Applied Algorithms

CSCI-B505 / INFO-I500

Lecture 7.

Amortized Analysis - 2

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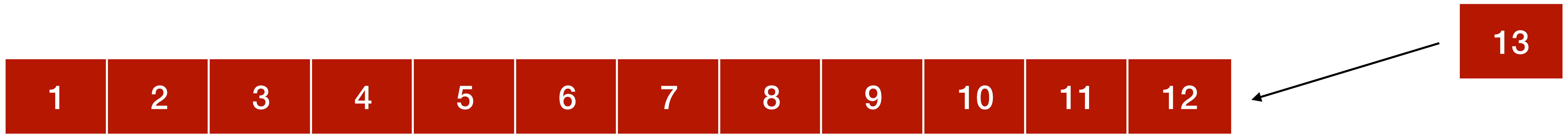
- Dynamic Array Allocation
- Amortized Dictionary Data Structure

Dynamic Arrays

- Remember array is a contiguous block in memory.
- Thus, its size should be definite at the time of creation.
- However, the size of an array can change frequently !
- Therefore, the **dynamic arrays**, whose size can be altered at run time is of our interest. *Notice that many modern programming languages make use of it.*
- **At the end we will see that, theoretically it is no different then static arrays !?**

Dynamic Array Allocation

Main Question: What to do when array is full, in other words, all cells are occupied, and we need to add a new element to this full array.



Allocate a larger size in the memory

Move **old** elements to the new array

Append the **new** item to the new array

Dynamic Array Allocation

Let's consider incrementing the size by a fixed amount each time we need to resize.

- Effort per item is $O(n^2)/n = O(n)$
- Not good, when compared to $O(1)$ in the regular static array.

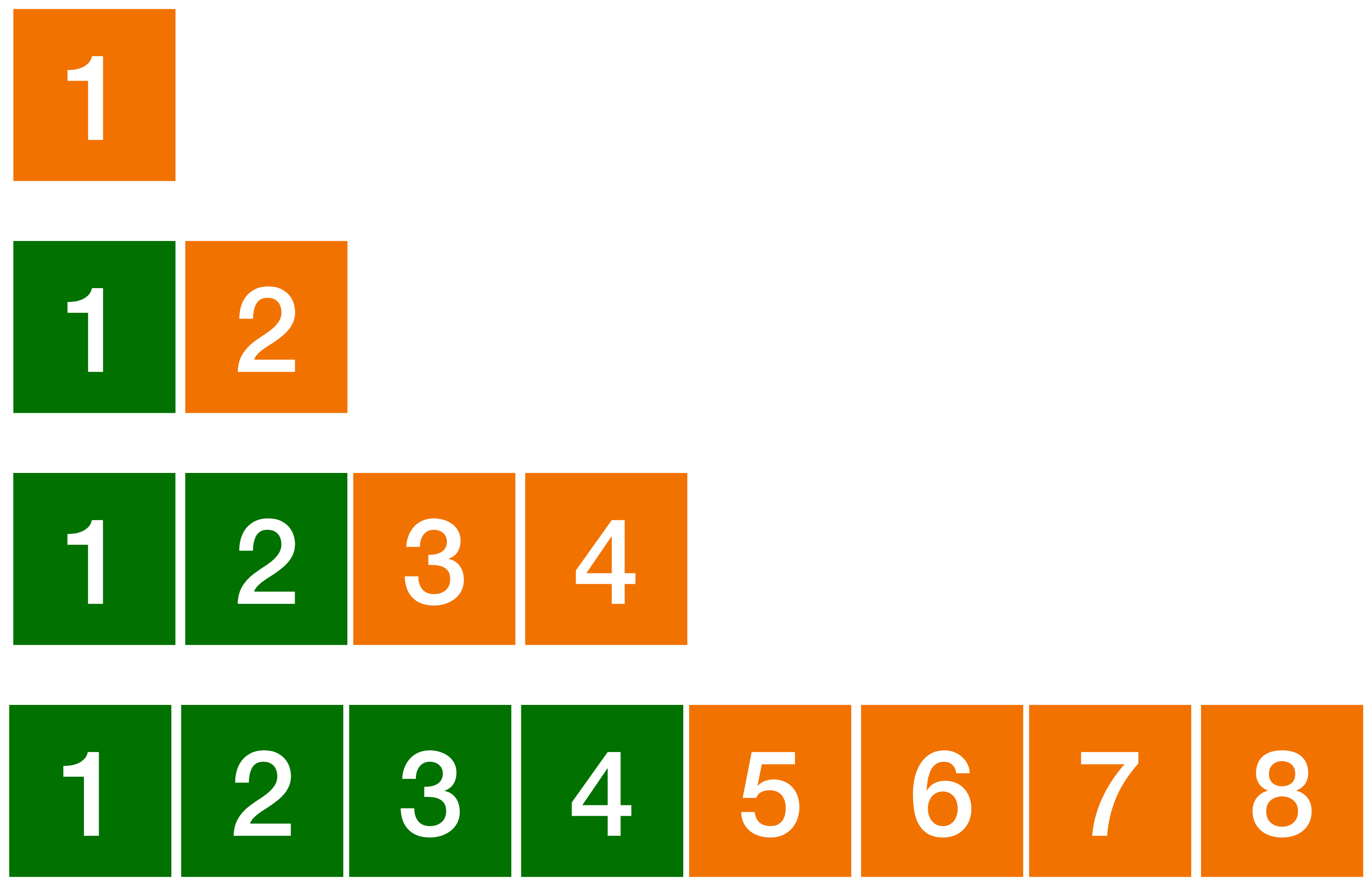
$$m = \left\lfloor \frac{n}{k} \right\rfloor$$

