SAIR

Spatial AI & Robotics Lab

CSE 473/573

L11: ALIGNMENT & FITTING

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What are Alignment and Fitting?

- Alignment
  - Find the parameters of a transformation that best aligns matched points
- Fitting
  - Find the parameters of a model that best fit the data

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Fitting and Alignment: Methods

- General Alignment
  - **Homographies**
  - **Rotational Panoramas**
  - Global Alignment
  - RANSAC
  - Warping
  - Blending
- Global optimization / Search for parameters
  - Least squares fit
  - Robust least squares
  - Other parameter search methods

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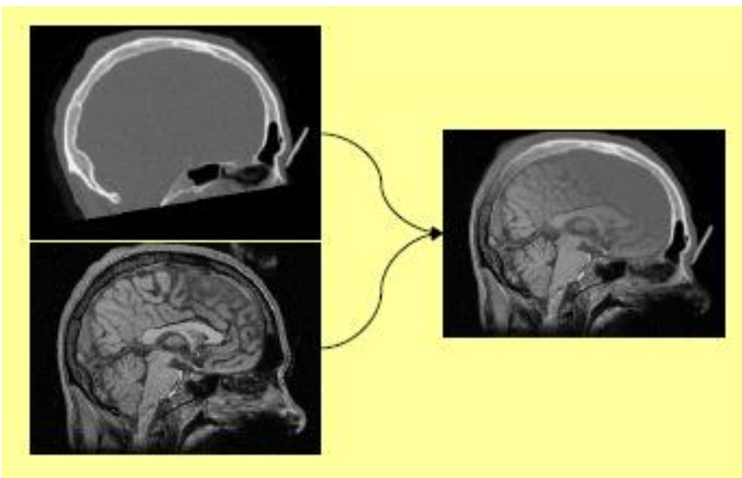
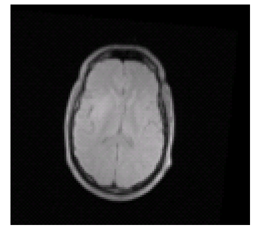
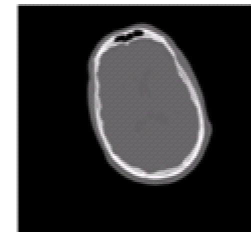
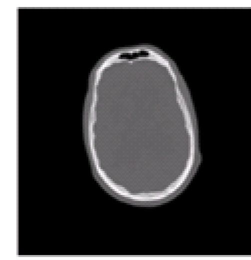
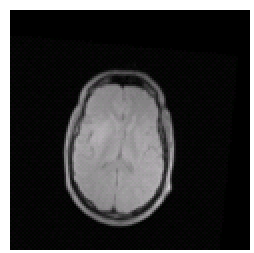
Motivation: Medical image registration

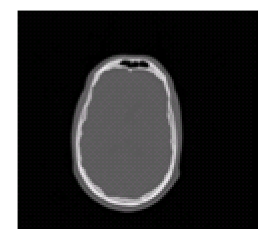












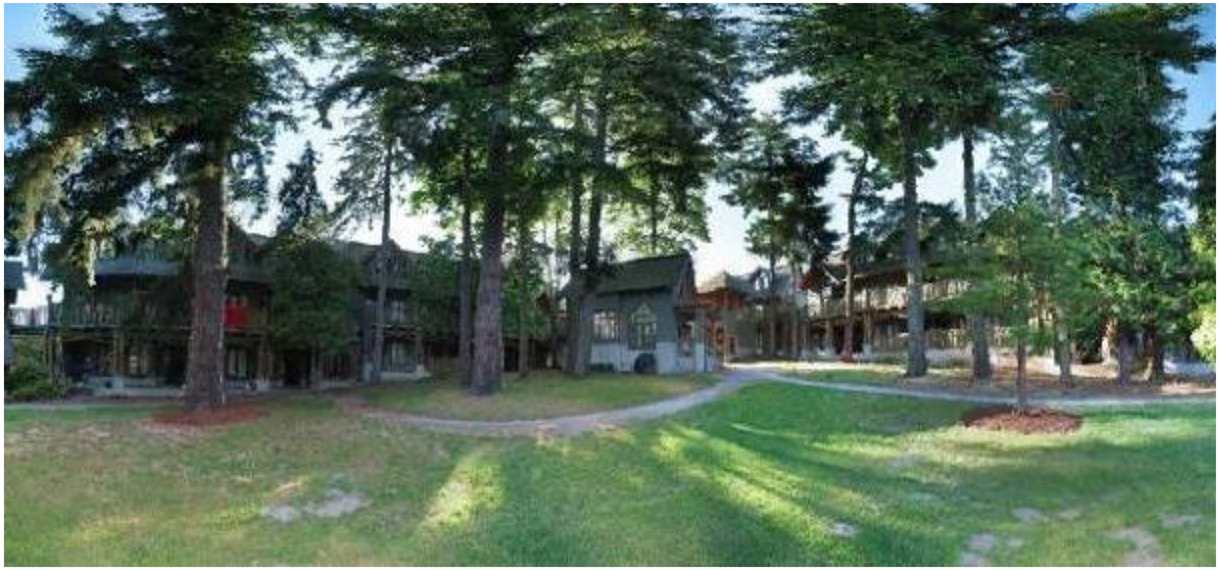
Motivation

- Getting the whole picture
  - Typical camera: 50° x 35°

Brown & Lowe 2003

Motivation

- Getting the whole picture
  - Typical camera: 50° x 35°
  - Human Vision: 176° x 135°



Brown & Lowe 2003

Motivation

- Getting the whole picture
  - Typical camera: $50^{\circ} \times 35^{\circ}$
  - Human Vision: $176^{\circ} \times 135^{\circ}$

Brown & Lowe 2003

Alignment

- Homography
- Rotational Panoramas
- RANSAC (Next Lecture)
- Global alignment
- Warping
- Blending

(a) (b)

Motion models

- What happens when we take two images with a camera and try to align them?
- translation?
- rotation?
- scale?
- affine?
- perspective?

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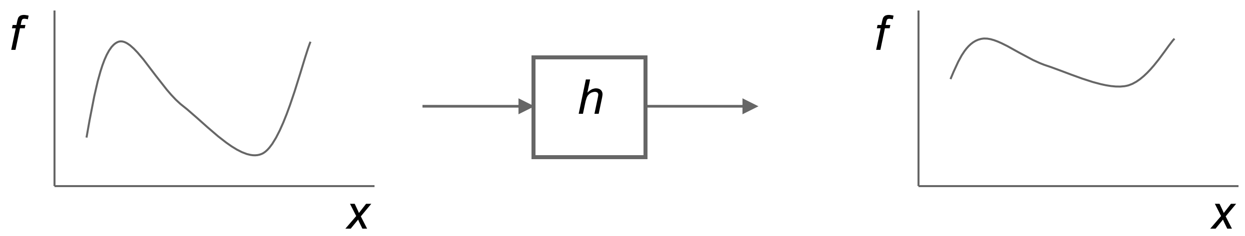
Szeliski

Image Warping (Recap)

