfrak{a}} f(H)e^{-\mathbf{i}r\boldsymbol{\alpha}(H)}dH = \int_{\mathbb{R}} f_{\boldsymbol{\alpha},+}(\nu)e^{-\mathbf{i}r\nu}d\nu. \tag{2.3.5.13}$$

The **Fourier normalized measure** is then defined to be

$$d\mu_{\mathbf{FOU}}(\boldsymbol{\lambda}) := \frac{dr}{2\pi}. \tag{2.3.5.14}$$

We define the **Harish-Chandra measure** on $\mathfrak{a}^{\vee}$ by

$$d\mu_{\mathbf{HC}}(\boldsymbol{\lambda}) := \frac{1}{|\mathbf{c}(\mathbf{i}\boldsymbol{\lambda})|^2}d\mu_{\mathbf{FOU}}(\boldsymbol{\lambda}) = \frac{r^2dr}{2\pi}, \quad \boldsymbol{\lambda} = s\boldsymbol{\alpha}. \tag{2.3.5.15}$$

We let $\boldsymbol{\Phi}_{\mathbf{P},\mathbf{HC}}$ be the integral operator having the kernel function $\Phi_{\mathbf{P}}(\mathbf{i}\boldsymbol{\lambda}, a)$ with respect to the Harish-Chandra measure $d\mu_{\mathbf{HC}}(\boldsymbol{\lambda})$, so for an even function h such that the integral is absolutely convergent, by definition

$$\begin{aligned} (\boldsymbol{\Phi}_{\mathbf{P},\mathbf{HC}}h)(a) &:= \int_{\mathfrak{a}^{\vee}} \Phi_{\mathbf{P}}(\mathbf{i}\boldsymbol{\lambda}, a)h(\mathbf{i}\boldsymbol{\lambda})\,d\mu_{\mathbf{HC}}(\boldsymbol{\lambda}) \\ &= \int_{\mathbb{R}} h_{\boldsymbol{\alpha}}(\mathbf{i}r)(-\mathbf{i}r)\frac{e^{\mathbf{i}r\nu} - e^{-\mathbf{i}r\nu}}{e^{\nu} - e^{-\nu}}\frac{dr}{2\pi} \quad \text{with } \nu = \boldsymbol{\alpha}(\ln a). \end{aligned} \tag{2.3.5.16}$$

Theorem 2.15

The integral operator $\boldsymbol{\Phi}_{\mathbf{P},\mathbf{HC}}$ is the inverse of $\mathbf{S}_{\mathbf{P}}$ on the Gauss spaces

$$\mathbf{S}_{\mathbf{P}} : \mathbf{Gauss}(\mathcal{G}) \longrightarrow \mathbf{Gauss}(\mathbf{i}\mathbb{R}).$$

The map $\mathbf{S}_{\mathbf{P}}$ is an L^2-isometry with respect to the polar measure on $\mathcal{G}$ and the Harish-Chandra measure on $\mathfrak{a}^{\vee}$, the integral scalar product taken on $\mathfrak{a}^{\vee}$. ♡

Proof. For given $\varphi_c \in \mathbf{Gauss}(\mathcal{G})$, let $h := \mathbf{S}_{\mathbf{P}}\varphi_c$. By (2.3.4.5),

$$h_{\boldsymbol{\alpha}}(\mathbf{i}r) = (2\pi)^{1/2}c^{3/2}e^{-r^2c/2}.$$

Then

$$\begin{aligned} (\boldsymbol{\Phi}_{\mathbf{P},\mathbf{HC}}h)(a) &= \int_{\mathbb{R}} (2\pi)^{1/2}c^{3/2}e^{-r^2c/2}(-\mathbf{i}r)\frac{e^{\mathbf{i}r\nu} - e^{-\mathbf{i}r\nu}}{e^{\nu} - e^{-\nu}}\frac{dr}{2\pi} \\ &= \int_{\mathbb{R}} \frac{(2\pi)^{-1/2}c^{3/2}}{\sinh\nu}(-\mathbf{i}r)e^{-r^2c/2+\mathbf{i}\nu r}dr \\ &= e^{-\nu^2/2c}\frac{\nu}{\sinh\nu} \;=\; \varphi_c(a). \end{aligned}$$

Thus $(\boldsymbol{\Phi}_{\mathbf{P},\mathbf{HC}} \circ \mathbf{S}_{\mathbf{P}})\varphi_c = \varphi_c$.

To prove the second statement, we first make explicit the L^2-scalar product on $\mathcal{G}$, for the polar measure. For two basic Gaussians $\varphi_c, \varphi_{c'}$, we get, integrating over $\mathfrak{a} \cong \mathbb{R}$:

$$\begin{aligned}\langle \varphi_c, \varphi_{c'} \rangle_{\mathbf{P}} &= \int_{\mathbb{R}} e^{-\nu^2/2c} \frac{2\nu}{e^{\nu}-e^{-\nu}} e^{-\nu^2/2c'} \frac{2\nu}{e^{\nu}-e^{-\nu}} \left(\frac{e^{\nu}-e^{-\nu}}{2}\right)^2 d\nu \\ &= \int_{\mathbb{R}} e^{-\nu^2(c+c')/2cc'} \nu^2 d\nu \;=\; (cc')^{3/2} \int_{\mathbb{R}} e^{-\nu^2(c+c')/2} \nu^2 d\nu.\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle \mathbf{S_P}\varphi_c, \mathbf{S_P}\varphi_{c'} \rangle_{\mathbf{HC}} &= \int_{\mathbb{R}} (\mathbf{S_P}\varphi_c)(\mathbf{i}r\boldsymbol{\alpha})(\mathbf{S_P}\varphi_{c'})(\mathbf{i}r\boldsymbol{\alpha}) \frac{r^2 dr}{2\pi} \\ &= \int_{\mathbb{R}} (2\pi)^{1/2} c^{3/2} e^{-r^2c/2} (2\pi)^{1/2} c'^{3/2} e^{-r^2c'/2} \frac{r^2 dr}{2\pi} \\ &= (cc')^{3/2} \int_{\mathbb{R}} e^{-r^2(c+c')/2} r^2 dr \;=\; \left(\frac{2cc'}{c+c'}\right)^{3/2} \Gamma(3/2).\end{aligned}$$

Thus $\langle \varphi_c, \varphi_{c'} \rangle_{\mathbf{P}} = \langle \mathbf{S_P}\varphi_c, \mathbf{S_P}\varphi_{c'} \rangle_{\mathbf{HC}}$. □

Jorgenson and Lang (2005) showed that the above isometry can be extended by continuity to $L^2(\mathcal{K} \setminus \mathcal{G}/\mathcal{K})$ because of the denseness of the Gauss space.

2.3.6 Point-pair invariants, the polar height, and the polar distance

Let $\varphi \in \mathbf{Fu}(\mathcal{K} \setminus \mathcal{G}/\mathcal{K})$ be a $\mathcal{K}$-bi-invariant function on $\mathcal{G}$. Following Selberg, we define its associated **point-pair invariant $\mathbf{K}_\varphi$** to be the function defined by

$$\mathbf{K}_\varphi(z,w) := \varphi(z^{-1}w), \quad z, w \in \mathcal{G}. \tag{2.3.6.1}$$

The $\mathcal{K}$-bi-invariance of φ implies that $\mathbf{K}_\varphi$ is defined on $\mathcal{G}/\mathcal{K} \times \mathcal{G}/\mathcal{K}$.

(1) **$\mathbf{K}_\varphi$ is symmetric**, i.e., $\mathbf{K}_\varphi(z,w) = \mathbf{K}_\varphi(w,z)$, because φ is even.

In higher dimensions, one has to assume the evenness condition in addition to the $\mathcal{K}$-bi-invariance.

(2) **$\mathbf{K}_\varphi$ is $\mathcal{G}$-invariant**, i.e., $\mathbf{K}_\varphi(gz, gw) = \mathbf{K}_\varphi(z,w)$.

A $\mathcal{K}$-bi-invariant function on $\mathcal{G}$ is determined by its values on $\mathcal{A}$ by the polar decomposition $\mathcal{G} = \mathcal{K}\mathcal{A}\mathcal{K}$. If $g = k_1 a k_2$, then b is uniquely determined up to a permutation of its diagonal elements. Hence a function on $\mathcal{A}$ that ia invariant under permutations extends to a $\mathcal{K}$-bi-invariant function on $\mathcal{G}$.

Recall that the polar height σ is the $\mathcal{K}$-bi-invariant function such that $\sigma \geq 0$ and

$$\sigma^2(g) = \sigma^2(b) = (\ln \boldsymbol{y}(b))^2 = (\ln b^{\boldsymbol{\alpha}})^2,$$

where $g = k_1 b k_2$ is the polar decomposition of $g \in \mathcal{G}$.

Proposition 2.5

Let $g = uak = k_1 b k_2$ be the Iwasawa, respectively polar, decomposition. Then

$$\sigma(a) \leq \sigma(b) \quad \textit{or} \quad |\ln a| \leq |\ln b|. \tag{2.3.6.2}$$

♡

In general reductive Lie group theory, this result is due to Harish-Chandra. The inequality (2.3.6.2) can be viewed from a differential geometric point of view. The space $\mathcal{G}/\mathcal{K}$ is a Cartan-Hadamard manifold, whose exponential map is metric semi-increasing. The $\mathcal{U}$-components are perpendicular to $\mathcal{A}$. The metric increasing property shows that the law of cosines with the inequality in the right direction is valid on $\mathcal{G}/\mathcal{K}$, and the desired inequality (2.3.6.2) then follows at once.

Proof. Let e_1, e_2 be the unit column vectors of $\mathbb{C}^2$. Let $a = \operatorname{diag}(a_1, a_2)$ and $b = \operatorname{diag}(b_1, b_2)$. We may without loss of generality that $a_1, b_1 \geq 1$. It suffices to prove that $a_1 \leq b_1$. From $\mathcal{U}\mathcal{A} = \mathcal{A}\mathcal{U}$, we have $ua = av$ for some $v \in \mathcal{U}$. Writing $av = ua = k_1 b k_2 k^{-1} = k_1 b k'$ with $k' := k_2 k^{-1} \in \mathcal{K}$, we have

$$a_1^2 = |ave_1|^2 = |k_1 bk' e_1|^2,$$

where $|\cdot|$ stands for he Euclidean metric on $\mathbb{C}^2$. Since k' is unitary, there are numbers $c_1, c_2 \in \mathbb{C}$ such that $k' e_1 = c_1 e_1 + c_2 e_2$ and $|c_1|^2 + |c_2|^2 = 1$. Then

$$|bk' e_1|^2 = |b_1 c_1|^2 + |b_2 c_2|^2 = b_1^2 |c_1|^2 + b_2^2 |c_2|^2.$$

It suffices to show that for $y \geq 1$ ($y := b_1^2$) we have

$$y|c_1|^2 + y^{-1}|c_2|^2 \leq y.$$

Using $|c_2|^2 = 1 - |c_1|^2$ yields

$$y|c_1|^2 + y^{-1}|c_2|^2 = \frac{|c_1|^2(y^2-1)+1}{y} \leq \frac{(y^2-1)+1}{y} = y$$

which proves the inequality (2.3.6.2). □

The following deep property is due to Cartan and Harish-Chandra.

Theorem 2.16. (Triangle inequality)

For $g, g' \in \mathcal{G}$, we have

$$\sigma(gg') \leq \sigma(g) + \sigma(g'). \tag{2.3.6.3}$$

♡

Lang (1999) provided an elementary self-contained treatment of Theorem 2.16 on Bruhat-Tits spaces.

Definition 2.1

We then define the **polar distance**

$$d_{\mathbf{P}}(z, w) := \sigma(z^{-1}w), \quad z, w \in \mathcal{G}, \tag{2.3.6.4}$$

which is the associated point-pair invariant. According to **Theorem 2.16**, *$d_{\mathbf{P}}$ is a distance defined on $\mathcal{G}/\mathcal{K}$.*

♣

Recall from **Theorem 4.4** that $\mathcal{G}/\mathcal{K} \approx \mathbf{Pos}_2$ by $g \mapsto gg^*$ (with $g^* = \bar{g}$). Let

$$\boldsymbol{y}_{\mathbf{P}}(g) = b^{\boldsymbol{\alpha}} = \frac{b_1}{b_2} = b_1^2$$

be the polar y-coordinate of $g = k_1 b k_2 \in \mathcal{G}$. Then

$$\sigma(g) = |\ln \boldsymbol{y}_{\mathbf{P}}(g)| = d_{\mathbf{P}}(g\mathcal{K}, e_{\mathcal{G}/\mathcal{K}}), \tag{2.3.6.5}$$

where $e_{\mathcal{G}/\mathcal{K}}$ is the unit coset of $\mathcal{G}/\mathcal{K}$.

Theorem 2.17

The function $d_{\mathbf{P}}$ is a distance function on $\mathcal{G}/\mathcal{K}$. ♡

Proof. Lang (1999) defined a distance on $\mathbf{Pos}_2(\mathbb{R})$. The analogous treatment for $\mathbf{Pos}_n(\mathbb{C})$ and $\mathbf{SPos}_n(\mathbb{C})$ runs exactly the same way. Thus we get a natural $\mathcal{G}$-invariant distance on $\mathcal{G}/\mathcal{K}$, which is a Riemannian manifold, with $\mathcal{G}$ acting by translation as a group of isometries. □

Chapter 3 Fundamental groups

Introduction

- *Homotopy*
- *H-spaces*
- $\pi_1(\mathbf{S}^1)$

3.1 Homotopy

Introduction

- *Homotopy relation*
- *The fundamental group π_1*
- *π_1(manifolds)*
- *π_1(product spaces)*
- *Retractions, deformation retractions*
- *Contractible spaces*

Recall the category **Top** of topological spaces and continuous maps. For any $f, g \in \mathbf{Hom}_{\mathbf{Top}}(X, Y)$, a **homotopy from f to g** is a continuous map $H : X \times [0,1] \to Y$, $(x,t) \mapsto H(x,t) \equiv H_t(x)$, such that

$$H_0 = f, \quad H_1 = g.$$

In this case, we also say that **f and g are homotopic** and write $f \simeq g$ or $H : f \simeq g$. If f is homotopic to a constant map, we say f is **null homotopic**.

3.1.1 Homotopy relation

It is clear that $f \simeq f$. Actually, we can prove that homotopy is an equivalence relation.

Proposition 3.1

(1) *Homotopy is an equivalence relation on* $\mathbf{Hom}_{\mathbf{Top}}(X, Y)$.

(2) *Homotopy relation is preserved by composition. That is, if*

$$f_0, f_1 \in \mathbf{Hom}_{\mathbf{Top}}(X, Y), \quad g_0, g_1 \in \mathbf{Hom}_{\mathbf{Top}}(Y, Z),$$

and $f_0 \simeq f_1$ *and* $g_0 \simeq g_1$, *then* $g_0 \circ f_0 \simeq g_1 \circ f_1$.

♡

Proof. (1) If $H : f \simeq g$, then $H^{-1} : g \simeq f$, where

$$\overline{H}(x,t) := H(x, 1-t), \quad (x,t) \in X \times [0,1].$$

If $F : f \simeq g$ and $G : g \simeq h$, then $H : f \simeq h$, where

$$H(x,t) := \begin{cases} F(x, 2t), & 0 \le t \le \frac{1}{2}, \\ G(x, 2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

Since $F(x,1)=g(x)=G(x,0)$, the map H is well-defined and hence, by **Theorem 1.12**, H is continuous.

(2) Let $F: f_0 \simeq f_1$ and $G: g_0 \simeq g_1$. Then we define $H_t := G_t \circ F_t$ which implies that $H: g_0 \circ f_0 \simeq g_1 \circ f_1$. □

Because $\simeq$ is an equivalence relation by **Proposition 3.1**, we denote by $[X,Y]$ the set of equivalence classes of continuous maps from X to Y, that is,

$$[X,Y] := \mathbf{Hom}_{\mathbf{Top}}(X,Y)/\simeq . \tag{3.1.1.1}$$

3.1.2 The fundamental group

Recall the category $\mathbf{Top}_*$ consists of pointed topological space (X,x) and point-preserving continuous maps. For any $(X,x)\in \mathbf{Ob}(\mathbf{Top}_*)$, denote by $\Omega(X,x)$ the set of all loops in X based at x, that is,

$$\Omega(X,x) := \{f \in [0,1] \to X | f \text{ is continuous and } f(0)=f(1)=x\}. \tag{3.1.2.1}$$

The set $\Omega(X,x)$ consists at least one element c_x given by

$$c_x(s) := x, \quad s\in[0,1]. \tag{3.1.2.2}$$

For $f\in\Omega(X,x)$, define its **reverse path** $\overline{f}$ by

$$\overline{f}(s) := f(1-s), \quad s\in[0,1].$$

Then $\overline{f}\in\Omega(X,x)$. We say $f,g\in\Omega(X,x)$ is **path homotopic** and write $f\sim g$, if there exists a continuous map $G:[0,1]\times[0,1]\to X$, $(s,t)\mapsto H(s,t)=H_t(s)$, such that

$$H_0=f, \quad H_1=g, \quad H_t(0)=H_t(1)=x \ (t\in[0,1]).$$

We also say that H is a **path homotopy from f to g** and write $H: f\sim g$.

Proposition 3.2

$\sim$ is an equivalence relation on $\Omega(X,x)$. ♡

Proof. The proof is almost the same as **Proposition 3.1**. □

We use the above proposition to define the **fundamental group of X based at x** as

$$\pi_1(X,x) := \Omega(X,x)/\sim = \{[f] | f\in\Omega(X,x)\}, \tag{3.1.2.3}$$

the set of path classes of loops based at x.

For $f,g\in\Omega(X,x)$, we can define their **product** $f*g:[0,1]\to X$ by

$$f*g(s) := \begin{cases} f(2s), & 0\le s\le \frac{1}{2}, \\ g(2s-1), & \frac{1}{2}\le s\le 1. \end{cases} \tag{3.1.2.4}$$

For $[f], [g] \in \pi_1(X, x)$, we can also define

$$[f] * [g] := [f * g]. \tag{3.1.2.5}$$

Firstly, we shall verify that the above definition (3.1.2.5) is well-defined, that is, if $F : f \sim f'$ and $G : g \sim g'$, then $H : f * g \sim f' * g'$. Indeed, define

$$H(s,t) = \begin{cases} F(2s,t), & 0 \le s \le \frac{1}{2},\ 0 \le t \le 1, \\ G(2s-1,t), & \frac{1}{2} \le s \le 1,\ 0 \le t \le 1. \end{cases}$$

Theorem 3.1

For any $(X, x) \in \mathbf{Ob}(\mathbf{Top}_*)$, $(\pi_1(X, x), *)$ is a group.

Proof. We shall verify

(1) $[c_x]$ is the identity element in $\pi_1(X, x)$:

$$[c_x] * [f] = [f] * [c_x] = [f].$$

(2) The inverse of $[f]$ is $[\overline{f}]$:

$$[f] * [\overline{f}] = [\overline{f}] * [f] = [c_x].$$

(3) $*$ is associative:

$$[f] * ([g] * [h]) = ([f] * [g]) * [h].$$

For (1): we have $H : f \sim c_x * f$, where

$$H(s,t) = \begin{cases} x, & t \ge 2s, \\ f\left(\frac{2s-t}{2-t}\right), & t \le 2s. \end{cases}$$

Also, we have $G : f \sim f * c_x$, where

$$G(s,t) = \begin{cases} f\left(\frac{2s}{2-t}\right), & t \le 2 - 2s, \\ x, & t \ge 2 - 2s. \end{cases}$$

For (2): we have $H : c_x \sim f * \overline{f}$, where

$$H(s,t) = \begin{cases} f(2s), & 0 \le s \le \frac{t}{2}, \\ f(t), & \frac{t}{2} \le s \le 1 - \frac{t}{2}, \\ f(2-2s), & 1 - \frac{t}{2} \le s \le 1. \end{cases}$$

Also, we have $G : c_x \sim \overline{f} * f$, where

$$H(s,t) = \begin{cases} f(1-2s), & 0 \le s \le \frac{t}{2}, \\ f(1-t), & \frac{t}{2} \le s \le 1 - \frac{t}{2}, \\ f(2s-1), & 1 - \frac{t}{2} \le s \le 1. \end{cases}$$

For (3): we have $H : (f * g) * h \sim f * (g * h)$. □

Exercise 3.1
Prove $(f * g) * h \sim f * (g * h)$ *in Theorem 3.1.*

Suppose now that X is path-connected. For any $x, y \in X$ we can define a natural map

$$\Phi_g : \pi_1(X, x) \longrightarrow \pi_1(X, y), \quad [f] \longmapsto [\overline{g} * f * g] = [\overline{g}] * [f] * [g], \tag{3.1.2.6}$$

where g is a path from x to y.

Proposition 3.3
If X is path-connected, then the map (3.1.2.6) is a group isomorphism with inverse $\Phi_{\overline{g}}$.

Proof. For any $[f_1], [f_2] \in \pi_1(X, x)$ we have

$$\begin{aligned} \Phi_g([f_1]) * \Phi_g([f_2]) &= ([\overline{g}] * [f_1] * [g]) * ([\overline{g}] * [f_2] * [g]) \\ = [\overline{g}] * [f_1] * ([g] * [\overline{g}]) * [f_2] * [g] &= [\overline{g}] * [f_1] * [f_2] * [g] = \Phi_g([f_1] * [f_2]). \end{aligned}$$

Hence Φ_g is a group homomorphism. To prove that Φ_g is an isomorphism we notice that

$$\Phi_g \circ \Phi_{\overline{g}}([f]) = \Phi_g([g] * [f] * [\overline{g}]) = [\overline{g}] * ([g] * [f] * [\overline{g}]) * [g] = [f].$$

Similarly one has $\Phi_{\overline{g}} \circ \Phi_g([f]) = [f]$. $\square$

As a consequence of **Proposition 3.3**, for any path-connected topological space X and any two points $x, y \in X$, we have that $\pi_1(X, x)$ is isomorphic to $\pi_1(X, y)$. Hence we can use the notation "$\pi_1(X)$" to denote "the" group $\pi_1(X, x)$, relative to any point $x \in X$.

We say a path-connected topological space is **simply-connected** if $\pi_1(X, x) = \{[c_x]\}$ (that is, $\pi_1(X)$ is trivial).

Example 3.1
(1) $\pi_1(\mathbf{R}^n)$ *is trivial. For any fixed point* $\boldsymbol{x} \in \mathbf{R}^n$ *and any* $f \in \Omega(\mathbf{R}^n, \boldsymbol{x})$*, we have*

$$f \sim c_{\boldsymbol{x}},$$

since

$$H(s, t) = (1 - t) f(s) + t c_x(s), \quad (s, t) \times [0, 1] \times [0, 1],$$

is a homotopic from f to c_x. Hence $[f] = [c_{\boldsymbol{x}}]$ for any $f \in \Omega(\mathbf{R}^n, \boldsymbol{x})$ and then $c_1(\mathbf{R}^n) = \{[c_{\boldsymbol{x}}]\}$.
(2) *For any convex domain $X \subseteq \mathbf{R}^n$, the fundamental group $c_1(X, x)$ is trivial. Indeed, for any $f \in \Omega(X, x)$ define H as in (1), so that $f \sim c_x$.*
(3) $\pi_1(\mathbf{S}^n)$ *is trivial, for any $n \geq 2$.*

Consider a morphism $\varphi \in \mathbf{Hom}_{\mathbf{Top}_*}((X, x), (Y, y))$. Then $\varphi : X \to Y$ is a continuous map with $\varphi(x) = y$. Define

$$\varphi_* : \pi_1(X, x) \longrightarrow \pi_1(Y, y), \quad [f] \longmapsto [\varphi \circ f]. \tag{3.1.2.7}$$

The same argument as for **Proposition 3.3** shows that φ_* is a group homomorphism. We call φ_* the **homomorphism induced by** φ.

The above discussion lets us to define a functor

$$\pi_1 : \mathbf{Top}_* \longrightarrow \mathbf{Group}, \quad (X, x) \longmapsto \pi_1(X, x), \tag{3.1.2.8}$$

with $\pi_1(\varphi) = \varphi_*$.

Theorem 3.2

π_1 defined in (3.1.2.8) is a functor from $\mathbf{Top}_*$ to $\mathbf{Group}$.

Proof. We shall verify the following two things:

(1) If $\varphi \in \mathbf{Hom}_{\mathbf{Top}_*}((X, x), (Y, y))$ and $\psi \in \mathbf{Hom}_{\mathbf{Top}_*}((Y, y), (Z, z))$, then

$$(\psi \circ \varphi)_* = \psi_* \circ \varphi_*.$$

(2) For any $x \in X$, we have

$$(\mathbf{1}_X)_* = \mathbf{1}_{\pi_1(X,x)}.$$

For (1),

$$(\psi \circ \varphi)_*([f]) = [\psi \circ \varphi \circ f] = \psi_*([\varphi \circ f]) = \psi_*(\varphi_*([f])).$$

For (2),

$$(\mathbf{1}_X)_*([f]) = [\mathbf{1}_X \circ f] = [f] = \mathbf{1}_{\pi_1(X,x)}([f]).$$

Consequently, π_1 is a functor. □

Corollary 3.1

If $\varphi : (X, x) \to (Y, y)$ is a homeomorphism in $\mathbf{Top}_*$, then the induced homomorphism $\varphi_* : \pi_1(X, x) \to \pi_1(Y, y)$ is a group isomorphism.

3.1.3 Fundamental groups of manifolds

Recall the Lebesgue number lemma, Theorem 1.31.

Theorem 3.3

The fundamental group of a topological manifold is countable.

Proof. Let $\mathcal{M}$ be a topological m-manifold and take any fixed point $x \in \mathcal{M}$. To prove $\pi_1(\mathcal{M}, x)$ is countable, we shall analyze any loop $f : [0, 1] \to \mathcal{M}$ at x.

(1) According to **Proposition 1.18**, we can choose a countable open cover $\mathscr{U}$ of $\mathcal{M}$ in such a way that each element $\mathcal{U} \in \mathscr{U}$ is isomorphic to some open ball in $\mathbf{R}^m$ (so that $\mathcal{U}$ is path-connected). For any given $\mathcal{U}, \mathcal{U}' \in \mathscr{U}$, the intersection $\mathcal{U} \cap \mathcal{U}'$ has at most countably many connected components $(\mathcal{U} \cap \mathcal{U}')_{[i]}$, $i \in I(\mathcal{U}, \mathcal{U}') \subseteq \mathbf{N}$; otherwise $\mathscr{U}$ is uncountable. For each

$i \in I(\mathcal{U}, \mathcal{U}')$, choose a point $x_i \in (\mathcal{U} \cap \mathcal{U}')_{[i]}$. Consider the set

$$\Delta := \left\{x_i \in (\mathcal{U} \cap \mathcal{U}')_{[i]} | i \in I(\mathcal{U}, \mathcal{U}'),\ \mathcal{U}, \mathcal{U}' \in \mathscr{U}\right\}.$$

Because $\mathscr{U}$ and $I(\mathcal{U}, \mathcal{U}')$ are countable, we see that the set Δ is also countable.

(2) For each $\mathcal{U} \in \mathscr{U}$ and each pair (x, x') in Δ with $x, x' \in \mathcal{U}$, we can take a path $h^{\mathcal{U}}_{x,x'}$ from x to x' in $\mathcal{U}$, since $\mathcal{U}$ is path-connected. Applying **Theorem 1.31** to the open cover

$$\mathscr{U}_f := f^{-1}\left(f([0,1]) \cap \mathscr{U}\right) = \left\{f^{-1}(\mathcal{U} \cap f([0,1]))\right\}$$

of the compact interval $[0,1]$, we obtain a number $\delta > 0$ such that any subset $A \subseteq [0,1]$ with diameter $\mathrm{diam}(A) < \delta$ is contained in some element of $\mathscr{U}_f$. In particular, taking $n \in \mathbf{N}$ with $n > 1/\delta$ yields that $f([(k-1)/n, k/n]) \subseteq \mathcal{U}_k \in \mathcal{U}$. Write

$$f_k(s) := f|_{[(k-1).n, k/n]}\left(\frac{s+k-1}{n}\right), \quad s \in [0,1].$$

Then $[f] = [f_1] \cdots [f_n]$.

(3) For each $k = 1, \cdots, n-1$, take a point $x_k \in \mathcal{U}_k \cap \mathcal{U}_{k+1}$. Since $f(k/n) \in \mathcal{U}_k \cap \mathcal{U}_{k+1}$ and $\mathcal{U}_k \cap \mathcal{U}_{k+1}$ is path-connected, it follows that we can choose a path g_k in $\mathcal{U}_k \cap \mathcal{U}_{k+1}$ from x_k to $f(k/n)$. Let

$$f'_k := g_{k-1} * f_k * \overline{g_k}.$$

For $k = 0$ or $k = n$, take $x_k = p$ and $g_k = c_x$ the constant path. Therefore

$$[f] = [f'_1] * \cdots * [f'_n].$$

Now f'_k and $h^{\mathcal{U}_k}_{x_{k-1}, x_k}$ both connect x_{k-1} to x_k in $\mathcal{U}_k$, using the simple-connectedness of $\mathcal{U}_k$, we see that f'_k is homotopic to $h^{\mathcal{U}_k}_{x_{k-1}, x_k}$. Consequently

$$[f] = \left[h^{\mathcal{U}_1}_{x_0, x_1}\right] * \cdots * \left[h^{\mathcal{U}_n}_{x_{n-1}, x_n}\right].$$

Hence $\pi_1(\mathcal{M}, x)$ is countable. □

The above argument also shows that the fundamental group of a compact manifold is finite.

To improve the result in **Theorem 3.3** we shall assume some curvature conditions.

Theorem 3.4. (Bonnet-Myers)

If $(\mathcal{M}, g)$ is a complete Riemannian m-manifold with $\mathrm{Ric}_g \geq (m-1)Kg$, where $K > 0$, then $\mathrm{diam}(\mathcal{M}, g) \leq \pi/\sqrt{K}$. In particular, $\mathcal{M}$ is compact and $\pi_1(\mathcal{M}) < +\infty$. ♡

Proof. We will give a proof in the course of Differential Manifold. □

If the Ricci curvature bound is replaced by the sectional sectional curvature bound, we have the following outstanding result.

Theorem 3.5. (Cheeger-Gromoll soul theorem)

If $(\mathcal{M}, g)$ is a complete Riemannian m-manifold with $\mathrm{Sec}_g > 0$ everywhere, then $\mathcal{M}$ is diffeomorphic to $\mathbf{R}^m$. ♡

In 1994, Perelman proved the so-called **soul conjecture**.

Theorem 3.6. (Soul conjecture, solved by Perelman)

If $(\mathcal{M}, g)$ is a complete Riemannian m-manifold with $\mathrm{Sec}_g \geq 0$ everywhere and > 0 at some point, then $\mathcal{M}$ is diffeomorphic to $\mathbf{R}^m$. ♡

For nonnegative Ricci curvature we have the following well-known conjecture.

Problem 3.1. (Milnor's conjecture)

If $(\mathcal{M}, g)$ is a complete Riemannian m-manifold with $\mathrm{Ric}_g \geq 0$, then $\pi_1(\mathcal{M})$ is finite generated. ♠

When $m = 3$, the above conjecture was recently solved by Gang Liu under the previous work of Schoen-Yau. Under an extra condition that g has maximum volume growth, Peter Li proved Problem 3.1 in 1986 (actually, he proved that $\pi_1(\mathcal{M})$ is finite). Recall that if $\mathrm{Ric}_g \geq 0$, then the volume radio

$$\frac{\mathrm{Vol}_g(\mathcal{B}_g(x,r))}{\omega_m r^m}$$

is nonincreasing. Then

$$\lim_{r\to\infty} \frac{\mathrm{Vol}_g(\mathcal{B}_g(x,r))}{\omega_m r^m} = \theta \in [0,1].$$

We say the Riemannian metric g has **maximum volume growth**, if $\theta \in (0,1]$. When $\theta = 1$, we can prove that in this case $(\mathcal{M}, g)$ is isometric to $\mathbf{R}^m$.

On the other hand, a simply-connected manifold with suitable curvature conditions has a rigidity result.

Theorem 3.7. (Rauch-Klingenberg-Berger topological sphere theorem)

If $(\mathcal{M}, g)$ is a complete, simply-connected Riemannian n-manifold with

$$\frac{1}{4} \leq \mathrm{Sec}_g(\Pi) \leq 1$$

for all 2-planes Π, then $\mathcal{M}$ is homeomorphic to $\mathbf{S}^m$. In particular, $\mathcal{M}^3$ is diffeomorphic to $\mathbf{S}^3$. ♡

How about the diffeomorphisms between $\mathcal{M}^m$ and $\mathbf{S}^m$ when the dimension $m \geq 4$? The answer is YES! It was proved by Brendle and Schoen by using the Ricci flow.

Theorem 3.8. (Diffeomorphic spheral theorem)

If $(\mathcal{M}, g)$ is a complete, simply-connected Riemannian n-manifold with

$$\frac{1}{4} \le \mathrm{Sec}_g(\Pi) \le 1$$

for all 2-planes Π, then $\mathcal{M}$ is diffeomorphic to $\mathbf{S}^m$. ♡

3.1.4 Fundamental groups of product spaces

Let $\{(X_i, \mathscr{T}_i)\}_{1\le i`n}$ be a finite family of topological spaces, and let

$$p_i : X_1 \times \cdots \times X_n \longrightarrow X_i, \quad (x_1, \cdots, x_n) \longmapsto x_i \tag{3.1.4.1}$$

denote the i-th projection. Then we get group homomorphisms

$$p_{i,*} : \pi_1(X_1 \times \cdots \times X_n, (x_1, \cdots, x_n)) \longrightarrow \pi_1(X_i, x_i) \tag{3.1.4.2}$$

by (3.1.2.7). Consequently, we can define a group homomorphism

$$\boldsymbol{p}_* : \pi_1(X_1 \times \cdots \times X_n, (x_1, \cdots, x_n)) \longrightarrow \pi_1(X_1, x_1) \times \cdots \times \pi_1(X_n, x_n) \tag{3.1.4.3}$$

given by

$$\boldsymbol{p}_*([f]) := (p_{1,*}[f], \cdots, p_{n,*}[f]). \tag{3.1.4.4}$$

Theorem 3.9

The map (3.1.4.4) is an isomorphism. ♡

Proof. We shall prove the injectivity and surjectivity of $\boldsymbol{p}_*$.

(1) $\boldsymbol{p}_*$ is injective. Suppose $\boldsymbol{p}_*([f]) = ([c_{x_1}], \cdots, [c_{x_n}])$. Write

$$f = (f_1, \cdots, f_n).$$

Then $p_i \circ f = f_i$ and

$$([c_{x_1}], \cdots, [c_{x_n}]) = ([f_1], \cdots, [f_n]).$$

Thus $f_i \sim c_{x_i}$, $1 \le i \le n$ and hence $f \sim c_{(x_1, \cdots, x_n)}$.

(2) $\boldsymbol{p}_*$ is surjective. For $([f_1], \cdots, [f_n] \in \pi_1(X_1, x_1) \times \cdots \times \pi_1(X_n, x_n)$, define

$$f(s) := (f_1(s), \cdots, f_n(s)), \quad s \in [0, 1].$$

Then $\boldsymbol{p}_*([f]) = ([f_1], \cdots, [f_n])$. □

3.1.5 Retractions and deformation retractions

Let $(X, \mathscr{T})$ be a topological space and $A \subseteq X$ a subset. Under the subspace topology $\mathscr{T}_A := \mathscr{T} \cap A$, we see that A is itself a topological space.

(1) A continuous map $r : X \to A$ is called a **retraction** if $r \circ \iota_A = \mathbf{1}_A$, where $\iota_A : A \to X$ is the inclusion. If there exists a retraction from X to A, we say that A is a **retract of X**.

It is clear that a retract of a connected space (resp., compact space) is connected (resp., compact). Moreover, if $A \subseteq B \subseteq X$, A is a retract of B, and B is a retract of X, then A is a retract of X.

(2) A retraction $r : X \to A$ is said to be a **deformation retraction** if $\iota_A \circ r$ is homotopic to $\mathbf{1}_X$. If there exists a deformation retraction from X to A, then A is said to be a **deformation retract of X**. That is, $A \subseteq X$ is a deformation retract of X if and only if there exists a homotopy $H : X \times [0,1] \to X$ satisfying

$$\begin{aligned} H(x,0) &\equiv H_0(x) = x, \quad x \in X, \\ H(x,1) &\equiv H_1(x) \in A, \quad x \in X, \\ H(a,1) &\equiv H_1(a) = a, \quad a \in A. \end{aligned}$$

Let $f \in \mathbf{Hom}_{\mathbf{Top}}(X,Y)$. A morphism $g \in \mathbf{Hom}_{\mathbf{Top}}(Y,X)$ is a **homotopy inverse for f** if

$$g \circ f \simeq \mathbf{1}_X, \quad f \circ g \simeq \mathbf{1}_Y.$$

If there exists a homotopy inverse for , then is called a **homotopy equivalence** and we say that ***X* is homotopic to *Y*** (write $X \simeq Y$).

Exercise 3.2

Homotopy equivalence is an equivalence relation on the class of all topological spaces $\mathbf{Ob}(\mathbf{Top})$.

If $r : X \to A$ is a deformation retraction, then, according to $r \circ \iota_A = \mathbf{1}_A$, r and ι_A are homotopy equivalences.

Proposition 3.4

(1) *If $r : X \to A$ is a retraction, then for any $x \in A$, $\iota_{A,*} : \pi_1(X,x) \to \pi_1(X,x)$ is injective and $r_* : \pi_1(X,x) \to \pi_1(A,x)$ is surjective.*

(2) *A retract of a simply-connected space is also simply-connected.*

Proof. (1) From $r \circ \iota_A = \mathbf{1}_A$ we obtain $\mathbf{1}_{\pi_1(A,x)} = r_* \circ \iota_{A,*}$ which implies the desired results.

(2) Since $\pi_1(X,x)$ is trivial and $\iota_{A,*} : \pi_1(A,x) \to \pi_1(X,x)$ is injective, it follows that $\pi_1(A,x)$ is also trivial. □

Example 3.2

(1) *The map $r : \mathbf{R}^m \setminus \{\mathbf{0}\} \to \mathbf{S}^{m-1}$ given by*

$$r(\boldsymbol{x}) := \frac{\boldsymbol{x}}{|\boldsymbol{x}|}$$

is a retraction. Since $\pi_1(\mathbf{S}^1) \cong \mathbf{Z}$ (we shall prove soon), it follows from Proposition 3.4 that $\mathbf{R}^2 \setminus \{\mathbf{0}\}$ is not simply-connected. Consequently, $\mathbf{R}^2 \setminus \{\mathbf{0}\}$ is not homeomorphic to $\mathbf{R}^2$.

(2) *Since the map* $r : \mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1 \to \mathbf{S}^1 \times \{1\}$ *given by*

$$r(x,y) := (x,1)$$

is a retraction, it follows that $\mathbf{T}^2$ *is not simply-connected and then* $\mathbf{T}^2$ *is not homeomorphic to* $\mathbf{S}^2$.

Let $X, Y \in \mathbf{Ob}(\mathbf{Top})$ and $A \subseteq X$. A homotopy H between $f, g \in \mathbf{Hom}_{\mathbf{Top}}(X,Y)$ is said to be **stationary on A** if

$$H(x,t) = f(x), \quad \forall\, (x,t) \in A \times [0,1].$$

If there exists such a homotopy H, we say that f and g are **homotopic relative to A**, written as

$$H : f \simeq g \ (\text{rel } A),$$

and H is called a **homotopy relative to A**. In this case, $f = g$ on A.

It is clear that homotopy relative to A is an equivalence relation in $\mathbf{Hom}_{\mathbf{Top}}(X,Y)$.

A retraction $r : X \to A$ is called a **strong deformation retraction** if $\mathbf{1}_X$ is homotopic to $\iota_A \circ r$ relative to A. That is, $A \subseteq X$ is a strong deformation retract of X if and only if there exists a homotopy $H : X \times [0,1] \to X$ satisfying

$$\begin{aligned}
H(x,0) &\equiv H_0(x) &=& \ x, \quad x \in X, \\
H(x,1) &\equiv H_1(x) &\in& \ A, \quad x \in X, \\
H(a,1) &\equiv H_1(a) &=& \ a, \quad a \in A, \\
H(a,t) &\equiv H_t(a) &=& \ a, \quad a \in A,\ t \in [0,1].
\end{aligned}$$

Example 3.3

(1) *For any* $m \geq 1$, $\mathbf{S}^{m-1}$ *is a strong deformation retract of* $\mathbf{R}^m \setminus \{\mathbf{0}\}$ *and of* $\overline{\mathbf{B}^m} \setminus \{\mathbf{0}\}$. *Indeed, consider the homotopy*

$$H : (\mathbf{R}^n \setminus \{\mathbf{0}\}) \times [0,1] \longrightarrow \mathbf{R}^m \setminus \{\mathbf{0}\}, \quad (\boldsymbol{x}, t) \longmapsto (1-t)\boldsymbol{x} + t\frac{\boldsymbol{x}}{|\boldsymbol{x}|}.$$

(2) *For* $m \geq 3$, $\mathbf{R}^m \setminus \{\mathbf{0}\}$ *and* $\overline{\mathbf{B}^m} \setminus \{\mathbf{0}\}$ *are both simply-connected.*

Next result generalizes Corollary 3.1.

Theorem 3.10. (Homotopy invariance)

If $\varphi : X \to Y$ *is a homotopy equivalence, then for any point* $x \in X$, $\varphi_* : \pi_1(X,x) \to \pi_1(Y, \varphi(x))$ *is an isomorphism.*

Proof. Let $\psi : Y \to X$ be a homotopy inverse for φ. Then, by definition, we have $\psi \circ \varphi \simeq \mathbf{1}_X$ and $\varphi \circ \psi \simeq \mathbf{1}_Y$. Moreover φ, ψ induce

$$\pi_1(X,x) \xrightarrow{\varphi_*} \pi_1(Y,\varphi(x)) \xrightarrow{\psi_*} \pi_1(X, \psi(\varphi(x))) \xrightarrow{\varphi_*} \pi_1(Y, \varphi(\psi(\varphi(x)))).$$

Lemma 3.1

If $\varphi, \psi \in \mathbf{Hom}_{\mathbf{Top}}(X, Y)$ and $H : \varphi \simeq \psi$ is a homotopy. For any $x \in X$, let h be the path in Y from $\varphi(x)$ to $\psi(x)$ defined by

$$h(t) := H(x, t), \quad t \in [0, 1],$$

and let $\Phi_h : \pi_1(Y, \varphi(x)) \to \pi_1(Y, \psi(x))$ be the isomorphism defined in (3.1.2.6). Then the following diagram

$$\begin{tikzcd} \pi_1(X, x) \arrow[r, "\varphi_*"] \arrow[d, equal] & \pi_1(Y, \varphi(x)) \arrow[d, "\Phi_h"] \\ \pi_1(X, x) \arrow[r, "\psi_*"] & \pi_1(Y, \psi(x)) \end{tikzcd}$$

is commutative. ♡

Proof. For $[f] \in \pi_1(X, x)$ we need to verify

$$\psi_*([f]) = \Phi_h(\varphi_*([f]))$$

which is equivalent to $h * (\psi \circ f) \sim (\varphi \circ f) * h$ or $(\varphi \circ f) * h \sim h * (\psi \circ f)$. Consider the map

$$F : [0, 1] \times [0, 1] \longrightarrow Y, \quad (s, t) \longmapsto H(f(s), t).$$

Then

$$F(s, 0) = H(f(s), 0) = \varphi(f(s)), \quad F(1, s) = H(f(1), s) = H(x, s) = h(s)$$

and

$$F(0, s) = H(f(0), s) = H(x, s) = h(s), \quad F(s, 1) = H(f(s), 1) = \psi(f(s)).$$

Lemma 3.2 shows that $(\varphi \circ f) * h \sim h * (\psi \circ f)$. □

Lemma 3.2. (Square lemma)

Let $F : [0, 1] \times [0, 1] \to X$ be a continuous map, and let f, g, h, k be the paths in X defined by

$$f(s) := F(s, 0), \quad g(s) := F(1, s), \quad h(s) := F(0, s), \quad k(s) := F(s, 1).$$

Then $f * g \sim h * k$. ♡

Proof. Consider the diagram map $d(s) := F(s, s)$. It is clear that $f * g \sim d$ (**Exercise 3.3**). Similarly, $h * k \sim d$. Hence $f * g \sim h * k$. □

Since $H : \mathbf{1}_X \simeq \psi \circ \varphi$, it follows that h defined by

$$h(t) := H(x, t), \quad t \in [0, 1],$$

is a path in X from x to $\psi(\varphi(x))$. By **Lemma 3.1**, we have the following commutative diagram:

$$\begin{array}{ccc} \pi_1(X,x) & \xrightarrow{\mathbf{1}_{X,*}} & \pi_1(X,x) \\ \| & & \downarrow \Phi_h \\ \pi_1(X,x) & \xrightarrow{(\psi\circ\varphi)_*} & \pi_1(Y,\psi(\varphi(x))) \end{array}$$

Thus

$$\Phi_h = \psi_* \circ \varphi_*.$$

Consequently, ψ_* is surjective and φ_* is injective.

Similarly, since $G : \mathbf{1}_Y \simeq \varphi \circ \psi$, it follows that g defined by

$$g(t) := G(\varphi(x), t), \quad t \in [0,1],$$

is a path in Y from $\varphi(x)$ to $\varphi(\psi(\varphi(x)))$. By **Lemma 3.1**, we have the following commutative diagram:

$$\begin{array}{ccc} \pi_1(Y,\varphi(x)) & \xrightarrow{\mathbf{1}_{Y,*}} & \pi_1(Y,\varphi(x)) \\ \| & & \downarrow \Phi_g \\ \pi_1(Y,\varphi(x)) & \xrightarrow{(\varphi\circ\psi)_*} & \pi_1(Y,\varphi(\psi(\varphi(x)))) \end{array}$$

Thus

$$\Phi_g = \varphi_* \circ \psi_*.$$

Consequently, φ_* is surjective and ψ_* is injective. Therefore φ_* is isomorphic. □

Exercise 3.3

Verify $f * g \sim d$ in **Lemma 3.2**.

3.1.6 Contractible spaces

A topological space X is said to be **contractible** if $\mathbf{1}_X$ is homotopic to some constant map. Equivalently, X is contractible, if there exists a point $x_0 \in X$ and a continuous map $H : X \times [0,1] \to X$ such that

$$H(x,0) = x \ \ (x \in X), \quad H(x,1) = x_0 \ \ (x \in X).$$

Thus, X is contractible if, intuitively, X can be continuously shrunk to a point.

(1) $\mathbf{R}^n$ is contractible, because we can take $H(\boldsymbol{x}, t) = (1-t)\boldsymbol{x}$.

(2) $[0,1]$ is contractible, because we can take $H(x,t) := (1-t)x$.

(3) Any convex subset X of $\mathbf{R}^n$ is contractible, because we can take $H(\boldsymbol{x},t) = (1-t)\boldsymbol{x} + t\boldsymbol{x}_0$.

(4) Any star-shaped subset X of $\mathbf{R}^n$ is contractible.

Proposition 3.5

(1) *If $f, g \in \mathbf{Hom}_{\mathbf{Top}}(X,Y)$ and Y is contractible, then $f \simeq g$.*

(2) *If X is contractible, then any two constant maps on X are homotopic and the identity*

map $\mathbf{1}_X$ is homotopic to any constant map on X.
(3) *A topological space X is contractible if and only if X is homotopic to a one-point space.*
(4) *Two contractible spaces are homotopic and any continuous map between contractible spaces is a homotopy equivalence.* ♡

Proof. (1) Assume $\mathbf{1}_Y \simeq c_{y_0}$, where c_{y_0} is a constant map on Y. Then $f = \mathbf{1}_Y f \simeq c_{y_0} f = c_{y_0} g \simeq \mathbf{1}_Y g = g$.

(2) Trivial.

(3) If X is homotopic to a one-point space $Y = \{y_0\}$, then we have a homotopy equivalence $f : X \to Y$ with homotopy inverse $g : Y \to X$. Then $\mathbf{1}_X \simeq gf = c_{x_0}$, where $x_0 := g(y_0)$. Thus X is contractible.

Conversely, assume that X is contractible. We can find a point $x_0 \in X$ and continuous map $H : X \times [0,1] \to X$ such that $H(x,0) = x$ and $F(x,1) = x_0$ for all $x \in X$. Set $Y := \{x_0\}$ and consider two continuous maps $f : X \to Y$ given by $f(x) := x_0$ and $g : Y \to X$ given by $g(x_0) = x_0$. Then

$$f \circ g = \mathbf{1}_Y, \quad g \circ f = c_{x_0} \simeq \mathbf{1}_X.$$

Therefore f is a homotopy equivalence from X to Y with homotopy inverse g.

(4) Let X and Y be two contractible spaces. By (3), X is homotopic to some one-point space $\{x_0\}$; thus $X \simeq \{x_0\}$. Similarly, $Y \simeq \{y_0\}$ for some $y_0 \in Y$. Since $\{x_0\} \simeq \{y_0\}$, it follows that $X \simeq Y$.

Next, take a map $f \in \mathbf{Hom}_{\mathbf{Top}}(X,Y)$ with X, Y being contractible. The first part gives two maps $f_1 \in \mathbf{Hom}_{\mathbf{Top}}(X,Y)$ and $g_1 \in \mathbf{Hom}_{\mathbf{Top}}(Y,X)$ such that $f_1 \circ g_1 \simeq \mathbf{1}_Y$ and $g_1 \circ f_1 \simeq \mathbf{1}_X$. Because $f \circ g_1 \in \mathbf{Hom}_{\mathbf{Top}}(Y,X)$ and $g_1 \circ f \in \mathbf{Hom}_{\mathbf{Top}}(X,Y)$. According to (1), we get $f \circ g_1 \simeq \mathbf{1}_Y$ and $g_1 \circ f \simeq \mathbf{1}_X$. Thus f is a homotopy equivalence. □

Example 3.4

Proposition 3.5 (1) *is not true in the case of relative homotopy. That is, if $f, g \in \mathbf{Hom}_{\mathbf{Top}}(X,Y)$, $f|_A = g|_A$ for some open subset $A \subseteq X$, and Y is contractible, then we may not have $f \simeq g$ $(\mathrm{rel}\ A)$.*

Consider the **comb space** *Y defined by*

$$Y = (\{0\} \times [0,1]) \cup ([0,1] \times \{0\}) \cup \left\{ (x,y) \in \mathbf{R}^2 \middle| y \in [0,1],\ x = \frac{1}{n},\ n = 1, 2, \cdots \right\}.$$

Let $H : Y \times [0,1] \to Y$ be defined by

$$H((x,y),t) := (x, (1-t)y).$$

Then H is a homotopy from $\mathbf{1}_Y$ to the projection of Y to the x-axis, and so $\mathbf{1}_Y$ is homotopic to some constant map on Y. Thus Y is contractible. Define $c : Y \to Y$, $(x,y) \mapsto (0,1)$.

Using **Proposition 3.5** *yields that* $\mathbf{1}_Y \simeq c$ *and* $\mathbf{1}_Y|_A = c|_A$ *where* $A := \{(0,1)\}$. *However we can not find a homotopy from* $\mathbf{1}_Y$ *to* c *relative to* A.

We say a map $f \in \mathbf{Hom}_{\mathbf{Top}}(X, Y)$ is **null homotopic** if it is homotopic to some constant map.

Theorem 3.11

Let $\boldsymbol{x}_0 \in \mathbf{S}^n$ *and let* $f \in \mathbf{Hom}_{\mathbf{Top}}(\mathbf{S}^n, Y)$. *Then TFAE:*

(1) f *is null homotopic.*

(2) f *can be continuously extend over* $\mathbf{D}^{n+1} := \overline{\mathbf{B}^{n+1}}$.

(3) f *is null homotopic relative to* $\boldsymbol{x}_0$, *that is,* f *is homotopic to some constant map relative to* $\boldsymbol{x}_0$.

Proof. $(1) \Rightarrow (2)$: Let $H : f \simeq c_{\boldsymbol{y}_0}$ for some $\boldsymbol{y}_0 \in Y$. Define an extension $\widetilde{f}$ of f over $\mathbf{D}^{n+1}$ by

$$\widetilde{f}(\boldsymbol{x}) := \begin{cases} \boldsymbol{y}_0, & 0 \le |\boldsymbol{x}| \le \frac{1}{2}, \\ H\left(\boldsymbol{x}/|\boldsymbol{x}|, 2(1-|\boldsymbol{x}|)\right), & \frac{1}{2} \le |\boldsymbol{x}| \le 1. \end{cases}$$

Then $\widetilde{f}$ is continuous.

$(2) \Rightarrow (3)$: Assume that f has a continuous extension $\widetilde{f} : \mathbf{D}^{n+1} \to Y$. Define $H : \mathbf{S}^n \times [0,1] \to Y$ by

$$H(\boldsymbol{x}, t) := \widetilde{f}\left((1-t)\boldsymbol{x} + t\boldsymbol{x}_0\right).$$

Then $H(\boldsymbol{x}, 0) = \widetilde{f}(\boldsymbol{x}) = f(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbf{S}^n$ and $H(\boldsymbol{x}, 1) = \widetilde{f}(\boldsymbol{x}_0) = c_{\boldsymbol{x}_0}$.

$(3) \Rightarrow (1)$: Trivial. □

Corollary 3.2

Any continuous map from $\mathbf{S}^n$ *to a contractible space has a continuous extension over* $\mathbf{D}^{n+1}$.

Proof. By **Proposition 3.5** (1) and **Theorem 3.11**. □

3.2 H-spaces

Introduction

- *Two notions on functors*
- *H-spaces*
- *A classical example of H-groups*
- *Suspension*
- *A classical example of H-cogroups*
- *Higher homotopgy groups*

The **homotopy category HTop** is the category whose objects are topological spaces and whose morphisms are homotopy classes of continuous maps. That is

$$\mathbf{Ob}(\mathbf{HTop})=\mathbf{Ob}(\mathbf{Top}),\quad \mathbf{Hom}_{\mathbf{HTop}}(X,Y):=[X,Y]$$

where the last one was defined in (3.1.1.1). By **Proposition 3.1**, we see that **HTop** is indeed a category and the composition is defined as

$$\mathbf{Hom}_{\mathbf{Htop}}(X,Y)\times\mathbf{Hom}_{\mathbf{HTop}}(Y,Z)\longrightarrow\mathbf{Hom}_{\mathbf{Htop}}(X,Z),\quad ([f],[g])\longmapsto[g\circ f].$$

Define a functor $[\cdot]:\mathbf{Top}\to\mathbf{HTop}$ to be

$$[\cdot]:\mathbf{Ob}(\mathbf{Top})\longrightarrow\mathbf{Ob}(\mathbf{HTop}),\quad X\longmapsto X$$

and

$$[\cdot]:\mathbf{Hom}_{\mathbf{Top}}(X,Y)\longrightarrow\mathbf{Hom}_{\mathbf{HTop}}(X,Y),\quad f\longmapsto[f].$$

Thus the above functor $[\cdot]$ is covariant.

We extend the homotopy category of topological spaces to the homotopy category of topological pairs.