$$k(1 + 2 + \ldots + (m-1)) = \frac{k \cdot m \cdot (m-1)}{2} \in O(k \cdot m^2) \to O(n^2)$$

Dynamic Array Allocation

Let's consider **doubling** the current size each time we need to resize.

- Now, effort per item is $O(n)/n = O(1)$
- Same with the regular static array
- But, where did our extra movements go? !
  - In the hidden constant of $O(1)$, which is 3 on the dynamic case ?



$$(1 + 2 + \ldots + 2^k) = 2^{k+1} - 1 \in O(2^k) \to O(n)$$

$$k = \lfloor \log n \rfloor \to 2^k \approx n$$

Dynamic Array Allocation

Aggregate analysis

insert	old capacity	new capacity	insert cost	copy cost
1	0	1	1	–
2	1	2	1	1
3	2	4	1	2
4	4	4	1	–
5	4	8	1	4
6	8	8	1	–
7	8	8	1	–
8	8	8	1	–
9	8	16	1	8
⋮	⋮	⋮	⋮	⋮

$\lfloor \log n \rfloor$

c_i is the cost of i^{th} insertion to the array

- If $i = 2^k + 1$, for some k>0, then $c_i = i$.
- Else, $c_i = 1$.

$$\sum_{i=1}^{n} c_i \leq n + \sum_{j=0}^{\lfloor \log n \rfloor} 2^j$$

$$< n + 2n$$

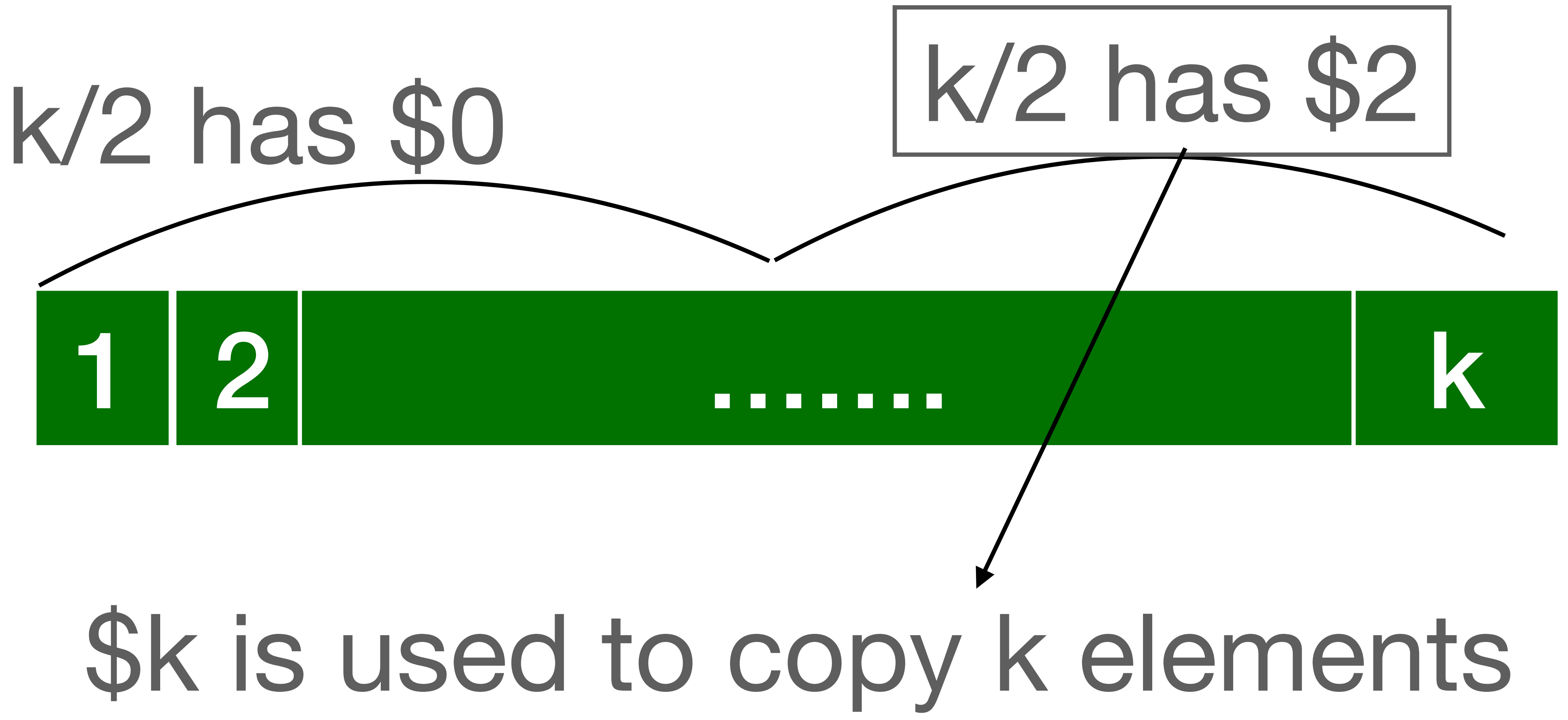
$$< 3n$$

- The cost per item is less than $\dfrac{3n}{n} = 3$
- Therefore, $O(1)$.

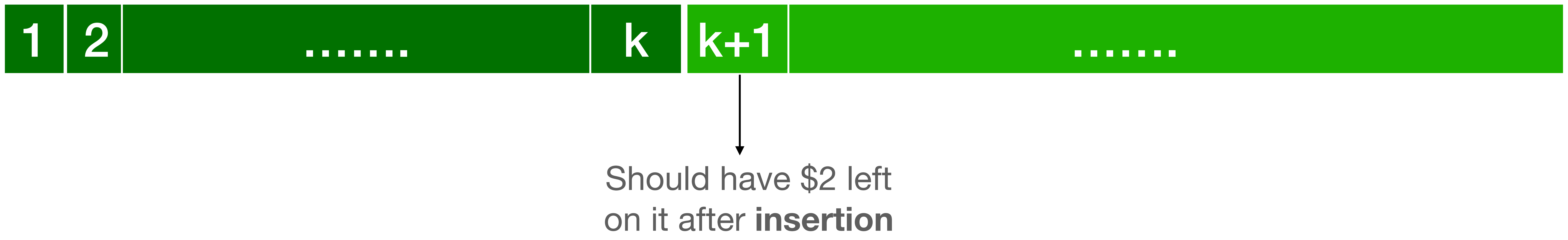
Dynamic Array Allocation

Accounting Method

$3 per each insert will guarantee to avoid bankruptcy.



If half of the elements have $2 on them, it is enough to copy the old items. Since regular $1 cost will be required per each, $3 will surely avoid bankruptcy.



Dynamic Array Allocation

Potential Method

$\phi(D_i) = 2 \cdot size(D_i) - capacity(D_i)$ $\qquad$ $\phi(D_0) = 0$

# of elements *Total space of the array*

$\forall i,\ \ \phi(D_i) \geq 0$, since after each resize capacity is 2 times the number of elements.

Case 1: There is space in the capacity, so no resize is required.

- Actual cost: $c_i = 1$
- $\phi(D_i) - \phi(D_{i-1}) = 2 \cdot size(D_i) - capacity(D_i) - 2 \cdot size(D_{i-1}) + capacity(D_{i-1}) = 2$, since

$$size(D_i) = size(D_{i-1}) + 1$$

$$capacity(D_i) = capacity(D_{i-1})$$

- Amortized cost : $\hat{c}_i = 1 + 2 = 3$

Dynamic Array Allocation

Potential Method

$\phi(D_i) = 2 \cdot \underbrace{size(D_i)}_{\text{\# of elements}} - \underbrace{capacity(D_i)}_{\text{Total space of the array}}$ $\qquad\qquad$ $\phi(D_0) = 0$

$\forall i,\ \ \phi(D_i) \geq 0$, since after each resize capacity is 2 times the number of elements.