- image filtering: change *range* of image
    - $g(x) = h(f(x))$



- image warping: change *domain* of image
    - $g(x) = f(h(x))$

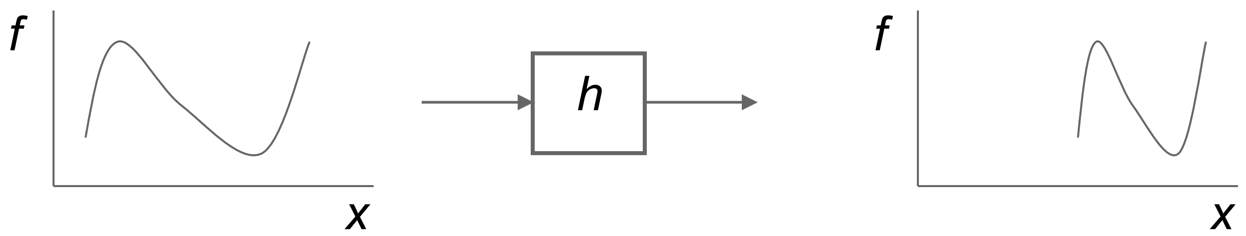


Image Warping

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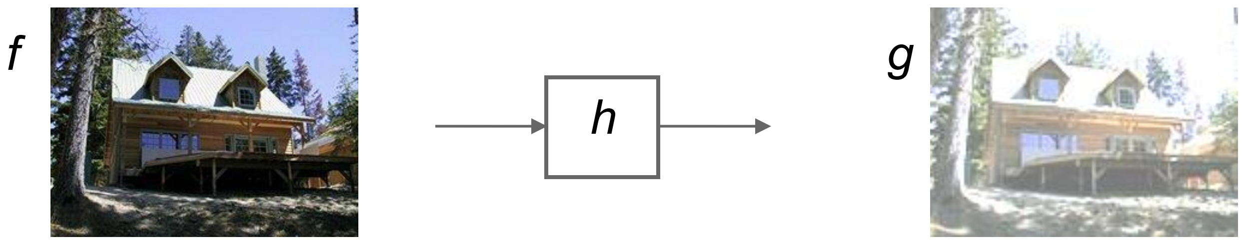
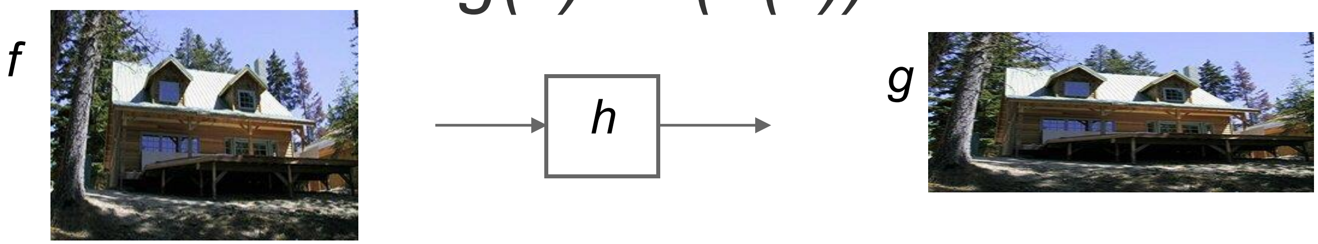


- image warping: change *domain* of image
  - $g(x) = f(h(x))$



Parametric (global) warping

- Examples of parametric warps:

translation

rotation

aspect

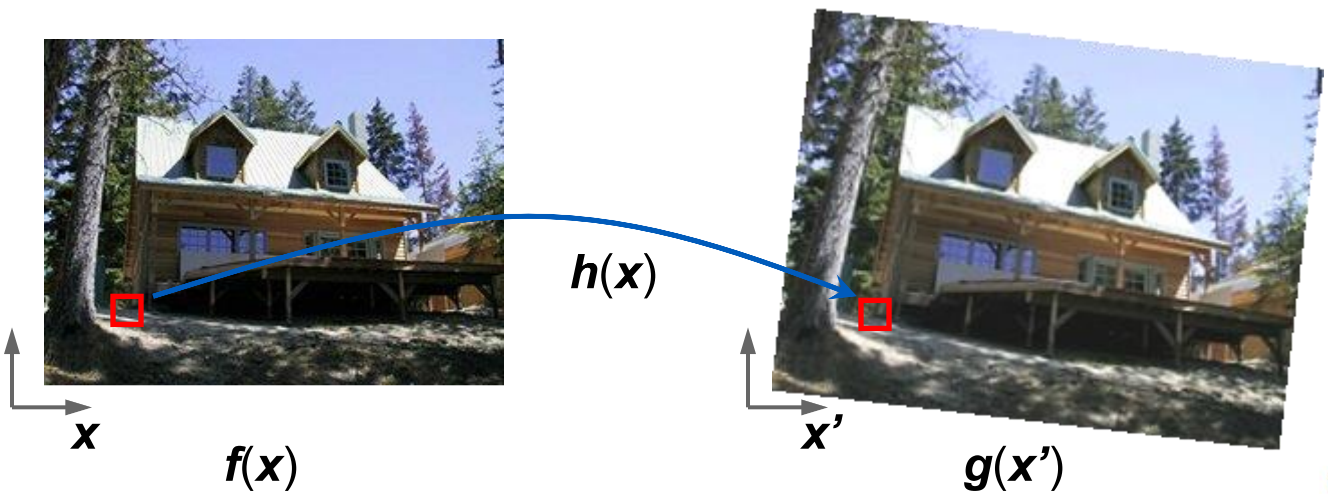
affine

perspective

cylindrical

Image Warping

- Given a coordinate transform $\boldsymbol{x'} = \boldsymbol{h}(\boldsymbol{x})$ and a source image $\boldsymbol{f}(\boldsymbol{x})$, how do we compute a transformed image $\boldsymbol{g}(\boldsymbol{x'}) = \boldsymbol{f}(\boldsymbol{h}(\boldsymbol{x}))$?

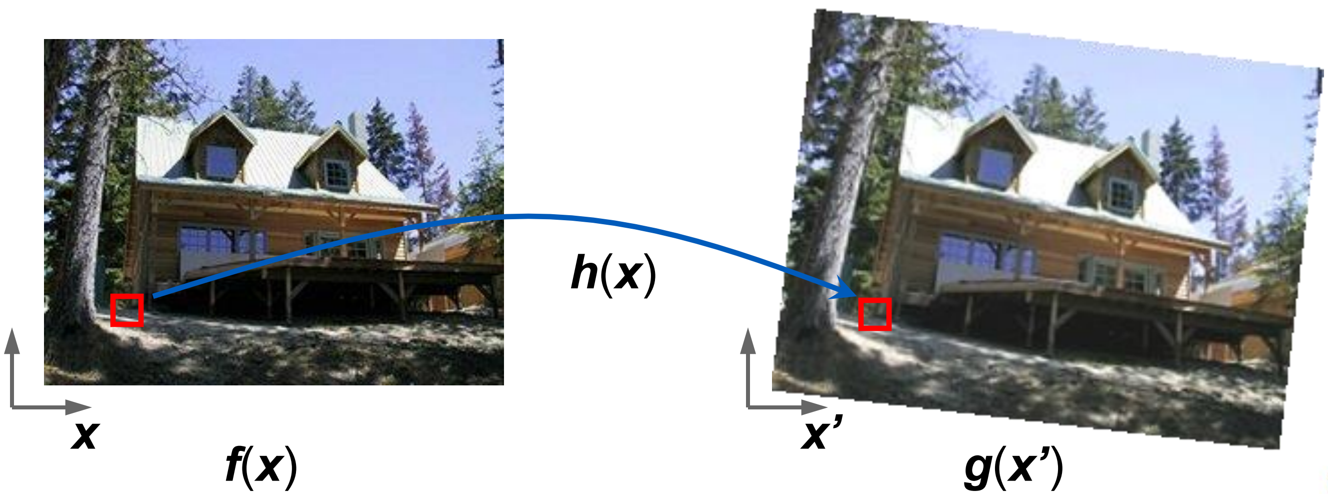


Szeliski

Forward Warping

- Send each pixel $\boldsymbol{f}(\boldsymbol{x})$ to its corresponding location $\boldsymbol{x'} = \boldsymbol{h}(\boldsymbol{x})$ in $\boldsymbol{g}(\boldsymbol{x'})$

