Matrix theory Assignment 16

K R Sai Pranav

Abstract—**This document contains the concept of dual basis**

Download all python codes from

https://github.com/saipranavkr/EE5609/codes

and latex-tikz codes from

https://github.com/saipranavkr/EE5609

1 Problem

Let $\mathbf{V}$ be the vector space of all polynomial functions p from $\mathbf{R}$ into $\mathbf{R}$ which have degree 2 or less:

$$p(x) = c_0 + c_1 x + c_2 x^2$$

Define three linear functionals on $\mathbf{V}$ by

$$f_1(p) = \int_0^1 p(x)\,dx;\ f_2(p) = \int_0^2 p(x)\,dx;$$
$$f_3(p) = \int_0^{-1} p(x)\,dx$$

Show that $\{f_1, f_2, f_3\}$ is a basis for $\mathbf{V}^*$ by exhibiting the basis for $\mathbf{V}$ of which it is the dual.

2 Theory

Given the basis $\mathbf{F}$ and corresponding dual basis $\mathbf{G}$, the defining property of the dual basis states that:

$$\mathbf{G}^T\mathbf{F} = \mathbf{I}$$
$$\implies \mathbf{G} = (\mathbf{F}^{-1})^T \tag{2.0.1}$$

3 Solution

$$f_1(p) = \int_0^1 p(x)\,dx = c_0 + \frac{1}{2}c_1 + \frac{1}{3}c_2$$
$$f_2(p) = \int_0^2 p(x)\,dx = 2c_0 + 2c_1 + \frac{8}{3}c_2$$
$$f_3(p) = \int_0^{-1} p(x)\,dx = -c_0 + \frac{1}{2}c_1 + \frac{-1}{3}c_2$$

Expressing $\{f_1, f_2, f_3\}$ as basis in terms of a matrix,

$$\mathbf{V} = \{f_1, f_2, f_3\} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 2 & 2 & \frac{8}{3} \\ -1 & \frac{1}{2} & \frac{-1}{3} \end{pmatrix} \tag{3.0.1}$$

Finding the dual basis for (3.0.1) using (2.0.1),

$$\mathbf{V}^* = (\mathbf{V}^{-1})^T$$
$$= \begin{pmatrix} 1 & 1 & \frac{-3}{2} \\ \frac{-1}{6} & 0 & \frac{1}{2} \\ \frac{-1}{3} & 1 & \frac{-1}{2} \end{pmatrix} \tag{3.0.2}$$

The dual basis (3.0.2) can be expressed as,

$$\mathbf{V}^* = \{\alpha_1, \alpha_2, \alpha_3\}$$

where,

$$\alpha_1 = 1 + x - \frac{3}{2}x^2 \tag{3.0.3}$$
$$\alpha_2 = \frac{-1}{6} + \frac{1}{2}x^2 \tag{3.0.4}$$
$$\alpha_3 = \frac{-1}{3} + x + \frac{-1}{2}x^2 \tag{3.0.5}$$

4 Proof

Let the indexed vector sets,

$$\mathbf{V} = \{f_1, f_2, f_3\};\ \mathbf{V}^* = \{\alpha_1, \alpha_2, \alpha_3\}$$

To prove (2.0.1), we have to prove that elements pair have an inner product equal to 1 if the indexes are equal, and equal to 0 otherwise.

$$\implies f_i^T\alpha_j = \delta_j^i = \begin{cases} 1; & if\ i=j \\ 0; & if\ i\neq j \end{cases} \tag{4.0.1}$$

where, δ_j^i is the kronecker delta symbol. Now,

$$f_1^T\alpha_1 = \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \end{pmatrix}^T \begin{pmatrix} 1 \\ 1 \\ -\frac{3}{2} \end{pmatrix} = 1 \tag{4.0.2}$$

$$f_1^T\alpha_2 = \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \end{pmatrix}^T \begin{pmatrix} -\frac{1}{6} \\ 0 \\ \frac{1}{2} \end{pmatrix} = 0 \tag{4.0.3}$$

$$f_1^T \alpha_3 = \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \end{pmatrix}^T \begin{pmatrix} -\frac{1}{3} \\ 1 \\ -\frac{1}{2} \end{pmatrix} = 0 \tag{4.0.4}$$

Similarly, the same can be proved for f_2, f_3 as well in the same manner as (4.0.2)-(4.0.4). Hence we can tell that,

$$\begin{aligned} &\mathbf{V}^T \mathbf{V}^* = \mathbf{I} \\ \implies &\mathbf{V}^* = (\mathbf{V}^T)^{-1} \end{aligned} \tag{4.0.5}$$