Matrix theory Assignment 17

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***Abstract*—This document contains the concept of natural isomorphism**

Download all python codes from

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https://github.com/saipranavkr/EE5609/codes
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and latex-tikz codes from

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https://github.com/saipranavkr/EE5609
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1 Problem

Let n be a positive integer and F a field. Let $\mathbf{W}$ be the set of all vectors $(x_1, \ldots, x_n)$ in F^n such that $x_1 + \ldots + x_n = 0$. Show that the dual space $\mathbf{W}^*$ of $\mathbf{W}$ can be 'naturally' identified with the linear functionals

$$f(x_1, \ldots, x_n) = c_1 x_1 + \ldots c_n x_n$$

on $\mathbf{F}^n$ which satisfy $c_1 + \ldots + c_n = 0$

2 Pictorial Representation

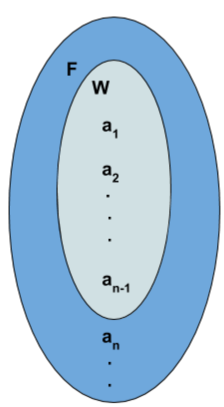

Fig. 0: $\mathbf{W}$ of dimension n-1, is the null space of $\mathbf{F}$, where $(a_1, \ldots, a_{n-1})$ are basis vectors for $\mathbf{W}$

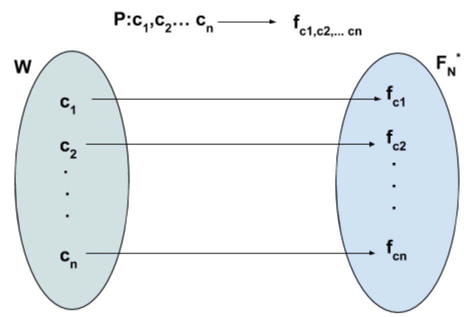

Fig. 0: Mapping from $\mathbf{W} \xrightarrow{P} \mathbf{F_N}^*$, where $P(c_1, \ldots, c_n) = f_{c_1,\ldots,c_n}$, $f_{c_1,\ldots,c_n}(x_1, \ldots, x_n) = c_1 x_1 + \ldots c_n x_n$

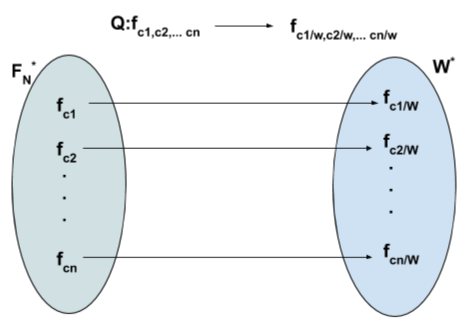

Fig. 0: Mapping from $\mathbf{F_N}^* \xrightarrow{Q} \mathbf{W}^*$, where $f_{c_1,\ldots,c_n}(x_1, \ldots, x_n) = c_1 x_1 + \ldots c_n x_n$ is the linear functional with $c_1 + \ldots + c_n = 0$

3 Solution

Given	$x_1 + \ldots + x_n = 0$ $(x_1, \ldots, x_n) \in \mathbf{W}$ F is a field $\mathbf{W}^*$ is dual space of $\mathbf{W}$
To prove	$\mathbf{W} \to \mathbf{W}^*$ is a natural isomorphism $f(x_1, \ldots, x_n) = c_1 x_1 + \ldots c_n x_n$ which satisfy $c_1 + \ldots + c_n = 0$
Proof	Let $\alpha_i = \epsilon_1 - \epsilon_{i+1}$ $i \in \{1, \ldots, n-1\}$ $\sum_{i=1}^{n-1} c_i \alpha_i = 0$ $\implies \left(\sum_{i=1}^{n-1} c_i\right) \epsilon_1 - \sum_{i=1}^{n-1} c_i \epsilon_{i+1} = 0$ $(\alpha_1, \ldots, \alpha_{n-1})$ are linearly independent and form a basis for $\mathbf{W}$ $\mathbf{W} \xrightarrow{P} (\mathbf{F}^n)^* \xrightarrow{Q} \mathbf{W}^*$ The function P is defined as $P(c_1, \ldots, c_n) = f_{c_1, \ldots, c_n}$; where, $f_{c_1, \ldots, c_n}(x_1, \ldots, x_n) = c_1 x_1 + \ldots c_n x_n$ Let $Q \circ P(c_1, \ldots, c_n) = 0$; $(c_1, \ldots, c_n) \in \mathbf{W}$ $Q(f_{c_1, \ldots, c_n}) = 0 \implies f_{c_1, \ldots, c_n \mid W} = 0$ $\implies f_{c_1, \ldots, c_n}(x_1, \ldots, x_n) = 0$ $f_{c_1, \ldots, c_n}(\alpha_i) = 0; \quad i = 1, \ldots, n-1$ $\implies c_1 = c_i; \quad i = 2, \ldots, n$ $\implies \sum_{i=2}^{n} c_i = (n-1) c_1$ since $(c_1, \ldots, c_n) \in \mathbf{W}$ $\sum_{i=1}^{n} c_i = 0$ $\implies c_1 = 0$ $\implies c_i = 0\ ; \quad i = 1, \ldots, n$ Hence, $f_{c_1, \ldots, c_n}$ is a zero function. Thus the mapping $\mathbf{W} \to \mathbf{W}^*$ is a natural isomorphism