Definition 3.1

A **topological pair** (X,A) *consists of a topological space X and a subspace $A\subseteq X$. If A is empty, we write simply X for $(X,\emptyset)$. A* **topological subpair** $(X',A')\subseteq(X,A)$ *consists of a topological pair (X',A') with $X'\subseteq X$ and $A'\subseteq A$. A* **topological pair map** *$f:(X,A)\to(Y,B)$ between topological pairs is a continuous map $f\in\mathbf{Hom}_{\mathbf{Top}}(X,Y)$ such that $f(A)\subseteq B$.* ♣

The above concept generalizes the pointed topological spaces and the pointed-preserving maps. Let $\mathbf{Top}_{**}$ denote the category of topological pairs and topological pair maps. Hence $\mathbf{Top}$ and $\mathbf{Top}_{*}$ are both (fully) subcategories of $\mathbf{Top}_{**}$.

Given a topological pair $(X,A)\in\mathbf{Ob}(\mathbf{Top}_{**})$, we let $(X,A)\times[0,1]$ denote the topological pair $(X\times[0,1],A\times[0,1])\in\mathbf{Ob}(\mathbf{Top}_{**})$.

(1) Let $X'\subseteq X$ and suppose $f_0,f_1\in\mathbf{Hom}_{\mathbf{Top}_{**}}((X,A),(Y,B))$ agree on X', that is, $f_0|_{X'}=f_1|_{X'}$. We say $\boldsymbol{f_0}$ **is homotopic to** $\boldsymbol{f_1}$ **relative to** $\boldsymbol{X'}$, denoted by $f_0\simeq f_1\ (\mathrm{rel}\ X')$, if there exists a map $F\in\mathbf{Hom}_{\mathbf{Top}_{**}}((X,A)\times[0,1],(Y,B))$ such that

$$F(x,0)=f_0(x),\quad F(x,1)=f_1(x),\quad x\in X$$

and

$$F(x,t)=f_0(x),\quad x\in X',\ t\in[0,1].$$

We also say F is a **homotopy relative to** $\boldsymbol{X'}$ **from** $\boldsymbol{f_0}$ **to** $\boldsymbol{f_1}$ and denote by $F:f_0\simeq f_1\ (\mathrm{rel}\ X')$.

(2) When $X'=\emptyset$, we return back to the previous definition of $f_0\simeq f_1$.

(3) Note that $f_0 \simeq f_1 \ (\mathrm{rel}\ X')$ implies $f_0 \simeq f_1 \ (\mathrm{rel}\ X'')$ whenever $X'' \subseteq X'$.

(4) We also write $F_t(x) := F(x,t)$ so that $F_t \in \mathbf{Hom}_{\mathbf{Top}_{**}}((X,A),(Y,B))$ is a continuous one-parameter family of topological pair maps from (X,A) to (Y,B).

Proposition 3.1 can be generalized to the following

Proposition 3.6

(1) *Homotopy relative to X' is an equivalence relation in*

$$\mathbf{Hom}_{\mathbf{Top}_{**}}((X,A),(Y,B)).$$

(2) *Homotopy relative to X' is preserved by composition. That is, if*

$$f_0, f_1 \in \mathbf{Hom}_{\mathbf{Top}_{**}}((X,A),(Y,B)), \quad g_0, g_1 \in \mathbf{Hom}_{\mathbf{Top}_{**}}((Y,B),(Z,C)),$$

and $f_0 \simeq f_1 \ (\mathrm{rel}\ X')$, $g_0 \simeq g_1 \ (\mathrm{rel}\ Y')$, *and* $f_0(X') = f_1(X') \subseteq Y'$, *then* $g_0 \circ f_0 \simeq g_1 \circ f_1 \ (\mathrm{rel}\ X')$.

♡

Exercise 3.4

Prove **Proposition 3.6**.

♠

The above proposition allows us to define the **homotopy class relative to X'**

$$[(X,A'),(Y,B)]_{X'} := \left\{[f]_{X'} : f \in \mathbf{Hom}_{\mathbf{Top}_{**}}((X,A),(Y,B))\right\}.$$

We also can define the **homotopy category of topological pairs $\mathbf{HTop}_{**}$** whose objects are topological pairs and whose morphisms are homotopy classes (relative to $\emptyset$). We also have a covariant functor from $\mathbf{Top}_{**}$ to $\mathbf{HTop}_{**}$.

(1) For any $(P,Q) \in \mathbf{Ob}(\mathbf{HTop}_{**})$, there is a covariant functor

$$[(P,Q),(\cdot,\cdot)] := \mathbf{Hom}_{\mathbf{HTop}_{**}}((P,Q),(\cdot,\cdot))$$

from $\mathbf{HTop}_{**}$ to $\mathbf{Set}$, satisfying

$$[(P,Q),(\cdot,\cdot)]((X,A)) := \mathbf{Hom}_{\mathbf{HTop}_{**}}((P,Q),(X,A))$$

and for $[f] \in \mathbf{Hom}_{\mathbf{HTop}_{**}}((X,A),(Y,B))$,

$$f_{\#} \equiv [(P,Q),(\cdot,\cdot)]([f]) \in \mathbf{Hom}_{\mathbf{Set}}\left([(P,Q),(X,A)],[(P,Q),(Y,B)]\right)$$

where

$$f_{\#}([g]) := [f \circ g], \quad [g] \in [(P,Q),(X,A)].$$

(2) A topological pair map $f : (X,A) \to (Y,B)$ is called a **homotopy equivalence** if $[f]$ is an equivalence in $\mathbf{HTop}_{**}$. Equivalently, there is a topological pair map $g : (Y,B) \to (X,A)$, called a **homotopy inverse of f**, such that

$$f \circ g \simeq \mathbf{1}_Y, \quad g \circ f \simeq \mathbf{1}_X.$$

Pairs (X,A) and (Y,B) are said to have the **same homotopy type** if there are equivalent in $\mathbf{HTop}$.

3.2.1 Two notions on functors

Let $\mathfrak{C}$ be a category and take an object $Y \in \mathfrak{C}$.

(1) Define a covariant functor $\pi_Y : \mathfrak{C} \to \mathbf{Set}$ by

$$\pi_Y := \mathbf{Hom}(Y, \cdot) \quad \text{on } \mathbf{Ob}(\mathfrak{C}) \tag{3.2.1.1}$$

and

$$\pi_Y(h) := h_\# : \mathbf{Hom}_{\mathfrak{C}}(Y, Z) \longrightarrow \mathbf{Hom}_{\mathfrak{C}}(Y', Z), \quad g \longmapsto h \circ g, \tag{3.2.1.2}$$

for any $h \in \mathbf{Hom}_{\mathfrak{C}}(Z, Z')$.

(2) Define a covariant functor $\pi^Y : \mathfrak{C} \to \mathbf{Set}$ by

$$\pi^Y := \mathbf{Hom}(\cdot, Y) \quad \text{on } \mathbf{Ob}(\mathfrak{C}) \tag{3.2.1.3}$$

and

$$\pi^Y(f) := h_\# : \mathbf{Hom}_{\mathfrak{C}}(X', Y) \longrightarrow \mathbf{Hom}_{\mathfrak{C}}(X, Y), \quad g' \longmapsto g' \circ f, \tag{3.2.1.4}$$

for any $f \in \mathbf{Hom}_{\mathfrak{C}}(X, X')$.

3.2.2 H-spaces

If $(X, x_0), (Y, y_0) \in \mathbf{Ob}(\mathbf{Top}_*)$, we denote by

$$[X, Y] := [(X, x_0), (Y, y_0)] \tag{3.2.2.1}$$

the set of morphisms from X to Y in $\mathbf{HTop}_*$.

Theorem 3.12

If P is a topological group with the identity element, π^P is a contravariant functor from $\mathbf{HTop}_*$ to $\mathbf{Group}$. ♡

Proof. (1) **$[X, P]$ is a group for any given pointed topological space X.** If $[g_1], [g_2] \in \mathbf{Hom}_{\mathbf{HTop}_*}(X, P)$, we define

$$[g_1][g_2] := [g_1 g_2]$$

with $g_1 g_2 : X \to P$ given by $(g_1 g_2)(x) := g_1(x) g_2(x)$. Clearly that under this multiplication, $[X, P]$ is a group.

(2) **π^P is a contravariant functor.** □

Exercise 3.5

Complete the proof of Theorem 3.12. ♠

It is clear that $[X, P]$ is Abelian if P is Abelian. For example, $[X, \mathbf{S}^1]$ is an Abelian group.

An ***H*-space** (P, μ) consists of a pointed topological space $P = (P, p_0)$ together with a continuous multiplication

$$\mu : P \times P \longrightarrow P \tag{3.2.2.2}$$

for which the constant map $c : P \to P$, given by $c = c_{p_0}$, is a **homotopy identity**, that is, each composite, $1 := \mathbf{1}_P$

$$P \xrightarrow{(c,1)} P \times P \xrightarrow{\mu} P, \qquad P \xrightarrow{(1,c)} P \times P \xrightarrow{\mu} P$$

is homotopic to 1.

(1) The multiplication μ is said to be **homotopy associative** if the square

$$\begin{tikzcd} P \times P \times P \arrow[r, "\mu\times 1"] \arrow[d, "1\times\mu"'] & P \times P \arrow[d, "\mu"] \\ P \times P \arrow[r, "\mu"'] & P \end{tikzcd}$$

is homotopy commutative, that is,

$$\mu \circ (\mu \times 1) \simeq \mu \circ (1 \times \mu).$$

(2) A continuous map $\varphi : P \to P$ is called a **homotopy inverse** for P and μ if each composite

$$P \xrightarrow{(\varphi,1)} P \times P \xrightarrow{\mu} P, \qquad P \xrightarrow{(1,\varphi)} P \times P \xrightarrow{\mu} P$$

is homotopic to c.

(3) An ***H*-group** (P, μ, φ) is a homotopy associative H-space with a homotopy inverse.

(4) Any topological group is an H-group.

(5) A multiplication μ in an H-space P is said to be **homotopy Abelian** if the triangle

$$\begin{tikzcd} P \times P \arrow[r, "T"] \arrow[d, "\mu"'] & P \times P \arrow[d, "\mu"] \\ P \arrow[r, equal] & P \end{tikzcd} \qquad T(p_1, p_2) := (p_2, p_1),$$

is homotopy commutative, that is,

$$\mu \circ T \simeq \mu.$$

(6) An H-group with homotopy Abelian multiplication is called an **Abelian *H*-group**.

(7) An ***H*-space homomorphism** between two H-spaces (P, μ) and (P', μ') is a continuous map $\alpha : P \to P'$ such that the square

$$\begin{tikzcd} P \times P \arrow[r, "\mu"] \arrow[d, "\alpha\times\alpha"'] & P \arrow[d, "\alpha"] \\ P' \times P' \arrow[r, "\mu'"'] & P' \end{tikzcd}$$

is homotopy commutative, that is,

$$\alpha \circ \mu \simeq \mu' \circ (\alpha \times \alpha).$$

Theorem 3.13

A pointed topological space having the same homotopy type as an H-space (or an H-group, or an Abelian H-group), is itself an H-space (or H-group, or an Abelian H-group) in such a way that the homotopy equivalence is an H-space homomorphism. ♡

Proof. (1) Let (P, μ) be an H-space, and (P', p_0') is a pointed topological space that has the same homotopy type as P. Let $f : P \to P'$ and $g : P' \to P$ be homotopy inverses. Define

$$P' \times P' \xrightarrow{g \times g} P \times P \xrightarrow{\mu} P \xrightarrow{f} P'$$

μ as

$$\mu' := f \circ \mu \circ (g \times g).$$

Then μ' is a continuous map. Next we shall verify that each composite

$$P' \xrightarrow{(c',1')} P' \times P' \xrightarrow{\mu'} P', \qquad P' \xrightarrow{(1',c')} P' \times P' \xrightarrow{\mu'} P'$$

is homotopic to $\mathbf{1}_{P'}$. Since

$$\begin{tikzcd} P' \arrow[r, "{(1',c')}"] \arrow[d, "g"'] & P' \times P' \arrow[r, "\mu'"] & P' \\ P \arrow[r, "{(1,c)}"] & P \times P \arrow[r, "\mu"] & P \arrow[u, "f"'] \end{tikzcd}$$

the above diagram shows $f \circ \mu \circ (1, c) \circ g = \mu' \circ (1, c')$ and $\mu \circ (1, c) \simeq \mathbf{1}_P$, it follows that

$$\mu' \circ (1', c') \simeq \mathbf{1}_{P'}.$$

Similarly, from

$$\begin{tikzcd} P' \arrow[r, "{(c',1')}"] \arrow[d, "g"'] & P' \times P' \arrow[r, "\mu'"] & P' \\ P \arrow[r, "{(c,1)}"] & P \times P \arrow[r, "\mu"] & P \arrow[u, "f"'] \end{tikzcd}$$

the above diagram shows $f \circ \mu \circ (c, 1) \circ g = \mu' \circ (c', 1')$ and $\mu \circ (c, 1) \simeq \mathbf{1}_P$, it follows that

$$\mu' \circ (c', 1') \simeq \mathbf{1}_{P'}.$$

Consequently, (P', μ') is an H-space.

(2) Consider the diagrams

$$\begin{tikzcd} P \times P \arrow[r, "\mu"] \arrow[d, "f \times f"'] & P \arrow[d, "f"] \\ P' \times P' \arrow[r, "\mu'"'] & P' \end{tikzcd} \qquad \begin{tikzcd} P' \times P' \arrow[r, "\mu'"] \arrow[d, "g \times g"'] & P' \arrow[d, "g"] \\ P \times P \arrow[r, "\mu"'] & P \end{tikzcd}$$

Because

$$\mu' \circ (f \times f) \ = \ f \circ \mu \circ (g \times g) \circ (f \times f)$$

$$= \ f \circ \mu \circ (g \circ f) \times (g \circ f) \ \simeq \ f \circ \mu \circ (\mathbf{1}_P \times \mathbf{1}_P) \ \simeq \ f \circ \mu$$

and

$$g \circ \mu' \;=\; g \circ f \circ \mu \circ (g \times g) \;\simeq\; \mu \circ (g \times g),$$

we see that g and f are both H-space homomorphisms.

(3) If μ is homotopy associative, so is μ'.

(4) If μ is homotopy Abelian, so is μ'.

(5) If $\varphi : P \to P$ is a homotopy inverse for P, then $\varphi' := f \circ \varphi \circ g : P' \to P'$ is a homotopy inverse for P'. □

Exercise 3.6

Verify (3), (4), *and* (5) *in* **Theorem 3.13**.

Given an H-space (P, μ) (with base point p_0) and a pointed topological space (X, x_0), for any $[g_1], [g_2] \in [X, P]$ define

$$[g_1] * [g_2] := [\mu \circ (g_1, g_2)]. \tag{3.2.2.3}$$

Clearly the above multiplication is well-defined. If (P, μ, φ) is an H-group, then $[X, P]$ is a group under (3.2.2.3).

Note that in (3.2.2.3), the multiplication $*$ is determined only by μ and then by P. If we consider $[g_1], [g_2] \in [Y, P]$, we still use the same notion $*$ for the multiplication of $[Y, P]$.

Exercise 3.7

Show that if (P, μ, φ) *is an* H*-group, then* $[X, P]$ *is a group under* (3.2.2.3).

We then have a contravariant functor

$$\pi^P : \mathbf{HTop}_* \longrightarrow \mathbf{Group}, \quad X \longmapsto [X, P]$$

and for any morphism $[f] \in \mathbf{Hom}_{\mathbf{HTop}_*}(X, Y)$,

$$f^\# := \pi^P([f]) \in \mathbf{Hom}_{\mathbf{Group}}([Y, P], [X, P]), \quad [Y, P] \ni g' \longmapsto g' \circ f.$$

Consequently, we obtain the following

Theorem 3.14

If $P \equiv (P, \mu, \varphi)$ *is an* H*-group, then* $\pi^P : \mathbf{HTop}_* \to \mathbf{Group}$ *is a contradvariant functor. If* P *is an Abelian* H*-group, then* π^P *is a contravariant functor from* $\mathbf{HTop}_*$ *into* $\mathbf{AG}$.

Theorem 3.15

If $P \equiv (P, p_0)$ *is a pointed topological space such that* π^P *takes values in* $\mathbf{Group}$*, then* P *is an* H*-group (or Abelian* H*-group if* π^P *takes values in* $\mathbf{AG}$*).*

Proof. Consider the pointed topological space $P \times P \equiv (P \times P, (p_0, p_0))$ and $\pi^P(P \times P) =$

$[P\times P, P] \in \mathbf{Ob}(\mathbf{Group})$. Denote by $*$ the group multiplication of $[P\times P, P]$. The projections

$$\begin{array}{ccc} P \times P & \xrightarrow{\mathrm{pr}_1} & P \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & \\ P & & \end{array}$$

gives rise to a continuous map $\mu : P \times P \to P$ such that

$$[\mathrm{pr}_1] * [\mathrm{pr}_2] = [\mu].$$

Write $c : P \to P$ the constant map c_{p_0}.

(1) (P, μ) is an H-space. Consider the following diagrams

$$P \underset{(1,c)}{\xrightarrow{(c,1)}} P \times P \xrightarrow{\mu} P$$

and try to prove $\mu \circ (c, 1) \simeq 1 \simeq \mu \circ (1, c)$, where $1 := \mathbf{1}_P$. We first establish a general formula for $[\mu \circ (f, g)]$, where $f, g \in \mathbf{Hom}_{\mathbf{Top}_*}(X, P)$. In fact,

$$\begin{aligned} [\mu \circ (f, g)] &= (f, g)^{\#}[\mu] = (f, g)^{\#}([\mathrm{pr}_1] * [\mathrm{pr}_2]) \\ &= \Big((f, g)^{\#}[\mathrm{pr}_1]\Big) * \Big((f, g)^{\#}[\mathrm{pr}_2]\Big) = [f] * [g]. \end{aligned}$$

Consequently,

$$[\mu \circ (1, c)] = [1] * [c].$$

Consider the following two continuous maps

$$f : P \longrightarrow \{p_0\}, \quad g : \{p_0\} \longrightarrow P$$

defined, respectively, by

$$f(p) := p_0, \quad g(p_0) := p_0.$$

Since π^P takes values in $\mathbf{Group}$, it follows that

$$\pi^P([f]) \in \mathbf{Hom}_{\mathbf{Group}}([\{p_0\}, P], [P, P]).$$

In particular,

$$\pi^P([f])(e_{[\{p_0\},P]}) = e_{[P,P]},$$

where e_G stands for the identity element of the given group G. Because $[\{p_0\}, P]$ consists only one element $[g]$, we must have $[g] = e_{[\{p_0\},P]}$. Hence

$$e_{[P,P]} = \pi^P([f])([g]) = [g \circ f] = [c].$$

Thus $[c]$ is the identity element in the group $[P, P]$. Consequently,

$$[\mu \circ (1, c)] = [1] * [c] = [1] \iff \mu \circ (1, c) \simeq 1.$$

Similarly, one has $\mu \circ (c, 1) \simeq 1$.

(2) (P,μ,φ) is an H-group. We next show that μ is homotopy associative, i.e.,

$$\begin{tikzcd} P\times P\times P \arrow[r, "\mu\times 1"] \arrow[d, "1\times\mu"'] & P\times P \arrow[d, "\mu"] \\ P\times P \arrow[r, "\mu"'] & P \end{tikzcd}, \qquad \mu\circ(\mu\times 1)\simeq\mu\circ(1\times\mu).$$

Let $q_1,q_2,q_3 : P\times P\times P\to P$ be three projections. Then

$$[\mu\circ(1\times\mu)] = (1\times\mu)^{\#}[\mu] = (1\times\mu)^{\#}([\mathrm{pr}_1]*[\mathrm{pr}_2])$$
$$= (1\times\mu)^{\#}[\mathrm{pr}_1] *_{P\times P\times P} (1\times\mu)^{\#}[\mathrm{pr}_2] = [q_1]*[\mu\circ(q_2,q_3)] = [q_1]*([q_2]*[q_3]).$$

Similarly,

$$[\mu\circ(\mu\times 1)] = ([q_1]*[q_2])*[q_3].$$

Since $*$ is associative, it follows that $[\mu\circ(1\times\mu)] = [\mu\circ(\mu\times 1)]$.

Thirdly, we show that P has a homotopy inverse. Let $[\varphi]\in[P,P]$ be the inverse of $[1]$ in $[P,P]$, i.e., $[1]*[\varphi]=[c]=[\varphi]*[1]$. Then $c\simeq\mu(1,\varphi)$ and $c\simeq\mu(\varphi,1)$.

(3) (P,μ,φ) is an Abelian H-group if π^P takes values in $\mathbf{AG}$. In this case, $*$ is Abelian. Consider the following diagram

$$\begin{tikzcd} P\times P \arrow[r, "T"] \arrow[d, "\mu"'] & P\times P \arrow[d, "\mu"] \\ P \arrow[r, equal] & P \end{tikzcd}, \qquad T(p_1,p_2)=(p_2,p_1).$$

Because $T=(\mathrm{pr}_2,\mathrm{pr}_1)$, we obtain

$$[\mu\circ T] = [\mu\circ(\mathrm{pr}_2,\mathrm{pr}_1)] = [\mathrm{pr}_2]*[\mathrm{pr}_1] = [\mathrm{pr}_1]*[\mathrm{pr}_2] = [\mu].$$

Thus P is Abelian. □

Corollary 3.3

Let $\alpha : P\to P'$ be a map between H-groups. Then $\alpha_{\#}$ is a natural transformation from π^P to $\pi^{P'}$ in **Group** *if and only if α is an H-space homomorphism.* ♡

Proof. For $[f]\in\mathbf{Hom}_{\mathbf{HTop}_*}(X,Y)$, consider the following diagram

$$\begin{tikzcd} \pi^P(X) \arrow[r, "\alpha_{\#}(X)"] & \pi^{P'}(X) \\ \pi^P(Y) \arrow[u, "\pi^P(f)"] \arrow[r, "\alpha_{\#}(Y)"'] & \pi^{P'}(Y) \arrow[u, "\pi^{P'}(f)"'] \end{tikzcd}$$

For any $[g]\in[Y,P]$, we have

$$\left(\pi^{P'}(f)\circ\alpha_{\#}(Y)\right)[g] \;=\; \pi^{P'}(f)[\alpha\circ g] \;=\; [(\alpha\circ g)\circ f]$$

and

$$\left(\alpha_{\#}(X)\circ\pi^P(f)\right)[g] \;=\; \alpha_{\#}(X)[g\circ f] \;=\; [\alpha\circ(g\circ f)].$$

Hence $\pi^{P'}(f)\circ\alpha_{\#}(Y) = \alpha_{\#}(X)\circ\pi^P(f)$. We need only to verify that $\alpha_{\#}(X) : [X,P]\to$

$[X, P']$ is a group homomorphism. For $[g_1], [g_2] \in [X, P]$ we have

$$\alpha_\#(X)([g_1] * [g_2]) \;=\; \alpha_\#(X)[\mu \circ (g_1, g_2)] \;=\; [\alpha \circ \mu \circ (g_1, g_2)]$$

and

$$\alpha_\#(X)[g_1] * \alpha_\#(X)[g_2] \;=\; [\alpha \circ g_1] * [\alpha \circ g_2] \;=\; [\mu' \circ (\alpha \circ g_1, \alpha \circ g_2)].$$

Consequently, $\alpha_\#(X)$ is a group homomorphism if and only if $\alpha \circ \mu \simeq \mu' \circ (\alpha \times \alpha)$, i.e., $\alpha_\#(X)$ is a group homomorphism for any $X \in \mathbf{Ob}(\mathbf{HTop}_*)$ if and only if α is a H-space homomorphism. □

3.2.3 A classical example of H-groups

A classical example of H-groups is as follows. Let $Y \equiv (Y, y_0)$ be a pointed topological space. The **loop space of Y (based at y_0)**, denoted by $\Omega(Y, y_0)$, is defined to be

$$\Omega(Y, y_0) := \left\{ \omega : ([0,1], \{0,1\}) \longrightarrow (Y, y_0) \,\middle|\, \begin{array}{c} \text{continuous functions} \\ \text{topologized by the} \\ \text{compact-open topology} \end{array} \right\}. \tag{3.2.3.1}$$

We recall the compact-open topology. If $(X, A), (Y, B) \in \mathbf{Ob}(\mathbf{Top}_{**})$, then

$$\mathbf{Hom}_{\mathbf{Top}_{**}}((X, A), (Y, B))$$

denotes the set of continuous functions $f : X \to Y$ such that $f(A) \subseteq B$. Let Y^X denote the space of continuous functions from X to Y, given the **compact-open topology** (which is the topology generated by the subbasis

$$\left\{ \mathbf{Hom}_{\mathbf{Top}_{**}}((X, K), (Y, U)) : K \text{ compact in } X \text{ and } U \text{ open in } Y \right\}. \tag{3.2.3.2}$$

Thus the compact-open topology is the collection of subsets consisting of Y^X, $\emptyset$, and all unions of finite intersections of elements of (3.2.3.2).

(1) Let

$$\mathbf{E} : Y^X \times X \longrightarrow Y, \quad (f, x) \longmapsto f(x), \tag{3.2.3.3}$$

denote the **evaluation map**. Given a function $g : Z \to Y^X$, the composite

$$Z \times X \xrightarrow{g \times 1} Y^X \times X \xrightarrow{\mathbf{E}} Y \tag{3.2.3.4}$$

is a function from $Z \times X$ to Y.

(2) **(Theorem of exponential correspondence)** If X is locally compact Hausdorff space and $Y, Z \in \mathbf{Ob}(\mathbf{Top})$, a map $g : Z \to Y^X$ is continuous if and only if $\mathbf{E} \circ (g \times 1) : Z \times X \to Y$ is continuous.

Theorem 3.16

$\Omega(Y, y_0)$ is an H-group. ♡

Proof. Define $\omega_0 : [0,1] \to Y$ to be $\omega_0(t) \equiv y_0$. Then $\omega_0 \in \Omega(Y, y_0)$ and then $\Omega(Y, y_0) \in \mathbf{Ob}(\mathbf{Top}_*)$.

(1) Define a map

$$\mu : \Omega(Y, y_0) \times \Omega(Y, y_0) \longrightarrow \Omega(Y, y_0)$$

by

$$\mu(\omega, \omega')(t) := (\omega * \omega')(t) = \begin{cases} \omega(2t), & 0 \le t \le \frac{1}{2}, \\ \omega'(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

To prove μ is continuous, we use the above Theorem of exponential correspondence. Let $\mathbf{E} : \Omega(Y, y_0) \times [0,1] \to Y$ be the evaluation map. It suffices to show that the composite

$$\Omega(Y, y_0) \times \Omega(Y, y_0) \times [0,1] \xrightarrow{\mu \times 1} \Omega(Y, y_0) \times [0,1] \xrightarrow{\mathbf{E}} Y$$

is continuous. By definition of μ, we see that $\mathbf{E} \circ (\mu \times 1)$ is continuous so that μ is continuous.

(2) To prove

$$\Omega(Y, y_0) \underset{(1,c)}{\xrightarrow{(c,1)}} \Omega(Y, y_0) \times \Omega(Y, y_0) \xrightarrow{\mu} \Omega(Y, y_0), \quad \mu \circ (c, 1) \simeq 1 \simeq \mu \circ (1, c)$$

where $1 := \mathbf{1}_{\Omega(Y, y_0)}$, we define a map

$$F : \Omega(Y, y_0) \times [0,1] \longrightarrow \Omega(Y, y_0), \quad F(\omega, t)(s) = \begin{cases} \omega(\frac{2s}{t+1}), & 0 \le s \le \frac{t+1}{2}, \\ y_0, & \frac{t+1}{2} \le s \le 1. \end{cases}$$

Since $\mathbf{E} \circ (F \times 1) : \Omega(Y, y_0) \times [0,1] \times [0,1] \to Y$ is continuous, it follows that F is continuous and then $F : \mu \circ (1, c) \simeq 1$. Similarly, we have $\mu \circ (c, 1) \sim 1$. Hence $(\Omega(Y, y_0), \mu)$ is an H-space.

(3) To show that μ is homotopy associative, consider the following diagram

$$\begin{tikzcd} \Omega(Y, y_0) \times \Omega(Y, y_0) \times \Omega(Y, y_0) \arrow[r, "\mu \times 1"] \arrow[d, "1 \times \mu"'] & \Omega(Y, y_0) \times \Omega(Y, y_0) \arrow[d, "\mu"] \\ \Omega(Y, y_0) \times \Omega(Y, y_0) \arrow[r, "\mu"'] & \Omega(Y, y_0) \end{tikzcd}$$

Define a homotopy by

$$G : \Omega(Y, y_0) \times \Omega(Y, y_0) \times \Omega(Y, y_0) \times [0,1] \longrightarrow \Omega(Y, y_0)$$

by

$$G(\omega, \omega', \omega'', t)(s) := \begin{cases} \omega(\frac{4s}{t+1}), & 0 \le s \le \frac{t+1}{4}, \\ \omega'(4s - t - 1), & \frac{t+1}{4} \le s \le \frac{t+2}{4}, \\ \omega''(\frac{4s-t-2}{2-t}), & \frac{t+2}{4} \le s \le 1. \end{cases}$$

Then G is continuous and $G : \mu \circ (\mu \times 1) \simeq \mu \circ (1 \times \mu)$.

(4) Define a homotopy inverse $\varphi : \Omega(Y, y_0) \to \Omega(Y, y_0)$ by

$$\varphi(\omega)(s) := \omega(1 - s), \quad s \in [0,1].$$

Define a homotopy by

$$H : \Omega(Y, y_0) \times [0,1] \longrightarrow \Omega(Y, y_0)$$

by

$$H(\omega,t)(s)=\begin{cases} y_0, & 0\le s\le \frac{t}{2},\\ \omega(2s-t), & \frac{t}{2}\le s\le \frac{1}{2},\\ \omega(2-2s-t), & \frac{1}{2}\le s\le 1-\frac{t}{2},\\ y_0, & 1-\frac{t}{2}\le s\le 1.\end{cases}$$

Then H is continuous and $H:\mu\circ(1,\varphi)\simeq c$. Similarly, we have $\mu\circ(\varphi,1)\simeq c$.

In summary, $(\Omega(Y,y_0),\mu,\varphi)$ is an H-group. □

Let $\mathbf{HGroup}$ denote the category of H-groups and continuous H-space homomorphisms.

Proposition 3.7

The loop functor Ω is a covariant functor from $\mathbf{Top}_*$ to $\mathbf{HGroup}$. ♡

Proof. For $(Y,y_0)\in\mathbf{Ob}(\mathbf{Top}_*)$, define $\Omega((Y,y_0)):=(\Omega(Y,y_0),\mu,\varphi)\in\mathbf{Ob}(\mathbf{HGroup})$. If $h\in\mathbf{Hom}_{\mathbf{Top}_*}(Y,Y')$ $(h(y_0)=y_0')$, we define

$$\Omega(h):\Omega(Y,y_0)\longrightarrow\Omega(Y',y_0')$$

by

$$\Omega(h)(\omega)(s):=h(\omega(s)),\quad s\in[0,1].$$

To verify $\Omega(h)\in\mathbf{Hom}_{\mathbf{HGroup}}(\Omega(Y,y_0),\Omega(Y',y_0'))$ we consider the following diagram

$$\begin{tikzcd} \Omega(Y,y_0)\times\Omega(Y,y_0) \arrow[r, "\mu"] \arrow[d, "\Omega(h)\times\Omega(h)"'] & \Omega(Y,y_0) \arrow[d, "\Omega(h)"] \\ \Omega(Y',y_0')\times\Omega(Y',y_0') \arrow[r, "\mu'"'] & >\Omega(Y',y_0') \end{tikzcd}$$

For any $\omega,\omega'\in\Omega(Y,y_0)$ we have

$$\begin{aligned}(\Omega(h)\circ\mu)(\omega,\omega') &= \Omega(h)(\mu(\omega,\omega')) = h\circ\mu(\omega,\omega') = h\circ(\omega*\omega')\\ \simeq (h\circ\omega)*(h\circ\omega') &= \mu'(h\circ\omega,h\circ\omega') = \mu'(\Omega(h)(\omega),\Omega(h)(\omega'))\\ &= \left(\mu'\circ(\Omega(h)\times\Omega(h))\right)(\omega,\omega').\end{aligned}$$

Thus $\Omega(h)\circ\mu\simeq\mu'\circ(\Omega(h)\times\Omega(h))$. In summary, Ω is a covariant functor. □

3.2.4 Suspension

If (X,x_0) and (Y,y_0) are pointed topological spaces, we define

$$X\vee Y:=(X\times\{y_0\})\cup\{x_0\}\times Y \tag{3.2.4.1}$$

the subspace of $X\times Y$. If $f\in\mathbf{Hom}_{\mathbf{Top}_*}(X,Z)$ and $g\in\mathbf{Hom}_{\mathbf{Top}_*}(Y,Z)$, then

$$(f,g):X\vee Y\longrightarrow Z \tag{3.2.4.2}$$

is given by $(f,g)|_X=f$ and $(f,g)|_Y=g$. Note that $X\vee Y$ can be regarded as a pointed topological space $(X\vee Y,(x_0,y_0))$.

Definition 3.2

Let $\mathfrak{C}$ be a category and $X_i \in \mathbf{Ob}(\mathfrak{C})$, $i \in I$. The **direct sum** *of $X_i (i \in I)$ is an object $\bigoplus_{i\in I} X_i$ together with a family of morphisms*

$$k_i : X_i \longrightarrow \bigoplus_{i\in I} X_i$$

with the universal property: For any object Y and any family of morphisms $f_i : X_i \to Y$, there exists a unique morphism $f : \bigoplus_{i\in I} X_i \to Y$ such that $f \circ k_i = f_i$ for any $i \in I$.

$$\begin{tikzcd} \bigoplus_{i\in I} X_i \arrow[d, "\exists! f"] & X_i \arrow[l, "k_i"'] \arrow[d, "f_i"] \\ C \arrow[r, equal] & C \end{tikzcd}$$

If the direct sum of X_i exists, it is unique up to unique morphism. ♣

We now show that $X \vee Y$, $X \equiv (X, x_0)$, $Y \equiv (Y, y_0)$ are pointed topological spaces, is the direct sum $X \oplus Y$ of X and Y in $\mathbf{Top}_*$. Consider

$$k_1 : X \to X \vee Y, \quad k_2 : Y \to X \vee Y$$

given, respectively, by $k_1(x) := (x, y_0)$ and $k_1(y) := (x_0, y)$. For any $Z \in \mathbf{Ob}(\mathbf{Top}_*)$ and $f_1 \in \mathbf{Hom}_{\mathbf{Top}_*}(X, Z)$ and $f_2 \in \mathbf{Hom}_{\mathbf{Top}_*}(Y, Z)$, define

$$f : X \vee Y \longrightarrow Z, \quad f(x, y) := \begin{cases} f_1(x), & (x, y) \in X \times \{y_0\}, \\ f_2(y), & (x, y) \in \{x_0\} \times Y. \end{cases}$$

Hence $f \circ k_i = f_i$, $i = 1, 2$ and therefore $X \vee Y \cong X \oplus Y$ in $\mathfrak{C}$.

An ***H*-cogroup** consists of a pointed topological space $Q \equiv (Q, q_0)$ together with a continuous multiplication

$$\nu : Q \longrightarrow Q \vee Q \tag{3.2.4.3}$$

for which the constant $c : P \to P$, given by $c = c_{q_0}$, is a **homotopy identity**, that is, each composite, $1 = 1_Q$,

$$Q \xrightarrow{\nu} Q \wedge Q \xrightarrow{(c,1)} Q, \quad Q \xrightarrow{\nu} Q \vee Q \xrightarrow{(1,c)} Q$$

is homotopic to 1.

(1) The multiplication ν is said to be **homotopy associative** if the square

$$\begin{tikzcd} Q \arrow[r, "\nu"] \arrow[d, "\nu"'] & Q \vee Q \arrow[d, "1\vee\nu"] \\ Q \vee Q \arrow[r, "\nu\vee 1"'] & Q \vee Q \vee Q \end{tikzcd}$$

is homotopy commutative, i.e.,

$$(1 \vee \nu) \circ \nu \simeq (\nu \vee 1) \circ \nu.$$

(2) A continuous map $\psi : Q \to Q$ is called a **homotopy inverse** for Q and ν if each composite

$$Q \xrightarrow{\nu} Q \vee Q \xrightarrow{(1,\psi)} Q, \quad Q \xrightarrow{\nu} Q \vee Q \xrightarrow{(\psi,1)} Q$$

is homotopic to c.

(3) An ***H*-cogroup** (Q, ν, ψ) is a homotopy associative H-cospace with a homotopy inverse.

(4) A multiplication ν in an H-cospace Q is said to be **homotopy Abelian** if the triangle

$$\begin{tikzcd} Q \arrow[r, "\nu"] \arrow[d, equal] & Q \vee Q \arrow[d, "T"] \\ Q \arrow[r, "\nu"'] & Q \vee Q \end{tikzcd} \qquad T(q_1, q_2) := (q_2, q_1),$$

is homotopy commutative, that is,

$$T \circ \nu \simeq \nu.$$

(5) An H-cogroup with homotopy Abelian multiplication is called an **Abelian *H*-cogroup**.

(6) An ***H*-cospace homomorphism** between two H-cospaces (Q, ν) and (Q', ν') is a continuous map $\beta : Q \to Q'$ such that he square

$$\begin{tikzcd} Q' \arrow[r, "\nu"] \arrow[d, "\beta"'] & Q \vee Q \arrow[d, "\beta\vee\beta"] \\ Q' \arrow[r, "\nu'"'] & Q' \vee Q' \end{tikzcd}$$

is homotopy commutative, that is,

$$(\beta \vee \beta) \circ \nu \simeq \nu' \circ \beta.$$

Theorem 3.17

A pointed topological space having the same homotopy type as an H-cospace (or an H-cogroup, or an Abelian H-cogroup), is itself an H-cospace (or an H-cogroup, or an Abelian H-cogroup) in such a way that the homotopy equivalence is an H-cospace homomorphism. ♡

Proof. Similar to **Theorem 3.13**. □

Given an H-cospace (Q, ν) (with base point q_0) and a pointed topological space (X, x_0), for any $[f_1], [f_2] \in [Q, X]$ define

$$[f_1] * [f_2] := [(f_1, f_2) \circ \nu]. \tag{3.2.4.4}$$

Clearly the above multiplication is well-defined. If (P, μ, φ) is an H-group, then $[X, P]$ is a group under (3.2.4.4).

We then have a covariant functor

$$\pi_Q : \mathbf{HTop}_* \longrightarrow \mathbf{Group}, \quad X \longmapsto [Q, X]$$

and for any morphism $[f] \in \mathbf{Hom}_{\mathbf{HTop}_*}(X, Y)$,

$$f_\# := \pi_Q([f]) \in \mathbf{Hom}_{\mathbf{Group}}([Q, X], [Q, Y]), \quad [Q, X] \ni g' \longmapsto f \circ g'.$$

Consequently, we obtain the following

Theorem 3.18

If $Q \equiv (Q, \nu, \psi)$ is an H-cogroup, then $\pi_Q : \mathbf{HTop}_* \to \mathbf{Group}$ is a covariant functor. If P is an Abelian H-cogroup, then π_Q is a covariant functor from $\mathbf{HTop}_*$ into $\mathbf{AG}$. ♡

Proof. Similar to Theorem 3.14. □

Theorem 3.19

If $Q \equiv (Q, q_0)$ is a pointed topological space such that π_Q takes values in $\mathbf{Group}$, then Q is an H-cogroup (or Abelian H-cogroup if π_Q takes values in $\mathbf{AG}$). ♡

Proof. Similar to Theorem 3.15. □

Corollary 3.4

Let $\beta : Q \to Q'$ be a map between H-cogroups. Then $\beta^\#$ is a natural transformation from $\pi_{Q'}$ to π_Q in $\mathbf{Group}$ if and only if β is an H-cospace homomorphism. ♡

Proof. Similar to Corollary 3.3. □

Exercise 3.8

Verify Theorem 3.17, Theorem 3.18, Theorem 3.19, *and* Corollary 3.4. ♠

3.2.5 A classical example of H-cogroups

A classical example of H-cogroups is as follows. Let $Z \equiv (Z, z_0)$ be a pointed topological space. The **suspension of Z (based at z_0)**, denoted by $\Xi(Z, z_0)$, is defined to be

$$\Xi(Z, z_0) := Z \times [0,1]/((Z \times \{0\}) \cup (\{z_0\} \times [0,1]) \cup (Z \times \{1\})). \tag{3.2.5.1}$$

The equivalence class of $(z,t) \in Z \times [0,1]$ is denoted by $[z,t]$ so that

$$[z,0] = [z',1] = [z_0,t], \quad \text{for all } z, z' \in Z \text{ and } t \in [0,1].$$

Moreover, $\Xi(Z, z_0)$ is a pointed topological space with base point $[z_0, 0]$. Define

$$\nu : \Xi(Z, z_0) \longrightarrow \Xi(Z, z_0) \vee \Xi(Z, x_0)$$

by

$$\nu([z,t]) := \begin{cases} ([z,2t],[z_0,0]), & 0 \le t \le \frac{1}{2}, \\ ([z_0,0],[z,2t-1]), & \frac{1}{2} \le t \le 1. \end{cases}$$

Theorem 3.20

$\Xi(Z, z_0)$ is an H-cogroup. ♡

Let **HCGroup** denote the category of H-cogroups and continuous H-cospace homomorphisms.

Proposition 3.8

The loop functor Ξ is a covariant functor from $\mathbf{Top}_*$ *to* **HCGroup**.

Lemma 3.3

For $n \geq 0$, $\Xi(\mathbf{S}^n)$ is homeomorphic to $\mathbf{S}^{n+1}$.

Proof. Let $\boldsymbol{p}_0 := (1, 0, \cdots, 0) \in \mathbf{S}^n$ be a base point of $\mathbf{S}^n$. Then

$$\mathbf{S}^n = \{\boldsymbol{z} \in \mathbf{R}^{n+2} | \boldsymbol{x} = (x^1, \cdots, x^{n+2}), \ |\boldsymbol{x}| = 1, \ x^{n+2} = 0\}$$

and

$$\mathbf{D}^{n+1} = \{\boldsymbol{x} \in \mathbf{R}^{n+2} | \boldsymbol{x} = (x^1, \cdots, x^{n+2}), \ |\boldsymbol{x}| \leq 1, \ x^{n+2} = 0\}.$$

It is clear that

$$\mathbf{S}^{n+1} = H_+ \cup H_-, \quad \mathbf{S}^n = H_+ \cap H_-,$$

where

$$H_\pm := \{\boldsymbol{x} \in \mathbf{S}^{n+1} | \pm x^{n+2} \geq 0\}.$$

The natural projection $\mathbf{R}^{n+2} \to \mathbf{R}^{n+1}$ gives rise two projections

$$p_\pm : H_\pm \longrightarrow \mathbf{D}^{n+1}, \quad \boldsymbol{x} = (x^1, \cdots, x^{n+2}) \longmapsto (x^1, \cdots, x^{n+1}).$$

Define $\boldsymbol{F} : \Xi(\mathbf{S}^n) \to \mathbf{S}^{n+1}$ to be, $(\boldsymbol{x}, t) \in \mathbf{S}^n \times [0, 1]$,

$$\boldsymbol{F}([\boldsymbol{x}, t]) := \begin{cases} p_-^{-1}(2t\boldsymbol{x} + (1 - 2t)\boldsymbol{p}_0), & 0 \leq t \leq \frac{1}{2}, \\ p_+^{-1}(2(1 - t)\boldsymbol{x} + (2t - 1)\boldsymbol{p}_0), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Since

$$\boldsymbol{F}([\boldsymbol{x}, 0]) = p_-^{-1}(\boldsymbol{p}_0) = (1, 0, \cdots, 0, 0) = p_+^{-1}(\boldsymbol{p}_0) = \boldsymbol{F}([\boldsymbol{x}, 1]) = \boldsymbol{F}([\boldsymbol{p}_0, t]) \in \mathbf{S}^{n+1},$$

it follows that $\boldsymbol{F} \in \mathbf{Hom}_{\mathbf{Top}_*}(\Xi(\mathbf{S}^n), \mathbf{S}^{n+1})$. Solving $\boldsymbol{F} \circ \boldsymbol{G} = \mathbf{1}$ and $\mathbf{G} \circ \mathbf{F} = \mathbf{1}$, we can construct the inverse $\boldsymbol{G} : \mathbf{S}^{n+1} \to \Xi(\mathbf{S}^n)$. So $\boldsymbol{F} : \Xi(\mathbf{S}^n) \to \mathbf{S}^{n+1}$ is homeomorphic. $\square$

Definition 3.3. (Higher homotopy groups)

For $n \geq 1$ the **n-th homotopy group functor** *π_n is the covariant functor on* $\mathbf{HTop}_*$ *defined by $\pi_n := \pi_{\mathbf{S}^n}$. For any $X \equiv (X, x_0) \in \mathbf{Ob}(\mathbf{HTop}_*)$, we call $\pi_n(X) \equiv \pi_n(X, x_0)$ the* **n-th homotopy group**.

Note π_1 is just the fundamental group functor proved in **Subsection 3.3.2**.

Proposition 3.9

A map $\alpha : \mathbf{S}^n \to X$ represents the trivial element of $\pi_n(X)$ for $n \geq 1$, where $X \in$ $\mathbf{Ob}(\mathbf{Top}_*)$, *if and only if α can be continuously extended over $\mathbf{D}^{n+1}$.*

Proof. By **Theorem 3.11**. □

3.2.6 Higher homotopy groups

If $P \equiv (P, \mu, \varphi) \in \mathbf{Ob}(\mathbf{HGroup})$ and $Q \equiv (Q, \nu, \psi) \in \mathbf{OB}(\mathbf{HCGroup})$, then we have two different group multiplication on $[P, Q]$ given, respectively, by (3.2.2.3) and (3.2.4.4).

Proposition 3.10

Let X and Y be two objects in a category $\mathfrak{C}$ and let $*$ and $*'$ be two laws of composition in $\mathbf{Hom}_{\mathfrak{C}}(X, Y)$ such that

(a) *$*$ and $*'$ have a common two-sided identity element, that is, there is a morphism $f_0 \in \mathbf{Hom}_{\mathfrak{C}}(X, Y)$ such that*

$$f * f_0 = f_0 * f = f = f *' f_0 = f_0 *' f,$$

for any $f \in \mathbf{Hom}_{\mathfrak{C}}(X, Y)$;

(b) *$*$ and $*'$ are mutually distributive, i.e., for any $f_1, f_2, g_1, g_2 \in \mathbf{Hom}_{\mathfrak{C}}(X, Y)$*

$$(f_1 * f_2) *' (g_1 * g_2) = (f_1 *' g_1) * (f_2 *' g_2).$$

Then $*$ and $*'$ are equal, and each is commutative and associative. ♡

Proof. For any $f, g \in \mathbf{Hom}_{\mathfrak{C}}(X, Y)$ one has

$$f * g = (f *' f_0) * (f_0 *' g) = (f * f_0) *' (f_0 * g) = f *' g$$

and

$$g * f = (f_0 *' g) * (f *' f_0) = (f_0 * f) *' (g * f_0) = f *' g.$$

Therefore $f * g = f *' g = g * f$. For any $f, g, h \in \mathbf{Hom}_{\mathfrak{C}}(X, Y)$ we have

$$(f * g) * h = (f * g) *' h = (f * g) *' (f_0 * h) = (f *' f_0) * (g *' h) = f * (g * h).$$

This proves the associativity of $*$ and then of $*'$. □

Corollary 3.5

If $P = (P, \mu)$ is an H-space and $Q \equiv (Q, \nu, \psi)$ is an H-cogroup, then $[Q, P]$ is an Abelian group and the group structure is defined by the multiplication map in P. ♡

Proof. Define two multiplications $*$ and $*'$ in $[Q, P] = \mathbf{Hom}_{\mathbf{HTop}_*}(Q, P)$ by

$$[f] * [g] := [\mu \circ (f, g)], \quad [f] *' [g] := [(f, g) \circ \nu], \quad [f], [g] \in [Q, P].$$

Because $\mu \circ (c, 1) \simeq 1 \simeq \mu \circ (1, c)$, with $c := c_{p_0}$, we get

$$\mu \circ (f, c) \simeq f \simeq \mu \circ (c, f) \quad \Longrightarrow \quad [f] * [c] = [f] = [c] * [f].$$

Similarly, using the definition of ν and noting that $f(q_0) = p_0$, we have

$$[f] *' [c] = [f] = [c] *' [f].$$

The mutual distribution of $*$ and $*'$ can be proved, see **Exercise 3.9**, in the same way.

According to **Proposition 3.10**, we can conclude that $*$ and $*'$ are equal, commutative and associative. $\square$

Corollary 3.6

If $P = (P, \mu)$ is an H-space, then $\pi_n(P)$ is Abelian for all $n \geq 1$. ♡

Proof. Recall from **Lemma 3.3** that

$$\pi_n(P) = \pi_{\mathbf{S}^n}(P) = [\mathbf{S}^n, P] = [\Xi(\mathbf{S}^{n-1}), P], \quad n \geq 1.$$

Using **Theorem 3.20** and **Corollary 3.5** yields that $\pi_n(P)$ is Abelian for any H-space and any $n \geq 1$. $\square$

To extend **Corollary 3.6** to general pointed topological spaces, we consider double suspensions

$$\pi_n(X) = [\Xi(\mathbf{S}^{n-1}), X] = \left[\Xi\left(\Xi(\mathbf{S}^{n-2})\right), X\right], \quad n \geq 2.$$

Define two multiplications $*$ and $*'$ on $[\Xi(\Xi(\mathbf{S}^{n-2})), P]$ as follows:

$$[f] * [g] := [f * g], \quad [f] *' [g] := [f *' g], \quad [f], [g] \in [\Xi(\Xi(\mathbf{S}^{n-2})), P],$$

where

$$(f * g)([[z, t], s]) := \begin{cases} f([[z, 2t], s]), & 0 \leq t \leq \frac{1}{2}, \\ g([[z, 2t-1], s]), & \frac{1}{2} \leq t \leq 1, \end{cases}$$

and

$$(f *' g)([[z, t], s]) := \begin{cases} f([[z, t], 2s]), & 0 \leq s \leq \frac{1}{2}, \\ g([[z, t], 2s-1]), & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Let $c : \Xi(\Xi(\mathbf{S}^{n-2})) \to P$ be the constant map, i.e., $c([[z, t], s]) = p_0$.

Corollary 3.7

For $n \geq 2$, $\pi_n : \mathbf{HTop}_* \to \mathbf{AG}$ is a covariant functor. ♡

Proof. (1) $[f] * [c] = [f] = [f] *' [c]$. We first prove that $f * c \simeq f$. Consider the map

$$F : \Xi(\Xi(\mathbf{S}^{n-2})) \times [0, 1] \longrightarrow P, \quad ([[z, t], s], u) \longmapsto \begin{cases} f([[z, \frac{2t}{u+1}], s]), & 0 \leq t \leq \frac{u+1}{2}, \\ p_0, & \frac{u+1}{2} \leq t \leq 1. \end{cases}$$

When $t = \frac{u+1}{2}$, we see that $f([[z, \frac{2t}{u+1}], s]) f([[z, 1], s]) = p_0$, since $[z, 1]$ is the base point of $\Xi(\mathbf{S}^{n-2})$. Hence F is continuous. Observe that

$$F([[z, t], s], 0) = f * c, \quad F([[z, t], s], 1) = f.$$

Hence $f * c \simeq f$. Similarly, one can prove $c * f \simeq f$ and $f *' c \simeq f \simeq c *' f$.

The mutual distribution of $*$ and $*'$ can be proved, see **Exercise 3.9**, in the same way.

According to **Proposition 3.10**, we can conclude that $*$ and $*'$ are equal, commutative and associative. Thus $\pi_n(X)$ is Abelian for all $n \geq 2$. $\square$

Exercise 3.9

Prove the mutual distribution in **Corollary 3.5** *and* **Corollary 3.7**.

3.3 $\pi_1(\mathbf{S}^1)$

Introduction

- *The fundamental groupoid*
- *The equivalent of* π_1
- $\pi_1(\mathbf{S}^1) = \mathbf{Z}$

Recall some facts from categories. A category $\mathfrak{C}$ is said to be **small** if $\mathbf{Ob}(\mathfrak{C})$ is a set. A morphism $f \in \mathbf{Hom}_{\mathfrak{C}}(X, Y)$ is called an **equivalence** if there is a morphism $g \in \mathbf{Hom}_{\mathfrak{C}}(Y, X)$ satisfying

$$g \circ f = \mathbf{1}_X, \quad f \circ g = \mathbf{1}_Y.$$

If $g' \in \mathbf{Hom}_{\mathfrak{C}}(Y, X)$ is another morphism satisfying

$$g' \circ f = \mathbf{1}_X, \quad f \circ g' = \mathbf{1}_Y,$$

then

$$g' = g' \circ \mathbf{1}_Y = g' \circ (f \circ g) = (g' \circ f) \circ g = \mathbf{1}_X \circ g = g.$$

Thus, g is unique and denoted by f^{-1}. A **groupoid** $\mathfrak{P}$ is a small category in which each morphism is an equivalence.

Proposition 3.11

Let $\mathfrak{P}$ be a groupoid.

(1) *For any $A, B \in \mathbf{Ob}(\mathfrak{P})$ define a relation*

$$A \sim B \iff \mathbf{Hom}_{\mathfrak{P}}(A, B) \neq \emptyset.$$

Then $\sim$ is an equivalence relation on the set $\mathbf{Ob}(\mathfrak{P})$. The equivalence classes are called the **components** *of $\mathfrak{P}$. The groupoid $\mathfrak{P}$ is said to be* **connected** *if it has only one component.*

(2) *For any $A \in \mathbf{Ob}(\mathfrak{P})$, the law of composition*

$$\mathbf{Hom}_{\mathfrak{P}}(A, A) \times \mathbf{Hom}_{\mathfrak{P}}(A, A) \longrightarrow \mathbf{Hom}_{\mathfrak{P}}(A, A), \quad (g, f) \longmapsto f \circ g$$

is a group operation in $\mathbf{Hom}_{\mathfrak{P}}(A, A)$.

(3) *There is a covariant functor $\mathbf{ad} : \mathfrak{P} \to \mathbf{Group}$ which assigns to an object $A \in \mathbf{Ob}(\mathfrak{P})$ the group $\mathbf{ad}(A) = \mathbf{Hom}_{\mathfrak{P}}(A, A)$ and to a morphism $f \in \mathbf{Hom}_{\mathfrak{P}}(A, B)$ the group*

homomorphism $\mathbf{ad}_f \in \mathbf{Hom}_{\mathbf{Group}}(\mathbf{ad}(A), \mathbf{ad}(B))$,

$$\mathbf{ad}_f : \mathbf{Hom}_{\mathfrak{P}}(A, A) \longrightarrow \mathbf{Hom}_{\mathfrak{P}}(B, B), \quad g \longmapsto \mathbf{ad}_f(g),$$

defined by $\mathbf{ad}_f(g) := f \circ g \circ f^{-1}$. *Actually,* $\mathbf{ad}_f$ *is a group isomorphism.*

(4) *If* $A \sim B$ *in* $\mathfrak{P}$, *the collection of isomorphisms* $\{\mathbf{ad}_f : f \in \mathbf{Hom}_{\mathfrak{P}}(A, B)\}$ *is a conjugacy class of isomorphisms* $\mathbf{Hom}_{\mathfrak{P}}(A, A) \to \mathbf{Hom}_{\mathfrak{P}}(B, B)$.

