

HILBERT SCHEMES OF POINTS ON FOLD-LIKE CURVES AND THEIR COMBINATORICS

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ABSTRACT. We investigate the Hilbert scheme of points on curves with n -fold singularities, that is curves that look locally around their singular points as the axis in an affine space. We describe the structure and number of its irreducible components, and provide a detailed analysis of their singularities, revealing rich combinatorial patterns governing its geometry.

CONTENTS

Introduction	1
Structure of the paper	3
Acknowledgments	3
Funding	3
Notations	4
1. Irreducible components of the punctual Hilbert scheme	4
2. The moment map and its combinatorics	9
3. From the punctual to the global Hilbert scheme	18
4. The local Hilbert scheme and its schematic structure	22
5. Singularities	30
5.1. Singular Locus	30
5.2. Local picture of the singularities	32
6. Smoothable and non-smoothable components	37
6.1. Non-smoothable components	37
6.2. Smoothable components	40
7. Ongoing work and future questions	41
Appendix A. The Hypersimplicial complex	42
Appendix B. Some Commutative Algebra	45
References	51

INTRODUCTION

The Hilbert scheme of m points on a variety X over \mathbb{C} , which in this paper will be denoted by $\mathcal{Hilb}^m(X)$, is one of the most studied objects in algebraic geometry because it not only parametrizes how points in X behave when they start to collide, but because of its beauty and complexity. Starting from curves, the geometry of Hilbert schemes of points tends to be very rich. In the case of smooth curves, these Hilbert schemes are isomorphic to the symmetric product of the curve, which is smooth. However, when we allow singularities on the curves, their geometry becomes much more complicated, cf. for example [Kiv19, Lua23] and references therein. If an integral curve has *locally planar singularities*, meaning that it is contained in a smooth surface, then the Hilbert scheme of points is irreducible [AIK77, BcGS81], and its singularities are in a deep relation with the compactified Jacobian of the curve, see e.g. [Est01, MRV17] and the survey [Mig20] for recent applications to knot theory.

When the curve ceases to be locally planar, there has been growing interest in obtaining new *invariants* [AN23, KNS25] that might serve as substitutes for the topological invariants which are effective in the case of plane curve singularities.

In this vein, the study of the geometry of the Hilbert scheme of points associated with these more general singularities becomes particularly meaningful.

One of the main features of Hilbert schemes of points is the so called *Murphy's law* [Vak06, Jel20] that essentially says that, in general, it is out of reach to understand them. Therefore, finding explicit descriptions for the Hilbert schemes of points for specific varieties usually yields to a good amount of geometry. Ran in [Ran05] studied the case of nodal singularities on curves, he gave a very precise description of the Hilbert scheme of points of an irreducible curve with nodal singularities, describing completely their structure. The case of curves which are not contained in smooth surfaces, to the knowledge of the authors, has not been explored yet.

In this work, we address the Hilbert scheme of points for a class of curves with *rational n -fold singularities*, for which we found fascinating geometry and combinatorics. Given a reduced curve C , a point $p \in C$ is a rational n -fold singularity of C if locally around p , the curve C is analytically isomorphic to the union of the axis in \mathbb{C}^n . Nodal singularities are the case $n = 2$ and when $n \geq 3$ they are no longer locally planar. Rational n -fold singularities have been studied because, as nodal singularities, they are *semi-normal* [Bom73, Dav78]. In a very recent work [HKS24] the authors construct an alternative compactification of the moduli space of curves by adding *stable and separating* fold-like curves, cf. [HKS24, Theorem 1.2], see also [Smy13].

With this motivation, and with the aim of describing explicitly their compactified Jacobians, we need to first study their Hilbert scheme of points. Suppose now that C is an irreducible curve whose unique singularity is a rational n -fold singularity. One of our main results is a precise characterization of the irreducible components of its Hilbert scheme of points.

Theorem A. *Let C be an irreducible curve with a unique rational n -fold singularity and denote by C_{sm} its smooth locus. The irreducible components of $\mathcal{Hilb}^m(C)$ are birational to*

$$\mathcal{Hilb}^m(C_{\text{sm}}) \quad \text{and} \quad \mathcal{Hilb}^{m-m'}(C_{\text{sm}}) \times \text{Gr}(n+1-m', n) \quad \text{for } 2 \leq m' \leq \min\{m, n-1\}.$$

In particular, the number of irreducible components of $\mathcal{Hilb}^m(C)$ is $\min\{n-1, m\}$.

A direct consequence of Theorem A is that the number of irreducible components of $\mathcal{Hilb}^m(C)$ is $n-1$ as long as $m \geq n-1$ (see Fig. 1). As far as the authors are aware, this phenomenon is rather unexpected, since it is typically observed that as the number of points increases, a non-irreducible Hilbert scheme of points tends to have an increasing number of components. Notice also that there exists components of different dimensions whenever $n \geq 4$ and $m \geq 2$. So in these cases, the Hilbert scheme of points is not Cohen-Macaulay by [Eis95, Corollary 18.11].

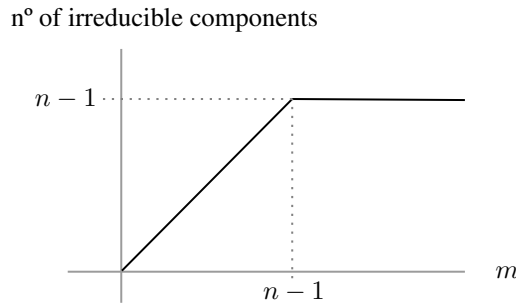


FIGURE 1. Graph of the number of irreducible components of the Hilbert scheme of m points on a curve with an n -fold singularity.

The strategy to prove Theorem A is to calculate the *elementary components* of $\mathcal{Hilb}^m(C)$. To do so, we first study the locus in $\mathcal{Hilb}^m(C)$ of those subschemes supported at the singularity, the so-called *punctual Hilbert scheme* $\mathcal{Hilb}_p^m(C)$. Since this is a local problem we can assume that we are studying the axis in \mathbb{C}^n , which we denote by X_n , and the singular point to be the origin $\mathbf{0}$. We classify the possible ideals that appear in $\mathcal{Hilb}_0^m(X_n)$. From there, we obtain the irreducible components of $\mathcal{Hilb}_0^m(X_n)$ and also the identification with the corresponding Grassmannians.

With this observation in place, we proceed to examine the structure of the Hilbert scheme of points. For this purpose, we make use of the combinatorial structures associated with these curves; in particular, we derive the following result concerning $\mathcal{Hilb}_0^m(X_n)$.

Theorem B. *The punctual Hilbert scheme $\mathcal{Hilb}_0^m(X_n)$ is a union of Grassmanians of the form $\text{Gr}(l, n)$ for $\max\{1, n + 1 - m\} \leq l \leq n - 1$, where $\text{Gr}(l, n)$ appears $\binom{l+m-2}{n-1}$ -times. Moreover, there is a well-defined moment map*

$$\mu_m : \mathcal{Hilb}_0^m(X_n) \longrightarrow (m - 1) \cdot \Delta_{n-1}.$$

The moment map allows us to study the geometry of $\mathcal{Hilb}_0^m(X_n)$ through the combinatorics of a *hypersimplicial complex* in $(m - 1) \cdot \Delta_{n-1}$. The strategy for the proof of Theorem B is motivated by the natural toric action on X_n . We identify which irreducible components of $\mathcal{Hilb}_0^m(X_n)$ are lifted to *elementary components* of $\mathcal{Hilb}^m(X_n)$ and relate them to the combinatorics of the moment map. To this end, we carry out a detailed analysis of the singularities and scheme structure of the local Hilbert scheme, employing deformation theory and the natural torus action to extend our results to the global setting. This leads to our other main theorem, which provides a detailed description of the singularities occurring in its components. Here, the combinatorics developed previously play a prominent role.

Theorem C. *The Hilbert scheme $\mathcal{Hilb}^m(X_n)$ is reduced. Moreover, each smoothable component of $\mathcal{Hilb}^m(X_n)$ is normal and has toric singularities. Each non-smoothable component is smooth.*

When we replace X_n by an *irreducible curve* C with a unique rational n -fold singularity, the smoothable component is no longer normal and its singularities are not toric. Moreover, the non-smoothable components are singular. However, their structure can be made explicit. This yields our final main result.

Theorem D. *Let C be an irreducible curve with a unique rational n -fold singularity. Then, $\mathcal{Hilb}^m(C)$ is reduced. The singularities of the smoothable components are locally unions of normal toric varieties, while those of the non-smoothable components are locally unions of affine spaces. In addition, there is an explicit description of the normalization of the non-smoothable components.*

Structure of the paper. In Section 1 the main result is Theorem 1.13, where we classify the irreducible components of $\mathcal{Hilb}_0^m(X_n)$. In Section 2 we construct a moment map for $\mathcal{Hilb}_0^m(X_n)$ leading to Theorem 2.1, obtaining Theorem B for the reduced structure of $\mathcal{Hilb}_0^m(X_n)$. In this section, we also explore the relation between the geometry of $\mathcal{Hilb}_0^m(X_n)$ and a hypersimplicial complex. Some of the combinatoric lemmas needed for this purpose are given in Section A. In Section 3, we prove Theorem 3.6 that establishes Theorem A for the reduced structure of $\mathcal{Hilb}^m(C)$. In Section 4 we deduce Theorem 4.11 that shows that the punctual Hilbert scheme is reduced. This completes the proof of Theorem B. Then by Section 3 we obtain that the same happens for the whole Hilbert scheme of points, completing the proof of Theorem A. This makes use of some amount of commutative algebra computations which are given in Section B. From there, in Section 5 we start studying the singular locus of $\mathcal{Hilb}^m(X_n)$ where, by using combinatoric methods explained in Section A, we characterize it completely. The main result of this section is Theorem 5.2. Afterwards in Section 6 we focus on the description of the smoothable and non-smoothable components. Propositions 6.5 and 6.9 complete the proof of Theorem C. Theorem 6.3 and Corollaries 6.6 and 6.10 lead to Theorem D. Finally in Section 7 we report some ongoing work and state some open questions and future research directions.

The theory presented in this paper is complemented by a variety of examples, intended to offer deeper insight into the problems under consideration and to illustrate the geometric structures that emerge from them.

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Notations. We will work over \mathbb{C} . All arguments remain valid over any algebraically closed field of characteristic zero. The case of positive characteristic is unknown to the authors. Vectors in \mathbb{C}^n are written in boldface. The origin in \mathbb{C}^n is denoted by $\mathbf{0}$ and the vector $(1, \dots, 1)$ as $\mathbf{1}$. The standard basis for \mathbb{C}^n is $\mathbf{e}_1, \dots, \mathbf{e}_n$. For a subset $S \subset [n]$, we set

$$\mathbf{e}_S := \sum_{i \in S} \mathbf{e}_i.$$

As is customary, $\binom{[n]}{l}$ denotes the set of subsets of $\{1, \dots, n\}$ with l elements.

- X_n : The union of the axis L_1, \dots, L_n , where $L_i := \mathbb{V}(x_j : j \neq i) \subseteq \mathbb{C}^n$ (Theorem 1.1).
- R_n : The coordinate ring of X_n (Eq. (1)).
- $\text{Sym}^m(C)$: The m -th symmetric product of C (Theorem 3.5).
- $\text{Hilb}^m(X_n)$: The reduced Hilbert scheme of m points of X_n (Theorem 1.2).
- $\mathcal{Hilb}_0^m(X_n)$: The punctual Hilbert scheme at the origin (Theorem 1.2).
- $\text{Hilb}_0^m(X_n)$: The reduced punctual Hilbert scheme at the origin (Theorem 1.2).
- $\Sigma(m, l, \mathbf{u})$: A subvariety of $\text{Hilb}_0^m(X_n)$ constructed by an integer $l \leq n$ and a partition \mathbf{u} (Theorem 1.5).
- $\Lambda_{\mathbf{u}}$: The vector subspace generated by monomials indexed by \mathbf{u} (Theorem 1.5).
- $\langle \Gamma \rangle$: The ideal generated by the vector subspace $\Gamma \subseteq \Lambda_{\mathbf{u}}$ (Eq. (2)).
- $\kappa(\mathbf{w}, k)$: The function that gives the indexes i for which $\mathbf{w}_i = k$ (Eq. (6)).
- $\mu_{l,n}$: The moment map (Eq. (10)).
- $\mu_{\mathbf{u},l}$: The moment map defined on $\Sigma(m, l, \mathbf{u})$ (Eq. (11)).
- μ_m : The moment map defined on $\text{Hilb}_0^m(X_n)$ (Theorem 2.1).
- $\mathcal{K}_n^{[m]}$: The (n, m) -hypersimplicial complex (Theorem A.2).
- $\mathcal{K}_n^{[m]}(S_1, S_2, l, \mathbf{u})$: The faces of the complex $\mathcal{K}_n^{[m]}$ (Eq. (14)).
- $\mathcal{K}_{l,n}^{[m]}$: subcomplex of $\mathcal{K}_n^{[m]}$ formed by hypersimplices of the form $\Delta_{l,n}$.
- \mathcal{G}_n^m : The variety obtained by gluing Grassmannians following $\mathcal{K}_n^{[m]}$ (Eq. (16)).
- $\mathcal{G}_{l,n}^m$: The subvariety of \mathcal{G}_n^m formed by the Grassmannians of the form $\text{Gr}(l, n)$.
- $\text{Hilb}^{m,m'}(X_n)$: The non-smoothable components of $\text{Hilb}^m(X_n)$ (Eq. (23)).
- \mathbb{S}_k and $\widehat{\mathbb{S}}_k$: The simplicial complexes describing the singularities of $\mathcal{Hilb}^m(X_n)$ (Theorem 5.8).
- $\mathcal{Hilb}^{m,m',u}(C)$: The strata of the non-smoothable components of $\mathcal{Hilb}^m(C)$ (Eq. (45)).

1. IRREDUCIBLE COMPONENTS OF THE PUNCTUAL HILBERT SCHEME

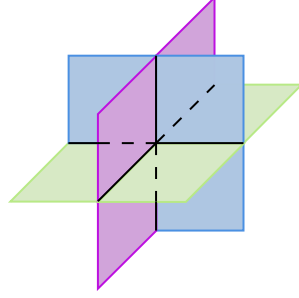
A classical strategy to analyze the irreducible components of Hilbert schemes of points is to focus on elementary components. Following [Iar73], an elementary component is an irreducible component of the Hilbert scheme of m points that parameterizes subschemes supported at a single point. In the case of a curve C with a rational n -fold singularity $p \in C$, an elementary component must parametrize length m subschemes supported at the singularity of p . We start this section defining this type of singularities.

Definition 1.1. Let X_n be the union of the axis L_1, \dots, L_n of \mathbb{C}^n , where $L_i := \mathbb{V}(x_j : j \neq i)$ and let $\mathbf{0} = (0, \dots, 0) \in \mathbb{C}^n$ be the singular point of X_n . A curve C has a rational n -fold singularity at $p \in C$ if, locally around p , C is analytically isomorphic to X_n around $\mathbf{0}$.

Algebraically, $p \in C$ is a rational n -fold singularity if there exists an isomorphism between the completed stalks $\widehat{\mathcal{O}}_{C,p} \simeq \widehat{\mathcal{O}}_{X_n,\mathbf{0}}$. We denote the coordinate ring of X_n by R_n , which is defined by

$$(1) \quad R_n = \mathbb{C}[x_1, \dots, x_n] / \langle x_i x_j : 1 \leq i < j \leq n \rangle.$$

A rational 2-fold singularity is a nodal singularity. However, for $n \geq 3$ such singularities have embedding dimension n ; in particular, they can no longer be embedded in a smooth surface. By the very definition of elementary components, we can replace C and p by X_n and $\mathbf{0}$, respectively. In this section, we analyze ideals in $\mathcal{Hilb}^m(X_n)$ supported at $\mathbf{0}$, and then extend the analysis to the full Hilbert scheme. We first perform this analysis on the reduced Hilbert scheme of points $\text{Hilb}^m(X_n)$, which is $\mathcal{Hilb}^m(X_n)$ endowed with its reduced structure.

FIGURE 2. Planes spanned by each pairs of lines in X_3 .

Definition 1.2. The punctual Hilbert scheme at $\mathbf{0}$, denoted by $\mathcal{Hilb}_0^m(X_n)$, is the locus in $\mathcal{Hilb}^m(X_n)$ of length m ideals in R_n supported at $\mathbf{0}$. The variety $\text{Hilb}_0^m(X_n)$ is the punctual Hilbert scheme endowed with its *reduced structure*.

Notice that an elementary component of $\mathcal{Hilb}^m(X_n)$ is an irreducible component of $\mathcal{Hilb}_0^m(X_n)$. We first study the irreducible components of $\mathcal{Hilb}_0^m(X_n)$, which can be done within $\text{Hilb}_0^m(X_n)$. In Section 3, we determine which of these lift to elementary components of $\mathcal{Hilb}^m(X_n)$.

Example 1.3. For $m = 2$, $\text{Hilb}_0^2(X_n) = \mathcal{Hilb}_0^2(X_n)$ is the projectivization of the tangent space of X_n at $\mathbf{0}$. In particular, $\text{Hilb}_0^2(X_n)$ is $\mathbb{P}(T_0 X_n) \simeq \mathbb{P}^{n-1}$ and we can identify ideals in $\text{Hilb}_0^2(X_n)$ with the tangent directions at $\mathbf{0}$. The intersection of $\text{Hilb}_0^2(X_n)$ and the smoothable component consists of all tangent directions at $\mathbf{0}$ that lie in the planes spanned by two of the lines L_1, \dots, L_n . This is also called the *tangent star* of X_n (see Fig. 2). We refer to [Rus16, Chapter 1] for the general definition of the tangent star. In particular, $\text{Hilb}_0^2(X_n) \simeq \mathbb{P}^{n-1}$ must be an irreducible component of $\text{Hilb}^2(X_n)$. This fact can also be derived by a dimension argument.

We now give a description of the generators of the ideals in $\text{Hilb}_0^m(X_n)$.

Proposition 1.4. Let $l \in [n]$ and $\mathbf{u} \in \mathbb{Z}_{\geq 1}^n$ be a strictly positive partition of $m + l - 1$. Consider a full rank matrix A of size $l \times n$ with no vanishing column. Then, the ideal generated by the polynomials

$$\begin{pmatrix} f_1 \\ \vdots \\ f_l \end{pmatrix} = A \begin{pmatrix} x_1^{u_1} \\ \vdots \\ x_n^{u_n} \end{pmatrix}.$$

lies in $\text{Hilb}_0^m(X_n)$. Moreover, all ideals in $\text{Hilb}_0^m(X_n)$ are of this form.

Proof. Let J be an ideal as above. First we check that J is supported at $\mathbf{0}$. Since A has not vanishing columns, for every $1 \leq i \leq n$ there exists $1 \leq j \leq l$ such that the entry $A_{j,i}$ of A is nonzero. Therefore it holds $x_i f_j = A_{j,i} x_i^{u_i+1}$. We deduce that J contains $x_1^{u_1+1}, \dots, x_n^{u_n+1}$ and therefore, J is supported at $\mathbf{0}$. We are left to show that J has length m . Since $x_1^{u_1+1}, \dots, x_n^{u_n+1} \in J$, the quotient R_n/J is generated by $1, x_1, \dots, x_1^{u_1}, \dots, x_n, \dots, x_n^{u_n}$. The generators of J induce l linearly independent linear relations among $x_1^{u_1}, \dots, x_n^{u_n}$. We deduce that

$$\dim R_n/J = 1 + u_1 + \dots + u_n - l = |\mathbf{u}| + 1 - l = m$$

as claimed.

We will now prove the second part of the proposition. Let J be an ideal of R_n supported at $\mathbf{0}$ of length m , and let f_1, \dots, f_l be minimal generators of J . Since J is supported at $\mathbf{0}$, then f_1, \dots, f_l do not have independent term. We write each f_i as

$$f_i = f_{i,1}(x_1) + \dots + f_{i,n}(x_n),$$

where $f_{i,j}(x_i)$ is a polynomial in x_i such that $f_{i,j}(0) = 0$. In the case of x_1 we get

$$f_{i,1} = \sum_{k=1}^{d_i} a_{i,j} x_1^k,$$

where $d_i = \deg f_{i,1}$ and $a_{i,j} \in \mathbb{C}$. Notice that if $f_{i,1}$ vanishes for all i , then the dimension of R_n/J as a \mathbb{C} -vector space is infinite, and this implies that J does not have finite length. Therefore we can assume that a_{1,d_1} is nonzero. Now, in R_n we have that

$$x_1 f_1 = x_1 f_{i,1} = \sum_{j=1}^{d_1} a_{1,j} x_1^{j+1}$$

is in J . Since J is supported at $\mathbf{0}$, the only root of $x_1 f_{i,1}$ is 0. We conclude that $f_{i,1}$ is the monomial $a_{i,d_i} x_1^{d_i+1}$. We can assume d_1 is the minimum between those d_1, \dots, d_n that are nonzero. By replacing f_i by $a_{1,d_1} f_i - a_{i,d_i} x_1^{d_i-d_1} f_1$, we may assume that $d_1 = d_2 = \dots = d_n$.

Let d_i be the degree of $f_{i,1}$ and suppose that there exists $2 \leq i \leq l$ such that $d_i > d_1$. In R_n we have that the monomial

$$x_1^{d_i-d_1} f_1 = x_1^{d_i-d_1} f_{1,1} = \sum_{j=1}^{d_1} a_{1,j} x_1^{j+d_i-d_1}$$

is in J . Since J is supported at $\mathbf{0}$, the only root of $x_1^{d_i-d_1} f_{1,1}$ is 0. We conclude that $f_{i,1}$ is the monomial $a_{1,d_1} x_1^{d_1}$. Repeating this process with the variables x_2, \dots, x_n we get that there exist d_1, \dots, d_n such that f_1, \dots, f_n are a linear combination of $x_1^{d_1}, \dots, x_n^{d_n}$. □

Notice that in the extremal case $l = n$ in Theorem 1.4 leads to the ideal $\langle x_1^{u_1}, \dots, x_n^{u_n} \rangle$.

Definition 1.5. Given $\mathbf{u} \in \mathbb{Z}_{\geq 1}^n$, let $\Lambda_{\mathbf{u}} := \langle x_1^{u_1}, \dots, x_n^{u_n} \rangle_{\mathbb{C}}$ be the \mathbb{C} -vector space generated by $x_1^{u_1}, \dots, x_n^{u_n}$. Fix $m \geq 1$. For $l \in [n]$ such that $|\mathbf{u}| = m + l - 1$, the subvariety $\Sigma(m, l, \mathbf{u}) \subseteq \text{Hilb}_{\mathbf{0}}^m(X_n)$ is the closure of the ideals of the form $\langle f_1, \dots, f_l \rangle$ where f_1, \dots, f_l are linearly independent elements of $\Lambda_{\mathbf{u}}$.

With the notation introduced above we immediately obtain the following.

Corollary 1.6. *Let m, n be positive integers, then there is a decomposition*

$$\text{Hilb}_{\mathbf{0}}^m(X_n) = \bigcup_{l=1}^n \bigcup_{\substack{\mathbf{u} \in \mathbb{Z}_{\geq 1}^n \\ |\mathbf{u}| = m + l - 1}} \Sigma(m, l, \mathbf{u}).$$

Hence the varieties $\Sigma(m, l, \mathbf{u})$ are the candidates to be irreducible components of the punctual Hilbert scheme. Next, we will describe the geometry of these varieties. For any $\Gamma \in \text{Gr}(l, \Lambda_{\mathbf{u}})$, let $\langle \Gamma \rangle$ be the ideal generated by Γ in R_n . Define the rational map

$$(2) \quad \begin{array}{ccc} \varphi_{l,\mathbf{u}} : & \text{Gr}(l, \Lambda_{\mathbf{u}}) & \dashrightarrow \Sigma(m, l, \mathbf{u}) \\ & \Gamma & \mapsto \langle \Gamma \rangle. \end{array}$$

Since $m + l - 1 = |\mathbf{u}|$, by Theorem 1.4 the map $\varphi_{l,\mathbf{u}}$ is well-defined in an open subset of the Grassmannian.

Lemma 1.7. *The base locus of $\varphi_{l,\mathbf{u}}$ is contained in the union*

$$(3) \quad \bigcup_{i=1}^n \mathcal{H}_i,$$

where $\mathcal{H}_i := \{\Gamma \in \text{Gr}(l, \Lambda_{\mathbf{u}}) : \Gamma \subseteq \langle x_j^{u_j} : j \neq i \rangle_{\mathbb{C}}\}$.

Proof. If $\Gamma \in \text{Gr}(l, \Lambda_{\mathbf{u}})$, then $\langle \Gamma \rangle$ is an ideal of R_n supported at $\mathbf{0}$. Hence, the base locus of $\varphi_{l,\mathbf{u}}$ coincide with the locus of $\Gamma \in \text{Gr}(l, \Lambda_{\mathbf{u}})$ such that $\langle \Gamma \rangle$ is not a length m ideal. Now, if $\Gamma \in \mathcal{H}_i$ for $1 \leq i \leq n$, then $\langle \Gamma \rangle$ has no finite length. Assume now that $\Gamma \notin \mathcal{H}_i$ for all $1 \leq i \leq n$. Then, $\langle \Gamma \rangle$ is generated by l polynomials f_1, \dots, f_l of the form

$$f_j = \sum_{i=1}^n a_{i,j} x_j^{u_i}.$$

Since $\Gamma \notin \mathcal{H}_i$ for all i , we deduce that for any $1 \leq i \leq n$, there exists $1 \leq j \leq l$ such that $a_{i,j} \neq 0$. In particular, $x_1^{u_1+1}, \dots, x_n^{u_n+1}$ are contained in $\langle \Gamma \rangle$, and hence, $\langle \Gamma \rangle$ has finite length. Moreover, this length is given by $u_1 + \dots + u_n + 1 - l = |\mathbf{u}| + 1 - l = m$. We conclude that $\varphi_{l,\mathbf{u}}$ is well-defined away from (3). □

In the next lemma we will show that the map $\varphi_{l,\mathbf{u}}$ can be extended along (3).

Lemma 1.8. *With the notation of Theorem 1.7, let $1 \leq k \leq n$ and assume $\Gamma \in \mathcal{H}_{i_1} \cap \cdots \cap \mathcal{H}_{i_k}$ where $1 \leq i_1 < \cdots < i_k \leq n$ and $\Gamma \notin \mathcal{H}_j$ for $j \neq i_1, \dots, i_k$. Then*

$$\varphi_{l,\mathbf{u}}(\Gamma) := \langle \Gamma \rangle + \langle x_{i_1}^{u_{i_1}+1}, \dots, x_{i_k}^{u_{i_k}+1} \rangle$$

extends $\varphi_{l,\mathbf{u}}$ to all $\text{Gr}(l, \Lambda_{\mathbf{u}})$.

Proof. Let J be an ideal in the image of $\varphi_{l,\mathbf{u}}$. By construction, $x_1^{u_1+1}, \dots, x_n^{u_n+1}$ are contained in J . In other words, $\langle x_1^{u_1+1}, \dots, x_n^{u_n+1} \rangle$ is contained in J . Since this containment is a closed condition in $\text{Hilb}^m(X_n)$, we deduce that for any J in the closure of the image of $\varphi_{l,\mathbf{u}}$, we have that $\langle x_1^{u_1+1}, \dots, x_n^{u_n+1} \rangle \subseteq J$.

Let Γ be an element of the base locus of $\varphi_{l,\mathbf{u}}$. By Theorem 1.7, there exist $1 \leq i_1 < \cdots < i_k \leq n$ such that $\Gamma \in \mathcal{H}_{i_1} \cap \cdots \cap \mathcal{H}_{i_k}$ for $1 \leq i_1 < \cdots < i_k \leq n$ such that $\Gamma \notin \mathcal{H}_j$ for $j \neq i_1, \dots, i_k$. In particular, $x_j^{u_j+1}$ is contained in $\langle \Gamma \rangle$ for $j \neq i_1, \dots, i_k$. Now, let C be a smooth curve in $\text{Gr}(l, \Lambda_{\mathbf{u}})$ passing through Γ and not contained in the base locus of $\varphi_{l,\mathbf{u}}$. Then, the restriction of $\varphi_{l,\mathbf{u}}$ to C extends to all C . Let I be the image of Γ via this extension. Note that by construction, $\langle \Gamma \rangle$ is contained in I . Therefore, we deduce that

$$(4) \quad \langle \Gamma \rangle + \langle x_{i_1}^{u_{i_1}+1}, \dots, x_{i_k}^{u_{i_k}+1} \rangle \subseteq I.$$

Both ideals in (4) are the same since both have length m . Then the proof follows from the fact that the ideal I does not depend on the curve C . \square

Remark 1.9. The varieties $\Sigma(m, l, \mathbf{u})$ in Theorem 1.5 are given as the closure of the ideals minimally generated by $f_1, \dots, f_l \in \Lambda_{\mathbf{u}}$. Using Theorem 1.8, there is a complete description of the elements in the boundary of this closure. This boundary is exactly the image of (3) through $\varphi_{l,\mathbf{u}}$.

Proposition 1.10. *The map $\varphi_{l,\mathbf{u}}$ extends uniquely to an isomorphism $\text{Gr}(l, \Lambda_{\mathbf{u}}) \cong \Sigma(m, l, \mathbf{u})$.*

Proof. The inverse of $\varphi_{l,\mathbf{u}}$ is the map

$$\psi_{l,\mathbf{u}}: \Sigma(m, l, \mathbf{u}) \dashrightarrow \text{Gr}(l, \Lambda_{\mathbf{u}})$$

that associates to an ideal J in $\Sigma(m, l, \mathbf{u})$ the linear subspace in $\Lambda_{\mathbf{u}}$ generated by its minimal set of generators, whenever this has dimension l . The only case where this does not happen is in the boundary of $\Sigma(m, l, \mathbf{u})$. By Theorem 1.9, such an ideal is, up to labeling, of the form

$$(5) \quad \langle f_1, \dots, f_l \rangle + \langle x_k^{u_k+1}, \dots, x_n^{u_n+1} \rangle,$$

where $f_1, \dots, f_l \in \langle x_1^{u_1}, \dots, x_{k-1}^{u_{k-1}} \rangle_{\mathbb{C}}$ are linearly independent and $1 \leq k \leq n$. Then, the extension of $\psi_{l,\mathbf{u}}$ sends J to $\langle f_1, \dots, f_l \rangle_{\mathbb{C}}$. \square

Using the map $\varphi_{l,\mathbf{u}}$, we can understand the intersection of two varieties of the form $\Sigma(m, l, \mathbf{u})$ and $\Sigma(m, l', \mathbf{v})$. Let $k \in \mathbb{Z}$ and $\mathbf{w} \in \mathbb{Z}^n$. Define

$$(6) \quad \kappa(\mathbf{w}, k) := \{i \in [n] : \mathbf{w}_i = k\} \subseteq [n].$$

Proposition 1.11. *Let $l, l' \in [n-1]$ and $\mathbf{u}, \mathbf{v} \in \mathbb{Z}_{\geq 1}^n$ such that $|\mathbf{u}| = m + l - 1$ and $|\mathbf{v}| = m + l' - 1$. Then, $\Sigma(m, l, \mathbf{u}) \cap \Sigma(m, l', \mathbf{v})$ is nonempty if and only if $\mathbf{u} - \mathbf{v} \in \{0, 1, -1\}^n \setminus \{\mathbf{1}, -\mathbf{1}\}$. In this case, the intersection $\Sigma(m, l, \mathbf{u}) \cap \Sigma(m, l', \mathbf{v})$ consists on the ideals of the form:*

$$(7) \quad \langle f_1, \dots, f_r \rangle + \langle x_i^{u_i} : i \in \kappa(\mathbf{u} - \mathbf{v}, 1) \rangle + \langle x_i^{u_i+1} : i \in \kappa(\mathbf{u} - \mathbf{v}, -1) \rangle,$$

with $f_1, \dots, f_r \in \langle x_i^{u_i} : i \in \kappa(\mathbf{u} - \mathbf{v}, 0) \rangle_{\mathbb{C}}$ linearly independent and $r = l - |\kappa(\mathbf{u} - \mathbf{v}, 1)|$.

Proof. Let $[I] \in \Sigma(m, l, \mathbf{u})$. Then by (5) there exist $S \subseteq [n]$ such that

$$I = \langle f_1, \dots, f_l \rangle + \langle x_i^{u_i+1} : i \in S \rangle,$$

with $f_1, \dots, f_l \in \langle x_i^{u_i} : i \notin S \rangle_{\mathbb{C}}$ linearly independent. With this presentation some of the generators f_1, \dots, f_l might be equal to $x_i^{u_i}$ for $i \notin S$. Rewrite I as

$$(8) \quad I = \langle x_i^{u_i+1} : i \in S \rangle + \langle x_i^{u_i} : i \in T \rangle + \langle f_1, \dots, f_r \rangle,$$

where $T \subset [n]$ is such that $S \cap T = \emptyset$, $r = l - |T|$ and $f_1, \dots, f_r \in \langle x_i^{u_i} : i \notin S \cup T \rangle$. Analogously, if $[I] \in \Sigma(m, l', \mathbf{v})$, then there exists $S', T' \subsetneq [n]$ disjoint such that

$$(9) \quad I = \langle x_i^{v_i+1} : i \in S' \rangle + \langle x_i^{v_i} : i \in T' \rangle + \langle f'_1, \dots, f'_{r'} \rangle,$$

with $r' = l' - |T'|$ and $f_1, \dots, f_{r'} \in \langle x_i^{v_i} : i \notin S' \cup T' \rangle$. From (8) and (9) we deduce that $r = r'$, $\langle f'_1, \dots, f'_{r'} \rangle = \langle f_1, \dots, f_r \rangle$ and $S \cup T = S' \cup T'$. Therefore, $u_i = v_i$ for $i \notin S \cup T$. For $i \in S$ we get that $u_i + 1 = v_i$ or $u_i + 1 = v_i + 1$. Similarly, $u_i = v_i$ or $u_i = v_i + 1$ for $i \in T$. Hence, $u_i - v_i \in \{0, 1, -1\}$ for any $i \in [n]$. Finally, notice that $S \neq [n]$, since the ideal $\langle x_1^{u_1+1}, \dots, x_n^{u_n+1} \rangle$ has length $|\mathbf{u}| + n + 1 - n = m + l$ which is greater than m and the same holds for S' . This shows the first inclusion.