Case 2: There is NO space in the capacity, so resizing is required.

- Actual cost: $c_i = 1 + capacity(D_{i-1})$
- $\phi(D_i) - \phi(D_{i-1}) = 2 \cdot size(D_i) - capacity(D_i) - 2 \cdot size(D_{i-1}) + capacity(D_{i-1})$

  $= 2 - capacity(D_{i-1}),$

  since $size(D_i) = size(D_{i-1}) + 1$ and $capacity(D_i) = 2 \cdot capacity(D_{i-1})$
- Amortized cost : $\hat{c}_i = 1 + capacity(D_{i-1}) + 2 - capacity(D_{i-1}) = 3$

Dynamic Array Allocation

- **What if we want to support resize on delete operation ?**
- When the number of items in the array become less, we shrink the array.
- What would be a good strategy?
    - Halve the array when items become 1/2 ???

Amortized Dictionary Data Structure

Dictionary structures are always in the heart of computing.
We have many alternatives. Here is one of those …

Problem: Given n items, provide an efficient way to support search and update.

Main idea:

- Instead of a single list, maintain a collection of arrays, say $A_0, A_1, \ldots, A_{k-1}$
- Array A_i has either exactly 2^i elements or empty with zero elements.
- Each array is sorted.
- There is no relation or order between the arrays.

Depending on the number n, how will we decide on the number of arrays (k=?), and how will we decide which ones will be empty and full ?

