Matrix theory Assignment 7

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***Abstract*—This document solves for a point on parabola at which tangent is parallel to a chord**

Download all python codes from

https://github.com/saipranavkr/EE5609/codes

and latex-tikz codes from

https://github.com/saipranavkr/EE5609

1 Problem

Find a point on the curve $y = (x-2)^2$ at which the tangent is parallel to the chord joining the points (2,0) and (4,4)

2 Solution

$y = (x-2)^2$ can be written as,

$$x^2 - 4x - y + 4 = 0 \tag{2.0.1}$$

From (2.0.1),

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \mathbf{u} = \begin{pmatrix} -2 \\ -\frac{1}{2} \end{pmatrix}; f = 4 \tag{2.0.2}$$

$$\left|V\right| = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 \tag{2.0.3}$$

(2.0.3) implies that the curve is a parabola. Now, finding the eigen values corresponding to the $\mathbf{V}$,

$$\left|V - \lambda I\right| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0$$

$$\implies \lambda = 0, 1 \tag{2.0.4}$$

Calculating the eigenvectors corresponding to $\lambda = 0, 1$ respectively,

$$\mathbf{Vx} = \lambda \mathbf{x}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} = 0; \implies \mathbf{p}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.5}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} = 0; \implies \mathbf{p}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.6}$$

By Eigen decomposition on $\mathbf{V}$,

$$\mathbf{V} = \mathbf{PDP}^T$$

$$where, \mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.0.7}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.8}$$

To find the vertex of the parabola,

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \tag{2.0.9}$$

$$where, \eta = \mathbf{u}^T \mathbf{p}_1 = -\frac{1}{2} \tag{2.0.10}$$

Substituting values from (2.0.2), (2.0.5) and (2.0.10) in (2.0.9),

$$\begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix} \tag{2.0.11}$$

Removing last row and representing (2.0.11) as augmented matrix and then converting the matrix to echelon form,

$$\begin{pmatrix} -2 & -1 & -4 \\ 1 & 0 & 2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{-2}} \begin{pmatrix} 1 & \frac{1}{2} & 2 \\ 1 & 0 & 2 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1}$$

$$\begin{pmatrix} 1 & \frac{1}{2} & 2 \\ 0 & -\frac{1}{2} & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow (-2R_2)} \begin{pmatrix} 1 & \frac{1}{2} & 2 \\ 0 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - \frac{R_2}{2}}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.12}$$

From (2.0.12) it can be observed that,

$$\mathbf{c} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{2.0.13}$$

Direction vector of the chord joining A(4,4) and B(2,0) can be calculated as,

$$\mathbf{m} = \mathbf{A} - \mathbf{B} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\implies \mathbf{m} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{2.0.14}$$

We know that,

$$\mathbf{m}^T\mathbf{n} = 0; \implies \mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{2.0.15}$$

To find the point of contact $\mathbf{q}$, which is intersection point for normal of the chord AB and also tangent of the curve,

$$\begin{pmatrix} \mathbf{u}^T + \kappa\mathbf{n}^T \\ \mathbf{V} \end{pmatrix}\mathbf{q} = \begin{pmatrix} -f \\ \kappa\mathbf{n} - \mathbf{u} \end{pmatrix} \tag{2.0.16}$$

$$where, \kappa = \frac{\mathbf{p_1}^T\mathbf{u}}{\mathbf{p_1}^T\mathbf{n}} = \frac{1}{2} \tag{2.0.17}$$

Substituting the values from (2.0.2),(2.0.15) and (2.0.17) in (2.0.16),

$$\begin{pmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{q} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} \tag{2.0.18}$$

Removing last row and representing (2.0.18) as augmented matrix and then converting the matrix to echelon form,

$$\begin{pmatrix} -1 & -1 & -4 \\ 1 & 0 & 3 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow (-R_1)} \begin{pmatrix} 1 & 1 & 4 \\ 1 & 0 & 3 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1}$$
$$\begin{pmatrix} 1 & 1 & 4 \\ 0 & -1 & -1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow (-R_2)} \begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & 1 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2}$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{pmatrix} \tag{2.0.19}$$

From (2.0.19), it can be observed,

$$\mathbf{q} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{2.0.20}$$

which is the required point of contact

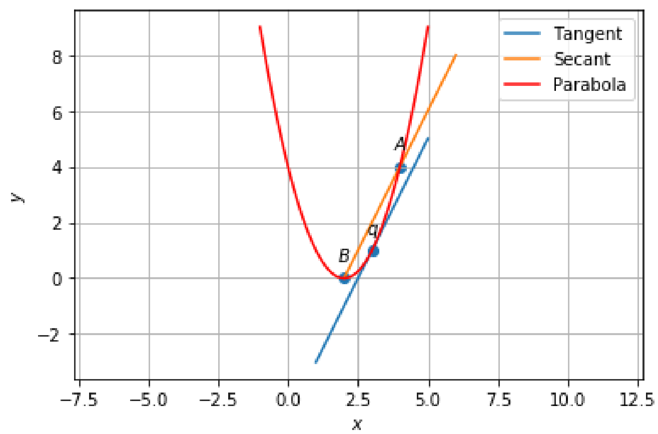

Fig. 1: Parabola with AB as chord, a tangent parallel to the chord