For the other inclusion, assume that $\mathbf{u} - \mathbf{v} \in \{0, 1, -1\}^n \setminus \{\mathbf{1}, -\mathbf{1}\}$ and let J be an ideal as in (7). Then $[J] \in \Sigma(m, l, \mathbf{u})$ since it is the image of

$$\langle f_1, \dots, f_r \rangle_{\mathbb{C}} + \langle x_i^{u_i} : i \in \kappa(\mathbf{u} - \mathbf{v}, 1) \rangle_{\mathbb{C}} \subseteq \Lambda_{\mathbf{u}}$$

via $\varphi_{l, \mathbf{u}}$. Similarly, we can rewrite J as

$$\langle f_1, \dots, f_r \rangle + \langle x_i^{u_i+1} : i \in \kappa(\mathbf{u} - \mathbf{v}, 1) \rangle + \langle x_i^{v_i} : i \in \kappa(\mathbf{u} - \mathbf{v}, -1) \rangle,$$

which is the image of

$$\langle f_1, \dots, f_r \rangle_{\mathbb{C}} + \langle x_i^{v_i} : i \in \kappa(\mathbf{u} - \mathbf{v}, -1) \rangle_{\mathbb{C}} \subseteq \Lambda_{\mathbf{v}}$$

via $\varphi_{\mathbf{v}, l'}$. We conclude that $[J] \in \Sigma(m, l, \mathbf{u}) \cap \Sigma(m, l', \mathbf{v})$. \square

Corollary 1.12. *Let $l, l' \in [n - 1]$ and $\mathbf{u}, \mathbf{v} \in \mathbb{Z}_{\geq 1}^n$ such that $|\mathbf{u}| = m + l - 1$ and $|\mathbf{v}| = m + l' - 1$ and $\mathbf{u} - \mathbf{v} \in \{0, 1, -1\}^n \setminus \{\mathbf{1}, -\mathbf{1}\}$. Then*

$$\Sigma(m, l, \mathbf{u}) \cap \Sigma(m, l', \mathbf{v}) \cong \text{Gr}(l - |\kappa(\mathbf{u} - \mathbf{v}, 1)|, |\kappa(\mathbf{u} - \mathbf{v}, 0)|),$$

where the intersection is taken with the reduced structure.

Proof. Let $U = \langle x_i^{u_i} : i \in \kappa(\mathbf{u} - \mathbf{v}, 0) \rangle_{\mathbb{C}}$. We get a closed embedding of $\text{Gr}(l - |\kappa(\mathbf{u} - \mathbf{v}, 1)|, U)$ into $\text{Gr}(l, \Lambda_{\mathbf{u}})$ by sending Γ to $\Gamma + \langle x_i^{u_i} : i \in \kappa(\mathbf{u} - \mathbf{v}, 1) \rangle_{\mathbb{C}}$. The composition of this closed embedding with $\varphi_{\mathbf{u}, l}$ gives the isomorphism. \square

Having described the varieties $\Sigma(m, l, \mathbf{u})$, we can finally identify which of these varieties are the irreducible components of $\text{Hilb}_0^m(X_n)$.

Theorem 1.13. *The irreducible components of $\text{Hilb}_0^m(X_n)$ are such $\Sigma(m, l, \mathbf{u})$ for which $\max\{1, n + 1 - m\} \leq l \leq n - 1$ and $\mathbf{u} \in \mathbb{Z}_{\geq 1}^n$ with $|\mathbf{u}| = m + l - 1$.*

Proof. Theorem 1.6 decomposes $\text{Hilb}_0^m(X_n)$ as the union of closed subvarieties, each of them irreducible, hence it suffices to show which $\Sigma(m, l, \mathbf{u})$ are irreducible components of $\text{Hilb}_0^m(X_n)$.

We will first show that $\Sigma(m, n, \mathbf{u})$ is not an irreducible component. If $[J] \in \Sigma(m, n, \mathbf{u})$ is generic, then it is minimally generated by f_1, \dots, f_n where f_1, \dots, f_n are linearly independent elements of $\langle x_1^{u_1}, \dots, x_n^{u_n} \rangle_{\mathbb{C}}$. Since the latter is an n -dimensional vector space we deduce that $J = \langle x_1^{u_1}, \dots, x_n^{u_n} \rangle$. Since $|\mathbf{u}| = m + n - 1$ and $m \geq 2$, there exists $1 \leq i \leq n$ such that $u_i \geq 2$. We claim that $[J] \in \Sigma(m, n - 1, \mathbf{u} - \mathbf{e}_i)$. Indeed, the family of length m ideals

$$J_{\lambda} := \langle x_k^{u_k} + \lambda x_i^{u_i-1} : k \in [n] \setminus \{i\} \rangle$$

with $\lambda \in \mathbb{C}^*$ satisfies $[J_{\lambda}] \in \Sigma(m, n - 1, \mathbf{u} - \mathbf{e}_i)$ and for $\lambda = 0$ this family extends uniquely to $J_0 = J$. We get that $\Sigma(m, n, \mathbf{u})$ is contained in $\Sigma(m, n - 1, \mathbf{u} - \mathbf{e}_i)$ and hence, it is not an irreducible component of $\text{Hilb}_0^m(X_n)$.

Now assume that $\Sigma(m, l, \mathbf{u})$ is not an irreducible component of $\text{Hilb}_0^m(X_n)$ for $1 \leq l \leq n - 1$. Then, there exists \mathbf{v} and l' such that $\Sigma(m, l', \mathbf{v})$ is an irreducible component and $\Sigma(m, l, \mathbf{u})$ is contained in $\Sigma(m, l', \mathbf{v})$. By Theorem 1.11, $\mathbf{u} - \mathbf{v} \in \{0, 1, -1\}^n \setminus \{\mathbf{1}, -\mathbf{1}\}$. Moreover, by Theorem 1.10 and Theorem 1.12 we get that

$$\text{Gr}(l, n) \simeq \Sigma(m, l, \mathbf{u}) = \Sigma(m, l, \mathbf{u}) \cap \Sigma(m, l', \mathbf{v}) \simeq \text{Gr}(l - |\kappa(\mathbf{u} - \mathbf{v}, 1)|, |\kappa(\mathbf{u} - \mathbf{v}, 0)|).$$

We deduce that $|\kappa(\mathbf{u} - \mathbf{v}, 0)| = n$. Hence, $\mathbf{u} = \mathbf{v}$ and $l = l'$ which is a contradiction. Therefore, $\Sigma(m, l, \mathbf{u})$ is an irreducible component of $\text{Hilb}_0^m(X_n)$. \square

Corollary 1.14. *The number of irreducible components of $\mathcal{Hilb}_0^m(X_n)$ is*

$$\begin{cases} \sum_{l=n-m+1}^{n-1} \binom{l+m-2}{n-1} = \frac{(m-1)(m+n-2)}{n} \binom{m+n-2}{n-1} & \text{if } n \geq m, \\ \sum_{l=1}^{n-1} \binom{m+l-2}{n-1} = \frac{(m-1)(m+n-2)}{n} \binom{m+n-2}{n-1} + \frac{(n-m)(m+n-2)}{n} \binom{m+n-2}{-1+n} & \text{if } n \leq m. \end{cases}$$

Proof. Since the number of irreducible components is topological we may work in $\text{Hilb}_0^m(X_n)$. By Theorem 1.13, the quantity on the left-hand side is determined, while the expression on the right-hand side follows from classical combinatorial identities. \square

From the above reasoning we deduce that $\text{Hilb}_0^m(X_n)$ is the union of some Grassmannians glued together through closed subvarieties. Theorem 1.15 shows how this gluing is done for $n = 2$.

Example 1.15. For $n = 2$, the only possible value of l is 1, and $\mathbf{u} = (u_1, u_2)$ is a strictly positive partition of m . Hence

$$\text{Hilb}_0^m(X_2) = \bigcup_{i=1}^{m-1} \Sigma(m, 1, (i, m-i))$$

has $m-1$ irreducible components. A generic point in $\Sigma(m, 1, (i, m-i))$ corresponds to an ideal of the form $\langle \lambda_1 x_1^i + \lambda_2 x_2^{m-i} \rangle$ for $\lambda_1, \lambda_2 \neq 0$. One can check that $\Sigma(m, 1, (i, m-i))$ is isomorphic to \mathbb{P}^1 , whose torus is identified with the ideals of the above form, and the torus invariant points are the ideals $\langle x_1^{i+1}, x_2^{m-i} \rangle$ and $\langle x_1^i, x_2^{m-i+1} \rangle$. Moreover, the ideal $\langle x_1^{i+1}, x_2^{m-i} \rangle$ is the intersection of $\Sigma(m, 1, (i, m-i))$ and $\Sigma(m, 1, (i+1, m-i_1))$. With further work, it can be shown that $\text{Hilb}_0^m(X_2)$ is a chain of rational curves with nodal singularities obtained by gluing consecutively $\Sigma(m, 1, (i, m-i))$ and $\Sigma(m, 1, (i+1, m-i_1))$ through the point associated to the ideal $\langle x_1^{i+1}, x_2^{m-i} \rangle$ (see Fig. 3). This is precisely [Ran05, Theorem 1].



FIGURE 3. The m rational irreducible components of $\text{Hilb}_0^m(X_2)$.

2. THE MOMENT MAP AND ITS COMBINATORICS

We keep the same notation as in Section 1. Our next aim is to describe how $\text{Hilb}_0^m(X_n)$ is obtained by gluing the Grassmannians $\Sigma(m, l, \mathbf{u})$ using the combinatorial framework developed in Section A. To encode the combinatorics of these Grassmannians, we make use of the moment map associated with the natural action of the algebraic torus on X_n , cf. e.g. [Aud04, Kir84] for more details about moment maps in symplectic and algebraic geometry. We consider the Plücker coordinates q_A for A in $\binom{[n]}{l}$. For $l < n$ the moment map $\mu_{l,n}$ is:

$$(10) \quad \begin{aligned} \mu_{l,n} : \quad \text{Gr}(l, n) &\longrightarrow \mathbb{R}^n \\ (q_A)_{A \in \binom{[n]}{l}} &\longmapsto \frac{1}{\sum_A |q_A|^2} \left(\sum_A |q_A|^2 \mathbf{e}_A \right). \end{aligned}$$

Note that the moment map is not algebraic and it is defined over the \mathbb{C} -points of $\text{Gr}(l, n)$. The image of the moment map is the hypersimplex $\Delta_{l,n}$, which lies in $l \cdot \Delta_{n-1}$. Here, $l \cdot \Delta_{n-1} \subset \mathbb{R}^n$ is the dilation by l of the $(n-1)$ -dimensional simplex. The definition of $\Delta_{l,n}$ and the description of its faces can be found in Section A. The vertices of $\Delta_{l,n}$ are exactly the vectors $\mathbf{e}_{i_1} + \dots + \mathbf{e}_{i_l}$ in \mathbb{R}^n for $1 \leq i_1 < \dots < i_l \leq n$, which correspond via $\mu_{l,n}$ to the point $[\langle \mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_l} \rangle_{\mathbb{C}}]$

in $\text{Gr}(l, n)$. More generally, a face $\Delta_{l,n}(S_1, S_2)$ for $S_1 \sqcup S_2 \subseteq [n]$ (see Eq. (49)) is isomorphic to the hypersimplex $\Delta_{l-|S_2|, n-r+1}$, and via the moment map, it corresponds to the variety

$$\{[E] \in \text{Gr}(l, n) : \langle \mathbf{e}_i : i \in S_2 \rangle_{\mathbb{C}} \subseteq E \subseteq \langle \mathbf{e}_i : i \notin S_1 \rangle_{\mathbb{C}}\},$$

which is isomorphic to $\text{Gr}(l - |S_2|, n - r + 1)$. The goal of this section is to provide a complete combinatorial description of $\text{Hilb}_0^m(X_n)$ by constructing a moment map on it. To do so, we first construct a moment map on each irreducible component. On $\Sigma(m, l, \mathbf{u}) \simeq \text{Gr}(l, \Lambda_{\mathbf{u}}) \simeq \text{Gr}(n - l, \Lambda_{\mathbf{u}}^*)$, define the following moment map:

$$(11) \quad \mu_{\mathbf{u}, l} : \Sigma(m, l, \mathbf{u}) \simeq \text{Gr}(n - l, \Lambda_{\mathbf{u}}^*) \longrightarrow \Delta_{n-l, n} + \mathbf{u} - \mathbf{1}$$

sending a point Γ to $\mu_{n-l, n}(\Gamma^*) + \mathbf{u} - \mathbf{1}$. In other words, the map $\mu_{\mathbf{u}, l}$ is the composition of the isomorphism $\Sigma(m, l, \mathbf{u}) \simeq \text{Gr}(n - l, n)$ with the moment map $\mu_{n-l, n}$ and the translation by the vector $\mathbf{u} - \mathbf{1}$. In the Plücker coordinates of $\text{Gr}(l, \Lambda_{\mathbf{u}})$, the moment map $\mu_{\mathbf{u}, l}$ is given by

$$(12) \quad \mu_{\mathbf{u}, l}((q_A)_{A \in \binom{[n]}{l}}) = \sum_I \frac{1}{|q_A|^2} \left(\sum_A |q_A|^2 \mathbf{e}_{[n] \setminus I} \right) + \mathbf{u} - \mathbf{1}.$$

Theorem 2.1. *For distinct pairs (\mathbf{u}, l) and (\mathbf{v}, l') , the moment maps $\mu_{\mathbf{u}, l}$ and $\mu_{\mathbf{v}, l'}$ coincide in the intersection of $\Sigma(m, l, \mathbf{u})$ and $\Sigma(m, l', \mathbf{v})$. Therefore, there is a well-defined moment map*

$$(13) \quad \mu_m : \text{Hilb}_0^m(X_n) \longrightarrow (m - 1)\Delta_{n-1}$$

whose restriction to each irreducible component $\Sigma(m, l, \mathbf{u})$ is $\mu_{\mathbf{u}, l}$.

Proof. Let $[I] \in \Sigma(m, l, \mathbf{u}) \cap \Sigma(m, l', \mathbf{v})$. By Theorem 1.11, $\mathbf{u} - \mathbf{v} \in \{0, 1, -1\}^n$ and

$$I = \langle f_1, \dots, f_r \rangle + \langle x_i^{u_i} : i \in \kappa(\mathbf{u} - \mathbf{v}, 1) \rangle + \langle x_i^{u_i+1} : i \in \kappa(\mathbf{u} - \mathbf{v}, -1) \rangle,$$

with $f_1, \dots, f_r \in \langle x_i^{u_i} : i \in \kappa(\mathbf{u} - \mathbf{v}, 0) \rangle_{\mathbb{C}}$ linearly independent and $r = l - |\kappa(\mathbf{u} - \mathbf{v}, 1)|$. As an element of $\text{Gr}(l, \Lambda_{\mathbf{u}})$, $[I]$ corresponds to the linear subspace

$$\langle f_1, \dots, f_r \rangle_{\mathbb{C}} + \langle x_i^{u_i} : i \in \kappa(\mathbf{u} - \mathbf{v}, 1) \rangle_{\mathbb{C}}.$$

We will compute explicitly the moment map in coordinates. Let M be the $l \times n$ matrix whose rows consists on the coefficients of $x_1^{u_1}, \dots, x_n^{u_n}$ in f_1, \dots, f_r and $x_i^{u_i}$ for $i \in \kappa(\mathbf{u} - \mathbf{v}, 1)$. In other words, M is a matrix of the form

$$M = \left(\begin{array}{c|c|c} \text{Id}_{|\kappa(\mathbf{u}-\mathbf{v}, 1)|} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & N & \mathbf{0} \end{array} \right),$$

$\underbrace{\hspace{10em}}_{\kappa(\mathbf{u}-\mathbf{v}, 1)} \quad \underbrace{\hspace{10em}}_{\kappa(\mathbf{u}-\mathbf{v}, 0)} \quad \underbrace{\hspace{10em}}_{\kappa(\mathbf{u}-\mathbf{v}, -1)}$

where N is the $r \times |\kappa(\mathbf{u} - \mathbf{v}, 0)|$ matrix

$$N = \begin{pmatrix} \cdots & f_1 & \cdots \\ & \vdots & \\ \cdots & f_r & \cdots \end{pmatrix}$$

whose rows are the coefficients of $x_i^{u_i}$ for $i \in \kappa(\mathbf{u} - \mathbf{v}, 0)$ in f_1, \dots, f_r . Thus, the Plücker coordinate q_A of $[I]$, where $A \in \binom{[n]}{l}$, corresponds to the $l \times A$ minor of M . We denote this minor by $\det(M, A)$. Note that if $A \cap \kappa(\mathbf{u} - \mathbf{v}, -1) \neq \emptyset$, the submatrix of M given by the columns in A has a vanishing column. Thus, we have that $q_A = 0$ if $A \cap \kappa(\mathbf{u} - \mathbf{v}, -1) \neq \emptyset$. Similarly, if $\kappa(\mathbf{u} - \mathbf{v}, 1)$ is not contained in A , the submatrix of M given by the columns in A has a vanishing row. Hence we get $q_A = 0$ if $\kappa(\mathbf{u} - \mathbf{v}, 1) \not\subseteq A$. Thus, the only nonvanishing Plücker coordinates of $[I]$ in $\text{Gr}(l, \Lambda_{\mathbf{u}})$ are such q_A with

$A = \kappa(\mathbf{u} - \mathbf{v}, 1) \cup B$ for some $B \in (\kappa(\mathbf{u} - \mathbf{v}, 0))$. In this case, q_A is the minor $r \times B$ of N , i.e. $q_A = \det(N, B)$. Using (11), we get that

$$\begin{aligned} \mu_{\mathbf{u}, l}([I]) &= \frac{1}{\sum_{B \in (\kappa(\mathbf{u} - \mathbf{v}, 0))} |\det(N, B)|^2} \left(\sum_{B \in (\kappa(\mathbf{u} - \mathbf{v}, 0))} |\det(N, B)|^2 \mathbf{e}_{[n] \setminus (B \cup \kappa(\mathbf{u} - \mathbf{v}, 1))} \right) + \mathbf{u} - \mathbf{1} \\ &= \frac{1}{\sum_B |\det(N, B)|^2} \left(\sum_B |\det(N, B)|^2 \mathbf{e}_{(\kappa(\mathbf{u} - \mathbf{v}, 0) \setminus B) \cup \kappa(\mathbf{u} - \mathbf{v}, -1)} \right) + \mathbf{u} - \mathbf{1} \\ &= \frac{1}{\sum_B |\det(N, B)|^2} \left(\sum_B |\det(N, B)|^2 (\mathbf{e}_{\kappa(\mathbf{u} - \mathbf{v}, 0) \cup \kappa(\mathbf{u} - \mathbf{v}, -1)} - \mathbf{e}_B) \right) + \mathbf{u} - \mathbf{1} \\ &= \mathbf{e}_{\kappa(\mathbf{u} - \mathbf{v}, 0) \cup \kappa(\mathbf{u} - \mathbf{v}, -1)} - \frac{1}{\sum_B |\det(N, B)|^2} \left(\sum_B |\det(N, B)|^2 \mathbf{e}_B \right) + \mathbf{u} - \mathbf{1} \\ &= \mathbf{e}_{\kappa(\mathbf{u} - \mathbf{v}, 0)} + \mathbf{e}_{\kappa(\mathbf{u} - \mathbf{v}, -1)} - \frac{1}{\sum_B |\det(N, B)|^2} \left(\sum_B |\det(N, B)|^2 \mathbf{e}_B \right) + \mathbf{u} - \mathbf{1}. \end{aligned}$$

A similar computation shows that

$$\mu_{\mathbf{v}, l'}([I]) = \mathbf{e}_{\kappa(\mathbf{u} - \mathbf{v}, 0)} + \mathbf{e}_{\kappa(\mathbf{u} - \mathbf{v}, 1)} - \frac{1}{\sum_B |\det(N, B)|^2} \left(\sum_B |\det(N, B)|^2 \mathbf{e}_B \right) + \mathbf{v} - \mathbf{1}.$$

The proof follows from the fact that $\mathbf{u} - \mathbf{v} = \mathbf{e}_{\kappa(\mathbf{u} - \mathbf{v}, 1)} - \mathbf{e}_{\kappa(\mathbf{u} - \mathbf{v}, -1)}$. \square

Example 2.2. Continuing Theorem 1.15, for $n = 2$, the $m - 1$ irreducible components of $\text{Hilb}_0^m(X_2)$ are $\Sigma(m, 1, (1, m - 1)), \dots, \Sigma(m, 1, (m - 1))$. Each of them is isomorphic to \mathbb{P}^1 . The moment map $\mu_{(i, m-i), 1}$ is defined as

$$\begin{aligned} \mu_{(i, m-i), 1} : \Sigma(m, 1, (i, m-i)) \simeq \mathbb{P}^1 &\longrightarrow (\mathbb{P}^1)^* \longrightarrow (m-1) \cdot \Delta_1 \\ [a_0, a_1] &\longmapsto [a_1, a_0] \longmapsto \left(\frac{|a_1|^2}{|a_0|^2 + |a_1|^2} + i - 1, \frac{|a_0|^2}{|a_0|^2 + |a_1|^2} + m - i - 1 \right). \end{aligned}$$

Using the above formula a direct computation yields

$$\begin{aligned} \mu_{(i, m-i), 1}(\langle x_1^{i+1}, x_2^{m-i} \rangle) &= \mu_{(i, m-i), 1}([0, 1]) = (i, m-i-1) = \mu_{(i+1, m-i-1), 1}([1, 0]) \\ &= \mu_{(i+1, m-i-1), 1}(\langle x_1^{i+1}, x_2^{m-i} \rangle). \end{aligned}$$

In particular, $\mu_{(i, m-i), 1}$ and $\mu_{(i+1, m-i-1), 1}$ coincide in the intersection of $\Sigma(m, 1, (i, m-i))$ and $\Sigma(m, 1, (i+1, m-i))$. The image of $\mu_{(i, m-i), 1}$ is the segment between $(i, m-i-1)$ and $(i-1, m-i)$. These segments form a subdivision of $(m-1) \cdot \Delta_1$, which is the image of the moment map μ_m . In Fig. 4, the image of μ_2, μ_3 and μ_4 is depicted.

The image of the moment map μ is the union of all hypersimplices $\Delta_{n-l, n} + \mathbf{u} - \mathbf{1}$ for $\max\{1, n+1-m\} \leq l \leq n-1$ and $\mathbf{u} \in \mathbb{Z}_{\geq 1}^n$ with $\mathbf{u} = m + l - 1$.

Definition 2.3. The hypersimplices $\Delta_{n-l, n} + \mathbf{u} - \mathbf{1}$ for $\max\{1, n+1-m\} \leq l \leq n-1$ and $\mathbf{u} \in \mathbb{Z}_{\geq 1}^n$ with $\mathbf{u} = m + l - 1$ form a hypersimplicial complex called the (n, m) -hypersimplicial complex. We denote this hypersimplicial complex by $\mathcal{K}_n^{[m]}$.

By Theorem A.2, the (n, m) -hypersimplicial complex $\mathcal{K}_n^{[m]}$ is indeed a hypersimplicial complex, and it subdivides $(m-1) \cdot \Delta_{n-1}$. Further properties of this complex are discussed in Section A. By the proof of Theorem 2.1, we have that

$$\mu(\Sigma(m, l, \mathbf{u}) \cap \Sigma(m, l', \mathbf{v}')) = (\Delta_{n-l, n} + \mathbf{u} - \mathbf{1}) \cap (\Delta_{n-l', n} + \mathbf{v} - \mathbf{1}).$$

Therefore, the faces of $\mathcal{K}_n^{[m]}$ encode the intersection of the distinct Grassmannian components of $\text{Hilb}_0^m(X_n)$. Fig. 4 depicts this hypersimplicial complex for the case $n = 2$.

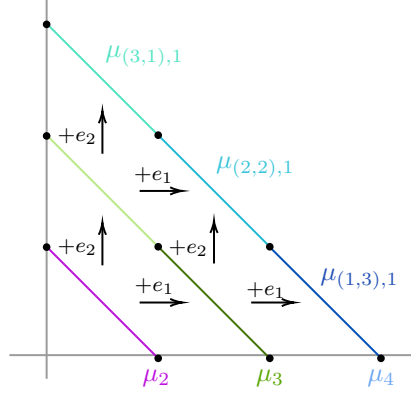


FIGURE 4. Decomposition of $(m-1) \cdot \Delta_1$ according to the moment map μ_m for $m = 2, 3, 4$ and $n = 2$. It corresponds to the hypersimplicial complexes $\mathcal{K}_2^{[2]}$, $\mathcal{K}_2^{[3]}$, and $\mathcal{K}_2^{[4]}$.

Example 2.4. For $n = 3$, the parameter l can be either be 1 or 2. In the case $l = 2$, we get that $\mathbf{u} = (u_1, u_2, u_3)$ is a partition of $m+1$. This leads to components of the form $\Sigma(m, 2, \mathbf{u}) \simeq \text{Gr}(2, 3) = (\mathbb{P}^2)^*$. In Plücker coordinates the moment map $\mu_{\mathbf{u},2}$ of these components is

$$\begin{aligned} \mu_{\mathbf{u},2} : \Sigma(m, 2, \mathbf{u}) \simeq (\mathbb{P}^2)^* &\longrightarrow (m-1)\Delta_{1,3} + \mathbf{u} - \mathbf{1} \\ [a_{23}, a_{13}, a_{12}] &\longrightarrow \left(\frac{|a_{23}|^2}{|a_{23}|^2 + |a_{13}|^2 + |a_{12}|^2}, \frac{|a_{13}|^2}{|a_{23}|^2 + |a_{13}|^2 + |a_{12}|^2}, \frac{|a_{12}|^2}{|a_{23}|^2 + |a_{13}|^2 + |a_{12}|^2} \right) + \mathbf{u} - \mathbf{1} \end{aligned}$$

The image of $\mu_{\mathbf{u},2}$ is the triangle defined by the vertices $(u_1, u_2 - 1, u_3 - 1)$, $(u_1 - 1, u_2, u_3 - 1)$ and $(u_1 - 1, u_2 - 1, u_3)$. These triangles are illustrated in blue in Fig. 5.

For $l = 1$, \mathbf{u} is a partition of m and we get components of the form $\Sigma(m, 1, \mathbf{u}) \simeq \text{Gr}(1, 3) = \mathbb{P}^2$. Since \mathbf{u} can not have zero entries, these type of components only appear for $m \geq 3$. In Plücker coordinates, the moment map $\mu_{\mathbf{u},1}$ of these components is

$$\begin{aligned} \mu_{\mathbf{u},1} : \Sigma(m, 1, \mathbf{u}) \simeq \mathbb{P}^2 &\longrightarrow (m-1)\Delta_{2,3} + \mathbf{u} - \mathbf{1} \\ [a_1, a_2, a_3] &\longrightarrow \left(\frac{|a_2|^2 + |a_3|^2}{|a_1|^2 + |a_2|^2 + |a_3|^2}, \frac{|a_1|^2 + |a_3|^2}{|a_1|^2 + |a_2|^2 + |a_3|^2}, \frac{|a_1|^2 + |a_2|^2}{|a_1|^2 + |a_2|^2 + |a_3|^2} \right) + \mathbf{u} - \mathbf{1}. \end{aligned}$$

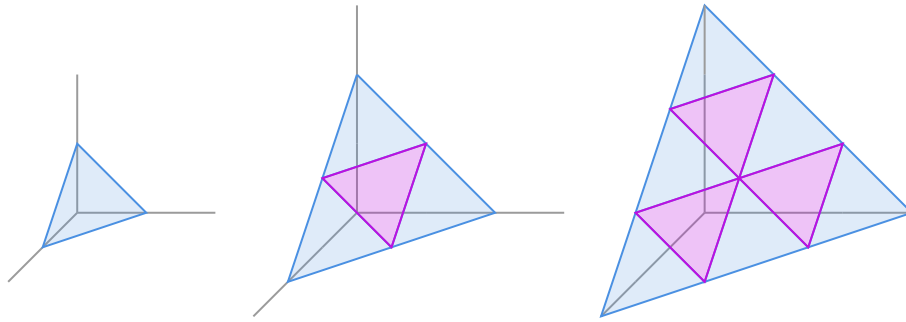


FIGURE 5. Hypersimplicial complex $\mathcal{K}_3^{[m]}$ subdividing $(m-1)\Delta_2$ for $m = 1, 2$ and 3 . The blue triangles correspond to the image of the maps $\mu_{\mathbf{u},2}$ whereas the purple triangles correspond to the image of the maps $\mu_{\mathbf{u},1}$.

The image of $\mu_{\mathbf{u},1}$ is the triangle defined by the vertices $(u_1, u_2, u_3 - 1)$, $(u_1 - 1, u_2, u_3)$ and $(u_1, u_2 - 1, u_3)$. These triangles are illustrated in purple in Fig. 5. These 2-dimensional hypersimplices form the complex $\mathcal{K}_3^{[m]}$ that subdivides $(m-1) \cdot \Delta_2$. Fig. 5 depicts this complex for $m = 2, 3, 4$ and Fig. 6 the case $m = 5$. We deduce that $\text{Hilb}_0^m(X_3)$ has $\binom{m}{2}$ components of the form $\Sigma(m, 2, \mathbf{u})$ and $\binom{m-1}{2}$ components of the form $\Sigma(m, 1, \mathbf{u})$. All these components are toric and

they intersect each other in the closure of toric orbits. The complex $\mathcal{K}_3^{[m]}$ encodes the toric representation of $\text{Hilb}_0^m(X_3)$ and how its components intersect.

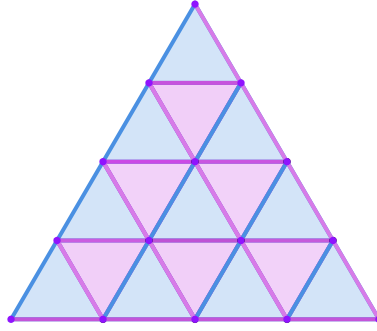


FIGURE 6. Hypersimplicial complex $\mathcal{K}_3^{[5]}$ encoding the toric representation of $\text{Hilb}_0^5(X_3)$ obtained by gluing together 16 copies of \mathbb{P}^2 .

Example 2.5. For $n = 4$, the possible values of l are $l = 1, 2, 3$. For $l = 1$, \mathbf{u} is a partition m and we get components of the form $\Sigma(m, 1, \mathbf{u}) \simeq \text{Gr}(1, 4) \simeq \mathbb{P}^3$. The image of $\mu_{\mathbf{u},1}$ is the translated hypersimplex $\Delta_{3,4} + \mathbf{u} - \mathbf{1}$. Note that since \mathbf{u} has no vanishing entries, these components only appear for $m \geq 4$. For $m = 4$, the only choice of \mathbf{u} is $\mathbf{1}$. In this case, the hypersimplex $\Delta_{3,4}$ is illustrated in purple in Fig. 8.

For $l = 2$, \mathbf{u} is a partition of $m + 1$ and we get components of the form $\Sigma(m, 2, \mathbf{u}) \simeq \text{Gr}(2, 4)$. The image of $\mu_{\mathbf{u},2}$ is the translated hypersimplex $\Delta_{2,4} + \mathbf{u} - \mathbf{1}$. These components appear for $m \geq 3$. For $m = 3$ we have that $\mathbf{u} = \mathbf{1}$. The hypersimplex $\Delta_{2,4}$ is depicted in purple in Fig. 7. For $m = 4$, the possible choices of \mathbf{u} are $\mathbf{1} + \mathbf{e}_i$ for $i \in [4]$. The hypersimplices $\Delta_{2,4} + \mathbf{e}_i$ are illustrated in green in Fig. 8.

Finally, for $l = 3$, \mathbf{u} is a partition of $m + 2$ and we get components of the form $\Sigma(m, 3, \mathbf{u}) \simeq \text{Gr}(3, 4) \simeq \mathbb{P}^3$. The image of $\mu_{\mathbf{u},3}$ is the translated hypersimplex $\Delta_{1,4} + \mathbf{u} - \mathbf{1}$. For $m = 2$, we have that $\mathbf{u} = \mathbf{1}$ and $\Delta_{1,4}$ is the usual simplex Δ_3 . For $m = 3$, $\mathbf{u} = \mathbf{1} + \mathbf{e}_i$ for $i \in [4]$. The hypersimplices $\Delta_{1,4} + \mathbf{e}_i$ are illustrated in blue in Fig. 7. For $m = 4$ the possible choices of \mathbf{u} are $\mathbf{u} = \mathbf{1} + \mathbf{e}_i + \mathbf{e}_j$ for $i, j \in [4]$. The 10 hypersimplices $\Delta_{1,4} + \mathbf{e}_i + \mathbf{e}_j$ for $i, j \in [4]$ are depicted in blue in Fig. 8.

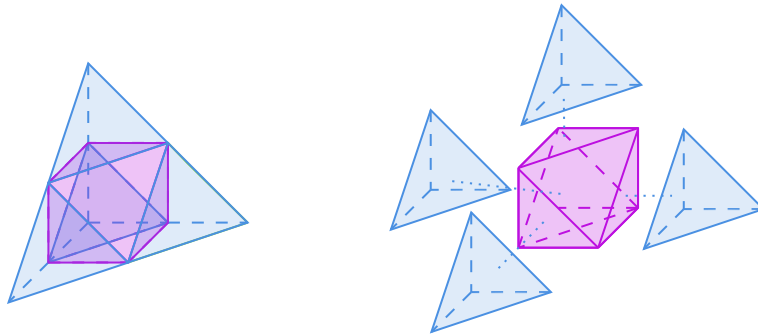


FIGURE 7. Hypersimplicial complex $\mathcal{K}_4^{[3]}$ encoding the intersection of the irreducible components of $\text{Hilb}_0^3(X_4)$.