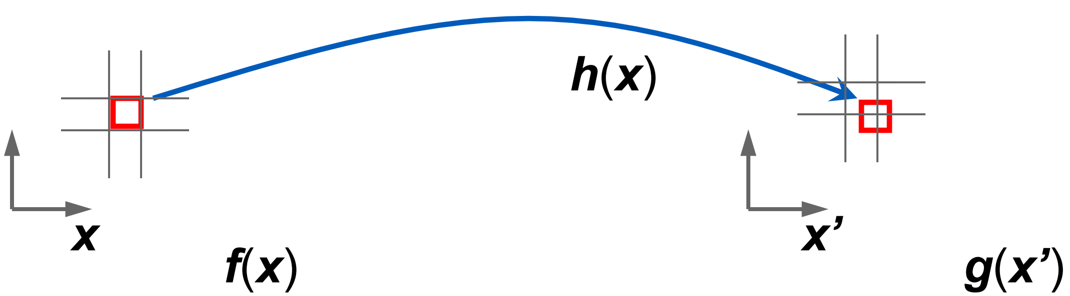
  - What if pixel lands "between" two pixels?



Forward Warping

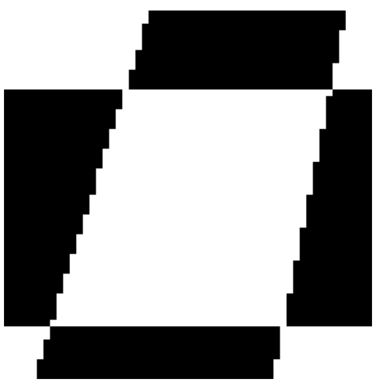
- Send each pixel $\boldsymbol{f}(\boldsymbol{x})$ to its corresponding location $\boldsymbol{x'} = \boldsymbol{h}(\boldsymbol{x})$ in $\boldsymbol{g}(\boldsymbol{x'})$

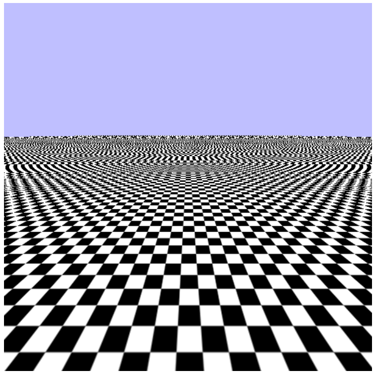
  - What if pixel lands "between" two pixels?
  - Answer: add "contribution" to several pixels, normalize later



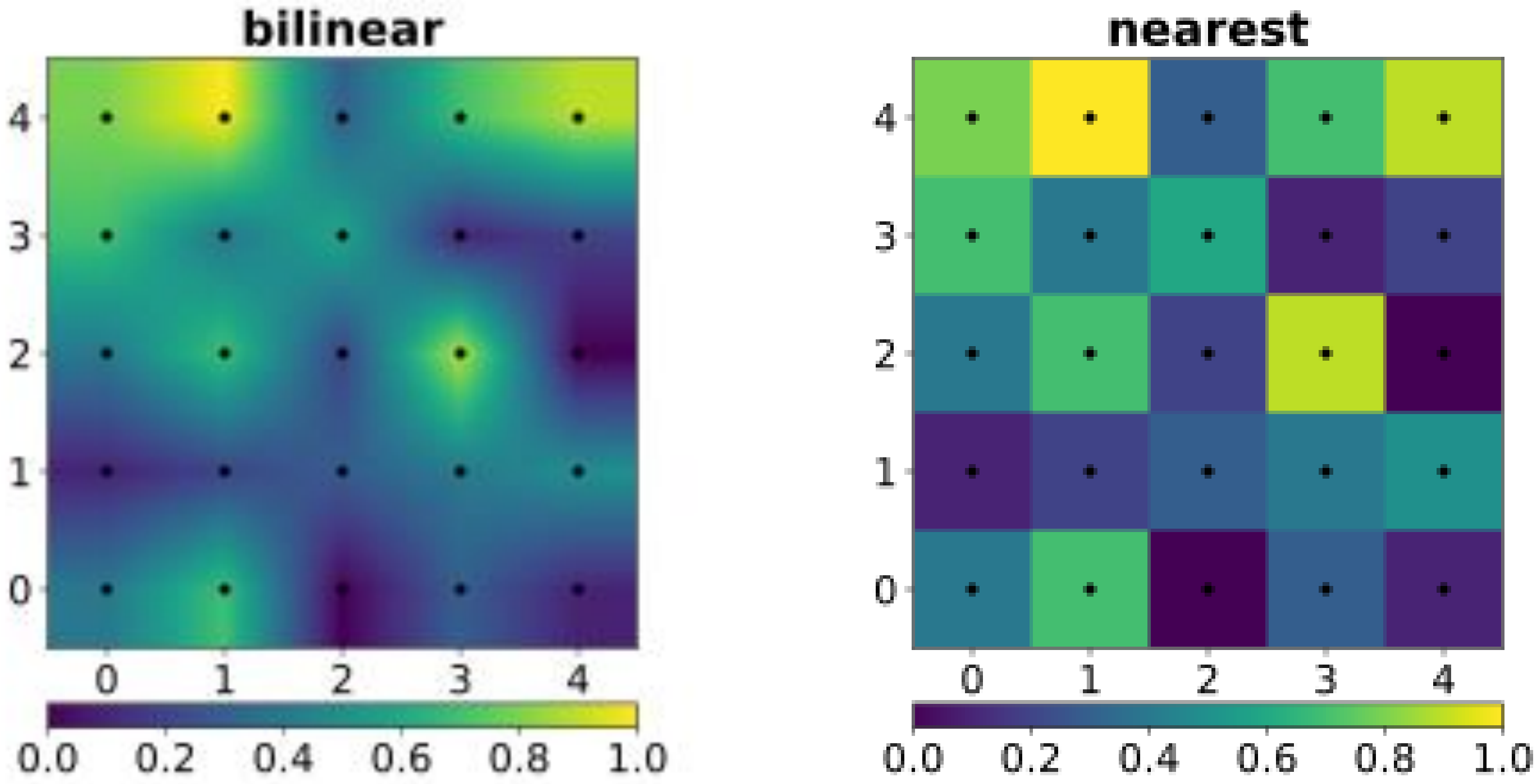
Interpolation

- Possible interpolation filters:
  - **Nearest Neighbor**
  - **Bilinear**
  - **Bicubic**
- Needed to prevent “jaggies” and “texture crawl”



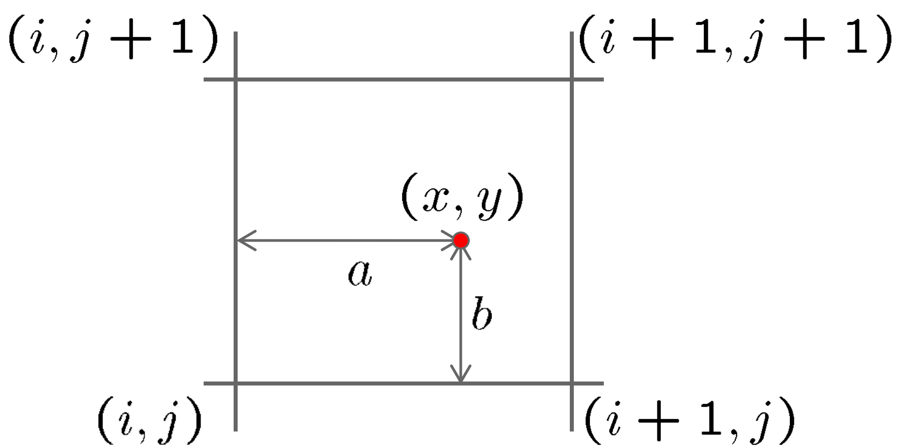


Bilinear interpolation



Bilinear interpolation

Sampling at *f(x,y):*



$$
\begin{aligned}
f(x,y) = \ & (1-a)(1-b) && f[i,j] \\
& +a(1-b) && f[i+1,j] \\
& +ab && f[i+1,j+1] \\
& +(1-a)b && f[i,j+1]
\end{aligned}
$$

