Matrix theory Assignment 17

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Abstract—**This document contains the concept of natural isomorphism**

Download all python codes from

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https://github.com/saipranavkr/EE5609/codes
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and latex-tikz codes from

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https://github.com/saipranavkr/EE5609
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1 Problem

Let n be a positive integer and F a field. Let $\mathbf{W}$ be the set of all vectors $(x_1, \ldots, x_n)$ in F^n such that $x_1 + \ldots + x_n = 0$. Show that the dual space $\mathbf{W}^*$ of $\mathbf{W}$ can be 'naturally' identified with the linear functionals

$$f(x_1, \ldots, x_n) = c_1 x_1 + \ldots c_n x_n$$

on $\mathbf{F}^n$ which satisfy $c_1 + \ldots + c_n = 0$

2 Solution

Given,

$$\begin{aligned} & x_1 + \ldots + x_n = 0; \ (x_1, \ldots, x_n) \in \mathbf{W} \\ & \implies dim(\mathbf{W}) = n - 1 \end{aligned} \tag{2.0.1}$$

We know that dual spaces have the same dimension. So,

$$dim(\mathbf{W}) = dim(\mathbf{W}^*) = n - 1 \tag{2.0.2}$$

Let,

$$\alpha_i = \epsilon_1 - \epsilon_{i+1}; \quad i \in \{1, \ldots, n-1\} \tag{2.0.3}$$

where, $(\epsilon_1, \ldots, \epsilon_n)$ are the standard basis for $\mathbf{F}^n$.

$$\begin{aligned} & \sum_{i=1}^{n-1} c_i \alpha_i = 0 \\ & \implies \left(\sum_{i=1}^{n-1} c_i \right) \epsilon_1 - \sum_{i=1}^{n-1} c_i \epsilon_{i+1} = 0 \end{aligned} \tag{2.0.4}$$

From (2.0.4), $(\alpha_1, \ldots, \alpha_{n-1})$ are linearly independent and form a basis for $\mathbf{W}$. Now, in order to identify an element of $\mathbf{W}^*$ with an unique element in $\mathbf{W}$, consider,

$$\mathbf{W} \xrightarrow{P} (\mathbf{F}^n)^* \xrightarrow{Q} \mathbf{W}^* \tag{2.0.5}$$

The function P is defined as,

$$P(c_1, \ldots, c_n) = f_{c_1, \ldots, c_n} \tag{2.0.6}$$

where,

$$f_{c_1, \ldots, c_n}(x_1, \ldots, x_n) = c_1 x_1 + \ldots c_n x_n \tag{2.0.7}$$

Let,

$$\begin{aligned} Q \circ P(c_1, \ldots, c_n) = 0; \quad (c_1, \ldots, c_n) \in \mathbf{W} \\ Q(f_{c_1, \ldots, c_n}) = 0 \implies f_{c_1, \ldots, c_n | W} = 0 \\ \implies f_{c_1, \ldots, c_n}(x_1, \ldots, x_n) = 0 \end{aligned} \tag{2.0.8}$$

Thus,

$$\begin{aligned} f_{c_1, \ldots, c_n}(\alpha_i) = 0; \quad i = 1, \ldots, n-1 \\ \implies c_1 = c_i; \quad i = 2, \ldots, n \\ \implies \sum_{i=2}^{n} c_i = (n-1) c_1 \end{aligned} \tag{2.0.9}$$

since $(c_1, \ldots, c_n) \in \mathbf{W}$

$$\begin{aligned} & \sum_{i=1}^{n} c_i = 0 \\ & \implies c_1 = 0 \\ & \implies c_i = 0 \ ; \quad i = 1, \ldots, n \end{aligned} \tag{2.0.10}$$

Hence, $f_{c_1, \ldots, c_n}$ is a zero function. Thus the mapping $\mathbf{W} \rightarrow \mathbf{W}^*$ is a natural isomorphism