(5) *Choose another groupoid* $\mathfrak{P}'$ *and let* $\mathbf{F}$ *be a covariant functor from* $\mathfrak{P}$ *to* $\mathfrak{P}'$. *Then* $\mathbf{F}$ *maps each component of* $\mathfrak{P}$ *into some component of* $\mathfrak{P}'$. ♡

Proof. (1) Obvious.

(2) The identity element of $\mathbf{Hom}_{\mathfrak{P}}(A, A)$ is the identity morphism

$$\mathbf{1}_A \in \mathbf{Hom}_{\mathfrak{P}}(A, A).$$

It is clearly that the group axioms are satisfied.

(3) For any $g_1, g_2 \in \mathbf{Hom}_{\mathfrak{P}}(A, A)$, one has

$$\mathbf{ad}_f(\mathbf{1}_A) = f \circ \mathbf{1}_A \circ f^{-1} = f \circ f^{-1} = \mathbf{1}_A$$

and

$$\begin{aligned}\mathbf{ad}_f(g_1 \circ g_2) &= f \circ (g_1 \circ g_2) \circ f^{-1} \\ &= (f \circ g_1 \circ f^{-1}) \circ (f \circ g_2 \circ f^{-1}) = \mathbf{ad}_f(g_1) \circ \mathbf{ad}_f(g_2).\end{aligned}$$

Thus $\mathbf{ad}_f$ is a group homomorphism. Hence $\mathbf{ad}$ is a covariant functor from $\mathfrak{P}$ to $\mathbf{Group}$. Since $\mathbf{ad}_f^{-1} = \mathbf{ad}_{f^{-1}}$, it follows that $\mathbf{ad}_f$ is a group isomorphism.

(4) It follows from (3).

(5) Obvious. □

Exercise 3.10

Complete the proof of **Proposition 3.11**. ♠

3.3.1 The fundamental groupoid

A path ω in a topological space X is a continuous map $\omega : [0, 1] \to X$; the point $\omega(0)$ is called the **origin** of ω, while the point $\omega(1)$ the **end** of ω. Two paths ω, ω' in X are said to be **homotopic** if they are homotopic relative to $\partial[0, 1] = \{0, 1\}$; in this case we denote by $\omega \simeq \omega'$ instead of $\omega \simeq \omega'(\text{rel } \{0, 1\})$. It is clear that "$\simeq$" is an equivalence relation so the resulting equivalence classes, denoted as $[\omega]$, are called **path classes**.

Recall the first fundamental group $(\pi_1(X, x), *)$ defined in (3.1.2.3). We now relax one point x to two points x_1, x_2, and then define a "fundamental group", called **fundamental groupoid**, which is a grouopoid.

Given a topological space $X \in \mathbf{Ob}(\mathbf{Top})$, we define a category $\mathfrak{P}(X)$ whose objects are the points of X, whose morphisms from x_1 to x_0, $x_0, x_1 \in X$, are the path class $[\omega]$ with x_0 as an origin and x_1 as end, and whose composite is the product of path classes:

$$[\omega] * [\omega'] = [\omega * \omega'], \quad [\omega] \in \mathbf{Hom}_{\mathfrak{P}(X)}(x_0, x_1), \quad [\omega'] \in \mathbf{Hom}_{\mathfrak{P}(X)}(x_1, x_2),$$

given by

$$(\omega * \omega')(s) := \begin{cases} \omega(2s), & 0 \le s \le \frac{1}{2}, \\ \omega(2s-1), & \frac{1}{2} \le s \le 1. \end{cases}$$

Theorem 3.21

$\mathfrak{P}(X)$ is a groupoid for any topological space $X \in \mathbf{Ob}(\mathbf{Top})$. ♡

Proof. (1) $\mathfrak{P}(X)$ is s category. To prove the associativity of the composite, consider

$$[\omega] \in \mathbf{Hom}_{\mathfrak{P}(X)}(x_0, x_1), \quad [\omega'] \in \mathbf{Hom}_{\mathfrak{P}(X)}(x_1, x_2), \quad [\omega''] \in \mathbf{Hom}_{\mathfrak{P}(X)}(x_2, x_3).$$

We should verify $(\omega * \omega') * \omega'' \simeq \omega * (\omega' * \omega'')$. Define the corresponding homotopy $F : [0,1] \times [0,1] \to X$, $(s,t) \mapsto F(s,t)$, by

$$F(s,t) = \begin{cases} \omega(\frac{4s}{t+1}), & 0 \le s \le \frac{t+1}{4}, \\ \omega'(4s-t-1), & \frac{t+1}{4} \le s \le \frac{t+2}{4}, \\ \omega''(\frac{4s-t-2}{2-t}), & \frac{t+2}{4} \le s \le 1. \end{cases}$$

For any $x \in X$, we should construct the identity morphism $\mathbf{1}_x \in \mathbf{Hom}_{\mathfrak{P}(X)}(x,x)$. Let $c_x : [0,1] \to X$ be the constant map given by $c_x(s) := x$ for any $s \in [0,1]$. We claim that

$$[c_x] = \mathbf{1}_x \in \mathbf{Hom}_{\mathfrak{P}(X)}(x,x), \quad x \in \mathbf{Ob}(\mathfrak{P}(X)).$$

For any $[\omega] \in \mathbf{Hom}_{\mathfrak{P}(X)}(x', x)$, define a homotopy $G : [0,1] \times [0,1] \to X$ by

$$G(s,t) = \begin{cases} \omega(\frac{2s}{t+1}), & 0 \le s \le \frac{t+1}{2}, \\ x, & \frac{t+1}{2} \le s \le 1. \end{cases}$$

Then $G : \omega * c_x \simeq \omega$. Thus $[\omega] * [c_x] = [\omega]$ for any $[\omega] \in \mathbf{Hom}_{\mathfrak{P}(X)}(x', x)$. Similarly, we have

$$[c_x] * [\omega] = [\omega], \quad \text{for all } [\omega] \in \mathbf{Hom}_{\mathfrak{P}(X)}(x, x').$$

(2) $\mathfrak{P}(X)$ is a groupoid. For any $[\omega] \in \mathbf{Hom}_{\mathfrak{P}(X)}(x, x')$ we should prove that $[\omega]^{-1} = [\omega^{-1}]$, where $\omega^{-1}(s) := \omega(1-s)$. Equivalently, we should prove that $\omega * \omega^{-1} \simeq c_x$ and $\omega^{-1} * \omega \simeq c_{x'}$. We give a proof of the first claim. The corresponding homotopy $H : [0,1] \to [0,1] \to X$ is

$$H(s,t) = \begin{cases} x, & 0 \le s \le \frac{t}{2}, \\ \omega(2s-t), & \frac{t}{2} \le s \le \frac{1}{2}, \\ \omega(2-2s-t), & \frac{1}{2} \le s \le 1 - \frac{t}{2}, \\ x, & 1 - \frac{t}{2} \le s \le 1. \end{cases}$$

In summary, $\mathfrak{P}(X)$ is a groupoid. $\square$

The category $\mathfrak{P}(X)$ is called the **category of path classes** of X or the **fundamental groupoid** of X. From the definition, we see that the components of the fundamental groupoid are path components of X.

Given a continuous map $f \in \mathbf{Hom}_{\mathbf{Top}}(X, Y)$, we can construct a covariant functor $f_\# : \mathfrak{P}(X) \to \mathfrak{P}(Y)$ as follows. For $x \in \mathbf{Ob}(\mathfrak{P}(X))$, let $f_\#(x) = f(x) \in \mathbf{Ob}(\mathfrak{P}(Y))$, and for $[\omega] \in \mathbf{Hom}_{\mathfrak{P}(X)}(x, x')$, let

$$f_\#([\omega]) := [f \circ \omega] \in \mathbf{Hom}_{\mathfrak{P}(Y)}(f(x), f(x')).$$

Then, $f_\#$ maps each path component of X into some path component of Y. Hence, we obtain a covariant functor

$$\pi_0 : \mathbf{Top} \longrightarrow \mathbf{Set} \tag{3.3.1.1}$$

with

$$\pi_0(X) = \{\text{path components of } X\}$$

and for $f \in \mathbf{Hom}_{\mathbf{Top}}(X, Y)$,

$$\pi_0(f) := f_\# : \pi_0(X) \longrightarrow \pi_0(Y)$$

maps the path component of x in X to the path component of $f(x)$ in Y. The above covariant functor (3.3.1.1) can be regarded as a covariant functor

$$\pi_0 : \mathbf{HTop} \longrightarrow \mathbf{Set} \tag{3.3.1.2}$$

with

$$\pi_0(X) = \{\text{path components of } X\}$$

and for $[f] \in \mathbf{Hom}_{\mathbf{HTop}}(X, Y)$,

$$\pi_0([f]) := f_\# : \pi_0(X) \longrightarrow \pi_0(Y).$$

Indeed, if $F : f \simeq f'$, then, for any $x \in X$, the path $\omega_x(t) := F(x, t)$ connects $f(x)$ to $f'(x)$. Therefore $f(x)$ and $f'(x)$ belong to the same path component of Y and then $f_\# = f'_\#$.

Proposition 3.12

If X is contractible, then π_X and π_0 are naturally equivalent functors on $\mathbf{HTop}$. ♡

Proof. Since X and one-point space P have the same homotopy type, it suffices to prove that π_P and π_0 are naturally equivalent functors on $\mathbf{HTop}$. For any $X \in \mathbf{Ob}(\mathbf{HTop})$, define

$$\psi(X)([g]) := g_\#(P), \quad \text{for all } [g] \in \pi_P(X) = [P, X].$$

Here we observe that $\pi_0(P)$ consists of the only path component P. Consider the digram

$$\begin{tikzcd} \pi_P(X) \arrow[r, "\psi(X)"] \arrow[d, "\pi_P([f])"'] & \pi_0(X) \arrow[d, "\pi_0([f])"] \\ \pi_P(Y) \arrow[r, "\psi(Y)"'] & \pi_0(Y) \end{tikzcd}$$

Because, for any $[g] \in \pi_P(X)$,

$$(\pi_0([f]) \circ \psi(X))\,([g]) \;=\; \pi_0([f])(g_\#(P)) \;=\; f_\#(g_\#(P)) \;=\; (f \circ g)_\#(P)$$

and

$$(\psi(Y) \circ \pi_P([f]))\,([g]) \;=\; \psi(Y)(f_\#([g])) \;=\; \psi(Y)([f \circ g]) \;=\; (f \circ g)_\#(P).$$

Thus ψ is a natural transformation. □

3.3.2 The equivalent of π_1

From **Lemma 3.1** we obtain a covariant functor

$$\pi : \mathbf{HTop}_* \longrightarrow \mathbf{Group} \tag{3.3.2.1}$$

given by, for $X \equiv (X, x_0) \in \mathbf{Ob}(\mathbf{HTop}_*)$,

$$\pi(X) := \pi_1(X, x_0)$$

and, for $[f] \in \mathbf{Hom}_{\mathbf{HTop}_*}(X, Y)$,

$$\pi([f]) := f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0).$$

Recall from **Definition 3.3** that $\pi_1 := \pi_{\mathbf{S}^1}$ is a covariant functor from $\mathbf{HTop}_*$ to $\mathbf{Group}$.

Theorem 3.22

Two covariant functors $\pi, \pi_1 : \mathbf{HTop}_* \to \mathbf{Group}$ are naturally equivalent. ♡

Proof. Define a continuous map $\lambda : [0,1] \to \Xi(\mathbf{S}^0)$ by $\lambda(s) := [-1, s]$, where $\mathbf{S}^0 = \{-1, 1\}$ and 1 is the base point of $\mathbf{S}^0$. Then λ induces, by the definition of $\Xi(\mathbf{S}^0) \cong \mathbf{S}^1$, a bijection

$$\lambda^\# : [\Xi(\mathbf{S}^0), X] = \pi_1(X) \longrightarrow \pi(X) = \pi_1(X, x_0), \quad [g] \longmapsto [g \circ \lambda].$$

Moreover we see that $\lambda^\#$ is a natural transformation between π_1 and π. □

A topological space $X \in \mathbf{Ob}(\mathbf{Top})$ is said to be ***n*-connected** for $n \geq 0$ if every continuous map $f : \mathbf{S}^k \to X$, for $k \leq n$, has a continuous extension over $\mathbf{D}^{k+1}$.

(1) A 1-connected space is also said to be **simply-connected**.

(2) If $0 \leq m \leq n$, an n-connected space is m-connected.

(3) According to **Proposition 3.9**, a topological space X is n-connected for $n \geq 1$ if and only if it is path connected and $\pi_k(X, x)$ is trivial for every base point $x \in X$ and $1 \leq k \leq n$.

(4) From **Corollary 3.2**, any contractible space is n-connected for every $n \geq 0$.

3.3.3 $\pi_1(\mathbf{S}^1) = \mathbf{Z}$

We compute the fundamental group $\pi(\mathbf{S}^1) = \pi_1(\mathbf{S}^1, 1)$ of $\mathbf{S}^1 = \{e^{\sqrt{-1}\theta}\}_{\theta\in[0,2\pi]}$ following Tucker (1945). Consider the exponential map

$$\exp : \mathbf{R} \longrightarrow \mathbf{S}^1, \quad t \longmapsto \exp(t) = e^{2\pi\sqrt{-1}t} \equiv \boldsymbol{e}(t). \tag{3.3.3.1}$$

Observe that

$$\boldsymbol{e}(t_1 + t_2) = \boldsymbol{e}(t_1)\boldsymbol{e}(t_2), \qquad \boldsymbol{e}(t_1) = \boldsymbol{e}(t_2) \iff t_1 - t_2 \in \mathbf{Z}.$$

Moreover, $\boldsymbol{e}|_{(-1/2,1/2)}$ is a homeomorphism of $(-\frac{1}{2}, \frac{1}{2})$ onto $\mathbf{S}^1 \setminus \{e^{\pi\sqrt{-1}}\}$. We then can define the inverse of $\boldsymbol{e}|_{(-1/2,1/2)}$ to be

$$\boldsymbol{e}^{-1} : \mathbf{S}^1 \setminus \{e^{\pi\sqrt{-1}}\} \longrightarrow \left(-\frac{1}{2}, \frac{1}{2}\right). \tag{3.3.3.2}$$

Recall that a subset $X \subseteq \mathbf{R}^n$ is **star-shaped with respect to $\boldsymbol{x}_0 \in X$** if $\overline{\boldsymbol{x}_0\boldsymbol{x}} \subseteq X$ whenever $\boldsymbol{x} \in X$.

Lemma 3.4

Let $X \subset \mathbf{R}^n$ be a star-shaped (with respect to $\boldsymbol{x}_0 \in X$), compact, and continuous set. For any given continuous map $f : X \to \mathbf{S}^1$ with $\boldsymbol{e}(t_0) = f(x_0)$ for some $t_0 \in \mathbf{R}$, we can find a unique continuous map $\widetilde{f} : X \to \mathbf{R}$ such that

$$\widetilde{f}(x_0) = t_0, \quad \boldsymbol{e} \circ \widetilde{f} = f. \tag{3.3.3.3}$$

Thus, f can be uniquely lifted to $\mathbf{R}$:

$$\begin{tikzcd} & \mathbf{R} \arrow[d, "\boldsymbol{e}"] \\ X \arrow[r, "f"'] \arrow[ur, dotted, "\exists! \widetilde{f}"] & \mathbf{S}^1 \end{tikzcd}$$

♡

Proof. (1) Uniqueness. Let $\widetilde{g} : X \to \mathbf{R}$ be another continuous map satisfying $\widetilde{g}(x_0) = t_0$ and $\boldsymbol{e} \circ \widetilde{g} = f$. Then $\boldsymbol{e} \circ (\widetilde{f} - \widetilde{g}) = 0$ and $(\widetilde{f} - \widetilde{g})(x_0) = 0$. Thus $\widetilde{f} - \widetilde{g}$ is a continuous map from X onto $\mathbf{Z}$. The continuity of X implies that $\widetilde{f} - \widetilde{g}$ is constant so that $\widetilde{f} = \widetilde{g}$ on X.

(2) Existence. Because the parallel translation is continuous, we may without loss of generality assume that $\boldsymbol{x}_0 = \mathbf{0}$. Since X is compact, it follows that f is uniformly continuous. There exists $\delta > 0$ such that

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| < 2 \quad \text{whenever} \quad |\boldsymbol{x} - \boldsymbol{y}| < \delta$$

(in particular, $f(\boldsymbol{x})$ and $f(\boldsymbol{y})$ are not antipodes in $\mathbf{S}^1$). The boundedness of X implies that there exists $n \in \mathbf{N}$ such that $|\boldsymbol{x}|/n < \delta$ for all $\boldsymbol{x} \in X$. For each $j \in \{0, 1, \cdots, n-1\}$ and all $x \in X$, we have

$$\left| f\left(\frac{j+1}{n}\boldsymbol{x}\right) - f\left(\frac{j}{n}\boldsymbol{x}\right) \right| < 2$$

because $|(j+1)\boldsymbol{x}/n - j\boldsymbol{x}/n| = |x|/n < \delta$. Particularly, $f((j+1)\boldsymbol{x}/n) \neq -f(j\boldsymbol{x}/n)$. Then we, for each $0 \leq j \leq n-1$, can define a continuous map

$$g_j : X \longrightarrow \mathbf{S}^1 \setminus \{e^{\pi\sqrt{-1}}\}, \quad x \longmapsto \frac{f((j+1)\boldsymbol{x}/n)}{f(j\boldsymbol{x}/n)}.$$

Consequently,

$$\prod_{0 \leq j \leq n-1} g_j(x) = \frac{f(\boldsymbol{x})}{f(0)} \quad \text{or} \quad f(\boldsymbol{x}) = f(0) \prod_{0 \leq j \leq n-1} g_j(\boldsymbol{x}).$$

Define $\widetilde{f} : X \to \mathbf{R}$ to be $\widetilde{f}(x) := \boldsymbol{e}^{-1}(f(\boldsymbol{x}))$; precisely,

$$\widetilde{f}(\boldsymbol{x}) := t_0 + \sum_{0 \leq j \leq n-1} \boldsymbol{e}^{-1}(g_j(\boldsymbol{x})).$$

Then $\widetilde{f}(0) = t_0$ and $\boldsymbol{e} \circ \widetilde{f} = f$. □

Taking $X = [0,1]$ in **Lemma 3.4** yields that for any loop $\alpha : [0,1] \to \mathbf{S}^1$ based at $p_0 = 1$, there exists a unique continuous map $\widetilde{\alpha} : [0,1] \to \mathbf{R}$ such that

$$\widetilde{\alpha}(0) = 0, \quad \boldsymbol{e} \circ \widetilde{\alpha} = \alpha.$$

In particular,

$$\boldsymbol{e}(\widetilde{\alpha}(1)) = \alpha(1) = p_0 = 1.$$

Thus $\widetilde{\alpha}(1) \in \mathbf{Z}$. The **degree** of α is now given by

$$\deg(\alpha) := \widetilde{\alpha}(1). \tag{3.3.3.4}$$

Lemma 3.5

For $[\alpha] = [\beta] \in \pi_1(\mathbf{S}^1, p_0)$*, we have*

$$\det(\alpha) = \deg(\beta). \tag{3.3.3.5}$$

Hence we have a well-defined function $\deg : \pi_1(\mathbf{S}^1, p_0) \to \mathbf{Z}$ *given by*

$$\deg([\alpha]) := \det(\alpha). \tag{3.3.3.6}$$

♡

Proof. Let $F : \alpha \simeq \beta$ be a homotopy relative to $\partial[0,1] = \{0,1\}$ from α to β. Then F is a continuous map from $[0,1] \times [0,1] \to \mathbf{S}^1$. By **Lemma 3.4**, there is a unique continuous map $\widetilde{F} : [0,1] \times [0,1] \to \mathbf{R}$ such that

$$\widetilde{F}(0,0) = 0, \quad \boldsymbol{r} \circ \widetilde{F} = F.$$

Because $\widetilde{F}(0,t) : [0,1] \to \mathbf{R}$ is continuous and $\boldsymbol{e}(\widetilde{F}(0,t)) = F(0,t) = p_0 = 1$, we obtain $\widetilde{F}(0,t)$ is constant for all $t \in [0,1]$ and then $\widetilde{F}(0,t) = \widetilde{F}(0,0) = 0$. Similarly one has $\widetilde{F}(1,t) = \widetilde{F}(1,1)$ for all $t \in [0,1]$. Since

$$\boldsymbol{e}(\widetilde{F}(s,0)) = F(s,0) = \alpha(s), \quad \boldsymbol{e}(\widetilde{F}(s,1)) = F(s,1) = \beta(s)$$

the uniqueness in **Lemma 3.4** implies that $\widetilde{\alpha}(s) = \widetilde{F}(s,0)$ and $\widetilde{\beta} = \widetilde{F}(s,1)$. In particular,

$$\widetilde{\alpha}(1) = \widetilde{F}(1,0) = \widetilde{F}(1,1) = \widetilde{\beta}(1).$$

Thus $\deg(\alpha) = \deg(\beta)$. □

Theorem 3.23

The function deg *is a group isomorphism*

$$\deg : \pi_1(\mathbf{S}^1, p_0) \longrightarrow \mathbf{Z}, \quad [\alpha] \longmapsto \deg(\alpha). \tag{3.3.3.7}$$

♡

Proof. For any $[\alpha], [\beta] \in \pi_1(\mathbf{S}^1, p_0)$, we have continuous maps $\widetilde{\alpha}, \widetilde{\beta} : [0,1] \to \mathbf{R}$ such thst

$$\widetilde{\alpha}(0) = \widetilde{\beta}(0) = 0, \quad \boldsymbol{e} \circ \widetilde{\alpha} = \alpha, \quad \boldsymbol{e} \circ \widetilde{\beta} = \beta.$$

Since $(\widetilde{\alpha} + \widetilde{\beta})(0) = 0$ and $\boldsymbol{e} \circ (\widetilde{\alpha} + \widetilde{\beta}) = \alpha\beta$, it follows from **Lemma 3.4** that $\widetilde{\alpha\beta} = \widetilde{\alpha} + \widetilde{\beta}$. Therefore

$$\deg([\alpha] * [\beta]) = \deg([\alpha\beta]) = \widetilde{\alpha\beta}(1) = \widetilde{\alpha}(1) + \widetilde{\beta}(1) = \deg([\alpha]) + \deg([\beta])$$

where the multiplication is determined by (3.2.2.3). Hence deg is a group homomorphism.

If $\deg([\alpha]) = 0$, then the lifted map $\widetilde{\alpha} : [0,1] \to \mathbf{R}$ is also a loop based at 0 and satisfies $\boldsymbol{e} \circ \widetilde{\alpha} = \alpha$. Since $\widetilde{\alpha}$ can be viewed as a continuous map from $\mathbf{S}^1$ to $\mathbf{R}$ and $\mathbf{R}$ is contractible, it follows from **Corollary 3.2** that $\widetilde{\alpha} \simeq c_0$ and hence $\boldsymbol{e} \circ \widetilde{\alpha} \simeq c_{p_0}$. Hence $\alpha \simeq c_{p_0}$. Thus deg is injective.

For any $n \in \mathbf{Z}$, define $\widetilde{\alpha}(s) := ns$ and $\alpha(s) := \boldsymbol{e}(\widetilde{\alpha}(s))$. Then

$$\alpha(0) = \boldsymbol{e}(\widetilde{\alpha}(0)) = \boldsymbol{e}(0) = p_0, \quad \alpha(1) = \boldsymbol{e}(\widetilde{\alpha}(1)) = \boldsymbol{e}(n) = p_0.$$

Hence $\deg(\alpha) = \widetilde{\alpha}(1) = n$. □

Chapter 4 Sheaves and schemes

Introduction

- *Category: II*
- *Sheaves: geometric aspect*
- *Sheaves: algebnraic aspect*
- *Schemes*

A **partial order** in a (nonempty) set A is a relation $\leq$ between elements of A which is reflexive and transitive. A **total order** in A is a partial order in A such that for $a, a' \in A$ either $a \leq a'$ or $a' \leq a$ and which is **anti-symmetric** (that is, $a \leq a'$ and $a' \leq a$ imply $a = a'$). A **partially ordered set** is a set with a partial order, and a **totally ordered set** is a set with a total order. An **inductively ordered set** is a partially ordered set in which every totally ordered subset has an upper bound (i.e., there exists $a \in A$ such that $a' \leq a$ for all $a' \in A$) states that any inductively ordered set contains maximal elements.

A **directed set** I is a nonempty set with a partial order relation $\leq$ such that for $i, j \in I$ there is $k \in I$ with $i \leq k$ and $j \leq k$. A **direct system of sets** $\{A_i, \phi_{ij}\}_{i \in I}$ consists of a collection of sets $\{A_i\}_{i \in I}$ indexed by a directed set I and a collection of maps $\phi_{ij} : A_i \to A_j$ for every pair $i \leq j$ such that

(1) $\phi_{ii} = 1_{A_i} : A_i \to A_i$ for all $i \in I$, and

(2) $\phi_{ik} = \phi_{jk} \circ \phi_{ij} : A_i \to A_k$ for $i \leq j \leq k$ in I.

$$\begin{tikzcd} A_i \arrow[r, "\phi_{ij}"] \arrow[dr, "\phi_{ik}"'] & A_j \arrow[d, "\phi_{jk}"] \\ & A_{jk} \end{tikzcd}$$

The **direct limit** of the direct system $\{A_i, \phi_{ij}\}_{i \in I}$, denoted by

$$\varinjlim_{i \in I} A_i := \coprod_{i \in I} A_i \Big/ \sim$$

is the set of equivalence classes of $\coprod_{i \in I} A_i$ with respect to the equivalence relation $x_i \sim x_j$ if there is $k \in I$ with $i \leq k$ and $j \leq k$ such that $\phi_{ik}(x_i) = \phi_{jk}(x_j)$. For each $i \in I$ there is a map $\iota_i : A_i \to \varinjlim_{i \in I} A_i$ defined by $\iota_i(x_i) := [x_i]$. If $i \leq j$, then $\iota_i = \iota_j \circ \phi_{ij}$.

$$\begin{tikzcd} A_i \arrow[r, "\phi_{ij}"] \arrow[d, "\iota_i"'] & A_j \arrow[d, "\iota_j"] \\ \varinjlim_{i \in I} A_i \arrow[r, equal] & \varinjlim_{i \in I} A_j \end{tikzcd}$$

Given a direct system of sets $\{A_i, \phi_{ij}\}_{i \in I}$ and given a set B and for every $i \in I$ a map

$g_i : A_i \to B$ such that $g_i = g_j \circ \phi_{ij}$ if $i \le j$, there is a unique map $g : \varinjlim_{i\in I} A_i \to B$ such that $g \circ \iota_i = g_i$ for all $i \in I$. We define $g([x_i]) := g_i(x_i)$. If $x_i \sim x_j$, then $\phi_{ik}(x_i) = f_{jk}(x_j)$ for some $k \in I$ with $i \le k$ and $j \le k$. Hence $g_i(a_i) = g_k \circ \phi_{ik}(a_i) = g_k \circ f_{jk}(x_j) = g_j(a_j)$. Thus g is well-defined and $g \circ \iota_i = g_i$ for all $i \in I$. The uniqueness is obvious.

An **inverse system of sets** $\{A_i, \phi_{ij}\}_{i\in I}$ consists of a collection of sets $\{A_i\}_{i\in I}$ indexed by a directed set I and a collection of maps $\phi_{ij} : A_j \to A_i$ for $i \le j$ such that

(1) $\phi_{ii} = 1_{A_i} : A_i \to A_i$ for all $i \in I$.

(2) $\phi_{ik} = \phi_{ij} \circ \phi_{jk} : A_k \to A_i$ for $i \le j \le k$ in I.

The **inverse limit** of the inverse system $\{A_i, \phi_{ij}\}_{i\in I}$, denoted by

$$\varprojlim_{i\in I} A_i := \left\{ (x_i)_{i\in I} \in \bigoplus_{i\in I} A_i : x_i = \phi_{ij}(x_j) \text{ if } i \le j \right\}$$

is the subset of $\bigoplus_{i\in I} A_i$ consisting of all points $\{x_i\}_{i\in I}$ such that if $i \le j$ then $x_i = \phi_{ij}(x_j)$. For each $i \in I$ there is a map $p_i : \varprojlim_{i\in I} A_i \to A_i$ defined by $p_i([x_i]) := x_i$. If $i \le j$, then $p_i = \phi_{ij} \circ p_j$.

$$\begin{tikzcd} A_j \arrow[r, "\phi_{ij}"] & A_i \\ \varprojlim_{i\in I} A_i \arrow[u, "p_j"'] \arrow[r, equal] & \varprojlim_{i\in I} A_i \arrow[u, "p_i"] \end{tikzcd}$$

Given an inverse system of sets $\{A_i, \phi_{ij}\}_{i\in I}$ and given a set B and fore every $i \in I$ a map $g_i : B \to A_i$ such that $g_i = \phi_{ij} \circ g_j$ if $i \le j$, there is a unique map $g : B \to \varprojlim_{i\in I} A_i$ such that $g_i = p_i \circ g$ for all $i \in I$.

4.1 Category: II

Introduction

- *Categories*
- *Direct products*
- *Direct sums*
- *Direct limits*
- *Direct limits on* **Set**
- *Direct limits on an Abelian category*
- *Inverse limit*
- *Additive category*
- *Kernel, cokernel, image and coimage*
- *Abelian category*
- *Exact sequences*
- *Functors*

4.1.1 Categories

A **category** $\mathcal{C}$ is a class of objects $(A, B, C, \cdots)$ together with

(1) A class $\mathbf{Mor}(\mathcal{C})$ of disjoint sets, denoted

$$\mathrm{Hom}_{\mathcal{C}}(A, B)$$

one for each pair of object in $\mathcal{C}$. An element f in $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is called a **morphism** from A to B and is denoted $f : A \to B$. The set of all objects is denoted $\mathbf{Ob}(\mathcal{C})$.

(2) For any triple (A, B, C) of objects of $\mathcal{C}$ a map

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, C), \quad (g, f) \longmapsto g \circ f$$

is called the **composite of f and g**, satisfying the axioms

(i) **Associativity**: $h \circ (g \circ f) = (h \circ g) \circ f$.

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[d, "g"] & \\ & C \arrow[r, "h"] & D \end{tikzcd}$$

(ii) **Identity**: For any $B \in \mathbf{Ob}(\mathcal{C})$ there exists $\mathbf{1}_B \in \mathrm{Hom}_{\mathcal{C}}(B, B)$ such that for any morphism $f : A \to B$ and morphism $g : B \to C$, one has

$$\mathbf{1}_B \circ f = f, \quad g \circ \mathbf{1}_B = g.$$

A morphism $f : A \to B$ in $\mathcal{C}$ is called a **monomorphism** or **injective** if for any morphisms

$$C \overset{\alpha}{\underset{\beta}{\longrightarrow}} A \xrightarrow{f} B$$

satisfying $f \circ \alpha = f \circ \beta$, we have $\alpha = \beta$. A morphism $f : A \to B$ is called an **epimorphism** or **surjective** if for any morphisms

$$A \xrightarrow{f} B \overset{\alpha}{\underset{\beta}{\longrightarrow}} C$$

satisfying $\alpha \circ f = \beta \circ f$, we have $\alpha = \beta$. If a morphism is both injective and surjective, we say that is is **bijective**. An **isomorphism** is a morphism with a two-sided inverse.

Note 4.1

(1) Isomorphism is bijective: If $f : A \to B$ is an isomorphism, there exists $g : B \to A$ such that $f \circ g = \mathbf{1}_B$ and $g \circ f = \mathbf{1}_A$. For morphisms $\alpha, \beta : C \to A$ with $f \circ \alpha = f \circ \beta$, we have $g \circ f \circ \alpha = g \circ f \circ \beta$, therefore $\alpha = \mathbf{1}_A \circ \alpha = \mathbf{1}_A \circ \beta = \beta$.

(2) Conversely, a bijective may not be an isomorphism. ♣

4.1.2 Direct product

Let $\{A_i\}_{i \in I}$ be a family of objects in a given category $\mathcal{C}$. The **direct product** of $\{A_i\}_{i \in I}$ is an object $\prod_{i \in I} A_i \in \mathbf{Ob}(\mathcal{C})$ together with a family of morphisms

$$p_i : \prod_{i \in I} A_i \longrightarrow A_i$$

with the following universal property: For any object C and any family of morphisms $f_i : C \to A_i$, $i \in I$, there exists a unique morphism $f : C \to \prod_{i \in I} A_i$ such that $p_i \circ f = f_i$ for any $i \in I$.

$$\begin{tikzcd} \prod_{i \in I} A_i \arrow[r, "p_i"] & A_i \\ C \arrow[u, "\exists! f"] \arrow[r, equal] & C \arrow[u, "f_i"'] \end{tikzcd}$$

If the direct product of $\{A_i\}_{i\in I}$ exists, then it is unique up to unique isomorphism: Let $\prod_{i\in I} A_i$ and $\prod'_{i\in I} A_i$ be two such direct products

$$\begin{tikzcd} \prod_{i\in I} A_i \arrow[r, "p_i"] & A_i \\ \prod'_{i\in I} A_i \arrow[u, "\exists! f"] \arrow[r, equal] & \prod'_{i\in I} A_i, \arrow[u, "p'_i"'] \end{tikzcd} \qquad \begin{tikzcd} \prod'_{i\in I} A_i \arrow[r, "p'_i"] & A_i \\ \prod_{i\in I} A_i \arrow[u, "\exists! g"] \arrow[r, equal] & \prod_{i\in I} A_i \arrow[u, "p_i"'] \end{tikzcd}$$

Since

$$p_i \circ (f \circ g) = p'_i \circ g = p_i, \quad p'_i \circ (g \circ f) = p_i \circ f = p'_i,$$

by the uniqueness, we must have

$$f \circ g = \mathbf{1}_{\prod_{i\in I} A_i}, \quad g \circ f = \mathbf{1}_{\prod'_{i\in I} A_i}.$$

So f is isomorphic.

4.1.3 Direct sum

The **direct sum** of $\{A_i\}_{i\in I}$ is an object $\bigoplus_{i\in I} A_i \in \mathbf{Ob}(\mathcal{C}$ together with a family of morphisms

$$k_i : A_i \longrightarrow \bigoplus_{i\in I} A_i$$

with the universal property: For any object $\mathcal{C}$ and any family of morphisms $f_i : A_i \to \mathcal{C}$, there exists a unique morphism $f : \bigoplus_{i\in I} A_i \to C$ such that $f \circ k_i = f_i$ for any $i \in I$.

$$\begin{tikzcd} \bigoplus_{i\in I} A_i \arrow[d, "\exists! f"'] & A_i \arrow[l, "k_i"'] \arrow[d, "f_i"] \\ C \arrow[r, equal] & C \end{tikzcd}$$

If the direct sum of A_i exists, it is unique up to unique morphism.

4.1.4 Direct limit

Let $(I, \leq)$ be a directed set. A **direct system** $\{A_i, \phi_{ij}\}_{i\in I}$ consists of a family of objects A_i and morphisms $\phi_{ij} : A_i \to A_j$ for pair $i \leq j$ such that

(i) $\phi_{ii} = \mathbf{1}_{A_i}$ for $i \in I$.

(ii) $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ for $i \leq j \leq k$.

A **direct limit** of a direct system $\{A_i, \phi_{ij}\}_{i\in I}$ is an object $\varinjlim_{i\in I} A_i$ together with morphisms

$$\phi_i \longrightarrow \varinjlim_{i\in I} A_i$$

satisfying

(i) $\phi_j \circ \phi_{ij} = \phi_i$ $(i \leq j)$.

(ii) universal property: For any object C and any morphisms $\psi_i : A_i \to C$ satisfying $\psi_j \circ \phi_{ij} = \psi_i$ $(i \leq j)$, there exists a unique morphism

$$\phi : \varinjlim_{i\in I} A_i \longrightarrow C$$

such that $\psi \circ \phi_i = \psi_i$ for any $i \in I$.

$$\begin{tikzcd} A_i \arrow[r, "\phi_i"] \arrow[d, "\psi_i"'] & \varinjlim_{i\in I} A_i \arrow[d, "\psi"] \\ C \arrow[r, equal] & C \end{tikzcd}$$

If the direct limit exists, it is unique up to a unique isomorphism.

A **morphism** from between direct systems $\{A_i, \phi_{ij}\}_{i\in I}$ and $\{A'_i, \phi'_{ij}\}_{i\in I}$ is a family of morphisms

$$u_i : A_i \longrightarrow A'_i$$

such that $\phi'_{ij} \circ u_i = u_j \circ \phi_{ij}$ for $i \leq j$.

$$\begin{tikzcd} A_i \arrow[r, "u_i"] \arrow[d, "\phi_{ij}"'] & A'_i \arrow[d, "\phi'_{ij}"] \\ A_j \arrow[r, "u_j"] & A'_j \end{tikzcd}$$

It induces a unique morphism $\varinjlim_{i\in I} u_i : \varinjlim_{i\in I} A_i \longrightarrow \varinjlim_{i\in I} A'_i$:

$$\begin{tikzcd} A_i \arrow[r, "\phi_i"] \arrow[d, "u_i"'] & \varinjlim_{i\in I} A_i \arrow[d, "\varinjlim_{i\in I} u_i"] \\ A'_i \arrow[r, "\phi'_i"] & \varinjlim_{i\in I} A'_i \end{tikzcd}$$

Since

$$\psi_j \circ \phi_{ij} = \phi'_i \circ u_i \circ \phi_{ij} = \phi'_i \circ \phi'_{ij} \circ u_i = \phi'_i \circ u_i = \psi_i$$

we have a unique morphism $\varinjlim_{i\in I} u_i$.

4.1.5 Direct limits on Set

On the category **Set** of sets , the two definitions of direct limit $\varinjlim_{i\in I} A_i$ are equivalent. Recall

$$\varinjlim_{i\in I} A_i = \coprod_{i\in I} A_i/\sim = \{[x_i] | i \in I\}.$$

Let

$$\phi_i : A_i \longrightarrow \varinjlim_{i\in I} A_i, \quad x_i \longmapsto [x_i];$$

then for any $i \leq j$ we have

$$\phi_j \circ \phi_{ij}(x_i) = [\phi_{ij}(x_i)] = [x_i] = \phi_i(x_i),$$

since $\phi_{jk}(\phi_{ij}(x_i)) = \phi_{ik}(x_i)$ implies $\phi_{ij}(x_i) \sim x_i$. For any object C and any morphisms $\psi_i : A_i \to C$ satisfying $\psi_i \circ \phi_{ij} = \psi_i$ $(i \leq j)$, we define the morphism

$$\psi : \varinjlim_{i\in I} A_i \longrightarrow C, \quad [x_i] \longmapsto \psi_i(x_i).$$

If $x_i \sim x_j$, then $\phi_{ik}(x_i) = \phi_{jk}(x_j)$ for some $k \in I$ with $i, j \leq k$. So

$$\psi_i(x_i) = \psi_k \circ \phi_{ik}(x_i) = \psi_k \circ \phi_{jk}(x_j) = \psi_j(x_j);$$

thus ψ is well-defined.

$$\begin{tikzcd} \varinjlim_{i\in I} A_i \arrow[r, "\psi"] & C \\ A_i \arrow[u, "\phi_i"] \arrow[r, equal] & A_i. \arrow[u, "\psi_i"'] \end{tikzcd}$$

From $\psi \circ \phi_i(x_i) = \psi([x_i]) = \psi_i(x_i)$, one has $\psi \circ \phi_i = \psi_i$. If there is another morphism $\psi' : \varinjlim_{i\in I} A_i \to C$ such that $\psi' \circ \phi_i = \psi_i$, then

$$\psi'([x_i]) = \psi' \circ \phi_i(x_i) = \psi_i(x_i) = \psi \circ \phi_i(x_i) = \psi([x_i]),$$

and hence $\psi' = \psi$.

4.1.6 Direct limits on an Abelian category

Let $\mathcal{C}$ be an Abelian category (defined 4.1.8). If $\{A_i, \phi_{ij}\}_{i\in I}$ is an direct system of Abelian groups, then

$$\varinjlim_{i\in I} A_i = A/B, \tag{4.1.6.1}$$

where

$$A := \bigoplus_{i\in I} A_i, \quad B := \{k_j(\phi_{ij}(x_i)) - k_i(x_i) | i \leq j,\ x_i \in A_i\}, \quad k_i : A_i \to A. \tag{4.1.6.2}$$

Let $\nu : A \to A/B$ denote the canonical homomorphism, we define

$$\phi_i := \nu \circ k_i. \tag{4.1.6.3}$$

4.1.7 Inverse limit

An **inverse system** $\{A_i, \phi_{ji}\}_{i\in I}$ consists of a family of objects $\{A_i\}_{i\in I}$ and morphisms $\phi_{ji} : A_j \to A_i$ $(i \leq j)$ such that

(i) $\phi_{ii} = \mathbf{1}_{A_i}$ for any $i \in I$.

(ii) $\phi_{ji} \circ \phi_{kj} = \phi_{ki}$ for $i \leq j \leq k$.

The **inverse limit** of an inverse system $\{A_i, \phi_{ji}\}_{i\in I}$ is an object $\varprojlim_{i\in I} A_i$ together with morphisms

$$\phi_i : \varprojlim_{i\in I} A_i \longrightarrow A_i$$

satisfying

(i) $\phi_{ji} \circ \phi_j = \phi_i$ for $i \leq j$,

(ii) universal property: For any object C and any morphisms $\psi_i : C \to A_i$ such that $\phi_{ji} \circ \psi_j = \psi_i$ $(i \leq j)$, there exists a unique morphism $\psi : C \to \varprojlim_{i\in I} A_i$ such that $\phi_i \circ \psi = \psi_i$:

$$\begin{tikzcd} A_i & \varprojlim_{i\in I} A_i \arrow[l, "\phi_i"'] \\ C \arrow[u, "\psi_i"] \arrow[r, equal] & C. \arrow[u, "\psi"'] \end{tikzcd}$$

A **morphism** between inverse systems $\{A_i, \phi_{ji}\}_{i\in I}$ and $\{A'_i, \phi'_{ji}\}_{i\in I}$ is a family of morphisms

$$u_i : A_i \longrightarrow A'_i$$

such that $\phi'_{ji} \circ u_j = u_i \circ \phi_{ji}$ $(i \leq j)$

$$\begin{tikzcd} A_j \arrow[r, "u_j"] \arrow[d, "\phi_{ji}"'] & A'_j \arrow[d, "\phi'_{ji}"] \\ A_i \arrow[r, "u_i"] & A'_i. \end{tikzcd}$$

It induces a unique morphism

$$\varprojlim_{i\in I} u_i : \varprojlim_{i\in I} A_i \longrightarrow \varprojlim_{i\in I} A'_i.$$

If $\{A_i, \phi_{ji}\}_{i\in I}$ is an inverse system of sets, then

$$\varprojlim_{i\in I} A_i = \left\{ (x_i)_{i\in I} \in \prod_{i\in I} Ai \middle| \phi_{ji}(x_j) = x_i,\ i \leq j \right\}. \tag{4.1.7.1}$$

4.1.8 Additive category

A category $\mathcal{C}$ is called an **additive category** if for any objects A, B, C in $\mathcal{C}$,

(1) the direct $A \times B$ exists,

(2) $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is an Abelian additive group, in particular, the zero element $\mathbf{0}_{AB}$ exists for any pair (A, B) and is called the **zero morphism**,

(3) the map

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, C), \quad (f, g) \longmapsto g \circ f$$

is a homomorphism.

Proposition 4.1

Let $\mathcal{C}$ be an additive category and let A and B be two objects in $\mathcal{C}$.

(i) Let

$$\begin{tikzcd} A \times B \arrow[r, "p_1"] \arrow[d, "p_2"] & A \\ B & \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[d, "k_1"] \\ B \arrow[r, "k_2"] & A \times B \end{tikzcd}$$

be the natural projections. Define $k_1 : A \to A \times B$ to be the unique morphism satisfying

$$p_1 \circ k_1 = \mathbf{1}_A, \quad p_2 \circ k_1 = \mathbf{0}_{AB},$$

and define $k_2 : B \to A \times B$ to be the unique morphism satisfying

$$p_1 \circ k_2 = \mathbf{0}_{AB}, \quad p_2 \circ k_2 = \mathbf{1}_B.$$

Then we have

$$k_1 \circ p_1 + k_2 \circ p_2 = \mathbf{1}_{A\times B}.$$

(ii) *Suppose that we have an object P and morphisms*

$$\begin{tikzcd} P \arrow[r, "p_1"] \arrow[d, "p_2"'] & A \arrow[d, "k_1"] \\ B \arrow[r, "k_2"] & P \end{tikzcd}$$

such that

$$p_1 \circ k_1 = \mathbf{1}_A, \quad p_2 \circ k_2 = \mathbf{1}_B, \quad k_1 \circ p_1 + k_2 \circ p_2 = \mathbf{1}_P.$$

Then (P, p_1, p_2) is the direct product of A and B, (P, k_1, k_2) is the direct sum of A and B. ♡

Proof. (i) It is obvious that

$$\begin{aligned} p_1 \circ (k_1 \circ p_1 + k_2 \circ p_2) &= \mathbf{1}_A \circ p_1 + \mathbf{0}_{BA} \circ p_2 = p_1 = p_1 \circ \mathbf{1}_{A\times B}, \\ p_2 \circ (k_1 \circ p_1 + k_2 \circ p_2) &= \mathbf{0}_{AB} \circ p_1 + \mathbf{1}_B \circ p_2 = p_2 = p_2 \circ \mathbf{1}_{A\times B}. \end{aligned}$$

By the uniqueness, we must have $k_1 \circ p_1 + k_2 \circ p_2 = \mathbf{1}_{A\times B}$.

$$\begin{tikzcd} A\times B \arrow[r, "p_1"] \arrow[d, "p_2"'] & A \arrow[r, equal] & A \arrow[d, equal] \\ B \arrow[d, equal] & A\times B \arrow[r, "p_1"'] \arrow[d, "p_2"] & A \\ B \arrow[r, equal] & B & \end{tikzcd}$$

(ii) First, we show that

$$p_1 \circ k_2 = \mathbf{0}_{BA}, \quad p_2 \circ k_1 = \mathbf{0}_{AB}.$$

Indeed,

$$\begin{aligned} p_1 \circ k_2 &= p_1 \circ \mathbf{1}_P \circ k_2 = p_1 \circ (k_1 \circ p_1 + k_2 \circ p_2) \circ k_2 \\ &= \mathbf{1}_A \circ p_1 \circ k_2 + p_1 \circ k_2 \circ \mathbf{1}_B = 2p_1 \circ k_2 \in \mathrm{Hom}_{\mathcal{C}}(B, A). \end{aligned}$$

Since $\mathrm{Hom}_{\mathcal{C}}(B, A)$ is an Abelian group, we have $p_1 \circ k_2 = \mathbf{0}_{BA}$. Next we prove that (P, p_1, p_2) is the direct product of A and B.

$$\begin{tikzcd} P \arrow[r, "p_1"] & A & P \arrow[r, "p_2"] & B \\ C \arrow[u, "f"] \arrow[r, equal] & C \arrow[u, "f_1"'] & C \arrow[u, "f"] \arrow[r, equal] & C \arrow[u, "f_2"'] \end{tikzcd}$$

For object C and morphisms $f_1 : C \to A$ and $f_2 : C \to B$, define

$$f := k_1 \circ f_1 + k_2 \circ f_2.$$

Then

$$\begin{aligned} p_1 \circ f &= p_1 \circ (k_1 \circ f_1 + k_2 \circ f_2) = \mathbf{1}_A \circ f_1 + \mathbf{0}_{BA} \circ f_2 = f_1, \\ p_2 \circ f &= p_2 \circ (k_1 \circ f_1 + k_2 \circ f_2) = \mathbf{0}_{AB} \circ f_1 + \mathbf{1}_B \circ f_2 = f_2. \end{aligned}$$

If $f' : C \to P$ is another morphism such that

$$p_1 \circ f' = f_1, \quad p_2 \circ f' = f_2.$$

Hence we get $f' = \mathbf{1}_P \circ f' = (k_1 \circ p_1 + k_2 \circ p_2) \circ f' = k_1 \circ f_1 + k_2 \circ f_2 = f$. Finally we show that (P, k_1, k_2) is the direct sum of A and B.

$$\begin{tikzcd} A \arrow[r, "k_1"] \arrow[d, "f_1"'] & P \arrow[d, "f"] \\ C \arrow[r, equal] & C \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "k_2"] \arrow[d, "f_2"'] & P \arrow[d, "f"] \\ C \arrow[r, equal] & C \end{tikzcd}$$

For any object C and morphisms $f_1 : A \to C$ and $f_2 : B \to C$, define

$$f := f_1 \circ p_1 + f_2 \circ p_2.$$

Then

$$\begin{aligned} f \circ k_1 &= (f_1 \circ p_1 + f_2 \circ p_2) \circ k_1 = f_1 \circ \mathbf{1}_A + f_2 \circ \mathbf{0}_{AB} = f_1, \\ f \circ k_2 &= (f_1 \circ p_1 + f_2 \circ p_2) \circ k_2 = f_1 \circ \mathbf{0}_{BA} + f_2 \circ \mathbf{1}_B = f_2. \end{aligned}$$

If $f' : P \to C$ is another morphism such that $f' \circ k_1 = f_1$ and $f' \circ k_2 = f_2$, then $f' = f$. □

4.1.9 Kernel, cokernel, image and coimage

Let $\mathcal{C}$ be an additive category and $f : A \to B$ a morphism in $\mathcal{C}$. We say a monomorphism $K \to A$ is the **kernel** of f, if

(i) the composite $K \to A \to B$ is $\mathbf{0}_{KB}$, and

(ii) for any morphism $K' \to A$ such that $K' \to A \to B$ is $\mathbf{0}_{K'B}$, there is a unique morphism $K' \to K$ such that

$$\begin{tikzcd} K' \arrow[d] \arrow[dr] & \\ K \arrow[r] & A \end{tikzcd}$$

Denote

$$K := \mathrm{Ker}(f) \tag{4.1.9.1}$$

and call it the kernel of f. Similarly, an **epimorphism** $B \to C$ is the **cokernel** of f if

(i) $A \to B \to C$ is $\mathbf{0}_{AC}$, and

(ii) for any morphism $B \to C'$ such that $A \to B \to C'$ is $0_{AC'}$, there exists a unique morphism $C \to C'$ such that

$$\begin{tikzcd} B \arrow[r] \arrow[dr] & C \arrow[d] \\ & C' \end{tikzcd}$$

Denote

$$C := \mathrm{Coker}(f) \tag{4.1.9.2}$$

and call it the cokernel of f. We define the **image** of f to be the kernel of the cokenel of f

$$\mathrm{Im}(f) := \mathrm{Ker}(\mathrm{Coker}(f)), \tag{4.1.9.3}$$

and define the **coimage** of f to be the cokernel of the kernel of f

$$\mathrm{Coim}(f) := \mathrm{Coker}(\mathrm{Ker}(f)). \tag{4.1.9.4}$$

Theorem 4.1

There exists a canonical morphism

$$\phi : \mathrm{Coim}(f) \longrightarrow \mathrm{Im}(f)$$

such that

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \uparrow \\ \mathrm{Coim}(f) & \xrightarrow[\phi]{} & \mathrm{Im}(f). \end{array}$$

♡

Proof. Let $K = \mathrm{Ker}(f)$ and $C = \mathrm{Coker}(f)$. Thus we have

$$\begin{array}{ccccc} K & \xrightarrow{f_1} & A & \xrightarrow{f} & B, \\ A & \xrightarrow{f} & B & \xrightarrow{f_2} & C, \\ K & \xrightarrow{f_1} & A & \xrightarrow{f_3} & \mathrm{Coim}(f), \\ \mathrm{Im}(f) & \xrightarrow{f_4} & B & \xrightarrow{f_2} & C, \end{array}$$

where

$$f \circ f_1 = \mathbf{0}_{AB}, \quad f_2 \circ f = \mathbf{0}_{AC}, \quad f_3 \circ f_1 = \mathbf{0}_{K,\mathrm{Coim}(f)}, \quad f_2 \circ f_4 = \mathbf{0}_{\mathrm{Im}(f),C}.$$

By the universal properties, there exists a unique morphisms

$$f_5 : \mathrm{Coim}(f) \longrightarrow B, \quad f_6 : A \longrightarrow \mathrm{Im}(f)$$

satisfying $f_4 \circ f_6 = f$ and $f_5 \circ f_3 = f$.

$$\begin{tikzcd} A \arrow[d, "f_6"'] \arrow[dr, "f"] & \\ \mathrm{Im}(f) \arrow[r, "f_4"'] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f_3"] \arrow[dr, "f"'] & \mathrm{Coim}(f) \arrow[d, "f_5"] \\ & B \end{tikzcd}$$

Therefore we have

$$\begin{tikzcd} K \arrow[r, "f_1"] & A \arrow[r, "f_3"] \arrow[d, equal] & \mathrm{Coim}(f) \arrow[r, "f_5"] \arrow[d, "\phi"] & B \arrow[r, "f_2"] \arrow[d, equal] & C \\ & A \arrow[r, "f_6"] & \mathrm{Im}(f) \arrow[r, "f_4"] & B & \end{tikzcd}$$

Since $f_2 \circ f_5 \circ f_3 = f_2 \circ f = \mathbf{0}_{AC}$ and f_3 is injective, it follows that $f_2 \circ f_5 = \mathbf{0}_{\mathrm{Coim}(f),C}$. By the uniqueness, there exists a unique morphism $\phi : \mathrm{Coim}(f) \to \mathrm{Im}(f)$ such that $\phi \circ f_3 = f_6$. On the other hand, $f_4 \circ \phi = f_5$ using the same argument. Therefore $f_4 \circ \phi \circ f_3 = f_4 \circ f_6 = f = f_5 \circ f_3$. □

Note 4.2

$f : A \to B$ is a morphism in the category $\mathbf{AB}$ *of Abelian groups, then*

$$\begin{aligned}
\mathrm{Ker}(f) &= \{a \in A | f(a) = 0\}, \\
\mathrm{Im}(f) &= \{b \in B | b = f(a) \text{ for some } a \in A\}, \\
\mathrm{Coker}(f) &= B/\mathrm{Im}(f), \\
\mathrm{Coim}(f) &= A/\mathrm{Ker}(f).
\end{aligned}$$

The canonical morphism from the coimage to the image is the canonical homomorphism $A/\mathrm{Ker}(f) \to \mathrm{Im}(f)$, which is an isomorphism. ♣

4.1.10 Abelian category

Let $\mathcal{C}$ be an additive category. A **zero object** $0_{\mathcal{C}}$ is $\mathcal{C}$ is an object such that $\mathrm{Hom}_{\mathcal{C}}(0_{\mathcal{C}}, 0_{\mathcal{C}}) = \{0\}$. It is equivalent to saying that the identity morphism of $0_{\mathcal{C}}$ is equal to the zero morphism. For any object X in $\mathcal{C}$, we have $\mathrm{Hom}_{\mathcal{C}}(X, 0_{\mathcal{C}}) = \{0\} = \mathrm{Hom}_{\mathcal{C}}(0_{\mathcal{C}}, X)$. In fact $f : X \to 0_{\mathcal{C}}$ is a morphism, then we have $\mathbf{1}_{0_{\mathcal{C}}} \circ f = f$ and hence $f = \mathbf{0}_{X,0_{\mathcal{C}}}$. All zero objects are isomorphic to each other.

