Matrix theory Assignment 16

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***Abstract*—This document contains the concept of dual basis**

Download all python codes from

https://github.com/saipranavkr/EE5609/codes

and latex-tikz codes from

https://github.com/saipranavkr/EE5609

1 Problem

Let $\mathbf{V}$ be the vector space of all polynomial functions p from $\mathbf{R}$ into $\mathbf{R}$ which have degree 2 or less:

$$p(x) = c_0 + c_1 x + c_2 x^2$$

Define three linear functionals on $\mathbf{V}$ by

$$f_1(p) = \int_0^1 p(x)\,dx;\ f_2(p) = \int_0^2 p(x)\,dx;$$
$$f_3(p) = \int_0^{-1} p(x)\,dx$$

Show that $\{f_1, f_2, f_3\}$ is a basis for $\mathbf{V}^*$ by exhibiting the basis for $\mathbf{V}$ of which it is the dual.

2 Theory

Given the basis $\mathbf{F}$ and corresponding dual basis $\mathbf{G}$, the defining property of the dual basis states that:

$$\mathbf{G}^T\mathbf{F} = \mathbf{I}$$
$$\implies \mathbf{G} = (\mathbf{F}^{-1})^T \tag{2.0.1}$$

3 Solution

$$f_1(p) = \int_0^1 p(x)\,dx = c_0 + \frac{1}{2}c_1 + \frac{1}{3}c_2$$
$$f_2(p) = \int_0^2 p(x)\,dx = 2c_0 + 2c_1 + \frac{8}{3}c_2$$
$$f_3(p) = \int_0^{-1} p(x)\,dx = -c_0 + \frac{1}{2}c_1 + \frac{-1}{3}c_2$$

Expressing $\{f_1, f_2, f_3\}$ as basis in terms of a matrix,

$$\mathbf{V} = \{f_1, f_2, f_3\} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 2 & 2 & \frac{8}{3} \\ -1 & \frac{1}{2} & \frac{-1}{3} \end{pmatrix} \tag{3.0.1}$$

Finding the dual basis for (3.0.1) using (2.0.1),

$$\mathbf{V}^* = (\mathbf{V}^{-1})^T$$
$$= \begin{pmatrix} 1 & 1 & \frac{-3}{2} \\ \frac{-1}{6} & 0 & \frac{1}{2} \\ \frac{-1}{3} & 1 & \frac{-1}{2} \end{pmatrix} \tag{3.0.2}$$

The dual basis (3.0.2) can be expressed as,

$$\mathbf{V}^* = \{\alpha_1, \alpha_2, \alpha_3\}$$

where,

$$\alpha_1 = 1 + x - \frac{3}{2}x^2 \tag{3.0.3}$$
$$\alpha_2 = \frac{-1}{6} + \frac{1}{2}x^2 \tag{3.0.4}$$
$$\alpha_3 = \frac{-1}{3} + x + \frac{-1}{2}x^2 \tag{3.0.5}$$