This PDF consists of solutions to chapter-2 Homework

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2.2

Given $f(x) = \begin{cases} 0 & \text{for } x < -1, \\ 1-|x| & \text{for } |x| \leq 1 \\ 0 & \text{for } 1 < x \end{cases}$; $[-2,2]$

$f(x)$ is even function.

$\Rightarrow b_0 = 0$ $\quad a_0 = \int_{-2}^{2} f(x)\,dx$

$$= \int_{-1}^{0} (x+1)\,dx + \int_{0}^{1} (1-x)\,dx$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

Evaluating fourier coeff :-

$$a_n = \int_{-1}^{0} (x+1)\cos\left(\frac{n\pi x}{2}\right) dx + \int_{0}^{1} (1-x)\cos\left(\frac{n\pi x}{2}\right) dx$$

$$b_n = \int_{-1}^{0} (x+1)\sin\left(\frac{n\pi x}{2}\right) dx + \int_{0}^{1} (1-x)\sin\left(\frac{n\pi x}{2}\right) dx$$

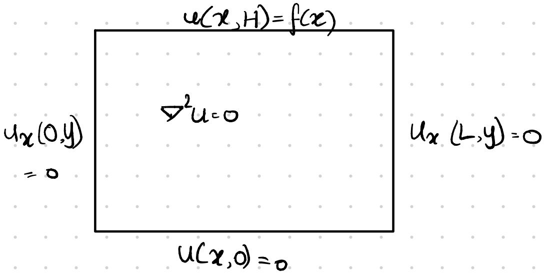
$$\rightarrow a_n = -2\left(\frac{4\cos\left(\frac{n\pi}{2}\right) - 4}{\pi^2 n^2}\right)$$

$$b_n = \left[\frac{4\sin\left(\frac{\pi n}{2}\right)}{\pi^2 n^2} - \frac{2}{\pi n}\right] + \left[\frac{2}{\pi n} - \frac{4\sin\left(\frac{\pi n}{2}\right)}{\pi^2 n^2}\right] = 0$$

$$\therefore f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \left(-2\left[\frac{4\cos\left(\frac{n\pi}{2}\right) - 4}{\pi^2 n^2}\right]\right) \cos(n\pi x)$$

2.5

→ 2D Heat distribution for Equilibrium Temp Distribution



Using seperation of variables:-

→ Assuming $u(x,y) = F(x)\,G(y)$

We have, $u_{xx} + u_{yy} = 0$

$$F_{xx}G + FG_{yy} = 0$$

$$\frac{F_{xx}}{F} = -\frac{G_{yy}}{G} = -\lambda \text{ (constant)}$$

NOW, we have ODE, $F_{xx} = \lambda F$ & $+G_{yy} = \lambda G$.

<u>BCs</u>:- $F_x(0) = 0$ & $F_x(L) = 0$ $G(0) = 0$ & $G(H) = f(x)$

→ Solving $F_{xx} = -\lambda F$ and find λ

<u>case (I)</u>:- $\lambda > 0$

<u>Assume</u>, $F = e^{\alpha x}$

$$F_{xx} = \alpha^2 e^{\alpha x}$$

$$\alpha^2 = -\lambda$$

$$\alpha = \pm i\sqrt{\lambda}$$

$$F(x) = k_1 \cos(\sqrt{\lambda}x) + ik_2 \sin(\sqrt{\lambda}x)$$

$$F_x(x) = -k_1 \sin(\sqrt{\lambda}x)\sqrt{\lambda} + ik_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

Taking $k_2 = 0$ $k_1 = 1$

$$F_x(x) = -k_1\sqrt{\lambda}\sin(\sqrt{\lambda}x)$$

satisfying boundary conditions,

$\sqrt{\lambda}x = n\pi \quad @ x = L$

$\sqrt{\lambda}L = n\pi$

$\lambda = \frac{n^2\pi^2}{L^2}$

$\Rightarrow F(x) = \cos\left(\frac{n\pi}{L}x\right)$

<u>case(2):-</u> $\lambda < 0$

F = exponential. B.C's are not satisfied.

