Matrix theory Assignment 9

K R Sai Pranav

***Abstract*—This document finds the coordinates of foot of perpendicular using Singular Value Decomposition.**

Download all python codes from

https://github.com/saipranavkr/EE5609/codes

and latex-tikz codes from

https://github.com/saipranavkr/EE5609

1 Problem

Find the coordinates of foot of perpendicular from (1,0,2) to the plane $2x - 3y + z = 0$ using SVD

2 Solution

First we find orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given plane $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m^T n} = 0$$
$$\implies \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0$$
$$\implies 2a - 3b + c = 0 \tag{2.0.1}$$

By substituting $a = 1; b = 0$ in (2.0.1),

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \tag{2.0.2}$$

By substituting $a = 0; b = 1$ in (2.0.1),

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \tag{2.0.3}$$

Now $\mathbf{M}$ can be written as,

$$\mathbf{M} = \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \tag{2.0.4}$$

such that solving $\mathbf{Mx} = \mathbf{b}$ gives the required solution.

$$\implies \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \tag{2.0.5}$$

Applying Singular Value Decomposition on $\mathbf{M}$,

$$\mathbf{M} = \mathbf{USV}^T \tag{2.0.6}$$

Where the columns of $\mathbf{V}$ are the eigenvectors of $\mathbf{M}^T\mathbf{M}$, the columns of $\mathbf{U}$ are the eigenvectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular values of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \tag{2.0.7}$$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \tag{2.0.8}$$

From (2.0.5) and (2.0.6),

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b}$$
$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \tag{2.0.9}$$

Where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$. Calculating eigenvalues of $\mathbf{MM}^T$,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0$$
$$\implies \begin{vmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 3 \\ -2 & 3 & 13-\lambda \end{vmatrix} = 0$$
$$\implies \lambda^3 + 15\lambda^2 - 14\lambda = 0$$

Hence eigenvalues of $\mathbf{MM}^T$ are,

$$\lambda_1 = 14; \quad \lambda_2 = 1; \quad \lambda_3 = 0 \tag{2.0.10}$$

And the corresponding eigenvectors are,

$$\mathbf{u_1} = \begin{pmatrix} \frac{-2}{13} \\ \frac{3}{13} \\ 1 \end{pmatrix}; \quad \mathbf{u_2} = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \end{pmatrix}; \quad \mathbf{u_3} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \tag{2.0.11}$$

Normalizing the above eigenvectors,

$$\mathbf{u_1} = \begin{pmatrix} \frac{-2}{\sqrt{182}} \\ \frac{3}{\sqrt{182}} \\ \frac{13}{\sqrt{182}} \end{pmatrix}; \quad \mathbf{u_2} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix}; \quad \mathbf{u_3} = \begin{pmatrix} \frac{2}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \end{pmatrix} \tag{2.0.12}$$

From (2.0.12) we obtain $\mathbf{U}$ as,

$$\mathbf{U} = \begin{pmatrix} \frac{-2}{\sqrt{182}} & \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{14}} \\ \frac{13}{\sqrt{182}} & 0 & \frac{1}{\sqrt{14}} \end{pmatrix} \tag{2.0.13}$$

Using values from (2.0.10),

$$\mathbf{S} = \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.14}$$

Calculating the eigenvalues of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0$$
$$\implies \begin{vmatrix} 5-\lambda & -6 \\ -6 & 10-\lambda \end{vmatrix} = 0$$
$$\implies \lambda^2 - 15\lambda + 14 = 0$$

Hence, eigenvalues of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = 14; \quad \lambda_5 = 1$$

And the corresponding eigenvectors are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix}; \quad \mathbf{v}_2 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$$

Normalizing the above eigenvectors,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix}; \quad \mathbf{v}_2 = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \tag{2.0.15}$$

From(2.0.15) we obtain $\mathbf{V}$ as,

$$\mathbf{V} = \begin{pmatrix} \frac{-2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \tag{2.0.16}$$

From (2.0.6) we get the Singular Value Decomposition of $\mathbf{M}$,

$$\mathbf{M} = \begin{pmatrix} \frac{-2}{\sqrt{182}} & \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{14}} \\ \frac{13}{\sqrt{182}} & 0 & \frac{1}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{-2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix}^T \tag{2.0.17}$$

Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.18}$$

From (2.0.9),

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{12\sqrt{2}}{\sqrt{91}} \\ \frac{3}{\sqrt{13}} \\ \frac{2\sqrt{2}}{7} \end{pmatrix}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{12}{7\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{3}{7} \\ \frac{6}{7} \end{pmatrix} \tag{2.0.19}$$

To verify the solution obtained from (2.0.19),

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{2.0.20}$$

Substituting the values from (2.0.7) in (2.0.20),

$$\begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix}\mathbf{x} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$$

Converting the above equation into augmented form and solving for $\mathbf{x}$,

$$\begin{pmatrix} 5 & -6 & -3 \\ -6 & 10 & 6 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{5R_2+6R_1}{14}} \begin{pmatrix} 5 & -6 & -3 \\ 0 & 1 & \frac{6}{7} \end{pmatrix}$$
$$\xleftrightarrow{R_1 \leftarrow \frac{R_1+6R_2}{5}} \begin{pmatrix} 1 & 0 & \frac{3}{7} \\ 0 & 1 & \frac{6}{7} \end{pmatrix} \tag{2.0.21}$$

From (2.0.21) it can be observed that,

$$\mathbf{x} = \begin{pmatrix} \frac{3}{7} \\ \frac{6}{7} \end{pmatrix} \tag{2.0.22}$$