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Q1.

Gaussian Quadrature with n = 2 (**Three point**) was followed to estimate the integrals. The following procedure was used:

(1) A **polynomial P(x) of degree 3** was constructed (Legendre polynomial).
(2) Then, the roots of the polynomial were considered as the nodes.
Roots = $[\frac{-\sqrt{3}}{\sqrt{5}}, \mathbf{0}, \frac{\sqrt{3}}{\sqrt{5}}]$
(3) Then, all the individual weights were found out using the formula:
$w_j = \int_a^b \ell_j(x)dx, j = 0 : n.$
(4) Required Estimate = $\mathbf{G_n(f) = w_0 f(x_0) + w_1 f(x_1) + \dots\dots\dots + w_n f(x_n)}$

Note: Since estimation with Legendre polynomials works best in the range **[-1,1]**, hence the x was substituted accordingly so that the range of integration becomes **[-1,1]**. For example, in part (a), **x was substituted with** $\frac{\mathbf{y+5}}{\mathbf{4}}$.
The results are as follows:

Part (a)
Estimated Value (with n = 2): **0.192259377256879**
Actual Value of Integral: 0.192259357732796
Error in the estimation: 1.9524082989219593 x $\mathbf{10^{-8}}$

Part (b)
Estimated Value (with n = 2): **-0.1768200178862206**
Actual Value of Integral: -0.176820020121789
Error in the estimation: 2.2355683970687323 x $\mathbf{10^{-9}}$

Part (c)
Estimated Value (with n = 2): **0.08875385361785668**
Actual Value of Integral: 0.08875528443525664
Error in the estimation: 1.4308173999638685 x $\mathbf{10^{-6}}$

We can see that the Gaussian quadrature predicts the integral correctly with great accuracy.

Q2.
Yes, such a function can be constructed.

Case 1 ($x_j \notin \{a, b\}$):
Define $r = \frac{1}{2}\min\{|x_{j+1} - x_j|, |x_j - x_{j-1}|\}$ and
define $f(x) = \begin{cases} -(x-(x_j - r))(x-(x_j + r)) & x \in [x_j - r, x_j + r] \\ 0 & \text{otherwise} \end{cases}$

Then $\sum w_j f(x_j) = \sum_{i \neq j} w_i f(x_i) + w_j f(x_j) = 0 + w_j f(x_j) < 0$. But clearly, $\int_a^b f(x)dx > 0$.

Case 2 ($x_j \in \{a, b\}$):
If $x_j = a$ then define $r = \frac{1}{2}|x_{j+1} - x_j|$ and

$$f(x) = \begin{cases} -(x-(x_j - r))(x-(x_j + r)) & x \in [x_j, x_j + r] \\ 0 & \text{otherwise} \end{cases}$$

If $x_j = b$ then define $r = \frac{1}{2}|x_j - x_{j-1}|$ and
$$f(x) = \begin{cases} -(x-(x_j - r))(x-(x_j + r)) & x \in [x_j - r, x_j + r] \\ 0 & \text{otherwise} \end{cases}$$
Then $\sum w_j f(x_j) = \sum_{i \neq j} w_i f(x_i) + w_j f(x_j) = 0 + w_j f(x_j) < 0$. But clearly, $\int_a^b f(x)dx > 0$
This shows that improper choice of weights may result in huge disparities.

Q3.
Here, the two-point Gaussian Quadrature (n=1) was applied to estimate the integral. Then, Simpson's Rule and Trapezoidal Rule was appropriately applied. The results are as follows:

```
Estimated Value through Gaussian Quadrature (two-point):
1.0909090909126549
Actual Value of Integral: 1.09861228866811
Error in the estimation: 0.007703197755455138

Estimated Value through Trapezoidal rule: 1.3333333333333333
Actual Value of Integral: 1.09861228866811
Error in the estimation: 0.23472104466522326

Estimated Value through Simpson's rule: 1.1111111111111112
Actual Value of Integral: 1.09861228866811
Error in the estimation: 0.012498822443001156
```

This shows that Gaussian Quadrature Formula is a **better** estimator than Simpson's Rule and Trapezoidal Rule.

Q4.

Here, the similar procedure was followed to estimate the integral **(Gaussian Quadrature with n=2).**

Estimated Value through Gaussian Quadrature (three-point):
0.6931216931216931
Actual Value of Integral: 0.6931471805599453
Error in the estimation: 2.54874382521475e **x** 10^{-5}

The following formula was used for Simpson's one third rule. (with h = 0.125)

$$\boxed{\int_a^b f(x)\,dx = h/3[(y_0+y_n) + 4(y_1+y_3+y_5+....+y_{n-1})+2(y_2+y_4+y_6+.....+y_{n-2})]}$$

Estimated Value through Simpson's 1/3 rule:
0.6931545306545307
Actual Value of Integral: 0.6931471805599453
Error in the estimation: 7.350094585412137e **x** 10^{-6}

Here, we can see that the error value in both cases is comparable. It is important to note that in Simpson's Rule, **n = 8**, but in the case of Gaussian Quadrature, **n = 2**.

Q5.

<u>**First Method:**</u>

The values of a, b and c have been calculated
(using sample functions 1, x, and x^2 (of degree < 3):
a = 0.1666666666666666
b = 0.6666666666666666
c = 0.1666666666666666

$$\boxed{\frac{(b-a)^5}{2880} f^{(4)}(\xi).}$$

Error Bound for **first** method = 0.00034722222222222224max|f^4(x)|

<u>**Second Method:**</u>

The values of α, β and γ have been calculated
(using sample functions 1, x, and x^2 (of degree < 3):
α = 0.6666666666666666
β = -0.3333333333333333
γ = 0.6666666666666666

$$\boxed{\frac{7(b-a)^5}{23040} f^{(4)}(\xi)}$$

Error Bound for **second** method = 0.0003038194444444445max|f^4(x)|

By observing the values of error, the error for **second** method is lesser for same value of **ξ**. So, the second method has a lower bound for maximum error.

Q6.

All values have been rounded up to 2 decimal places.

Estimated Value through Gaussian Quadrature (n = 1): -0.76
Actual Value of Integral: -0.915
Error in the estimation: 0.16

Estimated Value through Gaussian Quadrature (n = 2): -0.838
Actual Value of Integral: -0.915
Error in the estimation: 0.08

Estimated Value through Gaussian Quadrature (n = 3): -0.867
Actual Value of Integral: -0.915
Error in the estimation: 0.05

Estimated Value through Gaussian Quadrature (n = 4): -0.883
Actual Value of Integral: -0.915
Error in the estimation: 0.03

Estimated Value through Gaussian Quadrature (n = 5): -0.892
Actual Value of Integral: -0.915
Error in the estimation: 0.02

We can see that Error **decreases** with **increase** in the value of **n**.