Fourier and wavelet transforms

1. Fourier Series and transforms
2. Discrete Fourier transform (DFT) and Fast Fourier transform (FFT)
3. Transforming PDEs
4. Gabor transforms and Spectogram
5. Wavelets and multi-resolution analysis
6. 2D Transforms and image processing

A central concern of mathematical physics involves transformation of equations into a coordinate system where expressions simplify, decouple, and amenable to computation.

Data analysis: SVD helps

Dynamical Systems: Spectral decomposition into eigenvalues and eigenvectors

Control: defining coordinate systems by controllability and observability

Fourier has introduced most foundational and ubiquitous Coordinate transformation in early 1800s to theory of heat.

→ Sine and CoSine functions of increasing frequency Provide orthogonal basis for the Space of Solution functions.

→ Sine and Cosine functions Serve as eigenfunctions of heat equation, with Specific frequencies Serve as eigenvalues ← Determined by geometry

amplitude ← By boundary Conditions

- FFT play big role on real time image
  audio Compression
  Global Communication networks

→ Sine and CoSine functions serve as Tailored basis

2.1. Fourier Series and Fourier transforms

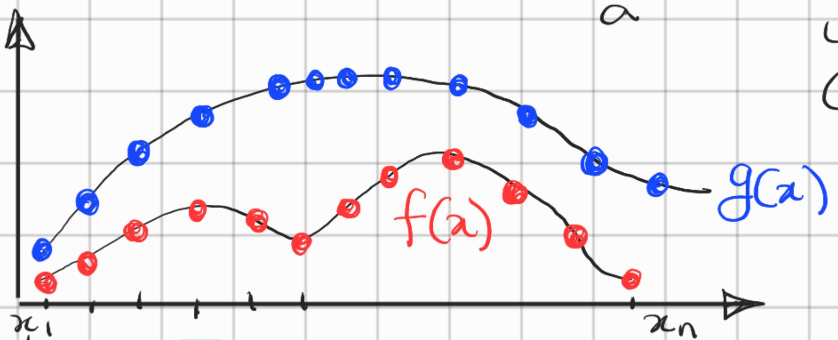
- Discrete and Continuous formulations Should match in the limit of data with infinitely fine resolution.

- Fourier Series and transform are intimately related to geometry of infinite-dimensional function spaces, or Hilbert Spaces, which generalize the notion of vector spaces to include functions with infinitely many degrees of freedom.

Inner products of functions and vectors

- Hermitian inner product of $f(x)$ and $g(x)$ defined for x on a domain $x \in [a, b]$:

$$\langle f(x), g(x) \rangle = \int_a^b f(x)\, \underbrace{\bar{g}(x)}_{\text{Complex Conjugate}}\, dx$$



a little motivation on innerproduct of vector data

discrete data
$$\underline{f} = [f_1, f_2, f_3 \cdots f_n]^T$$
$$\underline{g} = [g_1, g_2, g_3 \cdots g_n]^T$$

so $\langle \underline{f}, \underline{g} \rangle = g^* f = \sum_{k=1}^{n} f_k \bar{g}_k = \sum_{k=1}^{n} f(x_k)\bar{g}(x_k)$

- Add more and more data points and inner product grows

$\Delta x = (b-a)/(n-1)$

$$\frac{(b-a)}{(n-1)} \langle \underline{f}, \underline{g} \rangle = \sum_{k=1}^{n} f(x_k)\, \bar{g}(x_k)\, \Delta x$$

Riemann approximation

$\downarrow$

as $n \to \infty$, Vector inner Product Converges to inner Product of functions

- inner Product also induces a norm on functions

$$\|f\|_2 = \left(\langle f, f \rangle\right)^{1/2} = \left[\int_a^b f(x)\, \bar{f}(x)\, dx\right]^{1/2}$$

- Set of all functions with bounded norm define Set of square integrable functions: $L^2([a,b])$

$\downarrow$

Lebesgue integrable functions

$[a, b]$ Canbe $(-\infty, \infty)$

$[a, \infty]$

$[-\pi, \pi]$

Fourier Series

For Fourier analysis $f(x)$ is Periodic in 2π
Piecewise Smooth

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos(kx) + b_k \sin(kx) \right]$$

$$\underbrace{a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\, dx \qquad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\, dx}$$

