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Q1.

The **Euler's method** was followed to approximate the solution of the **Initial Value problems.**

(a) Number of nodes = 3, h = 0.5

x	Estimate value of y(x)
0	1
0.5	1.183940
1	1.436252

(b) Number of nodes = 3, h = 0.5

x	Estimate value of y(x)
1	2
1.5	2.333333
2	2.708333

(c) Number of nodes = 5, h = 0.25

x	Estimate value of y(x)
2	2
2.25	2.207107
2.5	2.490999
2.75	2.854680
3	3.302596

(d) Number of nodes = 5, h = 0.25

x	Estimate value of y(x)
2	2
2.25	1.227324
2.5	0.832150
2.75	0.570447
3	0.378827

Q2.

By using the given solutions to the IVP's, the corresponding error values was calculated

(a) Number of nodes = 3, h = 0.5

x	Estimate value of y(x)	Error
0	1	0
0.5	1.18394	0.030083
1	1.436252	0.053628

(b) Number of nodes = 3, h = 0.5

x	Estimate value of y(x)	Error
1	2	0
1.5	2.333333	0.020769
2	2.708333	0.033324

(c) Number of nodes = 5, h = 0.25

x	Estimate value of y(x)	Error
2	2	0
2.25	2.207107	0.037014
2.5	2.490999	0.073453
2.75	2.85468	0.110513
3	3.302596	0.148690

(d) Number of nodes = 5, h = 0.25

x	Estimate value of y(x)	Error
2	2	0
2.25	1.227324	0.175875
2.5	0.83215	0.18426
2.75	0.570447	0.167563
3	0.378827	0.150860

Q3.

Similarly, following as above, we have the following results:

(a)

Number of nodes = 21, h = 0.05

x	Estimate value of y(x)	Actual value of y(x)	Error value
1	-1	-1	0
1.05	-0.950000	-0.952381	0.002381
1.1	-0.904535	-0.909091	0.004555
1.15	-0.863007	-0.869565	0.006558
1.2	-0.824917	-0.833333	0.008416
1.25	-0.789848	-0.800000	0.010152
1.3	-0.757447	-0.769231	0.011784
1.35	-0.727415	-0.740741	0.013326
1.4	-0.699495	-0.714286	0.014791
1.45	-0.673467	-0.689655	0.016188
1.5	-0.649141	-0.666667	0.017525
1.55	-0.626350	-0.645161	0.018811
1.6	-0.604949	-0.625000	0.020051
1.65	-0.584812	-0.606061	0.021249
1.7	-0.565825	-0.588235	0.022410
1.75	-0.547890	-0.571429	0.023539
1.8	-0.530918	-0.555556	0.024637
1.85	-0.514832	-0.540541	0.025708
1.9	-0.499561	-0.526316	0.026754
1.95	-0.485043	-0.512821	0.027778
2	-0.47122	-0.500000	0.028780

(b)
Number of nodes was set to 21.
The function behavior at x = 1.052, y = 1.555 and x = 1.978 is as follows:

x	Estimated value of y(x) (Interpolation)	Actual value of y(x)	Error value
1.052	-0.948099	-0.950570	0.002472
1.555	-0.624150	-0.643086	0.018937
1.978	-0.477219	-0.505561	0.028342

Q4.

(i)

$\frac{dy}{dx}$ = f(x,y)

We assume that the solution exists and is twice differentiable. It is also given that $\frac{\partial f(x,y)}{\partial y} \leq 0$

Now, e_n = y(x_n) - y_n

As it is twice differentiable, by taylor's theorem,

y(x_{n+1}) = y(x_n)+h*f(x_n,y(x_n))+(h^2/2) $y''(\tau)$
(Equation No. 1)

Here, $\tau \in$ (x_n,x_{n+1}) .