For instance, for $m = 3$ there is no component of the form $\Sigma(3, 1, \mathbf{u})$, there is a component of the form $\Sigma(3, 2, \mathbf{1}) \simeq \text{Gr}(2, 4)$ and there are four components of the form $\Sigma(3, 3, \mathbf{u})$ for $\mathbf{u} = (2, 1, 1, 1), \dots, (1, 1, 1, 2)$. Fig. 7 illustrates the hypersimplicial complex $\mathcal{K}_4^{[3]}$ subdividing $2\Delta_3$. The maximal faces of the complex are the octahedron $\Delta_{2,4}$ in purple and the simplices $\Delta_3 + \mathbf{e}_1, \dots, \Delta_3 + \mathbf{e}_4$ in blue. As illustrated in Fig. 7, the gluing of the distinct components of $\text{Hilb}_0^3(X_4)$

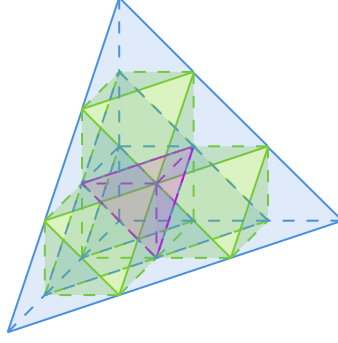


FIGURE 8. Hypersimplicial complex $\mathcal{K}_4^{[4]}$ encoding the intersection of the irreducible components of $\text{Hilb}_0^4(X_4)$.

is done through projective planes. In $\text{Gr}(2, 4)$, these projective planes correspond to the 2-dimensional linear subspaces in Equation (3). In the components isomorphic to \mathbb{P}^3 , the gluing is done through the torus-invariant projective planes.

For $m = 4$, $\text{Hilb}_0^4(X_4)$ has 15 irreducible components. One of them is $\Sigma(4, 1, 1) \simeq \mathbb{P}^3$. Four of them are of the form $\Sigma(4, 2, \mathbf{u}) \simeq \text{Gr}(2, 4)$ for $\mathbf{u} = (2, 1, 1, 1), \dots, (1, 1, 1, 2)$. Finally, there are 10 irreducible components of the form $\Sigma(4, 3, \mathbf{u}) \simeq \mathbb{P}^3$ where $\mathbf{u} = \mathbf{1} + \mathbf{e}_i + \mathbf{e}_j$ for $1 \leq i < j \leq 4$. The hypersimplicial complex $\mathcal{K}_4^{[4]}$ encoding the intersection of these components is illustrated in Fig. 8. The blue tetrahedron corresponds to the images of through the moment map of the components of the form $\Sigma(4, 3, \mathbf{1} + \mathbf{e}_i + \mathbf{e}_j)$. The four green octahedrons correspond to the images of the components $\Sigma(4, 2, \mathbf{1} + \mathbf{e}_i)$. Finally, the purple tetrahedron corresponds to the image of the component $\Sigma(4, 1, 1)$.

The faces of the hypersimplicial complex $\mathcal{K}_n^{[m]}$ are characterized in Section A. The $(n - r)$ -faces may be described as follows. Let $\max\{n - m + 1, 1\} \leq l \leq n - 1$, $\mathbf{u} \in \mathbb{Z}_{\geq 1}^n$ with $|\mathbf{u}| = m + l - 1$, and let S_1 and S_2 be two disjoint subsets of $[n]$ with $|S_1| + |S_2| = r - 1$ and $l \leq |S_1|$ and $|S_2| \leq n - l$. Then, the codimension r faces of $\mathcal{K}_n^{[m]}$ are of the form

$$(14) \quad \begin{aligned} \mathcal{K}_n^{[m]}(S_1, S_2, l, \mathbf{u}) &:= \text{Conv}(\mathbf{e}_{i_1} + \dots + \mathbf{e}_{i_{n-l-|S_2|}} : i_1, \dots, i_{n-l-|S_2|} \in [n] \setminus (S_1 \sqcup S_2) \text{ distinct}) + \sum_{i \in S_2} \mathbf{e}_i + \mathbf{u} - \mathbf{1} \\ &= \mathcal{K}_n^{[m]}(S_1, S_2, l, \mathbf{u}) = \left\{ \sum_{i \notin S_1 \sqcup S_2} \lambda_i \mathbf{e}_i + \sum_{i \in S_2} \mathbf{e}_i : 0 \leq \lambda_i \leq 1 \text{ and } \sum_{i \notin S_1 \sqcup S_2} \lambda_i = n - l - |S_2| \right\} + \mathbf{u} - \mathbf{1}. \end{aligned}$$

Note that the face $\mathcal{K}_n^{[m]}(S_1, S_2, l, \mathbf{u})$ is obtained by setting $\lambda_i = 0$ for $i \in S_1$ and $\lambda_i = 1$ for $i \in S_2$. The ideals $[J]$ in $\text{Hilb}_0^m(X_n)$ lying on the face $\mathcal{K}_n^{[m]}(S_1, S_2, l, \mathbf{u})$ are of the form

$$(15) \quad J = \langle x_i^{u_i} : i \in S_1 \rangle + \langle x_i^{u_i+1} : i \in S_2 \rangle + \langle g_1, \dots, g_{l-|S_1|} \rangle,$$

where $g_1, \dots, g_{l-|S_1|}$ are linearly independent polynomials in $\langle x_i^{u_i} : i \in [n] \setminus (S_1 \cup S_2) \rangle_{\mathbb{C}}$. Geometrically, these ideals form a Grassmannian $\text{Gr}(l - |S_1|, n - r + 1)$. Note that the ideals in (15) are of the same form as those described in Theorem 1.11. In particular, we have that

$$\begin{aligned} \mu(\Sigma(m, l, \mathbf{u}) \cap \Sigma(m, l', \mathbf{v})) &= (\Delta_{n-l, n} + \mathbf{u} - \mathbf{1}) \cap (\Delta_{n-l', n} + \mathbf{v} - \mathbf{1}) = \mathcal{K}_n^{[m]}(\kappa(\mathbf{u} - \mathbf{v}, 1), \kappa(\mathbf{u} - \mathbf{v}, -1), l, \mathbf{u}) \\ &= \mathcal{K}_n^{[m]}(\kappa(\mathbf{u} - \mathbf{v}, -1), \kappa(\mathbf{u} - \mathbf{v}, -1), l', \mathbf{v}). \end{aligned}$$

We conclude that the hypersimplicial complex $\mathcal{K}_n^{[m]}$ encodes the geometry of irreducible components of $\text{Hilb}_0^m(X_n)$: these components correspond to the Grassmannians associated with hypersimplices, and their intersections are likewise recorded. However, $\mathcal{K}_n^{[m]}$ cannot describe whether the intersection of these components is transversal or not. In Section 4, this question will be addressed. To do so, we associate to the hypersimplicial complex $\mathcal{K}_n^{[m]}$ the variety \mathcal{G}_n^m defined as follows. For $\max\{1, n - m + 1\} \leq l \leq n - 1$ and $\mathbf{u} \in \mathbb{Z}_{\geq 1}^n$ with $|\mathbf{u}| = m + l - 1$, we consider the Grassmannian $\text{Gr}(l, \Lambda_{\mathbf{u}})$ (see Eq. (2)). Recall that $\text{Gr}(l, \Lambda_{\mathbf{u}})$ is isomorphic to $\Sigma(m, l, \mathbf{u})$ via the map $\varphi_{l, \mathbf{u}}$ (see Eq. (2) and Theorem 1.10

). Using this map, we can consider the equivalence relation in the disjoint union

$$(16) \quad \bigsqcup_{l=\max\{1, n-m+1\}}^{n-1} \bigsqcup_{\substack{\mathbf{u} \in \mathbb{Z}_{\geq 1}^n \\ |\mathbf{u}| = m+l-1}} \text{Gr}(l, \Lambda_{\mathbf{u}})$$

given by $[E] \sim [E']$ for $[E] \in \text{Gr}(l, \Lambda_{\mathbf{u}})$ and $[E'] \in \text{Gr}(l', \Lambda_{\mathbf{v}})$ if and only if $\varphi_{l, \mathbf{u}}([E]) = \varphi_{l', \mathbf{v}}([E'])$. This condition occurs only in the intersection of $\Sigma(m, l, \mathbf{u})$ and $\Sigma(m, l', \mathbf{v})$, which by Theorem 1.12 is a Grassmannian. The variety \mathcal{G}_n^m is the variety obtained by quotienting (16) by this equivalence relation. In other words, \mathcal{G}_n^m is obtained by gluing the Grassmannians $\text{Gr}(l, \Lambda_{\mathbf{u}})$ via smaller Grassmannians. Since the intersections of the Grassmannians $\Sigma(m, l, \mathbf{u})$ are described by $\mathcal{K}_n^{[m]}$, we obtain that \mathcal{G}_n^m is obtained by gluing the Grassmannians $\text{Gr}(l, \Lambda_{\mathbf{u}})$ via the smaller Grassmannians corresponding to the faces of $\mathcal{K}_n^{[m]}$. Formally, this gluing may be done iteratively. In (16) we first glue together the points that corresponds to the same vertex in $\mathcal{K}_n^{[m]}$. Then, we glue the lines that correspond to the same edges of $\mathcal{K}_n^{[m]}$. Inductively on the dimension of the faces, we glue together the subgrassmannians that corresponds to the same face of the complex.

Example 2.6. Fix $n = 3$. For $m = 2$, the hypersimplicial complex $\mathcal{K}_3^{[2]}$ coincides with $\Delta_{1,3}$, and hence, $\mathcal{G}_3^2 \simeq \mathbb{P}^2$. For $m = 3$, $\mathcal{K}_3^{[3]}$ has four 2-dimensional hypersimplices as shown in Theorem 2.4. These hypersimplices are $\Delta_{2,3}$ and $\Delta_{1,3} + \mathbf{e}_i$ for $i \in [3]$, which are depicted in Fig. 5. The variety \mathcal{G}_3^3 is obtained by gluing 4 copies of \mathbb{P}^2 following the intersections of the corresponding hypersimplices in $\mathcal{K}_3^{[3]}$: at each of the three $(\mathbb{C}^*)^3$ -invariant lines of \mathbb{P}^2 we glue a copy of \mathbb{P}^2 through one of its invariant lines. Similarly, for $m = 4$, \mathcal{G}_3^4 is obtained by gluing 9 copies of \mathbb{P}^2 through torus invariant lines following the hypersimplicial complex $\mathcal{K}_3^{[4]}$ depicted in Fig. 5.

Example 2.7. Fix $n = 4$. For $m = 2$, the hypersimplicial complex $\mathcal{K}_4^{[2]}$ coincides with $\Delta_{1,4}$, and hence, $\mathcal{G}_4^2 \simeq \mathbb{P}^3$. For $m = 3$, $\mathcal{K}_4^{[3]}$ has five 3 dimensional hypersimplices as shown in Theorem 2.7. These hypersimplices are $\Delta_{2,4}$ and $\Delta_{1,4} + \mathbf{e}_i$ for $i \in [4]$, which are depicted in Fig. 7. The variety \mathcal{G}_4^3 is obtained by gluing 4 copies of \mathbb{P}^2 to $\text{Gr}(2, \langle x_1, x_2, x_3, x_4 \rangle_{\mathbb{C}}) \simeq \text{Gr}(2, 4)$ following the intersections of the corresponding hypersimplices in $\mathcal{K}_4^{[3]}$ (see Fig. 7). The hypersimplex $\Delta_{2,4}$ has eight 2-dimensional faces isomorphic to the simplex Δ_2 . Four of these faces coincide with a faces of each of the hypersimplices $\Delta_{1,4} + \mathbf{e}_i$ for $i \in [4]$. Geometrically, the associated Grassmannian $\text{Gr}(2, 4)$ has eight \mathbb{P}^2 embedded which are of the form:

$$Y_i := \{[E] \in \text{Gr}(2, 4) : E \subseteq \langle x_j : j \neq i \rangle_{\mathbb{C}}\} \quad \text{and} \quad Y_i^* := \{[E] \in \text{Gr}(2, 4) : \langle x_i \rangle_{\mathbb{C}} \subseteq E\},$$

for $i \in [4]$. The ideals associated to Y_i are of the form $\langle x_i^2, f_1, f_2 \rangle$ for $f_1, f_2 \in \langle x_j : j \neq i \rangle_{\mathbb{C}}$, and the ideals associated to Y_i^* are of the form $\langle x_i, f \rangle$ for $f \in \langle x_j : j \neq i \rangle_{\mathbb{C}}$. The varieties Y_i correspond to the four faces $\Delta_{2,4}(\emptyset, \{i\}, 2, 1)$ of $\Delta_{2,4}$ that intersect with the hypersimplices $\Delta_{1,4} + \mathbf{e}_i$. For each $i \in [4]$ we glue $\text{Gr}(2, 4)$ and \mathbb{P}^3 through Y_i and a toric invariant plane of \mathbb{P}^3 . The variety obtained by this glue is \mathcal{G}_4^3 .

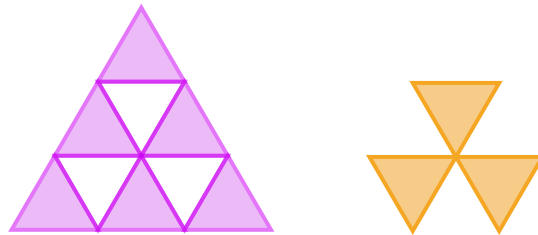


FIGURE 9. Hypersimplicial complex $\mathcal{K}_{3,1}^{[4]}$ and $\mathcal{K}_{3,2}^{[4]}$.

For $\max\{1, n - m + 1\} \leq l \leq n - 1$, we consider the subvariety $\mathcal{G}_{l,n}^m$ of \mathcal{G}_n^m given only by the components of \mathcal{G}_n^m that are Grassmannians of the form $\text{Gr}(l, n)$. Analogously, we consider the hypersimplicial complex $\mathcal{K}_{n-l,n}^{[m]}$, which is the hypersimplicial subcomplex of $\mathcal{K}_n^{[m]}$ given by the hypersimplices of the form $\Delta_{n-l,n} + \mathbf{u}$. We refer to Section A for

further details on $\mathcal{K}_{n-l,n}^{[m]}$. The variety $\mathcal{G}_{l,n}^m$ is the variety obtained by gluing the Grassmannian $\Sigma(m, l, \mathbf{u})$ via the faces of $\mathcal{K}_{n-l,n}^{[m]}$.

Example 2.8. The variety $\mathcal{G}_{2,3}^4$ consists of 6 copies of \mathbb{P}^2 that are glued together via torus invariant points as in the vertices of the hypersimplicial complex $\mathcal{K}_{1,3}^{[4]}$ illustrated in Fig. 9. Similarly, $\mathcal{G}_{1,3}^4$ is obtained by taking 3 copies of \mathbb{P}^2 and gluing together a torus invariant point on each of them. The corresponding hypersimplicial complex is $\mathcal{K}_{2,3}^{[4]}$ illustrated in Fig. 9.

Example 2.9. Fig. 10 depicts the hypersimplicial complexes $\mathcal{K}_{1,4}^{[3]}$ and $\mathcal{K}_{2,4}^{[4]}$. These two complexes represent the varieties $\mathcal{G}_{1,3}^3$ and $\mathcal{G}_{2,4}^4$. The variety $\mathcal{G}_{1,3}^3$ consists of 4 copies of \mathbb{P}^3 glued together via invariant torus points. Similarly, $\mathcal{G}_{2,4}^4$ consists of 4 copies of $\text{Gr}(2, 4)$ that are glued together via torus invariant lines.

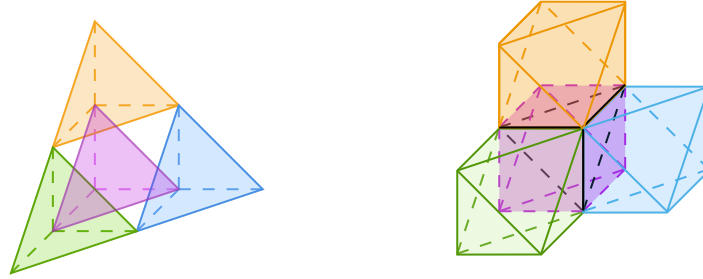


FIGURE 10. Hypersimplicial complexes $\mathcal{K}_{1,4}^{[3]}$ and $\mathcal{K}_{2,4}^{[4]}$.

Theorem A.8 described the intersection of $\mathcal{K}_n^{[m]}$ with a linear subspace of the form

$$(17) \quad H(S, \mathbf{a}) := \{\lambda_i = a_i : i \in S\} \text{ for } S \subseteq [n] \text{ and } \mathbf{a} \in \mathbb{Z}_{\geq 0}^n \text{ with } |\mathbf{a}| \leq m-1.$$

Such intersection is isomorphic to the hypersimplicial complex $\mathcal{K}_{n-|S|}^{[m-|\mathbf{a}|]}$. On the levels of ideals, $\mu([J])$ for $[J] \in \text{Hilb}_0^m(X_n)$ is contained in $H(S, \mathbf{a})$ if and only if $x_i^{a_i+1}$ for $i \in S$ among its minimal generators. Therefore, the intersection of $\text{Hilb}_0^m(X_n)$ with this condition is isomorphic to $\text{Hilb}_0^{m-\sum a_i}(X_{n-|S|})$. This intersection coincides with the image of the map

$$(18) \quad \begin{array}{ccc} \iota_{S,\mathbf{a}} \text{ Hilb}_0^{m-|\mathbf{a}|}(X_{n-|S|}) & \longrightarrow & \text{Hilb}_0^m(X_n) \\ [J] & \longmapsto & [J + \langle x_i^{a_i+1} : i \in S \rangle] \end{array}$$

which is an isomorphism onto its image. Moreover, such a map fits in the following commutative diagram

$$(19) \quad \begin{array}{ccc} \text{Hilb}_0^{m-|\mathbf{a}|}(X_{n-|S|}) & \longrightarrow & \text{Hilb}_0^m(X_n) \\ \downarrow & & \downarrow \\ (m-|\mathbf{a}|-1)\Delta_{n-|S|-1} & \longrightarrow & (m-1)\Delta_{n-1} \end{array},$$

where the vertical maps are the moment map and the map below is the translation by $\sum_{i \in S} a_i \mathbf{e}_i$.

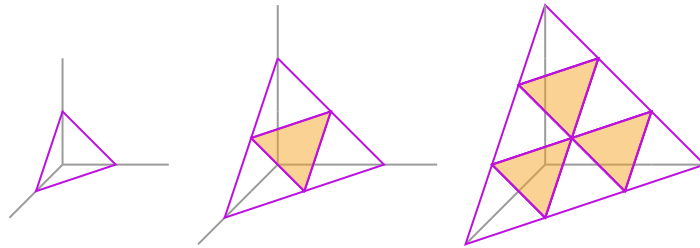
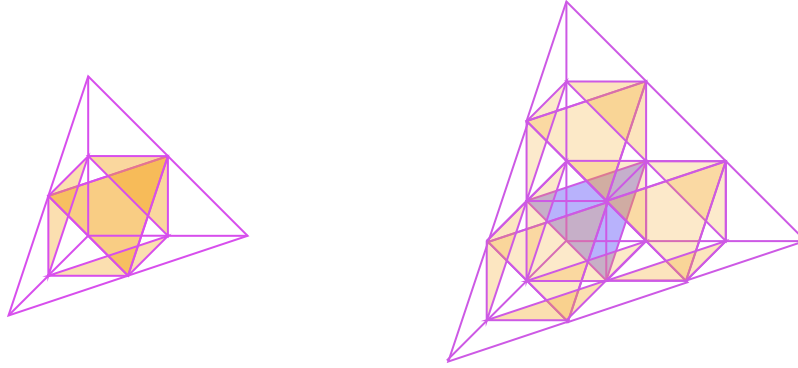


FIGURE 11. Smoothable faces of $\mathcal{K}_3^{[2]}$, $\mathcal{K}_3^{[3]}$ and $\mathcal{K}_3^{[4]}$

FIGURE 12. Smoothable faces of the complexes $\mathcal{K}_4^{[3]}$ and $\mathcal{K}_4^{[4]}$.

Now, we describe the ideals in $\text{Hilb}_0^m(X_n)$ that are mapped to a smoothable face of $\mathcal{K}_n^{[m]}$ via the moment map. We recall that a face Γ of $\mathcal{K}_n^{[m]}$ is smoothable face (see Theorem A.9) if one of the following conditions is satisfied

- $n = 0$ or $n = 1$.
- The face Γ is contained in $\Delta_{n-1,n} + \mathbf{v} - \mathbf{1}$ for certain \mathbf{v} .
- The face Γ is contained in a linear subspace $H(S, \mathbf{a})$ as in (17), and in the intersection of $H(S, \mathbf{a})$ and $\mathcal{K}_n^{[m]}$, the face Γ is smoothable.

In Section A equivalent definitions of smoothable faces are given. Fig. 11 illustrates the smoothable faces for $\mathcal{K}_3^{[m]}$ for $m = 2, 3, 4$. The smoothable faces of $\mathcal{K}_4^{[m]}$ for $m = 3, 4$ are illustrated in Fig. 12.

Proposition 2.10. *Let $[J] \in \text{Hilb}_0^m(X_n)$, then, $\mu([J])$ lies in a smoothable face of $\mathcal{K}_n^{[m]}$ if and only if $J = \langle x_i^{u_i} : i \in S \rangle + \langle f \rangle$ for $S \subset [n]$, $\mathbf{u} \in \mathbb{Z}_{\geq 1}^n$ with $|\mathbf{u}| = m + |S|$, and $f \in \langle x_i^{u_i} : i \notin S \rangle_{\mathbb{C}}$.*

Proof. Assume first that $J = \langle x_i^{u_i} : i \in S \rangle + \langle f \rangle$ and consider the integer vectors $\mathbf{u}_S = \sum_{i \in S} u_i \mathbf{e}_i$ and $\mathbf{u}_{[n] \setminus S} = \sum_{i \notin S} u_i \mathbf{e}_i$. Consider the ideal $J' = \langle f \rangle \subset \mathbb{C}[x_i : i \notin S]$. Then, J' is a length $|\mathbf{u}_{[n] \setminus S}|$ and via the map (25), we get that $[J] = \iota_{S, \mathbf{u}_S}([J'])$. Now, $[J'] \in \text{Hilb}_0^{|\mathbf{u}_{[n] \setminus S}|}(X_{n-|S|})$ is contained in the Grassmannian $\Sigma(|\mathbf{u}_{[n] \setminus S}|, 1, \mathbf{u}_{[n] \setminus S})$. Therefore, $\mu([J'])$ lies in $\Delta_{n-1,n} + \mathbf{u}_{[n] \setminus S} - \mathbf{1}$, and hence, $\mu([J'])$ is contained in a smoothable face. Using the commutative diagram (19), we deduce that $\mu([J])$ is contained in a smoothable face.

Assume now that $\mu([J])$ lies in a smoothable face Γ . Then, Γ is a hypersimplex of the form $\Delta_{n'-1,n'}$ for $n' \leq n$ (see Section A). By the commutative diagram (19), it is enough to check the case when $n' = n$. In other words, $\Gamma = \Delta_{n-1,n} + \mathbf{u} - \mathbf{1}$. In this case, $\mu([J])$ lies in Γ if and only if $[J] \in \Sigma(m, 1, \mathbf{u})$. The proof follows from the fact that any ideal in $\Sigma(m, 1, \mathbf{u})$ is of the form $\langle x_i^{u_i+1} : i \in S \rangle + \langle f \rangle$ for $S \subsetneq [n]$ and $f \in \langle x_i^{u_i} : i \notin S \rangle_{\mathbb{C}}$. \square

Analogously to the notion of smoothable face, the notion of singular face is introduced in Section A, which we recall here for convenience: A face Γ of $\mathcal{K}_n^{[m]}$ is *singular* if one of the following conditions is satisfied:

- The face Γ is in the intersection of two distinct maximal faces.
- The face Γ is smoothable of dimension at most $n - 2$, i.e. at most codimension 1.

In the following Proposition we describe the ideals that are contained in a singular face via the moment map.

Proposition 2.11. *Let $[J] \in \text{Hilb}_0^m(X_n)$. Then $\mu([J])$ is contained in a singular face if and only if one of the following conditions is satisfied*

- J admits a minimal generator of the form $x_i^{u_i}$ for $u_i \geq 2$.
- J admits a minimal representation of the form $\langle f, x_i : i \in S \rangle$ for $f \in \langle x_i^{u_i} : i \notin S \rangle_{\mathbb{C}}$ and $\emptyset \subsetneq S \subset [n]$.

Proof. Let $[J] \in \text{Hilb}_0^m(X_n)$ lying in the component $\Sigma(m, l, \mathbf{u})$. By Theorem 2.10, $\mu([J])$ lies in a smoothable face if and only if $J = \langle f, x_i : i \in S \rangle$ for $f \in \langle x_i^{u_i} : i \notin S \rangle_{\mathbb{C}}$ and $S \subset [n]$. Moreover, such a face has dimension $n - 1$ if and only if $S = \emptyset$. Therefore, the second condition of the definition of singular face corresponds to the second condition in Theorem 2.11.

Assume now that J contains a generator of the form $x_i^{u_i}$ with $u_i \geq 2$. Then $[J]$ also lies in the Grassmannian $\Sigma(m, r + |S| - 1, \mathbf{u} - \mathbf{e}_i)$. Therefore, $\mu([J])$ is singular, since it lies in the intersection of the hypersimplices corresponding to these two Grassmannians. Now assume that $\mu([J])$ is contained in the intersection of two distinct hypersimplices $\Delta_{l,n} + \mathbf{u} - \mathbf{1}$ and $\Delta_{l',n} + \mathbf{v} - \mathbf{1}$. Then, $[J]$ is contained in the intersection of the Grassmannians $\Sigma(m, l, \mathbf{u})$ and $\Sigma(m, l', \mathbf{v})$. By Theorem 1.11, J is as in (7). Since $\mathbf{u} \neq \mathbf{v}$, either $\kappa(\mathbf{u} - \mathbf{v}, 1)$ or $\kappa(\mathbf{u} - \mathbf{v}, -1)$ is nonempty. Without loss of generality assume that $\kappa(\mathbf{u} - \mathbf{v}, 1) \neq \emptyset$. Then, there exists $i \in [n]$ such that $x_i^{u_i} = x_i^{v_i+1}$ is a minimal generator of J . Since $v_i \geq 1$, we deduce that J admits a minimal generators of the form $x_i^{u_i}$ with $u_i \geq 2$. \square

In Section 5 and Section 6, we relate the notions of smoothable and singular faces with the smoothable ideals and the singular locus of $\text{Hilb}^m(X_n)$.

3. FROM THE PUNCTUAL TO THE GLOBAL HILBERT SCHEME

With the notation and definitions of the previous sections, we will study the relation behind the combinatorics and the geometry of the Hilbert scheme $\text{Hilb}^m(X_n)$. From this interplay we will get all the irreducible components of $\text{Hilb}^m(X_n)$.

For $2 \leq m' \leq \min\{m, n-1\}$, $l = n+1-m'$, and for $\mathbf{u} \in \mathbb{Z}_{\geq 0}^n$ with $|\mathbf{u}| = m - m'$, consider the isomorphism

$$(20) \quad \phi_{m', \mathbf{u}} : \Sigma(m', l, \mathbf{1}) \rightarrow \Sigma(m, l, \mathbf{u} + \mathbf{1})$$

given by the composition

$$\Sigma(m', l, \mathbf{1}) \xrightarrow{\varphi_{l, \mathbf{1}}^{-1}} \text{Gr}(l, \Lambda_{\mathbf{1}}) \longrightarrow \text{Gr}(l, \Lambda_{\mathbf{u}+\mathbf{1}}) \xrightarrow{\varphi_{l, \mathbf{u}}} \Sigma(m, l, \mathbf{u} + \mathbf{1}),$$

where the middle map is induced by the isomorphism of vector spaces $\Lambda_{\mathbf{1}} \rightarrow \Lambda_{\mathbf{u}+\mathbf{1}}$ that sends x_i to $x_i^{u_i+1}$. In other words, for $[J] \in \Sigma(m', l, \mathbf{1})$, its image $\phi_{m', \mathbf{u}}([J])$ is the ideal whose generators are obtained by replacing x_i by $x_i^{u_i+1}$. Moreover, the following diagram commutes

$$\begin{array}{ccc} \Sigma(m', l, \mathbf{1}) & \xrightarrow{\phi_{m', \mathbf{u}}} & \Sigma(m', l, \mathbf{u} + \mathbf{1}) \\ \mu \downarrow & & \downarrow \mu \\ \Delta_{n-l, n} & \xrightarrow{+\mathbf{u}} & \Delta_{n-l, n} + \mathbf{u} \end{array}.$$

In particular, the map $\phi_{m', \mathbf{u}-\mathbf{1}}$ is the geometric analogous to the translation $+\mathbf{u} - \mathbf{1}$ in the definition of the moment map.

Proposition 3.1. *For $2 \leq m' \leq \min\{m, n-1\}$ and for $\mathbf{u} \in \mathbb{Z}_{\geq 0}^n$ with $|\mathbf{u}| = m - m'$, consider the rational map*

$$(21) \quad \begin{aligned} \Sigma(m', n+1-m', \mathbf{1}) \times \prod_{i \in [n]} \text{Sym}^{u_i} L_i & \dashrightarrow \text{Hilb}^m(X_n) \\ ([J], q_1, \dots, q_n) & \longmapsto \mathbb{V}(J) \cup q_1 \cup \dots \cup q_n \end{aligned}$$

Let $[J] \in \Sigma(m', n+1-m', \mathbf{1})$ and let Z be a one parameter family in the domain of (21) that contains the point $z_0 = ([J], u_1 \cdot \mathbf{0}, \dots, u_n \cdot \mathbf{0})$ and the domain of definition of (21) intersect Z in a dense open subset. Then, the restriction of the map (21) to Z can be uniquely extended to z_0 and the image of z_0 is $\phi_{m', \mathbf{u}}([J])$, which does not depend on the family Z .

Proof. Let $z = ([J], q_1, \dots, q_n) \in Z$ lying in the domain of definition of (21). In particular, q_i represents u_i points in $L_i \setminus \{\mathbf{0}\}$. Let $I_i = \langle x_j : j \neq i \rangle + \langle f_i \rangle$ be the ideal of q_i . In other words, we can write f_i as $f_i = 1 + f'_i$ where f'_i is a degree u_i polynomial in x_i with $f'_i(0) = 0$. Then, the ideal of $q_1 \cup \dots \cup q_n$ in R_n is $\langle f_1 \cdots f_n \rangle = \langle 1 + \sum f'_j \rangle$. Let $g_1, \dots, g_{n+1-m'}$ be the generators of J , where $g_i = a_{i,1}x_1 + \dots, a_{i,n}x_n$. Therefore, the ideal of $\mathbb{V}(J) \cup q_1 \cup \dots \cup q_n$ is generated by

$$\langle g_1 \left(1 + \sum f_j\right), \dots, g_{n+1-m'} \left(1 + \sum f_j\right) \rangle.$$

Now, we have that

$$g_i \left(1 + \sum_{j=1}^n f_j\right) = g_i + \sum_{j=1}^n a_{i,j} x_j f'_j = \sum_{j=1}^n a_{i,j} x_j + a_{i,j} x_j f'_j = \sum_{j=1}^n a_{i,j} x_j (1 + f'_j) = \sum_{j=1}^n a_{i,j} x_j f_j.$$

Now, consider any limit of the form $\lim_{z \rightarrow z_0} z$. In other words, we are taking the limit when f_j goes to $x_j^{u_j}$ for all $j \in [n]$. Then, the image of such a limit is a length m ideal that must contained the ideal generated by

$$\sum_{j=1}^n a_{i,j} x_j^{u_j+1} \text{ for all } i \in [n].$$

Note that this ideal is exactly $\phi_{m', \mathbf{u}}([J])$. In particular, the length of this ideal is $|\mathbf{u}| + n + 1 - (n + 1 - m') = m$, and we conclude that $\phi_{m', \mathbf{u}}([J])$ is the image of the limit. \square

Remark 3.2. Theorem 3.1 allows us to relate the combinatorics studied in Section 2 and the geometry of the Hilbert scheme. Mainly, it shows the relation between the map (21) and the translations made in the definition of the moment map and in the hypersimplices of $\mathcal{K}_n^{[m]}$. Consider the translation by \mathbf{u} between the hypersimplices $\Delta_{m'-1, n}$ and $\Delta_{m'-1, n} + \mathbf{u}$ for $\mathbf{u} \in \mathbb{Z}_{\geq 0}^n$ and $|\mathbf{u}| = m - m'$. Let $[J] \in \Sigma(m', n + 1 - m', \mathbf{1})$, then $\mu([J])$ is in $\Delta_{m'-1, n}$. The translation $\mu([J]) + \mathbf{u}$ can be interpreted geometrically as follows. Let Z be the length m ideal obtained by adding u_i nonzero points in the line L_i to the scheme $\mathbb{V}(J)$ for all i . By Theorem 3.1, collapsing all these nonzero points to the singularity leads to the length m ideal $\phi_{m', \mathbf{u}}([J])$ in $\text{Hilb}_0^m(X_n)$. The image via the moment map of this ideal is exactly $\mu([J]) + \mathbf{u}$. Thus, the translation in the definition of the moment map is interpreted geometrically as adding to $\mathbb{V}(J)$ extra nonzero points in the lines of X_n and collapsing them to the singularity. This relation is explored in more detail in Section 6.1.

