Matrix theory Assignment 15

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***Abstract*—This document contains the concept similar matrices**

Download all python codes from

https://github.com/saipranavkr/EE5609/codes

and latex-tikz codes from

https://github.com/saipranavkr/EE5609

1 Problem

Let θ be a real number. Prove that the following two matrices are similar over the field of complex numbers:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

2 Theory

Two matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be similar iff there exists a invertible matrix $\mathbf{P}$ such that:

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} \tag{2.0.1}$$

3 Solution

Let,

$$\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}; \quad \mathbf{B} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \tag{3.0.1}$$

Finding the characteristic polynomial of $\mathbf{A}$,

$$\begin{aligned} \left|A - \lambda I\right| &= \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} \\ &= (\cos\theta - \lambda)^2 + \sin^2\theta \\ &= 1 + \lambda^2 + 2\lambda\cos\theta \end{aligned} \tag{3.0.2}$$

The eigenvalues can be calculated by equating the characteristic polynomial to zero. The eigenvalues are,

$$\lambda_1 = \cos\theta + i\sin\theta; \ \lambda_2 = \cos\theta - i\sin\theta \tag{3.0.3}$$

The eigenvectors corresponding to (3.0.3) are,

$$\alpha_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}; \ \alpha_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix} \tag{3.0.4}$$

$$\mathbf{P} = \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix} = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \tag{3.0.5}$$

Now,

$$\begin{aligned} \mathbf{P}^{-1}\mathbf{A}\mathbf{P} &= \frac{1}{2i}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{pmatrix} \\ &= \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \mathbf{B} \end{aligned} \tag{3.0.6}$$

Hence, from (3.0.6), $\mathbf{A}$ and $\mathbf{B}$ are similar matrices.