Matrices

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1 Introduction

You will have learned about arrays with patsy in the programming section. Now we will look at the concept of an array from a mathematical standpoint. In this context an array is known as a **matrix**. Matrices are extremely useful tools in mathematics since they can be used to move points around in space and they can be used to represent mathematical structures such as graphs. We'll also see that matrices can be equipped with arithmetic rules which make them behave a lot like numbers. One can hardly think of a more useful mathematical tool in engineering than a matrix and they can be used to solve a wide range of real world problems.

Definition 1.1. A **matrix** is a rectangular **array** of numbers.

Example

$$\begin{pmatrix} 1 & 3 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 4 & 5 & 7 \\ 1 & -2 & 11 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 1 & -2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 1 \\ 1 & -2 & 0 \\ 5 & 2 & -1 \end{pmatrix}$$

Matrices are made up of **rows** and **columns**. We often refer to a matrix as an $m \times n$-matrix where m is the number of rows and n is the number of columns. In the example above the matrices are $2 \times 2, 2 \times 3, 3 \times 2$ and 3×3 respectively.

Example

More examples of matrices

$$\begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 4 & 5 & 7 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -7 \end{pmatrix}$$

Fromm left to right the above matrices are $2 \times 1, 1 \times 3, 3 \times 1$ and 1×1 respectively.

Definition 1.2. If the number of rows and columns are equal, i.e. M is an $n \times n$ matrix, we call M a **square** matrix.

We often refer to the entries in a matrix in terms of their position in a row and a column. In a matrix A we refer to the entry in **row** i and **column** j as the A_{ij} entry of A. In this way, the subscript ij refers to the "address" in the matrix of a particular entry. This is exactly the same procedure used to reference positions in an array!

Example

Let

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & -2 & 0 \\ 5 & 2 & -1 \end{pmatrix}.$$

Then e.g. $A_{12} = 3, A_{23} = 0, A_{31} = 5$.

Exercise

Let

$$M = \begin{pmatrix} 1 & 3 \\ 1 & -2 \\ 0 & 1 \end{pmatrix}.$$

Write down M_{22}, M_{31}, M_{12}.

2 Matrix Arithmetic

Definition 2.1 (Matrix Addition)**.** If A and B are $m \times n$-matrices, i.e. they have the same shape, then we can add A and B entry by entry.

Example

Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 0 & 3 \\ 1 & -1 & 3 \end{pmatrix}.$$

Then $A + B$ is

$$\begin{aligned} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & -1 \end{pmatrix} + \begin{pmatrix} -2 & 0 & 3 \\ 1 & -1 & 3 \end{pmatrix} &= \begin{pmatrix} 1-2 & 2+0 & 3+3 \\ 3+1 & 0-1 & -1+3 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 2 & 6 \\ 4 & -1 & 2 \end{pmatrix}. \end{aligned}$$

Exercises

Compute each of the following matrix additions.

(i) $\begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix} + \begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix}$.

(ii) $\begin{pmatrix} 0 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix} + \begin{pmatrix} 3 & 0 & -1 \\ 0 & -1 & 2 \end{pmatrix}$.

(iii) $\begin{pmatrix} 1 & 3 & 1 \\ 1 & -2 & 0 \\ 5 & 2 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 3 & 1 \\ 1 & -2 & 0 \\ 5 & 2 & -1 \end{pmatrix}$.

Definition 2.2 (Matrix Subtraction)**.** If A and B are $m \times n$-matrices, i.e. they have the same shape, then we can subtract A and B entry by entry.

Example

Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 0 & 3 \\ 1 & -1 & 3 \end{pmatrix}.$$

Then $A - B$ is

$$\begin{aligned} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & -1 \end{pmatrix} - \begin{pmatrix} -2 & 0 & 3 \\ 1 & -1 & 3 \end{pmatrix} &= \begin{pmatrix} 1+2 & 2-0 & 3-3 \\ 3-1 & 0+1 & -1-3 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 2 & 0 \\ 2 & 1 & -4 \end{pmatrix}. \end{aligned}$$

Exercises

Compute each of the following matrix subtractions.

(i) $\begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix} - \begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix}$.

(ii) $\begin{pmatrix} 0 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix} - \begin{pmatrix} 3 & 0 & -1 \\ 0 & -1 & 2 \end{pmatrix}$.