Coordinates obtained by projecting the function onto orthogonal cosine and sine basis

<u>we can also written as</u>

$$a_k = \frac{1}{\underbrace{\|\cos(kx)\|^2}_{\pi}} \langle f(x), \cos(kx) \rangle \qquad b_k = \frac{1}{\underbrace{\|\sin(kx)\|^2}_{\pi}} \langle f(x), \sin(kx) \rangle$$

on L-Periodic function on $[0, L)$:-

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos\left(\frac{2\pi k x}{L}\right) + b_k \sin\left(\frac{2\pi k x}{L}\right) \right]$$

$$a_k = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{2\pi k x}{L}\right) dx \;\; \& \;\; b_k = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{2\pi k x}{L}\right) dx$$

→ Let us write it on **Complex Plane**

$$e^{ikx} = \cos(kx) + i \sin(kx) \qquad \text{with} \quad C_k = \alpha_k + i \beta_k$$

$$f(x) = \sum_{-\infty}^{\infty} c_k e^{ikx} = \sum_{-\infty}^{\infty} (\alpha_k + i\beta_k)\left[\cos(kx) + i\sin(kx)\right]$$

$$= (\alpha_0 + i\beta_0) + \sum_{k=1}^{\infty}\left[(\alpha_{-k}+\alpha_k)\cos(kx) + (\beta_{-k} - \beta_k)\sin(kx)\right]$$

$$+ i\sum_{k=1}^{\infty}\left[(\beta_{-k}+\beta_k)\cos(kx) - (\alpha_{-k}-\alpha_k)\sin(kx)\right]$$

- If $f(x)$ is real-valued, then $\alpha_{-k} = \alpha_k$ & $\beta_{-k} = -\beta_k$ so that $c_{-k} = \bar{c}_k$

- Thus, $\psi_k = e^{ikx}$ for $k \in \mathbb{Z}$ provide basis for periodic complex valued function on $[0, 2\pi)$

$$\langle \psi_j, \psi_k \rangle = \int_{-\pi}^{\pi} e^{ijx} e^{-ikx}\, dx = \int_{-\pi}^{\pi} e^{i(j-k)x}\, dx = \left[\frac{e^{i(j-k)x}}{i(j-k)}\right]_{-\pi}^{\pi} = \begin{cases} 0 & : j \neq k \\ 2\pi & : j = k \end{cases}$$

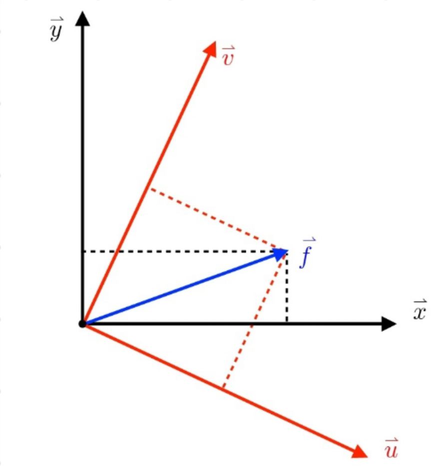
$$\Rightarrow \boxed{\langle \psi_j, \psi_k \rangle = 2\pi\, \delta_{jk}}$$

In principle: Fourier Series is just a change of coordinates of a function $f(x)$ into an infinite-dimensional orthogonal function space spanned by sines and cosines

$$\psi_k = e^{ikx} = \cos(kx) + i\sin(kx)$$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \psi_k(x) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \langle f(x), \psi_k(x)\rangle\, \psi_k(x)$$

where $c_k = \frac{1}{2\pi}\langle f(x), \psi_k(x)\rangle$ with $\|\psi_k(x)\|^2 = 2\pi$



$$\underline{f} = \langle \underline{f}, \underline{x} \rangle \frac{\underline{x}}{\|\underline{x}\|^2} + \langle \underline{f}, \underline{y} \rangle \frac{\underline{y}}{\|\underline{y}\|^2}$$

$$= \langle \underline{f}, \underline{u} \rangle \frac{\underline{u}}{\|\underline{u}\|^2} + \langle \underline{f}, \underline{v} \rangle \frac{\underline{v}}{\|\underline{v}\|^2}$$