We now calculate the irreducible components of $\text{Hilb}^m(X_n)$. The strategy is to use Theorem 3.1 to distinguish which irreducible components of the punctual Hilbert scheme $\text{Hilb}_0^m(X_n)$ lift to an elementary component of $\text{Hilb}^m(X_n)$. To do so, we first introduce the candidates to irreducible components of $\text{Hilb}^m(X_n)$. For $\mathbf{u} \in \mathbb{Z}_{\geq 1}^n$ with $|\mathbf{u}| = m + l - 1$, we define the rational map

$$(22) \quad X_n \times \Sigma(m, l, \mathbf{u}) \dashrightarrow \text{Hilb}^{m+1}(X_n)$$

that sends a point $q \in X_n$ and an ideal $[J] \in \Sigma(m, l, \mathbf{u})$ to the length- m subscheme $\{q\} \cup \mathbb{V}(J)$. The map (22) is not defined in $\mathbf{0} \times \Sigma(m, l, \mathbf{u})$. As a consequence, the image of (22) is not a closed subvariety. Theorem 3.1 allows us to extend (22) to a well-defined map. The image of the base locus is the union of the Grassmannians $\Sigma(m + 1, l, \mathbf{u} + \mathbf{e}_i)$ for $i \in [n]$. In general, for $2 \leq m' \leq \min\{n - 1, m\}$, we define the map

$$(23) \quad \begin{aligned} \text{Hilb}_{\text{sm}}^{m-m'}(X_n \setminus \{\mathbf{0}\}) \times \Sigma(m', n + 1 - m', \mathbf{1}) &\longrightarrow \text{Hilb}^m(X_n) \\ (Y, [J]) &\longmapsto Y \cup \mathbb{V}(J). \end{aligned}$$

The extension of (23) to the closure of its domain will be studied in Section 6.1. We denote the closure of the image of (23) by $\text{Hilb}^{m, m'}(X_n)$. In other words, $\text{Hilb}^{m, m'}(X_n)$ is the reduced version of $\mathcal{Hilb}^{m, m'}(X_n)$, defined as the closure of the locus of points $[J] \in \text{Hilb}^m(X_n)$ such that there exists J_0 in the primary decomposition of J supported at $\mathbf{0}$ with $[J_0] \in \Sigma(m', n + 1 - m', \mathbf{1})$. The following result describes which irreducible components of $\text{Hilb}_0^m(X_n)$ lift to elementary components of $\text{Hilb}^m(X_n)$ and which not. This allows us to compute the irreducible components of $\text{Hilb}^m(X_n)$.

Theorem 3.3. Fix $n \geq 1$ and $m \geq 1$. Let $l \in [n - 1]$ and let $\mathbf{u} \in \mathbb{Z}_{\geq 1}^n$ be partition of $m + l - 1$.

- (1) If $\mathbf{u} = \mathbf{1} = (1, \dots, 1)$ and $m = n$, then $l = 1$ and $\Sigma(n, 1, \mathbf{1})$ is contained in $\text{Hilb}_{\text{sm}}^m(X_n)$.
- (2) If $\mathbf{u} = \mathbf{1} = (1, \dots, 1)$ and $2 \leq m \leq n - 1$, then $2 \leq l = n + 1 - m \leq n - 1$. In this case, $\Sigma(m, n + 1 - m, \mathbf{1})$ is an irreducible component of $\text{Hilb}^m(X_n)$. In particular, $\Sigma(m, n + 1 - m, \mathbf{1})$ is an elementary component of $\text{Hilb}^m(X_n)$.
- (3) If there exists $1 \leq i \leq n$ such that $u_i \geq 2$ then $\Sigma(m, l, \mathbf{u})$ is contained in the closure of the image of the map

$$(24) \quad (L_i \setminus \{\mathbf{0}\}) \times \Sigma(m - 1, l, \mathbf{u} - \mathbf{e}_i) \longrightarrow \text{Hilb}^m(X_n)$$

sending a pair $(q, [I]) \in (L_i \setminus \{\mathbf{0}\}) \times \Sigma(m - 1, l, \mathbf{u} - \mathbf{e}_i)$ to the length m scheme $\{q\} \cup \mathbb{V}(I)$. In particular, $\Sigma(m, l, \mathbf{u})$ is not an elementary component of $\text{Hilb}^m(X_n)$.

Therefore, the irreducible components of $\text{Hilb}^m(X_n)$ are either an irreducible component of the smoothable component or an irreducible component of $\text{Hilb}^{m, m'}(X_n)$ for $2 \leq m' \leq \min\{m, n - 1\}$.

Proof. Assume first that $u_i \geq 2$ for some $1 \leq i \leq n$. Without loss of generality, we can assume that $u_1 \geq 2$. Let J be a generic ideal in $\Sigma(m, l, \mathbf{u})$. Then J is generated by l polynomials f_1, \dots, f_l of the form

$$f_i = a_{i,1}x_1^{u_1} + \dots + a_{i,n}x_n^{u_n}.$$

We can assume that $a_{1,1} = 1$. Replacing f_i by $a_{1,1}f_i - a_{i,1}f_1$ for $2 \leq i \leq l$, we can also assume that $a_{i,1} = 0$ for $2 \leq i \leq l$. We consider the ideal

$$J_\lambda = \langle f_1 + \lambda x_1^{u_1-1}, f_2, \dots, f_l \rangle$$

for $\lambda \in \mathbb{C}$. Note that $J_0 = J$ and J_λ is a length m ideal in R_n for all $\lambda \in \mathbb{C}$. As in Proposition 3.1, for $\lambda \neq 0$, we may write J_λ as

$$J_\lambda = \langle (\lambda + x_1)(x_1^{u_1-1} + \lambda^{-1}a_{1,2}x_2^{u_2} + \dots + \lambda^{-1}a_{1,n}x_n^{u_n}), (\lambda + x_1)f_2, \dots, (\lambda + x_1)f_n \rangle.$$

We deduce that for $\lambda \neq 0$, $\mathbb{V}(J_\lambda)$ is the union of the point $(-\lambda, 0, \dots, 0)$ and a length $m-1$ subscheme in $\Sigma(m, l, \mathbf{u} - \mathbf{e}_1)$ given by the ideal

$$J'_\lambda = \langle x_1^{u_1-1} + \lambda^{-1}a_{1,2}x_2^{u_2} + \dots + \lambda^{-1}a_{1,n}x_n^{u_n}, f_2, \dots, f_n \rangle.$$

Therefore, for $\lambda \neq 0$, $[J_\lambda]$ is the image via (24) of the tuple $((-\lambda, 0, \dots, 0), [J'_\lambda])$. By Theorem 3.1, we deduce that $[J]$ lies in the closure of the image of the map (24). Since $[J]$ is a generic element of $\Sigma(m, l, \mathbf{u})$, we conclude that $\Sigma(m, l, \mathbf{u})$ is also contained in this closure.

Now, assume that $l = 1$, $m = n$ and $\mathbf{u} = \mathbf{1}$. In this case, ideals in $\Sigma(n, 1, \mathbf{1})$ are generated by a linear form, so they correspond to hyperplanes passing through $\mathbf{0}$. Let J be a generic element in $\Sigma(n, 1, \mathbf{1})$ and let H be the corresponding hyperplane. Moreover, let v be a generic vector supported at $\mathbf{0}$ and consider the family of affine hyperplanes $H_t = H + tv$. Since H and v are generic, for $t \neq 0$, H_t intersects each of the lines L_1, \dots, L_n in a point distinct than $\mathbf{0}$. In particular, $H_t \cap X_n$ consists of n distinct points in X_n and we obtain a family of length n schemes $H_t \cap X_n$ that are smoothable for $t \neq 0$. We conclude that the scheme $H_0 \cap X_n = \mathbb{V}(J)$ is smoothable.

Assume that $2 \leq m \leq n-1$, $\mathbf{u} = (1, \dots, 1)$ and $l = n+1-m$. This implies that $n \geq 3$. We show that $\Sigma(m, n+1-m, \mathbf{1})$ is an irreducible component of $\text{Hilb}^m(X_n)$. For $m = 2$, we saw that $\Sigma(2, n-1, \mathbf{1})$ is an irreducible component of $\text{Hilb}^2(X_n)$ in Theorem 1.3 since the tangent star is properly contained in the tangent space $T_{\mathbf{0}}X_n$. Now assume that $m \geq 3$. For $0 \leq k \leq m$, we consider the map

$$(25) \quad \text{Hilb}^k(X_n \setminus \{\mathbf{0}\}) \times \text{Hilb}^{m-k}_{\mathbf{0}}(X_n) \dashrightarrow \text{Hilb}^m(X_n)$$

sending k distinct points $\{q_1, \dots, q_k\}$ in $X_n \setminus \{\mathbf{0}\}$ and a length $m-k$ scheme Z supported at $\mathbf{0}$ to $\{q_1, \dots, q_k\} \cup Z$. We denote the closure of the image of this map by $Y^m(n, k)$. Note that $Y^m(n, m)$ is the smoothable component of $\text{Hilb}^m(X_n)$.

To check that $\Sigma(m, n+1-m, \mathbf{1})$ is an elementary component of $\text{Hilb}^m(X_n)$ it is enough to show that it is not contained in $Y^m(n, k)$ for all $1 \leq k \leq m$. We argue by contradiction as follows. Let $[J]$ be a generic element in $\Sigma(m, n+1-m, \mathbf{1})$ and assume that $[J]$ lies in $Y^m(n, k)$ for some $1 \leq k \leq m$. Note that generic ideals in $\Sigma(m, n+1-m, \mathbf{1})$ are generated by $n+1-m$ linear forms and they correspond to generic $(m-1)$ -dimensional linear subspaces in \mathbb{A}^n passing through $\mathbf{0}$. Let Γ be the $(m-1)$ -dimensional subspace associated to J . Then there exists an irreducible reduced curve C , a one-dimensional family of length m schemes $\mathcal{Z} \rightarrow C$ and $t_0 \in C$ such that the fiber \mathcal{Z}_{t_0} is $\mathbb{V}(J)$ and the fibers \mathcal{Z}_t for $t \neq t_0$ lie in the image of (25). Since $1 \leq k$, there exists $1 \leq i \leq n$ such that \mathcal{Z}_t is contained in the image of the map

$$(L_i \setminus \{\mathbf{0}\}) \times \text{Hilb}^{k-1}(X_n \setminus \{\mathbf{0}\}) \times \text{Hilb}^{m-k}_{\mathbf{0}}(X_n) \dashrightarrow \text{Hilb}^m(X_n).$$

for $t \neq t_0$. Let Γ_t be the smallest linear subspace containing \mathcal{Z}_t . Assume first that $1 \leq k \leq m-1$. Since $k \leq m-1$, \mathcal{Z}_t contains $\mathbf{0}$ and a point in $L_i \setminus \{\mathbf{0}\}$ for $t \neq t_0$. Therefore, Γ_t is contained in L_i for $t \neq 0$. We deduce that $\Gamma_0 = \Gamma$ contains L_i . This is a contradiction since Γ is a generic linear subspace of dimension $m-1$ containing $\mathbf{0}$. Next, assume that $[J]$ lies in $Y^m(n, m)$, i.e., assume that J is smoothable. Then, \mathcal{Z}_t consists of m distinct points for $t \neq t_0$. Moreover, there exists i_1, \dots, i_m such that \mathcal{Z}_t is contained in the image of

$$(L_{i_1} \setminus \{\mathbf{0}\}) \times \dots \times (L_{i_m} \setminus \{\mathbf{0}\}) \dashrightarrow \text{Hilb}^m(X_n)$$

for $t \neq 0$. Let M_{i_1, \dots, i_k} be the affine subspace expand by L_{i_1}, \dots, L_{i_k} . Then \mathcal{Z}_t is contained in M_{i_1, \dots, i_k} for any $t \in C$. Therefore, Γ is contained in M_{i_1, \dots, i_k} . Since $m \leq n-1$, we have that $M_{i_1, \dots, i_k} \subsetneq \mathbb{A}^n$. We deduce that if J is smoothable, Γ would be contained in

$$\bigcup_{1 \leq i_1, \dots, i_k \leq n} M_{i_1, \dots, i_k} \subsetneq \mathbb{A}^n.$$

This is a contradiction since J and Γ are generic.

By the first statement of the theorem, we deduce that for $m \geq n$, $\text{Hilb}^m(X_n)$ does not have any elementary component. Similarly, the second statement of the theorem implies that for $2 \leq m \leq n-1$, the only elementary component of $\text{Hilb}^m(X_n)$ is $\Sigma(m, n+1-m, \mathbf{1})$. Therefore, we conclude that the irreducible components of $\text{Hilb}^m(X_n)$ are the smoothable components of the irreducible components of $\text{Hilb}^{m, m'}(X_n)$ for $2 \leq m' \leq \min\{m, n-1\}$. \square

The main consequences of Theorem 3.3 are the following results.

Corollary 3.4. *The number of irreducible components of $\text{Hilb}^m(X_n)$ is*

$$\binom{m+n-1}{m} + \sum_{m'=2}^{\min\{m, n-1\}} \binom{m-m'+n-1}{m-m'}.$$

Proof. By Theorem 3.3, the irreducible components of $\text{Hilb}^m(X_n)$ are the irreducible components of $\text{Hilb}_{\text{sm}}^m(X_n)$ and the irreducible components of $\text{Hilb}^{m, m'}(X_n)$ for $2 \leq m' \leq \min\{n-1, m\}$. The irreducible components of $\text{Hilb}_{\text{sm}}^m(X_n)$ are given by the possible distribution of m distinct points among the n of X_n . Therefore, $\text{Hilb}_{\text{sm}}^m(X_n)$ has $\binom{m+n-1}{m}$ components. Similarly, the components of $\text{Hilb}^{m, m'}(X_n)$ are in correspondence with the components of $\text{Hilb}_{\text{sm}}^{m-m'}(X_n)$ via the map (23). We deduce that the number of irreducible components of $\text{Hilb}^{m, m'}(X_n)$ is $\binom{m-m'+n-1}{m-m'}$. \square

Remark 3.5. The number of irreducible components of $\text{Hilb}^{m, m'}(X_n)$ is $\binom{m-m'+n-1}{m-m'}$. These components are birational to

$$\text{Sym}^{u_1-1}(L_1) \times \dots \times \text{Sym}^{u_n-1}(L_n) \times \Sigma(m', n+1-m', \mathbf{1}),$$

where $\mathbf{u} \in \mathbb{Z}_{\geq 1}^n$ with $|\mathbf{u}| = m - m' + n$. These components are in bijection with the hypersimplices of the form $\Delta_{m'-1, n} + \mathbf{u} - \mathbf{1}$ in $\mathcal{K}_n^{[m]}$.

Corollary 3.6. *Let C be an irreducible curve whose only singularity is a rational n -fold singularity. Then, the number of irreducible components of $\text{Hilb}^m(C)$ is $\min\{n-1, m\}$. Moreover, these irreducible components are $\text{Hilb}_{\text{sm}}^m(C)$ and $\text{Hilb}^{m, m'}(C)$ for $2 \leq m' \leq \min\{n-1, m\}$, which are birational to*

$$\text{Sym}^m(C) \quad \text{or} \quad \text{Sym}^{m-m'}(C) \times \text{Gr}(n+1-m', n) \quad \text{for } 2 \leq m' \leq \min\{n-1, m\}.$$

Proof. Follows from Theorem 3.3, since in this case the smoothable component is irreducible. \square

Remark 3.7. To the best of our knowledge, this is the first example of Hilbert schemes of m points where the number of irreducible components initially increases and then remains constant as m varies. The graph of the number of irreducible components for n fixed as m varies is illustrated in Fig. 1.

Example 3.8. Let C be an irreducible curve whose unique singularity is a rational 3-fold singularity. The number of irreducible components of $\text{Hilb}^m(C)$ is $\min\{2, m\}$. Assuming that $m \geq 2$, the two irreducible components of $\text{Hilb}^m(C)$ are the smoothable component and $\text{Hilb}^{m, 2}(C)$, which is birational to $\mathbb{P}^2 \times \text{Sym}^{m-2}(C)$.

Example 3.9. Let C be an irreducible curve whose unique singularity is a rational 4-fold singularity. The number of irreducible components of $\text{Hilb}^m(C)$ is $\min\{3, m\}$. For $m = 2$, the irreducible components are the smoothable component $\text{Hilb}^2(C_{\text{sm}})$ and a $\text{Hilb}^{2, 2}(C) \simeq \mathbb{P}^3$. For $m \geq 3$, the three irreducible components of $\text{Hilb}^m(C)$ are $\text{Hilb}^3(C_{\text{sm}}) \simeq_{\text{bir}} \text{Sym}^m(C)$, $\text{Hilb}^{m, 2}(C) \simeq_{\text{bir}} \text{Sym}^{m-2} C \times \mathbb{P}^3$ and $\text{Hilb}^{m, 3}(C) \simeq_{\text{bir}} \text{Sym}^{m-3} C \times \text{Gr}(2, 4)$. Note that two of the first two irreducible components have dimension m , while the third irreducible component has dimension $m+1$.

4. THE LOCAL HILBERT SCHEME AND ITS SCHEMATIC STRUCTURE

In the previous sections we have carried out a study of $\mathcal{Hilb}^m(X_n)$ from the perspective of algebraic varieties, and not considering the possibility of a non-reduced structure. This is reflected in Theorem 3.3 where the reduced structure of the irreducible components of $\mathcal{Hilb}^m(X_n)$ is given. The next goal is to analyse the reducedness of $\mathcal{Hilb}^m(X_n)$. We use the notation and classical results from deformation theory as presented in [Ser06], to which we refer the reader for further details. Consider the local Hilbert scheme H_J^n of $\mathbb{V}(J)$ in X_n . This is defined as the functor

$$H_J^n : \mathcal{A}rt \longrightarrow \text{Sets} \\ A \longmapsto \left\{ \begin{array}{c} \text{Deformations of } \mathbb{V}(J) \text{ in } X_n \\ \text{over } A \end{array} \right\}.$$

Then H_J^n is prorepresented by $\widehat{\mathcal{O}}_{\mathcal{Hilb}^m(X_n),[J]}$. We compute $\widehat{\mathcal{O}}_{\mathcal{Hilb}^m(X_n),[J]}$ by calculating the complete local ring prorepresenting the local Hilbert scheme using the same strategy as [Ran05]. We carry out this computation for the most singular points of $\mathcal{Hilb}^m(X_n)$, which are the ideals J such that $\mu_m(J)$ is a vertex of $\mathcal{K}_n^{[m]}$. In other words, $J = \langle x_1^{u_1}, \dots, x_n^{u_n} \rangle$. Up to labeling of the variables, we may assume $J = \langle x_1^{u_1}, \dots, x_k^{u_k}, x_{k+1}, \dots, x_n \rangle$, where $u_i \geq 2$ for $i \leq k$. Note that $k \geq 1$, since for $k = 0$, the ideal J has length 1. The value of k has the following combinatorial meaning. The hypersimplicial complex $\mathcal{K}_n^{[m]}$ fills the simplex $(m-1)\Delta_{n-1}$, which it can also be seen as a simplicial complex. Then, $k-1$ is the dimension of the face of $(m-1)\Delta_{n-1}$ where $\mu([J])$ lies. For instance, if $k = 1$, then $\mu([J])$ is a vertex of $(m-1)\Delta_{n-1}$. If $k = n$, then $\mu([J])$ is an interior point of $(m-1)\Delta_{n-1}$.

Consider the ring

$$S_k = \mathbb{C}[A_1, \dots, A_n, \alpha_{i,j,l} : i \in [n], j \in [k] \text{ and } l \in [u_j - 1]]$$

and the ideal of S_k given by

$$(26) \quad \begin{aligned} J_k = & \langle A_i A_j : \text{for } k+1 \leq j \leq n \text{ and } i \in [n] \setminus \{j\} \rangle \\ & + \langle A_i \alpha_{j,r,s} : \text{for } k+1 \leq j \leq n, i \in [n] \setminus \{j\}, r \in [k] \text{ and } s \in [u_r - 1] \rangle \\ & + \langle \alpha_{i,j,u_j-1} \alpha_{j,1} - A_i : \text{for } j \in [k] \text{ and } i \in [n] \setminus \{j\} \rangle \\ & + \langle \alpha_{i,j,u_j-1} A_j : \text{for } j \in [k] \text{ and } i \in [n] \setminus \{j\} \rangle \\ & + \langle \alpha_{i,j,u_j-1} \alpha_{j,j,l+1} - \alpha_{i,j,l} : \text{for } j \in [k], i \in [n] \setminus \{j\} \text{ and } l \in [u_j - 2] \rangle \\ & + \langle \alpha_{i,j,u_j-1} \alpha_{j,r,l} : \text{for } j \in [k], i \in [n] \setminus \{j\}, r \in [k] \setminus \{j\} \text{ and } l \in [u_j - 1] \rangle. \end{aligned}$$

Using this ideal we compute $\widehat{\mathcal{O}}_{\mathcal{Hilb}^m(X_n),[J]}$ in the following:

Theorem 4.1. *Let $0 \leq k \leq n$ and let $\mathbf{u} \in \mathbb{Z}_{\geq 1}^n$ such that $|\mathbf{u}| = m + n - 1$, $u_i \geq 2$ for $1 \leq i \leq k$ and $u_i = 1$ for $k+1 \leq i \leq n$. Then, the local Hilbert scheme of $J = \langle x_1^{u_1}, \dots, x_k^{u_k}, x_{k+1}, \dots, x_n \rangle$ in X_n is prorepresented by the completion of the quotient S_k/J_k localized at the origin.*

Proof. Let $S \in \mathcal{A}$ be a local Artinian \mathbb{C} -algebra with residue field \mathbb{C} , and let \mathfrak{m}_S its maximal ideal. Let $R_S := R \otimes_{\mathbb{C}} S$. A deformation of $\mathbb{V}(J)$ in $\text{Spec}(R_n)$ over $\text{Spec}(S)$ is an ideal J_S of R_S such that R_S/J_S is flat over S and $(R_S/J_S) \otimes_S \mathbb{C} = R/J$. In other words, J_S is generated by

$$\begin{aligned} f_1 &= x_1^{u_1} + A_1 + f_{1,1}(x_1) + \dots + f_{1,n}(x_n), \\ &\vdots \\ f_k &= x_k^{u_k} + A_k + f_{k,1}(x_1) + \dots + f_{k,n}(x_n), \\ f_{k+1} &= x_{k+1} + A_{k+1} + f_{k+1,1}(x_1) + \dots + f_{k+1,n}(x_n), \\ &\vdots \\ f_n &= x_n + A_n + f_{n,1}(x_1) + \dots + f_{n,n}(x_n), \end{aligned}$$

where A_i lies in \mathfrak{m}_S and $f_{i,j}(x_j)$ is a polynomial in x_j with coefficients in \mathfrak{m}_S and with no independent coefficient. By [Sta25, Tag 051G], R_S/J_S is a free S -module of rank m . By Nakayama's Lemma, R_S/J_S is freely generated by

$$1, x_1, \dots, x_1^{u_1-1}, \dots, x_k, \dots, x_k^{u_k-1}.$$

We can write x_i^d with $d_i \geq u_i$ as a linear combination of $1, x_1, \dots, x_1^{u_1-1}, \dots, x_k, \dots, x_k^{u_k-1}$ with coefficients in S . Therefore, we may assume that the degree of $f_{i,j}$ is at most $u_j - 1$. In particular, $f_{i,j} = 0$ for $k+1 \leq j \leq n$ and we get

$$f_{i,j} = \sum_{l=1}^{u_j-1} \alpha_{i,j,l} x_j^l$$

for $1 \leq j \leq k$. We deduce that J_S is generated by

$$\begin{aligned} f_1 &= x_1^{u_1} + A_1 + \sum_{l=1}^{u_1-1} \alpha_{1,1,l} x_1^l + \dots + \sum_{l=1}^{u_k-1} \alpha_{1,k,l} x_k^l, \\ &\vdots \\ f_k &= x_k^{u_k} + A_k + \sum_{l=1}^{u_1-1} \alpha_{k,1,l} x_1^l + \dots + \sum_{l=1}^{u_k-1} \alpha_{k,k,l} x_k^l, \\ f_{k+1} &= x_{k+1} + A_{k+1} + \sum_{l=1}^{u_1-1} \alpha_{k+1,1,l} x_1^l + \dots + \sum_{l=1}^{u_k-1} \alpha_{k+1,k,l} x_k^l, \\ &\vdots \\ f_n &= x_n + A_n + \sum_{l=1}^{u_1-1} \alpha_{n,1,l} x_1^l + \dots + \sum_{l=1}^{u_k-1} \alpha_{n,k,l} x_k^l. \end{aligned} \quad (27)$$

For $k+1 \leq j \leq n$ and $i \in [n] \setminus \{j\}$, we consider in R_S/J_S the relation

$$\begin{aligned} 0 = A_i f_j - x_j f_i &= A_i \left(x_j + A_j + \sum_{l=1}^{u_1-1} \alpha_{j,1,l} x_1^l + \dots + \sum_{l=1}^{u_k-1} \alpha_{j,k,l} x_k^l \right) - A_i x_j = \\ &= A_i A_j + \sum_{l=1}^{u_1-1} A_i \alpha_{j,1,l} x_1^l + \dots + \sum_{l=1}^{u_k-1} A_i \alpha_{j,k,l} x_k^l. \end{aligned} \quad (28)$$

Note that (28) is a relation among the free generators of R_S/J_S . We deduce that the coefficients of (28) vanish in S , and we get that for $1 \leq i \leq n$ and $k+1 \leq j \neq i \leq n$

$$\begin{aligned} A_i A_j &= 0 \\ A_i \alpha_{j,a,b} &= 0 \quad \text{for } 1 \leq a \leq k \text{ and } 1 \leq b \leq u_a - 1. \end{aligned} \quad (29)$$

Similarly, for $1 \leq j \leq k$ and $i \in [n] \setminus \{j\}$, we consider in R_S/J_S the relation

$$\begin{aligned} 0 = \alpha_{i,j,u_j-1} f_j - x_j f_i &= \alpha_{i,j,u_j-1} \left(x_j^{u_j} + A_j + \sum_{l=1}^{u_1-1} \alpha_{j,1,l} x_1^l + \dots + \sum_{l=1}^{u_k-1} \alpha_{j,k,l} x_k^l \right) - A_i x_j - \sum_{l=1}^{u_j-1} \alpha_{i,j,l} x_j^{l+1} \\ &= (\alpha_{i,j,u_j-1} \alpha_{j,j,1} - A_i) x_j + \sum_{l=1}^{u_j-2} (\alpha_{i,j,u_j-1} \alpha_{j,j,l+1} - \alpha_{i,j,l}) x_j^{l+1} + \alpha_{i,j,u_j-1} A_j + \sum_{a \neq j} \sum_{l=1}^{u_a-1} \alpha_{i,j,u_j-1} \alpha_{j,a,l} x_a^l. \end{aligned} \quad (30)$$

Again, (30) is a relation among the free generators of R_S/J_S . We deduce that for $1 \leq i \leq n$ and $1 \leq j \leq k$ with $i \neq j$ we have

$$\begin{aligned} \alpha_{i,j,u_j-1} \alpha_{j,j,1} - A_i &= 0 \\ \alpha_{i,j,u_j-1} A_j &= 0 \\ \alpha_{i,j,u_j-1} \alpha_{j,j,l+1} - \alpha_{i,j,l} &= 0 \quad \text{for } 1 \leq l \leq u_j - 2 \\ \alpha_{i,j,u_j-1} \alpha_{j,a,l} &= 0 \quad \text{for } 1 \leq a \neq j \leq k, \text{ and } 1 \leq l \leq u_a - 1. \end{aligned} \quad (31)$$

Therefore, if the R_S/J_S is a deformation over S , then the coefficients of f_1, \dots, f_n satisfy equations (29) and (31). Conversely, assume that the coefficients of f_1, \dots, f_n satisfy equations (29) and (31). Then, f_1, \dots, f_n satisfy the relations (28) and (30). Modulo $S/\mathfrak{m}_s \simeq \mathbb{C}$, these relations are exactly

$$x_i x_j^{u_j} = 0$$

for $i \neq j$, which are exactly the syzygies of J . By [Ser06, Corollary A.11], R_S/J_S is flat over S and we conclude that R_S/J_S is a deformation over S . Therefore, the local Hilbert scheme of $[J]$ is prorepresented by the completion of the stalk at the origin of the quotient of

$$S_k = \mathbb{C}[A_1, \dots, A_n, \alpha_{i,j,l} : 1 \leq i \leq n, 1 \leq j \leq k, 1 \leq l \leq u_j - 1]$$

by the ideal generated by the relations in (29) and (31), which coincide with J_k . \square

Theorem 4.1 computes the stalk $\widehat{\mathcal{O}}_{\mathcal{Hilb}^m(X_n), [J]}$ as the completion of the quotient S_k/J_k . However, the representation of this quotient is quite complicated. In Section B we give a better representation of this ring. In particular, Theorem B.3 and Theorem B.8 together with Theorem B.2 compute the irreducible components of the stalk of $\mathcal{Hilb}^m(X_n)$ at the point $[J] = [\langle x_1^{u_1}, \dots, x_k^{u_k}, x_{k+1}, \dots, x_n \rangle]$. Now, we identify each of these components with the corresponding components of $\mathcal{Hilb}^m(X_n)$.

Proposition 4.2. *For $k = 1$, $\widehat{\mathcal{O}}_{\mathcal{Hilb}^m(X_n), [J]}$ has $n + 1$ irreducible components and they correspond to the ideals in Theorem B.3. A generic point in the component corresponding to $\langle A_1, \alpha_{1,1} \rangle$ represents a length m scheme that is a point in*

$$(32) \quad \mathcal{Hilb}^{m-2}(L_1) \times \Sigma(2, n-1, \mathbf{1}).$$

A generic point in the component associated to $\langle \alpha_2, \dots, \alpha_n \rangle$ represents a smoothable length m scheme which is a point in $\mathcal{Hilb}^m(L_1)$. A generic point in $\langle A_1, \alpha_j : 2 \leq j \leq n \text{ and } j \neq i \rangle$ for $2 \leq i \leq n$ represents a smoothable length m scheme that is a point in $\mathcal{Hilb}^{m-1}(L_1) \times L_i$.

