Matrix theory Assignment 14

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***Abstract*—This document contains the concept of linear, one-one transformation**

Download all python codes from

https://github.com/saipranavkr/EE5609/codes

and latex-tikz codes from

https://github.com/saipranavkr/EE5609

1 Problem

Let $\mathbf{V}$ be the set of complex numbers regarded as a vector space over the field of real numbers. We define a function T from $\mathbf{V}$ into the space of 2×2 real matrices, as follows. If $z = x + iy$ with x and y real numbers, then

$$\mathbf{T}(z) = \begin{pmatrix} x+7y & 5y \\ -10y & x-7y \end{pmatrix}$$

Verify that T is a one-one (real) linear transformation of $\mathbf{V}$ into the space of 2×2 real matrices.

2 Theory

The kronecker product also called as matrix direct product is defined as

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{pmatrix} \tag{2.0.1}$$

Also,

$$\mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) = \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C} \tag{2.0.2}$$

$$\mathbf{A} \otimes (k\mathbf{B}) = k(\mathbf{A} \otimes \mathbf{B}) \tag{2.0.3}$$

3 Solution

Given,

$$\mathbf{T} : \mathbf{C} \to \mathbf{R}^{2\times2}$$

$$\mathbf{T}(x + iy) = \begin{pmatrix} x+7y & 5y \\ -10y & x-7y \end{pmatrix} \tag{3.0.1}$$

Let,

$$z = x + iy; \quad w = a + ib; \quad z, w \in \mathbf{C}$$

Also the RHS of (3.0.1) can be expressed as,

$$\begin{aligned} \mathbf{T}(\mathbf{z}) &= \left(\begin{pmatrix} 1 & 7 \\ 0 & -10 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \begin{pmatrix} 0 & 5 \\ 1 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right) \\ &= \begin{pmatrix} 1 & 7 & 0 & 5 \\ 0 & -10 & 1 & -7 \end{pmatrix} \begin{pmatrix} x & 0 \\ y & 0 \\ 0 & x \\ 0 & y \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \begin{pmatrix} \mathbf{x} & 0 \\ 0 & \mathbf{x} \end{pmatrix} \end{aligned} \tag{3.0.2}$$

where $\mathbf{A}$ and $\mathbf{B}$ are block matrices and,

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

The diagonal block matrix can be expressed as the kronecker product of $\mathbf{I}$ and $\mathbf{x}$

$$\mathbf{I} \otimes \mathbf{x} = \begin{pmatrix} \mathbf{x} & 0 \\ 0 & \mathbf{x} \end{pmatrix} \tag{3.0.3}$$

Where $\mathbf{I}$ is an identity matrix. (3.0.2) can be rewritten as,

$$\mathbf{T}(\mathbf{z}) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} (\mathbf{I} \otimes \mathbf{x}) \tag{3.0.4}$$

Consider,

$$\mathbf{T}(\alpha\mathbf{z} + \mathbf{w}) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} (\mathbf{I} \otimes (\alpha\mathbf{z} + \mathbf{w}))$$

Using properties (2.0.2), (2.0.3), the above equation can be expressed as,

$$\begin{aligned} \mathbf{T}(\alpha\mathbf{z} + \mathbf{w}) &= \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} (\mathbf{I} \otimes (\alpha\mathbf{z})) + \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} (\mathbf{I} \otimes \mathbf{w}) \\ &= \alpha \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} (\mathbf{I} \otimes \mathbf{z}) + \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} (\mathbf{I} \otimes \mathbf{w}) \\ &= \alpha\mathbf{T}(\mathbf{z}) + \mathbf{T}(\mathbf{w}) \end{aligned} \tag{3.0.5}$$

From (3.0.5), it can be proved that $\mathbf{T}$ is a linear operator.