

Matrix theory Assignment 10

K R Sai Pranav

Abstract—This document explains the conversion of a matrix into both row and column reduced echelon form.

Download all python codes from

<https://github.com/saipranavkr/EE5609/codes>

and latex-tikz codes from

<https://github.com/saipranavkr/EE5609>

1 PROBLEM

Let A be a $m \times n$ matrix. Show that by a finite number of elementary row and/or column operations one can pass from A to a matrix R which is both row-reduced echelon and column-reduced echelon, i.e., $R_{ij} = 0$ if $i \neq j$, $R_{ii} = 1$, $1 \leq i \leq r$, $R_{ii} = 0$, if $i > r$. Show that $R = PAQ$, where P is an invertible $m \times m$ matrix and Q is an invertible $n \times n$ matrix.

2 LEMMA

Every elementary matrix is invertible and the inverse is again an elementary matrix. If an elementary matrix E is obtained from I by using a certain row or column operation q , then E^{-1} is obtained from I by the "inverse" operation q^{-1} .

3 SOLUTION

Given A is a $m \times n$ matrix. Converting A into row reduced echelon form by performing a series of elementary row operations P . Let R' be the row reduced echelon matrix. Also, by using the lemma we can tell that P is invertible and order $m \times m$.

$$R' = PA \quad (3.0.1)$$

where,

$$R' = \begin{pmatrix} I & F \\ 0 & 0 \end{pmatrix}$$

I is an identity matrix, F is Free variables matrix and 0 represents a block of zeroes

R' is in row-reduced echelon form. To perform column operations, elementary matrices should be multiplied on the right side in order to convert the R' into column-reduced echelon form

$$R = R'Q \quad (3.0.2)$$

But performing column operations on a matrix is equivalent to performing row operations on the transposed matrix.

$$\begin{aligned} R^T &= (R'Q)^T \\ \Rightarrow R^T &= Q^T R'^T \end{aligned} \quad (3.0.3)$$

Hence, by using lemma it can be observed that Q^T is invertible and of the order $n \times n$. Converting R^T to row-reduced echelon is equivalent to converting R to column-reduced echelon.

$$R = PAQ \quad (3.0.4)$$

where,

$$R = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad (3.0.5)$$

I is an identity matrix and 0 represents a block of zeroes. Q is a upper triangular matrix. R in (3.0.4) is in both row and column reduced echelon form. Hence proved.

4 EXAMPLE

Let,

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 7 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad (4.0.1)$$

To convert (4.0.1) into row reduced echelon form, A has to be multiplied by P

$$P = \begin{pmatrix} -5 & 3 & 0 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad (4.0.2)$$

$$\mathbf{R}' = \mathbf{P}\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.0.3)$$

\mathbf{R}' is in row reduced echelon form. To convert (4.0.3) into column-reduced echelon form, elementary operations have to be performed on \mathbf{R}'^T . By multiplying all the elementary matrices,

$$\mathbf{Q}^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix} \quad (4.0.4)$$

$$\Rightarrow \mathbf{Q} = \begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.0.5)$$

So $\mathbf{P}\mathbf{A}\mathbf{Q}$ is in both row-reduced and column-reduced echelon form.

$$\mathbf{R} = \mathbf{P}\mathbf{A}\mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.0.6)$$

The inverses of \mathbf{P} and \mathbf{Q} are,

$$\mathbf{P}^{-1} = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 5 & 0 \\ 1 & 3 & 1 \end{pmatrix}; \quad \mathbf{Q}^{-1} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.0.7)$$