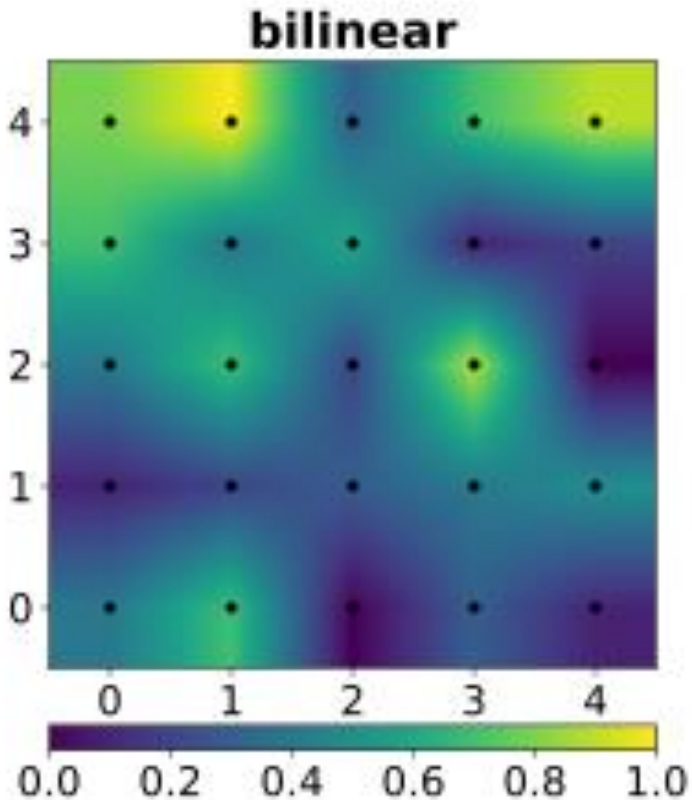
Summation of Weights is 1

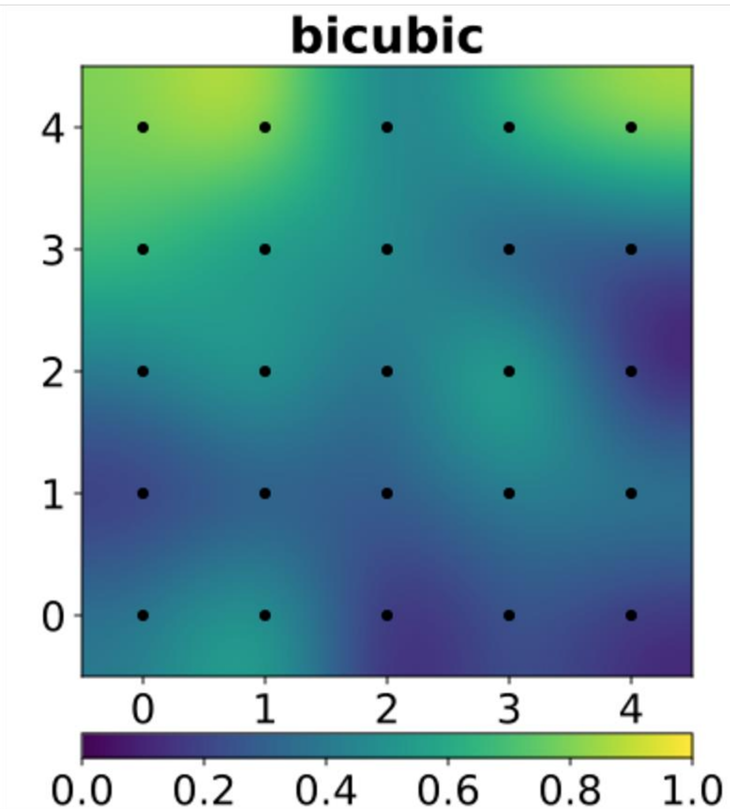
Slide from Alyosha Efros

[Reading more](#)

Bicubic interpolation

- Bilinear interpolation take pixel intensities into account.
- Bicubic interpolation also takes image gradients into account.





If the quality is of concern, bicubic would be the best choice.

Bicubic interpolation

$$p(x,y) = \sum_{i=0}^{3} \sum_{j=0}^{3} a_{ij} x^i y^j .$$

surface $p(x,y)$ on the unit square $[0,1] \times [0,1]$ that is continuous

This requires determining the 16 coefficients.

Consider 4 corners of the unit square. (0, 0) (1, 0) (0, 1) (1, 1)

1. $f(0,0) = p(0,0) = a_{00} ,$
2. $f(1,0) = p(1,0) = a_{00} + a_{10} + a_{20} + a_{30} ,$
3. $f(0,1) = p(0,1) = a_{00} + a_{01} + a_{02} + a_{03} ,$
4. $f(1,1) = p(1,1) = \sum_{i=0}^{3} \sum_{j=0}^{3} a_{ij} .$

Bicubic interpolation

We need following derivatives

$$p_x(x,y)=\sum_{i=1}^{3}\sum_{j=0}^{3}a_{ij}ix^{i-1}y^j,$$

$$p_y(x,y)=\sum_{i=0}^{3}\sum_{j=1}^{3}a_{ij}x^i jy^{j-1},$$

$$p_{xy}(x,y)=\sum_{i=1}^{3}\sum_{j=1}^{3}a_{ij}ix^{i-1}jy^{j-1}.$$

Likewise, eight equations for the derivatives in the x and the y directions:

1. $f_x(0,0)=p_x(0,0)=a_{10},$
2. $f_x(1,0)=p_x(1,0)=a_{10}+2a_{20}+3a_{30},$
3. $f_x(0,1)=p_x(0,1)=a_{10}+a_{11}+a_{12}+a_{13},$
4. $f_x(1,1)=p_x(1,1)=\sum_{i=1}^{3}\sum_{j=0}^{3}a_{ij}i,$
5. $f_y(0,0)=p_y(0,0)=a_{01},$
6. $f_y(1,0)=p_y(1,0)=a_{01}+a_{11}+a_{21}+a_{31},$
7. $f_y(0,1)=p_y(0,1)=a_{01}+2a_{02}+3a_{03},$
8. $f_y(1,1)=p_y(1,1)=\sum_{i=0}^{3}\sum_{j=1}^{3}a_{ij}j.$

And four equations for the xy mixed partial derivative:

1. $f_{xy}(0,0)=p_{xy}(0,0)=a_{11},$
2. $f_{xy}(1,0)=p_{xy}(1,0)=a_{11}+2a_{21}+3a_{31},$
3. $f_{xy}(0,1)=p_{xy}(0,1)=a_{11}+2a_{12}+3a_{13},$
4. $f_{xy}(1,1)=p_{xy}(1,1)=\sum_{i=1}^{3}\sum_{j=1}^{3}a_{ij}ij.$

Bicubic interpolation

Grouping the unknown parameters a_{ij} in a vector

$$\alpha = \begin{bmatrix} a_{00} & a_{10} & a_{20} & a_{30} & a_{01} & a_{11} & a_{21} & a_{31} & a_{02} & a_{12} & a_{22} & a_{32} & a_{03} & a_{13} & a_{23} & a_{33} \end{bmatrix}^T$$

and letting

$$x = \begin{bmatrix} f(0{,}0) & f(1{,}0) & f(0{,}1) & f(1{,}1) & f_x(0{,}0) & f_x(1{,}0) & f_x(0{,}1) & f_x(1{,}1) & f_y(0{,}0) & f_y(1{,}0) & f_y(0{,}1) & f_y(1{,}1) & f_{xy}(0{,}0) & f_{xy}(1{,}0) & f_{xy}(0{,}1) & f_{xy}(1{,}1) \end{bmatrix}^T,$$

the above system of equations can be reformulated into a matrix for the linear equation $A\alpha = x$.

