Problem Set 1

Buan 6340

Due: Monday, 2018-06-06-11:59pm

Linear Algebra Questions

Question 1 Define the following:

1. Define an operation which takes the mean of a vector using the inner product.
2. Define a function which takes a vector and outputs a difference between a vector and its associated mean.
3. Define the variance and covariance functions.

Question 2 An idempotent matrix is one where $AA = A$. Define $P = X\left(X'X\right)^{-1}X'$. Define $M = I_n - P$. Verify the following for the ordinary least squares linear model $y = X\beta + e$.

1. P is idempotent
2. M is idempotent
3. $\hat{y} = Py$
4. $\hat{e} = My$
5. $y = Py + My$
6. $\hat{y} \perp \hat{e}$

Programming Questions

Question 3 The hailstone sequence is one defined by

$$x_n = \begin{cases} x_{n-1}/2 & \text{if } x_{n-1} \text{ is even} \\ 3x_{n-1}+1 & \text{otherwise} \end{cases}$$

Eventually, for natural number values of x_1, the hailstone sequence will converge to the cycle $x_r = 1, x_{r+1} = 4, x_{r+2} = 2, x_{r+3} = 1, \ldots$. This is not immediately obvious, but it does happen for all the inputs $1, \ldots, 1M$. What I want you to consider in this question is the generalized hailstone defined by:

$$x_n = \begin{cases} x_{n-1}/2 & \text{if } x_{n-1} \text{ is even} \\ ax_{n-1}+b & \text{otherwise} \end{cases}$$

where a and b are integer valued in $\{0, \ldots, 10\}$. For each of these 121 possible combinations of a and b, find out whether or not the sequence converges for all x_1 values $1, \ldots, 1000$. If the sequence does converge for all these values, find out how many different cycles the sequence converges to. Once you have this done, try doing the same thing for the sequence where a_1, a_2, b_1, b_2 are in $\{0, \ldots, 10\}$:

$$x_n = \begin{cases} x_{n-1}/3 & \text{if } x_{n-1} \bmod 3 = 0 \\ a_1x_{n-1}+b_1 & \text{if } x_{n-1} \bmod 3 = 1 \\ a_2x_{n-1}+b_2 & \text{if } x_{n-1} \bmod 3 = 2 \end{cases}.$$