(iii) $\begin{pmatrix} 1 & 3 & 1 \\ 1 & -2 & 0 \\ 5 & 2 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 3 & 1 \\ 1 & -2 & 0 \\ 5 & 2 & -1 \end{pmatrix}$.

2.1 Scalar Multiplication

We can multiply a matrix by a scalar (fixed number) by simply multiplying every entry in the matrix by the scalar.

Example

Let
$$A = \begin{pmatrix} 3 & 2 \\ 5 & -1 \end{pmatrix}.$$
Then $4A$ is
$$4\begin{pmatrix} 3 & 2 \\ 5 & -1 \end{pmatrix} = \begin{pmatrix} 12 & 8 \\ 20 & -4 \end{pmatrix}.$$

Indeed, this also enables us to take out a common factor from each of the entries in a matrix.

Example

Suppose that
$$B = \begin{pmatrix} 5 & -5 \\ -15 & 5 \end{pmatrix}.$$
Then we can say that
$$\begin{pmatrix} 5 & -5 \\ -15 & 5 \end{pmatrix} = 5\begin{pmatrix} 1 & -1 \\ -3 & 1 \end{pmatrix},$$
i.e.
$$B = 5\begin{pmatrix} 1 & -1 \\ -3 & 1 \end{pmatrix}.$$

So far we have defined addition, subtractions and scalar multiplication of matrices. These operations are of an arithmetic nature so it should not come as a surprise to know that we can define a multiplication for matrices too.

2.2 Matrix Multiplication

Matrices are multiplied **row by column** as shown below. Let
$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Then
$$\begin{aligned} AB &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix} \end{aligned}$$

- We multiply the first row of A by the first column of B to obtain the entry in the first row and first column of the answer.
- We multiply the first row of A by the second column of B to obtain the entry in the first row and second column of the answer.
- And so on!

Example

(i)

$$\begin{pmatrix} 3 & 7 \end{pmatrix}\begin{pmatrix} 2 \\ 9 \end{pmatrix} = \begin{pmatrix} 3\times 2+7\times 9 \end{pmatrix}$$
$$= \begin{pmatrix} 69 \end{pmatrix}$$

(ii)

$$\begin{pmatrix} 4 & 2 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ 6 \\ 8 \end{pmatrix} = \begin{pmatrix} 4\times 3+2\times 6+5\times 8 \end{pmatrix}$$
$$= \begin{pmatrix} 64 \end{pmatrix}.$$

(iii)

$$\begin{pmatrix} 3 & 2 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} (3\times 1)+(2\times -3) & (3\times -1)+(2\times 1) \\ (5\times 1)+(-1\times -3) & (5\times -1)+(-1\times 1) \end{pmatrix}$$
$$= \begin{pmatrix} -3 & -1 \\ 8 & -6 \end{pmatrix}.$$

We can even multiply matrices with different shapes (well, sometimes).

Example

Let

$$A = \begin{pmatrix} 9 & -1 \\ 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 3 & 2 \\ 1 & 2 & -1 \end{pmatrix}.$$

Then AB is

$$\begin{aligned} AB &= \begin{pmatrix} 9 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 4 & 3 & 2 \\ 1 & 2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} (9)(4)+(-1)(1) & (9)(3)+(-1)(2) & (9)(2)+(-1)(-1) \\ (2)(4)+(3)(1) & (2)(3)+(3)(2) & (2)(2)+(3)(-1) \end{pmatrix} \\ &= \begin{pmatrix} 35 & 25 & 19 \\ 11 & 12 & 1 \end{pmatrix}. \end{aligned}$$

Can we compute BA? The answer is no! Check and see!

Definition 2.3. The product of two matrices A and B is only defined if the number of columns in A is equal to the number of rows in B.

So for example we can multiply

- a 2×3 matrix by a 3×1 matrix.

- a 2×2 matrix by a 2×1 matrix.
- a 3×1 matrix by a 1×5 matrix.

and so on! And indeed the product obtained each time will be

- a 2×3 by a 3×1 gives a 2×1 matrix.
- a 2×2 by a 2×1 gives a 2×1 matrix.
- a 3×1 by a 1×5 gives a 3×5 matrix.

and so on!