Proof. Let K be an ideal in the irreducible component corresponding to the ideal $\langle \alpha_2, \dots, \alpha_n \rangle$. We write the generators of K as in (27). Via the isomorphism in Theorem B.2, we deduce that $A_2 = \dots = A_n = 0$ and $\alpha_{2,1,u_1-1} = \dots = \alpha_{n,1,u_1-1} = 0$. Using the ideal (26) we deduce that the generators of K are

$$\begin{aligned} f_1 &= A_1 + \sum_{l=1}^{u_1-1} \alpha_{1,1,l} x_1^l + x_1^{u_1}, \\ f_2 &= x_2, \\ \vdots & \\ f_n &= x_n. \end{aligned}$$

Hence, K represents a length m ideal in $\mathcal{Hilb}^m(L_1)$. Assume now that K corresponds to a point in the ideal $\langle A_1, \alpha_j : 2 \leq j \leq n \text{ and } j \neq i \rangle$ for $2 \leq i \leq n$. As before, pulling back these conditions via the isomorphism build in Theorem B.2, we deduce that the generators of K are of the form

$$\begin{aligned} f_1 &= \sum_{l=1}^{u_1-1} \alpha_{1,1,l} x_1^l + x_1^{u_1} = x_1 \left(\sum_{l=0}^{u_1-2} \alpha_{1,1,l+1} x_1^l + x_1^{u_1-1} \right), \\ f_2 &= x_2, \\ \vdots & \\ f_i &= x_i + \alpha_{i,1,u_1-1} \left(\sum_{l=0}^{u_1-2} \alpha_{1,1,l+1} x_1^l + x_1^{u_1-1} \right), \\ \vdots & \\ f_n &= x_n. \end{aligned}$$

We deduce that the primary decomposition of K consists on the ideals $\langle \sum_{l=0}^{u_1-2} \alpha_{1,1,l+1} x_1^l + x_1^{u_1-1}, x_2, \dots, x_n \rangle$, which represents a point in $\mathcal{Hilb}^{m-1}(L_1)$ and $\langle x_1, x_2, \dots, x_i + \alpha_{i,1,u_1-1} \alpha_{1,1,1}, \dots, x_n \rangle$, which represents a point in L_i . Similarly, if K is in the component corresponding to $\langle A_1, \alpha_{1,1} \rangle$, the generators of K are

$$\begin{aligned} f_1 &= \sum_{l=2}^{u_1-1} \alpha_{1,1,l} x_1^l + x_1^{u_1} = x_1^2 \left(\sum_{l=0}^{u_1-3} \alpha_{1,1,l+2} x_1^l + x_1^{u_1-2} \right), \\ f_2 &= x_2 + \sum_{l=1}^{u_1-1} \alpha_{2,1,l} x_1^l = x_2 + \alpha_{2,1,u_1-1} x_1 \left(\sum_{l=0}^{u_1-3} \alpha_{1,1,l+2} x_1^l \right), \\ &\vdots \\ f_n &= x_n + \sum_{l=1}^{u_1-1} \alpha_{n,1,l} x_1^l = x_n + \alpha_{n,1,u_1-1} x_1 \left(\sum_{l=0}^{u_1-3} \alpha_{1,1,l+2} x_1^l \right). \end{aligned}$$

We deduce that the primary decomposition of K consists of two ideals. First, the ideal $\langle \sum_{l=0}^{u_1-3} \alpha_{1,1,l+2} x_1^l, x_2, \dots, x_n \rangle$ which leads to length $m-2$ scheme in L_1 . The second ideal is

$$\langle x_1^2, x_2 - \alpha_{2,1,u_1-1} \alpha_{1,1,2} x_1, \dots, x_n - \alpha_{n,1,u_1-1} \alpha_{1,1,2} x_1 \rangle,$$

which represent a length 2 scheme in $\Sigma(2, n-1, \mathbf{1})$. □

Proposition 4.3. *For $k \geq 2$, the number of irreducible components of $\hat{\mathcal{G}}_{\mathcal{Hilb}^m(X_n), [J]}$ is*

$$(33) \quad n + \sum_{i=1}^{\min\{k, n-2\}} \binom{k}{i} = \begin{cases} n + 2^k - 1 & \text{if } k \leq n-2 \\ n + 2^{n-1} - 2 & \text{if } k = n-1 \\ 2^k - 2 & \text{if } k = n \end{cases},$$

and each irreducible component corresponds to the ideals in Theorem B.8. A generic point in the components corresponding to \mathcal{K}_a represents length m schemes that are points in

$$\mathcal{Hilb}^{u_a}(L_a) \times \prod_{i \in [n] \setminus \{a\}} \mathcal{Hilb}^{u_i-1}(L_i).$$

In particular, points in the components corresponding to \mathcal{K}_a are smoothable. A generic point in the components corresponding to \mathcal{G}_S represents length m schemes that are points in

$$(34) \quad \prod_{i \in S} \mathcal{Hilb}^{u_i-2}(L_i) \times \prod_{i \in [k] \setminus S} \mathcal{Hilb}^{u_i-1}(L_i) \times \Sigma(|S| + 1, n - |S|, \mathbf{1}).$$

Proof. We proceed as in the proof of Theorem 4.2. We use the isomorphism of Theorem B.2 to translate the conditions imposed by the ideals in Theorem B.8 to the ring S_k/J_k and the generators (27). Let K be an ideal in the irreducible component corresponding to the ideal \mathcal{G}_S for S as in Theorem B.8. We may write the generators of K as in (27). To simplify these generators, we consider the polynomials

$$f'_i := \begin{cases} x_i^{u_i-2} + \sum_{l=2}^{u_i-1} \alpha_{i,i,l} x_i^{l-2} & \text{for } i \in S, \\ x_i^{u_i-1} + \sum_{l=1}^{u_i-1} \alpha_{i,i,l} x_i^{l-1} & \text{for } i \in [k] \setminus S. \end{cases}$$

Modulo the pullback of \mathcal{G}_S via the isomorphism in Theorem B.2, we may write the generators of K as

$$\begin{aligned} f_i &= x_i^{u_i} + \sum_{l=2}^{u_i-1} \alpha_{i,i,l} x_i^l = x_i^2 f'_i & \text{for } i \in S, \\ f_i &= x_i^{u_i} + \sum_{l=1}^{u_i-1} \alpha_{i,i,l} x_i^l + \sum_{j \in S} \alpha_{i,j,u_j-1} x_j f'_j & \text{for } i \in [k] \setminus S, \\ f_i &= x_i + \sum_{j \in S} \alpha_{i,j,u_j-1} x_j f'_j & \text{for } i \notin [k]. \end{aligned}$$

Now, we analyze the scheme defined by these equations. Assume first that $f'_i = 0$ for some $i \in S$. This implies that $0 = x_j f'_i = \alpha_{i,i,2} x_j$ for $j \neq i$. We get a component of K the form $\langle f'_i, x_j : j \neq i \rangle$ for $i \in S$ which represent a point in $\mathcal{Hilb}^{u_i-2}(L_i)$. Assume on the contrary that $f'_i \neq 0$ for every $i \in S$. Then, $x_i^2 = 0$ for every $i \in S$. The generators of K modulo this condition are

$$\begin{aligned} f_i &= x_i^{u_i} + \sum_{l=1}^{u_i-1} \alpha_{i,i,l} x_i^l + \sum_{j \in S} \alpha_{i,j,u_j-1} \alpha_{i,i,2} x_j & \text{for } i \in [k] \setminus S, \\ f_i &= x_i + \sum_{j \in S} \alpha_{i,j,u_j-1} \alpha_{i,i,2} x_j & \text{for } i \notin [k]. \end{aligned}$$

For $i \in [k] \setminus S$ we get that $0 = x_i f_i = x_i^2 f'_i$. If $f'_i = 0$, then, multiplying by x_j for $j \neq i$ we get that $x_j = 0$ for $j \neq i$. Therefore, we obtained the ideal $\langle f'_i, x_j : j \neq i \rangle$ for $i \in [k] \setminus S$, which represent a point in $\mathcal{Hilb}^{u_i-1}(L_i)$. If on the contrary $f'_i \neq 0$ for every $i \in [k]$. Since $x_i^2 f'_i = 0$, we deduce that $x_i^2 = 0$ for every $i \in [k]$. Therefore, we obtain the ideal

$$\begin{aligned} &\langle x_i^2 : i \in [k] \rangle + \langle \alpha_{i,i,1} x_i + \sum_{j \in S} \alpha_{i,j,u_j-1} \alpha_{i,i,2} x_j : i \in [k] \setminus S \rangle + \langle x_i + \sum_{j \in S} \alpha_{i,j,u_j-1} \alpha_{i,i,2} x_j : i \notin [k] \rangle \\ &= \langle \alpha_{i,i,1} x_i + \sum_{j \in S} \alpha_{i,j,u_j-1} \alpha_{i,i,2} x_j : i \in [k] \setminus S \rangle + \langle x_i + \sum_{j \in S} \alpha_{i,j,u_j-1} \alpha_{i,i,2} x_j : i \notin [k] \rangle, \end{aligned}$$

which represents a point in $\Sigma(|S| + 1, n - |S|, \mathbf{1})$. We conclude that K defines a length m scheme which corresponds to a point in

$$\prod_{i \in S} \mathcal{Hilb}^{u_i-2}(L_i) \times \prod_{i \in [k] \setminus S} \mathcal{Hilb}^{u_i-1}(L_i) \times \Sigma(|S| + 1, n - |S|, \mathbf{1}).$$

□

We can relate Theorem 4.2 and Theorem 4.3 with the combinatorics in Section 2. For $k = 1$, the ideal $J = \langle x_1^m, x_2, \dots, x_n \rangle$ is the intersection of the $n + 1$ irreducible components of the Hilbert scheme described in Theorem 4.2. The ideal J corresponds to the vertex $(m - 1)e_1$ of the simplex $(m - 1)\Delta_{n-1}$. The only hypersimplex of the hypersimplicial complex $\mathcal{K}_n^{[m]}$ containing this vertex is $\Delta_{1,n} + (m - 2)e_1$. The component of the form (32) corresponds to this translated hypersimplex. The translation by $(m - 2)e_1$ geometrically corresponds to the factor $\mathcal{Hilb}^{m-2}(L_1)$ of the irreducible components. The rest of the irreducible components containing the point $[J]$ are smoothable and cannot be seen from the complex $\mathcal{K}_n^{[m]}$. Therefore, these components do not come from the punctual Hilbert scheme. In particular, the number of irreducible components of $\mathcal{Hilb}^m(X_n)$ and $\mathcal{Hilb}_0^m(X_n)$ that contain $[J]$ is different. Similarly, for $k \geq 2$, we can associate to the components of the form (34) corresponding to \mathcal{G}_S the hypersimplex $\Delta_{n,|S|} + \mathbf{u} - \mathbf{e}_S - \mathbf{1}$. Such a hypersimplex corresponds to the component $\Sigma(m, n - |S|, \mathbf{u} - \mathbf{e}_S)$ of the punctual Hilbert scheme. This component is exactly the intersection of the punctual Hilbert scheme and the component (34). For $k = n - 1$ and $a \notin [k]$, we can associate to the component corresponding to \mathcal{K}_a in Theorem 4.3 the hypersimplex $\Delta_{n-1,n} + \mathbf{u} - \mathbf{e}_{[n] \setminus \{a\}} - \mathbf{1}$. Similarly, for $n = k$ and $a \in [k]$, we can associate to the component corresponding to \mathcal{K}_a in Theorem 4.3 the hypersimplex $\Delta_{n-1,n} + \mathbf{u} - \mathbf{e}_{[n] \setminus \{a\}} - \mathbf{1}$.

Remark 4.4. Fix the ideal $J = \langle x_1^{u_1}, \dots, x_k^{u_k}, x_{k+1}, \dots, x_n \rangle$ with $u_i \geq 2$ for $i \in [k]$ and $|\mathbf{u}| = m + n - 1$. Then, $\mu([J]) = \mathbf{u}$ is a vertex of $\mathcal{K}_n^{[m]}$ that is contained in the relative interior of a $(k - 1)$ -dimensional face of $(m - 1) \cdot \Delta_{n-1}$. The number of hypersimplices containing the vertex μ is exactly $2^k - 1$ if $k < n$ and $2^k - 2$ if $k = n$. Therefore, this number is the number of irreducible components of $\mathcal{Hilb}_0^m(X_n)$ containing $[J]$. On the other hand, by Theorem 4.2 and Theorem 4.3, the number of irreducible components of $\mathcal{Hilb}^m(X_n)$ containing $[J]$ is given by (33). For $k \leq n - 1$, this number of irreducible components of $\mathcal{Hilb}^m(X_n)$ and $\mathcal{Hilb}_0^m(X_n)$ differs by n if $k \leq n - 2$ and by $n - 1$ if $k = n - 1$. These extra components that do not appear in $\mathcal{Hilb}_0^m(X_n)$ are described in Theorem 4.3 and they are smoothable irreducible components of $\mathcal{Hilb}^m(X_n)$ that contain $[J]$. For $k = n$, the number of irreducible components of the local and punctual Hilbert scheme coincides.

Now, we show the local structure of $\mathcal{Hilb}_0^m(X_n)$ inside the stalk $\hat{\mathcal{O}}_{\mathcal{Hilb}^m(X_n), [J]}$.

Proposition 4.5. *Let $1 \leq k \leq n$ and let $\mathbf{u} \in \mathbb{Z}_{\geq 1}^n$ such that $|\mathbf{u}| = m + n - 1$, $u_i \geq 2$ for $1 \leq i \leq k$ and $u_i = 1$ for $k + 1 \leq i \leq n$. Then, the completion of the stalk of the punctual Hilbert scheme at $J = \langle x_1^{u_1}, \dots, x_k^{u_k}, x_{k+1}, \dots, x_n \rangle$, denoted by $\widehat{\mathcal{O}}_{\text{Hilb}_0^m(X_n), [J]}$ is isomorphic to the completion of the quotient*

$$(35) \quad \frac{\mathbb{Z}[\alpha_{i,j} : i \in [n], j \in [k] \setminus \{i\}]}{\langle \alpha_{i,j} \alpha_{j,r} : j \in [k], i \in [n] \setminus \{i\}, r \in [k] \setminus \{j\} \rangle}$$

localized at the origin. Through this isomorphism, the irreducible components of $\widehat{\mathcal{O}}_{\text{Hilb}_0^m(X_n), [J]}$ corresponds to the ideals of (35)

$$(36) \quad \mathcal{G}_{[k], T} := \langle \alpha_{i,j} : i, j \in [k] \setminus T, i \neq j \rangle + \langle \alpha_{i,j} : j \in T \text{ and } i \in [n] \setminus \{j\} \rangle$$

for $T \subsetneq [k]$ if $k \neq n$, or $\emptyset \neq T \subsetneq [k]$ if $k = n$. Moreover, the ideal $\mathcal{G}_{[k], T}$ represents the irreducible component

$$\Sigma(m, n - k + |T|, \mathbf{u} - \mathbf{e}_{[k] \setminus T})$$

of the punctual Hilbert scheme.

Proof. By Theorem 4.1, $\widehat{\mathcal{O}}_{\text{Hilb}_0^m(X_n), [J]}$ corresponds to the ideal of S_k/I_k describing the deformations of $\mathbb{V}(J)$ supported at the origin. Let K be an ideal corresponding to a point in S_k/J_k . In other words, the generators of K are given by (27). We need to check when the ideal K is an ideal supported at the origin. The polynomial $x_i f_i = x_i(x_i^{u_i} + A_i + \sum_{l=1}^{u_i-1} \alpha_{i,i,l} x_i^l)$ is a polynomial in K . If K is supported only at the origin, then $x_i = 0$ must be the only solution of $x_i f_i = 0$. We deduce that $A_i = \alpha_{i,i,1} = \dots = \alpha_{i,i,u_i-1} = 0$. Modulo this relation, we get that the generators of K are

$$f_i = x_i^{u_i} + \sum_{j \in [k] \setminus \{i\}} \alpha_{i,j,u_j-1} x_j^{u_j-1},$$

where $\alpha_{i,j,u_j-1} \alpha_{j,r,u_r-1} = 0$ for every $i \in [n]$, $j, r \in [k] \setminus \{i\}$ and $j \neq r$. Together with the generators of J_k and identifying α_{i,j,u_j-1} with $\alpha_{i,j}$ we get the ring (35). The primary decomposition of ideal in (35) is given by Theorem B.4 for $S = [k]$. Now fix an ideal $\mathcal{G}_{[k], T}$ in the primary decomposition of (35). Modulo $\mathcal{G}_{[k], T}$, the generators of the length m ideal K are

$$f_i = \begin{cases} x_i^{u_i} & \text{for } i \in [k] \setminus T, \\ x_i^{u_i} + \sum_{j \in [k] \setminus T} \alpha_{i,j} x_j^{u_j-1} & \text{for } i \in T, \\ x_i^{u_i} + \sum_{j \in [k] \setminus T} \alpha_{i,j} x_j^{u_j-1} & \text{for } i \notin [k]. \end{cases}$$

Note that the generators f_i for $i \in [k] \setminus T$ are not required since they are recovered from the multiplication $x_i f_j$ for $j \notin [k] \setminus T$. We deduce that the ideals in the component corresponding to $\mathcal{G}_{[k], T}$ are given by $n - k + |T|$ linearly independent polynomials in the vector space

$$\langle x_i^{u_i} : i \in T \rangle_{\mathbb{C}} + \langle x_i^{u_i-1} : i \in [k] \setminus T \rangle_{\mathbb{C}} + \langle x_i : i \notin [k] \rangle_{\mathbb{C}}.$$

Therefore, the component $\mathcal{G}_{[k], T}$ correspond with the irreducible component

$$\Sigma(m, n - k + |T|, \mathbf{u} - \mathbf{e}_{[k] \setminus T})$$

of the punctual Hilbert scheme. □

Theorem 4.5 allows us to carry out the local study of $\text{Hilb}_0^m(X_n)$, for the reducedness of this scheme and the transversality of the intersection of the irreducible components. To derive these results, we need first the following lemmas.

Lemma 4.6. *Let $[J] \in \text{Hilb}_0^m(X_n)$ such that $\mu([J])$ is a vertex of $\mathcal{K}_n^{[m]}$. Then, $[J]$ is a reduced point of $\text{Hilb}^m(X_n)$ and $\text{Hilb}_0^m(X_n)$.*

Proof. To show that $[J]$ is a reduced point of $\text{Hilb}^m(X_n)$, it is enough to show that the stalk $\mathcal{O}_{\text{Hilb}^m(X_n), [J]}$ is reduced. By Lemma [Sta25, Lemma 07NZ], it is enough to check that the completion of this stalk is reduced. By Theorem 4.1 and Theorem B.2, this completion is the quotient S_k/\mathcal{G}_k (see (55), (56) and (57) for the relevant definitions). Since the primary decomposition of the ideal \mathcal{G}_k calculated in Theorem B.3 and Theorem B.8 is given by prime ideals, the ideal \mathcal{G}_k is radical. Hence, S_k/\mathcal{G}_k is a reduced ring.

Similarly, to show that $[J]$ is a reduced point of $\text{Hilb}_0^m(X_n)$, it is enough to check that the ring (35) is reduced. The primary decomposition of the ideal in (35) is given in Equation (36) in Theorem 4.5. The proof follows from the fact that the ideals (36) are prime. \square

Lemma 4.7. *Let $[J] \in \mathcal{Hilb}^m(X_n)$, then there exists $[J_0] \in \mathcal{Hilb}_0^m(X_n)$ in the closure of the $(\mathbb{C}^*)^n$ -orbit of $[J]$ such that $\mu([J_0])$ is a vertex of $\mathcal{K}_n^{[m]}$.*

Proof. The statement of the Lemma is independent of the schematic structure of the Hilbert scheme. Therefore, we can replace $\mathcal{Hilb}^m(X_n)$ by $\text{Hilb}^m(X_n)$. Assume first that J is supported at $\mathbf{0}$, i.e. $[J] \in \text{Hilb}_0^m(X_n)$. By Theorem 1.13, it is enough to check the analogous statement for Grassmannians. Let $[E] \in \text{Gr}(l, n)$ be generated by the image of an $n \times l$ matrix $A = (a_{i,j})$. Without loss of generality, we may assume that A is a block matrix of the form

$$A = \begin{pmatrix} \text{Id}_l \\ A' \end{pmatrix}.$$

For $t = (t_1, \dots, t_n) \in (\mathbb{C}^*)^n$, $t \cdot [E]$ is the vector subspace generated by image of the matrix $t \cdot A = \text{Diag}(t_1, \dots, t_n) \cdot A$. Therefore, taking the limit when t_{l+1}, \dots, t_n goes to zero we get the linear subspace $\langle e_1, \dots, e_l \rangle$, which is a torus invariant point. Therefore, it corresponds to a vertex of $\Delta_{l,n}$ via the moment map, and it lies in the closure of the orbit of $[E]$.

Assume now that $[J] \notin \text{Hilb}_0^m(X_n)$. Since the statement of the lemma holds for ideals in the punctual Hilbert scheme, it is enough to check that in the closure of the $(\mathbb{C}^*)^n$ -orbit of $[J]$ there is a point in the punctual Hilbert scheme. Consider a one parameter family $\lambda : \mathbb{C}^* \rightarrow (\mathbb{C}^*)^n$ such that the limit of $\lambda(t)$ when t goes to 0 is $\mathbf{0}$. Then, for any point q in X_n , the limit of $\lambda(t) \cdot q$ when t goes to 0 is the singularity $\mathbf{0}$. Therefore, the limit of $\lambda(t) \cdot [J]$ when t goes to 0 is a length m ideal supported at $\mathbf{0}$. We conclude that in the closure of the $(\mathbb{C}^*)^n$ -orbit of $[J]$ there is a point in $\text{Hilb}_0^m(X_n)$. \square

Remark 4.8. Theorem 4.2 and Theorem 4.3 and Theorem 4.7 provide an alternative proof to Theorem 3.3: Let Z be an elementary component of $\mathcal{Hilb}^m(X_n)$. By Theorem 4.7, Z contains a point $[J] \in \text{Hilb}_0^m(X_n)$ corresponding to a vertex of $\mathcal{K}_n^{[m]}$. Since Z is an irreducible component of $\mathcal{Hilb}^m(X_n)$, it should correspond to an irreducible component of the completion of the stalk of $[J]$ calculated in Theorem 4.1. Theorem 4.2 and Theorem 4.3 give a geometrical interpretation to each of the irreducible components of this stalk. The only cases where these irreducible components are entirely contained in $\mathcal{Hilb}_0^m(X_n)$ are exactly the ones described in Theorem 3.3.

Proposition 4.9. *Let C be an irreducible curve whose singularities are all rational n -fold singularities. Then, for $m \geq 2$, the Hilbert scheme $\mathcal{Hilb}^m(C)$ is Cohen-Macaulay if and only if $n \leq 3$.*

Proof. First, note that for $n \geq 4$, the Hilbert scheme $\mathcal{Hilb}^m(C)$ is not equidimensional, and hence, it is not Cohen-Macaulay. Assume that $n \leq 3$. Without loss of generality, we may assume that $C = X_n$. Let $[J] \in \text{Hilb}^m(X_n)$, we need to check that the completion of the stalk of $[J]$ is Cohen-Macaulay. By Theorem 4.7, we may assume that $[J] \in \text{Hilb}_0^m(X_n)$ and $\mu([J])$ is a vertex of $\mathcal{K}_n^{[m]}$. By Theorem 4.1 and Theorem B.2, it is enough to check that the ring $\mathcal{S}_k/\mathcal{G}_k$ is Cohen-Macaulay for $n \leq 3$ and $k \in [n]$. A computation in Macaulay2 [GS] shows that $\mathcal{S}_k/\mathcal{G}_k$ is Cohen-Macaulay for $n \leq 3$ and $k \in [n]$. \square

Theorem 4.10. *The punctual Hilbert scheme $\mathcal{Hilb}_0^m(X_n)$ is reduced and isomorphic to \mathcal{G}_n^m .*

Proof. First, we prove that $\mathcal{Hilb}_0^m(X_n)$ is reduced. Assume on the contrary that $[J]$ is a nonreduced point in $\mathcal{Hilb}_0^m(X_n)$. This implies that $(\mathbb{C}^*)^n$ -orbit of $[J]$ is nonreduced, and hence, the closure of the $(\mathbb{C}^*)^n$ -orbit of $[J]$ is nonreduced. By Theorem 4.7, there exists an ideal $[J']$ in the closure of the orbit such that $\mu([J'])$ is a vertex of $\mathcal{K}_n^{[m]}$. This is a contradiction since by Theorem 4.6, $[J']$ is a reduced point.

We now check that $\mathcal{Hilb}_0^m(X_n)$ is isomorphic to \mathcal{G}_n^m . Note that by the universal property of pushouts, we have a map $\varphi : \mathcal{G}_n^m \rightarrow \mathcal{Hilb}_0^m(X_n)$ that on each component $\text{Gr}(l, \Lambda_{\mathbf{u}})$ of \mathcal{G}_n^m it is the map $\varphi_{l,\mathbf{u}}$. Therefore, φ is injective and its restriction to each irreducible component of \mathcal{G}_n^m and $\mathcal{Hilb}_0^m(X_n)$ leads to an isomorphism. Therefore, to check that φ is an isomorphism, it is enough to check what happens at the intersections. In other words, we need to check that in $\mathcal{Hilb}_0^m(X_n)$ the intersection of the Grassmannians $\Sigma(m, l, \mathbf{u})$ is locally the intersection of affine spaces. By Theorem 4.7

and using the torus action, it is enough to check this condition around an ideal $[J]$ corresponding to a vertex of $\mathcal{K}_n^{[m]}$. Then the proof follows from Theorem 4.5 since all the components of the completion of the stalk at $[J]$ are affine spaces. \square

Using the same technique as in Theorem 4.10 we derive that $\mathcal{Hilb}^m(X_n)$ is reduced.

Theorem 4.11. *The Hilbert scheme of points $\mathcal{Hilb}^m(X_n)$ is reduced.*

Proof. Let $[I] \in \mathcal{Hilb}^m(X_n)$ be a nonreduced point. Then the $(\mathbb{C}^*)^n$ -orbit of $[I]$ is nonreduced, and hence, the closure of the $(\mathbb{C}^*)^n$ -orbit is also nonreduced. By Theorem 4.7, the closure of this orbit contains an ideal $[J]$ in $\mathcal{Hilb}_0^m(X_n)$ associated to a vertex of \mathcal{K}_n^m via the moment map. This is a contradiction since by Theorem 4.6, $[J]$ is a reduced point of $\mathcal{Hilb}^m(X_n)$. Thus, we conclude that $[I]$ is a reduced point of $\mathcal{Hilb}^m(X_n)$. \square

As a consequence of Theorems 4.10 and 4.11, we have that $\text{Hilb}_0^m(X_n) = \mathcal{Hilb}_0^m(X_n)$ and $\text{Hilb}^m(X_n) = \mathcal{Hilb}^m(X_n)$. Therefore, for the rest of the paper, we will use the notation $\mathcal{Hilb}_0^m(X_n)$ and $\mathcal{Hilb}^m(X_n)$. The following improvement of Theorem 3.6 where we no longer require to take the reduced structure follows from Theorem 4.11 together with Theorem 3.3.

Corollary 4.12. *Let C be an irreducible curve with a unique rational n -fold singularity. Then, the irreducible components of $\mathcal{Hilb}^m(C)$ are*

$$\mathcal{Hilb}_{\text{sm}}^m(C) \text{ and } \mathcal{Hilb}^{m,m'}(C) \text{ for } 2 \leq m' \leq \min\{m, n-1\}.$$

The number of irreducible components is $\min\{n-1, m\}$. Moreover, these irreducible components are birational to

$$\text{Sym}^m(C) \quad \text{or} \quad \text{Sym}^{m-m'}(C) \times \text{Gr}(n+1-m', n) \quad \text{for } 2 \leq m' \leq \min\{m, n-1\}.$$

We can generalize Theorem 4.12 to irreducible curves with several rational fold like singularities. Given integers $k \in \mathbb{N}$, and $m, n_1, \dots, n_k \in \mathbb{Z}_{\geq 2}$, we define the number $\rho(k, m, n_1, \dots, n_k)$ as the cardinality of the set

$$S(k, m, n_1, \dots, n_k) := \{\mathbf{m} = (m_1, \dots, m_k) \in \mathbb{Z}_{\geq 0}^k : |\mathbf{m}| \leq m, m_i \neq 1 \text{ and } 0 \leq m_i \leq \min\{m, n_i-1\}\}.$$

Corollary 4.13. *Let C be an irreducible curve whose singularities are $p_1, \dots, p_k \in C$ where p_i is a rational n_i -fold singularity. Then, the number of irreducible components of $\mathcal{Hilb}^m(C)$ is $\rho(k, m, n_1, \dots, n_k)$. Moreover, $\mathcal{Hilb}^m(C)$ is reduced and its irreducible components of $\mathcal{Hilb}^m(C)$ are birational to*

$$\text{Sym}^m(C) \quad \text{and} \quad \text{Sym}^{m-|\mathbf{m}|}(C) \times \prod_{i=1}^k \Sigma(m_i, n_i + 1 - m_i, \mathbf{1})$$

for $\mathbf{m} \in S(k, m, n_1, \dots, n_k)$.

Proof. We first show that $\mathcal{Hilb}^m(C)$ is reduced. Given a point $[J] \in \mathcal{Hilb}^m(C)$, we can decompose $\mathbb{V}(J)$ as $\mathbb{V}(J) = \mathbb{V}(J_0) \cup \mathbb{V}(J_1) \cup \dots \cup \mathbb{V}(J_k)$ where J_0 is supported at the smooth locus of C and J_i is supported at p_i for every $i \in [k]$. Let m_i be the length of J_i . Around $[J]$, étale locally $\mathcal{Hilb}^m(C)$ is isomorphic to the product $\mathcal{Hilb}_{\text{sm}}^{m_0}(C) \times \mathcal{Hilb}^{m_1}(C) \times \dots \times \mathcal{Hilb}^{m_k}(C)$ for $([J_0], [J_1], \dots, [J_k])$ (cf. [Jel19]). Note that $[J_0]$ is reduced in $\mathcal{Hilb}_{\text{sm}}^{m_0}(C)$. The punctual Hilbert scheme $\mathcal{Hilb}_{p_i}^{m_i}(C)$ is isomorphic to $\mathcal{Hilb}_0^{m_i}(X_{n_i})$. Therefore, we may see $[J_i]$ also as a point in $\mathcal{Hilb}_0^{m_i}(X_{n_i}) \subset \mathcal{Hilb}^{m_i}(X_{n_i})$ through this isomorphism. Then, the completion of the stalk of $[J_i]$ in $\mathcal{Hilb}^{m_i}(C)$ is isomorphic to the completion of the stalk of $[J_i]$ in $\mathcal{Hilb}^{m_i}(X_{n_i})$ which is reduced by Theorem 4.11.

Next, we calculate the number of irreducible components. An elementary component of $\mathcal{Hilb}^m(C)$ parametrizes length m subschemes supported at one fixed singular point p_i . Therefore, elementary components of $\mathcal{Hilb}^m(C)$ are in bijection with elementary components of $\mathcal{Hilb}^m(X_{n_i})$ for $i \in [k]$. These components correspond to the vectors in $S(k, m, n_1, \dots, n_k)$ of the form $m\mathbf{e}_i$. By (23) we obtain that the non-elementary components of $\mathcal{Hilb}^m(C)$ correspond to the closure of the image of the map

$$(37) \quad \begin{aligned} \mathcal{Hilb}_{\text{sm}}^{m-|\mathbf{m}|}(C \setminus \{p_1, \dots, p_k\}) \times \prod_{i=1}^k \Sigma(m_i, n_i + 1 - m_i, \mathbf{1}) &\longrightarrow \mathcal{Hilb}^m(C) \\ ([J_0], [J_1], \dots, [J_k]) &\longmapsto [\mathbb{V}(J_0) \cup \dots \cup \mathbb{V}(J_k)] \end{aligned}$$

for $\mathbf{m} \in S(k, m, n_1, \dots, n_k)$ with $\mathbf{m} \neq m\mathbf{e}_i$ and $i \in [k]$. Here, $\Sigma(m_i, n_i + 1 - m_i, \mathbf{1})$ is seen as the corresponding elementary component in the punctual Hilbert scheme $\mathcal{Hilb}_{p_i}^{m_i}(C)$. Moreover, the map (37) is birational onto its image. Note that for $\mathbf{m} = \mathbf{0}$, the corresponding component is the smoothable component. In particular, we conclude that the irreducible components of $\mathcal{Hilb}^m(C)$ are in bijection with vectors in $S(k, m, n_1, \dots, n_k)$. \square

Remark 4.14. One can derive a formula for the cardinality of $\rho(k, m, n_1, \dots, n_k)$. To do so, we introduce the number $\chi(k, m, n_1, \dots, n_k)$ as the cardinality of the set

$$\{\mathbf{m} \in \mathbb{Z}_{\geq 0}^k : 0 \leq |\mathbf{m}| \leq m \text{ and } 0 \leq m_i \leq n_i\}.$$

We set $\chi(0, m) = 1$ for $k = 0$. Using the Exclusion-Inclusion formula, one may check that

$$\chi(k, m, n_1, \dots, n_k) = \sum_{J \subseteq [k]} (-1)^{|J|} \binom{m - \sum_{j \in J} (n_j + 1) + k}{k}.$$

Decompose the set $S(k, m, n_1, \dots, n_k)$ as

$$S = \bigsqcup_{J \subseteq [k]} S(k, m, n_1, \dots, n_k) \cap \{m_i = 0 : i \notin J\} \cap \{m_i \geq 2 : i \in J\}.$$

For $J \subseteq [k]$, the cardinality of the corresponding set in the above disjoint union is $\chi(|J|, m - 2, n_i - 2 : i \in J)$. We conclude that the number of irreducible components in Theorem 4.13 is

$$\rho(k, m, n_1, \dots, n_k) = \sum_{J \subseteq [k]} \chi(|J|, m - 2, n_i - 2 : i \in J) = \sum_{J \subseteq [k]} \sum_{I \subseteq J} (-1)^{|I|} \binom{m - 2 - \sum_{i \in I} (n_i - 1) + |J|}{|J|}.$$

5. SINGULARITIES

The goal of this section is to describe the singular locus of $\mathcal{Hilb}^m(X_n)$. For doing this, we will rely heavily in Combinatorics, in particular the hypersimplicial complex, as given in Sections A and 2.

5.1. Singular Locus. We will first compute the singular locus of $\mathcal{Hilb}^m(X_n)$. By a classical result in deformation theory (cf. [Ser06]), the dimension of the tangent space of $\mathcal{Hilb}^m(X_n)$ at the point $[J]$ is given by

$$\dim_{\mathbb{C}} T_{[J]} \mathcal{Hilb}^m(X_n) = \dim_{\mathbb{C}} \text{Hom}_R(J, R/J).$$

To apply this formula, we need the syzygies of J , which are computed in Theorem B.1. Using this lemma, we will describe the singular locus of $\mathcal{Hilb}^m(X_n)$ by the combinatorics of $\mathcal{K}_n^{[m]}$ using the notion of singular face introduced in Section 2.

Proposition 5.1. *Let $[J] \in \mathcal{Hilb}_0^m(X_n)$. Then the following are equivalent:*

- (1) $[J]$ is a singular point of $\mathcal{Hilb}^m(X_n)$.
- (2) $\mu_m([J])$ lies in a singular face of $\mathcal{K}_n^{[m]}$.
- (3) J admits a minimal generator of the form $x_i^{u_i}$ with $u_i \geq 2$ or J is minimally generated by $\langle f, x_i : i \in S \rangle$ for $f \in \langle x_j^{u_j} : j \notin S \rangle_{\mathbb{C}}$ and $\emptyset \subsetneq S \subset [n]$.