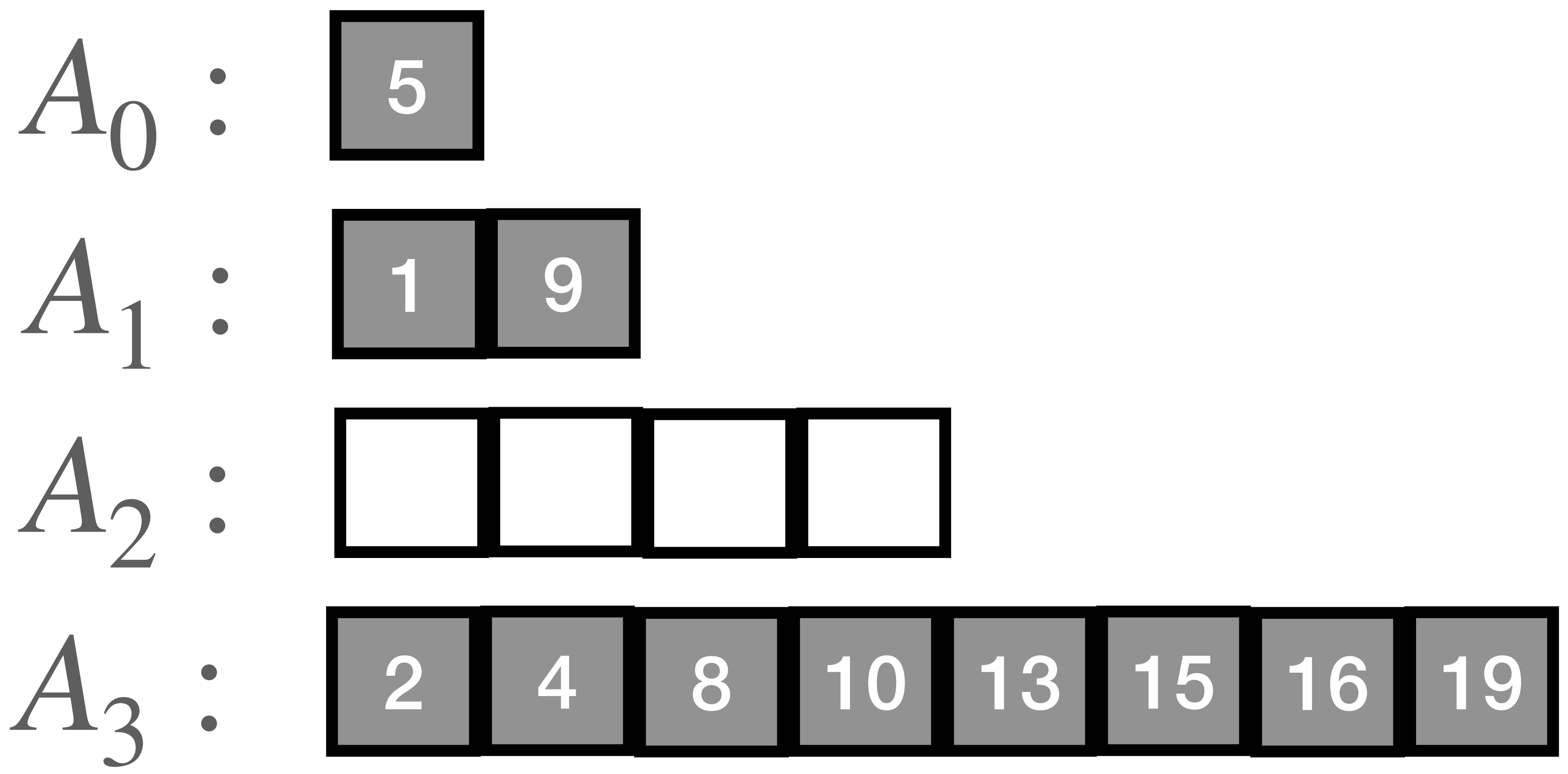
Amortized Dictionary Data Structure

Depending on the number n, how will we decide on the number of arrays (k=?), and how will we decide which ones will be empty and full ?

Each integer can be written as the sum of the powers of 2.
Actually, this is its binary representation !

$(n = 11) \rightarrow n = 1011$ **indicates n=8.1+4.0+2.1+1.1**

4 arrays A_3, A_2, A_1, A_0 **with sizes** $8,4,2,1$**, respectively.** A_2 **is empty.**