Exercises

Let

$$A = \begin{pmatrix} 9 & -1 \\ 2 & 3 \end{pmatrix}, B = \begin{pmatrix} 4 & 3 & 2 \\ 1 & 2 & -1 \end{pmatrix}, C = \begin{pmatrix} 3 & 6 \\ -3 & 1 \end{pmatrix}, D = \begin{pmatrix} -1 & 2 & -1 \\ 3 & -2 & 4 \end{pmatrix}.$$

Compute each of the following products; (i) AB, (ii) CD, (iii) AD, (iv) CA.

So far we have defined addition, subtraction, scalar multiplication and multiplication of matrices. One operation that we haven't yet discussed is that of matrix division, that's if it even exists at all! In order to explore this topic we'll begin by describing some special matrices.

2.3 Special Matrices

We've seen that matrices can be made to behave a like numbers. There are two special matrices that play similar roles to the numbers 0 and 1 in ordinary real arithmetic.

Definition 2.4. The **zero matrix** behaves just like the number 0. Let A be an $m \times n$-matrix. The corresponding $m \times n$-zero matrix 0_M has the property that $A + 0_M = A$.

Example

In ordinary real arithmetic we have that $a + 0 = a$ for every real number a e.g. $7 + 0 = 7, -5 + 0 = -5$ and so on. The zero matrix has the same property with respect to matrix addition.

$$\begin{aligned}
\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \\
\begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix} \\
\begin{pmatrix} -1 & 2 & -1 \\ 3 & -2 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} &= \begin{pmatrix} -1 & 2 & -1 \\ 3 & -2 & 4 \end{pmatrix}
\end{aligned}$$

Definition 2.5. Let A be an $n \times n$-matrix. The corresponding $n \times n$ **identity matrix** I has entries 1 on the diagonal and entries 0 everywhere else and satisfies $A \times I = I \times A = A$.

Example

In ordinary real arithmetic we have that $a \times 1 = a$ for every real number a e.g. $7 \times 1 = 7, -5 \times 1 = -5$ and so on. The identity matrix has the same property with respect to matrix multiplication. Note that the identity matrix is is always a square matrix i.e. it has the same number of rows as columns.

Let

$$A = \begin{pmatrix} 9 & -1 \\ 2 & 3 \end{pmatrix}.$$

The corresponding 2×2 identity matrix is

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We'll check that $AI = IA = A$.

$$\begin{aligned} AI &= \begin{pmatrix} 9 & -1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 9 & -1 \\ 2 & 3 \end{pmatrix} \end{aligned}$$

And that $IA = A$.

$$\begin{aligned} IA &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 9 & -1 \\ 2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 9 & -1 \\ 2 & 3 \end{pmatrix} \end{aligned}$$

The identity matrix is always square and has ones on the diagonal with zeros everywhere else. For example

- 2×2 Identity Matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,
- 3×3 Identity Matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,
- 4×4 Identity Matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

3 Matrix Inverse

The identity matrix brings to mind another concept from rel arithmetic. In real arithmetic we know that e.g.

$$7 \times \frac{1}{7} = 1.$$

We say that $\frac{1}{7}$ is the multiplicative inverse of 7. It has the property that when we multiply it by 7 we obtain the multiplicative identity. There is an equivalent concept of inverse for **square matrices**.

Definition 3.1. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The **inverse** of A denoted A^{-1} is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

And $AA^{-1} = A^{-1}A = I$.

Example

Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Then

$$\begin{aligned} A^{-1} &= \frac{1}{(2)(1) - (1)(1)} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \\ &= \frac{1}{1} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \end{aligned}$$

And we can check that

$$\begin{aligned} AA^{-1} &= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

and also

$$\begin{aligned} A^{-1}A &= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Multiplication by the inverse is equivalent to a notion of division. (Think of $7 \times \frac{1}{7} = 1$ i.e. $7 \div 7 = 1$.)

Exercises

Find the inverse of each of the following matrices. In each case verify that your answer is correct by multiplying the matrix by its inverse to obtain the identity.

1. $A = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}$.

2. $B = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$.

3. $C = \begin{pmatrix} 6 & 12 \\ 4 & 4 \end{pmatrix}$.

4. $D = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$.

In the next section we'll look at how we can solve problems with matrices.