Matrix theory Assignment 10

K R Sai Pranav

***Abstract*—This document explains the conversion of a matrix into both row and column reduced echelon form.**

Download all python codes from

https://github.com/saipranavkr/EE5609/codes

and latex-tikz codes from

https://github.com/saipranavkr/EE5609

1 Problem

Let A be a $m \times n$ matrix. Show that by a finite number of elementary row and/or column operations one can pass from A to a matrix R which is both row-reduced echelon and column-reduced echelon , i.e., $R_{ij} = 0$ if $i \neq j$, $R_{ii} = 1$, $1 \leq i \leq r$, $R_{ii} = 0$, if $i > r$. Show that $R = PAQ$, where P is an invertible $m \times m$ matrix and Q is an invertible $n \times n$ matrix.

2 Lemma

Every elementary matrix is invertible and the inverse is again an elementary matrix. If an elementary matrix E is obtained from I by using a certain row or column operation q, then E^{-1} is obtained from I by the "inverse" operation q^{-1}.

3 Solution

Given $\mathbf{A}$ is a $m \times n$ matrix. First converting $\mathbf{A}$ into row reduced echelon form by performing a series of elementary row operations. The elementary matrix that performs sequence of elementary operations is $\mathbf{P}$. Let $\mathbf{R'}$ be the row reduced echelon matrix. So, $\mathbf{P}$ is a $m \times m$ matrix. Also, by using the lemma we can tell that $\mathbf{P}$ is invertible.

$$\mathbf{R'} = \mathbf{PA} \tag{3.0.1}$$

A row of $\mathbf{R'}$ can be all zeroes, or can start with zeroes from the left then has a one, and can have non-zero entries after one. $\mathbf{R'}$ is in row-reduced echelon form. To perform column operations, elementary matrices should be multiplied on the right side in order to convert the $\mathbf{R'}$ into column-echelon form

$$\mathbf{R} = \mathbf{R'Q} \tag{3.0.2}$$

But performing column operations on a matrix is equivalent to performing row operations on the transposed matrix.

$$\begin{aligned} \mathbf{R}^T &= (\mathbf{R'Q})^T \\ \implies \mathbf{R}^T &= \mathbf{Q}^T\mathbf{R'}^T \end{aligned} \tag{3.0.3}$$

Hence, by using lemma it can be observed that $\mathbf{Q}^T$ is invertible and of the order $n \times n$. So, $\mathbf{Q}$ is also invertible. Converting $\mathbf{R}^T$ to row-reduced echelon is equivalent to converting $\mathbf{R}$ to column-reduced echelon.

$$\mathbf{R} = \mathbf{PAQ} \tag{3.0.4}$$

$\mathbf{R}$ in (3.0.4) is in both row and column reduced echelon form. Hence proved.