→ Solving $G_{yy} = \lambda G$ for $\lambda = \frac{n^2\pi^2}{L^2}$

$$G_{yy} = \frac{n^2\pi^2}{L^2}G \Rightarrow A_n e^{+\frac{n\pi}{L}y} + B_n e^{-\frac{n\pi}{L}y} = G(y)$$

$$\Rightarrow A_n + B_n = 0 \quad (BC: G(0) = 0)$$

$$A_n = -B_n$$

$$\Rightarrow G(y) = A_n\left[e^{\frac{n\pi y}{L}} - e^{-\frac{n\pi y}{L}}\right]$$

$$\boxed{G(y) = A_n \sinh\left(\frac{n\pi y}{L}\right)}$$

<u>Solution:-</u>

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi y}{L}\right)\cos\left(\frac{n\pi x}{L}\right)$$

<u>Solving the A_n's.</u>

$$G(x,H) = f(x) = \sum_{n=1}^{\infty} 2A_n \sinh\left(\frac{n\pi H}{L}\right)\cos\left(\frac{n\pi x}{L}\right)$$

$$\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)dx = \int_0^L \sum 2A_n \sinh\left(\frac{n\pi H}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi}{L}x\right)dx$$

orthogonal if $n \neq m$

$$\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx = A_n L \sinh\left(\frac{n\pi H}{L}\right)$$

$$\boxed{A_n = \frac{2}{L\sinh\left(\frac{n\pi H}{L}\right)} \int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx}$$

→ consider $f(x) = x$ (a nonconstant distribution)

$$\Rightarrow \quad A_n = \frac{2}{H\sinh\left(\frac{n\pi H}{L}\right)} \int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx$$

$$\boxed{\begin{gathered} A_n = \frac{2}{H\sinh\left(\frac{n\pi H}{L}\right)} \left[\frac{L^2(\pi n \sin(\pi n) + \cos(\pi n) - 1)}{\pi^2 n^2}\right] \\ \text{and } u(x,y) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi y}{L}\right)\cos\left(\frac{n\pi x}{L}\right) \end{gathered}}$$

→ If left and right boundary conditions were fixed at zero temperature, in the solution there would be a sin term instead of $\cos\left(\frac{n\pi x}{L}\right)$

2.7

Given PDE,

$$u_{tt} = c^2 u_{xx} \quad 0 \le x \le L,$$

with the initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = 0$$

and boundary conditions,

$$u(0,t) = 0, \quad u_x(L,t) = f(t)$$

→ Laplace wrt 't':

$$s^2 \bar{u}(x,s) - \underbrace{s u_t(x,0)}_{=0} - \underbrace{u(x,0)}_{=0} = c^2 \bar{u}_{xx}$$

$$\boxed{\bar{u}_{xx} = \frac{s^2}{c^2} \bar{u}}$$

(converted into a simple ODE)

$$\bar{u}(x,s) = \bar{A}(s) e^{-\frac{s}{c}x} + \bar{B}(s) e^{+\frac{s}{c}x}$$

$$\bar{u}(0,s) = 0 = \bar{A}(s) + \bar{B}(s) \Rightarrow \bar{B}(s) = -\bar{A}(s)$$

$$\bar{u}(x,s) = \bar{A}(s) \left[e^{-\frac{sx}{c}} - e^{\frac{sx}{c}} \right]$$

$$\bar{u}(L,s) = \bar{A}(s) \left[e^{-\frac{sL}{c}} - e^{\frac{sL}{c}} \right] = \bar{f}(s) = \mathcal{L}\{f(t)\}$$

$$\bar{u}(x,s) = \frac{\bar{f}(s)}{\left[e^{-\frac{sL}{c}} - e^{\frac{sL}{c}} \right]} \left[e^{-\frac{sx}{c}} - e^{\frac{sx}{c}} \right] = \frac{\bar{f}(s) \sinh\left(\frac{sx}{c}\right)}{\sinh\left(\frac{sL}{c}\right)}$$

→ Trying variable seperable

$$u = F(x) G(t)$$

& PDE is $u_{tt} = c^2 u_{xx}$

$$F G'' = c^2 F'' G$$

$$\frac{G''}{G} = c^2 \frac{F''}{F} = \text{constant} = \lambda$$

$G'' = \lambda G$ & $F'' = \frac{\lambda F}{c^2}$

$\rightarrow$ $(\lambda > 0)$ – Assume,

$G = e^{\sqrt{\lambda} t}$ $F = e^{\frac{\sqrt{\lambda}}{c} x}$

$$u = e^{\sqrt{\lambda} t} e^{\frac{\sqrt{\lambda}}{c} x}$$

B.C are not satisfied

2.10

a) $\mathcal{L}(e^{\lambda t}) = \frac{1}{s-\lambda}$

$$\mathcal{L}(e^{\lambda t}) = \int_0^\infty e^{\lambda t} e^{-st} dt = \int_0^\infty e^{(\lambda-s)t} dt = \left[\frac{1}{s-\lambda} e^{-(\lambda-s)\infty} \right]$$