$\underline{f}$ is Same on both (x,y) and (u,v) Coordinat

(1) Fourier Series of a Continuous hat function

$$f(x) = \begin{cases} 0 & \text{for } x \in [-\pi, \pi/2), \\ 1 + 2x/\pi & \text{for } x \in [-\pi/2, 0), \\ 1 - 2x/\pi & \text{for } x \in [0, \pi/2), \\ 0 & \text{for } x \in [\pi/2, \pi). \end{cases}$$

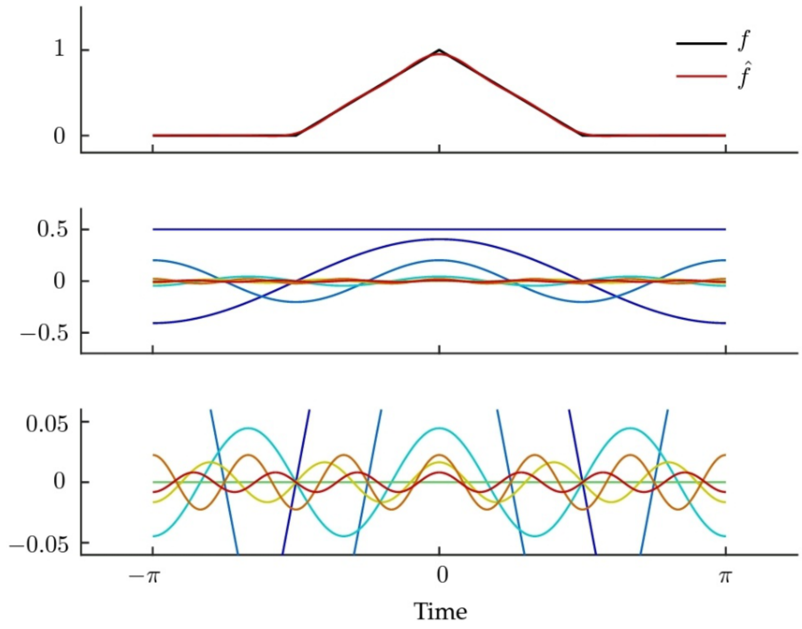


Figure 2.3: (top) Hat function and Fourier cosine series approximation for $n = 7$. (middle) Fourier cosines used to approximate the hat function. (bottom) Zoom in of modes with small amplitude and high frequency.

```python
# Define domain
dx = 0.001
L = np.pi
x = L * np.arange(-1+dx,1+dx,dx)
n = len(x)
nquart = int(np.floor(n/4))

# Define hat function
f = np.zeros_like(x)
f[nquart:2*nquart] = (4/n)*np.arange(1,nquart+1)
f[2*nquart:3*nquart] = np.ones(nquart) - (4/n)*np.arange(0,
    nquart)

# Compute Fourier series
A0 = np.sum(f * np.ones_like(x)) * dx
fFS = A0/2

A = np.zeros(20)
B = np.zeros(20)
for k in range(20):
    A[k] = np.sum(f * np.cos(np.pi*(k+1)*x/L)) * dx # Inner
        product
    B[k] = np.sum(f * np.sin(np.pi*(k+1)*x/L)) * dx
    fFS = fFS + A[k]*np.cos((k+1)*np.pi*x/L) + B[k]*np.sin((
        k+1)*np.pi*x/L)
```

```python
    ax.plot(x,fFS,'-')
```

- Run The code

(2) Fourier Series for discontinuous hat function

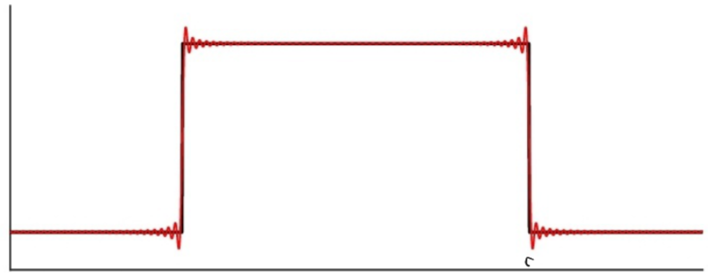


Figure 2.5: Gibbs phenomenon is characterized by high-frequency oscillations near discontinuities. The black curve is discontinuous, and the red curve is the Fourier approximation.