An **Abelian category** $\mathcal{C}$ is an additive category with the following properties:

(i) it has zero objects,

(ii) for any morphism f in $\mathcal{C}$, $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ exist, and

(iii) the canonical morphism $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism.

Theorem 4.2

In an Abelian category, a bijective morphism is an isomorphism. ♡

Proof. Let $f : A \to B$ be injective. Then $\mathrm{Ker}(f) = 0$ and $\mathrm{Coim}(f) = A$. Since $f : A \to B$ is surjective, we have $\mathrm{Coker}(f) = 0$ and $\mathrm{Im}(f) = B$. From the canonically commutative diagram one has $f = \phi : A \to B$; thus, f is an isomorphism. □

4.1.11 Exact sequences

Suppose $u : A \to B$ is a monomorphism in an Abelian category $\mathcal{C}$. We say A is a **subobject** of B. Let $B \to C$ be the cokernel of u. We call C the **quotient** of B by A and denote it by B/A. In an Abelian category $\mathcal{C}$, a sequence of morphisms

$$A \xrightarrow{u} B \xrightarrow{v} C$$

is called **exact** if $v \circ u = 0$ and the canonical morphism $\mathrm{Coim}(u) \to \mathrm{Ker}(v)$ is an isomorphism. An exact sequence of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is called a **short exact sequence**. This short exact sequence is called **split** if it is isomorphic to

$$0 \longrightarrow A \longrightarrow A \oplus C \longrightarrow C \longrightarrow 0,$$

where $A \to A \oplus C$ and $A \oplus C = A \times C \to C$ are the canonical morphisms.

Theorem 4.3

Let

$$0 \longrightarrow A_1 \xrightarrow{i_1} A \xrightarrow{p_2} A_2 \longrightarrow 0$$

be a short exact sequence in an Abelian category. The following conditions are equivalent:

(i) the above short exact sequence is split,

(ii) there exists a morphism $p_1 : A \to A_1$ such that $p_1 \circ i_1 = \mathbf{1}_{A_1}$,

(iii) there exists a morphism $i_2 : A_2 \to A$ such that $p_2 \circ i_2 = \mathbf{1}_{A_2}$.

♡

Proof. $(i) \Rightarrow (ii)$:

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r, "i_1"] \arrow[d, equal] & A \arrow[r, "p_2"] \arrow[d, equal] & A_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A_1 \arrow[r, "\iota_1"'] & A_1 \oplus A_2 \arrow[r, "\pi_2"'] & A_2 \arrow[r] & 0
\end{tikzcd}$$

Let $f : A \to A_1 \oplus A_2$ and $g : A_1 \oplus A_2 \to A$ be isomorphisms.

$$\begin{tikzcd}
A_1 \oplus A_2 = A_1 \times A_2 \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & A_1 \\
A_2 &
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[r, "\iota_1"] & A_1 \oplus A_2 \\
& A_2 \arrow[u, "\iota_2"']
\end{tikzcd}$$

Let $p_1 := \pi_1 \circ f : A \to A_1$. Then

$$p_1 \circ i_1 = \pi_1 \circ f \circ i_1 = \pi_1 \circ f \circ g \circ \iota_1 = \pi_1 \circ \mathrm{id}_{A_1 \oplus A_2} \circ \iota_1 = \pi_1 \circ \iota_1 = \mathbf{1}_{A_1}.$$

$(ii) \Rightarrow (iii)$: the proof is the same as before.

$(iii) \Rightarrow (i)$: Since A_2 is the cokernel of $i_1 : A_1 \to A$ and

$$(\mathrm{id}_A - i \circ p_1) \circ i_1 = i_1 - i_1 \circ (p_1 \circ i_1) = i_1 - i_1 = \mathbf{0}_{A_1 A}$$

it follows that there is a unique morphism $i_2 : A_2 \to A_1$ such that

$$i_2 \circ p_2 = \mathrm{id}_A - i_1 \circ p_1, \quad i_1 \circ p_1 + i_2 \circ p_2 = \mathbf{1}_A.$$

From 1.2.2 one has that (A, p_1, p_2) is the direct product of A_1 and A_2, (A, i_1, i_2) is the direct sum of A_1 and A_2. □

4.1.12 Functors

A **covariant functor** $F : \mathcal{C} \to \mathcal{D}$ between two categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of functions $(F(C), F(f))$ where $F(C) \in \mathbf{Ob}(\mathcal{C})$ for any $C \in \mathbf{Ob}(\mathcal{C})$, and $F(f) : F(C) \to F(D)$ for any morphism $f : C \to D$ in the category $\mathcal{C}$, satisfying

(1) $F(\mathbf{1}_C) = \mathbf{1}_{F(C)}$ for any object C in $\mathcal{C}$.

(2) $F(g \circ f) = F(g) \circ F(f)$ for any morphisms f and g.

A **contravariant functor** $G : \mathcal{C} \to \mathcal{D}$ between two categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of functions $(G(C), G(f))$ where $G(C) \in \mathbf{Ob}(\mathcal{C})$ for any $C \in \mathbf{Ob}(\mathcal{C})$, and $G(f) : G(D) \to G(C)$ for any morphism $f : C \to D$ in the category $\mathcal{C}$, satisfying

(1) $G(\mathbf{1}_C) = \mathbf{1}_{G(C)}$ for any object C in $\mathcal{C}$.

(2) $G(g \circ f) = G(f) \circ G(g)$ for any morphisms f and g.

Let $F, G : \mathcal{C} \to \mathcal{G}$ be two covariant functors between categories. A **natural transformation**

$$\alpha : F \longrightarrow G$$

is a function such that

(1) for any $C \in \mathbf{Ob}(\mathcal{C})$, there exists morphism $\alpha(C) : F(C) \to G(C)$.

(2) for any morphism $f : C \to D$, one has a commutative diagram

$$\begin{tikzcd} F(C) \arrow[r, "\alpha(C)"] \arrow[d, "F(f)"'] & G(C) \arrow[d, "G(f)"] \\ F(D) \arrow[r, "\alpha(D)"'] & G(D) \end{tikzcd}$$

If $\alpha(C)$ is an isomorphism for every C, then α is called a **natural isomorphism** of the functors F and G.

Let $F : \mathcal{C} \to \mathcal{D}$ be a covariant functor between Abelian categories. F is **additive** if for any objects A and B in $\mathcal{C}$, the map

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$$

is an Abelian group homomorphism.

Proposition 4.2

If $F : \mathcal{C} \to \mathcal{D}$ is a covariant functor between Abelian categories, then

$$F(A \oplus B) \cong F(A) \oplus F(B)$$

for any objects A and B in $\mathcal{C}$. Hence if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a split short exact sequence, then

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$$

is also a split short exact sequence. ♡

Proof. Since $A \oplus B = A \times B \in \mathbf{Ob}(\mathcal{C})$, we define natural morphisms

$$\begin{tikzcd} A \oplus B \arrow[r, "p_1"] \arrow[d, "p_2"'] & A \arrow[d, "k_1"] \\ B \arrow[r, "k_2"'] & A \oplus B \end{tikzcd}$$

such that

$$p_1 \circ k_1 = \mathbf{1}_A, \quad p_2 \circ k_2 = \mathbf{1}_B, \quad k_1 \circ p_1 + k_2 \circ p_2 = \mathbf{1}_{A \oplus B}.$$

Then

$$\begin{aligned} F(p_1) \circ F(k_1) &= \mathrm{id}_{F(A)}, \quad F(p_2) \circ F(k_2) = \mathrm{id}_{F(B)}, \\ F(k_1) \circ F(p_1) + F(k_1) \circ F(p_2) &= \mathrm{id}_{F(A \oplus B)}; \end{aligned}$$

thus $(F(A \oplus B, F(k_1), F(k_2))$ is the direct sum of $F(A)$ and $F(B)$. □

Note that the category $\mathbf{AB}$ of Abelian groups is an Abelian category.

4.2 Sheaves: geometric aspect

Introduction

- *Sheaf of germs of smooth functions*
- *sheaf of germs of holomorphic functions*
- *Sheaves*
- *Cohomology groups*
- *Exact sequences*
- *Fine sheaves*
- *De Rham's theorem*
- *Dolbeault's theorem*

4.2.1 Sheaf of germs of smooth functions

Let $\mathcal{M}$ be a smooth manifold of dimension m. By a **local C^∞ function** of $\mathcal{M}$ we mean a C^∞ function defined on a domain $\mathcal{U} \subset \mathcal{M}$. We denote by $D(f)$ the domain of f. f is called a **local C^∞ function at p** if $p \in D(f)$. We say that two local C^∞ functions f and g at p are **equivalent at p** if $f = g$ on some neighborhood of p. We denote it by $f \sim_p g$. Clearly that $\sim_p$ is an equivalence relation. By a **germ of C^∞ functions at p** we mean an equivalence class of local C^∞ functions at p. The equivalence class to which f belongs is called the **germ of f at p**, and denoted by f_p. The germ f_p can be regarded as a C^∞ function defined on an "infinitely" small neighborhood of p.

Let $\mathscr{A}_p$ be the set of germs of all complex-valued C^∞ functions at p. Let $\varphi, \psi \in \mathscr{A}_p$, and suppose that $\varphi = f_p$ and $\psi = g_p$. Then we define a linear combination of φ and ψ by setting

$$c_1\varphi + c_2\psi := (c_1 f + c_2 g)_p, \quad c_1, c_2 \in \mathbf{C}.$$

Clearly that this definition is independent of the choice of f and g. Thus $\mathscr{A}_p$ becomes a vector space over $\mathbf{C}$. Moreover we define the product of φ and ψ by

$$\varphi \cdot \psi := (f \cdot g)_p,$$

which makes $\mathscr{A}_p$ a ring. Let

$$\mathscr{A} := \bigcup_{p \in \mathcal{M}} \mathscr{A}_p \tag{4.2.1.1}$$

be the union of all $\mathscr{A}_p$, for each $p \in \mathcal{M}$.

We define a topology on $\mathscr{A}$ as follows: For a point $\varphi \in \mathscr{A}_p \subset \mathscr{A}$, choose any local C^∞ function f with $f_p = \varphi$, and an arbitrary open set $\mathcal{U}$ with $p \in \mathcal{U} \subset D(f)$, and put

$$\mathscr{U}(\varphi; f, \mathcal{U}) := \{f_q : q \in \mathcal{U}\}. \tag{4.2.1.2}$$

Note that $\varphi = f_p \in \mathscr{U}(\varphi; f, \mathcal{U})$.

(i) We introduce a topology on $\mathscr{A}$ by taking the family

$$\{\mathscr{U}(\varphi; f, \mathcal{U}) : f_p = \varphi,\ p \in \mathcal{U} \subset D(f)\}$$

consisting of all $\mathscr{U}(\varphi; f, \mathcal{U})$ with f and $\mathcal{U}$ satisfying the above conditions as the system of neighborhood of $\varphi \in \mathscr{A}$. Namely, a subset $\mathscr{U} \subset \mathscr{A}$ is defined to be open if and only if $\mathscr{U}$ contains some $\mathscr{U}(\varphi; f, \mathcal{U})$ for any $\varphi \in \mathscr{U}$.

(ii) We denote by $\mathfrak{D}$ the family of open sets of $\mathscr{A}$. Let $\mathscr{U}, \mathscr{V} \in \mathfrak{D}$ and $\varphi \in \mathscr{U} \cap \mathscr{V}$. Then there are neighborhood $\mathscr{U}(\varphi; f, \mathcal{U})$ and $\mathscr{U}(\varphi; g, \mathcal{V})$ such that $\mathscr{U}(\varphi; f, \mathcal{U}) \subset \mathscr{U}$ and $\mathscr{U}(\varphi; g, \mathcal{V}) \subset \mathscr{V}$. Since $f_p = \varphi = g_p$, $f = g$ in some neighborhood $\mathcal{W} \subset \mathcal{U} \cap \mathcal{V}$, hence $\mathscr{U}(\varphi; f, \mathcal{W}) \subset \mathscr{U}(\varphi; f, \mathcal{U}) \cap \mathscr{U}(\varphi; g, \mathcal{V}) \subset \mathscr{U} \cap \mathscr{V}$. Hence $\mathscr{U} \cap \mathscr{V} \subset \mathfrak{D}$. Thus $\mathscr{A}$ is a topological space with $\mathfrak{D}$ as the system of open sets. $\mathscr{U}(\varphi; f, \mathcal{U})$ is clearly open in the above sense.

(iii) For a distinct two points φ and ψ of $\mathscr{A}$, there is a neighborhood $\mathscr{U}(\varphi; f, \mathcal{U})$ of φ such that $\psi \notin \mathscr{U}(\varphi; f, \mathcal{U})$. In fact, suppose $\varphi \in \mathscr{A}_p$ and $\psi \in \mathscr{A}_q$. If $p \neq q$, then $q \notin \mathcal{U}$ for some neighborhood $\mathcal{U}$ of p. Consider the neighborhood $\mathscr{U}(\varphi; f, \mathcal{U})$. If $\psi \in \mathscr{U}(\varphi; f, \mathcal{U})$, we have $\psi = f_{p'}$ for some $p' \in \mathcal{U}$. Hence $\psi = \varphi$, a contradiction. Therefore $\psi \notin \mathscr{U}(\varphi; f, \mathcal{U})$. If $p = q$, we can consider any neighborhood $\mathscr{U}(\varphi; f, \mathcal{U})$ with $p \in \mathcal{U} \subset D(f)$, because of $\mathscr{A}_p \cap \mathscr{U}(\varphi; f, \mathcal{U}) = \{\varphi\}$.

(iv) $\mathscr{A}$ may not be Hausdorff. Let $\mathcal{M} = \mathbf{R}$ be the real line, and $u(x)$ and $v(x)$ C^∞ functions on $\mathbf{R}$ such that $u(x) = v(x)$ for $x \geq 0$ and that $u(x) < v(x)$ for $x < 0$. Put $\varphi = u_0$ and $\psi = v_0$. Then $\varphi \neq \psi$. But $\mathscr{U}(\varphi; f, \mathcal{U}) \cap \mathscr{U}(\psi; g, \mathcal{V}) \neq \emptyset$ for any $f, g, \mathcal{U}$ and $\mathcal{V}$. In fact, $f_0 = \varphi = u_0$ and $g_0 = \psi = v_0$ mean that for a sufficient small $\delta > 0$, $f(x) = u(x)$ and $g(x) = v(x)$ on $|x| < \delta$. Hence for $\epsilon \in \mathcal{U} \cap \mathcal{V}$ and $0 < \epsilon < \delta$, $f_\epsilon = u_\epsilon = v_\epsilon = g_\epsilon$.

Theorem 4.4

Define $\varpi : \mathscr{A} \to \mathcal{M}$ by $\varpi(\mathscr{A}_p) = p$. Then ϖ satisfies the following conditions:

(i) ϖ is a local homeomorphism, that is, for any $\varphi \in \mathscr{A}$ there exists a $\mathscr{U}(\varphi; f, \mathcal{U})$ such that $\varpi : \mathscr{U}(\varphi; f, \mathcal{U}) \to \mathcal{U}$ is a homeomorphism.

(ii) $\varpi^{-1}(p) = \mathscr{A}_p$.

(iii) The linear combination $c_1\varphi + c_2\psi$ depends continuously on φ and ψ, $\varpi(\varphi) = \varpi(\psi)$. ♡

Proof. (i) Since, restricted on $\mathscr{U}(\varphi; f, \mathcal{U})$, ϖ is represented by

$$\varpi : f_q \longmapsto q, \quad q \in \mathcal{U},$$

ϖ maps certainly $\mathscr{U}(\varphi; f, \mathcal{U})$ bijectively onto $\mathcal{U}$. For any $\psi = f_q \in \mathscr{U}(\varphi; f, \mathcal{U})$, a sufficiently small $\mathscr{U}(\psi; f, \mathcal{V})$, $q \in \mathcal{V} \subset \mathcal{U}$, is mapped bijectively by ϖ to the neighborhood $\mathcal{V}$ of q. Hence ϖ is a local homeomorphism.

(ii) Clearly.

(iii) Let $\chi = c_1\varphi + c_2\psi$ with $\varpi(\varphi) = \varphi(\psi) = p$. It suffices to prove that, for any $\mathscr{U}(\chi; f, \mathcal{U})$, there are sufficiently small $\mathscr{U}(\varphi; g, \mathcal{V})$ and $\mathscr{U}(\psi; h, \mathcal{V})$ such that $c_1\xi + c_2\eta \in \mathscr{U}(\chi; f, \mathcal{U})$ for $\xi \in \mathscr{U}(\varphi; g, \mathcal{V})$ and $\eta \in \mathscr{U}(\psi; h, \mathcal{V})$ with $\varpi(\xi) = \varpi(\eta)$. Since

$$(c_1 g + c_2 g)_p = c_1 g_p + c_2 h_p = c_1\varphi + c_2\psi = \chi = f_p,$$

f coincides with $c_1 g + c_2 h$ on some neighborhood $\mathcal{V} \subset \mathcal{U}$ of p. Put $q = \varpi(\xi) = \varpi(\eta) \in \mathcal{V}$. Then we have $c_1\xi + c_2\eta = c_1 g_q + c_2 h_q = f_q \in \mathscr{U}(\chi; f, \mathcal{U})$. □

$\mathscr{A}$ is called the **sheaf of germs of C^∞ functions over $\mathcal{M}$**. To emphasize the manifold, we usual write $\mathscr{A}$ as $\mathscr{A}_{\mathcal{M}}$.

4.2.2 Sheaf of germs of holomorphic functions

Let $\mathcal{X}$ be a complex manifold of complex dimension n. Using holomorphic functions instead of C^∞ functions, we obtain the definition of the **sheaf of germs of holomorphic functions** $\mathscr{O} \equiv \mathscr{O}_{\mathcal{X}} = \cup_{p\in\mathcal{X}}\mathscr{O}_p$. Let $\boldsymbol{z}_p = (z_p^1, \cdots, z_p^n)$ be local complex coordinates centered at $p \in \mathcal{X}$. Then a local holomorphic function at p, that is, a holomorphic function defined on some neighborhood of p, has a convergent power series expansion:

$$f = \mathscr{P}(\boldsymbol{z}_p) = \mathscr{P}(z_p^1, \cdots, z_p^n) = \sum_{m_1,\cdots,m_n \geq 0} a_{m_1,\cdots,m_n}(z_p^1)^{m_1}\cdots(z_p^n)^{m_n}.$$

f and g are equivalent if and only if f and g have the same power series expansion $\mathscr{P}(\boldsymbol{z}_p)$. Thus a germ of holomorphic functions at p corresponds in one-to-one manner to a convergent power series in $z_p^1, \cdots, z_p^n$, Hence we may identify $\mathscr{O}_p$ with the convergent power series ring $\mathbf{C}\{z_p^1, \cdots, z_p^n\}$:

$$\mathscr{O}_p = \mathbf{C}\{z_p^1, \cdots, z_p^n\}. \tag{4.2.2.1}$$

Let

$$\mathscr{U}(\varphi; f, \mathcal{U}) = \{f_q : q \in \mathcal{U}\}, \quad \varphi \in \mathscr{O}_p.$$

If $\mathcal{U}$ is assumed to be connected, then

(i) $\mathscr{U}(\varphi; f, \mathcal{U})$ is uniquely determined by φ and $\mathcal{U}$. For, $f = g$ on $\mathcal{U}$ if $f_p = \varphi = g_p$ since $\mathcal{U}$ is connected.

(ii) If $\varphi, \psi \in \mathscr{O}_p$ and $\varphi \neq \psi$,

$$\mathscr{U}(\varphi; f, \mathcal{U}) \cap \mathscr{U}(\psi; g, \mathcal{U}) = \emptyset.$$

In fact, if $f_q = g_q$ at some $q \in \mathcal{U}$, $f = g$ on all of $\mathcal{U}$, hence $\varphi = f_p = g_p = \psi$. Therefore the sheaf of germs of holomorphic functions $\mathscr{O}$ is Hausdorff.

Define $\varpi : \mathscr{O} \to \mathcal{X}$ by $\varpi(\mathscr{O}_p) = p$. Then ϖ is a local homeomorphism, and the linear combination $c_1\varphi + c_2\psi$, $\varpi(\varphi) = \varpi(\psi)$, depends continuously on φ and ψ.

4.2.3 Sheaves

Let $\mathcal{M}$ be a differentiable or a complex manifold. In what follows, we denote by $\mathbf{K}$ one of number fields $\mathbf{C}$, $\mathbf{R}$, and $\mathbf{Z}$.

Definition 4.1

A topological space $\mathscr{S}$ is called a **sheaf** *over $\mathcal{M}$ if the following conditions are satisfied:*

(i) A local homeomorphism ϖ of $\mathscr{S}$ onto $\mathcal{M}$ is defined.

(ii) For any $p \in \mathcal{M}$, $\varpi^{-1}(p)$ is a $\mathbf{K}$-module.

(iii) For $c_1, c_2 \in \mathbf{K}$, the linear combination $c_1\varphi + c_2\psi$ of $\varphi, \psi \in \mathscr{S}$ and $\varpi(\varphi) = \varpi(\psi)$, depends continuously on φ and ψ.

We call $\mathscr{S}_p := \varpi^{-1}(p)$ the **stalk of $\mathscr{S}$ over p**, *and ϖ the* **projection of $\mathscr{S}$**. ♣

$\mathscr{A}_\mathcal{M}$ and $\mathscr{O}_\mathcal{X}$ are examples of sheaves. We define the sheaf $\mathscr{A}^r_\mathcal{M}$ of germs of C^∞ r-forms on a differentiable manifold $\mathcal{M}$, and the sheaf $\Theta_\mathcal{X}$ of germs of holomorphic vector fields on a complex manifold $\mathcal{X}$ similarly.

Given a complex vector bundle $\mathcal{F}$ over a differentiable manifold $\mathcal{M}$, we define the sheaf of germs of all C^∞ sections of $\mathcal{F}$ as follows: By a **local C^∞ section** of $\mathcal{F}$ we mean a C^∞ section of $\mathcal{F}$ defined in a neighborhood of p. Two local C^∞ sections σ and τ are called **equivalent** if $\sigma = \tau$ in some neighborhood of p. We denote $\sigma {\sim_p} \tau$ if σ and τ are equivalent. Then $\sim_p$ is an equivalence relation. The equivalence class of local C^∞ sections is called a **germ of C^∞ sections of $\mathcal{F}$**. The class to which σ belongs is called a **germ of σ at p**, and denoted by σ_p. We denote the set of all local C^∞ sections of $\mathcal{F}$ at p by $\mathscr{A}_p(\mathcal{F})$. Put

$$\mathscr{A}_\mathcal{M}(\mathcal{F}) \equiv \mathscr{A}(\mathcal{F}) = \bigcup_{p\in\mathcal{M}} \mathscr{A}_p(\mathcal{F}).$$

Then $\mathscr{A}_\mathcal{M}(\mathcal{F})$ is a sheaf over $\mathcal{M}$ and is called the **sheaf of germs of C^∞ sections of $\mathcal{F}$**. Letting $\mathcal{F} = \wedge^r\mathbf{T}^*(\mathcal{M})$, we have $\mathscr{A}_\mathcal{M}(\wedge^r\mathbf{T}^*(\mathcal{M})) = \mathscr{A}^r_\mathcal{M}$. For a complex manifold $\mathcal{X}$,

$$\mathscr{A}^{p,q}_\mathcal{X} := \mathscr{A}_\mathcal{X}\left(\wedge^{p,q}\mathcal{X}\right) \tag{4.2.3.1}$$

is the sheaf of germs of C^∞ (p,q)-forms on $\mathcal{X}$.

Let $\mathcal{F}$ be a holomorphic vector bundle over a complex manifold $\mathcal{X}$. We define the sheaf of germs of holomorphic sections of $\mathcal{F}$ similarly:

$$\mathscr{O}_\mathcal{X}(\mathcal{F}) \equiv \mathscr{O}(\mathcal{F}) = \bigcup_{p\in\mathcal{X}} \mathscr{O}_p(\mathcal{F}).$$

When $\mathcal{F} = \mathbf{T}(\mathcal{X})$, $\mathscr{O}_{\mathcal{X}}(\mathbf{T}(\mathcal{X})) = \Theta_{\mathcal{X}}$ the sheaf of germs of holomorphic vector fields on $\mathcal{X}$. Moreover, $\mathscr{O}^r_{\mathcal{X}} = \mathscr{O}_{\mathcal{X}}(\wedge^r \mathbf{T}^*(\mathcal{X}))$ is the sheaf of germs of holomorphic r-forms on $\mathcal{X}$.

By a **locally constant function** we mean a local C^∞ function f with $f = c \in \mathbf{C}$ on $D(f)$. The **sheaf of germs of locally constant functions** is identified with the direct product $\mathcal{M} \times \mathbf{C}$ with the topology defined by the system of open sets $\mathfrak{D} = \{\mathcal{U} \times \mathbf{C} : \mathcal{U} \subset \mathcal{M},\ c \in \mathbf{C}\}$, where $\mathcal{U}$ is an open set of $\mathcal{M}$. In fact, if we identify a locally constant function f with its graph $D(f) \times c \subset \mathcal{M} \times \mathbf{C}$, then its germ f_p at $p \in D(f)$ is identified with $(p, c) \in \mathcal{M} \times \mathbf{C}$, and $\mathscr{U}(f_p; f, \mathcal{U}) = \{f_q : q \in \mathcal{U}\}$ with $\mathcal{U} \times c \subset \mathcal{M} \times \mathbf{C}$. We denote the sheaf $\mathcal{M} \times \mathbf{C}$ of germs of locally constant functions simply by $\underline{\mathbf{C}}_{\mathcal{M}}$. Similarly we denote by $\underline{\mathbf{Z}}_{\mathcal{M}}$ the sheaf $\mathcal{M} \times \mathbf{Z}$ of germs of $\mathbf{Z}$-valued locally constant functions.

Definition 4.2

Let $\mathscr{S}$ be a sheaf over $\mathcal{M}$, and $\mathcal{W}$ a subset of $\mathcal{M}$. A **section** *σ of $\mathscr{S}$ over $\mathcal{W}$ is a continuous map $\sigma : p \mapsto \sigma(p)$ of $\mathcal{W}$ into $\mathscr{S}$ with $\sigma(p) \in \mathscr{S}_p$.* ♣

Note 4.3

(1) Let $\mathcal{W} \subset \mathcal{M}$ be an open subset and f a C^∞ function on $\mathcal{W}$. Then the map $p \mapsto f_p$, $p \in \mathcal{W}$, defines a section of $\mathscr{A}_{\mathcal{M}}$ over $\mathcal{W}$. Conversely, any section $\sigma : p \mapsto \sigma(p)$ of $\mathscr{A}_{\mathcal{M}}$ over $\mathcal{W}$ is given by some C^∞ function f on $\mathcal{W}$ as $\sigma(p) = f_p$.

Proof. Each $\sigma(p)$ is the germ of some local C^∞ function $g^{[p]}$: $\sigma(p) = (g^{[p]})_p$. Define $f(p) := g^{[p]}(p)$. Then f is well-defined. Since σ is continuous, for a neighborhood $\mathscr{U}(\sigma(p); g^{[p]}, \mathcal{U})$ of $\sigma(p)$, there is a neighborhood $\mathcal{U}(p) \subset \mathcal{U}$ of p such that for $q \in \mathcal{U}(p)$,

$$\sigma(q) \in \mathscr{U}\left(\sigma(p); g^{[p]}, \mathcal{U}\right) = \left\{(g^{[p]})_q : q \in \mathcal{U}\right\},$$

which means $\sigma(q) = (g^{[p]})_q$. Hence $f(q) = g^{[p]}(q)$ for all $q \in \mathcal{U}(p)$. Thus for any $p \in \mathcal{W}$, f coincides with a C^∞ function $g^{[p]}$ on $\mathcal{U}(p)$. Hence f is C^∞, and $\sigma(p) = (g^{[p]})_p = f_p$. □

(2) Similarly, a section $\mathscr{O}_{\mathcal{X}}$ over an open subset $\mathcal{W} \subset \mathcal{X}$ corresponds in a one-to-one matter to a holomorphic function f defines on $\mathcal{W}$ by $\sigma(p) = f_p$. Note that for a section σ, $\sigma(p)$ is a power series $\mathscr{P}(z_p)$ whereas $f(p)$ is a complex number for a holomorphic function f.

(3) Let $\mathcal{F}$ denote a complex vector bundle over a differentiable manifold $\mathcal{M}$ or a holomorphic vector bundle over a complex manifold $\mathcal{X}$. We obtain a one-to-one correspondence between the sections σ of $\mathscr{A}_{\mathcal{M}}(\mathcal{F})$ over an open subset $\mathcal{W} \subset \mathcal{M}$ and the sections s of $\mathcal{F}$ over $\mathcal{W}$: let s be a section of $\mathcal{F}$ over $\mathcal{W}$. Put

$$\sigma(p) = s_p, \quad p \in \mathcal{W}.$$

Similarly, a section of $\mathscr{O}_{\mathcal{X}}(\mathcal{F})$ over an open subset $\mathcal{W} \subset \mathcal{X}$ may be regarded as a *holomorphic section of $\mathcal{F}$ defined there.* ♣

Let $\mathscr{S}$ be a sheaf over $\mathcal{M}$. For a section $\sigma : p \mapsto \sigma(p)$ over an open subset $\mathcal{W} \subset \mathcal{M}$, $\mathscr{W} = \sigma(\mathcal{W}) = \{\sigma(p) : p \in \mathcal{W}\}$ is an open subset of $\mathscr{S}$, and $\sigma : \mathcal{W} \to \mathscr{W}$ is a homeomorphism[1] . Conversely, an open subset $\mathscr{W}$ which intersects with each $\mathscr{S}_p$, $p \in \mathcal{W}$, at a single point $\sigma(p)$, defines a section σ over $\mathcal{W} = \varpi(\mathscr{W})$.

We denote by 0_p the identity of the Abelian group $\mathscr{S}_p$. Then the map $p \mapsto 0_p$ is a section[2] of $\mathscr{S}$ over $\mathcal{M}$.

Thus $p \mapsto 0_p$ is a section of $\mathscr{S}$ over $\mathcal{M}$, hence $\{0_p : p \in \mathcal{M}\}$ is an open subset of $\mathscr{S}$. Let $\mathcal{W}$ be a subset of $\mathcal{M}$ with $\mathcal{W} \neq \emptyset$. We denote the restriction of the map $p \mapsto 0_p$ to $\mathcal{W}$, which is a section over $\mathcal{W}$, by $0_{\mathcal{W}}$. For a sections σ, τ of $\mathscr{S}$ over $\mathcal{W}$, we define its linear combination by

$$c_1\sigma + c_2\tau : p \longmapsto c_1\sigma(p) + c_2\tau(p), \quad c_1, c_2 \in \mathbf{K}.$$

Then the set of all sections of $\mathscr{S}$ over $\mathcal{W}$ forms a $\mathbf{K}$-module.

Definition 4.3

We denote by $\Gamma(\mathcal{W}, \mathscr{S})$ the $\mathbf{K}$-module of all sections of $\mathscr{S}$ over $\mathcal{W}$. ♣

The identity of the module $\Gamma(\mathcal{W}, \mathscr{S})$ is the section $0_{\mathcal{W}} : p \mapsto 0_p$, $p \in \mathcal{W}$. If $\Gamma(\mathcal{W}, \mathscr{S})$ contains no elements other than $0_{\mathcal{W}}$, we denote $\Gamma(\mathcal{W}, \mathscr{S}) = 0$.

If $\mathcal{W}$ is open, then $\Gamma(\mathcal{W}, \mathscr{A}_{\mathcal{M}})$ can be identified with the vector space of all C^∞ functions on $\mathcal{W}$. Similarly, $\Gamma(\mathcal{W}, \mathscr{O}_{\mathcal{X}})$, $\Gamma(\mathcal{W}, \mathscr{A}_{\mathcal{M}}(\mathcal{F}))$ and $\Gamma(\mathcal{W}, \mathscr{O}_{\mathcal{X}}(\mathcal{F}))$ are identified with the vector space of all holomorphic functions on $\mathcal{W}$, all C^∞ sections of $\mathcal{F}$ over $\mathcal{W}$ and all holomorphic sections of $\mathcal{F}$ over $\mathcal{W}$ respectively.

Let $\mathcal{U} \subset \mathcal{W}$, and σ a section of $\mathscr{S}$ over $\mathcal{W}$. Restricting σ to $\mathcal{U}$, we obtain a section of $\mathscr{S}$ over $\mathcal{U}$, which is called the restriction of σ to $\mathcal{U}$, and denoted by $\mathrm{res}_{\mathcal{U}}\sigma$ or $\sigma|_{\mathcal{U}}$. $\mathrm{res}_{\mathcal{U}} : \sigma \mapsto \mathrm{res}_{\mathcal{U}}\sigma$ is a homomorphism of $\Gamma(\mathcal{W}, \mathscr{S})$ into $\Gamma(\mathcal{U}, \mathscr{S})$. Suppose given a finite number of sections $\sigma_\lambda \in \Gamma(\mathcal{W}_\lambda, \mathscr{S})$, $\lambda = 1, \cdots, \nu$. If $\mathcal{W}_{1,\cdots,\nu} := \cap_{\lambda=1}^{\nu} \mathcal{W}_\lambda \neq \emptyset$, for any $c_\lambda \in \mathbf{K}$, $\lambda = 1, \cdots, \nu$, the map $p \mapsto \sum_{\lambda=1}^{\nu} c_\lambda \sigma_\lambda(p)$, $p \in \mathcal{W}_{1,\cdots,\nu}$, is a section of $\mathscr{S}$ over $\mathcal{W}_{1,\cdots,\nu}$, which we denote by

$$\sum_{1 \le \lambda \le \nu} c_\lambda \mathrm{res}_{\mathcal{W}_{1,\cdots,\nu}} \sigma_\lambda = \sum_{1 \le \lambda \le \nu} c_\lambda \sigma_\lambda|_{\mathcal{W}_{1,\cdots,\nu}}. \tag{4.2.3.2}$$

[1]Since $\varpi : \mathscr{S} \to \mathcal{M}$ is locally homeomorphic, for any $q \in \mathcal{W}$, there is a neighborhood $\mathscr{V}$ of $\sigma(q)$ in $\mathscr{S}$ such that $\varpi : \mathscr{V} \to \mathcal{V} = \varpi(\mathscr{V})$ is a homeomorphism. Here $\mathcal{V}$ is a neighborhood of q. Since σ is continuous, there is a neighborhood $\mathcal{U} \subset \mathcal{V}$ of q such that $\mathscr{U} = \sigma(\mathcal{U}) \subset \mathscr{V}$. $\varpi(\sigma(p)) = p$ implies that $\varpi(\mathscr{U}) = \mathcal{U}$, and $\mathscr{U}$ is an open subset of $\mathscr{V}$ since ϖ is homeomorphic. thus for any $q \in \mathcal{W}$, there is a neighborhood $\mathcal{U} \subset \mathcal{W}$ such that $\mathscr{U} = \sigma(\mathcal{U}) \subset \mathscr{W}$ is an open set of $\mathscr{S}$. Hence $\mathscr{W}$ is an open in $\mathscr{S}$. Since $\varpi : \mathscr{U} \to \mathcal{U}$ is homeomorphic for each $\mathscr{U}$, $\varpi : \mathscr{W} \to \mathcal{W}$ is a homeomorphism. Hence its inverse $\sigma : \mathcal{W} \to \mathscr{W}$ is also a homeomorphism.

[2]It suffices to show that for any $q \in \mathcal{M}$, $p \mapsto 0_p$ is continuous in some neighborhood $\mathcal{U}(q)$ of q. Since ϖ is a local homeomorphism, for a sufficiently small $\mathcal{U}(q)$ there is a neighborhood $\mathscr{U}$ of 0_q such that $\varpi : \mathscr{U} \to \mathcal{U}$ is homeomorphic. consequently, its inverse $p \mapsto \sigma(p) \in \mathscr{U}$ is continuous, hence, by (iii) of **Definition 4.1**, $p \mapsto 0_p = \sigma(p) - \sigma(p)$ is continuous on $\mathcal{U}(q)$.

If $\mathcal{W}_{1,\cdots,\nu} = \emptyset$, we do not define $\sum_{1\le\lambda\le\nu} c_\lambda \mathrm{res}_{\mathcal{W}_{1,\cdots,\nu}}\sigma_\lambda$. In what follows, if we write $\sum_{1\le\lambda\le\nu} c_\lambda \mathrm{res}_{\mathcal{W}_{1,\cdots,\nu}}\sigma_\lambda$ we always assume that $\mathcal{W}_{1,\cdots,\nu} \neq \emptyset$.

4.2.4 Cohomology group

Let $\mathcal{M}$ be a differentiable manifold, $\mathscr{S}$ a sheaf over $\mathcal{M}$, and $\mathfrak{U} = \{\mathcal{U}_j\}_{j\in\mathbf{N}}$ an arbitrary locally finite open covering of $\mathcal{M}$.

A **0-cochain $c^0_{\mathfrak{U}}$ with respect to $\mathfrak{U}$** is a set $c^0_{\mathfrak{U}} = \{\sigma_j\}$ of sections where $\sigma_j \in \Gamma(\mathcal{U}_j, \mathscr{S})$ for each j. A **1-cochain $c^1_{\mathfrak{U}}$ with respect to $\mathfrak{U}$** is a set $c^1_{\mathfrak{U}} = \{\sigma_{jk}\}$ of sections $\sigma_{jk} \in \Gamma(\mathcal{U}_j \cap \mathcal{U}_k, \mathscr{S})$ for all indices j, k with $\mathcal{U}_{j,k} := \mathcal{U}_j \cap \mathcal{U}_k \neq \emptyset$ such that $\sigma_{jk} + \sigma_{kj} = 0$. In general, a ***q*-cochain $c^q_{\mathfrak{U}} = \{\sigma_{k_0\cdots k_q}\}$ with respect to $\mathfrak{U}$** is a set of sections $\sigma_{k_0\cdots k_q} \in \Gamma(\mathcal{U}_{k_0} \cap \cdots \cap \mathcal{U}_{k_q}, \mathscr{S})$, for all $(q+1)$-tuples of indices $k_0, \cdots, k_q$ with $\mathcal{U}_{k_0,\cdots,k_q} = \mathcal{U}_{k_0} \cap \cdots \cap \mathcal{U}_{k_q} \neq \emptyset$ which are skew-symmetric in the indices $k_0, \cdots, k_q$. Given two q-cochains $\{\sigma_{k_0\cdots k_q}\}$ and $\{\tau_{k_0\cdots k_q}\}$, we define their linear combination by

$$c_1\{\sigma_{k_0\cdots k_q}\} + c_2\{\tau_{k_0\cdots k_q}\} = \{c_1\sigma_{k_0\cdots k_q} + c_2\tau_{k_0\cdots k_q}\}, \quad c_1, c_2 \in \mathbf{K}.$$

The the set of all q-cochains with respect to $\mathfrak{U}$ forms a $\mathbf{K}$-module, which we denote by $C^q(\mathfrak{U}, \mathscr{S})$.

For a 0-cochain $c^0_{\mathfrak{U}} = \{\sigma_j\}$, the 1-cochain $c^1_{\mathfrak{U}} = \{\tau_{jk}\}$ defined by $\tau_{jk} := (\sigma_k - \sigma_j)|_{\mathcal{U}_{jk}}$ is called the **coboundary of $c^0_{\mathfrak{U}}$** and denoted by $\delta c^0_{\mathfrak{U}}$:

$$\delta\{\sigma_j\} = \{\tau_{jk}\}, \quad \tau_{jk} = (\sigma_k - \sigma_j)|_{\mathcal{U}_{jk}}. \tag{4.2.4.1}$$

For a 1-cochain $\{\sigma_{jk}\}$, we define its coboundary by

$$\delta\{\sigma_{jk}\} = \{\tau_{jk\ell}\}, \quad \tau_{jk\ell} = (\sigma_{k\ell} - \sigma_{j\ell} + \sigma_{jk})_{\mathcal{U}_{jk\ell}} = (\sigma_{jk} + \sigma_{k\ell} + \sigma_{\ell j})|_{\mathcal{U}_{jk\ell}}. \tag{4.2.4.2}$$

In general for a $(q-1)$-cochain $\{\sigma_{k_0\cdots k_{q-1}}\}$ with respect to $\mathfrak{U}$ we define its coboundary by

$$\delta\{\sigma_{k_0\cdots k_{q-1}}\} = \{\tau_{k_0\cdots k_q}\}, \quad \tau_{k_0\cdots k_q} = \sum_{0\le s\le q} (-1)^s \sigma_{k_0\cdots k_{s-1}k_{s+1}\cdots k_q}|_{\mathcal{U}_{k_0\cdots k_q}}. \tag{4.2.4.3}$$

If $\{\tau_{k_0\cdots k_q}\} = \delta\{\sigma_{k_0\cdots k_{q-1}}\}$, then we have $\delta\{\tau_{k_0\cdots k_q}\} = 0$. In fact

$$\begin{aligned}
\sum_{0\le s\le q+1} (-1)^s \tau_{k_0\cdots k_{s-1}k_{s+1}\cdots k_{q+1}} &= \sum_{0\le s\le q+1} (-1)^s \Bigg[\sum_{0\le t\le s-1} (-1)^t \\
\sigma_{k_0\cdots k_{t-1}k_{t+1}\cdots k_{s-1}k_{s+1}\cdots k_{q+1}} &+ \sum_{s+1\le t\le q+1} (-1)^{t-1} \sigma_{k_0\cdots k_{s-1}k_{s+1}\cdots k_{t-1}k_{t+1}\cdots k_{q+1}} \Bigg] \\
&= \sum_{0\le t<s\le q+1} (-1)^{s+t} \sigma_{k_0\cdots k_{t-1}k_{t+1}\cdots k_{s-1}k_{s+1}\cdots k_{q+1}} \\
&\quad - \sum_{0\le s<t\le q+1} (-1)^{s+t} \sigma_{k_0\cdots k_{s-1}k_{s+1}\cdots k_{t-1}k_{t+1}\cdots k_{q+1}} = 0.
\end{aligned}$$

Hence we have

$$\delta^2 = \delta\delta = 0. \tag{4.2.4.4}$$

If $\delta c^q_{\mathfrak{U}} = 0$, we call $c^q_{\mathfrak{U}} \in C^q(\mathfrak{U}, \mathscr{S})$ a **q-cocycle with respect to $\mathfrak{U}$**. We denote the set of q-cocycles with respect to $\mathfrak{U}$ by $Z^q(\mathfrak{U}, \mathscr{S})$:

$$Z^q(\mathfrak{U}, \mathscr{S}) = \left\{c^q_{\mathfrak{U}} \in C^q(\mathfrak{U}, \mathscr{S}) : \delta c^q_{\mathfrak{U}} = 0\right\} \tag{4.2.4.5}$$

is a submodule of $C^q(\mathfrak{U}, \mathscr{S})$. For $q \geq 1$, $\delta\delta C^{q-1}(\mathfrak{U}, \mathscr{S}) = 0$ by (4.2.4.4), hence $\delta C^{q-1}(\mathfrak{U}, \mathscr{S}) \subset Z^q(\mathfrak{U}, \mathscr{S})$. We put

$$H^q(\mathfrak{U}, \mathscr{S}) := \frac{Z^q(\mathfrak{U}, \mathscr{S})}{\delta C^{q-1}(\mathfrak{U}, \mathscr{S})}, \quad q \geq 1, \tag{4.2.4.6}$$

which we call the **cohomology group with coefficients in $\mathscr{S}$ with respect to $\mathfrak{U}$**. For $q = 0$, we define

$$H^0(\mathfrak{U}, \mathscr{S}) := Z^0(\mathfrak{U}, \mathscr{S}). \tag{4.2.4.7}$$

Lemma 4.1

For any locally finite open covering $\mathfrak{U}$ of $\mathcal{M}$, we have

$$H^0(\mathfrak{U}, \mathscr{S}) = \Gamma(\mathcal{M}, \mathscr{S}). \tag{4.2.4.8}$$

♡

Proof. If $\{\sigma_j\} \in Z^0(\mathfrak{U}, \mathscr{S})$, then $\sigma_j = \sigma_k$ on $\mathcal{U}_j \cap \mathcal{U}_k \neq \emptyset$, hence it defines an element $\sigma \in \Gamma(\mathcal{M}, \mathscr{S})$ with $\sigma = \sigma_j$ on $\mathcal{U}_j$. We have then $\sigma_j = \mathrm{res}_{\mathcal{U}_j}\sigma$. Thus we may identify $Z^0(\mathfrak{U}, \mathscr{S})$ with $\Gamma(\mathcal{M}, \mathscr{S})$. □

For $q \geq 1$, $H^q(\mathfrak{U}, \mathscr{S})$ does depend on $\mathfrak{U}$. Therefore we define the q-th cohomology group $H^q(\mathcal{M}, \mathscr{S})$ of $\mathcal{M}$ with coefficients in $\mathscr{S}$ as the "limit" of $H^q(\mathfrak{U}, \mathscr{S})$,

$$H^q(\mathcal{M}, \mathscr{S}) = \text{“}\lim_{\mathfrak{U}}\text{”} H^q(\mathfrak{U}, \mathscr{S}) \tag{4.2.4.9}$$

as each $\mathcal{U}_j$ of $\mathfrak{U}$ becomes "infinitely small". The limiting process will be explained now.

(i) We say that the locally finite open covering $\mathfrak{V} = \{\mathcal{V}_\lambda\}$ of $\mathcal{M}$ is a **refinement** of $\mathfrak{U} = \{\mathcal{U}_k\}$ if each $\mathcal{V}_\lambda$ is contained in some member $\mathcal{U}_k$ of $\mathfrak{U}$. In this case we write k as $k(\lambda)$. $\mathfrak{U} \prec \mathfrak{V}$ or $\mathfrak{V} \succ \mathfrak{U}$, if $\mathfrak{V}$ is a refinement of $\mathfrak{U}$. Then $\prec$ defines a partial order on the set of $\mathbf{LFOC}(\mathcal{M})$ of all locally finite open coverings of $\mathcal{M}$, which makes $\mathbf{LFOC}(\mathcal{M})$ a **directed set** since for any $\mathfrak{U}, \mathfrak{V} \in \mathbf{LFOC}(\mathcal{M})$, there is a $\mathfrak{W} \in \mathbf{LFOC}(\mathcal{M})$ such that $\mathfrak{V} \prec \mathfrak{W}$ and $\mathfrak{U} \prec \mathfrak{W}$.

(ii) Suppose $\mathfrak{V}, \mathfrak{U} \in \mathbf{LFOC}(\mathcal{M})$ with $\mathfrak{U} \prec \mathfrak{V}$. For each $\mathcal{V}_\lambda$, we have $\mathcal{V}_\lambda \subset \mathfrak{U}_{k(\lambda)}$. We define a homomorphism of $C^q(\mathfrak{U})$ into $C^q(\mathfrak{V})$,

$$\Pi^{\mathfrak{U}}_{\mathfrak{V}} : C^q(\mathfrak{U}) \longrightarrow C^q(\mathfrak{V}), \quad \{\sigma_{k_0\cdots k_q}\} \longmapsto \{\tau_{\lambda_0\cdots\lambda_q}\}, \tag{4.2.4.10}$$

by putting

$$\tau_{\lambda_0\cdots\lambda_q} := \mathrm{res}_{\mathcal{V}_{\lambda_0,\cdots,\lambda_q}}\sigma_{k(\lambda_0)\cdots k(\lambda_q)}, \quad \mathcal{V}_{\lambda_0,\cdots,\lambda_q} := \cap^q_{s=0}\mathcal{V}_{\lambda_s} \neq \emptyset.$$

Since

$$\delta \circ \Pi^{\mathfrak{U}}_{\mathfrak{V}} = \Pi^{\mathfrak{U}}_{\mathfrak{V}} \circ \delta,$$

$\Pi^{\mathfrak{U}}_{\mathfrak{V}}$ maps $Z^q(\mathfrak{U}, \mathscr{S})$ into $Z^q(\mathfrak{V}, \mathscr{S})$ and $\delta C^{q-1}(\mathfrak{U}, \mathscr{S})$ into $\delta C^{q-1}(\mathfrak{V}, \mathscr{S})$. Hence $\Pi^{\mathfrak{U}}_{\mathfrak{V}}$

defines a homomorphism of $H^q(\mathfrak{U}, \mathscr{S})$ into $H^q(\mathfrak{V}, \mathscr{S})$, which is also denoted by $\Pi^{\mathfrak{U}}_{\mathfrak{V}}$. $\Pi^{\mathfrak{U}}_{\mathfrak{V}} : C^q(\mathfrak{U}, \mathscr{S}) \to C^q(\mathfrak{V}, \mathscr{S})$ depend on the choice of $\mathcal{U}_{k(\lambda)} \supset \mathcal{V}_\lambda$ for each $\mathcal{V}_\lambda$, but $\Pi^{\mathfrak{U}}_{\mathfrak{V}} : H^q(\mathfrak{U}, \mathscr{S}) \to H^q(\mathfrak{V}, \mathscr{S})$ turns out to be independent of this choice.

Lemma 4.2

The homomorphism $\Pi^{\mathfrak{U}}_{\mathfrak{V}} : H^q(\mathfrak{U}, \mathscr{S}) \to H^q(\mathfrak{V}, \mathscr{S})$ is uniquely determined by $\mathfrak{U}$ and $\mathfrak{V}$. ♡

Proof. For each λ choose an arbitrary $j(\lambda)$ with $\mathcal{V}_\lambda \subset \mathcal{U}_{j(\lambda)}$, and put

$$\eta_{\lambda_0\cdots\lambda_q} = \mathrm{res}_{\mathcal{V}_{\lambda_0,\cdots,\lambda_q}} \sigma_{j(\lambda_0)\cdots j(\lambda_q)}, \quad \mathcal{V}_{\lambda_0,\cdots,\lambda_q} \neq \emptyset.$$

It suffices to show that for $\{\sigma_{k_0\cdots k_q}\} \in Z^q(\mathfrak{U}, \mathscr{S})$ we have

$$\{\eta_{\lambda_0\cdots\lambda_q}\} - \{\tau_{\lambda_0\cdots\lambda_q}\} \in \delta C^{q-1}(\mathfrak{V}, \mathscr{S}). \tag{4.2.4.11}$$

Put

$$\kappa_{\nu_1\cdots\nu_q} := \sum_{t=1}^{q} (-1)^{t-1} \mathrm{res}_{\mathcal{V}_{\nu_1,\cdots,\nu_q}} \sigma_{k(\nu_1)\cdots k(\nu_t) j(\nu_t)\cdots j(\nu_q)}, \tag{4.2.4.12}$$

where $\mathcal{V}_{\nu_1,\cdots,\nu_q} = \cap_{t=1}^q \mathcal{V}_{\nu_t} \neq \emptyset$. Though $\kappa_{\nu_1\cdots\nu_q}$ is not necessarily skew-symmetric in the indices, we define

$$(\delta\kappa)_{\lambda_0\cdots\lambda_q} := \sum_{s=0}^{q} (-1)^s \kappa_{\lambda_0\cdots\lambda_{s-1}\lambda_{s+1}\cdots\lambda_q}. \tag{4.2.4.13}$$

Then we see that

$$\eta_{\lambda_0\cdots\lambda_q} - \tau_{\lambda_0\cdots\lambda_q} = (\delta\kappa)_{\lambda_0\cdots\lambda_q}. \tag{4.2.4.14}$$

In fact, if we write k_s for $k(\lambda_s)$ and j_s for $j(\lambda_s)$, we have

$$\begin{aligned}
(\delta\kappa)_{\lambda_0\cdots\lambda_q} &= \sum_{0\le s\le q} (-1)^s \kappa_{\lambda_0\cdots\lambda_{s-1}\lambda_{s+1}\cdots\lambda_q} = \sum_{0\le s\le q} (-1)^s \Bigg[\sum_{0\le t\le s-1} (-1)^t \\
&\mathrm{res}_{\mathcal{V}_{\nu_0,\cdots,\nu_q}} \sigma_{k_0\cdots k_t j_t \cdots j_{s-1} j_{s+1}\cdots j_q} + \sum_{s+1\le t\le q} (-1)^{t-1} \mathrm{res}_{\mathcal{V}_{\nu_0,\cdots,\nu_q}} \sigma_{k_0\cdots k_{s-1}k_{s+1}\cdots k_t j_t\cdots j_q} \Bigg] \\
&= \sum_{0\le t\le q} (-1)^{t+1} \Bigg[\sum_{0\le s\le t-1} (-1)^s \mathrm{res}_{\mathcal{V}_{\nu_0,\cdots,\nu_q}} \sigma_{k_0\cdots k_{s-1}k_{s+1}\cdots k_t j_t \cdots j_q} \\
&\quad + \sum_{t+1\le s\le q} (-1)^{s+1} \mathrm{res}_{\mathcal{V}_{\nu_0,\cdots,\nu_q}} \sigma_{k_0\cdots k_t j_t\cdots j_{s-1}j_{s+1}\cdots j_q} \Bigg].
\end{aligned}$$

Since $\sigma \in Z^q(\mathfrak{U}, \mathscr{S})$, it follows

$$\begin{aligned}
&\sum_{0\le s\le t} (-1)^s \mathrm{res}_{\mathcal{V}_{\nu_0,\cdots,\nu_q}} \sigma_{k_0\cdots k_{s-1}k_{s+1}\cdots k_t j_t\cdots j_q} \\
&+ \sum_{t\le s\le q} (-1)^{s+1} \mathrm{res}_{\mathcal{V}_{\nu_0,\cdots,\nu_q}} \sigma_{k_0\cdots k_t j_t \cdots j_{s-1} j_{s+1}\cdots j_q} = 0.
\end{aligned}$$

Hence we have

$$\begin{aligned}(\delta\kappa)_{\lambda_0\cdots\lambda_q} &= \sum_{0\le t\le q} \mathrm{res}_{V_{\nu_0,\cdots,\nu_q}}\sigma_{k_0\cdots k_{t-1}j_t\cdots j_q} - \sum_{0\le t\le q} \mathrm{res}_{V_{\nu_0,\cdots,\nu_q}}\sigma_{k_0\cdots k_t j_{t+1}\cdots j_q}\\ &= \mathrm{res}_{V_{\nu_0,\cdots,\nu_q}}\sigma_{j_0\cdots j_q} - \mathrm{res}_{V_{\nu_0,\cdots,\nu_q}}\sigma_{k_0\cdots k_q} \;=\; \eta_{\lambda_0\cdots\lambda_q} - \tau_{\lambda_0\cdots\lambda_q}.\end{aligned}$$

The above $\kappa_{\nu_1\cdots\nu_q}$ is not necessarily skew-symmetric in $\nu_1,\cdots,\nu_q$ so we put

$$\widetilde{\kappa}_{\nu_1\cdots\nu_q} := \frac{1}{q!}\sum_{\mu_1,\cdots,\mu_q} \mathrm{sgn}\begin{pmatrix}\nu_1\cdots\nu_q\\ \mu_1\cdots\mu_q\end{pmatrix}\kappa_{\mu_1\cdots\mu_q}, \tag{4.2.4.15}$$

where the summation is taken over all the permutations $\mu_1,\cdots,\mu_q$ of $\nu_1,\cdots,\nu_q$. Then $\{\widetilde{\kappa}_{\nu_1\cdots\nu_q}\} \in C^{q-1}(\mathfrak{V},\mathscr{L})$ and

$$\eta_{\lambda_0\cdots\lambda_q} - \tau_{\lambda_0\cdots\lambda_q} = \sum_{0\le s\le q}(-1)^s\widetilde{\kappa}_{\lambda_0\cdots\lambda_{s-1}\lambda_{s+1}\cdots\lambda_q}.$$

Namely, $\{\eta_{\lambda_0\cdots\lambda_q}\} - \{\tau_{\lambda_0\cdots\lambda_q}\} = \delta\{\widetilde{\kappa}_{\lambda_0\cdots\lambda_{q-1}}\} \in \delta C^{q-1}(\mathfrak{V},\mathscr{S})$. □

(iii) By **Lemma 4.1** for $\mathfrak{U},\mathfrak{V}\in\mathbf{LFOC}(\mathcal{M})$ with $\mathfrak{U}\prec\mathfrak{V}$ a homomorphism $\Pi^{\mathfrak{U}}_{\mathfrak{V}} : H^q(\mathfrak{U},\mathscr{S}) \to H^q(\mathfrak{V},\mathscr{S})$ is defined. Clearly $\Pi^{\mathfrak{U}}_{\mathfrak{U}}$ is identity and if $\mathfrak{U}\succ\mathfrak{V}\succ\mathfrak{W}$, we have

$$\Pi^{\mathfrak{U}}_{\mathfrak{W}} = \Pi^{\mathfrak{V}}_{\mathfrak{W}}\circ\Pi^{\mathfrak{U}}_{\mathfrak{V}}. \tag{4.2.4.16}$$

Now we can define $\varinjlim_{\mathfrak{U}} H^q(\mathfrak{U},\mathscr{S})$ as follows. We define $g\succ 0$ for $g\in H^q(\mathfrak{U},\mathscr{S})$ if there exists $\mathfrak{V}\succ\mathfrak{U}$, $\mathfrak{V}\in\mathbf{LFOC}(\mathcal{M})$, such that $\Pi^{\mathfrak{U}}_{\mathfrak{V}}g = 0$. Set

$$N^q(\mathfrak{U},\mathscr{S}) := \{g\in H^q(\mathfrak{U},\mathscr{S}) : g\succ 0\}, \tag{4.2.4.17}$$

which is a subgroup of $H^q(\mathfrak{U},\mathscr{S})$. In fact, if $\Pi^{\mathfrak{U}}_{\mathfrak{V}}g = 0$ and $\Pi^{\mathfrak{U}}_{\mathfrak{W}}h = 0$, then taking $\mathfrak{X}\in\mathbf{LFOC}(\mathcal{M})$ with $\mathfrak{V}\prec\mathfrak{X}$ and $\mathfrak{W}\prec\mathfrak{X}$, we obtain by (4.2.4.16)

$$\Pi^{\mathfrak{U}}_{\mathfrak{X}}(c_1g + c_2h) = c_1\Pi^{\mathfrak{V}}_{\mathfrak{X}}\Pi^{\mathfrak{U}}_{\mathfrak{V}}g + c_2\Pi^{\mathfrak{W}}_{\mathfrak{X}}\Pi^{\mathfrak{U}}_{\mathfrak{W}}h = 0, \quad c_1,c_2\in\mathbf{K}.$$

Put

$$\overline{H}^q(\mathfrak{U},\mathscr{S}) := \frac{H^q(\mathfrak{U},\mathscr{S})}{N^q(\mathfrak{U},\mathscr{S})}, \tag{4.2.4.18}$$

and let

$$\Pi^{\mathfrak{U}} : H^q(\mathfrak{U},\mathscr{S}) \longrightarrow \overline{H}^q(\mathfrak{U},\mathscr{S}), \quad g\longmapsto\overline{g}, \tag{4.2.4.19}$$

be the canonical homomorphism.