$$= \frac{1}{s-\lambda}$$

b) $\mathcal{L}(af(t) + bf(t)) = a\bar{f}(s) + b\bar{f}(s)$ $\forall$ constants $a, b \in C$

$$\mathcal{L}(af(t) + bf(t)) = \int_0^\infty (af(t) + bf(t)) e^{st} dt$$

$$= a\int_0^\infty f(t) e^{st} dt + b\int_0^\infty f(t) e^{st} dt$$

$$= a\bar{f}(s) + b\bar{f}(s)$$ where $\bar{f}(s) = \mathcal{L}\{f(t)\}$

c) convolution:- $\mathcal{L}\{f*g\} = \bar{f}(s)\bar{g}(s)$

$$\mathcal{L}\{f*g\} = \int_0^\infty (f*g)\, e^{st} dt = \int_0^\infty \left[\int_0^t f(\lambda)\, g(t-\lambda)\, d\lambda\right] e^{-st} dt$$

$$= \int_0^\infty f(\lambda) \left(\int_0^\infty e^{-st} g(t-\lambda)\, dt \right) d\lambda$$

substituting $\tau = t - \lambda$

$$= \int_0^\infty f(\lambda) \left(\int_{-\lambda}^\infty g(\tau)\, e^{-s(\tau+\lambda)} d\tau \right) d\lambda$$

arranging integral limits for τ,

$$\mathcal{L}[f*g] = \int_0^\infty f(\lambda) \left(\int_0^\infty g(\tau)\, e^{-s(\tau+\lambda)} d\tau \right) d\lambda$$

$$= \int_0^\infty f(\lambda) e^{-s\lambda} \left(\int_0^\infty g(\tau)\, e^{-s\tau} d\tau \right) d\lambda$$

$$= \int_0^\infty f(\lambda)\, e^{-s\lambda} d\lambda \left(\int_0^\infty g(\tau)\, e^{-s\tau} d\tau \right)$$

$$= \bar{f}(s)\, \bar{g}(s)$$

d) Constant: $\mathcal{L}(1) = \frac{1}{s}$.

$$\int_0^\infty 1 e^{-st} dt = \left[\frac{-1}{s} e^{-st}\right]_0^\infty = \left[0 + \frac{1}{s}\right] = \frac{1}{s}$$

e) Delta function: $\mathcal{L}(\delta(t)) = 1$

we have delta function $\int_{a-\varepsilon}^{a+\varepsilon} f(x)\, \delta(x-a)\, dx = f(a)$ $\quad \varepsilon > 0$

replacing valiable

$$\int_{a-\varepsilon}^{a+\varepsilon} f(t)\, \delta(t-a)\, dt = f(a) \qquad \varepsilon > 0$$

$$\mathcal{L}\{\delta(t)\} = \int_0^\infty e^{-st} \delta(t)\, dt = e^{-s(0)} = 1$$

2.11

Given,

$$\ddot{x} + a\dot{x} + bx = u(t)$$

we have $\mathcal{L}(\ddot{x}) = -v_0 + s\mathcal{L}(\dot{x}) = -v_0 - sx_0 + s^2\bar{x}(s)$

& $\mathcal{L}(\dot{x}) = -x_0 + s\bar{x}(s)$

$\Rightarrow$ subs,

$$-v_0 - sx_0 + s^2\bar{x}(s) - ax_0 + as\bar{x}(s) + b\bar{x}(s) = \mathcal{L}(u(t))$$

$$= \bar{u}(s) \quad \left(\text{where } \bar{u}(s) = \mathcal{L}(u(t)) \right)$$

$$\Rightarrow \bar{x}(s)\left[s^2 + as + b\right] - (sx_0 + v_0 + ax_0) = \bar{u}(s)$$

$$\bar{x}(s) = \frac{\bar{u}(s) + sx_0 + v_0 + ax_0}{s^2 + as + b}$$

$\rightarrow$ If $u(t) = \delta(t)$ $\quad \bar{x}(s) = \dfrac{1 + sx_0 + v_0 + ax_0}{s^2 + as + b}$

$\rightarrow$ If $u(t) = 1$ $\quad \bar{x}(s) = \dfrac{\frac{1}{s} + sx_0 + v_0 + ax_0}{s^2 + as + b}$