Inverting the matrix gives the more useful linear equation $A^{-1}x = \alpha$, where

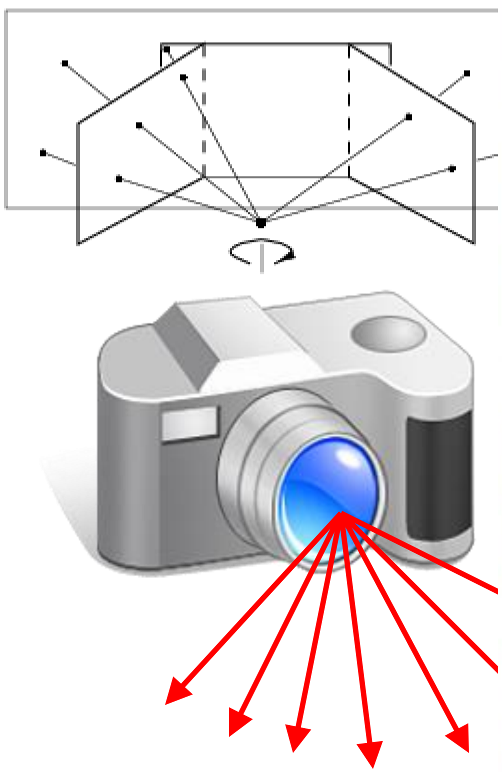
$$A^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3 & 3 & 0 & 0 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & -2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 3 & 0 & 0 & -2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 & 1 & 1 & 0 & 0 \\
-3 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -3 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & -1 & 0 \\
9 & -9 & -9 & 9 & 6 & 3 & -6 & -3 & 6 & -6 & 3 & -3 & 4 & 2 & 2 & 1 \\
-6 & 6 & 6 & -6 & -3 & -3 & 3 & 3 & -4 & 4 & -2 & 2 & -2 & -2 & -1 & -1 \\
2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
-6 & 6 & 6 & -6 & -4 & -2 & 4 & 2 & -3 & 3 & -3 & 3 & -2 & -1 & -2 & -1 \\
4 & -4 & -4 & 4 & 2 & 2 & -2 & -2 & 2 & -2 & 2 & -2 & 1 & 1 & 1 & 1
\end{bmatrix},$$

which allows α to be calculated quickly and easily.

A more compact form:

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} f(0,0) & f(0,1) & f_y(0,0) & f_y(0,1) \\ f(1,0) & f(1,1) & f_y(1,0) & f_y(1,1) \\ f_x(0,0) & f_x(0,1) & f_{xy}(0,0) & f_{xy}(0,1) \\ f_x(1,0) & f_x(1,1) & f_{xy}(1,0) & f_{xy}(1,1) \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 0 & 3 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix},$$

Panoramas



. . .

image from S. Seitz

Obtain a wide angle view by combining multiple images.

How to stitch together a panorama?

- Basic Procedure
  - Take a sequence of images from the same position
    - Rotate the camera about its **optical center.**
  - Compute transformation between 2nd and 1st image
  - Transform the 2nd image to overlap with the 1st
  - Blend the two together to create a mosaic
  - (If there are more images, repeat)
- Why should this work at all?
  - Why do we rotate w.r.t. optical center?

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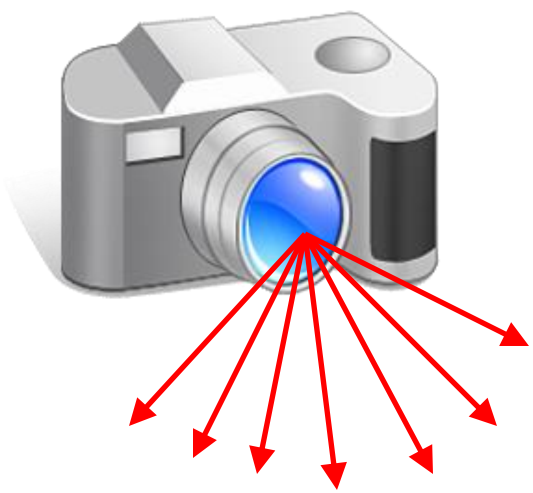
Source: Steve Seitz

