Matrix theory Assignment 19

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Abstract—**This document contains the concept of Jordan canonical form**

Download all python codes from

https://github.com/saipranavkr/EE5609/codes

and latex-tikz codes from

https://github.com/saipranavkr/EE5609

1 Problem

If A is a complex 5×5 matrix with characteristic polynomial $f = (x-2)^3(x+7)^2$ and minimal polynomial $p = (x-2)^2(x+7)$, what is the Jordan form for A?

2 Theory

Table 0 gives the overview of properties of a Jordan block based on characteristic polynomial, minimal polynomial, algebraic multiplicity and geometric multiplicity.

3 Solution

From the properties stated in table 0, the Jordan blocks for eigenvalues of $\mathbf{A}$ can be written as,

$$\mathbf{J_1} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}; \quad \mathbf{J_2} = \begin{pmatrix} -7 & 0 \\ 0 & -7 \end{pmatrix} \tag{3.0.1}$$

Where $\mathbf{J_1}$ and $\mathbf{J_1}$ are the Jordan blocks corresponding to $\lambda_1 = 2$ and $\lambda_1 = -7$ respectively. The Jordan form for $\mathbf{A}$ can be written as,

$$\mathbf{J} = \begin{pmatrix} \mathbf{J_1} & 0 \\ 0 & \mathbf{J_2} \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 0 & -7 \end{pmatrix} \tag{3.0.2}$$

4 Inference

An $n \times n$ matrix with with λ as diagonal elements, ones on the super diagonal and zeroes in all other entries is nilpotent with minimal polynomial $(A - \lambda I)^n$

5 Example

Let,

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.0.1}$$

(5.0.1) is nilpotent for minimal polynomial A^3

Feature	Effect on Jordan block	Example
characteristic polynomial or Algebraic multiplicity	The multiplicity of λ in the characteristic polynomial determines the size of the Jordan block for that eigenvalue. A_M = Size of Jordan block for λ	Let, $f = (x-2)^4$ be characteristic polynomial $\mathbf{J} = \begin{pmatrix} 2 & * & 0 & 0 \\ 0 & 2 & * & 0 \\ 0 & 0 & 2 & * \\ 0 & 0 & 0 & 2 \end{pmatrix}$ where $*$ can be either 1 or 0
Geometric multiplicity	The geometric multiplicity determines the total number of Jordan sub blocks for λ	If $A_M = 4$; $G_M = 2$; $\lambda = 2$ $\implies$ There should be 2 Jordan sub blocks for $\lambda = 2$. So, $\mathbf{J}$ has 2 possibilities $\mathbf{J} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$; or $\mathbf{J} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$
minimal polynomial	The multiplicity of λ in the minimal polynomial determines the size of the largest sub-block (Elementary Jordan block).	Let $p = (x-2)^3$ be minimal polynomial $\implies$ Size of largest sub-block is 3 Hence, one sub-block of size 3 and one sub-block of size 1 $\mathbf{J} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

TABLE 0: Properties of Jordan blocks and Jordan canonical form