Matrix theory Assignment 11

K R Sai Pranav

***Abstract*—This document explains the concept of vector space over a binary field**

Download all python codes from

https://github.com/saipranavkr/EE5609/codes

and latex-tikz codes from

https://github.com/saipranavkr/EE5609

1 Problem

Let V be a vector space over the field $F = \{0, 1\}$. Suppose α, β and γ are linearly independent vectors in V. Comment on $(\alpha + \beta)$, $(\beta + \gamma)$ and $(\gamma + \alpha)$

2 Solution

The addition of elements in the field $\mathbf{F}$ is defined as,

$$\begin{aligned} 0 + 0 &= 0 \\ 1 + 1 &= 0 \end{aligned} \tag{2.0.1}$$

A set are vectors $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ are linearly independent if

$$a\mathbf{v_1} + b\mathbf{v_2} + c\mathbf{v_3} = 0 \tag{2.0.2}$$

has only one trivial solution

$$a = b = c = 0 \tag{2.0.3}$$

Now,

$$a(\alpha + \beta) + b(\beta + \gamma) + c(\gamma + \alpha) = 0 \tag{2.0.4}$$

$$\implies (a + c)\alpha + (a + b)\beta + (b + c)\gamma = 0 \tag{2.0.5}$$

Writing (2.0.5) in matrix form,

$$\mathbf{x}^T\mathbf{V}\mathbf{y} = 0$$

$$\implies \begin{pmatrix} a \\ b \\ c \end{pmatrix}^T \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 0 \tag{2.0.6}$$

where,

$$\mathbf{x} = \begin{pmatrix} a & b & c \end{pmatrix}; \quad \mathbf{y} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

Let,

$$\mathbf{A_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \ \mathbf{A_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \ \mathbf{A_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\implies a\mathbf{A_1} + b\mathbf{A_2} + c\mathbf{A_3} =$$

$$a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.0.7}$$

Now,

$$a\mathbf{A_1} + b\mathbf{A_2} + c\mathbf{A_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.8}$$

$$\implies a = b = c = 0 \tag{2.0.9}$$

From (2.0.8), $\mathbf{A_1}, \mathbf{A_2}$ and $\mathbf{A_3}$ are linearly independent. Now, rewriting (2.0.5) as,

$$(a + c)\mathbf{A_1} + (a + b)\mathbf{A_2} + (b + c)\mathbf{A_3} =$$

$$(a + c)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + (a + b)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} +$$

$$(b + c)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} +$$

$$a\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} +$$

$$b\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\implies (a+c)\mathbf{A_1} + (a+b)\mathbf{A_2} + (b+c)\mathbf{A_3} = a\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix} + b\begin{pmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} + c\begin{pmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{pmatrix} \quad (2.0.10)$$

For $a = b = c = 1$, using properties from (2.0.1), (2.0.10) becomes,

$$(a+c)\mathbf{A_1} + (a+b)\mathbf{A_2} + (b+c)\mathbf{A_3} = \begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix} \quad (2.0.11)$$

Since it is not the trivial solution. Hence, $(\alpha+\beta)$, $(\beta+\gamma)$ and $(\gamma+\alpha)$ are linearly dependent.