Assume f(x_n,y) is a C^1 function of y after fixing x_n. Lagrange mean value theorem gives the below:

f(x_n,y(x_n)) = f(x_n,y_n) + f_y(x_n,η) (y(t_n) - y_n) where :

$\eta \in$ [y(t_n),y_n] or [y_n,y(t_n)] depending on y(t_n) $\leq y_n$ or vice versa.

f(x_n,y(x_n)) = f(x_n,y_n) + f_y(x_n,η)e_n

Therefore, substituting in 1, we get :

y(x_{n+1}) = y(x_n) + hf(x_n,y_n) + hf_y(x_n,η)e_n + (h^2/2)$y''(\tau)$

y(x_{n+1}) -y_{n+1} = y(x_n) + hf(x_n,y_n) + hf_y(x_n,η)e_n + (h^2/2)$y''(\tau)-y_{n+1}$
(Equation No. 2)

But, y_{n+1} = y_n + hf(x_n,y_n) —— By Euler's Method

Therefore, substituting in Equation 2, we get :

y(x_{n+1}) -y_{n+1} = y(x_n) + hf(x_n,y_n) + hf_y(x_n,η)e_n + (h^2/2)$y''(\tau)-y_n$ - hf(x_n,y_n)

e_{n+1} = e_n + hf_y(x_n,η)e_n + (h^2/2)$y''(\tau)$

$|e_{n+1}| = |e_n(1 + hf_y(x_n,\eta))| + (h^2/2)|y''(\tau)|$

$where : \tau \in [x_n,x_n + 1]$ and $\eta \in [y_n$,y$(x_n + 1)]$
Assuming f_y(x,y(x)) is bounded,

M = min(f_y(x,y(x))) ≤ 0, x $\in [x_n, x_{n+1}]$.

Then take $h = 0.5$ if M = 0

and take $h = -\frac{0.2}{M}$ if $M < 0$

in either case

$|e_{n+1}| = |e_n| + (h^2/2)|y''(\tau)|$

(ii)

$e_{n+1} < e_n + (h^2/2)\ y''(\tau)$ for some h
$|e_{n+1}| = |e_n(1 + hf_y(x_n,\eta))| + (h^2/2)|y''(\tau)|$
From 1, we have $|e_{n+1}| < |e_n| + \frac{h^2}{2}|Y''(z)|$ for some $h > 0$.

Also, $|e_{n+1} \leq |e_n||1 + hf_y(x_n, n)| + (h^2/2)|Y''(z_n)|$

Additionally assuming, $f_y(x, y(x))$ is a bounded function in $[x_0, x_n]$ we have $M = \min f_y(x, y(x)) \leq 0, \quad x \in [x_0, x_n)$ Then take

$$h = \begin{cases} 0.5 & if M = 0 \\ \frac{1}{5M} & if\ M < 0 \end{cases}$$

Then $|e_n| \leq |e_n| + \frac{h^2}{2}[\sum_{n=0}^{n-1} |Y''(\tau_n)|]$ (by telescoping sums)
Clearly $\frac{\sum_{n=0}^{n-1}|Y''(\tau_n)|}{n} \leq \max_{x\in[x_0,x_n]} |Y''(X)|$

Therefore, $|e_n| \leq |e_0| + h^2 nY$ where $Y = \frac{1}{2}\max|Y''(x)| \quad x \in [x_0, x_n]$

Q5.
λ = -20.
The value of h has been set to 0.5.
Number of Nodes =7

Function Approximation at **x = 3**:

```
For x = 3, the estimated value of y is: -785.2886498351329
For x = 3, the actual value of y is: 0.1411200080598672
The error between actual and real value is: 785.4297698431927
```

It can be seen that error is very large.
Now, if we decrease the value of h to **0.05**, we have the following results:

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Function Approximation at x = 3:
For x = 3, the estimated value of y is: 0.14133753721110437
For x = 3, the actual value of y is: 0.1411200080598672
The error between actual and real value is: 0.00021752915123715577
```

So, choosing a **lesser** value of **h** results in great reduction in error values.

h = 0.5
Now, Y = $\frac{1}{2}$ $\max\ |-\sin(x)|$ where x ϵ [0,3].
So, the maximum error bound is: err = nh^2Y **= 0.75** (h = 0.5)

The **actual absolute error** is much **greater** than the **theoretical upper bound** for error. However, this **does not contradict** the validity of theorem proved in Q4. In **Q4**, it was stated that **there exists a h** such that $|e_n| \leq |e_{n-1}| + \frac{h^2}{2}f''(\xi)$, with the help of which the theoretical upper bound was calculated. This theorem is **indeed valid for that particular value of h**, and

not for every possible value of h. in **Q5**, the value of h was fixed to **0.5**, fairly large value, which resulted in **large** error. However, when the value of h was **decreased**, the error became much **smaller**.