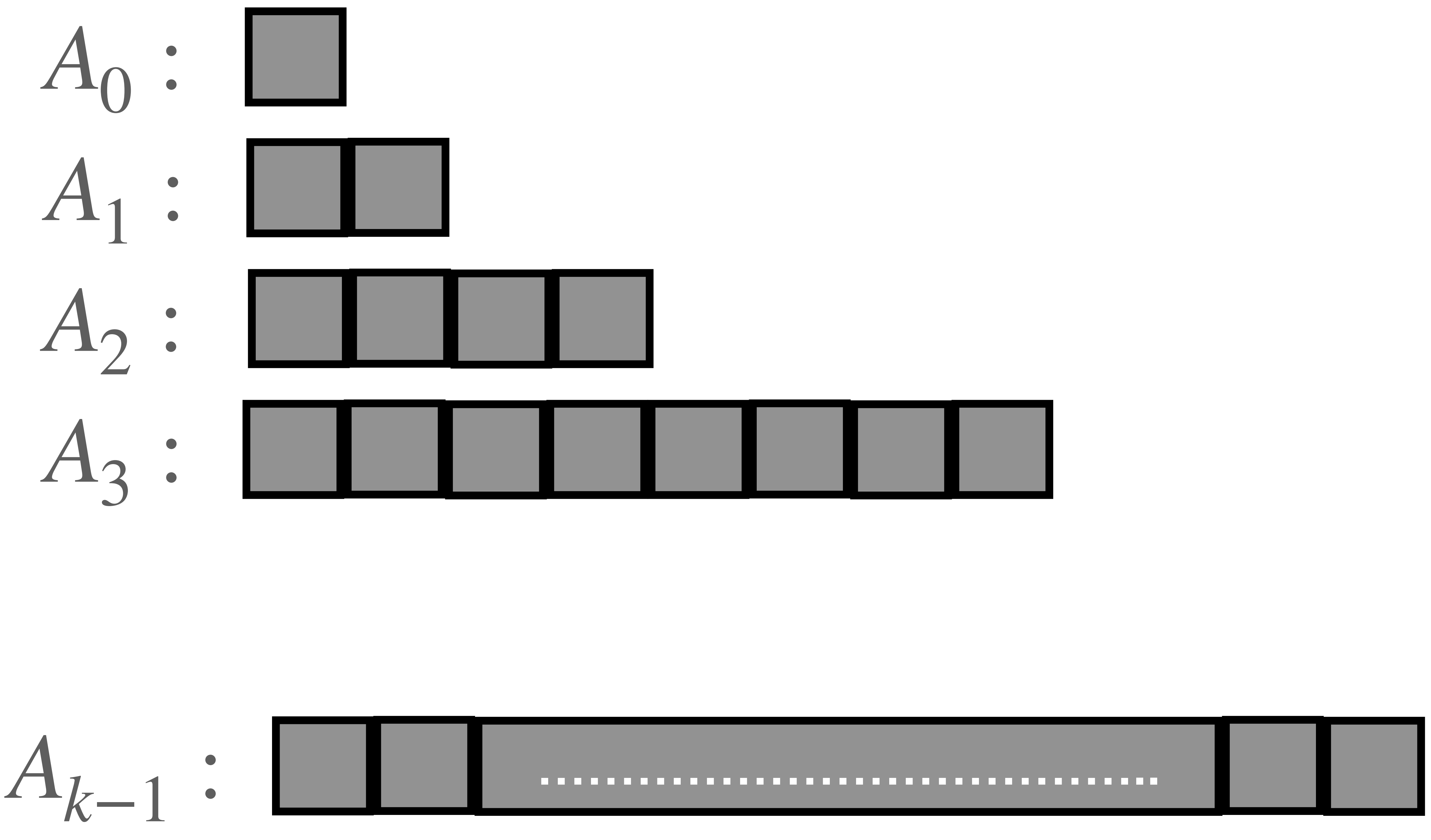


- Given n, we maintain $\lceil \log(n+1) \rceil$ arrays.
- Each has its corresponding size.
- The ones with a 1 bit are full, others empty

What do you think about the construction cost?

Amortized Dictionary Data Structure

Searching for a key on this dictionary which maintains n keys in total ?



