Matrix theory Assignment 16

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Abstract—**This document contains the concept of dual basis**

Download all python codes from

https://github.com/saipranavkr/EE5609/codes

and latex-tikz codes from

https://github.com/saipranavkr/EE5609

1 Problem

Let $\mathbf{V}$ be the vector space of all polynomial functions p from $\mathbf{R}$ into $\mathbf{R}$ which have degree 2 or less:

$$p(x) = c_0 + c_1 x + c_2 x^2$$

Define three linear functionals on $\mathbf{V}$ by

$$f_1(p) = \int_0^1 p(x)\,dx;\ f_2(p) = \int_0^2 p(x)\,dx;$$
$$f_3(p) = \int_0^{-1} p(x)\,dx$$

Show that $\{f_1, f_2, f_3\}$ is a basis for $\mathbf{V}^*$ by exhibiting the basis for $\mathbf{V}$ of which it is the dual.

2 Theory

Given the basis $\mathbf{F}$ and corresponding dual basis $\mathbf{G}$, the defining property of the dual basis states that:

$$\mathbf{G}^T\mathbf{F} = \mathbf{I}$$
$$\implies \mathbf{G} = (\mathbf{F}^{-1})^T \tag{2.0.1}$$

3 Solution

$$f_1(p) = \int_0^1 p(x)\,dx = c_0 + \frac{1}{2}c_1 + \frac{1}{3}c_2$$
$$f_2(p) = \int_0^2 p(x)\,dx = 2c_0 + 2c_1 + \frac{8}{3}c_2$$
$$f_3(p) = \int_0^{-1} p(x)\,dx = -c_0 + \frac{1}{2}c_1 + \frac{-1}{3}c_2$$

Expressing $\{f_1, f_2, f_3\}$ as basis in terms of a matrix,

$$\mathbf{V} = \{f_1, f_2, f_3\} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 2 & 2 & \frac{8}{3} \\ -1 & \frac{1}{2} & \frac{-1}{3} \end{pmatrix} \tag{3.0.1}$$

Finding the dual basis for (3.0.1) using (2.0.1),

$$\mathbf{V}^* = (\mathbf{V}^{-1})^T$$
$$= \begin{pmatrix} 1 & 1 & \frac{-3}{2} \\ \frac{-1}{6} & 0 & \frac{1}{2} \\ \frac{-1}{3} & 1 & \frac{-1}{2} \end{pmatrix} \tag{3.0.2}$$

The dual basis (3.0.2) can be expressed as,

$$\mathbf{V}^* = \{\alpha_1, \alpha_2, \alpha_3\}$$

where,

$$\alpha_1 = 1 + x - \frac{3}{2}x^2 \tag{3.0.3}$$
$$\alpha_2 = \frac{-1}{6} + \frac{1}{2}x^2 \tag{3.0.4}$$
$$\alpha_3 = \frac{-1}{3} + x + \frac{-1}{2}x^2 \tag{3.0.5}$$

4 Proof

Let the indexed vector sets,

$$\mathbf{V} = \{f_1, f_2, f_3\};\ \mathbf{V}^* = \{\alpha_1, \alpha_2, \alpha_3\}$$

1. Let,

$$\mathbf{p}' = \mathbf{c}^T\mathbf{x} \tag{4.0.1}$$

where,

$$\mathbf{c} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix}; \quad \mathbf{x} = \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix}$$

2. Representing the functionals as vector,

$$\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \tag{4.0.2}$$

3. Representing the integrations as vector,

$$\mathbf{I} = \begin{pmatrix} \int_0^1 dx \\ \int_0^2 dx \\ \int_0^{-1} dx \end{pmatrix} \tag{4.0.3}$$

4. So,

$$\mathbf{f} = \mathbf{I}\mathbf{c}^T\mathbf{x} = \mathbf{I}\mathbf{p}' \tag{4.0.4}$$

(4.0.4) can written in matrix format as,

$$\mathbf{f} = \mathbf{P}\mathbf{c} \tag{4.0.5}$$

where,

$$\mathbf{P} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 2 & 2 & \frac{8}{3} \\ -1 & \frac{1}{2} & \frac{-1}{3} \end{pmatrix} \tag{4.0.6}$$

5. **P** is one-one if it has a inverse. Calculating the determinant of **P**,

$$\implies \left|P\right| = -2 \tag{4.0.7}$$

From, (4.0.7), **P** is one-one. Also,

$$\mathbf{V} = \mathbf{f}^T = \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix} \tag{4.0.8}$$

From (4.0.5), (4.0.7) and (4.0.8), the rows of **P** are isomorphic to **V**. So, finding the dual basis by performing matrix operations on $\mathbf{P}^T$

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ \frac{1}{2} & 2 & \frac{1}{2} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{8}{3} & \frac{-1}{3} & 0 & 0 & 1 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 - \frac{R_1}{2}]{R_3 \leftarrow R_3 - \frac{R_1}{3}}$$

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{-1}{2} & 1 & 0 \\ 0 & 2 & 0 & \frac{-1}{3} & 0 & 1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow \frac{R_3 - 2R_2}{-2}}$$

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{-1}{2} & 1 & 0 \\ 0 & 0 & 1 & \frac{-1}{3} & 1 & \frac{-1}{2} \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 - R_3]{R_1 \leftarrow R_1 + R_3}$$

$$\begin{pmatrix} 1 & 2 & 0 & \frac{2}{3} & 1 & \frac{-1}{2} \\ 0 & 1 & 0 & \frac{-1}{6} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{-1}{3} & 1 & \frac{-1}{2} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - 2R_2}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & \frac{-3}{2} \\ 0 & 1 & 0 & \frac{-1}{6} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{-1}{3} & 1 & \frac{-1}{2} \end{pmatrix} \tag{4.0.9}$$

From (4.0.9), the dual of $\mathbf{V} \implies \mathbf{V}^*$ can be written as,

$$\mathbf{V}^* = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} \tag{4.0.10}$$

where,

$$\alpha_1 = 1 + x - \frac{3}{2}x^2 \tag{4.0.11}$$

$$\alpha_2 = \frac{-1}{6} + \frac{1}{2}x^2 \tag{4.0.12}$$

$$\alpha_3 = \frac{-1}{3} + x + \frac{-1}{2}x^2 \tag{4.0.13}$$