Matrix theory Assignment 6

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***Abstract*—This document solves for the angle between a pair of straight lines**

Download all python codes from

https://github.com/saipranavkr/EE5609/codes

and latex-tikz codes from

https://github.com/saipranavkr/EE5609

1 Problem

Prove that the following equation represents two straight lines; find also their point of intersection and the angle between them

$$6y^2 - xy - x^2 + 30y + 36 = 0$$

2 Solution

Given equation is,

$$x^2 + xy - 6y^2 - 30y - 36 = 0 \tag{2.0.1}$$

The general second order equation is given by,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.2}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \tag{2.0.4}$$

The equation (2.0.2) can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.5}$$

Comparing (2.0.1) with (2.0.2),

$$a = 1; b = \frac{1}{2}; c = -6; d = 0; e = -15; f = -36 \tag{2.0.6}$$

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & -6 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 0 \\ -15 \end{pmatrix} \tag{2.0.7}$$

The equation (2.0.5) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \tag{2.0.8}$$

Substituting values from (2.0.6) and (2.0.7) in (2.0.5),

$$\mathbf{x}^T \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & -6 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} 0 & -15 \end{pmatrix}\mathbf{x} - 36 = 0 \tag{2.0.9}$$

Substituting values from (2.0.6) and (2.0.7) in (2.0.8) to prove (2.0.1) represents a pair of straight lines,

$$D = \begin{vmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & -6 & -15 \\ 0 & -15 & -36 \end{vmatrix}$$

Solving the determinant,

$$D = 0 \tag{2.0.10}$$

From (2.0.10), it can be observed that (2.0.1) represents a pair of straight lines

$$Det(\mathbf{V}) = \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & -6 \end{vmatrix} = \frac{-25}{4} < 0 \tag{2.0.11}$$

(2.0.11) indicates that the pair of straight lines do intersect. Now, Let the pair of straight lines in vector form be given by,

$$\mathbf{n_1}^T\mathbf{x} = c_1 \tag{2.0.12}$$

$$\mathbf{n_2}^T\mathbf{x} = c_2 \tag{2.0.13}$$

Equating their product with (2.0.9),

$$(\mathbf{n_1}^T\mathbf{x} - c_1)(\mathbf{n_2}^T\mathbf{x} - c_2) = \mathbf{x}^T \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & -6 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} 0 & -15 \end{pmatrix}\mathbf{x} - 36 \tag{2.0.14}$$

$$\implies \mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 1 \\ 1 \\ -6 \end{pmatrix} \tag{2.0.15}$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\begin{pmatrix}0\\-15\end{pmatrix} \tag{2.0.16}$$
$$c_1c_2 = -36 \tag{2.0.17}$$

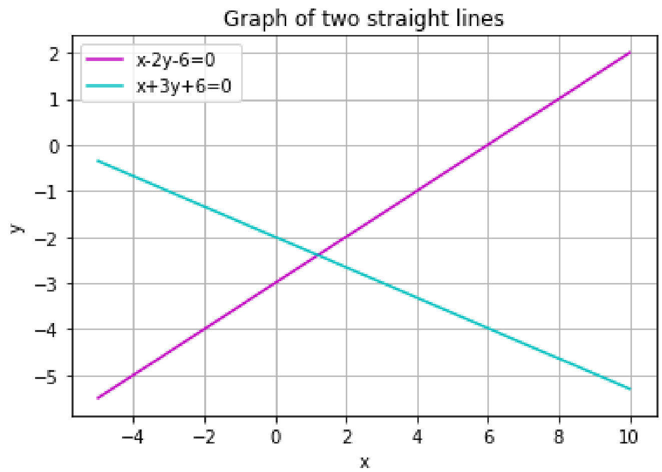

Fig. 1: Pair of straight lines

The slopes of the lines are given by the polynomials,

$$cm^2 + 2bm + a = 0$$
$$m_i = \frac{-b \pm \sqrt{-\left|\mathbf{V}\right|}}{c} \tag{2.0.18}$$
$$\mathbf{n}_i = k_i\begin{pmatrix}-m_i\\1\end{pmatrix}, \quad i = 1,2 \tag{2.0.19}$$

Substituting (2.0.6), (2.0.7) values in the above equations,

$$-6m^2 + m + 1 = 0$$
$$\implies m_i = \frac{\frac{-1}{2} \pm \frac{5}{2}}{-6}$$
$$\implies m_1 = \frac{1}{2}; \quad m_2 = \frac{-1}{3} \tag{2.0.20}$$

Substituting (2.0.20) values in (2.0.19),

$$\mathbf{n}_1 = k_1\begin{pmatrix}\frac{-1}{2}\\1\end{pmatrix}; \quad \mathbf{n}_2 = k_2\begin{pmatrix}\frac{1}{3}\\1\end{pmatrix} \tag{2.0.21}$$

Substituting (2.0.21) values in (2.0.15) and solving,

$$k_1k_2 = -6$$

Taking $k_1 = 2$ and $k_2 = -3$ and simplifying (2.0.21),

$$\mathbf{n}_1 = \begin{pmatrix}-1\\2\end{pmatrix}; \quad \mathbf{n}_2 = \begin{pmatrix}-1\\-3\end{pmatrix} \tag{2.0.22}$$

To verify if the values of $\mathbf{n}_1\mathbf{n}_2$ we compute convolution by representing $\mathbf{n}_1$ as Toeplitz matrix,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix}-1 & 0\\2 & -1\\0 & 2\end{pmatrix}\begin{pmatrix}-1\\-3\end{pmatrix} = \begin{pmatrix}1\\1\\-6\end{pmatrix} \tag{2.0.23}$$

It can be observed that (2.0.23) is equal to (2.0.15) From (2.0.16),

$$\begin{pmatrix}\mathbf{n}_1 & \mathbf{n}_2\end{pmatrix}\begin{pmatrix}c_2\\c_1\end{pmatrix} = -2\begin{pmatrix}0\\-15\end{pmatrix}$$
$$\implies \begin{pmatrix}-1 & -1\\2 & -3\end{pmatrix}\begin{pmatrix}c_2\\c_1\end{pmatrix} = -2\begin{pmatrix}0\\-15\end{pmatrix} \tag{2.0.24}$$

Converting (2.0.24) into row reduced echelon form,

$$\implies \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}c_2\\c_1\end{pmatrix} = \begin{pmatrix}6\\-6\end{pmatrix}$$
$$\implies c_1 = -6; \quad c_2 = 6 \tag{2.0.25}$$

(2.0.12) and (2.0.13) can be rewritten as,

$$\begin{pmatrix}-1 & 2\end{pmatrix}\mathbf{x} = -6 \tag{2.0.26}$$
$$\begin{pmatrix}-1 & -3\end{pmatrix}\mathbf{x} = 6 \tag{2.0.27}$$

3 Angle between the straight lines

Angle between pair of lines is,

$$\theta = \cos^{-1}\left(\frac{\mathbf{n_1}^T\mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|}\right) \tag{3.0.1}$$

$$\mathbf{n}_1^T\mathbf{n}_2 = \begin{pmatrix}-1 & 2\end{pmatrix}\begin{pmatrix}-1\\-3\end{pmatrix} = -5 \tag{3.0.2}$$

$$\|\mathbf{n}_1\| = \sqrt{1+4} = \sqrt{5}; \quad \|\mathbf{n}_2\| = \sqrt{1+9} = \sqrt{10} \tag{3.0.3}$$

Substituting the values from (3.0.3) and (3.0.2) in (3.0.1)

$$\theta = 135° \tag{3.0.4}$$

Hence, angle between the given pair of straight lines is 135°