$$f(x) = \begin{cases} 0 & \text{for } x \in [0, L/4), \\ 1 & \text{for } x \in [L/4, 3L/4), \\ 0 & \text{for } x \in [3L/4, L). \end{cases}$$

Applying Fourier Series to a discontinuous data results in 'ringing oscillations' – Gibbs phenomena

- Run the code with Step function

Fourier transform

- Fourier Series is defined for periodic functions
- Fourier transform integral is essentially the limit of a Fourier Series as the length of domain goes to infinity, which allows us to define a function defined on $(-\infty, \infty)$ without repeating

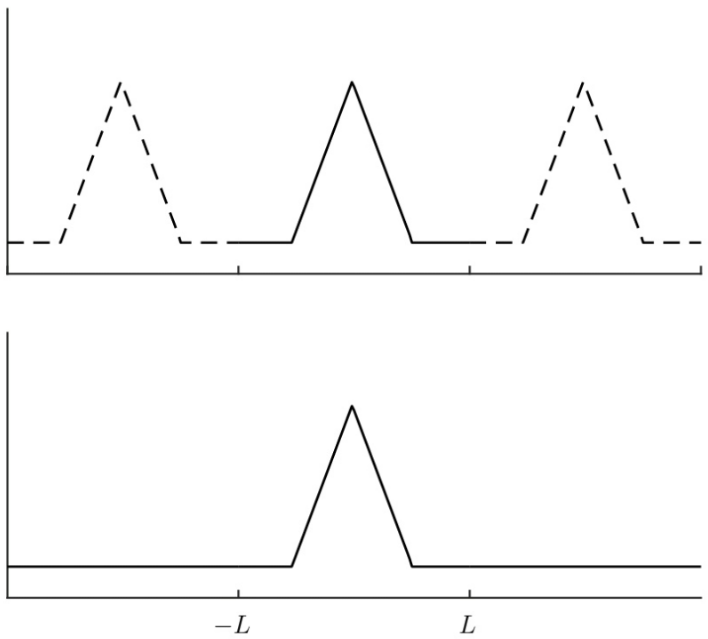


Figure 2.6: (top) Fourier series is only valid for a function that is periodic on the domain $[-L, L)$. (bottom) The Fourier transform is valid for generic non-periodic functions.

Fourier Series on a domain $x \in [-L, L)$ and then $L \to \infty$

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left[a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right)\right] = \sum_{-\infty}^{\infty} c_k e^{ik\pi x/L}$$

with coefficients

$$c_k = \frac{1}{2L}\langle f(x), \psi_k \rangle = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-ik\pi x/L}\, dx$$

$\to$ Now frequencies are not discrete, but continuous $\omega_k = \frac{k\pi}{L} \to \omega = \frac{k\pi}{L} \Rightarrow \boxed{\Delta\omega = \pi/L}$

$\Downarrow$

$L = \pi/\Delta\omega$

Note, Carefully we move on to a frequency domain

$$f(x) = \underset{\substack{\Delta\omega \to 0 \\ (=\pi/L)}}{Lt} \sum_{k=-\infty}^{\infty} \underbrace{\frac{\Delta\omega}{2\pi}}_{\frac{1}{2L}} \underbrace{\int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} f(\xi)\, e^{-ik\Delta\omega\xi}\, d\xi}_{\langle f(x), \psi_k(x) \rangle} \, e^{ik\Delta\omega x}$$

is the Fourier transform

$$\hat{f}(\omega) \triangleq \mathcal{F}(f(x))$$

- Summation with weight $\Delta\omega$ is Riemann integral

$$f(x) = \mathcal{F}^{-1}\left(\hat{f}(\omega)\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega x}\, d\omega$$

$$\hat{f}(\omega) = \mathcal{F}(f(x)) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