For $\mathfrak{U}\prec\mathfrak{V}$, we have $\Pi^{\mathfrak{U}}_{\mathfrak{V}}N^q(\mathfrak{U},\mathscr{S})\subset N^q(\mathfrak{V},\mathscr{S})$. In fact for $g\in N^q(\mathfrak{U},\mathscr{S})$, there is a $\mathfrak{W}\succ\mathfrak{U}$ with $\Pi^{\mathfrak{U}}_{\mathfrak{W}}g = 0$. For $\mathfrak{X}\in\mathbf{LFOC}(\mathcal{M})$ with $\mathfrak{V}\prec\mathfrak{X}$ and $\mathfrak{W}\prec\mathfrak{X}$, we have

$$\Pi^{\mathfrak{V}}_{\mathfrak{X}}\Pi^{\mathfrak{U}}_{\mathfrak{V}}g = \Pi^{\mathfrak{U}}_{\mathfrak{X}}g = \Pi^{\mathfrak{W}}_{\mathfrak{X}}\Pi^{\mathfrak{U}}_{\mathfrak{W}}g = 0,$$

hence $\Pi^{\mathfrak{U}}_{\mathfrak{V}}g\in N^q(\mathfrak{V},\mathscr{S})$. Consequently, $\Pi^{\mathfrak{U}}_{\mathfrak{V}} : H^q(\mathfrak{U},\mathscr{S}) \to H^q(\mathfrak{V},\mathscr{S})$ induces the homomorphism

$$\overline{\Pi}^{\mathfrak{U}}_{\mathfrak{V}} : \overline{H}^q(\mathfrak{U},\mathscr{S}) \longrightarrow \overline{H}^q(\mathfrak{V},\mathscr{S}), \quad \mathfrak{U}\prec\mathfrak{V}. \tag{4.2.4.20}$$

We can show that $\overline{\Pi}^{\mathfrak{U}}_{\mathfrak{V}}$ is *injective*. For if $\overline{\Pi}^{\mathfrak{U}}_{\mathfrak{V}}\overline{g} = 0$ with $\overline{g} = \Pi^{\mathfrak{U}}g$, $g\in H^q(\mathfrak{U},\mathscr{S})$, then $\Pi^{\mathfrak{U}}_{\mathfrak{V}}g\in N^q(\mathfrak{V},\mathscr{S})$, hence there is a $\mathfrak{W}\succ\mathfrak{V}$ with $\Pi^{\mathfrak{V}}_{\mathfrak{W}}\Pi^{\mathfrak{U}}_{\mathfrak{V}}g = 0$. Hence $\Pi^{\mathfrak{U}}_{\mathfrak{W}}g = 0$, which means $g \in N^q(\mathfrak{U}, \mathscr{S})$, that is $\overline{g} = 0$. Thus for $\mathfrak{U} \prec \mathfrak{V}$, $\overline{H}^q(\mathfrak{U}, \mathscr{S})$ may be identified with a submodule $\overline{\Pi}^{\mathfrak{U}}_{\mathfrak{V}} \overline{H}^q(\mathfrak{U}, \mathscr{S})$ via the inclusion $\overline{\Pi}^{\mathfrak{U}}_{\mathfrak{V}}$.

$$\begin{tikzcd} H^q(\mathfrak{U}, \mathscr{S}) \arrow[r, "\Pi^{\mathfrak{U}}"] \arrow[d, "\Pi^{\mathfrak{U}}_{\mathfrak{V}}"'] & \overline{H}^q(\mathfrak{U}, \mathscr{S}) \arrow[d, "\overline{\Pi}^{\mathfrak{U}}_{\mathfrak{V}}"] \\ H^q(\mathfrak{V}, \mathscr{S}) \arrow[r, "\Pi^{\mathfrak{V}}"'] & \overline{H}^q(\mathfrak{V}, \mathscr{S}). \end{tikzcd}$$

Taking the union $\cup_{\mathfrak{U} \in \mathbf{LFOC}(\mathcal{M})} \overline{H}^q(\mathfrak{U}, \mathscr{S})$ and considering it as

$$\varinjlim_{\mathfrak{U}} H^q(\mathfrak{U}, \mathscr{S}) := \bigcup_{\mathfrak{U} \in \mathbf{LFOC}(\mathcal{M})} \overline{H}^q(\mathfrak{U}, \mathscr{S}),$$

we define the **cohomology group of $\mathcal{M}$ with coefficients in $\mathscr{S}$** by

$$H^q(\mathcal{M}, \mathscr{S}) := \varinjlim_{\mathfrak{U}} H^q(\mathfrak{U}, \mathscr{S}) = \bigcup_{\mathfrak{U} \in \mathbf{LFOC}(\mathcal{M})} \overline{H}^q(\mathfrak{U}, \mathscr{S}). \tag{4.2.4.21}$$

For any $\mathfrak{U} \in \mathbf{LFOC}(\mathcal{M})$,

$$\Pi^{\mathfrak{U}} : H^q(\mathfrak{U}, \mathscr{S}) \longrightarrow \overline{H}^q(\mathfrak{U}, \mathscr{S}) \subset H^q(\mathcal{M}, \mathscr{S}) \tag{4.2.4.22}$$

is a homomorphism of $H^q(\mathfrak{U}, \mathscr{S})$ into $H^q(\mathcal{M}, \mathscr{S})$.

A q-cocycle $c^q_{\mathfrak{U}} \in Z^q(\mathfrak{U}, \mathscr{S})$ defines an element $g \in H^q(\mathfrak{U}, \mathscr{S})$, which in turn defines an element $\overline{g} = \Pi^{\mathfrak{U}} g \in H^q(\mathcal{M}, \mathscr{S})$. We call $\overline{g}$ the **cohomology class of a q-cocycle $c^q_{\mathfrak{U}}$**, and denote it by $[c^q_{\mathfrak{U}}] = \Pi^{\mathfrak{U}} g \in H^q(\mathcal{M}, \mathscr{S})$.

It is clear from **Lemma 4.1** that

$$H^0(\mathcal{M}, \mathscr{S}) = \Gamma(\mathcal{M}, \mathscr{S}). \tag{4.2.4.23}$$

Theorem 4.5

For $\mathfrak{U} \in \mathbf{LFOC}(\mathcal{M})$, $\Pi^{\mathfrak{U}} : H^1(\mathfrak{U}, \mathscr{S}) \to H^1(\mathcal{M}, \mathscr{S})$ is injective. Consequently,

$$H^1(\mathcal{M}, \mathscr{S}) = \bigcup_{\mathfrak{U} \in \mathbf{LFOC}(\mathcal{M})} H^1(\mathfrak{U}, \mathscr{S}). \tag{4.2.4.24}$$

♡

Proof. Since $H^1(\mathfrak{U}, \mathscr{S}) \to \overline{H}^1(\mathfrak{U}, \mathscr{S}) = H^1(\mathfrak{U}, \mathscr{S})/N^1(\mathfrak{U}, \mathscr{S}) \subset H^1(\mathcal{M}, \mathscr{S})$, it suffices to prove $N^1(\mathfrak{U}, \mathscr{S}) = 0$.

Let $g \in N^1(\mathfrak{U}, \mathscr{S})$. Then there is $\mathfrak{V} \succ \mathfrak{U}$, $\mathfrak{V} \in \mathbf{LFOC}(\mathcal{M})$, such that $\Pi^{\mathfrak{U}}_{\mathfrak{V}} g = 0$. We let $\mathfrak{U} = \{\mathcal{U}_j\}$, $\mathfrak{V} = \{\mathcal{V}_\lambda\}$ and put $\mathfrak{W} = \{\mathcal{W}_{j\lambda} : \mathcal{W}_{j\lambda} = \mathcal{U}_j \cap \mathcal{V}_\lambda \neq \emptyset\}$. Since $\mathcal{W}_{j\lambda} \subset \mathcal{V}_\lambda$, we have $\mathfrak{V} \prec \mathfrak{W}$, hence $\Pi^{\mathfrak{U}}_{\mathfrak{W}} g = \Pi^{\mathfrak{V}}_{\mathfrak{W}} \Pi^{\mathfrak{U}}_{\mathfrak{V}} g = 0$. Here by **Lemma 4.1**, we may assume that $\Pi^{\mathfrak{U}}_{\mathfrak{W}}$ is defined via the inclusion $\mathcal{W}_{j\lambda} \subset \mathcal{U}_j$.

Suppose that g is represented by a 1-cocycle $\{\sigma_{jk}\} \in Z^1(\mathfrak{U}, \mathscr{S})$. If we define $\Pi^{\mathfrak{U}}_{\mathfrak{W}} : Z^1(\mathfrak{U}, \mathscr{S}) \to Z^1(\mathfrak{W}, \mathscr{S})$ via the inclusion $\mathcal{W}_{j\lambda} \subset \mathcal{U}_j$, then

$$\Pi^{\mathfrak{U}}_{\mathfrak{W}}\{\sigma_{jk}\} = \{\tau_{j\lambda k\nu}\}, \quad \tau_{j\lambda k\nu} = \mathrm{res}_{\mathcal{W}_{j\lambda,k\nu}} \sigma_{jk} \quad \mathcal{W}_{j\lambda,k\nu} = \mathcal{W}_{j\lambda} \cap \mathcal{W}_{k\nu} \neq \emptyset.$$

Since $\Pi^{\mathfrak{U}}_{\mathfrak{W}} g = 0$, $\{\tau_{j\lambda k\nu}\} \in \delta C^0(\mathfrak{W}, \mathscr{S})$, hence

$$\tau_{j\lambda k\nu} = \tau_{k\nu} - \tau_{j\lambda}$$

for some $\{\tau_{j\lambda}\} \in C^0(\mathfrak{W}, \mathscr{S})$. Since $\sigma_{jj} = 0$, $\tau_{j\nu} - \tau_{j\lambda} = \tau_{j\lambda j\nu} = 0$ implying that $\tau_{j\nu} = \tau_{j\lambda}$

on $\mathcal{W}_{j\lambda} \cap \mathcal{W}_{j\nu} \neq \emptyset$. Therefore we can define a section $\tau_j \in \Gamma(\mathcal{U}_j, \mathscr{S})$ by setting

$$\tau_j = \tau_{j\lambda} \quad \text{on} \quad \mathcal{U}_j \cap \mathcal{V}_\lambda = \mathcal{W}_{j\lambda} \neq \emptyset.$$

On $\mathcal{U}_j \cap \mathcal{U}_k \cap \mathcal{V}_\lambda$, $\sigma_{jk} = \tau_{j\lambda k\lambda} = \tau_k - \tau_j$, hence $\sigma_{jk} = \tau_k - \tau_j$ on $\mathcal{U}_j \cap \mathcal{U}_k$, that is, $\{\sigma_{jk}\} = \delta\{\tau_j\} \in \delta C^0(\mathfrak{U}, \mathscr{S})$. Hence $g = 0$ and we obtain

$$\overline{H}^1(\mathfrak{U}, \mathscr{S}) = H^1(\mathfrak{U}, \mathscr{S}). \tag{4.2.4.25}$$

Therefore $\Pi^{\mathfrak{U}} : H^1(\mathfrak{U}, \mathscr{S}) \to \overline{H}^1(\mathfrak{U}, \mathscr{S}) \subset H^1(\mathcal{M}, \mathscr{S})$ is injective. □

Theorem 4.6

If $H^1(\mathcal{U}_j, \mathscr{S}) = 0$ for every $\mathcal{U}_j \in \mathfrak{U} \in \mathbf{LFOC}(\mathcal{M})$, then $H^1(\mathfrak{U}, \mathscr{S}) = H^1(\mathcal{M}, \mathscr{S})$. ♡

Proof. By **Theorem 4.5**, we have $H^1(\mathcal{M}, \mathscr{S}) = \cup_{\mathfrak{V} \in \mathbf{LFOC}(\mathcal{M})} H^1(\mathfrak{V}, \mathscr{S})$, hence it suffices to prove $H^1(\mathfrak{V}) \subset H^1(\mathfrak{U}, \mathscr{S})$ for any $\mathfrak{V} \in \mathbf{LFOC}(\mathcal{M})$. Let $\mathfrak{U} = \{\mathcal{U}_j\}$ and $\mathfrak{V} = \{\mathcal{V}_\lambda\}$, and put $\mathfrak{W} = \{\mathcal{W}_{j\lambda} : \mathcal{W}_{j\lambda} = \mathcal{U}_j \cap \mathcal{V}_\lambda \neq \emptyset\}$. By (4.2.4.22) and (4.2.4.25), $H^1(\mathfrak{V}, \mathscr{S}) \subset H^1(\mathfrak{W}, \mathscr{S})$, hence we have only to show that $H^1(\mathfrak{W}, \mathscr{S}) \subset H^1(\mathfrak{U}, \mathscr{S})$.

For this purpose, it suffices to prove that for any 1-cocycle $\{\tau_{j\lambda k\nu}\} \in Z^1(\mathfrak{W}, \mathscr{S})$, there is a 1-cocycle $\{\sigma_{jk}\} \in Z^1(\mathfrak{U}, \mathscr{S})$ and a 0-cochain $\{\tau_{j\lambda}\} \in C^0(\mathfrak{W}, \mathscr{S})$ such that

$$\tau_{j\lambda k\nu} = \mathrm{res}_{\mathcal{W}_{j\lambda} \cap \mathcal{W}_{k\nu}} \sigma_{jk} - (\tau_{k\nu} - \tau_{j\lambda}) \,. \tag{4.2.4.26}$$

In fact (4.2.4.26) means that $\{\tau_{j\lambda k\nu}\} = \Pi^{\mathfrak{U}}_{\mathfrak{W}}\{\sigma_{jk}\} - \delta\{\tau_{j\lambda}\}$, which implies $H^1(\mathfrak{W}, \mathscr{S}) \subset H^1(\mathfrak{U}, \mathscr{S})$.

For a fixed $\mathcal{U}_j$, $\mathfrak{W}_j = \{\mathcal{W}_{j\lambda}\}$ is a locally finite open covering of $\mathcal{U}_j$. Since by hypothesis $H^1(\mathcal{U}_j, \mathscr{S}) = 0$, we have $Z^1(\mathfrak{W}_j, \mathscr{S})/\delta C^0(\mathfrak{W}_j, \mathscr{S}) = H^1(\mathfrak{W}_j, \mathscr{S}) = 0$ by **Theorem 4.5**. Hence $\{\tau_{j\lambda j\nu}\} \in Z^1(\mathfrak{W}_j) = \delta C^0(\mathfrak{W}_j)$, i.e., there is a $\{\tau_{j\lambda}\} \in C^0(\mathfrak{W}_j)$ such that

$$\tau_{j\lambda j\nu} = \tau_{j\nu} - \tau_{j\lambda}$$

on $\mathcal{W}_{j\lambda} \cap \mathcal{W}_{j\nu} \neq \emptyset$. Put

$$\sigma_{j\lambda k\nu} := \tau_{j\lambda k\nu} - \tau_{k\nu} + \tau_{j\lambda}. \tag{4.2.4.27}$$

Then $\{\sigma_{j\lambda k\nu}\} \in Z^1(\mathfrak{V})$ and $\sigma_{j\lambda j\nu} = 0$. Since $\sigma_{j\lambda j\nu} + \sigma_{j\nu k\mu} + \sigma_{k\mu j\lambda} = 0$, and $\sigma_{j\lambda k\mu} = -\sigma_{k\mu j\lambda}$, it follows that $\sigma_{j\lambda k\mu} = \sigma_{j\nu k\mu}$. Similarly, $\sigma_{j\lambda k\mu} = \sigma_{j\lambda k\nu}$. Hence by putting

$$\sigma_{jk\mu} := \sigma_{j\lambda k\mu} \tag{4.2.4.28}$$

on $\mathcal{U}_j \cap \mathcal{U}_k \cap \mathcal{V}_\mu \cap \mathcal{V}_\lambda \neq \emptyset$, we define a section $\sigma_{jk\mu} \in \Gamma(\mathcal{U}_j \cap \mathcal{U}_k \cap \mathcal{V}_\mu, \mathscr{S})$. $\sigma_{j\lambda k\mu} = \sigma_{j\lambda k\nu}$ implies that $\sigma_{jk\mu} = \sigma_{jk\nu}$ on $\mathcal{U}_j \cap \mathcal{U}_k \cap \mathcal{V}_\mu \cap \mathcal{V}_\nu \neq \emptyset$. Hence by putting

$$\sigma_{jk} := \sigma_{jk\mu} \tag{4.2.4.29}$$

on $\mathcal{U}_j \cap \mathcal{U}_k \cap \mathcal{V}_\mu \neq \emptyset$, we obtain a section $\sigma_{jk} \in \Gamma(\mathcal{U}_j \cap \mathcal{U}_k, \mathscr{S})$. $\{\sigma_{jk}\}$ is a 1-cocycle since

$$\sigma_{jk} + \sigma_{k\ell} + \sigma_{\ell j} = \sigma_{j\lambda k\lambda} + \sigma_{k\lambda\ell\lambda} + \sigma_{\ell\lambda j\lambda} = 0$$

on each $\mathcal{U}_j \cap \mathcal{U}_k \cap \mathcal{U}_\ell \cap \mathcal{V}_\lambda \neq \emptyset$. Then (4.2.4.27) implies (4.2.4.26). □

Theorem 4.7

If $H^1(\mathcal{U}_j, \mathscr{S}) = 0$ for each $\mathcal{U}_j \in \mathfrak{U} \in \mathbf{LFOC}(\mathcal{M})$, then $\Pi^{\mathfrak{U}} : H^2(\mathfrak{U}, \mathscr{S}) \to H^2(\mathcal{M}, \mathscr{S})$ is injective. ♡

Proof. It suffices to prove $N^2(\mathfrak{U}, \mathscr{S}) = 0$. Let $g \in N^2(\mathfrak{U}, \mathscr{S})$, then there is a $\mathfrak{V} \succ \mathfrak{U}$ such that $\Pi^{\mathfrak{U}}_{\mathfrak{V}} g = 0$. Let $\mathfrak{U} = \{\mathcal{U}_j\}$ and $\mathfrak{V} = \{\mathcal{V}_\lambda\}$ and put $\mathfrak{W} = \{\mathcal{W}_{j\lambda} : \mathcal{W}_{j\lambda} = \mathcal{U}_j \cap \mathcal{V}_\lambda \neq \emptyset\}$. Then $\Pi^{\mathfrak{U}}_{\mathfrak{W}} g = \Pi^{\mathfrak{V}}_{\mathfrak{W}} \Pi^{\mathfrak{U}}_{\mathfrak{V}} g = 0$. Suppose that g is represented by a 2-cocycle $\{\sigma_{ijk}\} \in Z^2(\mathfrak{U}, \mathscr{S})$. If we define $\Pi^{\mathfrak{U}}_{\mathfrak{W}} : Z^2(\mathfrak{U}, \mathscr{S}) \to Z^2(\mathfrak{W}, \mathscr{S})$ via this inclusion $\mathcal{W}_{j\lambda} \subset \mathcal{U}_j$, we have

$$\Pi^{\mathfrak{U}}_{\mathfrak{W}} \{\sigma_{ijk}\} = \{\tau_{i\lambda j\mu k\nu}\}, \quad \tau_{i\lambda j\mu k\nu} = \mathrm{res}_{\mathcal{W}_{i\lambda} \cap \mathcal{W}_{j\mu} \cap \mathcal{W}_{k\nu}} \sigma_{ijk}.$$

Since $\Pi^{\mathfrak{U}}_{\mathfrak{W}} g = 0$, there is a 1-cochain $\{\tau_{j\mu k\nu}\} \in C^1(\mathfrak{W}, \mathscr{S})$ such that

$$\tau_{i\lambda j\mu k\nu} = \tau_{j\mu k\nu} + \tau_{k\nu i\lambda} + \tau_{i\lambda j\mu}. \tag{4.2.4.30}$$

Since for each j, $\sigma_{jjj} = 0$, we have

$$\tau_{j\mu j\nu} + \tau_{j\nu j\lambda} + \tau_{j\lambda j\mu} = \tau_{j\lambda j\mu j\nu} = 0,$$

which means $\{\tau_{j\lambda j\nu}\} \in Z^1(\mathfrak{W}_j, \mathscr{S})$ with $\mathfrak{W}_j = \{\mathcal{W}_{j\lambda}\} \in \mathbf{LFOC}(\mathcal{U}_j)$. Since $H^1(\mathcal{U}_j, \mathscr{S}) = 0$, it follows that there is a 0-cochain $\{\tau_{j\lambda}\} \in C^0(\mathfrak{W}_j, \mathscr{S})$ such that

$$\tau_{j\lambda j\nu} = \tau_{j\nu} - \tau_{j\lambda}.$$

Put

$$\sigma_{j\lambda k\nu} := \tau_{j\lambda k\nu} - \tau_{k\nu} + \tau_{j\lambda}, \tag{4.2.4.31}$$

then $\sigma_{j\lambda j\nu} = 0$ and by (4.2.4.30) we have

$$\tau_{i\lambda j\mu k\nu} = \sigma_{j\mu k\nu} + \sigma_{k\nu i\lambda} + \sigma_{i\lambda j\mu}. \tag{4.2.4.32}$$

Putting $i = j$ in (4.2.4.32) yields $\tau_{j\lambda j\mu k\nu} = \sigma_{j\mu k\nu} + \sigma_{k\nu j\lambda}$. Since $\sigma_{jjk} = 0$, $\tau_{j\lambda j\mu k\nu} = 0$. Therefore $\sigma_{j\lambda k\nu} = \sigma_{j\mu k\nu}$. Similarly $\sigma_{j\lambda k\mu} = \sigma_{j\lambda k\nu}$. As in the proof of **Theorem 4.6**, we can define a 0-cochain $\{\sigma_{jk}\} \in C^0(\mathfrak{U}, \mathscr{S})$ by putting

$$\sigma_{jk} := \sigma_{j\mu k\nu}$$

on $\mathcal{U}_j \cap \mathcal{U}_k \cap \mathcal{V}_\lambda \cap \mathcal{V}_\nu \neq \emptyset$. By putting $\mu = \nu = \lambda$ in (4.2.4.32), we see that

$$\sigma_{ijk} = \sigma_{jk} + \sigma_{ki} + \sigma_{ij}$$

on each $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k \cap \mathcal{V}_\lambda \neq \emptyset$. Hence $\{\sigma_{ijk}\} = \delta\{\sigma_{ij}\} \in \delta C^0(\mathfrak{U}, \mathscr{S})$, which means $g = 0$. □

4.2.5 Exact sequences

Let $\mathscr{S}$ be a sheaf over a differentiable manifold $\mathcal{M}$, and $\varpi : \mathscr{S} \to \mathcal{M}$ its projection.

Definition 4.4

A subset $\mathscr{S}' \subset \mathscr{S}$ is called a **subsheaf** *of $\mathscr{S}$ if the following conditions are satisfied:*

(i) $\mathscr{S}'$ *is an open subset of* $\mathscr{S}$.

(ii) $\varpi(\mathscr{S}') = \mathcal{M}$.

(iii) For any $p \in \mathcal{M}$, $\mathscr{S}'_p = \varpi^{-1}(p) \cap \mathscr{S}'$ is a $\mathbf{K}$-submodule of $\mathscr{S}_p$.

Clearly $\mathscr{S}'$ itself a sheaf over $\mathcal{M}$. ♣

Let $\mathscr{T}$ be a sheaf over $\mathcal{M}$ with the projection ϱ.

Definition 4.5

A **homomorphism** *h of $\mathscr{S}$ into $\mathscr{T}$ is defined to be a continuous map of $\mathscr{S}$ into $\mathscr{T}$ satisfying the following conditions:*

(i) $\varrho \circ h = \varpi$.

(ii) For any $p \in \mathcal{M}$, $h : \mathscr{S}_p \to \mathscr{T}_p$ is a $\mathbf{K}$-homomorphism.

$$\begin{array}{ccc} \mathscr{S} & \xrightarrow{h} & \mathscr{T} \\ {\scriptstyle\varpi}\downarrow & & \downarrow{\scriptstyle\varrho} \\ \mathcal{M} & = & \mathcal{M} \end{array}$$

Clearly that $h(\mathscr{S}_p) \subset \mathscr{T}_p$. Since ϖ and ϱ are local homeomorphisms, h is also a local homeomorphism by (i). ♣

Let $h : \mathscr{S} \to \mathscr{T}$ be a homomorphism. By $h(\varphi) = 0$, $\varphi \in \mathscr{S}_p$, we mean that $h(\varphi) = 0_{\mathscr{T}_p}$, where $0_{\mathscr{T}_p}$ is the zero of $\mathscr{T}_p$. Then

(i) $\mathscr{S}' = \{\varphi \in \mathscr{S} : h(\varphi) = 0\}$ is a subsheaf of $\mathscr{S}$.

Proof. $\mathscr{S}'$ obviously satisfies (ii) and (iii) in the definition of sheaves. Since h is continuous and $0_{\mathscr{T}} = \{0_{\mathscr{T}_p} : p \in \mathcal{M}\}$ is open in $\mathscr{T}$, $\mathscr{S}' = h^{-1}(0_{\mathscr{T}})$ is open in $\mathscr{S}$. □

We call this $\mathscr{S}'$ the **kernel** of h and denote by $\mathrm{Ker}(h)$.

(ii) $h(\mathscr{S})$ is a subsheaf of $\mathscr{T}$. In fact $h(\mathscr{S})$ is open in $\mathscr{T}$ since h is a local homeomorphism.

(iii) h is **injective** if and only if $\mathrm{Ker}(h) = 0_{\mathscr{T}}$.

Definition 4.6

$h : \mathscr{S} \to \mathscr{T}$ is called an **isomorphism** *of $\mathscr{S}$ onto $\mathscr{T}$ if h is a homomorphism of $\mathscr{S}$ onto $\mathscr{T}$ and if, for any $p \in \mathcal{M}$, $h : \mathscr{S}_p \to \mathscr{T}_p$ is a $\mathbf{K}$-isomorphism of $\mathscr{S}_p$ onto $\mathscr{T}_p$. If there exists an isomorphism of $\mathscr{S}$ onto $\mathscr{T}$, $\mathscr{S}$ and $\mathscr{T}$ are said to be* **isomorphic** *and is denoted by $\mathscr{S} \cong \mathscr{T}$.*

We often identify $\mathscr{S}$ with $\mathscr{T}$ if $\mathscr{S} \cong \mathscr{T}$. ♣

Let $\mathscr{S}' \subset \mathscr{S}$ be a subsheaf of $\mathscr{S}$. Let

$$\mathscr{Q}_p := \frac{\mathscr{S}_p}{\mathscr{S}'_p}$$

be the quotient group, and let

$$\mathscr{Q} := \bigcup_{p \in \mathcal{M}} \mathscr{Q}_p. \tag{4.2.5.1}$$

Define $\varrho : \mathscr{Q} \to \mathcal{M}$ by $\varrho(\mathscr{Q}_p) = p$. Let h_p be the natural map of $\mathscr{S}_p$ onto $\mathscr{Q}_p = \mathscr{S}_p/\mathscr{S}'_p$, and define

$$h : \mathscr{S} \longrightarrow \mathscr{Q}, \quad h(\varphi) := h_p(\varphi)$$

for $\varphi \in \mathscr{S}_p$. We give $\mathscr{Q}$ a topology by saying that $\mathscr{W}$ is open in $\mathscr{Q}$ if and only if $h^{-1}(\mathscr{W})$ is open in $\mathscr{S}$. Then $\mathscr{Q}$ is a sheaf[3] over $\mathcal{M}$.

Definition 4.7

$\mathscr{Q}$ is called the **quotient sheaf** *of $\mathscr{S}$ by $\mathscr{S}'$ and is denoted by $\mathscr{S}/\mathscr{S}'$. $h : \mathscr{S} \to \mathscr{Q}$ is a surjective homomorphism of $\mathscr{S}$ onto $\mathscr{Q}$.* ♣

For a homomorphism $h : \mathscr{S} \to \mathscr{T}$ of sheaves over $\mathcal{M}$, we put $\mathscr{S}' := \mathrm{Ker}(h)$. Then h maps $\mathscr{S}/\mathscr{S}'$ isomorphically onto $h(\mathscr{S})$, hence $h(\mathscr{S}) \cong \mathscr{S}/\mathscr{S}'$.

A homomorphism $h : \mathscr{S} \to \mathscr{T}$ of sheaves over $\mathcal{M}$ defines homomorphisms of cohomology groups

$$h : H^q(\mathcal{M}, \mathscr{S}) \longrightarrow H^q(\mathcal{M}, \mathscr{T}) \tag{4.2.5.2}$$

as follows.

(i) For a section $\sigma : p \mapsto \sigma(p)$ of $\mathscr{S}$ over a subset $\mathcal{W} \subset \mathcal{M}$,

$$h \circ \sigma : p \longmapsto h(\sigma(p))$$

is a section of $\mathscr{T}$ over $\mathcal{W}$, and $h : \sigma \mapsto h \circ \sigma$ is a $\mathbf{K}$-homomorphism of $\Gamma(\mathcal{W}, \mathscr{S})$ into $\Gamma(\mathcal{W}, \mathscr{T})$:

$$h : \Gamma(\mathcal{W}, \mathscr{S}) \longrightarrow \Gamma(\mathcal{W}, \mathscr{T}), \quad \sigma \longmapsto h \circ \sigma.$$

(ii) Let $\mathfrak{U} = \{\mathcal{U}_j\}$ be a locally finite open covering of $\mathcal{M}$. For any q-cochain $c^q_{\mathfrak{U}} = \{\sigma_{k_0 \cdots k_q}\} \in C^q(\mathfrak{U}, \mathscr{S})$, by defining $h \circ c^q_{\mathfrak{U}} = \{h \circ \sigma_{k_0 \cdots k_q}\}$, we obtain a homomorphism

$$h : C^q(\mathfrak{U}, \mathscr{S}) \longrightarrow C^q(\mathfrak{U}, \mathscr{T}), \quad c^q_{\mathfrak{U}} \longmapsto h \circ c^q_{\mathfrak{U}}.$$

Clearly that $h \circ \delta = \delta \circ h$. h maps $Z^q(\mathfrak{U}, \mathscr{S})$ into $Z^q(\mathfrak{U}, \mathscr{T})$ and $\delta C^{q-1}(\mathfrak{U}, \mathscr{S})$ into

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[3]First we prove that if for $\alpha, \beta \in \mathscr{S}_p$ we take sufficiently small neighborhoods $\mathscr{U}$ and $\mathscr{V}$ with $\alpha \in \mathscr{U} \subset \mathscr{S}$, $\beta \in \mathscr{V} \subset \mathscr{S}$, such that $\varpi(\mathscr{U}) = \varpi(\mathscr{V})$, then

$$\mathscr{U} + \mathscr{V} = \{\varphi + \psi : \varphi \in \mathscr{U}, \psi \in \mathscr{V}, \varpi(\varphi) = \varpi(\psi)\}$$

is a neighborhood of $\alpha + \beta$. For, if w put $\mathcal{U} = \varpi(\mathscr{U}) = \varpi(\mathscr{V})$, for sufficiently small $\mathscr{U}, \mathscr{V}$, $\varpi : \mathscr{U} \to \mathcal{U}$ and $\varpi : \mathscr{V} \to \mathcal{U}$ are both homeomorphisms. Their inverses $\sigma : p \mapsto \varphi = \sigma(p)$ and $\tau : p \mapsto \psi = \tau(p)$ are sections of $\mathscr{S}$ over $\mathcal{U}$. Hence $\sigma + \tau : p \mapsto \varphi + \psi = \sigma(p) + \tau(p)$ is also a section of $\mathscr{S}$ over $\mathcal{U}$. Hence $\mathscr{U} + \mathscr{V} = \{\sigma(p) + \tau(p) : p \in \mathcal{U}\}$ is open.

Since $\mathscr{S}'$ is open in $\mathscr{S}$, it follows that for any open $\mathscr{W} \subset \mathscr{S}$,

$$h^{-1}(h(\mathscr{W})) = \{\varphi + \psi : \varphi \in \mathscr{W}, \psi \in \mathscr{W}, \varpi(\varphi) = \varpi(\psi)\}$$

is open. Thus $h(\mathscr{W})$ is open in $\mathscr{Q}$, that is, h is an open map.

For any point $h(\alpha)$, $\alpha \in \mathscr{S}$, of $\mathscr{Q}$, take a neighborhood $\mathscr{U}$ of α such that $\varpi : \mathscr{U} \to \mathcal{U} = \varpi(\mathscr{U})$ is a homeomorphism. Then by the above argument $h(\mathscr{U})$ is an open subset of $\mathscr{Q}$. Moreover for any open subset $\mathscr{V} \subset \mathscr{U}$, $h(\mathscr{V})$ is an open subset of $h(\mathscr{U})$. Conversely, if $\overline{\mathscr{V}}$ is an open subset of $h(\mathscr{U})$, $\mathscr{V} = h^{-1}(\overline{\mathscr{V}}) \cap \mathscr{U}$ is an open subset of $\mathscr{U}$, and $h(\mathscr{V}) = \overline{\mathscr{V}}$. Hence $h : \mathscr{U} \to h(\mathscr{U})$ is a homeomorphism. Therefore $h : \mathscr{S} \to \mathscr{Q}$ is a local homeomorphism. Since $\varpi : \mathscr{U} \to \mathcal{U}$ is a homeomorphism and $\varrho \circ h = \varpi$, $\varrho : h(\mathscr{U}) \to \mathscr{U}$ is also a homeomorphism. Hence $\varrho : \mathscr{Q} \to \mathcal{M}$ is a local homeomorphism.

The continuity of $c_1\overline{\varphi} + c_2\overline{\psi}$ with $\varrho(\overline{\varphi}) = \varrho(\overline{\psi})$ with respect to $\overline{\varphi}, \overline{\psi} \in \mathscr{Q}$, follows from the continuity of $c_1\varphi + c_2\psi$ for $\varphi, \psi \in \mathscr{S}$. Hence $\mathscr{Q}$ is a sheaf over $\mathcal{M}$ with the projection ϱ.

$\delta C^{q-1}(\mathfrak{U}, \mathscr{T})$, hence h induces a homomorphism

$$h : H^q(\mathfrak{U}, \mathscr{S}) \longrightarrow H^q(\mathfrak{U}, \mathscr{T}).$$

(iii) Let $\mathfrak{V} \in \mathbf{LFOC}(\mathcal{M})$ with $\mathfrak{U} \prec \mathfrak{V}$. The following diagram

$$\begin{array}{ccc} C^q(\mathfrak{U}, \mathscr{S}) & \xrightarrow{h} & C^q(\mathfrak{U}, \mathscr{T}) \\ \Pi^{\mathfrak{U}}_{\mathfrak{V}} \big\downarrow & & \big\downarrow \Pi^{\mathfrak{U}}_{\mathfrak{V}} \\ C^q(\mathfrak{V}, \mathscr{S}) & \xrightarrow[h]{} & C^q(\mathfrak{V}, \mathscr{T}) \end{array}$$

is commutative, i.e., $h \circ \Pi^{\mathfrak{U}}_{\mathfrak{V}} = \Pi^{\mathfrak{U}}_{\mathfrak{V}} \circ h$. Hence the following diagram

$$\begin{array}{ccc} H^q(\mathfrak{U}, \mathscr{S}) & \xrightarrow{h} & H^q(\mathfrak{U}, \mathscr{T}) \\ \Pi^{\mathfrak{U}}_{\mathfrak{V}} \big\downarrow & & \big\downarrow \Pi^{\mathfrak{U}}_{\mathfrak{V}} \\ H^q(\mathfrak{V}, \mathscr{S}) & \xrightarrow[h]{} & H^q(\mathfrak{V}, \mathscr{T}) \end{array}$$

is also commutative. Therefore if $\Pi^{\mathfrak{U}}_{\mathfrak{V}} g = 0$ for $g \in H^q(\mathfrak{U}, \mathscr{S})$, then $\Pi^{\mathfrak{U}}_{\mathfrak{V}}(hg) = 0$. Thus h maps $N^q(\mathfrak{U}, \mathscr{S})$ into $N^q(\mathfrak{U}, \mathscr{T})$. Consequently h induces a homomorphism

$$h_{\mathfrak{U}} : \overline{H}^q(\mathfrak{U}, \mathscr{S}) \longrightarrow \overline{H}^q(\mathfrak{U}, \mathscr{T}).$$

(iv) If $\mathfrak{U} \prec \mathfrak{V}$, we have $\overline{H}^q(\mathfrak{U}, \mathscr{S}) \subset \overline{H}^q(\mathfrak{V}, \mathscr{S})$ and $\overline{H}^q(\mathfrak{U}, \mathscr{T}) \subset \overline{H}^q(\mathfrak{V}, \mathscr{T})$. It follows that $h_{\mathfrak{V}}$ coincides with $h_{\mathfrak{U}}$ on $\overline{H}^q(\mathfrak{U}, \mathscr{S})$. Since

$$H^q(\mathcal{M}, \mathscr{S}) = \bigcup_{\mathfrak{U} \in \mathbf{LFOC}(\mathcal{M})} \overline{H}^q(\mathfrak{U}, \mathscr{S}), \quad H^q(\mathcal{M}, \mathscr{T}) = \bigcup_{\mathfrak{U} \in \mathbf{LFOC}(\mathcal{M})} \overline{H}^q(\mathfrak{U}, \mathscr{T}),$$

by putting $h = h_{\mathfrak{U}}$ on each $\overline{H}^q(\mathfrak{U}, \mathscr{S})$, we obtain a homomorphism (4.2.5.2).

We write

$$\mathscr{S} \xrightarrow{h} \mathscr{T}$$

for a homomorphism $h : \mathscr{S} \to \mathscr{T}$.

Definition 4.8

A sequence of sheaves and sheaf homomorphisms over $\mathcal{M}$

$$\mathscr{S}_0 \xrightarrow{h_0} \cdots \longrightarrow \mathscr{S}_{m-1} \xrightarrow{h_{m-1}} \mathscr{S}_m \xrightarrow{h_m} \cdots \longrightarrow \mathscr{S}_\ell \tag{4.2.5.3}$$

is said to be **exact** *if*

$$h_{m-1}(\mathscr{S}_{m-1}) = \operatorname{Ker}(h_m), \quad m = 1, 2, \cdots, \ell - 1. \tag{4.2.5.4}$$

Similarly a sequence of $\mathbf{K}$*-modules and* $\mathbf{K}$*-homomorphisms*

$$H_0 \xrightarrow{h_0} \cdots \longrightarrow H_{m-1} \xrightarrow{h_{m-1}} H_m \xrightarrow{h_m} \cdots$$

is said to be **exact** *if*

$$h_{m-1}(H_{m-1}) = \operatorname{Ker}(h_m), \quad m = 1, 2, \cdots.$$

♣

Suppose given an exact sequence of sheaves over $\mathcal{M}$

$$0 \longrightarrow \mathscr{S}' \xrightarrow{\iota} \mathscr{S} \xrightarrow{h} \mathscr{S}'' \longrightarrow 0. \tag{4.2.5.5}$$

The exactness of (4.2.5.5) implies that $\operatorname{Ker}(\iota) = 0$, $\operatorname{Ker}(h) = \iota(\mathscr{S}')$, and $\mathscr{S}'' = h(\mathscr{S})$. Hence

$\mathscr{S}' = \iota(\mathscr{S}') \subset \mathscr{S}$ is a subsheaf and ι is the inclusion map. Hence also $\mathscr{S}'' \cong \mathscr{S}/\mathscr{S}'$. The homomorphism $\iota : \mathscr{S}' \to \mathscr{S}$ induces a homomorphism $\iota : H^q(\mathcal{M}, \mathscr{S}') \to H^q(\mathcal{M}, \mathscr{S})$ and $h : \mathscr{S} \to \mathscr{S}''$ induces $h : H^q(\mathcal{M}, \mathscr{S}) \to H^q(\mathcal{M}, \mathscr{S}'')$. Thus

$$H^q(\mathcal{M}, \mathscr{S}') \xrightarrow{\ \iota\ } H^q(\mathcal{M}, \mathscr{S}) \xrightarrow{\ h\ } H^q(\mathcal{M}, \mathscr{S}''). \tag{4.2.5.6}$$

Theorem 4.8

The exact sequence (4.2.5.5) induces the exact sequence of cohomology groups

$$\begin{aligned} 0 &\longrightarrow H^0(\mathcal{M}, \mathscr{S}') \xrightarrow{\ \iota\ } H^0(\mathcal{M}, \mathscr{S}) \xrightarrow{\ h\ } H^0(\mathcal{M}, \mathscr{S}'') \\ &\xrightarrow{\ \delta^*\ } H^1(\mathcal{M}, \mathscr{S}') \xrightarrow{\ \iota\ } \cdots \xrightarrow{\ \delta^*\ } H^q(\mathcal{M}, \mathscr{S}') \\ &\xrightarrow{\ \iota\ } H^q(\mathcal{M}, \mathscr{S}) \xrightarrow{\ h\ } H^q(\mathcal{M}, \mathscr{S}'') \xrightarrow{\ \delta^*\ } H^{q+1}(\mathcal{M}, \mathscr{S}') \xrightarrow{\ \iota\ } \cdots . \end{aligned} \tag{4.2.5.7}$$

♡

We explain first how δ^* is defined.

(i) $q = 0$: we are going to define δ^* so that the sequence

$$H^0(\mathcal{M}, \mathscr{S}) \xrightarrow{\ h\ } H^0(\mathcal{M}, \mathscr{S}'') \xrightarrow{\ \delta^*\ } H^1(\mathcal{M}, \mathscr{S}') \tag{4.2.5.8}$$

is exact. The exactness of (4.2.5.8) means that for a given $\sigma'' \in H^0(\mathcal{M}, \mathscr{S}'') = \Gamma(\mathcal{M}, \mathscr{S}'')$, there is a $\sigma \in H^0(\mathcal{M}, \mathscr{S}) = \Gamma(\mathcal{M}, \mathscr{S})$ such that $h\sigma = \sigma''$ if and only if $\delta^*\sigma'' = 0$.

Given a $\sigma'' \in H^0(\mathcal{M}, \mathscr{S}'') = \Gamma(\mathcal{M}, \mathscr{S}'')$, since $\varpi : \mathscr{S} \to \mathcal{M}$, $\varpi'' : \mathscr{S}'' \to \mathcal{M}$, and h are local homeomorphisms, we can take a sufficiently fine locally finite open covering $\mathfrak{U} = \{\mathcal{U}_j\}$ and a section $\sigma_j \in \Gamma(\mathcal{U}_j, \mathscr{S})$ for each j such that $h\sigma_j = \mathrm{res}_{\mathcal{U}_j}\sigma''$. Hence, if we put

$$c^0_{\mathfrak{U}} := \{\sigma_j\} \in C^0(\mathfrak{U}, \mathscr{S}),$$

then $hc^0_{\mathfrak{U}} = \{h\sigma_j\} = \{\mathrm{res}_{\mathcal{U}_j}\sigma''\} = \sigma''$. Now we show that $\delta c^0_{\mathfrak{U}} \in Z^1(\mathfrak{U}, \mathscr{S}')$. Put $\delta c^0_{\mathfrak{U}} = \{\tau'_{jk}\}$ with $\tau'_{jk} = \sigma_k - \sigma_j$. Since $h\tau'_{jk} = h\sigma_k - h\sigma_j = 0$ on $\mathcal{U}_j \cap \mathcal{U}_k \neq \emptyset$ and $\mathrm{Ker}(h) = \mathscr{S}'$, $\tau'_{jk} \in \Gamma(\mathcal{U}_j \cap \mathcal{U}_k, \mathscr{S}')$. We define $\delta^*\sigma'' \in H^1(\mathcal{M}, \mathscr{S}')$ as the cohomology class of the 1-cocycle $\delta c^0_{\mathfrak{U}}$:

$$\delta^*\sigma'' := [\delta c^0_{\mathfrak{U}}] \in H^1(\mathfrak{U}, \mathscr{S}'). \tag{4.2.5.9}$$

By **Theorem 4.5**, $H^1(\mathfrak{U}, \mathscr{S}') \subset H^1(\mathcal{M}, \mathscr{S}')$, we may consider

$$\delta^*\sigma'' = [\delta c^0_{\mathfrak{U}}] \in H^1(\mathcal{M}, \mathscr{S}').$$

$\delta^*\sigma''$ is determined uniquely by σ''. For, if $\sigma'' = 0$, then $h\sigma_j = 0$, hence $c^0_{\mathfrak{U}} = \{\sigma_j\} \in C^0(\mathfrak{U}, \mathscr{S}')$. $\delta^*\sigma''$ is also independent of the choice of an open covering.

If there is a $\sigma \in \Gamma(\mathcal{M}, \mathscr{S})$ with $h\sigma = \sigma''$, we have $\delta^*\sigma'' = 0$. In fact, if we put $c^0_{\mathfrak{U}} = \{\mathrm{res}_{\mathcal{U}_j}\sigma\}$, then $hc^0_{\mathfrak{U}} = \sigma''$ and hence $\delta c^0_{\mathfrak{U}} = 0$. Conversely, assume $\delta^*\sigma'' = 0$. Then $\delta c^0_{\mathfrak{U}} = \{\tau'_{jk}\} \in \delta C^0(\mathfrak{U}, \mathscr{S}')$, that is, $\sigma_k - \sigma_j = \tau'_{jk} = \sigma'_k - \sigma'_j$ with $\sigma'_j \in \Gamma(\mathfrak{U}, \mathscr{S}')$. For $\mathcal{U}_j \cap \mathcal{U}_k \neq \emptyset$, we have $\sigma_j - \sigma'_j = \sigma_k - \sigma'_k$. Putting $\sigma = \sigma_j - \sigma'_j$ on each $\mathcal{U}_j$ yields a section $\sigma \in \Gamma(\mathcal{M}, \mathscr{S})$. Since $h\sigma'_j = 0$ and $h\sigma_j = \mathrm{res}_{\mathcal{U}_j}\sigma''$, we obtain $h\sigma = \sigma''$. Hence (4.2.5.8) is exact.

(ii) $q \geq 1$: we must define δ^* so that

$$H^q(\mathcal{M}, \mathscr{S}) \xrightarrow{h} H^q(\mathcal{M}, \mathscr{S}'') \xrightarrow{\delta^*} H^{q+1}(\mathcal{M}, \mathscr{S}') \tag{4.2.5.10}$$

is exact.

- Let $\mathcal{X}$ be a complex manifold. Suppose given for every p a local coordinate system $\boldsymbol{z}_p = (z_p^1, \cdots, z_p^n)$ centered at p, and a coordinate polydisk

$$\mathcal{U}_{\epsilon(p)}(p) = \left\{q : |z_p^1(q)| < \epsilon(p), \cdots, |z_p^n(q)| < \epsilon(p)\right\}, \quad \epsilon(p) > 0.$$

Then we may choose, at most countably many coordinate polydisks $\mathcal{U}_\lambda$ with

$$\mathcal{U}_\lambda = \mathcal{U}_{\epsilon_\lambda}(p_\lambda) \subset \mathcal{U}_{\epsilon(p_\lambda)}(p_\lambda), \quad 0 < \epsilon_\lambda \leq \epsilon(p_\lambda), \quad \lambda = 1, 2, \cdots,$$

such that $\mathfrak{U} = \{\mathcal{U}_\lambda\}$ is a locally finite open covering of $\mathcal{X}$.

- Let $\mathcal{M}$ be a differentiable manifold. Using local C^∞ coordinates $\boldsymbol{x}_p = (x_p^1, \cdots, x_p^m)$ centered at p and an open set

$$\mathcal{U}_\epsilon(p) = \left\{q : |x_p^1(q)| < \epsilon, \cdots, |x_p^m(q)| < \epsilon\right\}, \quad \epsilon > 0,$$

we obtain the same result. Here we assume that the closure $\overline{\mathcal{U}_\epsilon(p)}$ is a compact subset of the domain of local C^∞ coordinates $(x_p^1, \cdots, x_p^m)$. We call $\mathcal{U}_\epsilon(p)$ a **coordinate multi-interval of radius ϵ**. For a complex manifold $\mathcal{X}$, we may consider coordinate multi-intervals, regarding $\mathcal{X}$ as a differentiable manifold as usual.

- Let $\mathfrak{U} = \{\mathcal{U}_j\}$ be a locally finite open covering of $\mathcal{M}$, and suppose $\{\mathcal{U}_j\}$ is compact. Then we may choose an open subset $\mathcal{W}_j$ of each $\mathcal{U}_j$ such that $\overline{\mathcal{W}_j} \subset \mathcal{U}_j$ and that $\mathfrak{W} = \{\mathcal{W}_j\}$ is already a covering of $\mathcal{M}$.

Proof. Since for $p \in \mathcal{M}$, there are only a finite number of $\mathcal{U}_j$ with $p \in \mathcal{U}_j$, taking $\epsilon(p) > 0$ sufficiently small, we may assume that $\overline{\mathcal{U}_{\epsilon(p)}(p)} \subset \mathcal{U}_j$ if $p \in \mathcal{U}_j$. Choose such $\mathcal{U}_{\epsilon(p)}(p)$ for each $p \in \mathcal{M}$, and take

$$\mathcal{V}_\lambda = \mathcal{U}_{\epsilon_\lambda}(p_\lambda) \subset \mathcal{U}_{\epsilon(p_\lambda)}(p_\lambda), \quad 0 < \epsilon_\lambda \leq \epsilon(p_\lambda), \quad \lambda = 1, 2, \cdots,$$

so that $\mathfrak{V} = \{\mathcal{V}_\lambda\}$ is a locally finite open covering of $\mathcal{M}$. Then p_λ belongs to one of $\mathcal{U}_j$ and if $p_\lambda \in \mathcal{U}_j$ then $\overline{\mathcal{V}_\lambda} \subset \overline{\mathcal{U}_{\epsilon(p_\lambda)}(p_\lambda)} \subset \mathcal{U}_j$. Thus for each λ, $\overline{\mathcal{V}_\lambda}$ is contained in some $\mathcal{U}_j$. Then if we put

$$\mathcal{W}_j := \bigcup_{\overline{\mathcal{V}_\lambda} \subset \mathcal{U}_j} \mathcal{V}_\lambda,$$

we have $\cup_j \mathcal{W}_j = \mathcal{M}$, hence $\mathfrak{W} = \{\mathcal{W}_j\}$ forms an open covering of $\mathcal{M}$. Since $\overline{\mathcal{U}_j}$ is compact, and $\{\mathcal{V}_\lambda\}$ is locally finite, there are only a finite number of $\mathcal{V}_\lambda$ with $\overline{\mathcal{V}_\lambda} \subset \mathcal{U}_j$. Hence $\overline{\mathcal{W}_j} = \cup_{\overline{\mathcal{V}_\lambda} \subset \mathcal{U}_j} \overline{\mathcal{V}_\lambda} \subset \mathcal{U}_j$. □

In what follows, we denote by $\mathfrak{U}, \mathfrak{V}, \mathfrak{W}, \cdots$ locally finite open covering of $\mathcal{M}$.