Correspondence

- Allows us to map image back to some real space



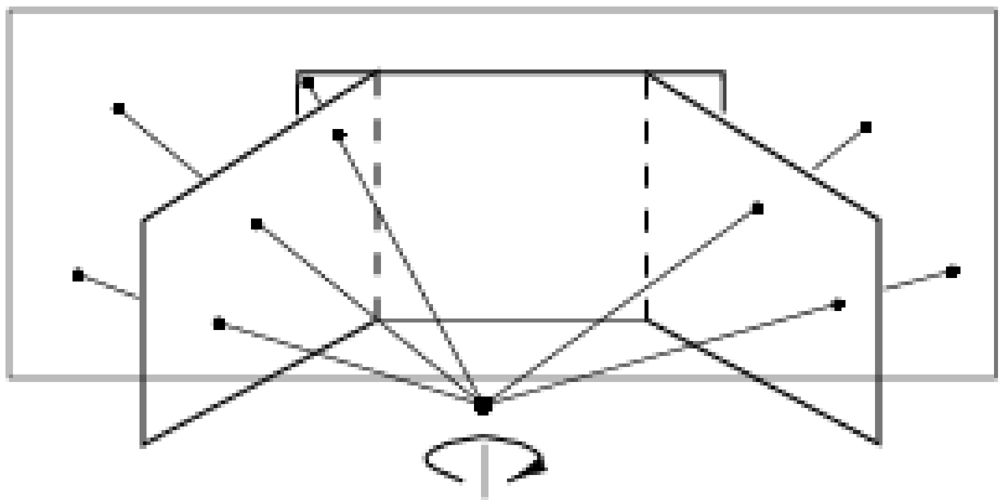
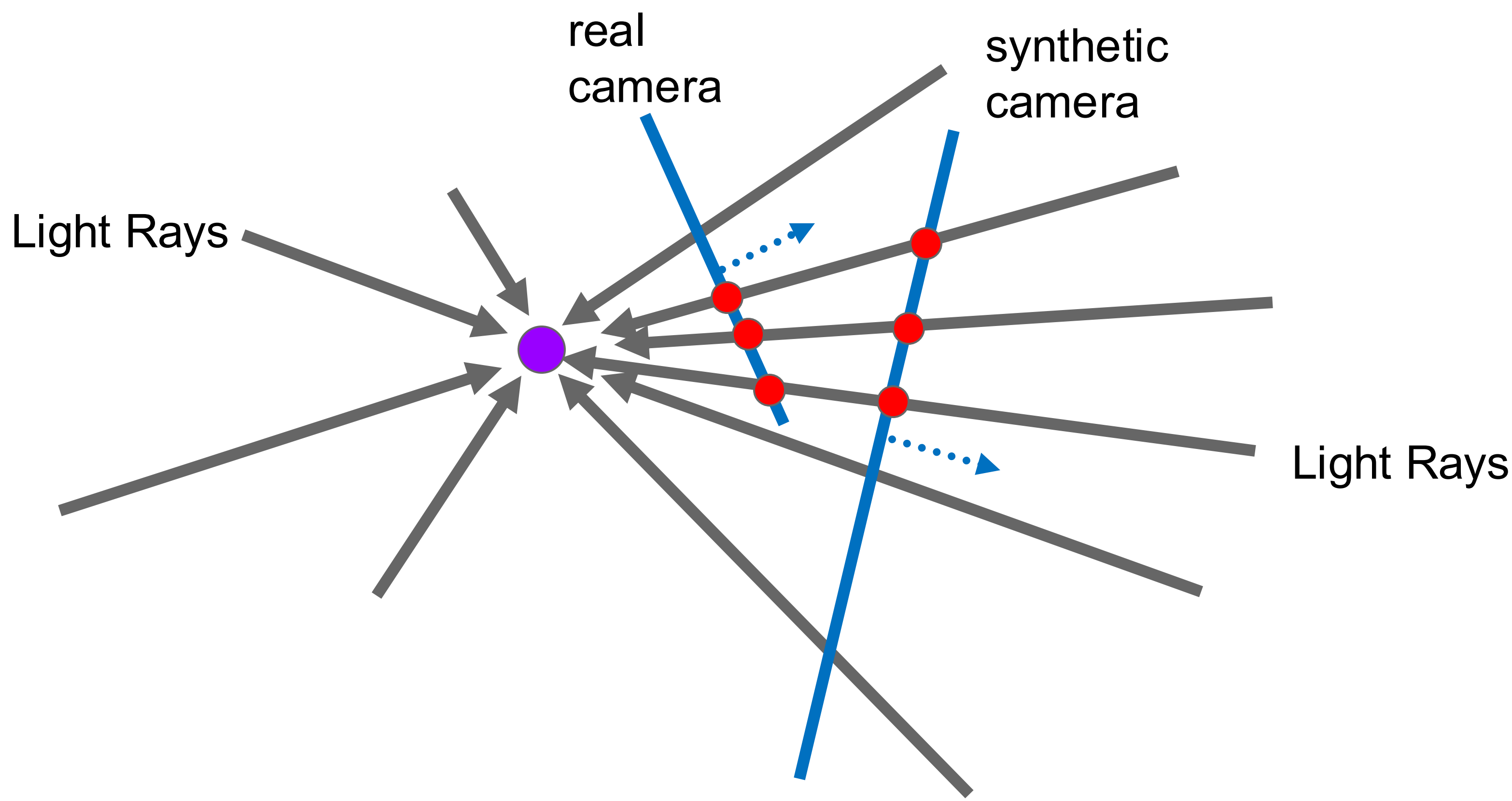


image from S. Seitz

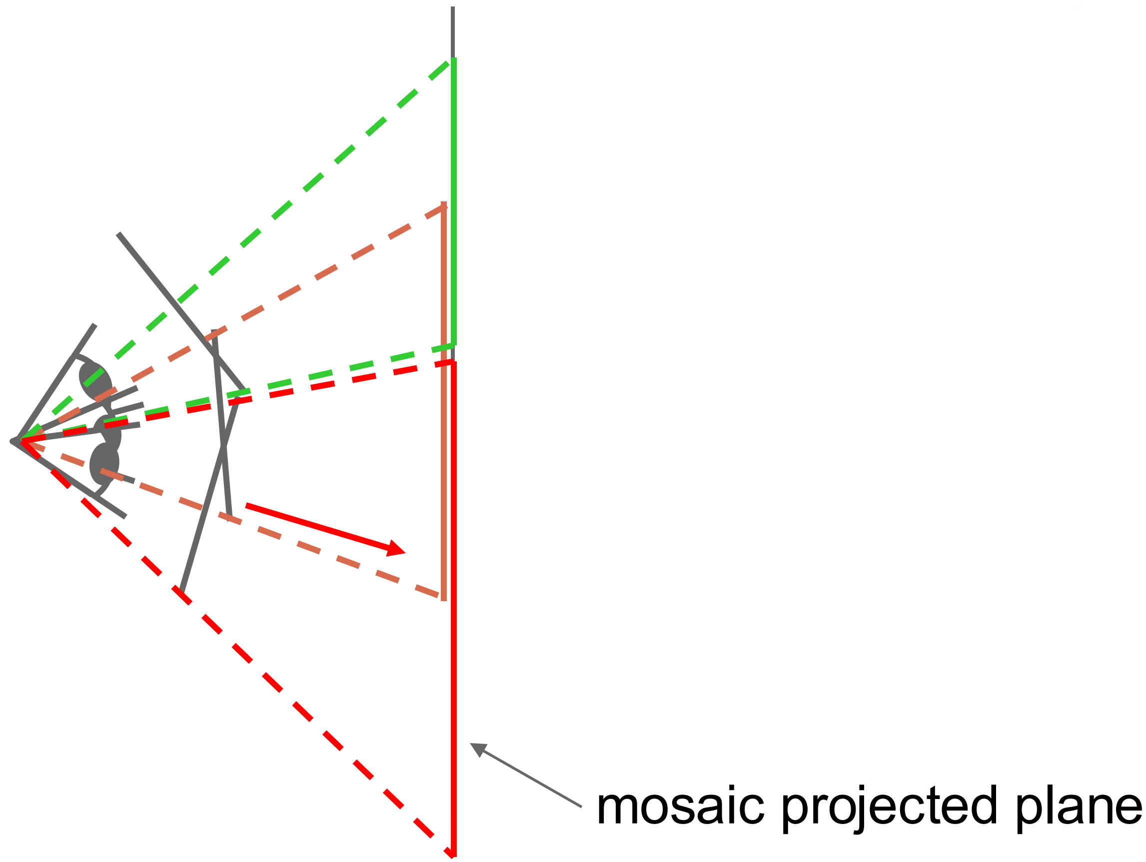
Panoramas: generating synthetic views



Can generate any synthetic camera view
as long as it has **the same center of projection!**

Source: Alyosha Efros

Image reprojection



- The mosaic has a natural interpretation in 3D
  - The images are reprojected onto a common plane
  - The mosaic is formed on this plane
  - Mosaic is a *synthetic wide-angle camera*

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Source: Steve Seitz

Recap: 2D coordinate transformations

- translation: $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{t}$ $\boldsymbol{x} = (x, y)$
- rotation: $\boldsymbol{x}' = \boldsymbol{R}\,\boldsymbol{x} + \boldsymbol{t}$
- similarity: $\boldsymbol{x}' = s\,\boldsymbol{R}\,\boldsymbol{x} + \boldsymbol{t}$
- affine: $\boldsymbol{x}' = \boldsymbol{A}\,\boldsymbol{x} + \boldsymbol{t}$
- perspective: $\underline{\boldsymbol{x}}' \cong \boldsymbol{H}\,\underline{\boldsymbol{x}}$ $\underline{\boldsymbol{x}} = (x,y,1)$

  ($\underline{\boldsymbol{x}}$ is a *homogeneous* coordinate)

- These all form a nested *group* (closed w/ inv.)

