

1. Start with the diode/BJT relation

$$I_C = I_S(T) \exp\left(\frac{qV_{be}}{k_B T}\right) \implies V_{be} = V_T \ln\left(\frac{I_C}{I_S}\right).$$

2. Differentiate V_{be} w.r.t. T (keep I_C constant):

$$\frac{\partial V_{be}}{\partial T} = \frac{\partial V_T}{\partial T} \ln\left(\frac{I_C}{I_S}\right) + V_T \frac{\partial}{\partial T} \left[\ln\left(\frac{I_C}{I_S}\right) \right].$$

Since I_C is constant, $\partial \ln(I_C/I_S)/\partial T = -\frac{1}{I_S} \frac{\partial I_S}{\partial T}$. Thus

$$\boxed{\frac{\partial V_{be}}{\partial T} = \frac{\partial V_T}{\partial T} \ln\left(\frac{I_C}{I_S}\right) - V_T \frac{1}{I_S} \frac{\partial I_S}{\partial T}} \quad (1.2)$$

3. Compute $\frac{1}{I_S} \frac{\partial I_S}{\partial T}$ for

$$I_S(T) = I_{C,0} T^\eta \exp\left(-\frac{qV_{gap,0}}{k_B T}\right).$$

Take the log and differentiate:

$$\ln I_S = \ln I_{C,0} + \eta \ln T - \frac{qV_{gap,0}}{k_B T},$$

so

$$\frac{1}{I_S} \frac{\partial I_S}{\partial T} = \frac{\partial \ln I_S}{\partial T} = \frac{\eta}{T} + \frac{qV_{gap,0}}{k_B T^2}.$$

Define $A = \frac{qV_{gap,0}}{k_B}$; equivalently note $\frac{qV_{gap,0}}{k_B T^2} = \frac{V_{gap,0}}{V_T T}$. For now keep the simple form:

$$\boxed{\frac{1}{I_S} \frac{\partial I_S}{\partial T} = \frac{\eta}{T} + \frac{qV_{gap,0}}{k_B T^2}} \quad (1.3)$$

4. Plug (1.3) into (1.2):

$$\frac{\partial V_{be}}{\partial T} = \frac{\partial V_T}{\partial T} \ln\left(\frac{I_C}{I_S}\right) - V_T \left(\frac{\eta}{T} + \frac{qV_{gap,0}}{k_B T^2} \right).$$

5. Use $\ln\left(\frac{I_C}{I_S}\right) = \frac{V_{be}}{V_T}$ and $\frac{\partial V_T}{\partial T} = \frac{k_B}{q} = \frac{V_T}{T}$:

$$\frac{\partial V_T}{\partial T} \ln\left(\frac{I_C}{I_S}\right) = \frac{V_T}{T} \cdot \frac{V_{be}}{V_T} = \frac{V_{be}}{T}.$$

Also note $V_T \cdot \frac{qV_{gap,0}}{k_B T^2} = \frac{V_{gap,0}}{T}$ because $V_T = \frac{k_B T}{q}$.

6. Collect terms \rightarrow final compact form (thesis eq. 1.5):

$$\frac{\partial V_{be}}{\partial T} = \frac{V_{be}}{T} - \frac{\eta V_T}{T} - \frac{V_{gap,0}}{T} = \frac{V_{be} - \eta V_T - V_{gap,0}}{T}.$$