Proof. By Proposition Theorem 2.11, (2) and (3) are equivalent. So it is enough to show that (1) and (3) are equivalent. By Theorem 1.4 there exists $1 \leq l \leq n$ and $\mathbf{u} \in \mathbb{Z}_{\geq 1}$ such that J is minimally generated by f_1, \dots, f_l where

$$\begin{pmatrix} f_1 \\ \vdots \\ f_l \end{pmatrix} = A \begin{pmatrix} x_1^{u_1} \\ \vdots \\ x_n^{u_n} \end{pmatrix}.$$

and A is a size $l \times n$ matrix as in Theorem 1.4. Every $\varphi \in \text{Hom}_R(J, R/J)$ is uniquely determined by l elements $\alpha_1, \dots, \alpha_l \in R/J$ satisfying the syzygies (52) of J , where $\alpha_j := \varphi(f_j)$ for $1 \leq j \leq l$. Write α_j as follows

$$\alpha_j = \alpha_{j,0} + \sum_{r=1}^n \sum_{s=1}^{u_r} \alpha_{j,r,s} x_r^s.$$

For $1 \leq i \leq n$ and $1 \leq j < k \leq l$ we have the following equalities:

$$(38) \quad 0 = A_{k,i}x_i\alpha_j - A_{j,i}x_i\alpha_k = A_{k,i}\alpha_{j,0}x_i + \sum_{s=1}^{u_i-1} A_{k,i}\alpha_{j,i,s}x_i^{s+1} - \sum_{s=1}^{u_i-1} A_{j,i}\alpha_{k,i,s}x_i^{s+1} =$$

$$(A_{k,i}\alpha_{j,0} - A_{j,i}\alpha_{k,0})x_i + \sum_{s=1}^{u_i-1} (A_{k,i}\alpha_{j,i,s} - A_{j,i}\alpha_{k,i,s})x_i^{s+1}.$$

We first assume that J does not admit a minimal generator of the form $x_i^{u_i}$. Then, none of the terms in (38) vanishes in R/J , and we obtain that $\alpha_1, \dots, \alpha_l$ satisfy the relations

$$(39) \quad \begin{aligned} A_{k,i}\alpha_{j,0} - A_{j,i}\alpha_{k,0} &= 0 & \text{for } 1 \leq j < k \leq l \text{ and } 1 \leq i \leq n, \\ A_{k,i}\alpha_{j,i,s} - A_{j,i}\alpha_{k,i,s} &= 0 & \text{for } 1 \leq j < k \leq l, 1 \leq i \leq n \text{ and } 2 \leq s \leq u_i. \end{aligned}$$

Rewrite the first type of relations in (39) as

$$(\alpha_{k,0}, -\alpha_{j,0}) \begin{pmatrix} A_{j,1} & \cdots & A_{j,n} \\ A_{k,1} & \cdots & A_{k,n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since the matrix A has maximum rank we get that the matrix

$$\begin{pmatrix} A_{j,1} & \cdots & A_{j,n} \\ A_{k,1} & \cdots & A_{k,n} \end{pmatrix}$$

has rank 2. This implies that $\alpha_{j,0} = 0$ for all $1 \leq j \leq l$. The second type of relations in (39) can be written as the 2×2 minors of the matrix

$$(40) \quad \begin{pmatrix} A_{1,i} & \alpha_{1,i,s} \\ \vdots & \vdots \\ A_{l,i} & \alpha_{l,i,s} \end{pmatrix}$$

for $1 \leq i \leq n$ and $2 \leq s \leq u_i$. Since A has no vanishing columns, there exists $1 \leq j \leq l$ such that $A_{j,i}$ is nonzero. Thus, the 2×2 minors of the matrix (40) give $l-1$ relations among $\alpha_{1,i,s}, \dots, \alpha_{l,i,s}$ for $1 \leq i \leq n$ and $2 \leq s \leq u_i$. Hence $\alpha_1, \dots, \alpha_l$ satisfy $l + \sum_{i=1}^n (u_i - 1)(l - 1) = l + (|\mathbf{u}| - n)(l - 1)$ relations. We conclude that the dimension of the tangent space at $[J]$ is

$$lm - l - (|\mathbf{u}| - n)(l - 1) = lm - l - (m + l - 1 - n)(l - 1) = l(n - l) + (m + l - 1 - n).$$

On the other hand, $[J]$ is contained in $\mathcal{Hilb}^{m,n+1-l}(X_n)$, which has dimension $l(n - l) + (m + l - 1 - n)$. Thus $[J]$ is a smooth point of $\mathcal{Hilb}^m(X_n)$.

Now, assume that J admits a generator of the form $x_r^{u_r}$. If $u_r \geq 2$, then J is in the intersection of two irreducible components of $\mathcal{Hilb}^m(X_n)$ and therefore J is singular a singular point of $\mathcal{Hilb}^m(X_n)$. Hence we can assume that $u_r = 1$ for every such generator. Let f_1, \dots, f_a be the minimal generators of J that are not of the form x_r . In other words, the minimal generators of J are $f_1, \dots, f_a, x_{n-l+a+1}, \dots, x_n$, and therefore we can write these generators as

$$\begin{pmatrix} f_1 \\ \vdots \\ f_a \\ x_{n-l+a+1} \\ \vdots \\ x_n \end{pmatrix} = \left(\begin{array}{c|c} A & \mathbf{0} \\ \hline \mathbf{0} & \text{Id}_{l-a} \end{array} \right) \begin{pmatrix} x_1^{u_1} \\ \vdots \\ x_{n-l+a}^{u_{n-l+a}} \\ x_{n-l+a+1} \\ \vdots \\ x_n \end{pmatrix},$$

where A has maximal rank and it has no vanishing row or column. Hence the relations among $\alpha_1, \dots, \alpha_l$ in R/J are

$$(41) \quad \begin{aligned} x_i\alpha_j &= 0 & \text{for } a+1 \leq j \leq l \text{ and } i \neq n-l+j, \\ x_i\alpha_j &= 0 & \text{for } 1 \leq j \leq a \text{ and } n-l+a+1 \leq i \leq n, \\ A_{k,i}x_i\alpha_j - A_{j,i}x_i\alpha_k &= 0 & \text{for } 1 \leq j < k \leq a \text{ and } 1 \leq i \leq n-l+a. \end{aligned}$$

Since $x_i \in J$ for $n - l + a + 1 \leq i \leq n$, the left-hand side of the second equation in (41) is zero in R/J and is therefore redundant. Similarly, the first relation is identically zero in R/J for $n - l + a + 1 \leq i \leq n$. We obtain

$$(42) \quad \begin{aligned} x_i \alpha_j &= 0 && \text{for } a + 1 \leq j \leq l \text{ and } 1 \leq i \leq n - l + a, \\ A_{k,i} x_i \alpha_j - A_{j,i} x_i \alpha_k &= 0 && \text{for } 1 \leq j < k \leq a \text{ and } 1 \leq i \leq n - l + a. \end{aligned}$$

From the first equation in (42), we get that

$$\alpha_{j,0} = 0 \text{ and } \alpha_{j,i,s} = 0 \text{ for } a + 1 \leq j \leq l, 1 \leq i \leq n - l + a \text{ and } 1 \leq s \leq u_i - 1.$$

In particular, there are $(l - a)(m + l - n)$ linear equations on $\alpha_{a+1}, \dots, \alpha_l$. As in the first part of the proof, the number of linearly independent equations obtained from the second equations in (42) is

$$a + \sum_{i=1}^{n-l+a} (a - 1)(u_i - 1) = a + (a - 1)(m + l - n - 1).$$

Now, assume first that $a = 1$. In other words, J is satisfy condition (3) of Theorem 5.1. Then, we have no equation of the second type in (42). In particular, the dimension of the tangent space is

$$ml - (l - 1)(m + l - n) = l(n - l) + m + l - n.$$

On the other hand, J lies in $\mathcal{Hilb}^{m,n+1-l}(X_n)$, which has dimension $l(n - l) + m + l - n - 1$. We conclude that $[J]$ is a singular point of $\mathcal{Hilb}^m(X_n)$.

Assume now that $a \geq 2$. Then, the dimension of the tangent space at $[J]$ is

$$ml - (l - a)(m + l - n) - a - (a - 1)(m + l - n - 1) = l(n - l) + m + l - n - 1.$$

Moreover, $[J]$ is contained in $\mathcal{Hilb}^{m,n+1-l}(X_n)$ which has also dimension $l(n - l) + m + l - n - 1$. We conclude that $[J]$ is a smooth point of $\mathcal{Hilb}^m(X_n)$. \square

Theorem 5.1 characterizes the points in $\mathcal{Hilb}_0^m(X_n)$ that are singular in $\mathcal{Hilb}^m(X_n)$. The study done in Section 3, allow us to move from the punctual Hilbert scheme to the global Hilbert scheme, giving a characterization of the singular locus of $\mathcal{Hilb}^m(X_n)$.

Theorem 5.2. *Let J be a length m ideal of R and let J_1, \dots, J_k be its primary decomposition, where J_a has length m_a . Then, $[J]$ is singular in $\mathcal{Hilb}^m(X_n)$ if and only if there exists $1 \leq a \leq k$ such that $[J_a]$ is contained in $\mathcal{Hilb}_0^{u_a}(X_n)$ and J_a satisfies one of the conditions in Theorem 5.1.*

Proof. Let $[J] \in \mathcal{Hilb}^m(X_n)$ and let J_1, \dots, J_k be its primary decomposition. Let Z be the subscheme of X_n defined by J and let Z_i be the subscheme defined by J_i for $1 \leq i \leq k$. Using [Ser06, Section 4.6.5], we get that

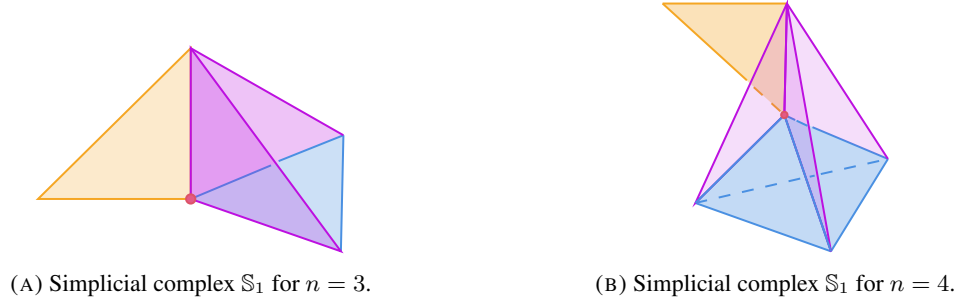
$$T_{[J]} \mathcal{Hilb}^m(X_n) = H^0(Z, N_{Z/X}) = \bigoplus_{1 \leq i \leq k} H^0(Z_i, N_{Z_i/X}) = \bigoplus_{1 \leq i \leq k} T_{[J_i]} \mathcal{Hilb}^m(X_n).$$

The proof then follows from Theorem 5.1. \square

Remark 5.3. Using the notation in Theorem 5.2, if $m_a = 1$, then $[J_a] \in \mathcal{Hilb}_0^1(X_n) = \{p\}$, which is singular. In particular, $\text{Sym}^{m-1}(X_n \setminus \{p\}) \times \{p\}$ is a subset of the singular locus of $\mathcal{Hilb}^m(X_n)$.

5.2. Local picture of the singularities. Next, we give a description of the singularities of $\mathcal{Hilb}^m(X_n)$. By Theorem 4.7, the closure of any torus orbit in $\mathcal{Hilb}^m(X_n)$ contains a point $[\mathbb{V}(J)] \in \mathcal{Hilb}_0^m(X_n)$ that corresponds to a vertex of $\mathcal{K}_n^{[m]}$ via the moment map. Therefore, it is enough to describe the singularities of $\mathcal{Hilb}^m(X_n)$ at points of the form $[J] = [\langle x_1^{u_1}, \dots, x_k^{u_k}, x_{k+1}, \dots, x_n \rangle]$ for $1 \leq k \leq n$ and $u_i \geq 2$ for $i \in [k]$. Using Theorem 4.1 and Theorem B.2, it is enough to analyze the singularity at the origin of the variety defined by the ideal \mathcal{G}_k . By Theorem B.3, Theorem B.8 and Theorem B.7, locally at the origin $\mathbb{V}(\mathcal{G}_k)$ is the union of normal toric varieties. We describe the singularity through the gluing of the polytopes associated to these toric varieties. To do so, we first do some simplification on the coordinate ring $\mathcal{S}_k/\mathcal{G}_k$. For $k = 1$, the variables $\alpha_{1,2}, \dots, \alpha_{1,m-1}$ do not appear in the ideal \mathcal{G}_1 . Therefore, we have that

$$\mathcal{S}_1/\mathcal{G}_1 \simeq \mathbb{C}[\alpha_{1,2}, \dots, \alpha_{1,m-1}] \otimes (\mathcal{S}'_1/\mathcal{G}_1),$$

FIGURE 13. Simplicial complex \mathbb{S}_1 for $n = 3, 4$.

where $\mathcal{S}'_1 = \mathbb{C}[\alpha_2, \dots, \alpha_n, A_1, \alpha_{1,1}]$ is the polynomial ring in the rest of the variables. Similarly, for $k \geq 2$, the variables $\beta_{1,s}$ for $i \in [k]$ and $2 \leq s \leq u_i - 1$ do not appear in the ideal \mathcal{G}_k . Therefore, we have that

$$\mathcal{S}_k/\mathcal{G}_k \simeq \mathbb{C}[\beta_{i,j} : 1 \leq i \leq k \text{ and } 2 \leq j \leq u_i - 1] \otimes (\mathcal{S}'_k/\mathcal{G}_k),$$

where $\mathcal{S}'_k = \mathbb{C}[\beta_1, \dots, \beta_k, \alpha_{i,j} : i \in [n], j \in [k], i \neq j]$ is the polynomial ring in the rest of the variables. In particular, the singularity type of $\mathcal{Hilb}^m(X_n)$ at $[J]$ is the same as the affine variety $\text{spec}(\mathcal{S}'_k/\mathcal{G}_k)$ at the origin.

We start by the case $k = 1$. In other words, assume that $J = \langle x_1^m, x_2, \dots, x_n \rangle$ and $\mu([J])$ is an edge of $(m-1) \cdot \Delta_{n-1}$. By Theorem B.3, the primary decomposition of \mathcal{G}_1 is given by the ideals $J_0 = \langle A_1, \alpha_{1,1} \rangle$, $J_1 = \langle \alpha_2, \dots, \alpha_n \rangle$ and $J_i = \langle A, \alpha_j : 2 \leq j \leq n, j \neq i \rangle$. Note that $\mathbb{V}(J_0)$ is a linear subspace of dimension $n-1$, and $\mathbb{V}(J_i)$ for $1 \leq i \leq n$ is a linear subspace of dimension 2. By Theorem 4.2, J_0 corresponds to the component $\mathcal{Hilb}^{m,2}(X_n)$ and J_i for $i \geq 1$ correspond to the smoothable component. Now, $\mathbb{V}(J_0)$ and $\mathbb{V}(J_1)$ intersect in the origin. Similarly, $\mathbb{V}(J_0)$ and $\mathbb{V}(J_i)$ for $2 \leq i \leq n$ intersect in the line corresponding to $\mathbb{C}[\alpha_i]$. Finally, $\mathbb{V}(J_i)$ and $\mathbb{V}(J_j)$ for $1 \leq i < j \leq n$ intersect in the line $\mathbb{C}[\alpha_{1,1}]$. We can associate to J_0 the simplex Δ_n whose vertices are labeled by v_0, v_2, \dots, v_n . Moreover, we associate to J_i the simplex $M_i := \Delta_2$ whose vertices are labeled by $w_{i,0}, w_{i,1}$ and $w_{i,2}$. We construct the simplicial complex \mathbb{S}_1 obtained by the following gluing:

- For $2 \leq i \leq n$, glue the edge $\overline{v_0, v_i}$ of Δ_n with the edge $\overline{w_{i,0}, w_{i,2}}$ of M_i .
- For $1 \leq i < j \leq n$, glue the edge $\overline{w_{i,0}, w_{i,1}}$ of M_i with the edge $\overline{w_{j,0}, w_{j,1}}$ of M_j .

In Fig. 13a and Fig. 13b, the simplicial complex \mathbb{S}_1 is depicted for $n = 3$ and $n = 4$ respectively. In these figures, the simplex in blue corresponds to Δ_n and J_0 , the simplex in orange represents M_1 and J_1 , the simplices in purple are M_i and J_i for $2 \leq i \leq n$. The simplicial complex \mathbb{S}_1 around the origin $\mathbf{0}$ describes how the components of \mathcal{G}_1 intersect at the origin. Here the origin $\mathbf{0}$ is the vertex obtained from the gluing of $v_0, w_{1,0}, \dots, w_{n,0}$. In particular, the singularity type is described by the complex $\hat{\mathbb{S}}_1$.

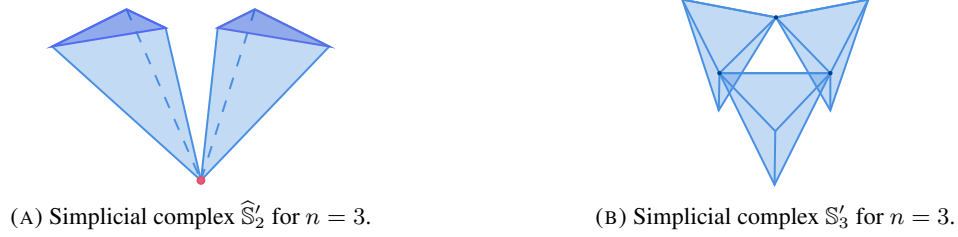
For $k \geq 2$, the situation is a bit more complicated. The primary decomposition of \mathcal{G}_k is given in Theorem B.8, and it has two types of ideals: \mathcal{Q}_i for $i \in [n]$ and \mathcal{G}_S for $S \subsetneq [k]$ with $1 \leq |S| \leq \min\{k, n-2\}$. By Theorem 4.3, \mathcal{Q}_i corresponds to the smoothable component and \mathcal{G}_S corresponds to the component $\mathcal{Hilb}^{m,|S|+1}(C)$. First, we construct a simplicial complex that describes how the ideals \mathcal{G}_S intersect. We can associate to the ring \mathcal{S}'_k the affine space \mathbb{A}^{nk} and the real vector space \mathbb{R}^{nk} . We denote the standard vectors of \mathbb{R}^{nk} by $\mathbf{b}_1, \dots, \mathbf{b}_k, \mathbf{a}_{i,j}$ for $i \in [n], j \in [k]$ with $i \neq j$, where \mathbf{b}_j and $\mathbf{a}_{i,j}$ are the standard vectors associated to the variables β_j and $\alpha_{i,j}$ respectively. Then, we have that

$$\mathbb{V}_{\mathbb{R}}(\mathcal{G}_S) = \text{Span}(\mathbf{b}_j : j \in [k] \setminus S, \mathbf{a}_{i,j} : i \in [n] \setminus S, j \in S).$$

as a real linear subspace. In \mathbb{R}^{nk} , we consider the simplices $\Delta_{nk-1} := \text{Conv}(\mathbf{b}_j : j \in [k], \mathbf{a}_{i,j} : i \in [n], j \in [k], i \neq j)$ and $\hat{\Delta}_{nk} := \text{Conv}(\mathbf{0}, \Delta_{nk-1})$. Moreover, we associated to \mathcal{G}_S the simplices

$$(43) \quad \begin{aligned} \Delta_S &:= \text{Conv}(\mathbf{b}_j : j \in [k] \setminus S, \mathbf{a}_{i,j} : i \in [n] \setminus S, j \in S) = \mathbb{V}(\mathcal{G}_S) \cap \Delta_{nk-1}, \\ \hat{\Delta}_S &:= \text{Conv}(\mathbf{0}, \Delta_S) = \mathbb{V}(\mathcal{G}_S) \cap \hat{\Delta}_{nk}. \end{aligned}$$

Note that $\hat{\Delta}_S$ is the cone of Δ_S . Geometrically, Δ_S is the polytope associated to the projective space defined by \mathcal{G}_S in \mathbb{P}^{nk-1} and $\hat{\Delta}_S$ is the polytope associated to the cone over this projective space, i.e. the polytope associated to $\mathbb{V}(\mathcal{G}_S)$ in \mathbb{A}^{nk} .

FIGURE 14. Simplicial complex \widehat{S}_2 and S'_3 for $n = 3$.

Lemma 5.4. *The simplices Δ_S for $S \subsetneq [k]$ with $1 \leq |S| \leq \min\{k, n-2\}$ form a simplicial complex denoted by S'_k . Similarly, the simplices $\widehat{\Delta}_S$ for $S \subsetneq [k]$ with $1 \leq |S| \leq \min\{k, n-2\}$ form a simplicial complex denoted by \widehat{S}'_k . Moreover, we have that*

$$\Delta_S \cap \Delta_{S'} = \mathbb{V}_{\mathbb{R}}(\mathcal{G}_S + \mathcal{G}_{S'}) \cap \Delta_{nk-1} \quad \text{and} \quad \widehat{\Delta}_S \cap \widehat{\Delta}_{S'} = \mathbb{V}_{\mathbb{R}}(\mathcal{G}_S + \mathcal{G}_{S'}) \cap \widehat{\Delta}_{nk}$$

for $S, S' \subsetneq [k]$ with $1 \leq |S|, |S'| \leq \min\{k, n-2\}$.

Proof. Let $S, S' \subsetneq [k]$ with $1 \leq |S|, |S'| \leq \min\{k, n-2\}$. Then, the linear subspace $\mathbb{V}(\mathcal{G}_S + \mathcal{G}_{S'})$ is generated by the vectors e_i for $i \in [k] \setminus (S \cup S')$ and $f_{i,j}$ for $i \in [n] \setminus (S \cup S')$ and $j \in S \cap S'$. Using (43), we get that

$$\begin{aligned} \Delta_S \cap \Delta_{S'} &= \mathbb{V}(\mathcal{G}_S) \cap \mathbb{V}(\mathcal{G}_{S'}) \cap \Delta_{nk-1} = \mathbb{V}(\mathcal{G}_S + \mathcal{G}_{S'}) \cap \Delta_{nk-1} = \\ &= \text{Conv}(\mathbf{b}_i : i \in [k] \setminus (S \cup S'), \mathbf{a}_{i,j} : i \in [n] \setminus (S \cup S') \text{ and } j \in S \cap S'). \end{aligned}$$

In particular, $\Delta_S \cap \Delta_{S'}$ is a face of both Δ_S and $\Delta_{S'}$, and hence they form a simplicial complex. For \widehat{S}'_k , the proof follows from the fact that it is the simplicial complex obtained by taking the cone of S'_k over the origin. \square

From Theorem 5.4, we conclude that the simplicial complex \widehat{S}'_k around $\mathbf{0}$ describes how the linear spaces $\mathbb{V}(\mathcal{G}_S)$ intersect. Since \widehat{S}'_k is the cone of S'_k over the origin, such combinatorics are also encoded in S'_k .

Example 5.5. Fix $n = 3, m = 3$ and $k = 2$, and focus on the singular point $[\langle x_1^2, x_2^2, x_3 \rangle]$. This point corresponds to the middle point in the bottom edge of $\mathcal{K}_3^{[3]}$ in Fig. 5. The corresponding ring is $\mathcal{S}_k = \mathbb{C}[\beta_1, \beta_2, \alpha_{1,2}, \alpha_{2,1}, \alpha_{3,1}, \alpha_{3,2}]$, and we have two possible ideals of the type \mathcal{G}_S :

$$\mathcal{G}_{\{1\}} = \langle \beta_1, \alpha_{1,2}, \alpha_{3,2} \rangle \quad \text{and} \quad \mathcal{G}_{\{2\}} = \langle \beta_2, \alpha_{2,1}, \alpha_{3,1} \rangle.$$

Then, $\mathbb{V}(\mathcal{G}_{\{1\}}) \cap \mathbb{V}(\mathcal{G}_{\{2\}})$ is the origin $\mathbf{0}$. Similarly, the simplices associated to them are

$$\widehat{\Delta}_{\{1\}} = \text{Conv}(\mathbf{0}, \mathbf{b}_2, \mathbf{a}_{2,1}, \mathbf{a}_{3,1}) \quad \text{and} \quad \widehat{\Delta}_{\{2\}} = \text{Conv}(\mathbf{0}, \mathbf{b}_1, \mathbf{a}_{2,1}, \mathbf{a}_{3,2}).$$

In this case, the corresponding simplicial complex \widehat{S}'_2 is depicted in Fig. 14a. The simplicial complex S'_2 is illustrated in dark blue in Fig. 14a as a subcomplex of \widehat{S}'_2 .

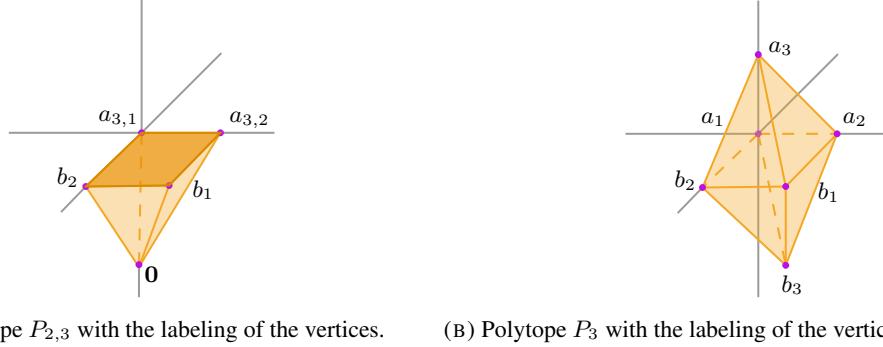
Example 5.6. Fix $n = 3, m = 4$ and $k = 3$, and focus on the singular point $[\langle x_1^2, x_2^2, x_3^2 \rangle]$. Such a point corresponds to the middle vertex of $\mathcal{K}_3^{[3]}$ in Fig. 5. The corresponding ring is $\mathcal{S}_k = \mathbb{C}[\beta_1, \beta_2, \beta_3, \alpha_{1,2}, \alpha_{1,3}, \alpha_{2,1}, \alpha_{2,3}, \alpha_{3,1}, \alpha_{3,2}]$, and we have three possible ideals of the type \mathcal{G}_S :

$$\mathcal{G}_{\{1\}} = \langle \beta_1, \alpha_{1,2}, \alpha_{1,3}, \alpha_{2,3}, \alpha_{3,2} \rangle, \quad \mathcal{G}_{\{2\}} = \langle \beta_2, \alpha_{2,1}, \alpha_{2,3}, \alpha_{1,3}, \alpha_{3,1} \rangle \quad \text{and} \quad \mathcal{G}_{\{3\}} = \langle \beta_3, \alpha_{3,1}, \alpha_{3,2}, \alpha_{1,2}, \alpha_{2,1} \rangle.$$

Then $\mathbb{V}(\mathcal{G}_{\{a\}}) \cap \mathbb{V}(\mathcal{G}_{\{b\}})$ is $\text{Spec}(\mathbb{C}[\beta_c])$ for $\{a, b, c\} = [3]$. Similarly, the simplices associated to them are

$$\Delta_{\{1\}} = \text{Conv}(\mathbf{b}_2, \mathbf{b}_3, \mathbf{a}_{2,1}, \mathbf{a}_{3,1}), \quad \Delta_{\{2\}} = \text{Conv}(\mathbf{b}_1, \mathbf{b}_3, \mathbf{a}_{1,2}, \mathbf{a}_{3,2}) \quad \text{and} \quad \Delta_{\{3\}} = \text{Conv}(\mathbf{b}_1, \mathbf{b}_2, \mathbf{a}_{1,3}, \mathbf{a}_{2,3}).$$

In this case, the simplicial complex S_2 is depicted in Fig. 14b. The simplicial complex \widehat{S}'_3 is the cone over S'_2 .

FIGURE 15. Polytopes $P_{2,3}$ and P_3 with the labeling of their vertices.

Next, we add to our simplicial complexes the polytopes associated to the ideals of the form \mathcal{Q}_i in Theorem B.3. For $i \notin [k]$, the ring $\mathcal{S}'_k/\mathcal{Q}_i$ is the ring

$$\mathbb{C}[\alpha_{i,1}, \dots, \alpha_{i,k}, \beta_1, \dots, \beta_k] / \langle \alpha_{i,r}\beta_r - \alpha_{i,s}\beta_s : 1 \leq r < s \leq k \rangle.$$

The ideal of this quotient corresponds to the ideal (65). Since the ideal is homogeneous and toric, we can associate to \mathcal{Q}_i the polytopes $P_{i,k}$, corresponding to the projective toric variety, and $\widehat{P}_{i,k}$, which is the cone over $P_{i,k}$ and corresponds to the corresponding affine toric variety. By Theorem B.6, $\widehat{P}_{i,k}$ is the polytope P_k and $\widehat{P}_{i,k}$ is the cone over P_k . We refer to (66) for the definition of P_k . We label the vertices of $P_{i,k}$ and $\widehat{P}_{i,k}$ slightly different than the label done for P_k . The vertex $\mathbf{0}$ of P_k is denoted by $a_{i,1}$ in $P_{k,i}$. Similarly, the vertices \mathbf{e}_1 , \mathbf{e}_2 and $\mathbf{e}_1 + \mathbf{e}_2$ are labeled by b_2 , $a_{i,2}$ and b_1 respectively. The vertices \mathbf{e}_j and $\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_j$ are labeled by $a_{i,j}$ and b_j for $3 \leq j \leq k$. The extra point we get after taking the cone over P_k is denoted by $\mathbf{0}$.

Example 5.7. Fix $n = 3$ and $k = 2$. Then, P_2 is the convex hull of $\mathbf{0}$, \mathbf{e}_1 , \mathbf{e}_2 and $\mathbf{e}_1 + \mathbf{e}_2$. The polytope $P_{2,3}$ associated to the ideal \mathcal{Q}_3 is the cone over P_2 . In Fig. 15a, the polytope $P_{2,3}$ is illustrated together with the labeling of its vertices. Moreover, the polytope P_2 is also illustrated in dark orange in Fig. 15a.

For $n = 4$ and $k = 3$, the polytope P_3 is the convex hull of $\mathbf{0}$, \mathbf{e}_1 , \mathbf{e}_2 , $\mathbf{e}_1 + \mathbf{e}_2$, \mathbf{e}_3 and $\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$. The polytope $P_{3,4}$ associated to the ideal \mathcal{Q}_4 is the cone associated to P_3 . In Fig. 15b, the polytope P_3 is depicted with the labelling induced by $P_{3,4}$.

Now, let $i \in [k]$. Without loss of generality, we may assume that $i = k$. Then, the coordinate ring $\mathcal{S}/\mathcal{Q}_k$ is the ring

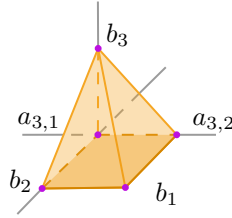
$$\frac{\mathbb{C}[\alpha_{k,1}, \dots, \alpha_{k,k-1}, \beta_1, \dots, \beta_k]}{\langle \alpha_{k,1}\beta_1 - \alpha_{k,s}\beta_s : r \in [k-1] \rangle} \simeq \frac{\mathbb{C}[\alpha_{k,1}, \dots, \alpha_{k,k-1}, \beta_1, \dots, \beta_{k-1}]}{\langle \alpha_{k,1}\beta_1 - \alpha_{k,s}\beta_s : r \in [k-1] \rangle} \otimes \mathbb{C}[\beta_k].$$

Therefore, the variety defined by \mathcal{Q}_i is isomorphic to $\mathbb{V}(I_{k-1}) \times \mathbb{A}_{\mathbb{C}}^1$ where I_{k-1} is defined in (65). Note that I_a is defined in (65) for $a \geq 2$. We set $I_1 = \langle 0 \rangle$. In particular, the polytope P_1 associated to I_1 is the one dimensional simplex. As before, let $P_{k,k}$ and $\widehat{P}_{k,k}$ be the polytopes associated to the affine and projective varieties defined by \mathcal{Q}_i respectively. By construction, $P_{k,k}$ is the cone of the polytope P_{k-1} . Explicitly, we embed $P_{k-1} \subset \mathbb{R}^{k-1}$ in \mathbb{R}^k . Then, $P_{k,k}$ is the convex hull of P_{k-1} and \mathbf{e}_k . Then, $\widehat{P}_{k,k}$ is the cone of $P_{k,k}$. As before, we slightly change the labeling of the vertices of $P_{k,k}$ and $\widehat{P}_{k,k}$ for $i \in [k]$. We label the vertices corresponding to $\mathbf{0}$, \mathbf{e}_1 , \mathbf{e}_2 and $\mathbf{e}_1 + \mathbf{e}_2$ by $a_{k,1}$, b_2 , $a_{k,2}$, and b_1 respectively. Similarly, the vertices \mathbf{e}_j and $\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_j$ for $3 \leq j \leq k-1$ are labeled by $a_{k,j}$ and b_j respectively. The extra vertex \mathbf{e}_k is labeled by b_k , and the vertex of the cone is denoted by $\mathbf{0}$. In Fig. 16, the convex hull of P_2 and \mathbf{e}_3 is depicted with the labeling induced by \mathcal{Q}_3 . The polytope $P_{3,3}$ is the cone over the polytope in Fig. 16.

Now, we build the polyhedral complexes \mathcal{S}_k and $\widehat{\mathcal{S}}_k$ by adding to the complex \mathcal{S}'_k and $\widehat{\mathcal{S}}'_k$ the polytopes $P_{i,k}$ and $\widehat{P}_{i,k}$ respectively. By Theorem B.6, for $i \notin [k]$, the facets of $P_{i,k}$ are simplices whose vertices are

$$(44) \quad \{a_{i,j} : j \in S\} \cup \{b_j : j \in [k] \setminus S\}$$

for $S \subseteq \{1, \dots, k\}$. We denote such a facet by F_S . We can associate to the facet F_S of $P_{i,k}$ a face of the simplicial complex \mathcal{S}'_k as follows. We consider the face of Δ_S spanned by the vectors \mathbf{b}_j for $j \in [k] \setminus S$ and $\mathbf{a}_{i,j}$ for $j \in S$. Such

FIGURE 16. The convex hull of P_2 and e_3 with the labeling induced by $P_{3,3}$.

face is isomorphic to F_S by identifying the vertices b_j and $a_{i,j}$ of F_S with the vectors \mathbf{b}_j and $\mathbf{a}_{i,j}$ respectively. Similarly, for $i \in [k]$ we can identify each facet of $P_{i,k}$ with an isomorphic face of the complex \mathbb{S}'_k by identifying the vertices $a_{i,j}$ and b_j with the vectors $\mathbf{a}_{i,j}$ and \mathbf{b}_j respectively. The polyhedral complex \mathbb{S}_k is the complex obtained by adding to the simplicial complex \mathbb{S}'_k the polytopes $P_{i,k}$ through the above identification. Finally, the polyhedral complex $\widehat{\mathbb{S}}_k$ is the complex obtained by taking the cone over the complex \mathbb{S}_k . The complex $\widehat{\mathbb{S}}_k$ can also be constructed by adding, similarly to the construction of \mathbb{S}_k , the polytopes $\widehat{P}_{i,k}$ to the complex $\widehat{\mathbb{S}}'_k$.

Proposition 5.8. *Let I, J be two ideals among the ideals in the primary decomposition of \mathcal{G}_k , and let Q_I and Q_J be their corresponding polytopes in the complex $\widehat{\mathbb{S}}_k$. Then, the intersection of $\mathbb{V}(I)$ and $\mathbb{V}(J)$ is the closure of a toric orbit that corresponds to the intersection of Q_I and Q_J in $\widehat{\mathbb{S}}_k$.*

Proof. If I and J are both of the form \mathcal{G}_S for $S \subseteq [k]$, then the proof follows from Theorem 5.4. Assume now that $I = \mathcal{Q}_i$ and $J = \mathcal{Q}_j$ for $i, j \in [n]$. Then

$$\mathcal{Q}_i + \mathcal{Q}_j = \langle \alpha_{i,j} : i \in [n], j \in [k] \setminus \{i\} \rangle.$$

Therefore, the coordinate ring of $\mathbb{V}(\mathcal{Q}_i) \cap \mathbb{V}(\mathcal{Q}_j)$ is $\mathbb{C}[\beta_1, \dots, \beta_k]$. The face in $\widehat{P}_{i,k}$ corresponding to this intersection is the convex hull of b_1, \dots, b_k , which is exactly the intersection between $\widehat{P}_{i,k}$ and $\widehat{P}_{j,k}$ in $\widehat{\mathbb{S}}_k$.