→ Both integrals Converge as long as

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} |\hat{f}(\omega)|\, d\omega < \infty$$

$$f, \hat{f} \in L^1(-\infty, \infty)$$

1. Derivative of Functions

$$\mathcal{F}\left(\frac{d}{dx} f(x)\right) = \int_{-\infty}^{\infty} \underbrace{f'(x)}_{dv} \underbrace{e^{-i\omega x}}_{u} dx$$

$$= \underbrace{\left[f(x)\, e^{-i\omega x}\right]_{-\infty}^{\infty}}_{uv} - \int_{-\infty}^{\infty} \underbrace{f(x)}_{v} \underbrace{\left[-i\omega\, e^{-i\omega x}\right]}_{du} dx$$

$$= i\omega \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= i\omega\, \mathcal{F}(f(x))$$

Very important property

$$\underset{\text{PDE}}{u_{tt} = c\, u_{xx}} \longrightarrow \underset{\text{ODE}}{\hat{u}_{tt} = -c\omega^2 \hat{u}}$$

2. Linearity: $\mathcal{F}(\alpha f(x) + \beta g(x)) = \alpha \mathcal{F}(f) + \beta \mathcal{F}(g)$

$$\mathcal{F}^{-1}(\alpha \hat{f} + \beta \hat{g}) = \alpha \mathcal{F}^{-1}(\hat{f}) + \beta \mathcal{F}^{-1}(\hat{g})$$

3. Parseval's theorem: $\displaystyle\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega = 2\pi \int_{-\infty}^{\infty} |f(x)|^2 dx$

Fourier transforms maintain L_2 norm, upto a Constant, So that two functions retain the inner Product

4. Convolution: $$f * g = \int_{-\infty}^{\infty} f(x-\xi)\, g(\xi)\, d\xi$$

$$\mathcal{F}^{-1}(\hat{f}\hat{g}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}\,\hat{g}\, e^{i\omega x}\, d\omega$$

$$= \int_{-\infty}^{\infty} \hat{f}\, e^{i\omega x}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} g(y)\, e^{-i\omega y}\, dy\right) d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y)\, \hat{f}\, e^{i\omega(x-y)}\, d\omega\, dy$$

$$= \int_{-\infty}^{\infty} g(y) \underbrace{\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}\, e^{i\omega(x-y)}\, d\omega\right)}_{f(x-y)} dy$$

$$= \int_{-\infty}^{\infty} g(y)\, f(x-y)\, dy = g * f = f * g$$

In Control Systems, it is a useful Property That multiplying in frequency domain is same as Convolving in Spatial domain.

2.2. Discrete Fourier transform

- Reality data is discrete
- Discrete Fourier transforms doesnot scale well for large $n \gg 1$.

→ Simple formulation involves multiplication by a dense $n \times n$ matrix requiring $O(n^2)$ operations

- Cooley (IBM) and Tukey (Princeton) developed revolutionary fast Fourier transform (FFT) in $\underbrace{\text{1965 which scales as } O(n \log(n))}$

Algorithm based on fractal Symmetry which reduces # of Computations

$$\hat{f}_k = \sum_{j=0}^{n-1} f_j \, e^{-i \frac{2\pi j k}{n}}$$

and the inverse transform is

$$f_k = \frac{1}{n} \sum_{j=0}^{n-1} \hat{f}_j \, e^{i \frac{2\pi j k}{n}}$$

Generate real Part of that matrix

$$\begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_n \end{bmatrix} = \underbrace{\overset{\underline{\underline{F}}}{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & & \omega_n^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & & \omega_n^{(n-1)^2} \end{bmatrix}}}_{\text{DFT – Vandermode matrix}} \underbrace{\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}}_{\text{input}}$$

Fast Fourier transform

Advantage is $O(n\log(n))$ instead $O(n^2)$

- Audio is generally Sampled at 44.1 KHz

→ For 10 Sec audio, The vector $\underline{f}$ will have dimension $n = 4.41 \times 10^5$

→ Computing DFT using matrix require 2×10^{11} or 200 billion multiplications

→ FFT requires only 2×10^6 multiplications

- Transmission, Storage and Decoding