$$k = \lceil \log(n+1) \rceil$$

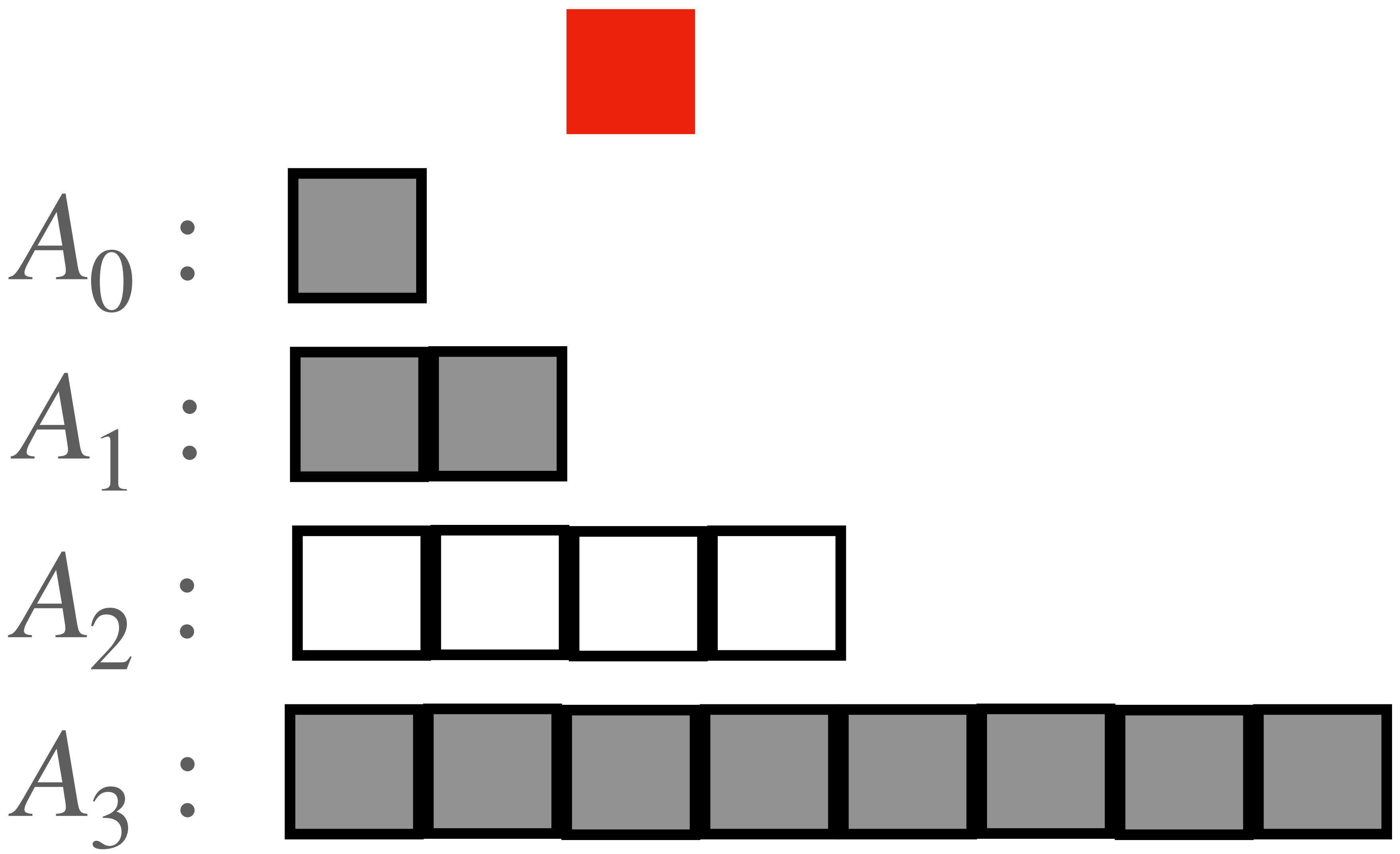
- Investigate each array one-by-one.
- We have $k = \lceil \log(n+1) \rceil$ arrays.
- Search on a sorted array of t elements is $O(\log t)$ via binary search.
- Longest array size $\leq n$.
- At most k arrays will be searched.
- Each search is $O(\log n)$ time.

