Matrix theory Assignment 11

K R Sai Pranav

***Abstract*—This document explains the concept of vector space over a binary field**

Download all python codes from

https://github.com/saipranavkr/EE5609/codes

and latex-tikz codes from

https://github.com/saipranavkr/EE5609

1 Problem

Let V be a vector space over the field $F = \{0, 1\}$. Suppose α, β and γ are linearly independent vectors in V. Comment on $(\alpha + \beta)$, $(\beta + \gamma)$ and $(\gamma + \alpha)$

2 Solution

The addition of elements in the field $\mathbf{F}$ is defined as,

$$\begin{aligned} 0 + 0 &= 0 \\ 1 + 1 &= 0 \end{aligned} \tag{2.0.1}$$

A set are vectors $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ are linearly independent if

$$a\mathbf{v_1} + b\mathbf{v_2} + c\mathbf{v_3} = 0 \tag{2.0.2}$$

has only one trivial solution

$$a = b = c = 0 \tag{2.0.3}$$

Now,

$$a(\alpha + \beta) + b(\beta + \gamma) + c(\gamma + \alpha) = 0 \tag{2.0.4}$$

$$\implies (a + c)\alpha + (a + b)\beta + (b + c)\gamma = 0 \tag{2.0.5}$$

Writing (2.0.5) in matrix form,

$$\begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.0.6}$$

where,

$$\mathbf{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\mathbf{x}^T \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 0 \tag{2.0.7}$$

Since α, β and γ are linearly independent vectors,

$$\mathbf{x}^T \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \tag{2.0.8}$$

Transposing on both sides,

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.0.9}$$

By using the properties from (2.0.1) and reducing (2.0.9) to row echelon form,

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} &\xleftrightarrow{R_2 \leftarrow R_1 + R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\ &\xleftrightarrow{R_3 \leftarrow R_2 + R_3} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{2.0.10}$$

Expressing (2.0.10) as a linear combination of vectors,

$$a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\implies \begin{pmatrix} a + c \\ b + c \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\implies a + c = 0; \quad b + c = 0 \tag{2.0.11}$$

The solutions to (2.0.11) are,

$$a = b = c = 0; \quad a = b = c = 1 \tag{2.0.12}$$

Since there is no trivial solution, $(\alpha+\beta)$, $(\beta+\gamma)$ and $(\gamma + \alpha)$ are linearly dependent