Matrix theory Assignment 10

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Abstract—**This document explains the conversion of a matrix into both row and column reduced echelon form.**

Download all python codes from

https://github.com/saipranavkr/EE5609/codes

and latex-tikz codes from

https://github.com/saipranavkr/EE5609

1 Problem

Let A be a $m \times n$ matrix. Show that by a finite number of elementary row and/or column operations one can pass from A to a matrix R which is both row-reduced echelon and column-reduced echelon , i.e., $R_{ij} = 0$ if $i \neq j$, $R_{ii} = 1$, $1 \leq i \leq r$, $R_{ii} = 0$, if $i > r$. Show that $R = PAQ$, where P is an invertible $m \times m$ matrix and Q is an invertible $n \times n$ matrix.

2 Lemma

Every elementary matrix is invertible and the inverse is again an elementary matrix. If an elementary matrix E is obtained from I by using a certain row or column operation q, then E^{-1} is obtained from I by the "inverse" operation q^{-1}.

3 Solution

Given $\mathbf{A}$ is a $m \times n$ matrix. Converting $\mathbf{A}$ into row reduced echelon form by performing a series of elementary row operations $\mathbf{P}$. Let $\mathbf{R'}$ be the row reduced echelon matrix. Also, by using the lemma we can tell that $\mathbf{P}$ is invertible and order $m \times m$.

$$\mathbf{R'} = \mathbf{PA} \tag{3.0.1}$$

where,

$$\mathbf{R'} = \begin{pmatrix} \mathbf{I} & \mathbf{F} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$\mathbf{I}$ is an identity matrix, $\mathbf{F}$ is Free variables matrix and $\mathbf{0}$ represents a block of zeroes

$\mathbf{R'}$ is in row-reduced echelon form. To perform column operations, elementary matrices should be multiplied on the right side in order to convert the $\mathbf{R'}$ into column-reduced echelon form

$$\mathbf{R} = \mathbf{R'Q} \tag{3.0.2}$$

But performing column operations on a matrix is equivalent to performing row operations on the transposed matrix.

$$\mathbf{R}^T = (\mathbf{R'Q})^T$$
$$\implies \mathbf{R}^T = \mathbf{Q}^T\mathbf{R'}^T \tag{3.0.3}$$

Hence, by using lemma it can be observed that $\mathbf{Q}^T$ is invertible and of the order $n \times n$. Converting $\mathbf{R}^T$ to row-reduced echelon is equivalent to converting $\mathbf{R}$ to column-reduced echelon.

$$\mathbf{R} = \mathbf{PAQ} \tag{3.0.4}$$

where,

$$\mathbf{R} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{3.0.5}$$

$\mathbf{I}$ is an identity matrix and $\mathbf{0}$ represents a block of zeroes. $\mathbf{Q}$ is a upper triangular matrix. $\mathbf{R}$ in (3.0.4) is in both row and column reduced echelon form. Hence proved.

4 Example

Let,

$$\mathbf{A} = \begin{pmatrix} 1 & 5 & 6 & 2 \\ 2 & 12 & 7 & 5 \end{pmatrix} \tag{4.0.1}$$

To convert (4.0.1) into row reduced echelon form, $\mathbf{A}$ has to be multiplied by $\mathbf{P}$

$$\mathbf{P} = \begin{pmatrix} 1 & -5 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 6 & -\frac{5}{2} \\ -1 & \frac{1}{2} \end{pmatrix} \tag{4.0.2}$$

$$\mathbf{R'} = \mathbf{PA} = \begin{pmatrix} 1 & 0 & \frac{37}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{5}{2} & \frac{1}{2} \end{pmatrix} \tag{4.0.3}$$

$\mathbf{R}'$ is in row reduced echelon form. To convert (4.0.3) into column-reduced echelon form, elementary operations have to be performed on $\mathbf{R}'^T$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-37}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{37}{2} & -\frac{5}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -\frac{5}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -\frac{5}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -\frac{5}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{5}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -\frac{5}{2} \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.0.4}$$

By multiplying all the elementary matrices,

$$\mathbf{Q}^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{37}{2} & \frac{5}{2} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 1 \end{pmatrix} \tag{4.0.5}$$
$$\implies \mathbf{Q} = \begin{pmatrix} 1 & 0 & -\frac{37}{2} & \frac{1}{2} \\ 0 & 1 & \frac{5}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.0.6}$$

So $\mathbf{PAQ}$ is in both row-reduced and column-reduced echelon form.

$$\mathbf{R} = \mathbf{PAQ} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{4.0.7}$$

The inverses of $\mathbf{P}$ and $\mathbf{Q}$ are,

$$\mathbf{P}^{-1} = \begin{pmatrix} 1 & 5 \\ 2 & 12 \end{pmatrix}; \quad \mathbf{Q}^{-1} = \begin{pmatrix} 1 & 0 & \frac{37}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{5}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.0.8}$$