Matrix theory Assignment 6

K R Sai Pranav

***Abstract*—This document solves for the angle between a pair of straight lines**

Download all python codes from

https://github.com/saipranavkr/EE5609/codes

and latex-tikz codes from

https://github.com/saipranavkr/EE5609

1 Problem

Prove that the following equation represents two straight lines; find also their point of intersection and the angle between them

$$6y^2 - xy - x^2 + 30y + 36 = 0$$

2 Solution

Given equation is,

$$x^2 + xy - 6y^2 - 30y - 36 = 0 \tag{2.0.1}$$

From the above equation, then following values can be found,

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & -6 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 0 \\ -15 \end{pmatrix} \quad f = -36 \tag{2.0.2}$$

Substituting values from (2.0.2) into the given equation,

$$\mathbf{x}^T \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & -6 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} 0 & -15 \end{pmatrix} \mathbf{x} - 36 = 0 \tag{2.0.3}$$

To prove (2.0.1) represents a pair of straight lines,

$$D = \begin{vmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & -6 & -15 \\ 0 & -15 & -36 \end{vmatrix}$$

Solving the determinant,

$$D = 0 \tag{2.0.4}$$

From (2.0.4), it can be observed that (2.0.1) represents a pair of straight lines

$$Det(\mathbf{V}) = \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & -6 \end{vmatrix} = \frac{-25}{4} < 0 \tag{2.0.5}$$

(2.0.5) indicates that the pair of straight lines do intersect. Now, Let the pair of straight lines in vector form be given by,

$$\mathbf{n_1}^T \mathbf{x} = c_1 \tag{2.0.6}$$

$$\mathbf{n_2}^T \mathbf{x} = c_2 \tag{2.0.7}$$

Equating their product with (2.0.3),

$$(\mathbf{n_1}^T \mathbf{x} - c_1)(\mathbf{n_2}^T \mathbf{x} - c_2) = \mathbf{x}^T \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & -6 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} 0 & -15 \end{pmatrix} \mathbf{x} - 36 \tag{2.0.8}$$

$$\implies \mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 1 \\ 1 \\ -6 \end{pmatrix} \tag{2.0.9}$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\begin{pmatrix} 0 \\ -15 \end{pmatrix} \tag{2.0.10}$$

$$c_1 c_2 = -36 \tag{2.0.11}$$

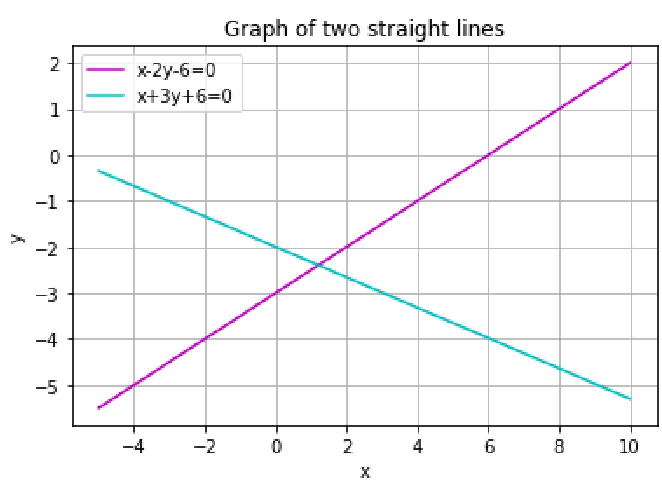

Fig. 1: Pair of straight lines

The slopes of the lines are given by the polynomials,

$$cm^2 + 2bm + a = 0$$

$$m_i = \frac{-b \pm \sqrt{-\left|\mathbf{V}\right|}}{c} \tag{2.0.12}$$

$$\mathbf{n}_i = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix}, \quad i = 1, 2 \tag{2.0.13}$$

Substituting (2.0.2) values in the above equations,

$$-6m^2 + m + 1 = 0$$

$$\implies m_i = \frac{\frac{-1}{2} \pm \frac{5}{2}}{-6}$$

$$\implies m_1 = \frac{1}{2}; \quad m_2 = \frac{-1}{3} \tag{2.0.14}$$

Substituting (2.0.14) values in (2.0.13),

$$\mathbf{n}_1 = k_1 \begin{pmatrix} \frac{-1}{2} \\ 1 \end{pmatrix}; \quad \mathbf{n}_2 = k_2 \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \tag{2.0.15}$$

Substituting (2.0.15) values in (2.0.9) and solving,

$$k_1 k_2 = -6$$

Taking $k_1 = 2$ and $k_2 = -3$ and simplifying (2.0.15),

$$\mathbf{n}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \quad \mathbf{n}_2 = \begin{pmatrix} -1 \\ -3 \end{pmatrix} \tag{2.0.16}$$

To verify if the values of $\mathbf{n}_1 \mathbf{n}_2$ we compute convolution by representing $\mathbf{n}_1$ as Toeplitz matrix,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} -1 & 0 \\ 2 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -6 \end{pmatrix} \tag{2.0.17}$$

It can be observed that (2.0.17) is equal to (2.0.9) From (2.0.10),

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} 0 \\ -15 \end{pmatrix}$$

$$\implies \begin{pmatrix} -1 & -1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} 0 \\ -15 \end{pmatrix} \tag{2.0.18}$$

Converting (2.0.18) into row reduced echelon form,

$$\implies \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$$

$$\implies c_1 = -6; \quad c_2 = 6 \tag{2.0.19}$$

(2.0.6) and (2.0.7) can be rewritten as,

$$\begin{pmatrix} -1 & 2 \end{pmatrix} \mathbf{x} = -6 \tag{2.0.20}$$

$$\begin{pmatrix} -1 & -3 \end{pmatrix} \mathbf{x} = 6 \tag{2.0.21}$$

3 Angle between the straight lines

Angle between pair of lines is,

$$\theta = \cos^{-1}\left(\frac{\mathbf{n_1}^T \mathbf{n_2}}{\|\mathbf{n_1}\| \, \|\mathbf{n_2}\|}\right) \tag{3.0.1}$$

$$\mathbf{n}_1^T \mathbf{n}_2 = \begin{pmatrix} -1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ -3 \end{pmatrix} = -5 \tag{3.0.2}$$

$$\|\mathbf{n}_1\| = \sqrt{1+4} = \sqrt{5}; \quad \|\mathbf{n}_2\| = \sqrt{1+9} = \sqrt{10} \tag{3.0.3}$$

Substituting the values from (3.0.3) and (3.0.2) in (3.0.1)

$$\theta = 135^\circ \tag{3.0.4}$$

Hence, angle between the given pair of straight lines is 135°