Recap: Basic 2D Transformations

- Basic 2D transformations as 3x3 matrices

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translate

$$\begin{bmatrix} \boldsymbol{x'} \\ \boldsymbol{y'} \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{s_x} & 0 & 0 \\ 0 & \boldsymbol{s_y} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ 1 \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotate

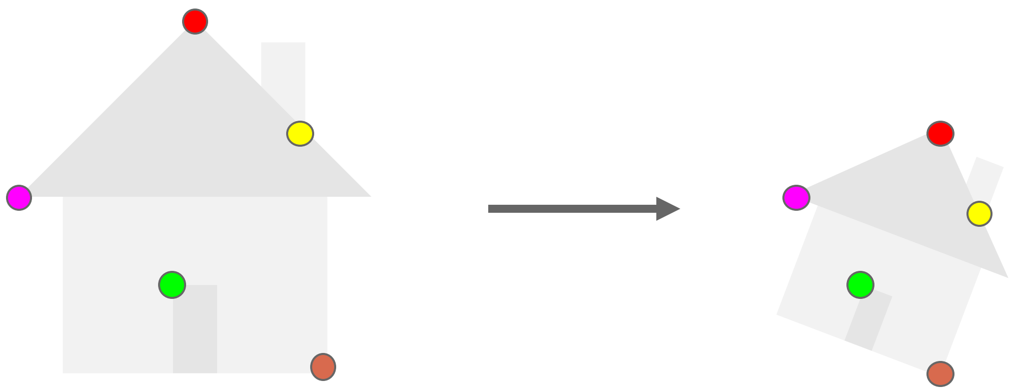
$$\begin{bmatrix} \boldsymbol{x'} \\ \boldsymbol{y'} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \boldsymbol{sh_x} & 0 \\ \boldsymbol{sh_y} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ 1 \end{bmatrix}$$

Shear (Skew)

Source: Alyosha Efros

Image alignment

- Two broad approaches:
  - Direct (pixel-based) alignment
    - Search for alignment where most pixels agree
  - Feature-based alignment
    - Search for alignment where *extracted features* agree
    - Can be verified using pixel-based alignment



L. Lazebnik

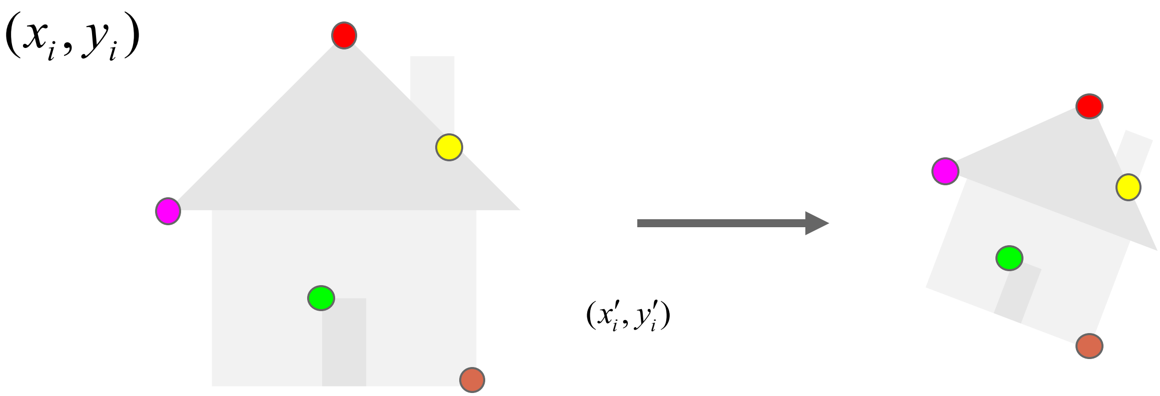
Fitting an affine transformation

Affine model approximates perspective projection of planar objects.

Figures from David Lowe, ICCV 1999

Fitting an affine transformation

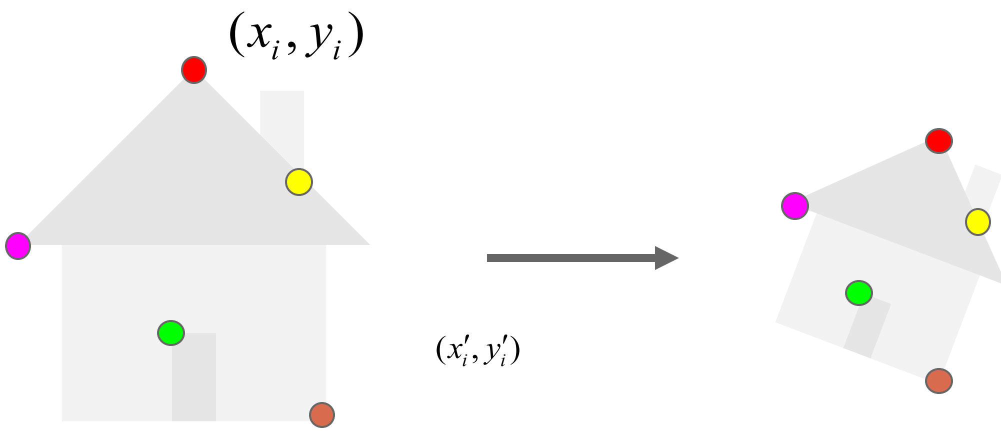
- Assuming we know the correspondences, how do we get the transformation?



$$\begin{bmatrix} x_i' \\ y_i' \end{bmatrix} = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

Fitting an affine transformation

- Assuming we know the correspondences, how do we get the transformation?



$$\begin{bmatrix} x'_i \\ y'_i \end{bmatrix} = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

$$\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

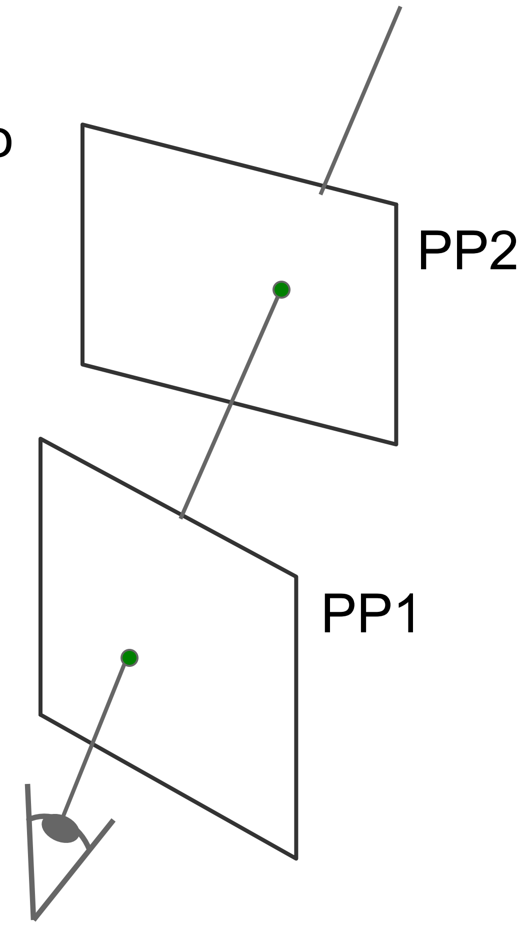
Fitting an affine transformation

$$\begin{bmatrix} & & \cdots & & & \\ x_i & y_i & 0 & 0 & 1 & 0 \\ 0 & 0 & x_i & y_i & 0 & 1 \\ & & \cdots & & & \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} \cdots \\ x_i' \\ y_i' \\ \cdots \end{bmatrix}$$

- How many matches (correspondence pairs) do we need to solve for the transformation parameters?
- Once we have solved for the parameters, how do we compute the coordinates of the corresponding point for (x_{new}, y_{new})?