Then, overall cost of search is

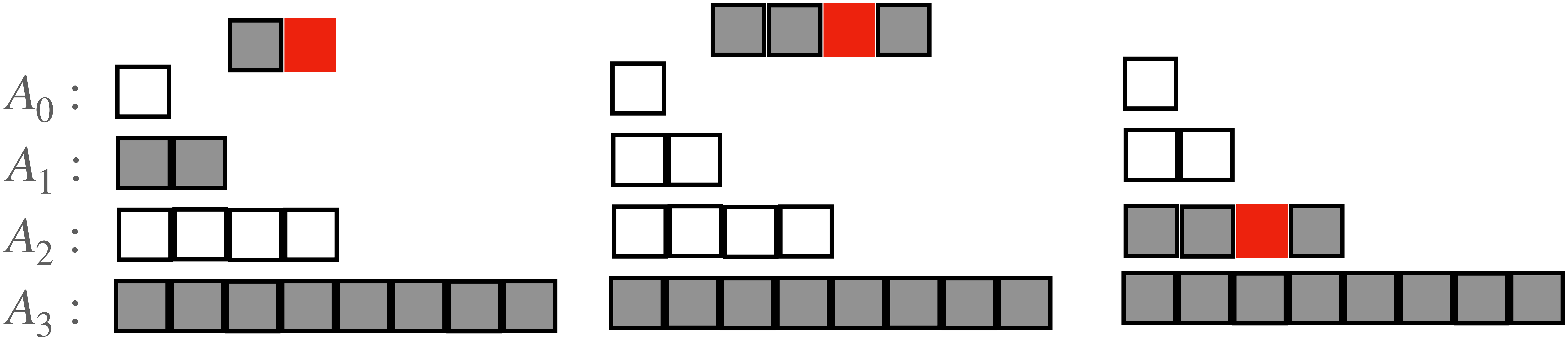
$$k \cdot \log n \in O(\log^2 n)$$

Amortized Dictionary Data Structure

INSERTING a new key



- Put new element into array H of size 1.
- $i = 0$
- Check A_i. If empty, copy H to A_i and stop. Else, $H \leftarrow merge(A_i, H)$ and $i \leftarrow i + 1$ and repeat.