Lemma 4.3

Given a q-cochain $c''^q_{\mathfrak{U}} \in C^q(\mathfrak{U}, \mathscr{S}'')$, we can find a locally finite refinement $\mathfrak{V}$ of $\mathfrak{U}$ and $c^q_{\mathfrak{V}} \in C^q(\mathfrak{V}, \mathscr{S})$ such that $\Pi^{\mathfrak{U}}_{\mathfrak{V}} c''^q_{\mathfrak{U}} = hc^q_{\mathfrak{V}}$. ♡

Proof. Put $\mathfrak{U} = \{\mathcal{U}_j\}$. We give here only the proof for the case $q = 2$, but the generalization is straightforward. Since for $\mathfrak{V} \succ \mathfrak{W} \succ \mathfrak{U}$, $\Pi^{\mathfrak{U}}_{\mathfrak{V}} = \Pi^{\mathfrak{W}}_{\mathfrak{V}} \circ \Pi^{\mathfrak{U}}_{\mathfrak{W}}$, replacing $\mathfrak{U}$ by its appropriate refinement id necessary, we may assume that the closure $\overline{\mathcal{U}_j}$ of each $\mathcal{U}_j$ is compact. Then there is an open covering $\mathfrak{W} = \{\mathcal{W}_j\}$ with $\overline{\mathcal{W}_j} \subset \mathcal{U}_j$.

Let $c''^2_{\mathfrak{U}} = \{\sigma''_{ijk}\} \in C^2(\mathfrak{U}, \mathscr{S}'')$, $\sigma''_{ijk} \in \Gamma(\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k, \mathscr{S}'')$. Take a point $p \in \mathcal{M}$, and denote by $\mathcal{U}_\epsilon(p)$ the coordinate multi-interval of radius $\epsilon > 0$ with center p. Suppose $p \in \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$, then since ϖ, ϖ'' and h are local homeomorphisms, for a sufficiently small $\mathcal{U}_\epsilon(p) \subset \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$, there is an element $\tau \in \Gamma(\mathcal{U}_\epsilon(p), \mathscr{S})$ such that $h\tau = \mathrm{res}_{\mathcal{U}_\epsilon(p)} \sigma''_{ijk}$. Since $\mathfrak{U} = \{\mathcal{U}_j\}$ is locally finite, there are only a finite number of $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$ with $p \in \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$ for a fixed p. Hence we can find for each p a sufficiently small $\mathcal{U}_\epsilon(p)$ satisfying the following conditions:

(i) If $p \in \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$, $\mathcal{U}_\epsilon(p) \subset \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$, and there is a $\tau \in \Gamma(\mathcal{U}_\epsilon(p), \mathscr{S})$ with $\mathrm{res}_{\mathcal{U}_\epsilon(p)} \sigma''_{ijk} = h\tau$.

(ii) If $p \in \mathcal{W}_j$, then $\mathcal{U}_\epsilon(p) \subset \mathcal{W}_j$.

(iii) If $\overline{\mathcal{W}_j} \cap \mathcal{U}_\epsilon(p) \neq \emptyset$, then $\mathcal{U}_\epsilon(p) \subset \mathcal{U}_j$.

Choose coordinate multi-intervals

$$\mathcal{V}_\lambda = \mathcal{U}_{\epsilon_\lambda}(p_\lambda) \subset \mathcal{U}_{\epsilon(p_\lambda)}(p_\lambda), \quad 0 < \epsilon_\lambda \leq \epsilon(p_\lambda), \quad \lambda = 1, 2, \cdots,$$

such that $\mathfrak{V} = \{\mathcal{V}_\lambda\}$ is a locally finite open covering of $\mathcal{M}$. By (ii), each $\mathcal{V}_\lambda$ is contained in some $\mathcal{W}_j$. For each λ, choose an arbitrary $\mathcal{W}_{j(\lambda)}$ with $\mathcal{V}_\lambda \subset \mathcal{U}_{\epsilon(p_\lambda)}(p_\lambda) \subset \mathcal{W}_{j(\lambda)}$. Then $\mathcal{V}_\lambda \subset \mathcal{W}_{j(\lambda)} \subset \mathcal{U}_{j(\lambda)}$, hence $\mathfrak{V} \prec \mathfrak{U}$. Define

$$\Pi^{\mathfrak{U}}_{\mathfrak{V}} : C^q(\mathfrak{U}, \mathscr{S}'') \longrightarrow C^q(\mathfrak{V}, \mathscr{S}'')$$

via the inclusion $\mathcal{V}_\lambda \subset \mathcal{U}_{j(\lambda)}$. For $c''^2_{\mathfrak{U}} = \{\sigma''_{ijk}\}$, $\Pi^{\mathfrak{U}}_{\mathfrak{V}} c''^2_{\mathfrak{U}} = \{\tau''_{\lambda\mu\nu}\}$ is given by

$$\tau''_{\lambda\mu\nu} = \mathrm{res}_{\mathcal{V}_{\lambda,\mu,\nu}} \sigma''_{j(\lambda)j(\mu)j(\nu)}, \quad \mathcal{V}_{\lambda,\mu,\nu} = \mathcal{V}_\lambda \cap \mathcal{V}_\mu \cap \mathcal{V}_\nu \neq \emptyset.$$

We now prove the existence of $\tau_{\lambda\mu\nu} \in \Gamma(\mathcal{V}_{\lambda,\mu,\nu}, \mathscr{S})$ such that $h\tau_{\lambda\mu\nu} = \tau''_{\lambda\mu\nu}$ for each $\tau''_{\lambda\mu\nu}$. Since $\mathcal{V}_\lambda \subset \mathcal{W}_{j(\lambda)}, \mathcal{V}_\mu \subset \mathcal{W}_{j(\mu)}, \mathcal{V}_\nu \subset \mathcal{W}_{j(\nu)}$ and $\mathcal{V}_\lambda \cap \mathcal{V}_\mu \cap \mathcal{V}_\nu \neq \emptyset$, $\mathcal{V}_\lambda \cap \mathcal{W}_{j(\mu)} \neq \emptyset$. Therefore

$$\mathcal{W}_{j(\mu)} \cap \mathcal{U}_{\epsilon(p_\lambda)}(p_\lambda) \supset \mathcal{W}_{j(\mu)} \cap \mathcal{V}_\lambda \neq \emptyset,$$

hence by (iii), $\mathcal{U}_{\epsilon(p_\lambda)}(p_\lambda) \subset \mathcal{U}_{j(\mu)}$. Similarly $\mathcal{U}_{\epsilon(p_\lambda)}(p_\lambda) \subset \mathcal{U}_{j(\nu)}$. Hence we have

$$\mathcal{V}_\lambda \subset \mathcal{U}_{\epsilon(p_\lambda)}(p_\lambda) \subset \mathcal{U}_{j(\lambda)} \cap \mathcal{U}_{j(\mu)} \cap \mathcal{U}_{j(\nu)}.$$

Consequently by (i), there is a $\tau \in \Gamma(\mathcal{V}_\lambda, \mathscr{S})$ with $h\tau = \mathrm{res}_{\mathcal{V}_\lambda} \sigma''_{ijk}$. Put

$$\tau_{\lambda\mu\nu} := \mathrm{res}_{\mathcal{V}_{\lambda,\mu,\nu}} \tau.$$

Then we have $h\tau_{\lambda\mu\nu} = \mathrm{res}_{\mathcal{V}_{\lambda,\mu,nu}} \sigma''_{ijk} = \tau''_{\lambda\mu\nu}$ as desired. □

(iii) (continuous): For a sheaf $\mathscr{S}$ we denote the cohomology class with respect to $\mathfrak{U}$ of a q-cocycle $c^q_{\mathfrak{U}} \in Z^q(\mathfrak{U}, \mathscr{S})$ by

$$[c^q_{\mathfrak{U}}]_{\mathfrak{U}} \in H^q(\mathfrak{U}, \mathscr{S}) = \frac{Z^q(\mathfrak{U}, \mathscr{S})}{\delta C^{q-1}(\mathfrak{U}, \mathscr{S})}.$$

Then the cohomology class $[c^q_{\mathfrak{U}}]$ of $c^q_{\mathfrak{U}}$ is given by

$$[c^q_{\mathfrak{U}}] = \Pi^{\mathfrak{U}} [c^q_{\mathfrak{U}}]_{\mathfrak{U}} \in H^q(\mathcal{M}, \mathscr{S}),$$

where $\Pi^{\mathfrak{U}} : H^q(\mathfrak{U}, \mathscr{S}) \to \overline{H}^q(\mathfrak{U}, \mathscr{S}) \subset H^q(\mathcal{M}, \mathscr{S})$. The cohomology class of c^q does not depend on the choice of refinement of a locally finite open covering. For $\mathfrak{U} \prec \mathfrak{V}$, $\Pi^{\mathfrak{U}}_{\mathfrak{V}} : H^q(\mathfrak{U}, \mathscr{S}) \to H^q(\mathfrak{V}, \mathscr{S})$ induces an inclusion $\overline{H}^q(\mathfrak{U}, \mathscr{S}) \to \overline{H}^q(\mathfrak{V}, \mathscr{S})$. Hence

$$\Pi^{\mathfrak{U}} [c^q_{\mathfrak{U}}]_{\mathfrak{U}} = \Pi^{\mathfrak{V}} \Pi^{\mathfrak{U}}_{\mathfrak{V}} [c^q_{\mathfrak{U}}]_{\mathfrak{U}} = \Pi^{\mathfrak{V}} \left[\Pi^{\mathfrak{U}}_{\mathfrak{V}} c^q_{\mathfrak{U}}\right]_{\mathfrak{V}}.$$

We then see that

$$[c^q_{\mathfrak{U}}] = \left[\Pi^{\mathfrak{U}}_{\mathfrak{B}} c^q_{\mathfrak{U}}\right]. \tag{4.2.5.11}$$

Lemma 4.4

For any $g'' \in H^q(\mathcal{M}, \mathscr{S}'')$, by a suitable choice of $\mathfrak{U}$, we can find a q-cocycle $c''^q_{\mathfrak{U}}$ and a q-cochain $c^q_{\mathfrak{U}}$ such that

$$g'' = [c''^q_{\mathfrak{U}}], \quad hc^q_{\mathfrak{U}} = c''^q_{\mathfrak{U}} \in Z^q(\mathfrak{U}, \mathscr{S}''), \quad c^q_{\mathfrak{U}} \in C^q(\mathfrak{U}, \mathscr{S}). \tag{4.2.5.12}$$

♡

Proof. Let g'' be given as $g'' = [b''^q_{\mathfrak{V}}]$ by some q-cocycle $b''^q_{\mathfrak{V}} \in Z^q(\mathfrak{V}, \mathscr{S}'')$. By **Lemma 4.3**, taking a suitable refinement $\mathfrak{U} \succ \mathfrak{V}$, we can find $c^q_{\mathfrak{U}} \in C^q(\mathfrak{U}, \mathscr{S})$ such that $hc^q_{\mathfrak{U}} = \Pi^{\mathfrak{V}}_{\mathfrak{U}} b''^q_{\mathfrak{V}}$. Put $c''^q_{\mathfrak{U}} := \Pi^{\mathfrak{V}}_{\mathfrak{U}} b''^q_{\mathfrak{V}} \in Z^q(\mathfrak{U}, \mathscr{S}'')$. Then by (4.2.5.11) we see that (4.2.5.12) holds. □

(iv) (continuous): For any $g'' \in H^q(\mathcal{M}, \mathscr{S}'')$, take $c''^q_{\mathfrak{U}} \in Z^q(\mathfrak{U}, \mathscr{S}'')$ and $c^q_{\mathfrak{U}} \in C^q(\mathfrak{U}, \mathscr{S})$ as in **Lemma 4.3**. Then since $h\delta c^q_{\mathfrak{U}} = \delta hc^q_{\mathfrak{U}} = \delta c''^q_{\mathfrak{u}} = 0$ and $\mathrm{Ker}(h) = \mathscr{S}'$, it follows that $\delta c^q_{\mathfrak{U}} \in Z^{q+1}(\mathfrak{U}, \mathscr{S}')$. We define

$$\delta^* g'' := [\delta c^q_{\mathfrak{U}}] \in H^{q+1}(\mathcal{M}, \mathscr{S}'). \tag{4.2.5.13}$$

Then $\delta^* g''$ is determined uniquely by g'' and independent of the choice of $\mathfrak{U}$, $c''^q_{\mathfrak{U}}$, and $c^q_{\mathfrak{U}}$ used above.

Proof. Take another $b''^q_{\mathfrak{V}} \in Z^q(\mathfrak{V}, \mathscr{S}'')$ and $b^q_{\mathfrak{V}} \in C^q(\mathfrak{V}, \mathscr{S})$ such that $g'' = [b''^q_{\mathfrak{V}}]$ and $hb^q_{\mathfrak{V}} = b''^q_{\mathfrak{V}}$. We must show that $[\delta b^q_{\mathfrak{V}}] = [\delta c^q_{\mathfrak{U}}]$ in $H^{q+1}(\mathcal{M}, \mathscr{S}')$. Take an arbitrary $\mathfrak{W}$ with $\mathfrak{W} \succ \mathfrak{U}$ and $\mathfrak{W} \succ \mathfrak{V}$. Then by (4.2.5.11),

$$g'' = \left[\Pi^{\mathfrak{V}}_{\mathfrak{W}} b''^q_{\mathfrak{V}}\right] = \left[\Pi^{\mathfrak{U}}_{\mathfrak{W}} c''^q_{\mathfrak{U}}\right], \quad [\delta b^q_{\mathfrak{V}}] = \left[\delta \Pi^{\mathfrak{V}}_{\mathfrak{W}} b^q_{\mathfrak{V}}\right], \quad h\Pi^{\mathfrak{V}}_{\mathfrak{W}} b^q_{\mathfrak{V}} = \Pi^{\mathfrak{V}}_{\mathfrak{W}} b''^q_{\mathfrak{V}}.$$

Moreover $h\Pi^{\mathfrak{U}}_{\mathfrak{W}} c^q_{\mathfrak{U}} = \Pi^{\mathfrak{U}}_{\mathfrak{W}} c''^q_{\mathfrak{U}}$ and $[\delta c^q_{\mathfrak{U}}] = [\delta \Pi^{\mathfrak{U}}_{\mathfrak{W}} c^q_{\mathfrak{U}}]$. If we put $a''_{\mathfrak{W}} := \Pi^{\mathfrak{V}}_{\mathfrak{W}} b''^q_{\mathfrak{V}} - \Pi^{\mathfrak{U}}_{\mathfrak{W}} c''^q_{\mathfrak{U}}$ and $a_{\mathfrak{W}} := \Pi^{\mathfrak{V}}_{\mathfrak{W}} b^q_{\mathfrak{V}} - \Pi^{\mathfrak{U}}_{\mathfrak{W}} c^q_{\mathfrak{U}}$, then we have

$$[a''_{\mathfrak{W}}] = 0, \quad ha_{\mathfrak{W}} = a''_{\mathfrak{W}} \in Z^q(\mathfrak{W}, \mathscr{S}''), \quad a_{\mathfrak{W}} \in C^q(\mathfrak{W}, \mathscr{S}).$$

We must prove $[\delta a_{\mathfrak{W}}] = 0$ in $H^{q+1}(\mathcal{M}, \mathscr{S}')$. Since $[a''_{\mathfrak{W}}] = 0$, taking a suitable $\mathfrak{X} \succ \mathfrak{W}$, we can find $e''_{\mathfrak{X}} \in C^{q-1}(\mathfrak{X}, \mathscr{S}'')$ such that $\Pi^{\mathfrak{W}}_{\mathfrak{X}} a''_{\mathfrak{W}} = \delta e''_{\mathfrak{X}}$. By **Lemma 4.3**, we can find

$e_{\mathfrak{Y}} \in C^{q-1}(\mathfrak{Y}, \mathscr{S})$ such that $he_{\mathfrak{Y}} = \Pi^{\mathfrak{X}}_{\mathfrak{Y}} e''_{\mathfrak{X}}$, where $\mathfrak{Y}$ is a suitable refinement of $\mathfrak{X}$. Since

$$h\Pi^{\mathfrak{W}}_{\mathfrak{Y}} a_{\mathfrak{W}} = h\Pi^{\mathfrak{X}}_{\mathfrak{Y}}\Pi^{\mathfrak{W}}_{\mathfrak{X}} a_{\mathfrak{W}} = \Pi^{\mathfrak{X}}_{\mathfrak{Y}} \delta e''_{\mathfrak{X}} = \delta h e_{\mathfrak{Y}} = h\delta e_{\mathfrak{Y}},$$

it follows that $ha'_{\mathfrak{Y}} = 0$ where $a'_{\mathfrak{Y}} = \Pi^{\mathfrak{W}}_{\mathfrak{Y}} a_{\mathfrak{W}} - \delta e_{\mathfrak{Y}}$. On the other hand, $\mathrm{Ker}(h) = \mathscr{S}'$, we have $a' \in C^q(\mathfrak{Y}, \mathscr{S}')$ and hence

$$\Pi^{\mathfrak{W}}_{\mathfrak{Y}} \delta a_{\mathfrak{W}} = \delta \Pi^{\mathfrak{W}}_{\mathfrak{Y}} a_{\mathfrak{W}} = \delta a'_{\mathfrak{Y}} \in C^q(\mathfrak{Y}, \mathscr{S}').$$

Therefore $[\delta a_{\mathfrak{W}}] = [\Pi^{\mathfrak{W}}_{\mathfrak{Y}} \delta a_{\mathfrak{W}}] = [\delta a'_{\mathfrak{W}}] = 0$ in $H^{q+1}(\mathcal{M}, \mathscr{S}')$. □

(v) (continuous): we prove the exactness of (4.2.5.10). We must prove that for $g'' \in H^q(\mathcal{M}, \mathscr{S}'')$, there exists $g \in H^q(\mathcal{M}, \mathscr{S})$ with $g'' = hg$ if and only if $\delta^* g'' = 0$.

Proof. Suppose $g'' = hg$ with $g \in H^q(\mathcal{M}, \mathscr{S})$. Then there exists a $c^q_{\mathfrak{U}} \in Z^q(\mathfrak{U}, \mathscr{S})$ with $[c^q_{\mathfrak{U}}] = g$. Setting $c''^q_{\mathfrak{U}} = hc^q_{\mathfrak{U}}$ yields $g'' = [c''^q_{\mathfrak{U}}]$. Hence by (4.2.5.12) and (4.2.5.13), we have $\delta^* g'' = [\delta c^q_{\mathfrak{U}}] = 0$.

Conversely, suppose $\delta^* g'' = 0$. Let $g'' = [c''^q_{\mathfrak{U}}]$, and $hc^q_{\mathfrak{U}} = c''^q_{\mathfrak{U}} \in Z^q(\mathfrak{U}, \mathscr{S})$ with $c^q_{\mathfrak{U}} \in C^q(\mathfrak{U}, \mathscr{S})$ by **Lemma 4.4**. Then $[\delta c^q_{\mathfrak{U}}] = 0$ in $H^{q+1}(\mathcal{M}, \mathscr{S}')$. In other words, for a suitable $\mathfrak{V} \succ \mathfrak{U}$ and a suitable $c'^q_{\mathfrak{V}} \in C^q(\mathfrak{V}, \mathscr{S}')$ we have $\Pi^{\mathfrak{U}}_{\mathfrak{V}} \delta c^q_{\mathfrak{U}} = \delta c'^q_{\mathfrak{V}}$. Since $\delta(\Pi^{\mathfrak{U}}_{\mathfrak{V}} c^q_{\mathfrak{U}} - c'^q_{\mathfrak{V}}) = 0$, we see that $\Pi^{\mathfrak{U}}_{\mathfrak{V}} c^q_{\mathfrak{U}} - c'^q_{\mathfrak{V}} \in Z^q(\mathfrak{V}, \mathscr{S})$. Hence if we put $g := [\Pi^{\mathfrak{U}}_{\mathfrak{V}} c^q_{\mathfrak{U}} - c'^q_{\mathfrak{V}}] \in H^q(\mathcal{M}, \mathscr{S})$, we get

$$hg = \left[\Pi^{\mathfrak{U}}_{\mathfrak{V}} h c^q_{\mathfrak{U}}\right] = \left[\Pi^{\mathfrak{U}}_{\mathfrak{V}} c''^q_{\mathfrak{U}}\right] = [c''^q_{\mathfrak{U}}] = g'',$$

since $hc'^q_{\mathfrak{V}} = 0$. □

Proof of **Theorem 4.8***:* We may assume that $\iota : \mathscr{S}' \to \mathscr{S}$ is the inclusion. Then $H^0(\mathcal{M}, \mathscr{S}') = \Gamma(\mathcal{M}, \mathscr{S}')$ is a submodule of $H^0(\mathcal{M}, \mathscr{S}) = \Gamma(\mathcal{M}, \mathscr{S})$. Hence the exactness of

$$\begin{aligned} 0 \longrightarrow H^0(\mathcal{M}, \mathscr{S}') &\xrightarrow{\ \iota\ } H^0(\mathcal{M}, \mathscr{S}) \\ \xrightarrow{\ h\ } H^0(\mathcal{M}, \mathscr{S}'') &\xrightarrow{\ \delta^*\ } H^1(\mathcal{M}, \mathscr{S}') \end{aligned}$$

follows from the exactness of (4.2.5.8). Since (4.2.5.10) is exact, in order to prove the exactness of (4.2.5.7), it suffices to verify the exactness of

$$H^{q-1}(\mathcal{M}, \mathscr{S}'') \xrightarrow{\ \delta^*\ } H^q(\mathcal{M}, \mathscr{S}') \xrightarrow{\ \iota\ } H^q(\mathcal{M}, \mathscr{S}), \tag{4.2.5.14}$$

and

$$H^q(\mathcal{M}, \mathscr{S}') \xrightarrow{\ \iota\ } H^q(\mathcal{M}, \mathscr{S}) \xrightarrow{\ h\ } H^q(\mathcal{M}, \mathscr{S}'') \tag{4.2.5.15}$$

for each $q = 1, 2, \cdots$.

Take a $g' \in H^q(\mathcal{M}, \mathscr{S}')$. If $g' = \delta^* g''$ with $g'' \in H^{q-1}(\mathcal{M}, \mathscr{S}'')$, then $g' = [\delta c^{q-1}_{\mathfrak{U}}] \in H^q(\mathcal{M}, \mathscr{S}')$ for some $c^{q-1}_{\mathfrak{U}} \in C^{q-1}(\mathfrak{U}, \mathscr{S})$. Here

$$g'' = \left[c''^{q-1}_{\mathfrak{U}}\right], \quad hc^{q-1}_{\mathfrak{U}} = c''^{q-1}_{\mathfrak{U}} \in Z^{q-1}(\mathfrak{U}, \mathscr{S}), \quad c^{q-1}_{\mathfrak{U}} \in C^{q-1}(\mathfrak{U}, \mathscr{S}),$$

by **Lemma 4.4**. Since ι is the inclusion, $\iota g' = [\delta c^{q-1}_{\mathfrak{U}}]$ is the cohomology class of $\delta c^{q-1}_{\mathfrak{U}}$ regarded as an element of $Z^q(\mathfrak{U}, \mathscr{S})$. Hence $\iota g' = 0$. Conversely, suppose $\iota g' = 0$ in $H^q(\mathcal{M}, \mathscr{S})$. Let $g' = [c'^q_{\mathfrak{U}}]$ with $c'^q_{\mathfrak{U}} \in Z^q(\mathfrak{U}, \mathscr{S}')$. Then since $[c'^q_{\mathfrak{U}}] = \iota g' = 0$ means that $[c'^q_{\mathfrak{U}}] = 0$ as an element of $H^q(\mathcal{M},\mathscr{S})$, there are a refinement $\mathfrak{V} \prec \mathfrak{U}$ and an element $c_{\mathfrak{U}}^{q-1} \in C^{q-1}(\mathfrak{V},\mathscr{S})$ such that $\Pi_{\mathfrak{V}}^{\mathfrak{U}} c_{\mathfrak{U}}'^q = \delta c_{\mathfrak{V}}^{q-1}$. Putting $c_{\mathfrak{V}}''^{q-1} := h c_{\mathfrak{V}}^{q-1}$, we obtain

$$\delta c_{\mathfrak{V}}''^{q-1} = h\delta c_{\mathfrak{V}}^{q-1} = \Pi_{\mathfrak{V}}^{\mathfrak{U}} h c_{\mathfrak{U}}'^q = 0.$$

Hence $c_{\mathfrak{V}}''^{q-1} \in Z^{q-1}(\mathfrak{V},\mathscr{S}'')$. Therefore, putting $g'' := [c_{\mathfrak{V}}''^{q-1}] \in H^{q-1}(\mathcal{M},\mathscr{S}'')$ yields

$$\delta^* g'' = \left[\delta c_{\mathfrak{V}}^{q-1}\right] = \left[\Pi_{\mathfrak{V}}^{\mathfrak{U}} c_{\mathfrak{U}}'^q\right] = [c_{\mathfrak{U}}'^q] = g'.$$

Thus (4.2.5.14) is exact.

Take any $g \in H^q(\mathcal{M},\mathscr{S})$. If $h = \iota g'$ with $g' \in H^q(\mathcal{M},\mathscr{S}')$, clearly $hg = 0$. Conversely, suppose $hg = 0$. Let $g = [c_{\mathfrak{U}}^q]$ with $c_{\mathfrak{U}}^q \in Z^q(\mathfrak{U},\mathscr{S})$. Then there are a refinement $\mathfrak{V} \prec \mathfrak{U}$ and an element $c_{\mathfrak{V}}''^{q-1} \in C^{q-1}(\mathfrak{V},\mathscr{S}'')$ such that $\Pi_{\mathfrak{V}}^{\mathfrak{U}} h c_{\mathfrak{U}}^q = \delta c_{\mathfrak{V}}''^{q-1}$. By **Lemma 4.3**, we can find a locally finite refinement $\mathfrak{W}$ of $\mathfrak{V}$ and $c_{\mathfrak{W}}^{q-1} \in C^{q-1}(\mathfrak{W},\mathscr{S})$ such that $h c_{\mathfrak{W}}^{q-1} = \Pi_{\mathfrak{W}}^{\mathfrak{V}} c_{\mathfrak{V}}''^{q-1}$. Hence we have

$$\Pi_{\mathfrak{W}}^{\mathfrak{U}} h c_{\mathfrak{U}}^q = \Pi_{\mathfrak{W}}^{\mathfrak{V}} \delta c_{\mathfrak{V}}''^{q-1} = \delta h c_{\mathfrak{W}}^{q-1},$$

that is, $h(\Pi_{\mathfrak{W}}^{\mathfrak{U}} c_{\mathfrak{U}}^q - \delta c_{\mathfrak{W}}^{q-1}) = 0$ and hence $\Pi_{\mathfrak{W}}^{\mathfrak{U}} c_{\mathfrak{U}}^q - \delta c_{\mathfrak{W}}^{q-1} \in Z^q(\mathfrak{W},\mathscr{S}')$. Consequently, setting $g' = [\Pi_{\mathfrak{W}}^{\mathfrak{U}} c_{\mathfrak{U}}^q - \delta c_{\mathfrak{W}}^{q-1}] \in H^q(\mathfrak{W},\mathscr{S}')$, we obtain

$$\iota g' = \left[\Pi_{\mathfrak{W}}^{\mathfrak{U}} c_{\mathfrak{U}}^q - \delta c_{\mathfrak{W}}^{q-1}\right] = \left[\Pi_{\mathfrak{W}}^{\mathfrak{U}} c_{\mathfrak{U}}^q\right] = [c_{\mathfrak{U}}^q] = g$$

since $[\delta c_{\mathfrak{W}}^{q-1}] = 0$ in $H^q(\mathcal{M},\mathscr{S})$. Thus (4.2.5.15).

For given **K**-modules $H_1,\cdots,H_\nu,\cdots$, $L_1,\cdots,L_\nu,\cdots$ and **K**-homomorphisms $h_\nu : H_\nu \to H_{\nu+1}$, $k_\nu : L_\nu \to L_{\nu+1}$, $\psi_\nu : H_\nu \to L_\nu$, the diagram

$$\begin{array}{ccccccccc} H_1 & \xrightarrow{h_1} & H_2 & \longrightarrow & \cdots & \longrightarrow & H_\nu & \xrightarrow{h_\nu} & H_{\nu+1} & \longrightarrow & \cdots \\ \downarrow{\scriptstyle\psi_1} & & \downarrow{\scriptstyle\psi_2} & & & & \downarrow{\scriptstyle\psi_\nu} & & \downarrow{\scriptstyle\psi_{\nu+1}} & & \\ L_1 & \xrightarrow{k_1} & L_2 & \longrightarrow & \cdots & \longrightarrow & L_\nu & \xrightarrow{k_\nu} & L_{\nu+1} & \longrightarrow & \cdots \end{array} \tag{4.2.5.16}$$

is called **commutative** if

$$k_\nu \circ \psi_\nu = \psi_{\nu+1} \circ h_\nu, \quad \nu = 1,2,3,\cdots. \tag{4.2.5.17}$$

If (4.2.5.16) is commutative and each row is exact, we call (4.2.5.16) an **exact commutative diagram**. Similarly we define an exact commutative diagram of sheaves and sheaf homomorphisms. For example,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{S}' & \xrightarrow{\iota} & \mathscr{S} & \xrightarrow{h} & \mathscr{S}'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\psi'} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi''} & & \\ 0 & \longrightarrow & \mathscr{T}' & \xrightarrow{\iota} & \mathscr{T} & \xrightarrow{k} & \mathscr{T}'' & \longrightarrow & 0 \end{array} \tag{4.2.5.18}$$

is an exact commutative diagram if each row is exact, $\iota \circ \psi' = \psi \circ \iota$ and $k \circ \psi = \psi'' \circ h$.

Theorem 4.9

An exact commutative diagram of sheaves

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{S}' \arrow[r, "\iota"] \arrow[d, "\psi'"] & \mathscr{S} \arrow[r, "h"] \arrow[d, "\psi"] & \mathscr{S}'' \arrow[r] \arrow[d, "\psi''"] & 0 \\ 0 \arrow[r] & \mathscr{T}' \arrow[r, "\iota"] & \mathscr{T} \arrow[r, "k"] & \mathscr{T}'' \arrow[r] & 0 \end{tikzcd}$$

induces an exact commutative diagram of cohomology groups

$$\begin{tikzcd} 0 \arrow[r] & H^0(\mathcal{M}, \mathscr{S}') \arrow[r, "\iota"] \arrow[d, "\psi'"] & H^0(\mathcal{M}, \mathscr{S}) \arrow[r, "h"] \arrow[d, "\psi"] & H^0(\mathcal{M}, \mathscr{S}'') \arrow[r, "\delta^*"] \arrow[d, "\psi''"] & {} \\ 0 \arrow[r] & H^0(\mathcal{M}, \mathscr{T}') \arrow[r, "\iota"] & H^0(\mathcal{M}, \mathscr{T}) \arrow[r, "k"] & H^0(\mathcal{M}, \mathscr{T}'') \arrow[r, "\delta^*"] & {} \end{tikzcd} \tag{4.2.5.19}$$

♡

Proof. It suffices to verify the commutativity of (4.2.5.19). Since it is clear that $k \circ \psi = \psi'' \circ h$ and $\iota \circ \psi' = \psi \circ \iota$, we have only to show that the diagram

$$\begin{tikzcd} H^q(\mathcal{M}, \mathscr{S}'') \arrow[r, "\delta^*"] \arrow[d, "\psi''"'] & H^{q+1}(\mathcal{M}, \mathscr{S}') \arrow[d, "\psi'"] \\ H^q(\mathcal{M}, \mathscr{T}'') \arrow[r, "\delta^*"'] & H^{q+1}(\mathcal{M}, \mathscr{T}') \end{tikzcd}$$

is commutative for $q = 0, 1, \cdots$. For any $g'' \in H^q(\mathcal{M}, \mathscr{S}'')$. By **Lemma 4.4**, there are $c''^q_{\mathfrak{U}} \in Z^q(\mathfrak{U}, \mathscr{S}'')$ and $c^q_{\mathfrak{U}} \in C^q(\mathfrak{U}, \mathscr{S})$ with $g'' = [c''^q_{\mathfrak{U}}]$, $hc^q_{\mathfrak{U}} = c''^q_{\mathfrak{U}}$ such that $\delta^* g'' = [\delta c^q_{\mathfrak{U}}]$ where $\delta c^q_{\mathfrak{U}} \in Z^{q+1}(\mathfrak{U}, \mathscr{S}')$. Hence $\psi' \delta^* g'' = [\psi' \delta c^q_{\mathfrak{U}}]$. Since ι is the inclusion, $\iota \circ \psi' = \psi \circ \iota$ means that $\psi = \psi'$ on $\mathscr{S}' \subset \mathscr{S}$. Therefore $\psi' \delta c^q_{\mathfrak{U}} = \psi \delta c^q_{\mathfrak{U}}$, hence $\psi' \delta^* g'' = [\psi \delta c^q_{\mathfrak{U}}]$. On the other hand, we have $\psi'' g'' = [\psi'' c''^q_{\mathfrak{U}}]$ with $\psi'' c''^q_{\mathfrak{U}} \in Z^q(\mathfrak{U}, \mathscr{T}'')$. Since $\psi'' \circ h = k \circ \psi$, we also have

$$\psi'' c''^q_{\mathfrak{U}} = \psi'' h c^q_{\mathfrak{U}} = k \psi c^q_{\mathfrak{U}}.$$

Hence $\delta^* \psi'' g'' = [\delta \psi c^q_{\mathfrak{U}}]$, which implies that $\delta^* \psi'' g'' = \psi' \delta^* g''$. Since g'' is an arbitrary element of $H^q(\mathcal{M}, \mathscr{S}'')$, we obtain $\delta^* \circ \psi'' = \psi' \circ \delta^*$ as desired. □

4.2.6 Fine sheaves

Let $h : \mathscr{S} \to \mathscr{T}$ be a homomorphism of sheaves over $\mathcal{M}$. The closure of the set of points $p \in \mathcal{M}$ with $h(\mathscr{S}_p) \neq 0_{\mathscr{T}_p}$ is called the **support of** h and is denoted by $\operatorname{supp}(h)$:

$$\operatorname{supp}(h) = \overline{\{p \in \mathcal{M} : h(\mathscr{S}_p) \neq 0_{\mathscr{T}_p}\}}. \tag{4.2.6.1}$$

Definition 4.9

Let $\mathscr{S}$ be a sheaf over $\mathcal{M}$. $\mathcal{M}$ is called a **fine sheaf** *if for any locally finite open covering $\mathfrak{U} = \{\mathcal{U}_j\}$, there is a family of homomorphisms $\{h_j\}$ with $h_j : \mathscr{S} \to \mathscr{S}$, such that*

(i) $\operatorname{supp}(h_j) \subset \mathcal{U}_j$,

(ii) $\sum_j h_j = 1$ *the identity, i.e.,* $\sum_j h_j(s) = s$ *for any* $s \in \mathscr{S}$.

Note that by (i), $h_j(s) \neq 0$ for $s \in \mathscr{S}_p$ implies $p \in \mathcal{U}_j$. Consequently the summation in (ii) makes sense since $\mathfrak{U}$ is locally finite.

♣

Example 4.1

$\mathscr{A}_{\mathcal{M}}$ is a finite sheaf.

Proof. Let $\mathfrak{U} = \{\mathcal{U}_j\}$ be a locally finite open covering of $\mathcal{M}$, and $\{\rho_j\}$ a partition of unity subordinate to $\mathfrak{U}$. For $s = f_p \in \mathscr{A}_{\mathcal{M},p}$ with f a local C^∞ function at p, define $h_j s$ by

$$h_j s := (\rho_j f)_p\,, \quad s = f_p.$$

$h_j s$ is independent of the choice of f. In order to see that h_j is a homomorphism, it suffices to verify that h_j is continuous. Given a neighborhood $\mathscr{U}(h_j s; g, \mathcal{U})$ of $h_j s$, since $g_p = h_j s = (\rho_j f)_p$, there is a neighborhood $\mathcal{V} \subset \mathcal{U}$ such that $g = \rho_j f$ on $\mathcal{V}$. Hence for $t \in \mathscr{U}(s; f, \mathcal{V})$, $q = \varpi(t) \in \mathcal{V}$,

$$h_j t = h_j f_q = (\rho_j f)_q = g_q \in \mathscr{U}(h_j s; g, \mathcal{U}).$$

Thus h_j is continuous, hence a homomorphism. Clearly that $\mathrm{supp}(h_j) = \mathrm{supp}(\rho_j) \subset \mathcal{U}_j$. Moreover $\sum_j h_j = 1$ follows from $\sum_j \rho_j = 1$. □

The essential point of the above argument is in the fact that if f is a local C^∞ function, so is $\rho_j f$. Consequently by a similar argument, we see that $\mathscr{A}^r_{\mathcal{M}}$, $\mathscr{A}^{p,q}_{\mathcal{X}}$ and $\mathscr{A}_{\mathcal{M}}(\mathcal{F})$ are fine sheaves where $\mathcal{F}$ is a C^∞ vector bundle over $\mathcal{M}$.

However, $\mathscr{O}_{\mathcal{X}}$ is not fine. In fact, if otherwise, $h_j 1 \in \Gamma(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ is holomorphic, where 1 is a holomorphic function which is identically one on $\mathcal{X}$. Since $h_j 1$ is equal to 0 on $\mathcal{X} - \mathrm{supp}(h_j)$, by the analytic continuation, $h_j 1 = 0$ on $\mathcal{X}$, which contradicts $\sum_j h_j 1 = 1$. ♠

Theorem 4.10

If $\mathscr{S}$ is a fine sheaf over $\mathcal{M}$,

$$H^q(\mathcal{M}, \mathscr{S}) = 0 \tag{4.2.6.2}$$

for $q \geq 1$. ♡

Proof. Since $H^q(\mathcal{M}, \mathscr{S}) = \cup_{\mathfrak{U} \in \mathbf{LFOC}(\mathcal{M})} \overline{\Pi}^{\mathfrak{U}} H^q(\mathfrak{U}, \mathscr{S})$, it suffices to prove that

$$H^q(\mathfrak{U}, \mathscr{S}) = \frac{Z^q(\mathfrak{U}, \mathscr{S})}{\delta C^{q-1}(\mathfrak{U}, \mathscr{S})} = 0$$

for $q \geq 1$ and any locally finite open covering $\mathfrak{U}$. We give the proof for the case $q = 2$. Since $\mathscr{S}$ is fine, there is a family $\{h_j\}$, $h_j : \mathscr{S} \to \mathscr{S}$ of homomorphisms satisfying the above (i) and (ii). Each h_i induces a homomorphism $h_i : \Gamma(\mathcal{W}, \mathscr{S}) \to \Gamma(\mathcal{W}, \mathscr{S})$ for any open set $\mathcal{W} \subset \mathcal{M}$.

Let $c^2_{\mathfrak{U}} = \{\sigma_{ijk}\} \in Z^2(\mathfrak{U}, \mathscr{S})$ be a 2-cocycle. For $\mathcal{U}_j \cap \mathcal{U}_k \cap \mathcal{U}_i \neq \emptyset$, $h_i\sigma_{ijk} : p \mapsto (h_i\sigma_{ijk})(p) = h_i(\sigma_{ijk}(p))$ is a section of $\mathscr{S}$ over $\mathcal{U}_j \cap \mathcal{U}_k \cap \mathcal{U}_i$, and if $p \notin \mathrm{supp}(h_i)$, then $(h_i\sigma_{ijk})(p) = 0$. Since $\mathrm{supp}(h_i)$ is closed in $\mathcal{U}_j$, putting

$$\widetilde{\sigma}_{ijk}(p) := \begin{cases} (h_i\sigma_{ijk})(p), & p \in \mathcal{U}_j \cap \mathcal{U}_k \cap \mathcal{U}_i, \\ 0, & p \in \mathcal{U}_j \cap \mathcal{U}_k - \mathrm{supp}(h_i), \end{cases}$$

we get a section $\widetilde{\sigma}_{ijk} : p \mapsto \widetilde{\sigma}_{ijk}(p)$ of $\mathscr{S}$ on $\mathcal{U}_j \cap \mathcal{U}_k$. $\widetilde{\sigma}_{ijk}$ is an extension of $h_i\sigma_{ijk}$. For a fixed

$p \in \mathcal{U}_j \cap \mathcal{U}_k$, $\widetilde{\sigma}_{ijk}(p) \neq 0$ for only finite i's. Consequently, putting

$$\tau_{jk} := \sum_i \widetilde{\sigma}_{ijk},$$

we obtain $\tau_{jk} \in \Gamma(\mathcal{U}_j \cap \mathcal{U}_k, \mathscr{S})$ with $\tau_{jk} + \tau_{kj} = 0$, that is, $c^1_{\mathfrak{U}} = \{\tau_{jk}\} \in C^1(\mathfrak{U}, \mathscr{S})$.

It suffices then to check $c^2_{\mathfrak{U}} = \delta c^1_{\mathfrak{U}}$. Since $\delta c^2_{\mathfrak{U}} = 0$, we have

$$\sigma_{jk\ell} = \sigma_{ik\ell} - \sigma_{ij\ell} + \sigma_{ijk} \quad \text{on} \quad \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k \cap \mathcal{U}_\ell \neq \emptyset.$$

Hence

$$h_i\sigma_{jk\ell} = h_i\sigma_{ik\ell} - h_i\sigma_{ij\ell} + h_i\sigma_{ijk} = \widetilde{\sigma}_{ik\ell} - \widetilde{\sigma}_{ij\ell} + \widetilde{\sigma}_{ijk}$$

on $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k \cap \mathcal{U}_\ell \neq \emptyset$. Hence we have

$$\sigma_{jk\ell} = \sum_i h_i\sigma_{jk\ell} = \tau_{k\ell} - \tau_{j\ell} + \tau_{jk}$$

on $\mathcal{U}_j \cap \mathcal{U}_k \cap \mathcal{U}_\ell$ and consequently $c^2_{\mathfrak{U}} = \delta c^1_{\mathfrak{U}}$. □

Consider the sheaf $\mathscr{A}^r_{\mathcal{M}}$ over a C^∞ manifold $\mathcal{M}$. For a germ $s = \varphi_p$ of a local C^∞ r-form φ at p, we define its exterior differential by

$$ds := (d\varphi)_p, \quad s = \varphi_p.$$

$d : s \mapsto ds$ gives a homomorphism $d : \mathscr{A}^r_{\mathcal{M}} \to \mathscr{A}^{r+1}_{\mathcal{M}}$ of sheaves, where we put $\mathscr{A}^0_{\mathcal{M}} = \mathscr{A}_{\mathcal{M}}$. The sheaf of locally constant functions $\underline{\mathbf{C}}_{\mathcal{M}}$ is a subsheaf of $\mathscr{A}^0_{\mathcal{M}}$. Then

The sequence

$$0 \longrightarrow \underline{\mathbf{C}}_{\mathcal{M}} \xrightarrow{\iota} \mathscr{A}^0_{\mathcal{M}} \xrightarrow{d} \mathscr{A}^1_{\mathcal{M}} \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{A}^m_{\mathcal{M}} \longrightarrow 0 \tag{4.2.6.3}$$

is exact[4], where ι is the inclusion.

Definition 4.10

An exact sequence of sheaves over $\mathcal{M}$

$$0 \longrightarrow \mathscr{S} \xrightarrow{\iota} \mathscr{B}^0 \xrightarrow{d} \mathscr{B}^1 \xrightarrow{d} \mathscr{B}^2 \xrightarrow{d} \cdots \tag{4.2.6.5}$$

is called a **fine resolution of** $\mathscr{S}$ *if each $\mathscr{B}^i$ is fine.* ♣

Example 4.1 shows that (4.2.6.3) is a fine resolution of $\underline{\mathbf{C}}_{\mathcal{M}}$.

---

[4]For $s = f_p \in \mathscr{A}^0_{\mathcal{M}}$, $ds = (df)_p = 0$ implies $df = 0$ on some neighborhood of p, i.e., f is a constant there. Thus

$$0 \longrightarrow \underline{\mathbf{C}}_{\mathcal{M}} \xrightarrow{\iota} \mathscr{A}^0_{\mathcal{M}} \xrightarrow{d} \mathscr{A}^1_{\mathcal{M}}$$

is exact. The exactness of

$$\mathscr{A}^{r-1}_{\mathcal{M}} \xrightarrow{d} \mathscr{A}^r_{\mathcal{M}} \xrightarrow{d} \mathscr{A}^{r+1}_{\mathcal{M}}, \quad r \geq 1, \tag{4.2.6.4}$$

follows from **Theorem 4.5**. In fact, take any $s = \varphi_p \in \mathscr{A}^r_{\mathcal{M}}$, where φ is a local C^∞ r-form at p. $ds = (d\varphi)_p = 0$ implies that $d\varphi = 0$ on some neighborhood $\mathcal{U}$ of p. We may assume that $\mathcal{U}$ is a coordinate multi-interval $\mathcal{U} = \{q : |x^1_p(q)| < \epsilon, \cdots, |x^m_p(q)| < \epsilon\}$ with center p. Then by **Theorem 4.5** there is a C^∞ $(r-1)$-form ψ on $\mathcal{U}$ such that $\varphi = d\psi$. Putting $t = \psi_p \in \mathscr{A}^{r-1}_{\mathcal{M}}$, we obtain $s = \varphi = (d\psi)_p = dt$. Thus if $ds = 0$, there is a $t \in \mathscr{A}^{r-1}_{\mathcal{M}}$ with $dt = s$.

Conversely, if $s = dt$ with $t = \psi_p \in \mathscr{A}^{r-1}_{\mathcal{M}}$, $ds = d(d\psi)_p = (dd\psi)_p = 0$. Consequently (4.2.6.4) is exact.

Let $d : \mathscr{B} \to \mathscr{C}$ be a homomorphism of sheaves over $\mathcal{M}$. Put

$$\mathscr{S} := \mathrm{Ker}(d).$$

$\mathscr{S}$ is a subsheaf of $\mathscr{B}$. We denote the inclusion $\mathscr{S} \to \mathscr{B}$ by ι. Then

$$0 \longrightarrow \mathscr{S} \xrightarrow{\iota} \mathscr{B} \xrightarrow{d} d\mathscr{B} \longrightarrow 0 \tag{4.2.6.6}$$

is exact.

Lemma 4.5

If $\mathscr{B}$ is fine, then

$$H^1(\mathcal{M}, \mathscr{S}) \cong \frac{H^0(\mathcal{M}, d\mathscr{B})}{dH^0(\mathcal{M}, \mathscr{B})}, \tag{4.2.6.7}$$

$$H^q(\mathcal{M}, \mathscr{S}) \cong H^{q-1}(\mathcal{M}, d\mathscr{B}), \quad q = 2, 3, \cdots. \tag{4.2.6.8}$$

♡

Proof. By **Theorem 4.8**, we get the exact sequence of cohomology groups

$$\begin{aligned} 0 &\longrightarrow H^0(\mathcal{M}, \mathscr{S}) \xrightarrow{\iota} H^0(\mathcal{M}, \mathscr{B}) \xrightarrow{d} H^0(\mathcal{M}, d\mathscr{B}) \\ &\xrightarrow{\delta^*} H^1(\mathcal{M}, \mathscr{S}) \xrightarrow{\iota} H^1(\mathcal{M}, \mathscr{B}) \xrightarrow{d} \cdots \\ &\longrightarrow H^{q-1}(\mathcal{M}, \mathscr{B}) \xrightarrow{d} H^{q-1}(\mathcal{M}, d\mathscr{B}) \xrightarrow{\delta^*} H^q(\mathcal{M}, \mathscr{S}) \\ &\xrightarrow{\iota} H^q(\mathcal{M}, \mathscr{B}) \xrightarrow{d} \cdots \end{aligned}$$

Since $\mathscr{B}$ is fine, $H^q(\mathcal{M}, \mathscr{B}) = 0$ for $q \geq 1$ by **Theorem 4.10**. Therefore we obtain the exact sequence

$$0 \to H^0(\mathcal{M}, \mathscr{S}) \to H^0(\mathcal{M}, \mathscr{B}) \to H^0(\mathcal{M}, d\mathscr{B}) \to H^1(\mathcal{M}, \mathscr{S}) \to 0, \tag{4.2.6.9}$$

and

$$0 \to H^{q-1}(\mathcal{M}, d\mathscr{B}) \to H^q(\mathcal{M}, \mathscr{S}) \to 0, \quad q = 2, 3 \cdots. \tag{4.2.6.10}$$

Hence we get (4.2.6.7) and (4.2.6.8). □

Theorem 4.11

If

$$0 \longrightarrow \mathscr{S} \xrightarrow{\iota} \mathscr{B}^0 \xrightarrow{d} \mathscr{B}^1 \xrightarrow{d} \mathscr{B}^2 \xrightarrow{d} \cdots$$

is a fine resolution of $\mathscr{S}$ over $\mathcal{M}$, then

$$H^q(\mathcal{M}, \mathscr{S}) \cong \frac{\Gamma(\mathcal{M}, d\mathscr{B}^{q-1})}{d\Gamma(\mathcal{M}, \mathscr{B}^{q-1})}, \quad q = 1, 2, \cdots. \tag{4.2.6.11}$$

♡

Proof. From the exact sequence

$$0 \longrightarrow \mathscr{S} \xrightarrow{\iota} \mathscr{B}^0 \xrightarrow{d} \mathscr{B}^1 \xrightarrow{d} \cdots$$

we obtain the exact sequence

$$0 \longrightarrow \mathscr{S} \xrightarrow{\iota} \mathscr{B}^0 \xrightarrow{d} d\mathscr{B}^0 \longrightarrow 0$$

and

$$0 \longrightarrow d\mathscr{B}^{r-1} \longrightarrow \mathscr{B}^r \xrightarrow{d} d\mathscr{B}^r \longrightarrow 0, \quad r = 1, 2, \cdots.$$

By hypothesis, each $\mathscr{B}^i$ is fine. Hence according to **Lemma 4.5**, we have

$$\begin{aligned} H^1(\mathcal{M}, \mathscr{S}) &\cong \frac{H^0(\mathcal{M}, d\mathscr{B}^0)}{dH^0(\mathcal{M}, \mathscr{B}^0)}, \\ H^q(\mathcal{M}, \mathscr{S}) &\cong H^{q-1}(\mathcal{M}, d\mathscr{B}^0), \\ H^1(\mathcal{M}, d\mathscr{B}^{r-1}) &\cong \frac{H^0(\mathcal{M}, d\mathscr{B}^r)}{dH^0(\mathcal{M}, d\mathscr{B}^r)}, \\ H^q(\mathcal{M}, d\mathscr{B}^{r-1}) &\cong H^{q-1}(\mathcal{M}, d\mathscr{B}^r), \quad q = 2, 3, \cdots. \end{aligned}$$

Therefore we have

$$\begin{aligned} H^q(\mathcal{M}, \mathscr{S}) &\cong H^{q-1}(\mathcal{M}, d\mathscr{B}^0) \cong H^{q-2}(\mathcal{M}, d\mathscr{B}^1) \\ &\cong \cdots \cong H^1(\mathcal{M}, d\mathscr{B}^{q-2}) \cong \frac{H^0(\mathcal{M}, d\mathscr{B}^{q-1})}{dH^0(\mathcal{M}, \mathscr{B}^{q-1})}. \end{aligned}$$

On the other hand, $\Gamma(\mathcal{M}, d\mathscr{B}^{q-1}) = H^0(\mathcal{M}, d\mathscr{B}^{q-1})$ and $\Gamma(\mathcal{M}, \mathscr{B}^{q-1}) = H^0(\mathcal{M}, \mathscr{B}^{q-1})$, we obtain (4.2.6.11). □

4.2.7 De Rham's theorem

By **Theorem 4.11**, the exact sequence (4.2.6.3) gives

Theorem 4.12

If $\mathcal{M}$ is a smooth manifold, then

$$H^q(\mathcal{M}, \underline{\mathbf{C}}_{\mathcal{M}}) \cong \frac{\Gamma(\mathcal{M}, d\mathscr{A}^{q-1}_{\mathcal{M}})}{d\Gamma(\mathcal{M}, \mathscr{A}^{q-1}_{\mathcal{M}})}, \quad q = 1, 2, \cdots. \tag{4.2.7.1}$$

♡

We usually write

$$H^q(\mathcal{M}, \mathbf{C}) := H^q(\mathcal{M}, \underline{\mathbf{C}}_{\mathcal{M}}). \tag{4.2.7.2}$$

Note that $H^q(\mathcal{M}, \mathbf{C})$ is the cohomology group of $\mathcal{M}$ with complex coefficients, depending only on the topology of $\mathcal{M}$. In the proof of **Theorem 4.11**, we have

$$H^q(\mathcal{M}, \mathbf{C}) \cong H^{q-m}(\mathcal{M}, d\mathscr{A}^{m-1}_{\mathcal{M}}) = H^{q-m}(\mathcal{M}, \mathscr{A}^m_{\mathcal{M}}) = 0$$

for $q - m \geq 1$, since $\mathscr{A}^m_{\mathcal{M}}$ is fine.

Since for $q \geq 1$, $0 \to d\mathscr{A}^{q-1}_{\mathcal{M}} \to \mathscr{A}^q_{\mathcal{M}} \to \mathscr{A}^{q+1}_{\mathcal{M}} \to 0$ is exact, $d\mathscr{A}^{q-1}_{\mathcal{M}}$ is the sheaf of local closed C^∞ q-forms. In fact, $d\mathscr{A}^{q-1}_{\mathcal{M}} = \mathrm{Ker}(d)$ consists of germs $\varphi_p \in \mathscr{A}^q_{\mathcal{M}}$ with $(d\varphi)_p = d(\varphi_p) = 0$, which means that $d\varphi = 0$ on some neighborhood of p. Thus $\Gamma(\mathcal{M}, d\mathscr{A}^{q-1}_{\mathcal{M}})$ is the linear space of all d-closed C^∞ q-forms on $\mathcal{M}$, while $d\Gamma(\mathcal{M}, \mathscr{A}^{q-1}_{\mathcal{M}})$ is the linear subspace of $\Gamma(\mathcal{M}, d\mathscr{A}^{q-1}_{\mathcal{M}})$ consisting of all exact q-forms on $\mathcal{M}$. We call the quotient space

$$H^q_d(\mathcal{M}) := \frac{\Gamma(\mathcal{M}, d\mathscr{A}^{q-1}_{\mathcal{M}})}{d\Gamma(\mathcal{M}, \mathscr{A}^{q-1})}, \quad q = 1, 2, \cdots. \tag{4.2.7.3}$$

the d-**cohomology group of** $\mathcal{M}$. Then De Rham's theorem

$$H^q_d(\mathcal{M}) \cong H^q(\mathcal{M}, \mathbf{C}), \quad q = 1, 2, \cdots. \tag{4.2.7.4}$$

implies that the d-cohomology group $H^q_d(\mathcal{M})$, which is defined with respect to the differentiable structure of $\mathcal{M}$, actually depends only on the topology of $\mathcal{M}$.