Homography

- Take as a 2D **image warp** using projective transform**.**
- A **projective transform** is a mapping between any two PPs with the same center of projection
  - rectangle should map to arbitrary quadrilateral
  - parallel lines aren't preserved.
  - but straight lines are preserved.
- Called **Homography**

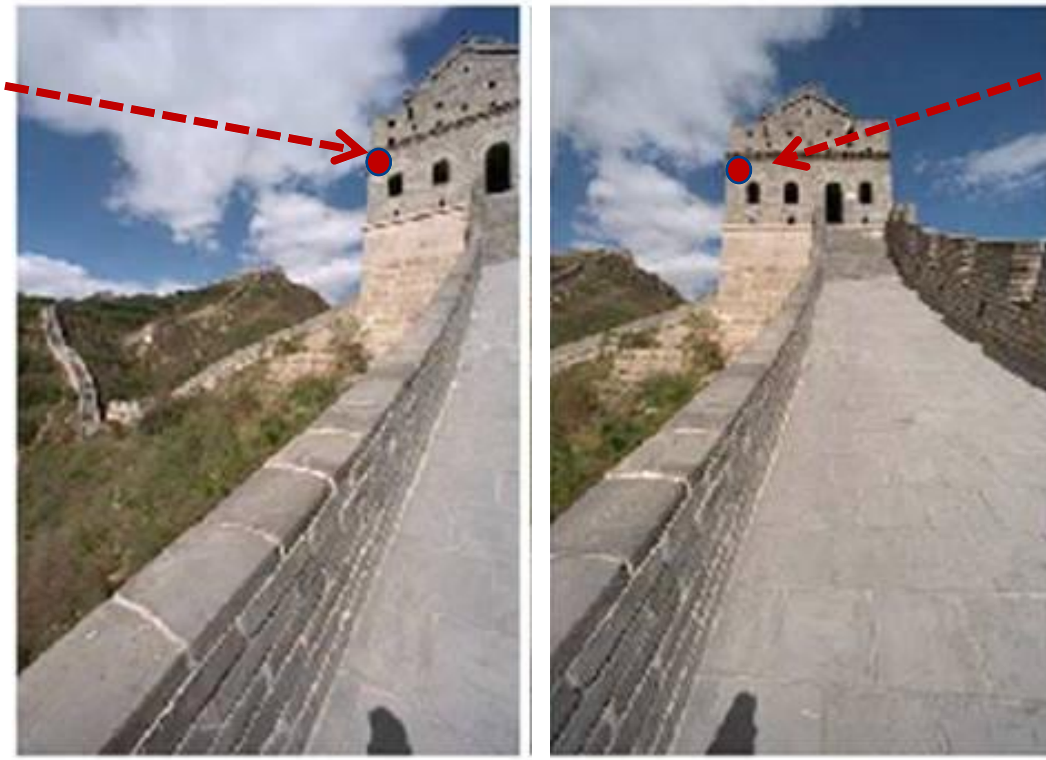


$$\underset{\mathbf{p'}}{\begin{bmatrix} wx' \\ wy' \\ w \end{bmatrix}} = \underset{\mathbf{H}}{\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}} \underset{\mathbf{p}}{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}$$

Source: Alyosha Efros

Solving for homographies

$$(x, y)$$



$$\left({wx'}/{w}, {wy'}/{w} \right) = (x', y')$$

To **apply** a given homography **H**

- Compute **p'** = **Hp** (regular matrix multiply)
- Convert **p'** from homogeneous to image coordinates

$$\underbrace{\begin{bmatrix} wx' \\ wy' \\ w \end{bmatrix}}_{\mathbf{p'}} = \underbrace{\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}_{\mathbf{p}}$$

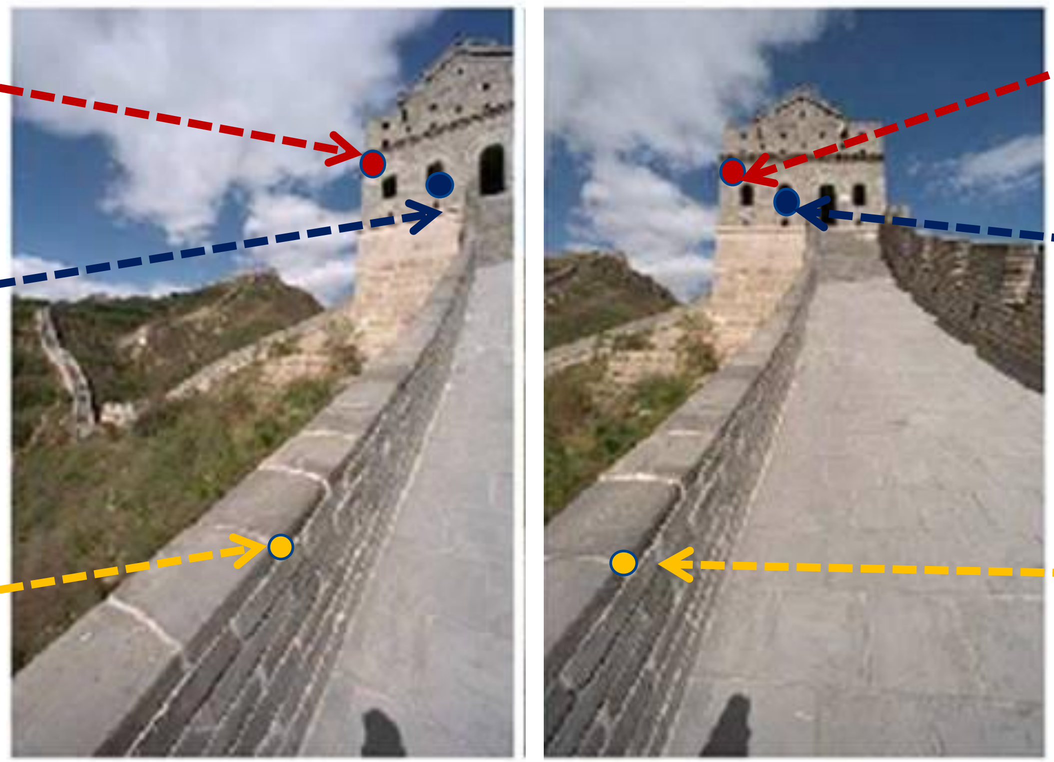
Solving for homographies

(x_1, y_1) (x'_1, y'_1)

(x_2, y_2) (x'_2, y'_2)

$\vdots$ $\vdots$

(x_n, y_n) (x'_n, y'_n)



To **compute** the homography given pairs of corresponding points, we need to set up an equation where the parameters of **H** are the unknowns…

Grauman

Solving for homographies

p' = Hp

$$\begin{bmatrix} wx' \\ wy' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Can set scale factor $i = 1$ or $||H|| = 1$. So, there are 8 unknowns.
- Set up a system of linear equations:
  - **Ah = b**

  where vector of unknowns h = [a, b, c, d, e, f, g, h]^T
- Need at least 8 equations (4 points), but the more the better…
- Solve for H. If over-constrained, solve using least-squares:

$$\min \left\| Ah - b \right\|^2$$

$$h = (A^T A)^{-1} A^T b$$

Grauman

Proof of least squares

- $F(h) = ||Ah - b||^2 = (Ah - b)^T\ (Ah - b)$
- $F(h) = h^T A^T A h - h^T A^T b - b^T A h + b^T b$
- $\frac{\partial}{\partial h} F(h) = 2A^T A h - A^T b - (b^T A)\text{^}T$
- Setting derivative to 0: $\frac{\partial}{\partial h} F(h)$=0
- $A^T A h = A^T b$
- $h = (A^T A)^{-1} A^T b$

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How to stitch together a panorama?

- Take a sequence of images from the same position
  - Rotate the camera about its optical center
- Compute transformation between second image and first
- Transform the second image to overlap with the first
- Blend the two together to create a mosaic
- If there are more images, repeat

Source: Steve Seitz

Content

- Stitching
  - Alignment
    - Interpolation
    - Homography
  - Fitting
    - Solving for homographies

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