Assume now that $I = \mathcal{Q}_i$ and $J = \mathcal{G}_S$. We distinguish two cases. Assume first that $i \in S$, then $\mathcal{Q}_i + \mathcal{G}_S = \langle \beta_s : s \in [k] \setminus S, \alpha_{r,s} : r \in [n], s \in [k] \setminus \{r\} \rangle$ and the coordinate ring of the intersection is $\mathbb{C}[\beta_r : r \in [k] \setminus S]$. Therefore, the face of Δ_S corresponding to this intersection is the convex hull of the vertices \mathbf{b}_r for $r \in [k] \setminus S$. Such a face in $P_{i,k}$ is the one given by the vertices b_r for $r \in [k] \setminus S$. This face coincides with the intersection of Δ_S and $P_{i,k}$ in \mathbb{S}_k .

Assume now that $i \notin S$. Then, the coordinate ring of the intersection $\mathbb{V}(\mathcal{Q}_i + \mathcal{G}_S)$ is $\mathbb{C}[\beta_i : i \in [k] \setminus S, \alpha_{i,s} : s \in S]$. The face of Δ_S corresponding to this intersection is the convex hull of the vertices \mathbf{b}_r for $r \in [k] \setminus S$ and $\mathbf{a}_{i,s}$ for $s \in S$. Such a face in $P_{i,k}$ is the one given by the vertices b_r for $r \in [k] \setminus S$ and $a_{i,s}$ for $s \in S$. As before, this face coincides with the intersection of Δ_S and $P_{i,k}$ in \mathbb{S}_k . \square

From Theorem 5.8, we deduce that singularity type of $\mathcal{Hilb}^m(X_n)$ at $[J]$ is described via the complex $\widehat{\mathbb{S}}_k$ locally around the vertex of the cone.

Example 5.9. Fix $n = 3$ and $k = 2$. The complex $\widehat{\mathbb{S}}_2$ describes the singularity type of $\mathcal{Hilb}^m(X_3)$ at a point of the form $[(x_1^{m-i+1}, x_2^i, x_3)]$ for $1 \leq i \leq m$. The ring \mathcal{S}'_k is $\mathbb{C}[\beta_1, \beta_2, \alpha_{1,2}, \alpha_{2,1}, \alpha_{3,1}, \alpha_{3,2}]$. The primary decomposition of \mathcal{G}_2 is given by the ideals

$$\begin{aligned} \mathcal{Q}_1 &= \langle \alpha_{2,1}, \alpha_{3,1}, \alpha_{3,2} \rangle, \quad \mathcal{Q}_2 = \langle \alpha_{1,2}, \alpha_{3,1}, \alpha_{3,2} \rangle, \quad \mathcal{Q}_3 = \langle \alpha_{1,2}, \alpha_{2,1}, \alpha_{3,1}\beta_1 - \alpha_{3,2}\beta_2 \rangle, \\ \mathcal{G}_{\{1\}} &= \langle \beta_1, \alpha_{1,2}, \alpha_{3,2} \rangle, \quad \text{and} \quad \mathcal{G}_{\{2\}} = \langle \beta_2, \alpha_{2,1}, \alpha_{3,1} \rangle, \end{aligned}$$

To each of these ideals, we associate two polytopes. To \mathcal{Q}_i , we associate the polytopes $P_{i,2}$ and $\widehat{P}_{i,2}$. The polytopes $P_{1,2}$ and $P_{2,2}$ are the 2-dimensional simplices with set of vertices $\{b_1, b_2, a_{1,2}\}$ and $\{b_1, b_2, a_{2,1}\}$ respectively. These polytopes are illustrated in purple in Fig. 17. On the other hand, $P_{3,2}$ is a square and its set of vertices is $\{b_1, b_2, a_{3,1}, a_{3,2}\}$. In Fig. 17, $P_{3,2}$ is depicted in orange. The polytope $\widehat{P}_{i,2}$ is the cone over $P_{i,k}$. Similarly, $\mathcal{G}_{\{i\}}$ we associate the polytopes $\Delta_{\{i\}}$ and $\widehat{\Delta}_{\{i\}}$. The polytope $\Delta_{\{1\}}$ is the 2 dimensional simplex spanned by the vertices $\mathbf{b}_2, \mathbf{a}_{3,1}$ and $\mathbf{a}_{2,1}$. Analogously, $\Delta_{\{2\}}$ is the 2 dimensional simplex spanned by the vertices $\mathbf{b}_1, \mathbf{a}_{3,2}$ and $\mathbf{a}_{1,2}$. Both $\Delta_{\{1\}}$ and $\Delta_{\{2\}}$ are illustrated in blue in Fig. 17. The complex \mathbb{S}_2 , which is illustrated in Fig. 17, is obtained by gluing the polytopes $P_{1,2}, P_{2,2}, P_{3,2}, \Delta_{\{1\}}$ and $\Delta_{\{2\}}$ through the faces spanned by vertices with the same labeling. The complex $\widehat{\mathbb{S}}_2$ is the cone over the complex \mathbb{S}_2 .

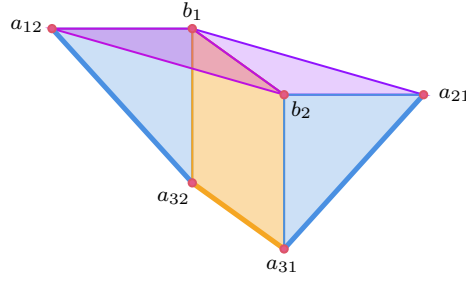


FIGURE 17. Complex \mathbf{S}_2 for $n = 3$. The simplices in blue correspond to $\Delta_{\{1\}}$ and $\Delta_{\{2\}}$. The simplices $P_{1,2}$ and $P_{2,2}$ are illustrated in purple. The orange square corresponds to $P_{3,2}$.

Remark 5.10. Since the punctual Hilbert scheme $\mathcal{Hilb}_0^m(X_n)$ is invariant by the torus action, we can also illustrate it locally around $[J]$ through the complexes \mathbf{S}_k and $\widehat{\mathbf{S}}_k$. The irreducible components of $\mathcal{Hilb}_0^m(X_n)$ that contain $[J]$ correspond to faces of the complex. Theorem 4.5, allows us to carry out such identification. Given a maximal face Δ_S of \mathbf{S}_k (see (43) for the definition of Δ_S), the face $\text{Conv}(\mathbf{a}_{i,j} : i \in [n] \setminus S, j \in S)$ corresponds to the hypersimplex $\Delta_{|S|,n} + \mathbf{u} - \mathbf{e}_S - \mathbf{1}$ and the Grassmannian $\Sigma(m, n - |S|, \mathbf{u} - \mathbf{e}_S)$. Similarly, given a maximal face of the form $P_{i,k}$, then the face spanned by the vertices $a_{i,j}$ for $j \in [k] \setminus \{i\}$ corresponds to the hypersimplex $\Delta_{n-1,n} + \mathbf{u} + \mathbf{e}_i - \mathbf{1}$ and the Grassmannian $\Sigma(m, 1, \mathbf{u} + \mathbf{e}_i - \mathbf{1})$. In particular, the part of \mathbf{S}_k associated to the punctual Hilbert scheme is the subcomplex formed by all the faces spanned by vertices of the form $a_{i,j}$. Analogously, for $\widehat{\mathbf{S}}_k$, these faces are those spanned by the vertex of the cone and the vertices of the form $a_{i,j}$. For instance, following Theorem 5.9, for $n = 3$ and $k = 2$, the faces of \mathbf{S}_2 corresponding to the punctual Hilbert scheme are the edges $\overline{a_{1,2}a_{3,2}}$, $\overline{a_{3,2}a_{3,1}}$ and $\overline{a_{2,1}a_{3,1}}$. These edges are represented in Fig. 17 with a thick line. The cone over these three edges is exactly the local picture of $\mathcal{K}_2^{[m]}$ around the vertex corresponding to $[J]$. Such a vertex is an interior point on an edge of $(m-1) \cdot \Delta_2$.

Note that the complex \mathbf{S}_k and the ring $\mathcal{S}'_k/\mathcal{G}_k$ depend only on n and k . Therefore, the singularity types that appear in $\mathcal{Hilb}^m(X_n)$ depend only on k . We conclude this section with the following result.

Corollary 5.11. *For $m \geq 2$, any singularity type appearing in $\mathcal{Hilb}^m(X_n)$ also appears in $\mathcal{Hilb}^{n+1}(X_n)$*

Proof. Let $[J] \in \mathcal{Hilb}^m(X_n)$ be a singular point. By Theorem 4.7, there exists $[J_0] \in \mathcal{Hilb}_0^m(X_n)$ lying in the closure of the $(\mathbb{C}^*)^n$ -orbit of $[J]$ such that $\mu([J_0])$ is a vertex of $\mathcal{K}_n^{[m]}$. Let k be the dimension of the face of $(m-1) \cdot \Delta_n$ where $\mu([J_0])$ lies. Then, we can see $[J]$ as a point in the complex $\widehat{\mathbf{S}}_k$ and the singularity type of $[J]$ only depends on the relative interior cell of $\widehat{\mathbf{S}}_k$ where it lies. Since for fixed n , the complex $\widehat{\mathbf{S}}_k$ depends only on k , the possible singularity types of $\mathcal{Hilb}^m(X_n)$ depend only on the possible values of k . So, it is enough to check that for every $k \in [n]$, there exists $[J_0] \in \mathcal{Hilb}_0^{n+1}(X_n)$ such that $\mu([J_0])$ is a vertex of $\mathcal{K}_n^{[n+1]}$ lying in the interior of a k -dimensional face of $n \cdot \Delta_{n-1}$. Note that the vertices of $\mathcal{K}_n^{[n+1]}$ coincide with the integer points of $n \cdot \Delta_{n-1}$. Then, the proof follows from the fact that the interior of every face of $n \cdot \Delta_{n-1}$ has an integer point. Indeed, the point $(n-k+1)\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_k$ lies in the interior of a k -dimensional face of $n \cdot \Delta_{n-1}$. Note that $n \cdot \Delta_{n-1}$ is the first simplex of the form $m \cdot \Delta_{n-1}$ with such a property. \square

6. SMOOTHABLE AND NON-SMOOTHABLE COMPONENTS

With the results of the previous sections at hand, we proceed to analyze the irreducible components of the Hilbert scheme, which display strikingly different behaviors. Interestingly, the smoothable component turns out to be the most singular, whereas the non-smoothable ones exhibit remarkably well-behaved geometry.

6.1. Non-smoothable components. Let C be a genus g irreducible curve whose unique singularity $p \in C$ is a rational n -fold singularity. In this Subsection, we give a detailed description of the non-smoothable components of $\mathcal{Hilb}^m(C)$ and their normalization. For $2 \leq m' \leq \min\{n-1, m\}$, we consider the irreducible component $\mathcal{Hilb}^{m,m'}(C)$ of $\mathcal{Hilb}^m(C)$. By Theorem 4.12, we may see $\text{Sym}^{m-m'}(C \setminus \{p\}) \times \Sigma(m', n+1-m', 1)$ as an open subset of this component. We now give a stratification of such a component where this open subset is the biggest strata.

Let $\nu : \tilde{C} \rightarrow C$ be the normalization of C , and let p_1, \dots, p_n be the n preimages of the singularity p . In particular, ν gives an isomorphism between $\tilde{C} \setminus \{p_1, \dots, p_n\}$ and $C \setminus \{p\}$. We consider the following stratification of the symmetric product $\text{Sym}^m(\tilde{C})$. For $0 \leq u \leq m$ and for a partition $\mathbf{u} \in \mathbb{Z}_{\geq 0}^n$ with $|\mathbf{u}| = u$, we consider the locus

$$\text{Sym}^m(\tilde{C})_{\mathbf{u}} = \left\{ \sum_{i=1}^n u_i p_i + q : \text{for } q \in \text{Sym}^{m-u}(\tilde{C} \setminus \{p_1, \dots, p_n\}) \right\} \simeq \text{Sym}^{m-u}(\tilde{C} \setminus \{p_1, \dots, p_n\}).$$

Then, the locally closed subvarieties $\text{Sym}^m(\tilde{C})_{\mathbf{u}}$ form a stratification of $\text{Sym}^m(\tilde{C})$. For fixed $0 \leq u \leq m$, we also consider the variety

$$\text{Sym}^m(\tilde{C})_u = \bigcup_{\mathbf{u} \in \mathbb{Z}_{\geq 0}^n, |\mathbf{u}|=u} \text{Sym}^m(\tilde{C})_{\mathbf{u}}.$$

Note that this union is actually a disjoint union. We can use this stratification of \tilde{C} to describe the irreducible component $\mathcal{Hilb}^{m,m'}(C)$. For $\mathbf{u} \in \mathbb{Z}_{\geq 0}^n$ with $0 \leq |\mathbf{u}| \leq m - m'$, we consider the map

$$\begin{aligned} \psi_{m',\mathbf{u}} : \text{Sym}^{m-m'}(\tilde{C})_{\mathbf{u}} \times \Sigma(m', n+1-m', \mathbf{1}) &\longrightarrow \mathcal{Hilb}^{m,m'}(C) \\ (\sum_{i=1}^n u_i p_i + q, [J]) &\longmapsto \nu(q) \cup \mathbb{V}(\phi_{m',\mathbf{u}}([J])). \end{aligned}$$

Here ν also denotes the lift of the normalization map ν to the symmetric product of the curve, and the map $\phi_{m',\mathbf{u}}$ is defined in (20). The image of $\psi_{m',\mathbf{u}}$ is

$$\mathcal{Hilb}^{m,m',\mathbf{u}}(C) := \left\{ [Z \cup \mathbb{V}(J)] : Z \in \text{Sym}^{m-m'-|\mathbf{u}|}(C \setminus \{0\}) \text{ and } [J] \in \Sigma(m' + |\mathbf{u}|, n+1-m', \mathbf{u} + \mathbf{1}) \right\},$$

which is isomorphic to $\text{Sym}^{m-m'-|\mathbf{u}|}(C \setminus \{0\}) \times \Sigma(m' + |\mathbf{u}|, n+1-m', \mathbf{u} + \mathbf{1})$. Using that $C \setminus \{0\} \simeq \tilde{C} \setminus \{p_1, \dots, p_n\}$, we deduce that $\psi_{m',\mathbf{u}}$ is an isomorphism onto $\mathcal{Hilb}^{m,m',\mathbf{u}}(C)$. Note that the varieties $\mathcal{Hilb}^{m,m',\mathbf{u}}(C)$ do not provide a stratification of $\mathcal{Hilb}^{m,m'}(C)$ since they are not disjoint. Indeed, for $\mathbf{u}, \mathbf{v} \in \mathbb{Z}_{\geq 0}^n$ with $|\mathbf{u}| = |\mathbf{v}|$, we have that the intersection $\mathcal{Hilb}^{m,m',\mathbf{u}}(C) \cap \mathcal{Hilb}^{m,m',\mathbf{v}}(C)$ is the product of $\text{Sym}^{m-m'-|\mathbf{u}|}(C \setminus \{0\})$ with

$$\Sigma(m' + |\mathbf{u}|, n+1-m', \mathbf{u} + \mathbf{1}) \cap \Sigma(m' + |\mathbf{v}|, n+1-m', \mathbf{v} + \mathbf{1}).$$

The above intersection is done in $\mathcal{G}_{n+1-m',n}^{m'+|\mathbf{u}|} \subseteq \mathcal{Hilb}_0^{m,m'}(C)$ and might not be empty. To solve this problem, for $0 \leq u \leq m - m'$, we consider the map

$$\psi_{m',u} : \text{Sym}^{m-m'}(\tilde{C})_u \times \Sigma(m', n+1-m', \mathbf{1}) \longrightarrow \mathcal{Hilb}^{m,m'}(C)$$

whose restriction to each connected component $\text{Sym}^{m-m'}(\tilde{C})_{\mathbf{u}}$ is $\psi_{m',\mathbf{u}}$. Note that $\psi_{m',0}$ is the birational morphism between $\text{Sym}^{m-m'}(C) \times \Sigma(m', n+1-m', \mathbf{1})$ and $\mathcal{Hilb}^{m,m'}(C)$. The image of $\psi_{m',u}$ is the locally closed subvariety

$$(45) \quad \mathcal{Hilb}^{m,m',u}(C) := \left\{ [Z \cup \mathbb{V}(J)] \in \mathcal{Hilb}^{m,m'}(X_n) : Z \in \text{Sym}^{m-m'-u}(\tilde{C} \setminus \{p_1, \dots, p_n\}) \text{ and } [J] \in \mathcal{G}_{n+1-m',n}^{m'+u} \right\}$$

Using Theorem 4.10, we may see $\mathcal{G}_{n+1-m',n}^{m'+u}$ as the subvariety of $\mathcal{Hilb}_0^{m'+u}(C)$ given by the union of the components of the form $\Sigma(m', n+1-m', \mathbf{u} + \mathbf{1})$ with $|\mathbf{u}| = u$. Note that for $u = m - m'$, we get that $\mathcal{Hilb}^{m,m',m-m'}(C) \simeq \mathcal{G}_{n+1-m',n}^m$. From (45), we get that

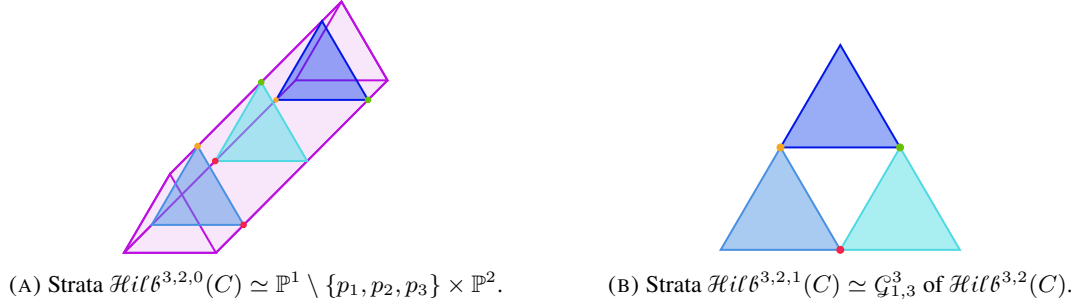
$$(46) \quad \mathcal{Hilb}^{m,m',u}(C) \simeq \text{Sym}^{m-m'-u}(\tilde{C} \setminus \{p_1, \dots, p_n\}) \times \mathcal{G}_{n+1-m',n}^{m'+u}.$$

Moreover, note that $\mathcal{Hilb}^{m,m',u}(C)$ and $\mathcal{Hilb}^{m,m',v}(C)$ are disjoint for $0 \leq u < v \leq m - m'$ and, by Theorem 3.1, we deduce that

$$\mathcal{Hilb}^{m,m'}(C) = \bigsqcup_{0 \leq u \leq m-m'} \mathcal{Hilb}^{m,m',u}(C).$$

Therefore, the varieties $\mathcal{Hilb}^{m,m',u}(C)$ provide a stratification of $\mathcal{Hilb}^{m,m'}(C)$.

Example 6.1. Let C be an irreducible genus 3 curve with a rational 3-fold singularity at $p \in C$. The normalization of C is \mathbb{P}^1 and the preimage of p consists of 3 points $p_1, p_2, p_3 \in \mathbb{P}^1$. By Theorem 4.12, the component $\mathcal{Hilb}^{3,2}(C)$ is birational to $\mathbb{P}^1 \times \mathbb{P}^2$. We can stratify $\mathcal{Hilb}^{3,2}(C)$ by $\mathcal{Hilb}^{3,2,0}(C)$ and $\mathcal{Hilb}^{3,2,1}(C)$. The strata $\mathcal{Hilb}^{3,2,0}(C)$ is the open subset of $\mathcal{Hilb}^{3,2}(C)$ of the form $(\mathbb{P}^1 \setminus \{p_1, p_2, p_3\}) \times \mathbb{P}^2$, which is depicted in Fig. 18a. The strata $\mathcal{Hilb}^{3,2,1}(C)$ is the variety $\mathcal{G}_{2,3}^3$, that is obtained by gluing three \mathbb{P}^2 by toric invariant points as illustrated in Fig. 18b.

FIGURE 18. Stratification of $\mathcal{Hilb}^{3,2}(C)$ for a genus 3 curve with a rational 3-fold singularity.

Example 6.2. Let C be as in Theorem 6.1, then $\mathcal{Hilb}^{4,2}(C)$ is stratified by $\mathcal{Hilb}^{4,2,0}(C)$, $\mathcal{Hilb}^{4,2,1}(C)$ and $\mathcal{Hilb}^{4,2,2}(C)$. The strata $\mathcal{Hilb}^{4,2,0}(C)$ is $\text{Sym}^2(\mathbb{P}^1 \setminus \{p_1, p_2, p_3\}) \times \mathbb{P}^2 \simeq \mathbb{P}^2 \setminus (l_1 \cup l_2 \cup l_3) \times \mathbb{P}^2$ where l_i corresponds to the line in $\text{Sym}^2(\mathbb{P}^1) \simeq \mathbb{P}^2$ of the form $p_i + q$ for $q \in \mathbb{P}^1$. These lines l_i and l_j intersect in the point $p_i + p_j$. Moreover, the line l_i has an extra marked point $2p_i$. Then, the strata $\mathcal{Hilb}^{4,2,1}(C)$ is

$$\mathcal{Hilb}^{4,2,1}(C) = (\mathbb{P}^1 \setminus \{p_1, p_2, p_3\}) \times \mathcal{G}_{2,3}^3.$$

We see that $\mathcal{Hilb}^{4,2,1}(C)$ has three components of the form $(\mathbb{P}^1 \setminus \{p_1, p_2, p_3\}) \times \Sigma(3, 2, \mathbf{1} + \mathbf{e}_i)$ for $i \in [3]$ coming from the three components of $\mathcal{G}_{2,3}^3$ (see Fig. 18b). Via the map $\psi_{2,1}$, these three components are in correspondence with the three components of

$$\text{Sym}^2(\mathbb{P}^1)_1 = l_1 \cup l_2 \cup l_3 \setminus \{p_i + p_j : 1 \leq i \leq j \leq 3\}.$$

The final strata is $\mathcal{Hilb}^{4,2,1}(C) = \mathcal{G}_{2,3}^4$ which is obtained by gluing 6 projective planes as illustrated in Fig. 9. Each of these 6 projective planes corresponds to a hypersimplex in $\mathcal{K}_3^{[4]}$ of the form $\Delta_{1,3} + \mathbf{e}_i + \mathbf{e}_j$. We can associate such hypersimplex to the point $p_i + p_j$ among the 6 special points in the lines l_1, l_2, l_3 via the map $\psi_{2,2}$.

The above stratification of $\mathcal{Hilb}^{m,m'}(C)$ allows us to calculate its normalization.

Theorem 6.3. The birational map $\psi_{m',0} : \text{Sym}^{m-m'}(\tilde{C}) \times \Sigma(m', n+1-m', \mathbf{1}) \dashrightarrow \mathcal{Hilb}^{m,m'}(C)$ extends uniquely to a finite map

$$(47) \quad \psi_{m'} : \text{Sym}^{m-m'}(\tilde{C}) \times \Sigma(m', n+1-m', \mathbf{1}) \rightarrow \mathcal{Hilb}^{m,m'}(C)$$

such that the restriction of $\psi_{m'}$ to $\text{Sym}^{m-m'}(\tilde{C})_{\mathbf{u}} \times \Sigma(m', n+1-m', \mathbf{1})$ is $\psi_{m',\mathbf{u}}$ for $0 \leq \mathbf{u} \leq m-m'$. In particular, the map (47) is the normalization of $\mathcal{Hilb}^{m,m'}(C)$.

Proof. First, we construct a map $\psi_{m',\text{top}}$ at the level of topological spaces that extends continuously $\psi_{m',0}$. We construct $\psi_{m',\text{top}}$ as the map from topological spaces whose restriction to $\text{Sym}^{m-m'}(\tilde{C})_{\mathbf{u}} \times \Sigma(m', n+1-m', \mathbf{1})$ is $\psi_{m',\mathbf{u}}$. We claim that $\psi_{m',\text{top}}$ is continuous. Let $z_0 = (q, [J])$ be a point in $\text{Sym}^{m-m'}(\tilde{C})_{\mathbf{u}} \times \Sigma(m', n+1-m', \mathbf{1})$, and let Z be a one parameter family in $\text{Sym}^{m-m'}(\tilde{C}) \times \Sigma(m', n+1-m', \mathbf{1})$ passing through z_0 . Let $\mathbf{v} \in \mathbb{Z}_{\geq 0}^n$ be the integer vector with highest $|\mathbf{v}|$ such that $Z \subseteq \text{Sym}^{m-m'}(\tilde{C})_{\mathbf{v}} \times \Sigma(m', n+1-m', \mathbf{1})$. Then $|\mathbf{v}| \leq |\mathbf{u}|$ and $\mathbf{u} - \mathbf{v} \in \mathbb{Z}_{\geq 0}^n$. By Theorem 3.1, we get that

$$\lim_{z \rightarrow z_0} \psi_{m',\text{top}}(z) = \lim_{z \rightarrow z_0} \psi_{m',\mathbf{v}}(z) = q \cup \mathbb{V}(\phi_{m',\mathbf{u}}([J])) = \psi_{m',\mathbf{u}}(z_0).$$

Therefore, we conclude that $\psi_{m',\text{top}}$ is continuous. Moreover, by Theorem 3.1, $\psi_{m',\text{top}}$ is the only possible extension of $\psi_{m',0}$ at the level of topological spaces.

Next, we show that $\psi_{m',0}$ extends not only at the level of topological spaces, but also at the scheme-theoretical level. Let Γ be the graph of $\psi_{m',0}$ and let $\bar{\Gamma}$ be its closure. Consider the projection $\pi : \bar{\Gamma} \rightarrow \text{Sym}^{m-m'}(\tilde{C}) \times \Sigma(m', n+1-m', \mathbf{1})$. Now, let $y \in \text{Sym}^{m-m'}(\tilde{C}) \times \Sigma(m', n+1-m', \mathbf{1})$ be a point outside the domain of definition of $\psi_{m',0}$. Since $\psi_{m',\text{top}}$ is the unique topological extension of $\psi_{m',0}$, we deduce that $\pi^{-1}(y)$ is the point $(y, \psi_{m',\text{top}}(y))$. Then, by [Kan21, Theorem 2], $\psi_{m',0}$ extends to the map (47)

Now, we show that (47) is finite. By [Sta25, Tag 01W6], $\psi_{m'}$ is proper. Therefore, it is enough to check that $\psi_{m'}$ is quasi-finite. First of all, note that $\psi_{m'}^{-1}(\mathcal{Hilb}^{m,m',u}(C)) = \text{Sym}^{m-m'}(\tilde{C})_u \times \Sigma(m', n+1-m', \mathbf{1})$. Since $\mathcal{Hilb}^{m,m',u}(C)$ is a stratification of $\mathcal{Hilb}^{m,m'}(C)$, it is enough to check that the map $\psi_{m',u}$ has finite fibers for all $0 \leq u \leq m-m'$. Let $(q, [J]) \in \mathcal{Hilb}^{m,m',u}(C)$. Since for each $\mathbf{u} \in \mathbb{Z}_{\geq 0}^n$ with $|\mathbf{u}| = u$, $\psi_{m',\mathbf{u}}$ is an isomorphism onto $\mathcal{Hilb}^{m,m',\mathbf{u}}(C)$ the number of fibers of $(q, [J])$ is exactly the number of distinct $\mathcal{Hilb}^{m,m',\mathbf{u}}(C)$ containing this point. This is equivalent to the number of Grassmannians in $\mathcal{G}_{n+1-m',n}^{m'+u}$ containing $[J]$. In terms of the combinatorics, this is equivalent to counting how many hypersimplices of the form $\Delta_{m'-1,n} + \mathbf{u}$ contain $\mu([J])$. We conclude that $\psi_{m'}$ is finite. Moreover, since $\psi_{m'}$ is finite, birational and its domain is normal, we conclude that (47) is the normalization of $\mathcal{Hilb}^{m,m'}(C)$. \square

Remark 6.4. Theorem 6.3 states that $\text{Sym}^{m-m'}(\tilde{C}) \times \Sigma(m', n+1-m', \mathbf{1})$ is the normalization of $\mathcal{Hilb}^{m,m'}(C)$. Since this normalization is smooth, the map (47) is also a resolution of singularities of $\mathcal{Hilb}^{m,m'}(C)$. Moreover, we can describe the locus where the normalization map $\psi_{m'}$ is not injective. Consider the strata $\mathcal{Hilb}^{m,m',u}(C)$ of $\mathcal{Hilb}^{m,m'}(C)$. By (46), a point in $\mathcal{Hilb}^{m,m',u}(C)$ is a tuple $(q, [J])$ where $q \in \text{Sym}^{m-m'}(\tilde{C} \setminus \{p_1, \dots, p_n\})$ and $[J] \in \mathcal{G}_{n+1-m',n}^{m'+u}$. Then $(q, [J])$ lies in the birational locus of $\psi_{m'}$ if and only if $[J]$ is contained in only one irreducible component of $\mathcal{G}_{n+1-m',n}^{m'+u}$. In other words, $\mu([J])$ is contained in a unique hypersimplex of $\mathcal{K}_{n+1-m',n}^{[m'+u]}$. In terms of the fiber, the degree of the fiber $\psi_{m'}^{-1}((q, [J]))$ is the number of hypersimplices containing $\mu([J])$ in $\mathcal{K}_{n+1-m',n}^{[m'+u]}$.

Now, we replace C by X_n . In this case, the geometry of the non-smoothable components of $\mathcal{Hilb}^m(X_n)$ is simpler than for irreducible curves.

Proposition 6.5. *The non-smoothable components of $\mathcal{Hilb}^m(X_n)$ are isomorphic to*

$$(48) \quad \text{Sym}^{u_1}(L_1) \times \dots \times \text{Sym}^{u_n}(L_n) \times \Sigma(m', n+1-m', \mathbf{1})$$

for $2 \leq m' \leq \min\{m, n-1\}$ and $\mathbf{u} = (u_1, \dots, u_n)$ partition of $m-m'$. In particular, the non-smoothable components of $\mathcal{Hilb}^m(X_n)$ are smooth.

Proof. Let Z be a non-smoothable component of $\mathcal{Hilb}^m(X_n)$. By Theorem 3.3, the non-smoothable components of $\mathcal{Hilb}^m(X_n)$ are birational to (48). By Theorem 4.7, it is enough to check the smoothness at a point $[J]$ supported at $\mathbf{0}$ such that $\mu([J])$ is a vertex of $\mathcal{K}_n^{[m]}$. By Theorem 3.1, $[J]$ lies in $\Sigma(m, n+1-m', \mathbf{1} + \mathbf{u})$. By Theorem 4.2 and Theorem 4.3, the only component in the completion of the stalk of $\mathcal{Hilb}^m(X_n)$ at $[J]$ corresponding to Z is a component associated to an ideal of the form \mathcal{I}_S . Since the ideal \mathcal{I}_S defines an affine space (see Theorem B.8), we conclude that Z is smooth.

Now, in this case, the normalization map (47) in Theorem 6.3 maps (48) to Z . By the uniqueness of the normalization we deduce that Z is isomorphic to (48). \square

Note that Theorem 6.5 is no longer true if we replace X_n by an irreducible curve C with a rational n -fold singularity at p . From the same arguments used in Theorem 6.5, we obtain the following.

Corollary 6.6. *Let C be an irreducible curve whose only singularity p is a rational n -fold singularity. Then, the singularities of the non-smoothable components $\mathcal{Hilb}^{m,m'}(C)$ are locally union of affine spaces.*

6.2. Smoothable components. For this subsection, we will use the notion of smoothable face, cf. Theorem A.9. We start with the following.

Proposition 6.7. *Let $[J] \in \mathcal{Hilb}_0^m(X_n)$. Then $[J]$ is smoothable if and only if $\mu([J])$ lies in a smoothable face of $\mathcal{K}_n^{[m]}$.*

Proof. Let $[J] \in \mathcal{Hilb}_0^m(X_n)$ such that $\mu([J])$ lies in a smoothable face of $\mathcal{K}_n^{[m]}$. By Theorem 2.10, there exists $S \subset [n]$ and $\mathbf{u} \in \mathbb{Z}_{\geq 1}^n$ with $|\mathbf{u}| = m + |S|$ such that $J = \langle f, x_i^{u_i} : i \in S \rangle$ where $f = \sum_{i \notin S} a_i x_i^{u_i}$. By Theorem 3.1, $[J]$ can be obtained as a limit of length m schemes of the form $q \cup \mathbb{V}(J')$ where $q \in (X_n \setminus \{0\})^{m-n+|S|}$ and $J' = \langle f', x_i : i \in S \rangle$ and $f' = \sum_{i \notin S} a_i x_i$. Therefore, to check that $[J]$ is smoothable, it is enough to check that $[J']$ is smoothable. Consider the ideal $\tilde{J} = \langle f' \rangle$ of the ring $R_{n-|S|} = \mathbb{C}[x_i : i \notin S] / \langle x_i x_j : i \neq j \rangle$. Then, $[J']$ is smoothable if and only if $[\tilde{J}]$ is smoothable in $\mathcal{Hilb}^n(X_{n-|S|})$. Now, by Theorem 3.3, $[J']$ is smoothable since it lies in $\Sigma(n, 1, \mathbf{1})$.