4.2.8 Dolbeault's theorem

Let $\mathcal{X}$ be an n-dimensional complex manifold, and $\mathscr{A}_{\mathcal{X}}^{p,q}$ the sheaf of germs of C^∞ (p,q)-forms over $\mathcal{X}$. The sheaf $\mathscr{O}_{\mathcal{X}}$ of germs of holomorphic functions on $\mathcal{X}$ is a subsheaf of $\mathscr{A}_{\mathcal{X}} = \mathscr{A}_{\mathcal{X}}^{0,0}$. A local C^∞ function f is holomorphic if and only if $\overline{\partial} f = 0$. Thus

$$0 \longrightarrow \mathscr{O}_{\mathcal{X}} \xrightarrow{\iota} \mathscr{A}_{\mathcal{X}}^{0,0} \xrightarrow{\overline{\partial}} \mathscr{A}_{\mathcal{X}}^{0,1} \tag{4.2.8.1}$$

is exact, where $\iota : \mathscr{O}_{\mathcal{X}} \to \mathscr{A}_{\mathcal{X}}^{0,0}$ is the inclusion. Similarly, $\mathscr{O}_{\mathcal{X}}^p$ is a subsheaf of $\mathscr{A}_{\mathcal{X}}^{p,0}$, and a local C^∞ $(p,0)$-form φ is a holomorphic p-form if and only if $\overline{\partial}\varphi = 0$. Thus

$$0 \longrightarrow \mathscr{O}_{\mathcal{X}}^p \xrightarrow{\iota} \mathscr{A}_{\mathcal{X}}^{p,0} \xrightarrow{\overline{\partial}} \mathscr{A}_{\mathcal{X}}^{p,1} \tag{4.2.8.2}$$

is also exact.

By **Theorem 4.5**, a C^∞ (p,q)-form φ, $q \geq 1$, on a polydisk is $\overline{\partial}$-closed if and only if there is a C^∞ $(p,q-1)$-form ψ such that $\varphi = \overline{\partial}\psi$. Consequently

$$\mathscr{A}^{p,q-1} \xrightarrow{\overline{\partial}} \mathscr{A}_{\mathcal{X}}^{p,q} \xrightarrow{\overline{\partial}} \mathscr{A}_{\mathcal{X}}^{p,q+1}, \quad q \geq 1 \tag{4.2.8.3}$$

is exact. Thus by (4.2.8.2) and (4.2.8.3), we obtain the exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathcal{X}}^p \xrightarrow{\iota} \mathscr{A}_{\mathcal{X}}^{p,0} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathscr{A}_{\mathcal{X}}^{p,n} \longrightarrow 0 \tag{4.2.8.4}$$

is exact. Since $\mathscr{A}_{\mathcal{X}}^{p,q}$ is fine, (4.2.8.4) gives a fine resolution of $\mathscr{O}_{\mathcal{X}}^p$ over $\mathcal{X}$. For $p = 0$ we obtain a fine resolution of $\mathscr{O}_{\mathcal{X}}$:

$$0 \longrightarrow \mathscr{O}_{\mathcal{X}} \xrightarrow{\iota} \mathscr{A}_{\mathcal{X}}^{0,0} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathscr{A}_{\mathcal{X}}^{0,n} \longrightarrow 0. \tag{4.2.8.5}$$

Applying **Theorem 4.5**, we obtain

Theorem 4.13

If $\mathcal{X}$ is a complex manifold, then

$$H^q(\mathcal{X}, \mathscr{O}_{\mathcal{X}}^p) \cong \frac{\Gamma(\mathcal{X}, \overline{\partial}\mathscr{A}_{\mathcal{X}}^{p,q-1})}{\overline{\partial}\Gamma(\mathcal{X}, \mathscr{A}_{\mathcal{X}}^{p,q-1})}, \quad q \geq 1. \tag{4.2.8.6}$$

♡

In (4.2.8.6), we see that $\overline{\partial}\mathscr{A}_{\mathcal{X}}^{p,q-1}$ is the sheaf of germs of $\overline{\partial}$-closed C^∞ (p,q)-forms on $\mathcal{X}$, $\Gamma(\mathcal{X}, \overline{\partial}\mathscr{A}_{\mathcal{X}}^{p,q-1})$ is the linear space of all $\overline{\partial}$-closed C^∞ (p,q)-forms on $\mathcal{X}$, and $\overline{\partial}\Gamma(\mathcal{X}, \mathscr{A}_{\mathcal{X}}^{p,q-1})$ its subspace. The quotient space

$$H_{\overline{\partial}}^{p,q}(\mathcal{X}) := \frac{\Gamma(\mathcal{X}, \overline{\partial}\mathscr{A}_{\mathcal{X}}^{p,q-1})}{\overline{\partial}\Gamma(\mathcal{X}, \mathscr{A}_{\mathcal{X}}^{p,q-1})} \tag{4.2.8.7}$$

is called the **$\overline{\partial}$-cohomology group of $\mathcal{X}$**. Thus Dolbeault's theorem says that

$$H^q(\mathcal{X}, \mathscr{O}_{\mathcal{X}}^p) \cong H_{\overline{\partial}}^{p,q}(\mathcal{X}). \tag{4.2.8.8}$$

In particular for $p = 0$, (4.2.8.8) reads

$$H^q(\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \cong H_{\overline{\partial}}^{0,q}(\mathcal{X}). \tag{4.2.8.9}$$

Dolbeault's theorem can be extended for the sheaf $\mathscr{O}_{\mathcal{X}}(\mathcal{F})$ of germs of holomorphic sections of $\mathcal{F}$ over $\mathcal{X}$, where $\mathcal{F}$ is a holomorphic vector bundle over $\mathcal{X}$. Recall that

$$\mathscr{A}_{\mathcal{X}}^{p,q} = \mathscr{A}\left(\wedge^p \mathbf{T}^*(\mathcal{X}) \otimes \wedge^q \overline{\mathbf{T}}^*(\mathcal{X})\right).$$

Similarly, we define $\mathscr{A}_{\mathcal{X}}^{0,q}(\mathcal{F})$ by

$$\mathscr{A}_{\mathcal{X}}^{0,q}(\mathcal{F}) := \mathscr{A}\left(\mathcal{F} \otimes \wedge^q \overline{\mathbf{T}}^*(\mathcal{X})\right), \tag{4.2.8.10}$$

where $\mathscr{A}_{\mathcal{X}}^{0,0}(\mathcal{F}) = \mathscr{A}_{\mathcal{X}}(\mathcal{F})$.

We shall show that $\mathscr{O}_{\mathcal{X}}(\mathcal{F})$ has a fine resolution

$$0 \longrightarrow \mathscr{O}_{\mathcal{X}}(\mathcal{F}) \xrightarrow{\ \iota\ } \mathscr{A}_{\mathcal{X}}^{0,0}(\mathcal{F}) \xrightarrow{\ \overline{\partial}\ } \cdots \xrightarrow{\ \overline{\partial}\ } \mathscr{A}_{\mathcal{X}}^{0,n}(\mathcal{F}) \longrightarrow 0. \tag{4.2.8.11}$$

First we investigate the structure of $\mathscr{A}_{\mathcal{X}}^{0,q}(\mathcal{F})$ and define $\overline{\partial}$.

Let $\{e_{j1}, \cdots, e_{j\nu}\}$ be a basis of $\mathcal{F}$ over $\mathcal{U}_j$ where $\mathfrak{U} = \{\mathcal{U}_j\}$ is a suitable locally finite open covering of $\mathcal{X}$. We denote a point of $\mathcal{X}$ by z and $(z_j^1, \cdots, z_j^n)$ the local complex coordinates on $\mathcal{U}_j$. Then each $e_{j\lambda} : z \mapsto e_{j\lambda}(z)$ is a holomorphic section of $\mathcal{F}$ over $\mathcal{U}_j$, and $\{e_{j1}(z), \cdots, e_{j\nu}(z)\}$ form a basis of $\mathcal{F}_z$ for $z \in \mathcal{U}_j$. Let $\boldsymbol{f}_{jk}(z) = (f_{jk\mu}^{\lambda}(z))_{1\leq\lambda,\mu\leq\nu}$ be the transition function of $\mathcal{F}$. Then on $\mathcal{U}_j \cap \mathcal{U}_k \neq \emptyset$,

$$e_{k\mu}(z) = \sum_{1\leq\lambda\leq\nu} f_{jk\mu}^{\lambda}(z) e_{j\lambda}(z), \tag{4.2.8.12}$$

where $f_{jk\mu}^{\lambda}(z)$ is holomorphic. Assume each $\mathcal{U}_j$ is a coordinate neighborhood. So $\nu\binom{n}{q}$ sections

$$e_{j\lambda} \otimes dz_j^{\overline{\alpha_1}} \wedge \cdots \wedge dz_j^{\overline{\alpha_q}}, \quad \lambda = 1, \cdots, \nu, \quad 1 \leq \alpha_1, \cdots, \alpha_q \leq n,$$

form a basis of $\mathcal{F} \otimes \wedge^q \overline{T}^*(\mathcal{X})$ over $\mathcal{U}_j$. Consequently a C^∞-section φ of $\mathcal{F} \otimes \wedge^q \overline{T}^*(\mathcal{X})$ over an open subset $\mathcal{W} \subset \mathcal{X}$ is given on each $\mathcal{U}_j \cap \mathcal{W} \neq \emptyset$ by

$$\varphi = \sum_{1\leq\lambda\leq\nu} \frac{1}{q!} \sum_{1\leq\alpha_1,\cdots,\alpha_q\leq n} \varphi_{j\overline{\alpha_1}\cdots\overline{\alpha_q}}^{\lambda}(z) e_{j\lambda}(z) \otimes dz_j^{\overline{\alpha_1}} \wedge \cdots \wedge dz_j^{\overline{\alpha_q}},$$

where $\varphi_{j\overline{\alpha_1}\cdots\overline{\alpha_q}}^{\lambda}(z)$ is a C^∞-function on $\mathcal{W} \cap \mathcal{U}_j$. Putting

$$\varphi_j^{\lambda}(z) := \frac{1}{q!} \sum_{1\leq\alpha_1,\cdots,\alpha_q\leq n} \varphi_{j\overline{\alpha_1}\cdots\overline{\alpha_q}}^{\lambda}(z) dz_j^{\overline{\alpha_1}} \wedge \cdots \wedge dz_j^{\overline{\alpha_q}},$$

we can write φ as

$$\varphi = \sum_{1\leq\lambda\leq\nu} e_{j\lambda}(z) \otimes \varphi_j^{\lambda}(z) = \sum_{1\leq\lambda\leq\nu} e_{j\lambda} \otimes \varphi_j^{\lambda},$$

where φ_j^{λ} is a C^∞ $(0, q)$-form on $\mathcal{W} \cap \mathcal{U}_j$. On $\mathcal{W} \cap \mathcal{U}_j \cap \mathcal{U}_k \neq \emptyset$,

$$\varphi = \sum_{1\leq\lambda\leq\nu} e_{j\lambda} \otimes \varphi_j^{\lambda} = \sum_{1\leq\lambda\leq\nu} e_{k\mu} \otimes \varphi_k^{\mu}$$

hence

$$\varphi_j^{\lambda}(z) = \sum_{1\leq\mu\leq\nu} f_{jk\mu}^{\lambda}(z) \varphi_k^{\mu}(z). \tag{4.2.8.13}$$

If we denote $\sum_{1\leq\lambda\leq\nu} e_{j\lambda} \otimes \varphi_j^{\lambda}$ by the vector $(\varphi_j^1, \cdots, \varphi_j^{\nu})$, whose components are C^∞ $(0, q)$-forms, φ is represented on each $\mathcal{W} \cap \mathcal{U}_j$ by the vector $\varphi = \left(\varphi_j^1, \cdots, \varphi_j^{\nu}\right)$, which satisfies (4.2.8.13) on $\mathcal{W} \cap \mathcal{U}_j \cap \mathcal{U}_k$. Conversely, suppose given on each $\mathcal{W} \cap \mathcal{U}_j \neq \emptyset$ a vector $(\varphi_j^1, \cdots, \varphi_j^{\nu})$ with φ_j^{λ} a C^∞ $(0, q)$-form, satisfying (4.2.8.13) on $\mathcal{W} \cap \mathcal{U}_j \cap \mathcal{U}_k$. Then there is a C^∞ section φ of $\mathcal{F} \otimes \wedge^q \overline{\mathbf{T}}^*(\mathcal{X})$ over $\mathcal{W}$ such that $\varphi = \sum_{1\leq\lambda\leq\nu} e_{j\lambda} \otimes \varphi_j^{\lambda}$ on $\mathcal{W} \cap \mathcal{U}_j \neq \emptyset$. We call a

C^∞-section $\varphi = (\varphi_j^1, \cdots, \varphi_j^\nu)$ of $\mathcal{F} \otimes \wedge^q \overline{\mathbf{T}}^*(\mathcal{X})$ a **C^∞ $(0,q)$-form with coefficients in $\mathcal{F}$**, and $\mathscr{A}^{0,q}_{\mathcal{X}}(\mathcal{F})$ the **sheaf of germs of C^∞ $(0,q)$-forms with coefficients in $\mathcal{F}$**. For $q = 0$, each φ_j^λ is a C^∞-function. For a C^∞ $(0,q)$-form $\varphi = (\varphi_j^1, \cdots, \varphi_j^\nu)$ with coefficients in $\mathcal{F}$, we define

$$\overline{\partial}\varphi := \left(\overline{\partial}\varphi_j^1, \cdots, \overline{\partial}\varphi_j^\nu\right). \tag{4.2.8.14}$$

Since $\overline{\partial} f^\lambda_{jk\mu}(z) = 0$, applying $\overline{\partial}$ to (4.2.8.14) we obtain

$$\overline{\partial}\varphi_j^\lambda = \sum_{1 \le \mu \le \nu} f^\lambda_{jk\mu}(z)\overline{\partial}\varphi_k^\mu.$$

Therefore $\overline{\partial}\varphi$ is a C^∞ $(0, q+1)$-form with coefficients in $\mathcal{F}$. Thus $\overline{\partial}$ gives a homomorphism

$$\overline{\partial} : \mathscr{A}^{0,q}_{\mathcal{X}}(\mathcal{F}) \longrightarrow \mathscr{A}^{0,q+1}_{\mathcal{X}}(\mathcal{F}).$$

Lemma 4.6

$\mathscr{O}_{\mathcal{X}}(\mathcal{F})$ has a fine resolution (4.2.8.11). ♡

Proof. Since $\mathscr{A}^{0,q}_{\mathcal{X}}(\mathcal{F})$ is fine, it suffices to prove the exactness of (4.2.8.11). For this purpose we have only to prove that at each point $z \in \mathcal{X}$, the sequence of the stalks

$$0 \longrightarrow \mathscr{O}_{\mathcal{X}}(\mathcal{F})_z \xrightarrow{\ \iota\ } \mathscr{A}^{0,0}_{\mathcal{X}}(\mathcal{F})_z \xrightarrow{\ \overline{\partial}\ } \mathscr{A}^{0,1}_{\mathcal{X}}(\mathcal{F})_z \xrightarrow{\ \overline{\partial}\ } \cdots$$

is exact. Thus we have only to prove the exactness of (4.2.8.11) on each $\mathcal{U}_j$. But on $\mathcal{U}_j$, a local C^∞ $(0,q)$-form $\varphi = (\varphi_j^1, \cdots, \varphi_j^\nu)$ with coefficients in $\mathcal{F}$ may be regarded as a ν-tuple of C^∞ $(0,q)$-forms φ_j^λ, $\lambda = 1, \cdots, \nu$. Moreover $\overline{\partial}\varphi = (\overline{\partial}\varphi_j^1, \cdots, \overline{\partial}\varphi_j^\nu)$. Therefore the exactness of (4.2.8.11) follows from the exactness of (4.2.8.5). □

Theorem 4.14

If $\mathcal{F}$ is a holomorphic vector bundle over a complex manifold $\mathcal{X}$, then

$$H^q(\mathcal{X}, \mathscr{O}_{\mathcal{X}}(\mathcal{F})) \cong \frac{\Gamma(\mathcal{X}, \overline{\partial}\mathscr{A}^{0,q-1}_{\mathcal{X}}(\mathcal{F}))}{\overline{\partial}\Gamma(\mathcal{X}, \mathscr{A}^{0,q-1}_{\mathcal{X}}(\mathcal{F}))}, \quad q \ge 1. \tag{4.2.8.15}$$

♡

Let φ be a C^∞ section of $\mathcal{F} \otimes \wedge^p \mathbf{T}^*(\mathcal{X}) \wedge^q \overline{\mathbf{T}}^*(\mathcal{X})$ over an open set $\mathcal{W} \subset \mathcal{X}$. Then φ is represented on each $\mathcal{W} \cap \mathcal{U}_j \ne \emptyset$ by a vector $\varphi = (\varphi_j^1, \cdots, \varphi_j^\nu)$ where each φ_j^λ is a C^∞ (p,q)-form, and its transition relation on $\mathcal{W} \cap \mathcal{U}_j \cap \mathcal{U}_k \ne \emptyset$ is given by (4.2.8.13). We call φ a **C^∞ (p,q)-form with coefficients in $\mathcal{F}$**. We denote by $\mathscr{A}^{p,q}_{\mathcal{X}}(\mathcal{F})$ the sheaf of germs of C^∞ (p,q)-forms with coefficients in $\mathcal{F}$:

$$\mathscr{A}^{p,q}_{\mathcal{X}}(\mathcal{F}) := \mathscr{A}_{\mathcal{X}}\left(\mathcal{F} \otimes \wedge^p T^*(\mathcal{X}) \wedge^q \overline{T}^*(\mathcal{X})\right). \tag{4.2.8.16}$$

If each φ_j^λ of $\varphi = (\varphi_j^1, \cdots, \varphi_j^\nu)$ is a holomorphic p-form, we call φ a **holomorphic p-form with coefficients in $\mathcal{F}$**. We denote by $\mathscr{O}^p_{\mathcal{X}}(\mathcal{F})$ the sheaf of germs of holomorphic p-forms with coefficients in $\mathcal{F}$:

$$\mathscr{O}^p_{\mathcal{X}}(\mathcal{F}) := \mathscr{O}_{\mathcal{X}}\left(\mathcal{F} \otimes \wedge^p T^*(\mathcal{X})\right). \tag{4.2.8.17}$$

From (4.2.8.15) we have

$$H^q(\mathcal{X}, \mathscr{O}^p_{\mathcal{X}}(\mathcal{F})) \cong \frac{\Gamma(\mathcal{X}, \overline{\partial}\mathscr{A}^{p,q-1}_{\mathcal{X}}(\mathcal{F}))}{\overline{\partial}\Gamma(\mathcal{X}, \mathscr{A}^{p,q-1}_{\mathcal{X}}(\mathcal{F}))}, \quad q \ge 1, \tag{4.2.8.18}$$

where $\Gamma(\mathcal{X}, \overline{\partial}\mathscr{A}_{\mathcal{X}}^{p,q-1}(\mathcal{F}))$ is the linear space of all $\overline{\partial}$-closed C^∞ (p,q)-forms with coefficients in $\mathcal{F}$ over $\mathcal{X}$. The quotient space

$$H_{\overline{\partial}}^{p,q}(\mathcal{X}, \mathcal{F}) := \frac{\Gamma(\mathcal{X}, \overline{\partial}\mathscr{A}_{\mathcal{X}}^{p,q-1}(\mathcal{F}))}{\overline{\partial}\Gamma(\mathcal{X}, \mathscr{A}_{\mathcal{X}}^{p,q-1}(\mathcal{F}))} \tag{4.2.8.19}$$

is called the **$\overline{\partial}$-cohomology group of $\mathcal{X}$ with coefficients in $\mathcal{F}$**. Consequently,

$$H^q(\mathcal{X}, \mathscr{O}_{\mathcal{X}}^p(\mathcal{F})) \cong H_{\overline{\partial}}^{p,q}(\mathcal{X}, \mathcal{F}). \tag{4.2.8.20}$$

4.3 Sheaves: algebraic aspect

Introduction

- *Presheaves*
- *Sheaves*
- *Stalks and germs*
- *Morphisms*
- *Direct images and inverse images*
- *The additive functor $f^{-1}f_*$*
- *The additive functor f_*f^{-1}*
- *Maps $\alpha_{\mathcal{F},\mathcal{G}}$ and $\beta_{\mathcal{F},\mathcal{G}}$*
- *Left adjoint and right adjoint*
- $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$
- $(f^{-1}\mathcal{G})_P = \mathcal{G}_{f(P)}$
- *The five lemma*

4.3.1 Presheaves

Definition 4.11

A **presheaf** *$\mathcal{F}$ of sets on a topological space X consists of*

(a) For every nonempty open subset U of X, we have a set $\mathcal{F}(U)$ whose elements are called **sections** *of $\mathcal{F}$ over U.*

(b) For every inclusion $V \subset U$ of nonempty open subsets of X, we have a map $\rho_{U,V}^{\mathcal{F}} : \mathcal{F}(U) \to \mathcal{F}(V)$, called the **restriction**.

These data satisfy

(i) $\rho_{U,U}^{\mathcal{F}} = \mathrm{id}_{\mathcal{F}(U)}$.

(ii) If $W \subset V \subset U$ are open subsets of X, then $\rho_{U,W}^{\mathcal{F}} = \rho_{V,W}^{\mathcal{F}} \circ \rho_{U,V}^{\mathcal{F}}$.

For any section $s \in \mathcal{F}(U)$ and $V \subset U$, we denote $\rho_{U,V}^{\mathcal{F}}(d)$ by $s|_V$. Elements in $\mathcal{F}(U)$ are denoted by (s, U). We often make the convention that $\mathcal{F}(\emptyset) = \{0\}$ for any presheaf of sets $\mathcal{F}$. Similarly, we define a presheaf $\mathcal{F}$ of Abelian groups (resp. rings) on X. ♣

Example 4.2

(1) Let X be a topological space and A an Abelian group. For every nonempty open subset U of X, define $\mathcal{F}(U) = A$, and for every inclusion $V \subset U$ of nonempty open subsets, define $\rho_{U,V}^{\mathcal{F}} = \mathrm{id}_A$. Then $\mathcal{F}$ is a presheaf of Abelian groups, called the **constant presheaf associated to** *A, and denoted by $\underline{A}_X$.*

(2) Let X be a topological space. For every open subset U of X, define $\mathcal{C}(U)$ to be the

ring of complex-valued continuous functions on U, and for every inclusion $V \subset U$ of nonempty open subsets, define $\rho^{\mathcal{C}}_{U,V} : \mathcal{C}(U) \to \mathcal{C}(V)$ to be the restriction of functions. Then $\mathcal{C}$ is a presheaf of rings.

(3) Let $\pi : Y \to X$ be a continuous map of topological spaces. For every nonempty open subset U of X, define $\mathcal{S}(U)$ to be the set of continuous sections of π over U:

$$\mathcal{S}(U) = \{s : U \to \pi^{-1}(U) | \pi \circ s = \mathrm{id}_U \text{ and } s \text{ is continuous}\},$$

and for every inclusion $V \subset U$ of nonempty open subsets, define $\rho^{\mathcal{S}}_{U,V} : \mathcal{S}(U) \to \mathcal{S}(V)$ to be the restriction of sections. Then $\mathcal{S}$ is a presheaf of sets. ♠

4.3.2 Sheaves

A presheaf $\mathcal{F}$ of sets(resp. Abelian groups, resp. rings) is a **sheaf** if it satisfies

(iii) Let $s, t \in \mathcal{F}(U)$ be two sections. If there exists an open covering $\{U_i\}_{i \in I}$ of U such that $s|_{U_i} = t|_{U_i}$ for any i, then $s = t$.

(iv) Suppose that $\{U_i\}_{i \in I}$ is an open covering of U and $s_i \in \mathcal{F}(U_i)$ are some sections satisfying $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for any $i, j \in I$. Then there exists a section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for any $i \in I$.

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\rho^{\mathcal{F}}_{U,U_i}"] \arrow[d, "\rho^{\mathcal{F}}_{U,U_i}"'] & \mathcal{F}(U_i) \arrow[d, "\rho^{\mathcal{F}}_{U_i,U_i \cap U_j}"] \\
\mathcal{F}(U_i) \arrow[r, "\rho^{\mathcal{F}}_{U_j,U_i \cap U_j}"'] & \mathcal{F}(U_i \cap U_j)
\end{tikzcd}$$

Note 4.4

(1) By (iii), such s appeared in (iv) is unique.

(2) A presheaf $\mathcal{F}$ of Abelian groups is a sheaf if and only if for any open covering $\{U_i\}_{i \in I}$ of any open subset U, the sequence

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\alpha} \prod_{i \in I} \mathcal{F}(U_i) \xrightarrow{\beta} \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j) \tag{4.3.2.1}$$

is exact, where

$$\alpha(s) = \prod_{i \in I} (s|_{U_i}), \tag{4.3.2.2}$$

$$\beta\left(\prod_{i \in I} s_i\right) = \prod_{i,j \in I} (s_j|_{U_i \cap U_j} - s_i|_{U_i \cap U_j}). \tag{4.3.2.3}$$

♣

4.3.3 Stalks and germs

Let X be a topological space and P a point in X. For any two neighborhoods U and V of P, we say $V \leq U$ if $U \subset V$. Let N_P be the set of all neighborhoods of P. Then $(N_P, \leq)$ is a direct set. For any presheaf $\mathcal{F}$ on X, define the **stalk** $\mathcal{F}_P$ of $\mathcal{F}$ at P by

$$\mathcal{F}_P := \varinjlim_{P \in U} \mathcal{F}(U) = \{[(s,U)] | s \in \mathcal{F}(U),\ P \in U \subset N_P\}. \tag{4.3.3.1}$$

Two sections $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ define the same element in $\mathcal{F}_P$ if and only if there exists a neighborhood W of P such that $W \subset U \cap V$ and $s|_W = t|_W$. For any neighborhood U of P, we have a canonical map

$$\mathcal{F}(U) \longrightarrow \mathcal{F}_P, \quad s \longmapsto [(s,U)] := s_P. \tag{4.3.3.2}$$

The image of a section $s \in \mathcal{F}(U)$ in $\mathcal{F}_P$ is called the **germ** of s at P.

4.3.4 Morphisms

Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves of Abelian groups on X. A **morphism of presheaves** $\phi : \mathcal{F} \to \mathcal{G}$ consists of a homomorphism of Abelian groups $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for every open subset U such that for every inclusion $V \subset U$ of open subsets, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\phi(U)"] \arrow[d, "\rho^{\mathcal{F}}_{U,V}"'] & \mathcal{G}(U) \arrow[d, "\rho^{\mathcal{G}}_{U,V}"] \\
\mathcal{F}(V) \arrow[r, "\phi(V)"'] & \mathcal{G}(V).
\end{tikzcd}$$

For any point $P \in X$, ϕ induces a homomorphism on stalk

$$\phi_P : \mathcal{F}_P \longrightarrow \mathcal{G}_P, \quad [(s,U)] \longmapsto [(\phi(U)(s), U)]. \tag{4.3.4.1}$$

Note that ϕ_P is well-defined: If $(s,U) \sim (t,V)$, then there exists an open neighborhood $W \subset U \cap V$ such that $s|_W = t|_W$,

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\rho^{\mathcal{F}}_{U,W}"] \arrow[d, "\phi(U)"] & \mathcal{F}(W) \arrow[d, "\phi(W)"] & \mathcal{F}(V) \arrow[l, "\rho^{\mathcal{F}}_{V,W}"'] \arrow[d, "\phi(V)"] \\
\mathcal{G}(U) \arrow[r, "\rho^{\mathcal{G}}_{U,W}"] & \mathcal{G}(W) & \mathcal{G}(V). \arrow[l, "\rho^{\mathcal{G}}_{V,W}"']
\end{tikzcd}$$

one has

$$(\phi(U)(s))|_W = \phi(W)(s|_W) = \phi(W)(t|_W) = (\phi(V)(t))|_W.$$

For any presheaf, we have the identity morphism $\mathrm{id}_{\mathcal{F}}$. Given two morphisms of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{G} \to \mathcal{H}$, we can define their composite $\psi \circ \phi : \mathcal{F} \to \mathcal{H}$. A morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ is called an **isomorphism** if it has a two-sided inverse, that is, there exists a morphism of presheaves $\psi : \mathcal{G} \to \mathcal{F}$ such that $\psi \circ \phi = \mathrm{id}_{\mathcal{F}}$ and $\phi \circ \psi = \mathrm{id}_{\mathcal{G}}$. This is equivalent to saying that $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism for every open subset U. We define morphisms of sheaves as morphisms of presheaves.

Proposition 4.3

Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on a topological space X. Then ϕ is an isomorphism if and only if the induced map on stalks $\phi_P : \mathcal{F}_P \to \mathcal{G}_P$ is an isomorphism *for every $P \in X$.* ♡

Proof. (1) For any $P \in X$, there exists an open subset U of P such that $U \subset X$. One has

$$\begin{aligned}\psi_P \circ \phi_P([(s, U)]) &= \psi_P([(\phi(U)(s), U)]) \\ &= [(\psi(U)(\phi(U)(s)), U)] = [(\mathrm{id}_{\mathcal{F}}(U)(s), U)] = [(s, U)];\end{aligned}$$

similarly, $\phi_P \circ \psi_P = \mathrm{id}_{\mathcal{G}_P}$.

(2) Suppose that $\phi_P : \mathcal{F}_P \to \mathcal{G}_P$ is bijective for each $P \in X$.

(2.1): $\phi(U)$ is injective.

$$\begin{array}{ccc}\mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_P & \xrightarrow[\phi_P]{} & \mathcal{G}_P\end{array}$$

Let $s, s' \in \mathcal{F}(U)$ such that $\phi(U)(s) = \phi(U)(s')$. Then

$$\phi_P(s_P) = \phi_P([(s, U)]) = [(\phi(U)(s), U)] = [(\phi(U)(s'), U)] = \phi_P([(s', U)]) = \phi_P(s'_P).$$

Since ϕ_P is injective, we have $s_P = s'_P$ and hence there exists an open subset $P \in U_P \subset U$ such that $s|_{U_P} = s'|_{U_P}$. Therefore $s = s'$ since $\{U_P\}_{P \in U}$ is an open covering of U and $\mathcal{F}$ is a sheaf.

(2.2): $\phi(U)$ is surjective. For any $t \in \mathcal{G}(U)$ and any $P \in U$, one has $\phi_P(s_P) = t_P$ for some $s_P \in \mathcal{F}_P$. Write $s_P = [(s, V_P)]$ with $s \in \mathcal{F}(V_P)$. Since

$$[(t, U)] = t_P = \phi_P(s_P) = \phi_P([(s, V_P)]) = [(\phi(V_P)(s), V_p)]$$

we must have $U_P \subset U$ so that $\phi(V_p)(s)|_{U_P} = t|_{U_P}$, i.e.,

$$\phi(U_P)(s|_{U_P}) = t|_{U_P}.$$

Now we consider two points $P, Q \in U$:

$$\phi(U_P)(s|_{U_P}) = t|_{U_P}, \quad \phi(U_Q)(s|_{U_Q}) = t|_{U_Q}.$$

The commutative diagram

$$\begin{array}{ccc}\mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(U_P) & \xrightarrow{\phi(U_P)} & \mathcal{G}(U_P) \\ \downarrow & & \downarrow \\ \mathcal{F}(U_P \cap U_Q) & \xrightarrow{\phi(U_P \cap U_Q)} & \mathcal{G}(U_P \cap U_Q)\end{array}$$

says that

$$\phi(U_P \cap U_Q)((s|_{U_P})|_{U_P \cap U_Q}) = (t|_{U_P})|_{U_P \cap U_Q} = \phi(U_P \cap U_Q)((s|_{U_Q})|_{U_P \cap U_Q}).$$

By the injectivity of $\phi(U_P \cap U_Q)$ we have

$$(s|_{U_P})|_{U_P \cap U_Q} = (s|_{U_Q})|_{U_P \cap U_Q}.$$

Since $\{U_P\}_{P\in U}$ is an open covering of U, it follows that there is an element $s \in \mathcal{F}(U)$ such that

$$(s, U)|_{U_P} = (s, U_P).$$

Moreover, $\phi(U)(s)|_{U_P} = \phi(U_P)(s|_{U_P}) = t|_{U_P}$, hence $\phi(U)(s) = t$. $\square$

Proposition 4.4

Let X be a topological space. Then the category of presheaves of Abelian groups on X is an Abelian category. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves of Abelian groups. Then the kernel, cokernel and image of ϕ are the presheaves of Abelian groups defined by

$$\begin{aligned}
(\mathrm{Ker}^\circ(\phi))(U) &= \mathrm{Ker}(\phi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)),\\
(\mathrm{Coker}^\circ(\phi))(U) &= \mathrm{Coker}(\phi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)),\\
(\mathrm{Im}^\circ(\phi))(U) &= \mathrm{Im}(\phi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U))
\end{aligned}$$

for every open subset U of X. The stalks of these presheaves at a point $P \in X$ are

$$\begin{aligned}
(\mathrm{Ker}^\circ(\phi))_P &= \mathrm{Ker}(\phi_P : \mathcal{F}_P \longrightarrow \mathcal{G}_P),\\
(\mathrm{Coker}^\circ(\phi))_P &= \mathrm{Coker}(\phi_P : \mathcal{F}_P \longrightarrow \mathcal{G}_P),\\
(\mathrm{Im}^\circ(\phi))_P &= \mathrm{Im}(\phi_P : \mathcal{F}_P \longrightarrow \mathcal{G}_P).
\end{aligned}$$

♡

Proof. Define the direct product $\mathcal{F} \times \mathcal{G}$ by

$$(\mathcal{F} \times \mathcal{G})(U) := \mathcal{F}(U) \times \mathcal{G}(U)$$

that is an Abelian group. If $V \subset U$ are two open subsets, we define

$$\rho_{U,V}^{\mathcal{F}\times\mathcal{G}} := \rho_{U,V}^{\mathcal{F}} \circ \pi_1 + \rho_{U,V}^{\mathcal{G}} \circ \pi_2 : (\mathcal{F} \times \mathcal{G})(U) \longrightarrow (\mathcal{F} \times \mathcal{G})(V)$$

where π_1 and π_2 are the natural projections

$$\begin{tikzcd}
(\mathcal{F} \times \mathcal{G})(U) \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & \mathcal{F}(U) \\
\mathcal{G}(U) &
\end{tikzcd}$$

For $(s_1, s_2) \in \mathcal{F}(U) \times \mathcal{G}(U)$, one has

$$\rho_{U,V}^{\mathcal{F}\times\mathcal{G}}(s_1, s_2) = (\rho_{U,V}^{\mathcal{F}}(s_1), \rho_{U,V}^{\mathcal{G}}(s_2)).$$

Hence $\mathcal{F} \times \mathcal{G}$ is a presheaf of Abelian groups on X.

For the morphism $\phi : \mathcal{F} \to \mathcal{G}$, we shall show that $\mathrm{Ker}^\circ(\phi)$ and $\mathrm{Coker}^\circ(\phi)$ exist and $\mathrm{Coim}^\circ(\phi) \to \mathrm{Im}^\circ(\phi)$ is isomorphic. For a pair $V \subset U$ of open subsets, we define

$$\rho_{U,V}^{\mathrm{Ker}^\circ(\phi)} := \rho_{U,V}^{\mathcal{F}} : (\mathrm{Ker}^\circ(\phi))(U) \longrightarrow (\mathrm{Ker}^\circ(\phi))(V).$$

For $s \in (\mathrm{Ker}^\circ(\phi))(U)$, we have

$$\phi(V)(\rho_{U,V}^{\mathcal{F}}(s)) = \rho_{U,V}^{\mathcal{G}}(\phi(U)(s)) = \rho_{U,V}^{\mathcal{G}}(0_{\mathcal{G}(U)}) = 0_{\mathcal{F}(U)}.$$

Thus $\mathrm{Ker}^\circ(\phi)$ is a presheaf of Abelian group on X. Similarly, $\mathrm{Coim}^\circ(\phi)$ is a presheaf. Moreover,

$$(\mathrm{Coim}^\circ(\phi))(U) = \mathrm{Coim}(\phi(U)) = \mathcal{F}(U)/\mathrm{Ker}(\phi(U)) = \mathrm{Im}(\phi(U)) = (\mathrm{Im}^\circ(\phi))(U).$$

So $\mathrm{Coim}^\circ(\phi) \to \mathrm{Im}^\circ(\phi)$ is isomorphic. For any point $P \in X$,

$$(\mathrm{Ker}^\circ(\phi))_P = \varinjlim_{P\in U} (\mathrm{Ker}^\circ(\phi))(U) = \varinjlim_{P\in U} \mathrm{Ker}(\phi(U)) = \mathrm{Ker}(\varinjlim_{P\in U} \phi(U)) = \mathrm{Ker}(\phi_P).$$

Others can be proved similarly. □

Proposition 4.5

Let $\mathcal{F}$ be a presheaf on X. There exists a pair $(\mathcal{F}^+, \theta)$ consisting of a sheaf $\mathcal{F}^+$ and a morphism $\theta : \mathcal{F} \to \mathcal{F}^+$ of presheaves such that for any sheaf $\mathcal{G}$ and any morphism $\phi : \mathcal{F} \to \mathcal{G}$ of sheaves, there is a unique morphism ψ

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\theta"] \arrow[dr, "\phi"'] & \mathcal{F}^+ \arrow[d, "\exists!\psi"] \\ & \mathcal{G} \end{tikzcd}$$

of sheaves satisfying $\phi = \psi \circ \theta$. The pair $(\mathcal{F}^+, \theta)$ is unique up to unique isomorphism. For any point $P \in X$, $\theta_P : \mathcal{F}_P \to (\mathcal{F}^+)_P$ is an isomorphism. We call $\mathcal{F}^+$ the **sheaf associated to the presheaf** $\mathcal{F}$. ♡

Proof. For any open subset U of X, define $\mathcal{F}^+(U)$ to be the set of functions $s^+ : U \mapsto \coprod_{P\in X} \mathcal{F}_P$ satisfying the following two conditions:

(a) For any $P \in U$, we have $s^+(P) \in \mathcal{F}_P$.

(b) For any $P \in U$, there exists a neighborhood U_P of P contained in U and a section $t^P \in \mathcal{F}(U_P)$ such that $s^+(Q) = t^P_Q$ for any $Q \in U_P$.

By definition we can check that $\mathcal{F}^+$ is indeed a sheaf: For open subsets $V \subset U$, we define

$$\rho^{\mathcal{F}^+}_{U,V} : \mathcal{F}^+(U) \longrightarrow \mathcal{F}^+(V), \quad s \longmapsto \rho^{\mathcal{F}^+}_{U,V}(s) := s|_V.$$

Then $\rho^{\mathcal{F}^+}$ defines a presheaf. The extra conditions for sheaves are easily satisfied. The canonical morphism

$$\theta : \mathcal{F} \longrightarrow \mathcal{F}^+$$

is given by

$$\theta(U) : \mathcal{F}(U) \longrightarrow \mathcal{F}^+(U), \quad s \longmapsto s^+ : U \longrightarrow \coprod_{P\in X} \mathcal{F}_P, \quad Q \longmapsto s_Q.$$

For open subsets $V \subset U$,

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\theta(U)"] \arrow[d, "\rho^{\mathcal{F}}_{U,V}"'] & \mathcal{F}^+(U) \arrow[d, "\rho^{\mathcal{F}^+}_{U,V}"] \\ \mathcal{F}(V) \arrow[r, "\theta(V)"'] & \mathcal{F}^+(V) \end{tikzcd}$$

and every element $s \in \mathcal{F}(U)$, one has

$$\rho^{\mathcal{F}^+}_{U,V} \circ \theta(U)(s) = \rho^{\mathcal{F}^+}_{U,V}(s^+) = s^+|_V = (s|_V)^+ = \theta(V)(s|_V) = \theta(V) \circ \rho^{\mathcal{F}}_{U,V}(s).$$

This completes the proof. □

Note 4.5

(1) Sections in $\mathcal{F}^+$ locally come from sections of $\mathcal{F}$: For any section $s^+ \in \mathcal{F}^+(U)$ any point $P \in U$, there exist an open subset $U_P \subset U$ and a section $t^P \in \mathcal{F}(U_P)$ such that

$$s^+(Q) = (t^P)_Q$$

for any point $Q \in U_P$. So

$$\theta(U_P)(t^P) = s^+|_{U_P}.$$

(2) Two sections of $\mathcal{F}$ which are locally equal are identified in $\mathcal{F}^+$: For $s, t \in \mathcal{F}(U)$, then there is an open covering $\{U_P\}_{P\in U}$ of U satisfying $s|_{U_P} = t|_{U_P}$ for any $P \in U$, if and only if $\theta(U)(s) = \theta(U)(t)$. Indeed, $\theta(U)(s) = \theta(U)(t)$ if and only if $s^+ = t^+$, i.e., $s_P = t_P$ for any $P \in U$; thus if and only if there exists an open covering $\{U_P\}_{P\in U}$ with $s|_U = t|_U$.

(3) If $\mathcal{F}$ is a sheaf, then $\theta : \mathcal{F} \to \mathcal{F}^+$ is an isomorphism of sheaves. ♣

Proposition 4.6

The category of sheaves of Abelian groups on X is an Abelian category. Let $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of Abelian groups. Then

(1) $\mathrm{Ker}(\phi) := \mathrm{Ker}^\circ(\phi)$ *is the sheaf.*

(2) $\mathrm{Coker}(\phi)$ *is the sheaf associated to the presheaf* $\mathrm{Coker}^\circ(\phi)$.

(3) $\mathrm{Im}(\phi)$ *is the sheaf associated to the presheaf* $\mathrm{Im}^\circ(\phi)$.

Moreover, for any $P \in X$, we have

$$(\mathrm{Ker}(\phi))_P = \mathrm{Ker}(\phi_P), \quad (\mathrm{Coker}(\phi))_P = \mathrm{Coker}(\phi_P), \quad (\mathrm{Im}(\phi))_P = \mathrm{Im}(\phi_P).$$ ♡

Proof. (1) is clearly. For (2) one considers

$$\begin{tikzcd}
 & & \mathrm{Coim}(\phi) \arrow[d] & \\
\mathrm{Ker}(\phi) \arrow[r] & \mathcal{F} \arrow[r, "\phi"] \arrow[d] & \mathcal{G} \arrow[r] & \mathrm{Coker}(\phi) \arrow[d] \\
 & \mathrm{Im}(\phi) \arrow[r] & (\mathrm{Im}(\phi))^+ \arrow[u, "\exists!"'] \arrow[r] & (\mathrm{Coker}(\phi))^+.
\end{tikzcd}$$

Since $\mathrm{Im}(\phi) \to \mathcal{G} \to \mathrm{Coker}(\phi)$ is zero, it follows that $\mathrm{Im}(\phi) \to \mathcal{G} \to (\mathrm{Coker}(\phi))^+$ is zero and $(\mathrm{Im}(\phi))^+ \to \mathcal{G} \to (\mathrm{Coker}(\phi))^+$ is also zero. By the universal property, there exist a unique morphism

$$(\mathrm{Im}(\phi))^+ \longrightarrow \mathrm{Ker}(\mathcal{G} \to (\mathrm{Coker}(\phi))^+).$$

For each $P \in X$, we have

$$\begin{aligned}
(\mathrm{Ker}(\mathcal{G} - (\mathrm{Coker}(\phi))^+))_P &= \mathrm{Ker}(\mathcal{G}_P \to (\mathrm{Coker}(\phi))_P) \\
= \mathrm{Ker}(\mathcal{G}_P \to \mathrm{Coker}(\phi_P)) &= \mathrm{Im}(\phi_P) = (\mathrm{Im}(\phi))_P = ((\mathrm{Im}(\phi))^+)_P
\end{aligned}$$

and hence $\mathrm{Im}(\phi)$ is the sheaf associated to the presheaf $\mathrm{Im}^\circ(\phi)$. □

Lemma 4.7

If $f : A \to B$ is a morphism in an Abelian category, then

(i) f is injective if and only if $\mathrm{Ker}(f) = 0$.

(ii) f is surjective if and only if $\mathrm{Coker}(f) = 0$.

(iii) f is surjective if and only if $\mathrm{Im}(f) = B$. ♡

Proof. Suppose first that f is injective. Consider the maps

$$\mathrm{Ker}(f) \xrightarrow{f_1} A \xrightarrow{f} B$$

where f_1 is injective. Choosing an arbitrary element $g \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{Ker}(f), \mathrm{Ker}(f))$ we have

$$f \circ f_1 \circ g = 0 \circ g = 0 = f \circ f_1;$$

hence $f_1 \circ g = f_1$. By the uniqueness, one must have $g = 0$; thus $\mathrm{Ker}(f) = 0$. Conversely,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ \mathrm{Coim}(f) \arrow[r, "\phi"] & \mathrm{Im}(f) \arrow[u] \end{tikzcd}$$

if $\mathrm{Ker}(f) = 0$, then, since $\mathrm{Coim}(f) \cong \mathrm{Im}(f)$, then $A \to \mathrm{Coim}(f)$ is isomorphic. Therefore $A \to B$ is injective. Similarly, we can prove the rest parts. □

Corollary 4.1

Suppose that X is a topological space.

(i) In the category of presheaves of Abelian groups on X, $\phi : \mathcal{F} \to \mathcal{G}$ is injective (resp., surjective, bijective) if and only if $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective (resp., surjective, bijective).

(ii) In the category of sheaves of Aelian groups on X, $\phi : \mathcal{F} \to \mathcal{G}$ is injective if and only if $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective.

(iii) In the category of sheaves of Abelian groups on X, $\phi : \mathcal{F} \to \mathcal{G}$ is injective (resp., surjective, bijective) if and only if $\phi_P : \mathcal{F}_P \to \mathcal{G}_P$ is injective (resp., surjective, bijective) for each point $P \in X$.

(iv) In the category of sheaves of Abelian groups on X, $\phi : \mathcal{F} \to \mathcal{G}$ is surjective if and only if for any open subset $U \subset X$ and any section $t \in \mathcal{G}(U)$ there exist an open covering $\{U_i\}_{i \in I}$ of U and sections $s_i \in \mathcal{F}(U_i)$ such that $\phi(U_i)(s_i) = t|_{U_i}$ for any $i \in I$. ♡

Proof. (i) $\phi : \mathcal{F} \to \mathcal{G}$ is injective if and only if $\mathrm{Ker}(\phi) = 0$; equivalently, if and only if $\mathrm{Ker}(\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)) = 0$ for any open subset $U \subset X$.

(ii) The same as in (i).

(iii) $\phi : \mathcal{F} \to \mathcal{G}$ is surjective if and only if $\mathrm{Im}(\phi) \cong \mathcal{G}$; thus if and only if $\mathrm{Im}(\phi_P)(\mathrm{Im}(\phi))_P \cong \mathcal{G}_P$

for each point $P \in X$.

(iv) $\phi : \mathcal{F} \to \mathcal{G}$ is surjective, then $(\mathrm{Im}(\phi))_P \cong \mathcal{G}_P$. Therefore one can find an open covering $\{U_P\}_{P \in U}$ such that $\phi(U_P)(t^P) = t|_{U_P}$ for some $t^P \in \mathcal{F}(U_P)$. □

Corollary 4.2

Let X be a topological space.

(i) In the category of presheaves of Abelian groups on X, a sequence

$$\mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$$

is exact if and only if

$$\mathcal{F}(U) \xrightarrow{\phi(U)} \mathcal{G}(U) \xrightarrow{\psi(U)} \mathcal{H}(U)$$

is exact for any open subset U of X.

(ii) In the category of sheaves of Abelian groups on X, a sequence

$$\mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$$

is exact if and only if

$$\mathcal{F}_P \xrightarrow{\phi_P} \mathcal{G}_P \xrightarrow{\psi_P} \mathcal{H}_P$$

is exact for all point $P \in X$. ♡

Proof. In the Abelian category, the word "bijective" is equivalent to the word "isomorphic". □

4.3.5 Direct images and inverse images

Let $f : X \to Y$ be a continuous map of topological spaces.

- For any sheaf $\mathcal{F}$ on X, the **direct image** $f_*\mathcal{F}$ of $\mathcal{F}$ is the sheaf on Y:

$$f_*\mathcal{F}(V) := \mathcal{F}(f^{-1}(V)) \tag{4.3.5.1}$$

  for an open subset $V \subset Y$.

- For any sheaf $\mathcal{G}$ on Y, the **inverse image** $f^{-1}\mathcal{G}$ of $\mathcal{G}$ is the sheaf on X:

$$f^{-1}\mathcal{G} := \left(U \mapsto \varinjlim_{f(U) \subset V} \mathcal{G}(V) \right)^+ . \tag{4.3.5.2}$$

- If f is an embedding map and $\mathcal{G}$ is a sheaf on Y, then we denote

$$\mathcal{G}|_X := f^{-1}\mathcal{G} \tag{4.3.5.3}$$

  the **restriction** of $\mathcal{G}$ on X.

Frequently, we use $\mathfrak{S}_{\mathbf{Ab}}(X)$ to denote the category of sheaves of Abelian groups on X.

Proposition 4.7

$f_*\mathcal{F}$ and $f^{-1}\mathcal{G}$ are indeed the sheaves. Moreover,

(i) $f_* : \mathfrak{S}_{\mathbf{Ab}}(X) \to \mathfrak{S}_{\mathbf{Ab}}(Y)$ is an additive functor.