Amortized Dictionary Data Structure

INSERTING a new key

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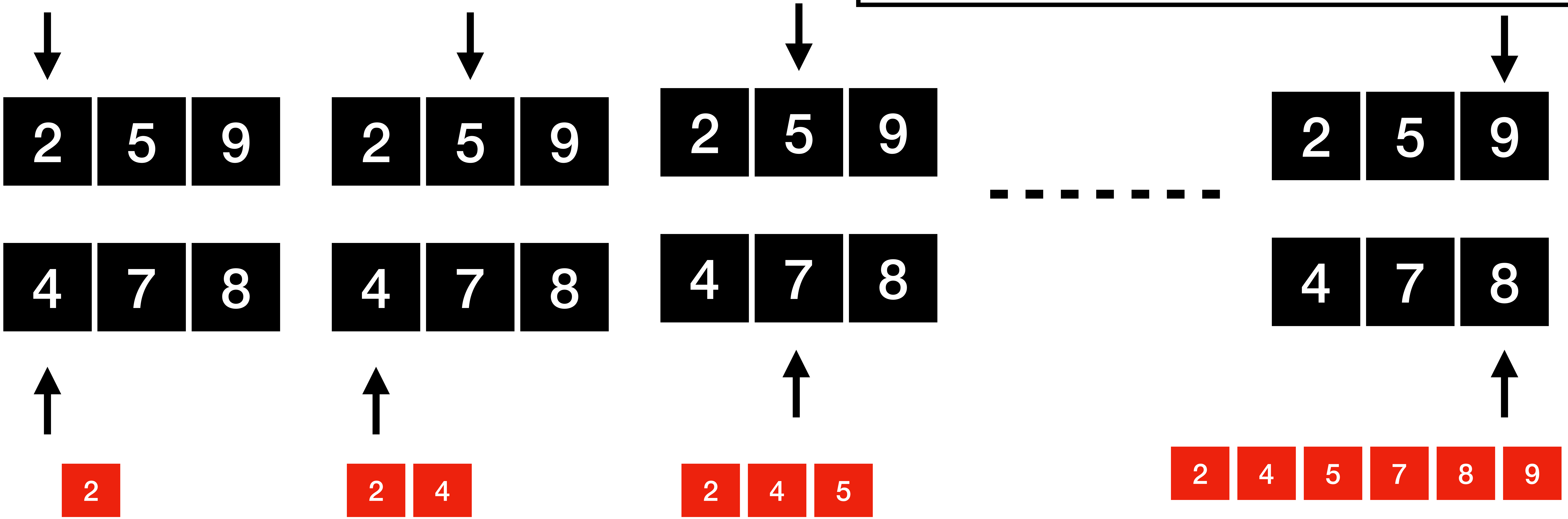
Merging two sorted list, each with size ℓ requires **less than** 2ℓ comparisons !



Amortized Dictionary Data Structure

INSERTING a new key

- Worst case: We visit and merge all arrays in the dictionary, e.g. $n = 2^k - 1$ elements in the dictionary for some k, and we are adding the $2^k th$ element
- What will be the cost of this worst case?
- Merging two sorted list, each with size ℓ requires less than 2ℓ comparisons !
- Therefore, $C = 2 + 4 + 8 + ... + 2^k = 2^{k+1} - 1$. Since $k \in O(\log n)$, $C \in O(n)$.

Once such a worst case happens, can it appear repeatedly ?

NO!

So, regular worst case analysis is **not tight**! We can try an amortized approach by computing the cost of , say t, consecutive insert operations.

Amortized Dictionary Data Structure

INSERTING a new key has $O(\log n)$ amortized complexity.

The merge cost of A_i is at most $2 \cdot 2^i$ as merging two list each with ℓ costs 2ℓ.

During n insertion operations,

- A_0 will be subject to merge $n/2$ times with a cost of 2.
- A_1 will be subject to merge $n/4$ times with a cost of 4.
- A_2 will be subject to merge $n/8$ times with a cost of 8.
- Totally $O(\log n)$ arrays will be subject to merge each with $O(n)$ cost.
- **Therefore, this makes total cost $O(n \log n)$ for n insertions, which makes amortized cost of insertion $O(\log n)$.**

$$\frac{n}{2} \cdot 2 + \frac{n}{4} \cdot 4 + \frac{n}{8} \cdot 8 + \ldots \approx n \cdot \log n$$

This is exactly the same with the binary counter amortized analysis with one difference as the cost of flipping k^{th} bit is 2^k instead of a constant 1 unit.

Amortized Dictionary Data Structure

DELETING a key

- Assume we will be deleting an item from the array A_i that includes 2^i elements.
- Split A_i into small arrays of length $1,2,4,\ldots,2^{i-1}$. Notice that $1 + 2 + 4 + \ldots + 2^{i-1} = 2^i - 1$, which is exactly the number of remaining elements in A_i. Delete all items from A_i.
- For each of these small arrays, insert it into the dictionary again. Insert operation starts with the corresponding list length, i.e., small array of size 1, start with A_0, size 2 start with A_1, and continue accordingly.

There can be at most $\log n$ small arrays after deleting an element. The amortized cost of insertion process is $O(\log n)$ as we showed previously. So the cost of deleting an element in the worst case is $O(\log^2 n)$ with the proposed method.

There might be other ways of deletion as well ?

Reading assignment

- Read chapter 17 Amortized Analysis from Cormen and also related chapters from other text books or resources on the internet.

- Next week we will study recursions and divide-and-conquer type algorithms.