Cayley-Hamilton Theorem

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***Abstract*—This document explains Cayley-Hamilton theorem with proof**

Download all python codes from

https://github.com/saipranavkr/EE5609/codes

and latex-tikz codes from

https://github.com/saipranavkr/EE5609

1 Problem

Proof for Cayley-Hamilton theorem

2 Theorem Statement

Every square matrix satisfies it's own characteristic equation. Suppose a square matrix A of size nxn, then it should satisfy it's characteristic equation $\left|\mathbf{A} - \lambda I\right| = 0$

$$\implies \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

$$\implies a_0 + a_1\mathbf{A} + a_2\mathbf{A^2} + \cdots + a_n\mathbf{A^n} = 0 \quad (2.0.1)$$

3 Proof

We know that,

$$\mathbf{A}^{-1} = \frac{Adj(\mathbf{A})}{\left|\mathbf{A}\right|} \implies \left|\mathbf{A}\right|\mathbf{A}^{-1} = Adj(\mathbf{A})$$

Multiplying the above equation with $\mathbf{A}$ on both sides,

$$\left|\mathbf{A}\right|\mathbf{I} = \mathbf{A}Adj(\mathbf{A}) \quad (3.0.1)$$

Substituting $\mathbf{A} = Adj(\mathbf{A})$ in the equation (3.0.1),

$$\left|\mathbf{A} - \lambda\mathbf{I}\right|\mathbf{I} = (\mathbf{A} - \lambda\mathbf{I})Adj(\mathbf{A} - \lambda\mathbf{I}) \quad (3.0.2)$$

Let,

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n = V(\lambda) \quad (3.0.3)$$

Also, adjoint matrix can be expanded as,

$$Adj(\mathbf{A} - \lambda\mathbf{I}) = B_0 + B_1\lambda + B_2\lambda^2 + \cdots + B_{n-1}\lambda^{n-1} \quad (3.0.4)$$

Multiplying (3.0.4) with $(\mathbf{A} - \lambda\mathbf{I})$,

$$\begin{aligned} \implies (\mathbf{A} - \lambda\mathbf{I})(Adj(\mathbf{A} - \lambda\mathbf{I})) &= (B_0 + B_1\lambda + B_2\lambda^2 + \\ &\cdots + B_{n-1}\lambda^{n-1})(\mathbf{A} - \lambda\mathbf{I}) \\ \implies (\mathbf{A} - \lambda\mathbf{I})(Adj(\mathbf{A} - \lambda\mathbf{I})) &= AB_0 + AB_1\lambda + \\ AB_2\lambda^2 + \cdots + AB_{n-1}\lambda^{n-1} &- B_0\lambda - B_1\lambda^2 - \\ &B_2\lambda^3 - \cdots - B_{n-1}\lambda^n \\ \implies (\mathbf{A} - \lambda\mathbf{I})(Adj(\mathbf{A} - \lambda\mathbf{I})) &= AB_0 + (AB_1 - B_0)\lambda + \\ &(AB_2 - B_1)\lambda^2 + \cdots - B_{n-1}\lambda^n \end{aligned} \quad (3.0.5)$$

Substituting (3.0.3) and (3.0.5) in (3.0.2),

$$\begin{aligned} a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n &= AB_0 + (AB_1 - B_0)\lambda + \\ &(AB_2 - B_1)\lambda^2 + \cdots - B_{n-1}\lambda^n \end{aligned} \quad (3.0.6)$$

Comparing the common terms on both sides of (3.0.6),

$$a_0 = AB_0 \quad (3.0.7)$$
$$a_1 = AB_1 - B_0 \quad (3.0.8)$$
$$a_2 = AB_2 - B_1 \quad (3.0.9)$$
$$\vdots$$
$$a_n = -B_{n-1} \quad (3.0.10)$$

Multiplying (3.0.7) by $\mathbf{I}$, (3.0.8) by $\mathbf{A}$, (3.0.9) by $\mathbf{A^2} \cdots$, (3.0.10) by $\mathbf{A^n}$

$$a_0 = AB_0 \quad (3.0.11)$$
$$a_1A = A^2B_1 - AB_0 \quad (3.0.12)$$
$$a_2A^2 = A^3B_2 - A^2B_1 \quad (3.0.13)$$
$$\vdots$$
$$a_nA^n = -A^nB_{n-1} \quad (3.0.14)$$

Summing equations (3.0.11) to (3.0.14),

$$\implies a_0 + a_1\mathbf{A} + a_2\mathbf{A^2} + \cdots + a_n\mathbf{A^n} = 0 \quad (3.0.15)$$

It can be observed that (3.0.15) is same as (2.0.1). Hence, Cayley-Hamilton theorem is proved.