(ii) $f^{-1} : \mathfrak{S}_{\mathbf{Ab}}(Y) \to \mathfrak{S}_{\mathbf{Ab}}(X)$ is an additive functor. ♡

Proof. $f_*\mathcal{F}$ is a sheaf. For open subsets $V \subset U$, we define

$$\rho^{f_*\mathcal{F}}_{U,V} := \rho^{\mathcal{F}}_{f^{-1}(U),f^{-1}(V)}. \tag{4.3.5.4}$$

Then $\rho^{f_*\mathcal{F}}_{U,U} = \rho^{\mathcal{F}}_{f^{-1}(U),f^{-1}(U)} = \mathrm{id}_{\mathcal{F}(f^{-1}(U))} = \mathrm{id}_{f_*\mathcal{F}(U)}$. For open subsets $W \subset V \subset U$, one has

$$\rho^{f_*\mathcal{F}}_{U,W} = \rho^{\mathcal{F}}_{f^{-1}(U),f^{-1}(W)} = \rho^{\mathcal{F}}_{f^{-1}(V),f^{-1}(W)} \circ \rho^{\mathcal{F}}_{f^{-1}(U),f^{-1}(V)} = \rho^{f_*\mathcal{F}}_{V,W} \circ \rho^{f_*\mathcal{F}}_{U,V}.$$

$f^{-1}\mathcal{G}$ is the sheaf. We denote by $\mathcal{F}$ the presheaf whose associated sheaf is $f^{-1}\mathcal{G}$. For an open subset $U \subset X$, we define

$$\mathcal{F}(U) := \varinjlim_{f(U)\subset V} \mathcal{G}(V) = \{[(t, U_t)] | t \in \mathcal{G}(U_t),\ f(U) \subset U_t\},$$

and

$$\rho^{\mathcal{F}}_{U,V} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V), \quad s = [(t, U_t)] \longmapsto [(\rho^{\mathcal{G}}_{U_t,f(V)}(t), f(V))]$$

where $V \subset U$ are open subsets such that $f(V) \subset f(U) \subset U_t$. If $[(t_1, U_{t_1})] = [(t_2, U_{t_2})] \in \mathcal{F}(U)$, then $t_1|_W = t_2|_W$ for some open subset W with $f(U) \subset W \subset U_{t_1} \cap U_{t_2}$. Hence $t_1|_{f(V)} = t_2|_{f(V)}$. It shows that $\rho^{\mathcal{F}}_{U,V}$ is well-defined. Since $f(U) \subset U_t$, it follows that

$$\rho^{\mathcal{F}}_{U,U}([(t, U_t)]) = [(\rho^{\mathcal{G}}_{U_t,f(U)}(t), f(U))] = [(t|_{f(U)}, f(U))] = [(t, U_t)].$$

For $W \subset V \subset U$ and $[(t, U_t)] \in \mathcal{F}(U)$ we obtain

$$\begin{aligned} \rho^{\mathcal{F}}_{V,W} \circ \rho^{\mathcal{F}}_{U,V}([(t, U_t)]) &= \rho^{\mathcal{F}}_{V,W}([\rho^{\mathcal{G}}_{U_t,f(V)}(t), f(V)]) \\ = [\rho^{\mathcal{G}}_{f(V),f(W)} \circ \rho^{\mathcal{G}}_{U_t,f(V)}(t), f(W)] &= [(\rho^{\mathcal{G}}_{U_t,f(W)}(t), f(W))] = \rho^{\mathcal{G}}_{U,W}([(t, U_t)]). \end{aligned}$$

f_* is an additive functor. For any set $\mathrm{Hom}_{\mathfrak{S}_{\mathbf{Ab}}(X)}(\mathcal{F}, \mathcal{G})$ we define

$$\mathrm{Hom}(f_*) : \mathrm{Hom}_{\mathfrak{S}_{\mathbf{Ab}}(X)}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Hom}_{\mathfrak{S}_{\mathbf{Ab}}(Y)}(f_*\mathcal{F}, f_*\mathcal{G}), \quad \phi \longmapsto f_*\phi \tag{4.3.5.5}$$

where $f_*\phi(U) = \phi(f^{-1}(U))$. For $V \subset U$,

$$\begin{array}{ccc} (f_*\mathcal{F})(U) & \xrightarrow{(f_*\phi)(U)} & (f_*\mathcal{G})(U) \\ {\scriptstyle \rho^{f_*\mathcal{F}}_{U,V}}\downarrow & & \downarrow{\scriptstyle \rho^{f_*G}_{U,V}} \\ (f_*\mathcal{F})(V) & \xrightarrow[(f_*\phi)(V)]{} & (f_*\mathcal{G})(V) \end{array}$$

we show that $f_*\phi \in \mathrm{Hom}_{\mathfrak{S}_{\mathbf{Ab}}(Y)}(f_*\mathcal{F}, f_*\mathcal{G})$:

$$\begin{aligned} \rho^{f_*\mathcal{G}}_{U,V} \circ (f_*\phi)(U) &= \rho^{\mathcal{G}}_{f^{-1}(U),f^{-1}(V)} \circ \phi(f^{-1}(U)) \\ &= \phi(f^{-1}(V)) \circ \rho^{\mathcal{F}}_{f^{-1}(U),f^{-1}(V)} = (f_*\phi)(V) \circ \rho^{f_*\mathcal{F}}_{U,V}. \end{aligned}$$

f_* is a covariant functor: $f_*(\mathrm{id}_{\mathcal{F}})(U) = \mathrm{id}_{\mathcal{F}}(f^{-1}(U)) = f^{-1}(U) = \mathrm{id}_{f_*\mathcal{F}}(U)$. For $\phi \in \mathrm{Hom}_{\mathfrak{S}_{\mathbf{Ab}}(X)}(\mathcal{F}, \mathcal{G})$ and $\psi \in \mathrm{Hom}_{\mathfrak{S}_{\mathbf{Ab}}(X)}(\mathcal{G}, \mathcal{H})$ we have

$$\begin{aligned} f_*(\psi \circ \phi)(U) &= (\psi \circ \phi)(f^{-1}(U)) = \psi(f^{-1}(U)) \circ \phi(f^{-1}(U)) \\ &= (f_*\psi)(U) \circ (f_*\phi)(U) = (f_*\psi \circ f_*\phi)(U) \end{aligned}$$

which implies $f_*(\psi \circ \phi) = f_*\psi \circ f_*\phi$. f_* is an additive functor:

$$\mathrm{Hom}(f_*)(\phi + \psi) = f_*(\phi + \psi) = f_*\phi + f_*\psi = \mathrm{Hom}(f_*)(\phi) + \mathrm{Hom}(f_*)(\psi).$$

The second argument is proved similarly. □

4.3.6 The additive functor $f^{-1}f_*$

Let $f : X \to Y$ be a continuous map of topological spaces and $\mathcal{F}$ a sheaf on X. We have an additive functor $f^{-1}f_* : f^{-1}f_*\mathcal{F} \to \mathcal{F}$:

$$\begin{array}{ccccc} f^{-1}f_*\mathcal{F} & \xleftarrow{f^{-1}} & f_*\mathcal{F} & \xleftarrow{f_*} & \mathcal{F} \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y & \xleftarrow{f} & X \end{array}$$

Since $f^{-1}f_*\mathcal{F}$ is the sheaf associated to the presheaf

$$U \longmapsto \varinjlim_{f(U)\subset V} (f_*\mathcal{F})(V) = \varinjlim_{f(U)\subset V} \mathcal{F}(f^{-1}(V)),$$

it is sufficient to define a canonical morphisms from the presheaf

$$U \longmapsto \varinjlim_{U\subset f^{-1}(V)} \mathcal{F}(f^{-1}(V))$$

to the sheaf $\mathcal{F}$. We define

$$\alpha(U) : \varinjlim_{U\subset f^{-1}(V)} \mathcal{F}(f^{-1}(V)) \longrightarrow \mathcal{F}(U), [(s, f^{-1}(V))] \longmapsto \rho^{\mathcal{F}}_{f^{-1}(V),U}(s). \tag{4.3.6.1}$$

If $(s, f^{-1}(V_1))$ and $(s_2, f^{-1}(V_2))$ have the same image in $\varinjlim_{U\subset f^{-1}(V)} \mathcal{F}(f^{-1}(V))$, then there exists an open subset $U \subset W \subset f^{-1}(V_1) \cap f^{-1}(V_2)$ such that $s_1|_W = s_2|_W$. Hence

$$\rho^{\mathcal{F}}_{f^{-1}(V_1),U}(s_1) = \rho^{\mathcal{F}}_{W,U} \circ \rho^{\mathcal{F}}_{f^{-1}(V_1),W}(s_1) = \rho^{\mathcal{F}}_{W,U} \circ \rho^{\mathcal{F}}_{f^{-1}(V_2),W}(s_2) = \rho^{\mathcal{F}}_{f^{-1}(V_2),U}(s_2).$$

Thus α is well-defined. For any open subsets $V \subset U \subset X$, one has

$$\begin{array}{ccc} \varinjlim_{U\subset f^{-1}(U_1)} \mathcal{F}(f^{-1}(U_1)) & \xrightarrow{\alpha(U)} & \mathcal{F}(U) \\ \Big\downarrow \rho_{U,V} & & \Big\downarrow \rho^{\mathcal{F}}_{U,V} \\ \varinjlim_{V\subset f^{-1}(V_1)} \mathcal{F}(f^{-1}(V_1)) & \xrightarrow[\alpha(V)]{} & \mathcal{F}(V) \end{array}$$

where

$$\rho_{U,V}([(s, f^{-1}(U_1))]) := [(\rho^{\mathcal{F}}_{f^{-1}(U_1),f^{-1}(V_1)}(s), f^{-1}(V_1))].$$

Therefore

$$\begin{aligned} \rho^{\mathcal{F}}_{U,V} \circ \alpha(U)([(s, f^{-1}(U_1))]) &= \rho^{\mathcal{F}}_{U,V}(\rho^{\mathcal{F}}_{f^{-1}(U_1),U}(s)) \\ = \rho^{\mathcal{F}}_{f^{-1}(U_1),V}(s) &= \alpha(V)([(s, f^{-1}(V_1))])] = \alpha(V) \circ \rho_{U,V}([(s, f^{-1}(U_1))]) \end{aligned}$$

4.3.7 The additive functor f_*f^{-1}

Let $f : X \to Y$ be a continuous map of topological spaces and $\mathcal{G}$ a sheaf on Y.

$$\begin{array}{ccccc} \mathcal{G} & \xrightarrow{f^{-1}} & f^{-1}\mathcal{G} & \xrightarrow{f_*} & f_*f^{-1}\mathcal{G} \\ \downarrow & & \downarrow & & \downarrow \\ Y & \xleftarrow{f} & X & \xrightarrow{f} & Y \end{array}$$

Define a canonical morphism

$$f_*f^{-1} : \mathcal{G} \longrightarrow f_*f^{-1}\mathcal{G} \tag{4.3.7.1}$$

as follows: For any open subset $W \subset Y$, $f(f^{-1}(W)) \subset W$; we have a morphism

$$\mathcal{G} \longrightarrow \varinjlim_{f(f^{-1}(W))\subset V} \mathcal{G}(V), \quad t \longmapsto \left[(\rho^{\mathcal{G}}_{W,f(f^{-1}(W))}(t), f(f^{-1}(W)))\right].$$

On the other hand,

$$\varinjlim_{f(f^{-1}(W))\subset V} \mathcal{G}(V) \longrightarrow f^{-1}\mathcal{G}(f^{-1}(W)) = (f_*f^{-1}\mathcal{G})(W).$$

4.3.8 Maps $\alpha_{\mathcal{F},\mathcal{G}}$ and $\beta_{\mathcal{F},\mathcal{G}}$

Let $f : X \to Y$ be a continuous map of topological spaces. For any sheaf $\mathcal{F}$ on X and any sheaf $\mathcal{G}$ on Y,

$$\begin{tikzcd} \mathcal{F} \arrow[d] & \mathcal{G} \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

we define a map

$$\alpha_{\mathcal{F},\mathcal{G}} : \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F}) \to \mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F}), \phi \mapsto \alpha_{\mathcal{F},\mathcal{G}}(\phi) := f^{-1}f_* \circ f^{-1}\phi. \tag{4.3.8.1}$$

That is

$$f^{-1}\mathcal{G} \xrightarrow{f^{-1}\phi} f^{-1}f_*\mathcal{F} \xrightarrow{f^{-1}f_*} \mathcal{F}.$$

On the other hand, we define

$$\beta_{\mathcal{F},\mathcal{G}} : \mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F}) \to \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F}), \psi \mapsto \beta_{\mathcal{F},\mathcal{G}}(\psi) := f_*\psi \circ f_*f^{-1}. \tag{4.3.8.2}$$

That is

$$\mathcal{G} \xrightarrow{f_*f^{-1}} f_*f^{-1}\mathcal{G} \xrightarrow{f_*\psi} f_*\mathcal{F}.$$

Theorem 4.15

$\alpha_{\mathcal{F},\mathcal{G}}$ and $\beta_{\mathcal{F},\mathcal{G}}$ are inverse to each other, and they are functorial with respect to $\mathcal{F}$ and $\mathcal{G}$, that is, for any morphism $\phi : \mathcal{F}_1 \to \mathcal{F}_2$ of sheaves on X and any morphism $\psi : \mathcal{G}_1 \to \mathcal{G}_2$ of sheaves on Y, the following diagram commutes:

$$\begin{tikzcd} \mathrm{Hom}(\mathcal{G}_2, f_*\mathcal{F}_1) \arrow[r, "\alpha_{\mathcal{F}_1,\mathcal{G}_2}"] \arrow[d, "\mathrm{Hom}(\psi, f_*\phi)"] & \mathrm{Hom}(f^{-1}\mathcal{G}_2, \mathcal{F}_1) \arrow[r, "\beta_{\mathcal{F}_1,\mathcal{G}_2}"] \arrow[d, "\mathrm{Hom}(f^{-1}\psi, \phi)"] & \mathrm{Hom}(\mathcal{G}_2, f_*\mathcal{F}_1) \arrow[d, "\mathrm{Hom}(\psi, f_*\phi)"] \\ \mathrm{Hom}(\mathcal{G}_1, f_*\mathcal{F}_2) \arrow[r, "\alpha_{\mathcal{F}_2,\mathcal{G}_1}"] & \mathrm{Hom}(f^{-1}\mathcal{G}_1, \mathcal{F}_2) \arrow[r, "\beta_{\mathcal{F}_2,\mathcal{G}_1}"] & \mathrm{Hom}(\mathcal{G}_1, f_*\mathcal{F}_2) \end{tikzcd}$$

♡

Proof. Recall the following two commutative diagrams:

$$\begin{tikzcd} f^{-1}f_*\mathcal{F}_1 \arrow[r, "f^{-1}f_*"] \arrow[d, "f^{-1}f_*\phi"'] & \mathcal{F}_1 \arrow[d, "\phi"] \\ f^{-1}f_*\mathcal{F}_2 \arrow[r, "f^{-1}f_*"'] & \mathcal{F}_2 \end{tikzcd} \qquad \begin{tikzcd} \mathcal{G}_1 \arrow[r, "\psi"] \arrow[d, "f_*f^{-1}"'] & \mathcal{G}_2 \arrow[d, "f_*f^{-1}"] \\ f_*f^{-1}\mathcal{G}_1 \arrow[r, "f_*f^{-1}\psi"'] & f_*f^{-1}\mathcal{G}_2 \end{tikzcd}$$

Hence, for any $\varphi \in \mathrm{Hom}(\mathcal{G}_2, f_*\mathcal{F}_1)$ we have

$$\begin{aligned}\alpha_{\mathcal{F}_2,\mathcal{G}_1} \circ \mathrm{Hom}(\psi, f_*\phi)(\varphi) &= \alpha_{\mathcal{F}_2,\mathcal{G}_1}(f_*\phi \circ \varphi \circ \psi) = f^{-1}f_* \circ f^{-1}(f_*\phi \circ \varphi \circ \psi) \\ &= f^{-1}f_* \circ f^{-1}f_*\phi \circ f^{-1}\varphi \circ f^{-1}\psi = \phi \circ f^{-1}f_* \circ f^{-1}\varphi \circ f^{-1}\psi \\ &= \mathrm{Hom}(f^{-1}\psi, \phi)(f^{-1}f_* \circ f^{-1}\varphi) = \mathrm{Hom}(f^{-1}\psi, \phi) \circ \alpha_{\mathcal{F}_1,\mathcal{G}_2}(\varphi).\end{aligned}$$

Similarly, for $\varphi \in \mathrm{Hom}(f^{-1}\mathcal{G}_2, \mathcal{F}_1)$ we have

$$\begin{aligned}\beta_{\mathcal{F}_2,\mathcal{G}_1} \circ \mathrm{Hom}(f^{-1}\psi, \phi)(\varphi) &= \beta_{\mathcal{F}_2,\mathcal{G}_1}(\phi \circ \varphi \circ f^{-1}\psi) = f_*(\phi \circ \varphi \circ f^{-1}\psi) \circ f_*f^{-1} \\ &= f_*\phi \circ f_*\varphi \circ f_*f^{-1}\psi \circ f_*f^{-1} = f_*\phi \circ f_*\varphi \circ f_*f^{-1} \circ \psi \\ &= \mathrm{Hom}(\psi, f_*\phi)(f_*\varphi \circ f_*f^{-1}) = \mathrm{Hom}(\psi, f_*\phi) \circ \beta_{\mathcal{F}_1,\mathcal{G}_2}(\varphi).\end{aligned}$$

From the definitions, one gets

$$f^{-1}f_* \circ f^{-1}f_*f^{-1} = \mathrm{id}_{f^{-1}\mathcal{G}}, \quad f_*f^{-1}f_* \circ f_*f^{-1} = \mathrm{id}_{f_*\mathcal{F}}.$$

Here $f^{-1}f_*f^{-1} := f^{-1}(f_*f^{-1})$ and $f_*f^{-1}f_* := f_*(f^{-1}f_*)$:

$$\begin{aligned} f^{-1}\mathcal{G} &\xrightarrow{f^{-1}f_*f^{-1}} f^{-1}f_*f^{-1}\mathcal{G} \xrightarrow{f^{-1}f_*} f^{-1}\mathcal{G}, \\ f_*\mathcal{F} &\xrightarrow{f_*f^{-1}} f_*f^{-1}f_*\mathcal{F} \xrightarrow{f_*f^{-1}f_*} f_*\mathcal{F}.\end{aligned}$$

For any $\phi \in \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F})$ and $\psi \in \mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F})$, it follows that

$$\begin{aligned}\alpha_{\mathcal{F},\mathcal{G}} \circ \beta_{\mathcal{F},\mathcal{G}}(\psi) &= \alpha_{\mathcal{F},\mathcal{G}}(f_*\psi \circ f_*f^{-1}) = f^{-1}f_* \circ f^{-1}(f_*\psi \circ f_*f^{-1}) \\ = f^{-1}f_* \circ f^{-1}f_*\psi \circ f^{-1}f_*f^{-1} &= \psi \circ f^{-1}f_* \circ f^{-1}f_*f^{-1} = \psi \circ \mathrm{id}_{f^{-1}\mathcal{G}} = \psi.\end{aligned}$$

Finally, we check that

$$\begin{aligned}\beta_{\mathcal{F},\mathcal{G}} \circ \alpha_{\mathcal{F},\mathcal{G}}(\phi) &= \beta_{\mathcal{F},\mathcal{G}}(f^{-1}f_* \circ f^{-1}\phi) = f_*(f^{-1}f_* \circ f^{-1}\phi) \circ f_*f^{-1} \\ &= f_*f^{-1}f_* \circ f_*f^{-1}\phi \circ f_*f^{-1} = f_*f^{-1}f_* \circ f_*f^{-1} \circ \phi = \phi\end{aligned}$$

which implies the required result. □

4.3.9 Left adjoint and right adjoint

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. Let $u : \mathcal{C} \to D$ and $v : \mathcal{D} \to \mathcal{C}$ be two functors. We say that u is **left adjoint to** v or v **is right adjoint to** u if for any object C in $\mathcal{C}$ and any object $D \in \mathcal{D}$, we have a bijection

$$\alpha_{C,D} : \mathrm{Hom}_{\mathcal{C}}(C, v(D)) \cong \mathrm{Hom}_{\mathcal{D}}(u(C), D) \tag{4.3.9.1}$$

which is functorial in C and D, that is, if $f : C_1 \to C_2$ and $g : D_1 \to D_2$ are morphisms, then we have

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(C_2, v(D_1)) \arrow[r, "\alpha_{C_2,D_1}"] \arrow[d, "\mathrm{Hom}(f,v(g))"] & \mathrm{Hom}_{\mathcal{D}}(u(C_2), D_1) \arrow[r, "\alpha^{-1}_{C_2,D_1}"] \arrow[d, "\mathrm{Hom}(u(f),g)"] & \mathrm{Hom}_{\mathcal{C}}(C_2, v(D_1)) \arrow[d, "\mathrm{Hom}(f,v(g))"] \\ \mathrm{Hom}_{\mathcal{C}}(C_1, v(D_2)) \arrow[r, "\alpha_{C_1,D_2}"] & \mathrm{Hom}_{\mathcal{D}}(u(C_1), D_2) \arrow[r, "\alpha^{-1}_{C_1,D_2}"] & \mathrm{Hom}_{\mathcal{C}}(C_1, v(D_2)) \end{tikzcd}$$

a commutative diagram. If $\phi \in \mathrm{Hom}_{\mathcal{C}}(C_2, v(D_1))$ and $\psi \in \mathrm{Hom}_{\mathcal{D}}(u(C_2), D_1)$, the commutativity of diagram tells us that

$$\begin{aligned} g \circ \alpha_{C_2,D_1}(\phi) \circ u(f) &= \alpha_{C_1,D_1}(v(g) \circ \phi \circ f), \\ v(g) \circ \alpha^{-1}_{C_2,D_1}(\phi) \circ f &= \alpha^{-1}_{C_1,D_2}(g \circ \psi \circ u(f)), \end{aligned}$$

where

$$\mathrm{Hom}(f, v(g))(\phi) := v(g) \circ \phi \circ f, \quad \mathrm{Hom}(u(f), g)(\psi) := g \circ \psi \circ u(f). \tag{4.3.9.2}$$

Note 4.6

(1) The functors f^{-1} and f_* are adjoint for each other, hence

$$\mathrm{Hom}_{\mathfrak{S}_{\mathbf{Ab}}(X)}(\cdot, f_*(\cdot)) \cong \mathrm{Hom}_{\mathfrak{S}_{\mathbf{Ab}}(Y)}(f^{-1}(\cdot), \cdot) \tag{4.3.9.3}$$

where $f^{-1} : \mathfrak{S}_{\mathbf{Ab}}(Y) \to \mathfrak{S}_{\mathbf{Ab}}(X)$ and $f_* : \mathfrak{S}_{\mathbf{Ab}}(X) \to \mathfrak{S}_{\mathbf{Ab}}(Y)$.

(2) The functor $+ : \mathcal{F} \to \mathcal{F}^+$ from the category of presheaves to the category of sheaves is left adjoint to the inclusion functor from the category of sheaves to the category of presheaves. ♣

Proposition 4.8

If a left adjoint to v exists, then it is unique up to isomorphism. Similarly, if a right adjoint to u exists, it is unique up to isomorphism. ♡

Proof. Let u and u' be left adjoint to v. For a morphism $f : C_1 \to C_2$ and a morphism $g : D_1 \to D_2$ in the categories $\mathcal{C}$ and $\mathcal{D}$, we have

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{D}}(u'(C_2), D_1) & \xrightarrow{\alpha'^{-1}_{C_2,D_1}} & \mathrm{Hom}_{\mathcal{C}}(C_2, v(D_1)) & \xrightarrow{\alpha_{C_2,D_1}} & \mathrm{Hom}_{\mathcal{D}}(u(C_2), D_1) \\ \downarrow{\scriptstyle \mathrm{Hom}(u'(f),g)} & & \downarrow{\scriptstyle \mathrm{Hom}(f,v(g))} & & \downarrow{\scriptstyle \mathrm{Hom}(u(f),g)} \\ \mathrm{Hom}_{\mathcal{D}}(u'(C_1), D_2) & \xrightarrow{\alpha'^{-1}_{C_1,D_2}} & \mathrm{Hom}_{\mathcal{C}}(C_1, v(D_2)) & \xrightarrow{\alpha_{C_1,D_2}} & \mathrm{Hom}_{\mathcal{D}}(u(C_1), D_2) \end{array}$$

and

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{D}}(u(C_2), D_1) & \xrightarrow{\alpha^{-1}_{C_2,D_1}} & \mathrm{Hom}_{\mathcal{C}}(C_2, v(D_1)) & \xrightarrow{\alpha'_{C_2,D_1}} & \mathrm{Hom}_{\mathcal{D}}(u'(C_2), D_1) \\ \downarrow{\scriptstyle \mathrm{Hom}(u(f),g)} & & \downarrow{\scriptstyle \mathrm{Hom}(f,v(g))} & & \downarrow{\scriptstyle \mathrm{Hom}(u'(f),g)} \\ \mathrm{Hom}_{\mathcal{D}}(u(C_1), D_2) & \xrightarrow{\alpha^{-1}_{C_1,D_2}} & \mathrm{Hom}_{\mathcal{C}}(C_1, v(D_2)) & \xrightarrow{\alpha'_{C_1,D_2}} & \mathrm{Hom}_{\mathcal{D}}(u'(C_1), D_2) \end{array}$$

we get

$$\begin{aligned} \mathrm{Hom}(u(f), g) \circ \alpha_{C_2,D_1} \circ \alpha'^{-1}_{C_2,D_1} &= \alpha_{C_1,D_2} \circ \alpha'^{-1}_{C_1,D_2} \circ \mathrm{Hom}(u'(f), g), \\ \mathrm{Hom}(u'(f), g) \circ \alpha'_{C_2,D_1} \circ \alpha^{-1}_{C_2,D_1} &= \alpha'_{C_1,D_2} \circ \alpha^{-1}_{C_1,D_2} \circ \mathrm{Hom}(u(f), g). \end{aligned}$$

Hence for morphisms $f : u(C) \to u'(C)$ and $g : u'(C) \to u(C)$ one concludes that

$$f \circ \alpha'_{C,u(C)} \circ \alpha^{-1}_{C,u(C)}(g) \circ \mathrm{id}_{u(C)} = \alpha'_{C,u(C)} \circ \alpha^{-1}_{C,u'(C)} \circ (f \circ g \circ \mathrm{id}_{u'(C)}),$$

thus

$$f \circ \alpha'_{C,u(C)} \circ \alpha^{-1}_{C,u(C)} = \alpha'_{C,u'(C)} \circ \alpha^{-1}_{C,u'(C)} \circ f.$$

similarly, we get

$$g \circ \alpha_{C,u'(C)} \circ \alpha'^{-1}_{C,u'(C)} = \alpha_{C,u(C)} \circ \alpha'^{-1}_{C,u(C)} \circ g.$$

For any object $C \in \mathcal{C}$ and any object $D \in \mathcal{D}$, we have

$$\mathrm{Hom}_{\mathcal{D}}(u(C), D) \cong \mathrm{Hom}_{\mathcal{C}}(C, v(D)) \cong \mathrm{Hom}_{\mathcal{D}}(u'(C), D).$$

Taking $D = u(C)$ gives a unique morphism $\phi(C) : u'(C) \to u(C)$ defined by

$$\phi(C) := \alpha'_{C,u(C)} \circ \alpha^{-1}_{C,u(C)}(\mathrm{id}_{u(C)});$$

taking $D = u'(C)$ gives another unique morphism $\psi(C) : u(C) \to u'(C)$ given by

$$\psi(C) := \alpha_{C,u'(C)} \circ \alpha'^{-1}_{C,u'(C)}(\mathrm{id}_{u'(C)}).$$

Therefore

$$\begin{aligned}\psi(C) \circ \phi(C) &= \alpha_{C,u'(C)} \circ \alpha'^{-1}_{C,u'(C)}(\alpha'_{C,u(C)}(\alpha^{-1}_{C,u(C)}(\mathrm{id}_{u(C)}))) \\ &= \alpha'_{C,u'(C)} \circ \alpha^{-1}_{C,u'(C)} \circ \alpha_{C,u'(C)} \circ \alpha'^{-1}_{C,u'(C)}(\mathrm{id}_{u'(C)}) \;=\; \mathrm{id}_{u'(C)}.\end{aligned}$$

Using the same argument, one concludes $\phi(C) \circ \psi(C) = \mathrm{id}_{u(C)}$. For any morphism $g : u(C_2) \to u(C_1)$, we have $g \circ \phi(C_1) \circ u'(f) = \alpha'_{C_1,u(C_1)}(\alpha^{-1}_{C_1,u(C_1)}(g \circ u(f)))$ implying $a(f) \circ \phi(C_1) = \phi(C_2) \circ u'(f)$. □

4.3.10 $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Let $f : X \to Y$ and $g : Y \to Z$ be two continuous maps between topological spaces.

(1) Let $\mathcal{F}$ be any sheaf on X and $\mathcal{G}$ be any sheaf on Y. For an open subset $U \subset Z$, one has

$$\begin{aligned}((g \circ f)_*\mathcal{F})(U) &= \mathcal{F}((g \circ f)^{-1}(U)) \\ &= \mathcal{F}(f^{-1}(g^{-1}(U))) \;=\; f_*\mathcal{F}(g^{-1}(U)) \;=\; (g_*(f_*\mathcal{F}))(U).\end{aligned}$$

(2) Let $\mathcal{H}$ be a sheaf on Z. Hence

$$\begin{aligned}\mathrm{Hom}((g \circ f)^{-1}\mathcal{H}, \mathcal{F}) &= \mathrm{Hom}(\mathcal{H}, (g \circ f)_*\mathcal{F}) \;=\; \mathrm{Hom}(\mathcal{H}, (g_* \circ f_*)(\mathcal{F})) \\ &= \mathrm{Hom}(g^{-1}\mathcal{H}, f_*\mathcal{F}) \;=\; \mathrm{Hom}(f^{-1}(g^{-1}\mathcal{H}), \mathcal{F});\end{aligned}$$

thus

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}. \tag{4.3.10.1}$$

4.3.11 $(f^{-1}\mathcal{G})_P = \mathcal{G}_{f(P)}$

Let P be a point in a topological space X and $i : P \hookrightarrow X$ be the inclusion map. For any sheaf $\mathcal{F}$ on X we have

$$i^{-1}\mathcal{F} = \mathcal{F}_P. \tag{4.3.11.1}$$

Now we consider a continuous map $f : X \to Y$ between topological spaces and a sheaf $\mathcal{G}$ on Y. Then for any point $P \in X$, we get

$$(f^{-1}\mathcal{G})_P = i^{-1}(f^{-1}\mathcal{G}) = (f \circ i)^{-1}\mathcal{G} = \mathcal{G}_{f(P)}. \tag{4.3.11.2}$$

4.3.12 The five lemma

At the end of this section, we mention a useful and powerful lemma

Lemma 4.8. (The five lemma)

Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

be a commutative diagram in an Abelian category such that the two rows are exact. Then there exists a morphism

$$\delta : \mathrm{Ker}(w) \longrightarrow \mathrm{Coker}(u)$$

satisfying

$$0 \to \mathrm{Ker}(u) \to \mathrm{Ker}(v) \to \mathrm{Ker}(w) \to \mathrm{Coker}(u) \to \mathrm{Coker}(v) \to \mathrm{Coker}(w) \to 0.$$

Corollary 4.3

Let

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

be a commutative diagram in an Abelian category such that the two rows are exact. If f_1, f_2, f_4, and f_5 are isomorphisms, then f_3 is also an isomorphism.

4.4 Schemes

Introduction

- *Localization of rings*
- *Spectrum*
- *Zariski topology*
- *Schemes*

Throughout this section all rings are commutative ring with identity 1, and all homomorphisms of rings are assumed to be homomoprhism mapping 1 to 1.

4.4.1 Localization of rings

Let S be a multiplicative system of a ring A. Let $S^{-1}A$ be the localization of ring A relative to S. That is

$$S^{-1}A = S \times A/ \sim = \{a/s | a \in A,\ s \in S\}. \qquad (4.4.1.1)$$

Here the equivalence relation is given as follows: $(a, s) \sim (a', s')$ if and only if $s''(s'a - sa') = 0$ for some $s'' \in S$. We usually write $[(a, s)]$ as a/s.

If $\mathfrak{p} \in \operatorname{Spec}(A)$ be a prime ideal of A, setting $S = A - \mathfrak{p}$, we obtain a local ring

$$A_{\mathfrak{p}} := S^{-1}A. \tag{4.4.1.2}$$

For any element $f \in A$, we set $S = \{f^n\}_{n\geq 0}$ and get a local ring

$$A_f := S^{-1}A. \tag{4.4.1.3}$$

Lemma 4.9

If f is nilpotent, then $A_f = 0$. ♡

Proof. Suppose $f^{n+1} = 0$ for the least natural number n. Then

$$S = \{1, f, \cdots, f^n\}, \quad A_f = \{a/f^i | a \in A,\ 1 \leq i \leq n\}.$$

For any element $a/f^i \in A_f$, $f^{n+1}(a_i \cdot 1 - 0 \cdot f^i) = f^{n+1}a_i = 0$; hence $a/f_i = 0$. □

4.4.2 Zariski topology

For every ring A and every ideal $\mathfrak{a}$ of A, we define

$$V(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Spec}(A) | \mathfrak{a} \subset \mathfrak{p}\}. \tag{4.4.2.1}$$

Proposition 4.9

(i) $V(0) = \operatorname{Spec}(A)$ *and* $V(A) = \varnothing$.

(ii) If $\mathfrak{a} \subset \mathfrak{b}$, *then* $V(\mathfrak{a}) \supset V(\mathfrak{b})$.

(iii) $\cap_{i\in I} V(\mathfrak{a}_i) = V(\sum_{i\in I} \mathfrak{a}_i)$ *for any family of ideals* $\mathfrak{a}_i$ $(i \in I)$ *of* A.

(iv) $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b})$. ♡

This proposition shows $\operatorname{Spec}(A)$ with this topology, called the **Zariski topology**, is a topological space. We shall construct a sheaf $\mathcal{O}_{\operatorname{Spec}(A)}$ of rings over $\operatorname{Spec}(A)$.

Proposition 4.10

(i) $\operatorname{Spec}(A) = \varnothing$ *if and only if* $0 = 1$ *in* A.

(ii) For any ideal $\mathfrak{a}$ *of* A, *then* $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$.

(iii) $\sqrt{\mathfrak{a}} = \cap_{\mathfrak{p}\in V(\mathfrak{a})} \mathfrak{p}$.

(iv) For any ideals $\mathfrak{a}$ *and* $\mathfrak{b}$ *of* A, $V(\mathfrak{a}) \subset V(\mathfrak{b})$ *if and only if* $\sqrt{\mathfrak{a}} \supset \sqrt{\mathfrak{b}}$. ♡

Proposition 4.11

For any $f \in A$, *the subset*

$$D(f) := \operatorname{Spec}(A) - V((f)) = \{\mathfrak{p} \in \mathfrak{A} | f \notin \mathfrak{p}\} \tag{4.4.2.2}$$

is open. Open subsets of this type form a basis for the Zariski topology of $\operatorname{Spec}(A)$, *that is, any open subset of* $\operatorname{Spec}(A)$ *is a union of such open subsets. Moreover,* $D(f)$ *is quasi-compact.* ♡

Proof. Since $D(f)$ is the complement of $V((f))$, it is open. Let U be an open subset of $\mathrm{Spec}(A)$ with $U = \mathrm{Spec}(A) - V(\mathfrak{a})$ for some ideal $\mathfrak{a}$. For any prime ideal $\mathfrak{p} \in U$, we have $\mathfrak{a} \notin \mathfrak{p}$ and hence there exists an element $f \in \mathfrak{a} - \mathfrak{p}$. Therefore $\mathfrak{p} \in D(f) \subset U$ since $(f) \subset \mathfrak{a}$. Thus

$$U = \cup_{\mathfrak{p} \in U} D((f)).$$

Now, we prove that $D(f)$ is quasi-compact. Suppose that $D(f) \in \cup_{i \in I} D(f_i)$ for some family of elements $f_i (i \in I)$ in A. Then we have

$$V((f)) \supset \cap_{i \in I} V((f_i)) = V\left(\sum_{i \in I} (f_i)\right).$$

So $\sqrt{(f)} \subset \sqrt{\sum_{i \in I}(f_i)}$ and there is a natural number n such that

$$f^n = a_{i_1} f_{i_1} + \cdots + a_{i_k} f_{i_k}, \quad i_1, \cdots, i_k \in I, \quad a_{i_1}, \cdots, a_{i_k} \in A.$$

This implies that $D(f) \subset \cup_{j=1}^{k} D(f_{i_j})$. □

4.4.3 Spectrum

Let A be a ring. Define a sheaf $\mathcal{O}_{\mathrm{Spec}(A)}$ on $\mathrm{Spec}(A)$ as follows: For any open subset U of $\mathrm{Spec}(A)$, $\mathcal{O}_{\mathrm{Spec}(A)}(U)$ consists of functions $s : U \to \coprod_{\mathfrak{p} \in \mathrm{Spec}(A)} A_{\mathfrak{p}}$ satisfying

(a) For any $\mathfrak{p} \in U$, we have $s(\mathfrak{p}) \in A_{\mathfrak{p}}$.

(b) For any $\mathfrak{p} \in U$, there exist a neighborhood $U_{\mathfrak{p}}$ of $\mathfrak{p}$ contained in U and $a, f \in A$ such that for any $\mathfrak{q} \in U_{\mathfrak{p}}$, we have $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \frac{a}{f}$ in $A_{\mathfrak{q}}$.

For any inclusion of open subsets $V \subset U$, we define $\mathcal{O}_{\mathrm{Spec}(A)}(U) \to \mathcal{O}_{\mathrm{Spec}(A)}(V)$ to be the restriction of functions. We call the pair $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ the **spectrum** of A.

Proposition 4.12

(i) For any $\mathfrak{p} \in \mathrm{Spec}(A)$*, we have a canonical isomorphism* $\mathcal{O}_{\mathrm{Spec}(A),\mathfrak{p}} \cong A_{\mathfrak{p}}$.

(ii) For any $f \in A$*, we have a canonical isomorphism* $\mathcal{O}_{\mathrm{Spec}(A)}(D(f)) \cong A_f$*. In particular,* $\mathcal{O}_{\mathrm{Spec}(A)}(\mathrm{Spec}(A)) \cong A$.

♡

Proof. (i) For any open neighborhood U of $\mathfrak{p}$, we have a homomorphism

$$\mathcal{O}_{\mathrm{Spec}(A)}(U) \longrightarrow A_{\mathfrak{p}}, \quad s \longmapsto s(\mathfrak{p}).$$

Then it induces a homomorphism

$$\mathcal{O}_{\mathrm{Spec}(A),\mathfrak{p}} = \varinjlim_{\mathfrak{p} \in U} \mathcal{O}_{\mathrm{Spec}(A)}(U) \longrightarrow A_{\mathfrak{p}}, \quad [(s, U)] \longmapsto s(\mathfrak{p}).$$

It is surjective: For $a/f \in A_{\mathfrak{p}}$ $(a \in A, f \notin \mathfrak{p})$, we define $s = a/f \in \mathcal{O}_{\mathrm{Spec}(A),\mathfrak{p}}$ such that $s(\mathfrak{q}) = a/f$ for any $\mathfrak{q} \in U$. It is injective: Assume that $[(s, U)] \in \mathcal{O}_{\mathrm{Spec}(A),\mathfrak{p}}$ and $s(\mathfrak{p}) = 0$. We show that $(s, U) \sim (0, V)$ for some neighborhood of $\mathfrak{p}$. Since there exists an open neighborhood $U_{\mathfrak{p}}$ of $\mathfrak{p}$ contained in U and $a, f \in A$ such that $s(\mathfrak{q}) = a/f$ in $A_{\mathfrak{q}}$ with $f \notin \mathfrak{q}$ for any $\mathfrak{q}$ $U_{\mathfrak{q}}$, and $s(\mathfrak{p}) = 0$, it follows that $s(\mathfrak{p}) = 0 = a/f$ in $A_{\mathfrak{p}}$. Hence $ta = 0$ for some $t \notin \mathfrak{p}$. Then for any

$\mathfrak{q} \in D(f) \cap D(t)$, we have

$$\frac{a}{f} = \frac{ta}{tf} = 0$$

in $A_{\mathfrak{q}}$. Thus, $s|_{D(f)\cap D(t)} = 0$ and $[(s, U)] \sim [(0, D(f) \cap D(t))]$.

(ii) Define the homomorphism

$$A_f \longrightarrow \mathcal{O}_{\mathrm{Spec}(A)}(D(f)), \quad \frac{a}{f^k} \longmapsto s : D(f) \longrightarrow \coprod_{\mathfrak{p}\in\mathrm{Spec}(A)} A_{\mathfrak{p}}, \quad \mathfrak{p} \longmapsto \frac{a}{f^k}.$$

The map is injective: Suppose that $a/f^k = 0$ in $A_{\mathfrak{p}}$ with $\mathfrak{p} \in D(f)$. There exists an element $t \notin \mathfrak{p}$ such that $ta = 0$ in $A_{\mathfrak{p}}$. Hence $\mathrm{Ann}(\mathfrak{a}) \nsubseteq \mathfrak{p}$ for any $\mathfrak{p} \in D(f)$; consequently, $V(\mathrm{Ann}(\mathfrak{a})) \subset V((f))$. It follows that

$$f \in (f) \subset \sqrt{(f)} \subset \sqrt{\mathrm{Ann}(a)} \Longrightarrow f^n a = 0$$

for some n. So $\frac{a}{f^k} = \frac{f^n a}{f^{n+k}} = 0$ in A_f.

The map is surjective: Let $s : D(f) \to \coprod_{\mathfrak{p}\in\mathrm{Spec}(A)} A_{\mathfrak{p}}$ be a section in $\mathcal{O}_{\mathrm{Spec}(A)}(D(f))$. Then there exists an open covering $\{U_i\}_{i\in I}$ of $D(f)$ and $a_i, f_i \in A$ such that

$$s(\mathfrak{q}) = \frac{a_i}{f_i}$$

in $A_{\mathfrak{q}}$ for any $\mathfrak{q} \in U_i$ and $f_i \notin \mathfrak{q}$.

(A) We may assume that $u_i = D(g_i)$ where $g_i \in A$ and $|I| < \infty$. Hence $D(g_i) \subset D(f_i)$ and $\sqrt{(f_i)} \subset \sqrt{(g_i)}$. It follows that $g_i^{k_i} = h_i f_i$ for some k_i and $h_i \in A$. For $\mathfrak{q} \in D(g_i)$ we have

$$h_i f_i = g_i^{k_i} \notin \mathfrak{q}, \quad s(\mathfrak{q}) = \frac{a_i}{f_i} = \frac{h_i a_i}{h_i f_i} = \frac{h_i a_i}{g_i^{k_i}}$$

in $A_{\mathfrak{q}}$. Since $D(g_i) = D(g_i^{k_i})$, we may assume that we have a finite covering $\{U_i\}_{i\in I}$ ($|I| < \infty$, $U_i = D(g_i)$) of $D(f)$ and $a_i \in A$ such that

$$s(\mathfrak{q}) = \frac{a_i}{g_i}$$

in $A_{\mathfrak{q}}$ for any $\mathfrak{q} \in D(g_i)$.

(B) For any $\mathfrak{q} \in D(g_i) \cap D(g_j) = D(h_i g_j)$, we have

$$\frac{a_i}{g_i} = \frac{a_j}{g_j}$$

in $A_{\mathfrak{q}}$; so for any $\mathfrak{q} \in D(g_i g_j)$ there exists an element $t \notin \mathfrak{q}$ such that

$$t(a_i g_j - g_j g_i) = 0$$

in $a_{\mathfrak{q}}$. Hence for $\mathfrak{q} \in D(q_i q_j)$, we have $a_i g_j - a_j g_i \in \mathrm{Ann}(t)$ and

$$\mathrm{Ann}(a_i g_j - a_j g_i) \nsubseteq \mathfrak{q} \Longrightarrow V(\mathrm{Ann}(a_i g_j - a_j g_i)) \subset V((g_i g_j)).$$

That is, if $\mathfrak{q} \in \mathrm{Spec}(A) - V((g_i g_j))$, then

$$\mathfrak{q} \in \mathrm{Spec}(A) - V(\mathrm{Ann}(a_i g_j - a_j g_i)) \Longrightarrow g_i g_j \in \sqrt{\mathrm{Ann}(a_i g_j - a_j g_i)}.$$

There have k_{ij} such that

$$(g_i g_j)^{k_{ij}} (a_i g_j - a_j g_i) = 0.$$

Let $k = \sum_{i,j\in I} k_{ij} < \infty$. We get

$$(g_i g_j)^k (a_i g_j - a_j g_i) = 0 \Longrightarrow g_j^{k+1}(a_i g_i^k) - g_i^{k+1}(a_j g_j^k) = 0.$$

We can assume that $\{D(g_i)\}_{i\in I}$ is a finite covering of $D(f)$ and for any $\mathfrak{q} \in D(g_i)$ we have

$$s(\mathfrak{q}) = \frac{a_i}{g_i}$$

in $A_{\mathfrak{q}}$ and $a_i g_j = a_j g_i$ for any i, j.

From $D(f) = \cup_{i\in I} D(g_i)$ we get $V((f)) = \cap_{i\in I} V((g_i)) = V(\sum_{i\in I}(g_i))$. So $f^k = \sum_{i\in I} b_i g_i$ with $b_i \in A$ where $|I| < \infty$. For each j, we have

$$a_j f^k = a_j \sum_{i\in I} b_i g_i = \sum_{i\in I} b_i a_i g_j = a g_j, \quad s = \sum_{i\in I} a_i b_i$$

hence

$$\frac{a_j}{g_j} = \frac{a}{f^k}$$

for every $i \in I$. Hence under the homomorphism $A_f \to \mathrm{Spec}(\mathcal{A})(D(f))$ defined as the beginning, $\frac{\sum_{i\in I} b_i a_i}{f^k}$ is mapped to the section $s \in \mathcal{O}_{\mathrm{Spec}(A)}(D(f))$. so $A_f \to \mathcal{O}_{\mathrm{Spec}(A)}(D(f))$ is surjective. □

4.4.4 Schemes

A **ringed space** is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X. We say that $(X, \mathcal{O}_X)$ is a **locally ringed space** if for any $P \in X$, the stalk $\mathcal{O}_{X,P}$ of $\mathcal{O}_X$ at P is a local ring. Let $\mathfrak{m}_{X,P}$ be the maximal ideal of $\mathcal{O}_{X,P}$. We call $k_{X,P} = \mathcal{O}_{X,P}/\mathfrak{m}_{X,P}$ the **residue field** of $(X, \mathcal{O}_X)$ at P.

For any ring A, the spectrum $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ is a locally ringed space. If X is a topological space and $\mathcal{C}_X$ is the sheaf of continuous functions on X, then $(X, \mathcal{O}_X)$ is also a locally ringed space.

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two ringed spaces. A **morphism** from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\sharp)$ consisting of a continuous map $f : X \to Y$ and a morphism of sheaves $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$. For any point $P \in X$, we have a homomorphism $(f_*\mathcal{O}_X)_{f(P)} \to \mathcal{O}_{X,P}$ defined by

$$(f_*\mathcal{O}_X)_{f(P)} = \varinjlim_{f(P)\subset V} (f_*\mathcal{O}_X)(V) = \varinjlim_{P\in f^{-1}(V)} \mathcal{O}_X(f^{-1}(V)) \longrightarrow \varinjlim_{P\in U} \mathcal{O}_X(U) = \mathcal{O}_{X,P}.$$

Then we get a homomorphism

$$f_P^\sharp : \mathcal{O}_{Y,f(P)} \longrightarrow (f_*\mathcal{O}_X)_{f(P)} \longrightarrow \mathcal{O}_{X,P}. \tag{4.4.4.1}$$

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two locally ringed spaces. A **morphism of locally ringed spaces** $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces such that for any $P \in X$, $f_X^\sharp : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ is a local homomorphism, that is,

$$f_P^\sharp(\mathfrak{m}_{Y,f(P)}) \subset \mathfrak{m}_{X,P} \Longleftrightarrow (f_P^\sharp)^{-1}(\mathfrak{m}_{X,P}) = \mathfrak{m}_{Y,f(P)}.$$

An **isomorphism of locally ringed spaces** $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism with a two-sided inverse. This is equivalent to saying that $f : X \to Y$ is a homeomorphism of topological spaces and $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism of sheaves. The composite of morphisms of locally ringed spaces is also a locally ringed space.

Any locally ringed space that is isomorphic to $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ for some ring A is called an **affine scheme**. A **scheme** $(X, \mathcal{O}_X)$ is a locally ringed space such that there exists an open covering $\{U_i\}_{i\in I}$ of X such that each $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme. We call X the **underlying topological space** and $\mathcal{O}_X$ the **structure sheaf**. We define **morphisms of schemes** as morphisms of locally ringed spaces.

Proposition 4.13

(i) If $\phi : A \to B$ is a homomorphism of rings, then ϕ induces a morphisms of locally ringed spaces

$$(f, f^\sharp) : (\mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}(B)}) \longrightarrow (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}).$$

(ii) Any morphism $(f, f^\sharp) : (\mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}(B)}) \to (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ of locally ringed spaces is obtained from this way. ♡

Proof. (i) For any morphism $\phi : A \to B$, we define

$$f : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A), \quad \mathfrak{q} \longmapsto \phi^{-1}(\mathfrak{q})$$

for any prime ideal $\mathfrak{q} \in \mathrm{Spec}(B)$. For any ideal $\mathfrak{a}$ of A, we have

$$f^{-1}(V(\mathfrak{a})) = V(\mathfrak{a}B)$$

where $\mathfrak{a}B$ is the ideal generated by $\phi(\mathfrak{a})$. So f is continuous.

For any open subset V of $\mathrm{Spec}(A)$ and and section

$$s : V \longrightarrow \coprod_{\mathfrak{p}\in\mathrm{Spec}(A)} A_{\mathfrak{p}}$$

we define a morphism of sheaves

$$f^\sharp(V) : \mathcal{O}_{\mathrm{Spec}(A)}(V) \longrightarrow \mathcal{O}_{\mathrm{Spec}(B)}(f^{-1}(V)), \quad s \longmapsto f^\sharp(V)(s)$$

where

$$f^*(V)(s) : f^{-1}(V) \longrightarrow \coprod_{\mathfrak{q}\in\mathrm{Spec}(B)} B_{\mathfrak{q}}, \quad \mathfrak{q} \longmapsto \phi_{\mathfrak{q}}(s(f(\mathfrak{q}))).$$

The homomorphism $\phi_{\mathfrak{q}} : A_{\phi^{-1}(\mathfrak{q})} \to B_{\mathfrak{q}}$ is induced by $\phi : A \to B$:

$$\phi_{\mathfrak{q}}(a/f) := \frac{\phi(a)}{\phi(f)}.$$

Since $f \notin \phi^{-1}(\mathfrak{q})$ it follows that $\phi(f) \notin \mathfrak{q}$ and hence the above homomorphism is well-defined. If $a_1/f_1 \sim a_2/f_2$, then $f(a_1f_2 - a_2f_1) = 0$ for some $f \in A$. Since ϕ is a homomorphism,

$$\phi(f)(\phi(a_1)\phi(f_1) - \phi(a_2)\phi(f_1)) = 0 \Longrightarrow \phi(a_1)/\phi(f_1) \sim \phi(a_2) \sim \phi(f_2).$$

Thus ϕ is a local homomorphism. For any open subsets $V \subset U$, one has

$$\begin{tikzcd}
\mathcal{O}_{\mathrm{Spec}(A)}(U) \arrow[r, "f^\sharp(U)"] \arrow[d, "\rho_{U,V}^{\mathcal{O}_{\mathrm{Spec}(A)}}"'] & \mathcal{O}_{\mathrm{Spec}(B)}(f^{-1}(U)) \arrow[d, "\rho_{f^{-1}(U),f^{-1}(V)}^{\mathcal{O}_{\mathrm{Spec}(B)}}"] \\
\mathcal{O}_{\mathrm{Spec}(A)}(V) \arrow[r, "f^\sharp(V)"'] & \mathcal{O}_{\mathrm{Spec}(B)}(f^{-1}(V))
\end{tikzcd}$$

For any $s \in \mathcal{O}_{\mathrm{Spec}(A)}(U)$, we have

$$\rho_{f^{-1}(U),f^{-1}(V)}^{\mathcal{O}_{\mathrm{Spec}(B)}} \circ f^\sharp(U)(s) = f^\sharp(U)|_{f^{-1}(V)}(s) = f^\sharp(U)(s)|_{f^{-1}(V)}$$

and

$$f^\sharp(V) \circ \rho_{U,V}^{\mathcal{O}_{\mathrm{Spec}(A)}}(s) = f^\sharp(V)(s|_V).$$

Thus $f^\sharp : \mathcal{O}_{\mathrm{Spec}(A)} \to f_*\mathcal{O}_{\mathrm{Spec}(B)}$ is a morphism of sheaves. Hence $f^\sharp_{\mathfrak{q}}$ is a local homomorphism.

$$\begin{tikzcd}
\mathcal{O}_{\mathrm{Spec}(A),f(\mathfrak{q})} \arrow[r, "f^\sharp_{\mathfrak{q}}"] \arrow[d, "\cong"'] & \mathcal{O}_{\mathrm{Spec}(B),\mathfrak{q}} \arrow[d, "\cong"] \\
A_{f(\mathfrak{q})} = A_{\phi^{-1}(\mathfrak{q})} \arrow[r, "\phi_{\mathfrak{q}}"'] & B_{\mathfrak{q}}.
\end{tikzcd}$$

(ii) Suppose that $(f, f^\sharp) : (\mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}(B)}) \to (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ is a morphism of locally ringed spaces. Define $\phi : A \to B$ as follows:

$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[d, "\cong"'] & B \arrow[d, "\cong"] \\
\mathcal{O}_{\mathrm{Spec}(A)}(\mathrm{Spec}(A)) \arrow[r, "f^\sharp"'] & (f_*\mathcal{O}_{\mathrm{Spec}(B)})(\mathrm{Spec}(A)) = \mathcal{O}_{\mathrm{Spec}(B)}(\mathrm{Spec}(B)).
\end{tikzcd}$$

For $\mathfrak{q} \in \mathrm{Spec}(B)$, define $\phi'_{\mathfrak{q}} : A_{f(\mathfrak{q})} \to B_{\mathfrak{q}}$ so that the diagram

$$\begin{tikzcd}
A_{f(\mathfrak{q})} \arrow[r, "\phi'_{\mathfrak{q}}"] \arrow[d, "\cong"] & B_{\mathfrak{q}} \\
\mathcal{O}_{\mathrm{Spec}(A),f(\mathfrak{q})} \arrow[r, "f^\sharp_{\mathfrak{q}}"'] & \mathcal{O}_{\mathrm{Spec}(B),\mathfrak{q}} \arrow[u, "\cong"']
\end{tikzcd}$$

commutes. Since $f^\sharp_{\mathfrak{q}}$ is a local homomorphism, $\phi'_{\mathfrak{q}}$ is local. the following diagram

$$\begin{tikzcd}
\mathcal{O}_{\mathrm{Spec}(A)}(\mathrm{Spec}(A)) \arrow[r, "f^\sharp"] \arrow[d] & \mathcal{O}_{\mathrm{Spec}(B)}(\mathrm{Spec}(B)) \arrow[d] \\
\mathcal{O}_{\mathrm{Spec}(A),f(\mathfrak{q})} \arrow[r, "f^\sharp_{\mathfrak{q}}"'] & \mathcal{O}_{\mathrm{Spec}(B),\mathfrak{q}}
\end{tikzcd}$$

is commutative, i.e, for $s \in \mathcal{O}_{\mathrm{Spec}(A)}(\mathrm{Spec}(A))$, $f^\sharp_{\mathfrak{q}}([(s, \mathrm{Spec}(A))]) = [(f^\sharp(s), \mathrm{Spec}(B))]$. It induces a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[d, "p_A"'] & B \arrow[d, "p_B"] \\
A_{f(\mathfrak{q})} \arrow[r, "\phi'_{\mathfrak{q}}"'] & B_{\mathfrak{q}}.
\end{tikzcd}$$

Hence

$$\phi^{-1}(\mathfrak{q}) = \phi^{-1}(p_B^{-1}(\mathfrak{q}B_\mathfrak{q})) = p_A^{-1}(\phi'_\mathfrak{q})^{-1}(\mathfrak{q}B_\mathfrak{q}) = p_A^{-1}((\mathfrak{q})A_{f(\mathfrak{q})}) = f(\mathfrak{q}).$$

Now we proved that $\phi'_\mathfrak{q} = \phi_\mathfrak{q}$.

For any open subset V of $\mathrm{Spec}(A)$ and any $\mathfrak{q} \in f^{-1}(V)$, we have a commutative diagram:

$$\begin{array}{ccc}
\mathcal{O}_{\mathrm{Spec}(A)}(V) & \xrightarrow{f^\sharp(V)} & \mathcal{O}_{\mathrm{Spec}(B)}(f^{-1}(V)) \\
\downarrow & & \downarrow \\
\mathcal{O}_{\mathrm{Spec}(A),f(\mathfrak{q})} & \xrightarrow{f^\sharp_\mathfrak{q}} & \mathcal{O}_{\mathrm{Spec}(B),\mathfrak{q}} \\
\downarrow & & \downarrow \\
A_{\phi^{-1}(\mathfrak{q})} & \xrightarrow{\phi_\mathfrak{q}} & B_\mathfrak{q}.
\end{array}$$

For any section $s : V \to \coprod_{\mathfrak{p}\in\mathrm{Spec}(A)} A_\mathfrak{p}$, the section $f^\sharp(V)(s) : f^{-1}(V) \to \coprod_{\mathfrak{q}\in\mathrm{Spec}(B)} B_\mathfrak{q}$ is given by

$$f^\sharp(V)(s)(\mathfrak{q}) = \phi_\mathfrak{q}(s(\phi^{-1}(\mathfrak{q}))).$$

Indeed, $[(f^\sharp(V)(s), f^{-1}(V))] = f^\sharp_\mathfrak{q}([(s,V)])$ and $\phi_\mathfrak{q}(s(f(\mathfrak{q}))) = f^\sharp(V)(s)(\mathfrak{q})$. □

Proposition 4.14

For any $f \in A$, we have a canonical isomorphism of locally ringed space

$$(D(f), \mathcal{O}_{\mathrm{Spec}(A)}|_{D(f)}) \cong (\mathrm{Spec}(A_f), \mathcal{O}_{\mathrm{Spec}(A_f)}) \tag{4.4.4.2}$$

♡

Proof. Let $\pi : A \to A_f, a \mapsto a/1$ be the canonical morphism. It induces a morphism

$$(\varphi, \varphi^\sharp) : (\mathrm{Spec}(A_f), \mathcal{O}_{\mathrm{Spec}(A_f)}) \longrightarrow (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}).$$

The map $\varphi : \mathrm{Spec}(A_f) \to \mathrm{Spec}(A)$ induces an isomorphism

$$\varphi : \mathrm{Spec}(A_f) \longleftrightarrow D(f), \quad \mathfrak{q} \longmapsto \varphi(\mathfrak{q}) = \pi^{-1}(\mathfrak{q}) \nsubseteq V((f)).$$

Since for any prime ideal $\mathfrak{p} \in D(f)$,

$$\begin{aligned}
(\mathcal{O}_{\mathrm{Spec}(A)}|_{D(f)})_\mathfrak{p} &= \mathcal{O}_{\mathrm{Spec}(A),\mathfrak{p}} \cong A_\mathfrak{p} \\
&= (A_f)_{\varphi^{-1}(\mathfrak{p})} = \mathcal{O}_{\mathrm{Spec}(A_f),\varphi^{-1}(\mathfrak{p})} = (\varphi_*\mathcal{O}_{\mathrm{Spec}(A_f)})_\mathfrak{p}.
\end{aligned}$$

Therefore $\mathcal{O}_{\mathrm{Spec}(A)}|_{D(f)} \cong \varphi_*\mathcal{O}_{\mathrm{Spec}(A_f)}$. □

Corollary 4.4

Let $(X, \mathcal{O}_X)$ be a scheme and U an open subset of X. Then $(U, \mathcal{O}_X|_U)$ is a scheme. We call this scheme an **open subscheme** *of $(X, \mathcal{O}_X)$.*

♡

Proof. Suppose X is covered by affine open subschemes $(U_i, \mathcal{O}_X|_{U_i})$ such that $(U_i, \mathcal{O}_X|_{U_i}) \cong (\mathrm{Spec}(A_i), \mathcal{O}_{\mathrm{Spec}(A_i)})$. Then $U = \cap_{i\in I} U \cap U_i$. Since $U \cap U_i$ are open subsets of U_i, there exist $f_{ij} \in A_i$ such that $U \cap U_i = \cup_{j\in J_i} D(f_{ij})$. Hence

$$(D(f_{ij}), (\mathcal{O}_X|_{U_i})|_{D(f_{ij})}) \cong (\mathrm{Spec}(A_i)_{f_{ij}}, \mathcal{O}_{\mathrm{Spec}(A_i)_{f_{ij}}})$$

is an affine schemes and $(U, \mathcal{O}_X|_U)$ is also a scheme. □

Proposition 4.15

Let X be a scheme and A a ring. Then there is a one-to-one correspondence between the family of morphisms of schemes from X to $\operatorname{Spec}(A)$ and the family of homomorphisms of rings from A to $\mathcal{O}_X(X)$.