Assume now that $[J]$ is a smoothable ideal. We apply induction on n . For $n = 2$, $\mu([J])$ lies in a smoothable face since all faces of $\mathcal{K}_2^{[m]}$ are smoothable. Assume now that the statement holds for all $n' < n$. Let $\Delta_{l,n} + \mathbf{u} - \mathbf{1}$ be a hypersimplex containing $[J]$. By Theorem 3.3 and Theorem 3.2, we may assume that $\mathbf{u} = \mathbf{1}$. In particular, we have that $m = l + 1 \leq n$. If $[J]$ is smoothable, then $[J]$ is the limit of m distinct points q_1, \dots, q_m in X_n . If $m < n$, then, along the limit, q_1, \dots, q_m are contained in at most m of the lines of X_n . In particular, $[J]$ lies in the Hilbert scheme of points of those m lines. By induction, $\mu([J])$ is contained in a smoothable face. Assume now that $m = n$. Then $[J]$ is the limit of n distinct points q_1, \dots, q_n in X_n . As before, if along such limit, there is a line of X_n not containing any of the points q_1, \dots, q_n , then, we can apply induction. Therefore, we may assume that q_i lies in L_i for each $i \in [n]$. Then, q_1, \dots, q_n are the intersection of a hyperplane $\mathbb{V}(a_0 + \sum a_i x_i)$ with X_n . As in the proof of Theorem 3.3, we deduce that such limit is $\mathbb{V}(\sum a_i x_i)$. Therefore, $[J] = [\langle \sum a_i x_i \rangle]$ is contained in $\Sigma(n, 1, \mathbf{1})$ and $\mu([J])$ lies in $\Delta_{n-1,n}$ which is smoothable. \square

From Theorem 6.7, we derive the following result.

Corollary 6.8. *Let C be a curve whose unique singularity p is a rational n -fold singularity. Let $\mathbb{V}(J)$ be a length m subscheme of C , and let J_0, J_1, \dots, J_k be the ideals in the primary decomposition of J such that J_0 is supported at the singularity p and has length m' . Then $[J]$ is smoothable if and only if $\mu([J_0])$ lies in a smoothable face of $\mathcal{K}_n^{[m']}$.*

We finish this section by stating more properties of the singularities of the smoothable components.

Proposition 6.9. *Each smoothable component of $\mathcal{Hilb}^m(X_n)$ is normal and has toric singularities.*

Proof. Let Z be a smoothable component of $\mathcal{Hilb}^m(X_n)$. Then Z is birational to

$$\mathrm{Sym}^{u_1} L_1 \times \cdots \times \mathrm{Sym}^{u_n} L_n$$

for some $\mathbf{u} = (u_1, \dots, u_n)$ partition of m . By Theorem 4.7, it is enough to check the statement for $[J] \in Z$ supported at $\mathbf{0}$ such that $\mu([J])$ is a vertex of $\mathcal{K}_n^{[m]}$. The completion of the stalk of $\mathcal{Hilb}^m(X_n)$ at $[J]$ is computed in Theorem 4.1. The irreducible components of this stalk are calculated in Theorem B.3 and Theorem B.8. In Theorem 4.2 and Theorem 4.3, we identify which irreducible components of the stalk at $[J]$ correspond to each irreducible component of $\mathcal{Hilb}^m(X_n)$. For X_n , this correspondence associates Z to a unique component of the stalk, which corresponds to an ideal of the form \mathcal{Q}_i . By Theorem B.5 and Theorem B.7, \mathcal{Q}_i is normal and toric. We conclude that the completion of the stalk of Z at $[J]$ is normal. By [Sta25, Tag 07QU], the stalk of Z at $[J]$ is an excellent ring, and we deduce that the stalk is normal by [Mat89, Theorem 79]. We conclude that Z is normal and its singularities are toric. \square

As in Theorem 6.6, Theorem 6.9 is no longer true if we replace X_n with an irreducible curve C with a rational n -fold singularity. By the same techniques used in Proposition 6.9, we obtain the following result.

Corollary 6.10. *Let C be an irreducible curve whose only singularity p is a rational n -fold singularity. Then, the singularities of the smoothable components are locally union of normal toric varieties.*

7. ONGOING WORK AND FUTURE QUESTIONS

This paper sits in the theory of a combinatorial study of certain properties of Hilbert schemes of curves, not necessarily planar. It naturally leads to questions such as how to go beyond fold-like curves and investigate configurations of lines in projective space. This, in turn, gives rise to the following natural questions.

Question 1. *When the number of lines in \mathbb{C}^n is greater than n , do we still have a combinatorial description of the Hilbert scheme of points?*

The first interesting case would be four lines in \mathbb{C}^3 . One possible idea to approach this question is to study degenerations of configurations of lines to a lower dimensional affine space. The authors plan to come back to this problem in the future. Following this line of thought, this leads to.

Question 2. *Are there combinatorial descriptions of Hilbert schemes of points for configurations of planes or higher dimensional linear subspaces in \mathbb{C}^n ?*

Again, by analogy with the case treated in this paper, we believe that for *transversal* unions of planes, similarly to the situation of fold-like curves, rich geometric and combinatorial structures will emerge. Another direction of ongoing work concerns the study of the Quot scheme, which naturally arises in connection with the compactified Jacobian. Indeed, for curves with non-locally planar singularities, the appropriate source of the Abel map is the Quot scheme rather than the Hilbert scheme of points, leading us to investigate the following.

Question 3. *Is there a nice combinatorial description of the Quot scheme for fold-like curves?*

The authors expect that such a description can be useful for a concrete study of the alternate compactification of the moduli space of curves described in [HKS24].

APPENDIX A. THE HYPERSIMPLICIAL COMPLEX

In this Appendix we present the proofs of the combinatorial properties of hypersimplicial complex $\mathcal{K}_n^{[m]}$ used in Section 2. We refer to [GM82] for further details on the relation between Grassmannians, hypersimplices and their combinatorics. The hypersimplex $\Delta_{l,n}$ is defined as

$$\Delta_{l,n} := \text{Conv}\{\mathbf{e}_{i_1} + \dots + \mathbf{e}_{i_l} : 1 \leq i_1 < \dots < i_l \leq n\} = \left\{ \sum_{i=1}^n \lambda_i \mathbf{e}_i : 0 \leq \lambda_i \leq 1, \text{ and } \sum_{i=1}^n \lambda_i = l \right\}.$$

By definition, the hypersimplex $\Delta_{l,n}$ is contained in the dilated simplex $l \cdot \Delta_{n-1}$. The vertices of $\Delta_{l,n}$ are exactly the vectors $\mathbf{e}_{i_1} + \dots + \mathbf{e}_{i_l}$ for $1 \leq i_1 < \dots < i_l \leq n$. The number of vertices of $\Delta_{l,n}$ is $\binom{n}{l}$. For $0 \leq r \leq n-1$, the $(n-r)$ -faces of the hypersimplex $\Delta_{l,n}$ are of the form

$$(49) \quad \Delta_{l,n}(S_1, S_2) := \left\{ \sum_{i \notin S_1 \sqcup S_2} \lambda_i \mathbf{e}_i + \sum_{i \in S_2} \mathbf{e}_i : 0 \leq \lambda_i \leq 1 \text{ and } \sum_{i \notin S_1 \sqcup S_2} \lambda_i = l - |S_2| \right\} = \\ \text{Conv} \left(\mathbf{e}_{i_1} + \dots + \mathbf{e}_{i_{l-|S_2|}} : i_1, \dots, i_{l-|S_2|} \notin S_1 \sqcup S_2 \text{ distinct} \right) + \sum_{i \in S_2} \mathbf{e}_i$$

for $S_1 \sqcup S_2 \subseteq [n]$ and $|S_1| + |S_2| = r-1$. The face $\Delta_{l,n}(S_1, S_2)$ is obtained by setting $\lambda_i = 0$ for all $i \in S_1$ and $\lambda_i = 1$ for all $i \in S_2$. Moreover, such a face is isomorphic to the hypersimplex $\Delta_{n-r+1, l-|S_2|}$. In particular, every hypersimplex can be seen as a hypersimplicial complex. Recall that a hypersimplicial complex is a polyhedral complex whose faces are hypersimplices. See [GM82, Section 2.1.3] for these details on hypersimplices.

Lemma A.1. *Let $m \geq 2$ and $1 \leq l, l' \leq \min\{n-1, m-1\}$. Consider two integer vectors $\mathbf{u}, \mathbf{v} \in \mathbb{Z}_{\geq 0}^n$ with $|\mathbf{u}| = m-1-l$ and $|\mathbf{v}| = m-1-l'$. Then, the intersection of $(\Delta_{l,n} + \mathbf{u})$ and $(\Delta_{l',n} + \mathbf{v})$ is nonempty if and only if $\mathbf{u} - \mathbf{v} \in \{0, 1, -1\}^n$. Moreover, in this case we have that*

$$(\Delta_{l,n} + \mathbf{u}) \cap (\Delta_{l',n} + \mathbf{v}) := \Delta_{l,n}(\kappa(\mathbf{u} - \mathbf{v}, 1), \kappa(\mathbf{u} - \mathbf{v}, -1)) + \mathbf{u} = \Delta_{l',n}(\kappa(\mathbf{u} - \mathbf{v}, -1), \kappa(\mathbf{u} - \mathbf{v}, 1)) + \mathbf{v}.$$

where the function κ is defined as in (6).

Proof. We can write the translated hypersimplices $\Delta_{l,n} + \mathbf{u}$ and $\Delta_{l',n} + \mathbf{v}$ as

$$(50) \quad \Delta_{l,n} + \mathbf{u} = \left\{ \sum_{i=1}^n \lambda_i \mathbf{e}_i : \sum \lambda_i = |\mathbf{u}| - l \text{ and } u_i \leq \lambda_i \leq u_i + 1 \text{ for all } i \in [n] \right\}, \\ \Delta_{l',n} + \mathbf{v} = \left\{ \sum_{i=1}^n \lambda_i \mathbf{e}_i : \sum \lambda_i = |\mathbf{v}| - l' \text{ and } v_i \leq \lambda_i \leq v_i + 1 \text{ for all } i \in [n] \right\}.$$

Let $\lambda = \sum \lambda_i \mathbf{e}_i$ be a point in the intersection of $\Delta_{l,n} + \mathbf{u}$ and $\Delta_{l',n} + \mathbf{v}$. By (50), we have that λ_i is contained in the intersection of the intervals $[u_i, u_i + 1] \cap [v_i, v_i + 1]$. This intersection is nonempty if and only if $u_i - v_i \in \{0, 1, -1\}$. We deduce that the intersection of $\Delta_{l,n} + \mathbf{u}$ and $\Delta_{l',n} + \mathbf{v}$ is nonempty if and only if $\mathbf{u} - \mathbf{v} \in \{0, 1, -1\}^n$.

Assume now that $\mathbf{u} - \mathbf{v} \in \{0, 1, -1\}^n$. Then, we can write the intersection of the hypersimplices as

$$\begin{aligned} (\Delta_{l,n} + \mathbf{u}) \cap (\Delta_{l',n} + \mathbf{v}) &= \left\{ \sum_{i=1}^n \lambda_i \mathbf{e}_i : \sum \lambda_i = m-1 \text{ and } \begin{cases} \lambda_i = u_i \text{ for } i \in \kappa(\mathbf{u} - \mathbf{v}, 1), \\ \lambda_i = u_i + 1 \text{ for } i \in \kappa(\mathbf{u} - \mathbf{v}, -1), \\ u_i \leq \lambda_i \leq u_i + 1 \text{ for } i \in \kappa(\mathbf{u} - \mathbf{v}, 0) \end{cases} \right\} = \\ &= \left\{ \sum_{i \in \kappa(\mathbf{u} - \mathbf{v}, 0)} \lambda_i \mathbf{e}_i : \sum_{i \in \kappa(\mathbf{u} - \mathbf{v}, 0)} \lambda_i = m-1 - |\mathbf{u}| - |\kappa(\mathbf{u} - \mathbf{v}, -1)| \text{ and } 0 \leq \lambda_i \leq 1 \right\} + \mathbf{e}_{\kappa(\mathbf{u} - \mathbf{v}, -1)} + \mathbf{u} = \\ &= \left\{ \sum_{i \in \kappa(\mathbf{u} - \mathbf{v}, 0)} \lambda_i \mathbf{e}_i : \sum_{i \in \kappa(\mathbf{u} - \mathbf{v}, 0)} \lambda_i = l - |\kappa(\mathbf{u} - \mathbf{v}, -1)| \text{ and } 0 \leq \lambda_i \leq 1 \right\} + \mathbf{e}_{\kappa(\mathbf{u} - \mathbf{v}, -1)} + \mathbf{u}. \end{aligned}$$

Using (49), we conclude that

$$(\Delta_{l,n} + \mathbf{u}) \cap (\Delta_{l',n} + \mathbf{v}) = \Delta_{l,n}(\kappa(\mathbf{u} - \mathbf{v}, 1), \kappa(\mathbf{u} - \mathbf{v}, -1)) + \mathbf{u}.$$

□

Using Theorem A.1, we can construct a polyhedral complex using the translated hypersimplices $\Delta_{l,n} + \mathbf{u}$.

Proposition and Definition A.2. For $n \geq 2$ and $m \geq 2$, the set of all faces of the translated hypersimplices $\Delta_{l,n} + \mathbf{u}$ for $1 \leq l \leq \min\{n-1, m-1\}$ and $\mathbf{u} \in \mathbb{Z}_{\geq 0}^n$ with $|\mathbf{u}| = m-1-l$ form a hypersimplicial complex $\mathcal{K}_n^{[m]}$ called the (n, m) -hypersimplicial complex. Moreover, $\mathcal{K}_n^{[m]}$ forms a subdivision of $(m-1) \cdot \Delta_{n-1}$.

Proof. To verify that the translated hypersimplices $\Delta_{l,n} + \mathbf{u}$ form a polyhedral complex, it is enough to show that any two such hypersimplices intersect in a face of both hypersimplices. This follows from Theorem A.1. It remains to show that all these hypersimplices cover $(m-1) \cdot \Delta_{n-1}$. Let $\lambda = \sum \lambda_i \mathbf{e}_i$ be a point in $(m-1) \cdot \Delta_{n-1}$. In other words, $0 \leq \lambda_i \leq m-1$ and $\sum \lambda_i = m-1$. Now, consider the integer vector $\mathbf{u} = (u_1, \dots, u_n)$ where $u_i = \lfloor \lambda_i \rfloor$ for every $i \in [n]$. Then \mathbf{u} is in $\mathbb{Z}_{\geq 0}^n$ and $|\mathbf{u}| \leq m-1$. We write λ as

$$\lambda = \sum (\lambda_i - u_i) \mathbf{e}_i + \mathbf{u}.$$

Note that $0 \leq \lambda_i - u_i \leq 1$ and $\sum (\lambda_i - u_i) = m-1 - |\mathbf{u}|$. By construction $m-1 - |\mathbf{u}| \leq n$. Assume first that $m-1 - |\mathbf{u}| \leq n-1$. In this case $m-1 - |\mathbf{u}| \leq \min\{n-1, m-1\}$ and λ is contained in the translated hypersimplex $\Delta_{m-1-|\mathbf{u}|,n} + \mathbf{u}$. If $m-1 - |\mathbf{u}| = n$, then $\lambda = \mathbf{u}$ is an integer point in $(m-1) \cdot \Delta_{n-1}$, and there exists $i \in [n]$ such that $\lambda_i \geq 1$. Then λ is contained in $\Delta_{1,n} + \lambda - \mathbf{e}_i$. Therefore, we conclude that any point in $(m-1) \Delta_{n-1}$ is contained in a face of $\mathcal{K}_n^{[m]}$. □

Example A.3. For $n = 2$, the only hypersimplex is $\Delta_{1,2} = \Delta_1$. Therefore, the maximal faces $\mathcal{K}_2^{[m]}$ are the segments between the points $(m-1-i, i)$ and $(m-1-i-1, i+1)$ for $0 \leq i \leq m-2$. Fig. 4 illustrates the hypersimplicial complexes $\mathcal{K}_2^{[2]}$, $\mathcal{K}_2^{[3]}$, and $\mathcal{K}_2^{[4]}$.

Example A.4. For $n = 3$, we have two types of hypersimplices: $\Delta_{1,3} = \Delta_2$ and $\Delta_{2,3}$. For $m = 2$, $\mathcal{K}_3^{[2]}$ coincide with $\Delta_{1,3}$ as a simplicial complex. For $m = 3$, $\mathcal{K}_3^{[3]}$ has 4 maximal faces: $\Delta_{2,3}$ and $\Delta_{1,3} + \mathbf{e}_i$ for $i \in [3]$. Similarly, $\mathcal{K}_3^{[4]}$ has 9 maximal faces: $\Delta_{1,3} + \mathbf{e}_i + \mathbf{e}_j$ for $i, j \in [3]$ and $\Delta_{2,3} + \mathbf{e}_i$ for $i \in [3]$. In Fig. 5, the hypersimplicial complexes $\mathcal{K}_3^{[2]}$, $\mathcal{K}_3^{[3]}$, and $\mathcal{K}_3^{[4]}$ are depicted.

In general, $\mathcal{K}_3^{[m]}$ has $\binom{m}{2}$ hypersimplices of the form $\Delta_{1,3}$, and $\binom{m-1}{2}$ hypersimplices of the form $\Delta_{2,3}$. The first type of hypersimplices corresponds to the triangles given by the vertices $(u_1, u_2, u_3) + \mathbf{e}_i$ for $i \in [3]$ and $u_1 + u_2 + u_3 = m-2$. Similarly, the hypersimplices of the form $\Delta_{2,3}$ are triangles whose vertices are $(u_1, u_2, u_3) + \mathbf{e}_i + \mathbf{e}_j$ for $1 \leq i < j \leq 3$ and $u_1 + u_2 + u_3 = m-3$.

Example A.5. For $n = 4$, there are 3 possible hypersimplices: $\Delta_{1,4}$, $\Delta_{2,4}$ and $\Delta_{3,4}$. For instance, the maximal faces of $\mathcal{K}_4^{[3]}$ are $\Delta_{2,4}$ and $\Delta_{1,4} + \mathbf{e}_i$ for $i \in [4]$. Fig. 7 illustrates $\mathcal{K}_4^{[3]}$. Similarly, the maximal faces of $\mathcal{K}_4^{[4]}$ are the ten simplices $\Delta_{1,4} + \mathbf{e}_i + \mathbf{e}_j$ for $i, j \in [4]$, the four hypersimplices $\Delta_{2,4} + \mathbf{e}_i$ for $i \in [4]$ and the simplex $\Delta_{3,4}$. The hypersimplicial complex $\mathcal{K}_4^{[4]}$ is illustrated in Fig. 8.

For $1 \leq l \leq \min\{n-1, m-1\}$ fixed, we also consider the hypersimplicial subcomplex $\mathcal{K}_{l,n}^{[m]}$ of $\mathcal{K}_n^{[m]}$ whose maximal faces are the translated hypersimplices $\Delta_{l,n} + \mathbf{u}$ for $\mathbf{u} \in \mathbb{Z}_{\geq 0}^n$ with $|\mathbf{u}| = m-1-l$. In particular, $\mathcal{K}_n^{[m]}$ is the union of the complexes $\mathcal{K}_{l,n}^{[m]}$ for $1 \leq l \leq \min\{n-1, m-1\}$. The number of maximal faces in $\mathcal{K}_{l,n}^{[m]}$ coincides with the number of integer vectors $\mathbf{u} \in \mathbb{Z}_{\geq 0}^n$ with $|\mathbf{u}| = m-1-l$, which is $\binom{m+n-l-2}{n-1}$. Therefore, the number of maximal faces of $\mathcal{K}_n^{[m]}$ is

$$(51) \quad \sum_{l=1}^{\min\{n-1, m-1\}} \binom{m+n-l-2}{n-1}.$$

Example A.6. For $n = 3$ and $m = 4$, we have two possible hypersimplicial complexes: $\mathcal{K}_{1,3}^{[4]}$ and $\mathcal{K}_{2,3}^{[4]}$. The maximal cells of $\mathcal{K}_{1,3}^{[4]}$ are the 6 hypersimplices $\Delta_{1,3} + \mathbf{e}_i + \mathbf{e}_j$ for $i, j \in [3]$. Fig. 9 illustrates in purple the complex $\mathcal{K}_{1,3}^{[4]}$. Similarly, the complex $\mathcal{K}_{2,3}^{[4]}$ is depicted in orange in Fig. 9 and its maximal cells are the hypersimplices $\Delta_{2,3} + \mathbf{e}_i$ for $i \in [3]$.

Example A.7. The maximal cells of $\mathcal{K}_{1,4}^{[3]}$ are the 3 hypersimplices $\Delta_{1,4} + \mathbf{e}_i$ for $i \in [4]$. Fig. 9 illustrates on the left the complex $\mathcal{K}_{1,3}^{[4]}$. The complex $\mathcal{K}_{2,4}^{[3]}$ is exactly the hypersimplex $\Delta_{2,4}$. Similarly, the maximal cells of $\mathcal{K}_{2,4}^{[4]}$ are the four hypersimplices $\Delta_{2,4} + \mathbf{e}_i$ for $i \in [4]$. This complex is depicted in the right side of in Fig. 9.

By construction, the $(n-r)$ -dimensional faces of the (n, m) -hypersimplicial complex are the $(n-r)$ -dimensional faces of the translated hypersimplices. Using (49), we may describe these faces as follows. For $1 \leq l \leq \min\{n-1, m-1\}$, $\mathbf{u} \in \mathbb{Z}_{\geq 0}^n$ with $|\mathbf{u}| = m-1-l$, and for $S_1 \sqcup S_2 \subseteq [n]$ with $|S_1| + |S_2| = r-1$, we get the face

$$\mathcal{K}_n^{[m]}(S_1, S_2, l, \mathbf{u}) := \Delta_{l,n}(S_1, S_2) + \mathbf{u}.$$

Using this notation, we can write the intersection in Theorem A.1 as

$$(\Delta_{l,n} + \mathbf{u}) \cap (\Delta_{l',n} + \mathbf{v}) = \mathcal{K}_n^{[m]}(\emptyset, \emptyset, l, \mathbf{u}) \cap \mathcal{K}_n^{[m]}(\emptyset, \emptyset, l', \mathbf{v}) = \mathcal{K}_n^{[m]}(\kappa(\mathbf{u} - \mathbf{v}, 1), \kappa(\mathbf{u} - \mathbf{v}, -1), l, \mathbf{u}) = \mathcal{K}_n^{[m]}(\kappa(\mathbf{u} - \mathbf{v}, -1), \kappa(\mathbf{u} - \mathbf{v}, 1), l', \mathbf{v}).$$

Note that the face $\mathcal{K}_n^{[m]}(S_1, S_2, l, \mathbf{u})$ is contained in the intersection of $\mathcal{K}_n^{[m]}$ with the linear subspace $\{\lambda_i = u_i : \text{for } i \in S_1\}$ and $\{\lambda_i = u_i + 1 : \text{for } i \in S_2\}$. Here, $\lambda_1, \dots, \lambda_n$ are the coordinates of \mathbb{R}^n . In the following result, we study the intersection of such type of linear subspaces with the hypersimplicial complex.

Proposition A.8. For $S \subseteq [n]$ and $\mathbf{a} = (a_i)_{i \in S} \in \mathbb{Z}_{\geq 0}^{|S|}$ with $|\mathbf{a}| \leq m-1$, consider the linear subspace

$$H(S, \mathbf{a}) := \{\lambda_i = a_i : \text{for } i \in S\}.$$

Then, the intersection of $\mathcal{K}_n^{[m]}$ is isomorphic to the hypersimplicial complex $\mathcal{K}_{n-|S|}^{[m-|\mathbf{a}|]}$.

Proof. Without loss of generality assume that $S = \{n-|S|+1, \dots, n\}$, and consider the linear projection

$$\pi_S : \mathbb{R}^n \rightarrow \mathbb{R}^n / \langle \mathbf{e}_i : i \in S \rangle \simeq \mathbb{R}^{n-|S|}.$$

Note that the restriction of π_S to $H(S, \mathbf{a})$ is an isomorphism. We claim that the projection of the intersection of $\mathcal{K}_n^{[m]}$ and $H(S, \mathbf{a})$ is $\mathcal{K}_{n-|S|}^{[m-|\mathbf{a}|]}$. Note that the intersection of $\mathcal{K}_n^{[m]}$ and $H(S, \mathbf{a})$ consists of all the faces of the form $\mathcal{K}_n^{[m]}(S_1, S_2, l, \mathbf{u})$ such that $S \subseteq S_1 \sqcup S_2$, and $u_i = a_i$ for every $i \in S \cap S_1$ and $u_i + 1 = a_i$ for every $i \in S \cap S_2$. The projection of such a face is

$$\left\{ \sum_{i=1}^{n-|S|} \lambda_i \mathbf{e}_i : \sum \lambda_i = m-1-|\mathbf{a}| \text{ and } \begin{array}{l} \lambda_i = u_i \text{ for } i \in S_1 \setminus S, \\ \lambda_i = u_i + 1 \text{ for } i \in S_1 \setminus S, \\ u_i \leq \lambda_i \leq u_i + 1 \text{ for } i \notin S_1 \sqcup S_2 \end{array} \right\} = \\ \left\{ \sum_{i \notin S_1 \sqcup S_2} \lambda_i \mathbf{e}_i : \sum \lambda_i = m-1-|\mathbf{a}|-|S_2 \setminus S| \text{ and } 0 \leq \lambda_i \leq 1 \text{ for } i \notin S_1 \sqcup S_2 \right\} + \mathbf{e}_{S_2 \setminus S} + \mathbf{u} = \\ \Delta_{m-1-|\mathbf{a}|-|S_2 \setminus S|, n-|S|}(S_1 \setminus S, S_2 \setminus S) + \mathbf{u},$$

which is a maximal face of $\mathcal{K}_{n-|S|}^{[m-|\mathbf{a}|]}$. It remains to show that any face of $\mathcal{K}_{n-|S|}^{[m-|\mathbf{a}|]}$ is achieved from the projection π_S . This follows from the fact that the image of $H(S, \mathbf{a}) \cap \mathcal{K}_n^{[m]}$ through the projection π_S is the simplex $(m - |\mathbf{a}| - 1) \cdot \Delta_{n-|S|-1}$. \square

Using Theorem A.8, we introduce the notion of smoothable face.

Definition A.9. A face Γ of $\mathcal{K}_n^{[m]}$ is smoothable if one of the following conditions holds:

- $n = 1$ or $n = 2$.
- Γ is contained in $\Delta_{n-1,n} + \mathbf{u}$ for $\mathbf{u} \in \mathbb{Z}_{\geq 0}^n$ with $|\mathbf{u}| = m - n$.
- Γ is contained in a linear subspace $H(S, \mathbf{u})$ and in the intersection of $H(S, \mathbf{u})$ and $\mathcal{K}_n^{[m]}$, Γ is a smoothable face.

A first remark from Theorem A.9 is that all vertices and edges of $\mathcal{K}_n^{[m]}$ are smoothable. The third condition in Theorem A.9 is recursive, since by Theorem A.8 the intersection of $H(S, \mathbf{u})$ and $\mathcal{K}_n^{[m]}$ is the hypersimplicial complex $\mathcal{K}_{n-|S|}^{[m-|\mathbf{u}|]}$. Note that the faces of $\Delta_{n-1,n}$ are all hypersimplices of the form $\Delta_{n'-1,n'}$ for $n' \leq n$. Therefore, a face of $\mathcal{K}_n^{[m]}$ is smoothable if and only if it is a hypersimplex of the form $\Delta_{n'-1,n'}$ for $n' \leq n$. In other words, a face of the form $\mathcal{K}_n^{[m]}(S_1, S_2, l, \mathbf{u})$ is smoothable if and only if $n - l - |S_2| = n - |S_1| - |S_2| - 1$, which is equivalent to $l = |S_1| + 1$.

Example A.10. • For $n = 3$ and $m = 3$, $\Delta_{2,3}$ is a smoothable face of $\mathcal{K}_3^{[3]}$ by definition. Therefore, for $m \geq 3$, the smoothable faces are the vertices, the edges and the hypersimplices $\Delta_{2,3} + \mathbf{u} - 1$ for $|\mathbf{u}| = m$, which are illustrated in Fig. 11.

• For $n = 4$ and $m = 2$, the smoothable faces of $\mathcal{K}_4^{[2]}$ are the vertices and the edges. For $n = 4$ and $m = 3$, aside from the vertices and the edges, $\mathcal{K}_4^{[2]}$ has four 2-dimensional faces that are smoothable. They are faces of $\Delta_{2,4}$ and they arise from the 2-dimensional smoothable face of $\mathcal{K}_3^{[3]}$. In Fig. 12, these four faces are depicted in orange. For $n = 4$ and $m = 4$, $\mathcal{K}_4^{[4]}$ has 16 smoothable faces of dimension 2. They correspond to the translation $y \mathbf{e}_i$ for $i \in [4]$ of the four smoothable faces of $\Delta_{2,4}$ in $\mathcal{K}_4^{[3]}$. Moreover, $\Delta_{3,4}$ is a 3-dimensional smoothable face of $\mathcal{K}_4^{[4]}$. In Fig. 12 all the smoothable faces of $\mathcal{K}_4^{[3]}$ and $\mathcal{K}_4^{[4]}$ are illustrated.

Analogously to the notion of smoothable face, we introduce the notion of singular face.

Definition A.11. We say that a face Γ of $\mathcal{K}_n^{[m]}$ is *singular* if one of the following conditions is satisfied:

- Γ is in the intersection of two distinct maximal faces.
- Γ is smoothable of dimension at most $n - 2$.

Note that vertices are always singular faces and, for $n \geq 3$, edges are always singular. For example, for $n = 3$ the vertices and edges are exactly the only singular faces.

Example A.12. For $n = 4$ and $m = 2$, the singular faces of $\mathcal{K}_4^{[2]}$ are the vertices and edges. For $n = 4$ and $m = 2$, the singular faces of $\mathcal{K}_4^{[2]}$ are the vertices, the edges and the two dimensional faces of $\Delta_{2,4}$.

APPENDIX B. SOME COMMUTATIVE ALGEBRA

In this technical Appendix we will give first describe the first syzygies of the ideal representing a point in $\mathcal{Hilb}_0^m(X_n)$, secondly a presentation of the ideal in (26) that will be useful for thirdly compute the primary decompositions of the ideals we are interested in. We follow the notations of the previous sections.

Lemma B.1. Let $[J] \in \mathcal{Hilb}_0^m(X_n)$ and let f_1, \dots, f_l minimal generators of J as in Theorem 1.4. The beginning of the minimal free resolution of J is

$$(52) \quad \begin{array}{ccccc} R(-1)^{n \cdot \binom{l}{2}} & \longrightarrow & R^l & \longrightarrow & 0 \\ e_{i,j,k} & \longmapsto & A_{k,i}x_i e_j - A_{j,i}x_i e_k & & \end{array}$$

where $1 \leq i \leq n, 1 \leq j < k \leq l$ and $A_{k,i}$ denotes the entry (k, i) of the matrix A in Theorem 1.4.

Proof. By construction of the matrix A we have $A_{k,i}x_i f_j = A_{k,i}A_{j,i}x_i^{u_i+1}$ and $A_{j,i}x_i f_k = A_{j,i}A_{k,i}x_i^{u_i+1}$. Hence relations in (52) are syzygies of J . Consider a syzygy of the form

$$g_1 f_1 + \cdots + g_l f_l = 0,$$

where

$$g_a = \sum_{i=1}^n \sum_{b=1}^d B_{a,i,b} x_i^b.$$

Note that g_a has no independent coefficient since f_1, \dots, f_l are linearly independent. We write the syzygy as follows

$$\sum_a g_a f_a = \sum_{i=1}^n \sum_{b=1}^d \sum_{a=1}^l B_{a,i,b} A_{a,i} x_i^{u_i+b} = 0.$$

We get that for every $1 \leq i \leq n$ and $1 \leq b \leq d$,

$$(53) \quad \sum_{a=1}^l B_{a,i,b} A_{a,i} x_i^{u_i+b} = x_i^{u_i+b} \sum_{a=1}^l B_{a,i,b} A_{a,i} = 0.$$

Using that $A_{a,i} x_i^{u_i+1} = x_i^1 f_a$, we get that

$$(54) \quad \sum_{a=1}^l B_{a,i,b} x_i f_a = 0$$

for every $1 \leq i \leq n$ and $1 \leq b \leq d$. Thus, to check that the syzygy $g_1 f_1 + \cdots + g_l f_l = 0$ can be obtained from the syzygies of the form (52), it is enough to show that (54) is a linear combination of the syzygies (52) for every $1 \leq i \leq n$ and $1 \leq b \leq d$. This is equivalent to showing that the vector $(B_{1,i,b}, \dots, B_{l,i,b})$ is contained in the linear subspace spanned by the vectors $A_{k,i}e_k - A_{j,i}e_j$ for $1 \leq j < k \leq l$. Note that this linear subspace is exactly the kernel of the matrix $(A_{1,i}, \dots, A_{l,i})$. On the other hand, using (53), we get that

$$(A_{1,i}, \dots, A_{l,i}) \begin{pmatrix} B_{1,i,b} \\ \vdots \\ B_{l,i,b} \end{pmatrix} = 0.$$

We conclude that $(B_{1,i,b}, \dots, B_{l,i,b})$ is a linear combination of the vectors $A_{k,i}e_k - A_{j,i}e_j$ for $1 \leq j < k \leq l$, and hence, the syzygy $g_1 f_1 + \cdots + g_l f_l = 0$ can be obtained from the syzygies (52). \square

Following the notation introduced in Section 4 for $k = 1$ we consider the ring

$$\mathcal{S}_1 = \mathbb{C}[\alpha_2, \dots, \alpha_n, A_1, \alpha_{1,1}, \dots, \alpha_{1,u_1-1}]$$

and the ideal of \mathcal{G}_1 given by

$$(55) \quad \mathcal{G}_1 = \langle A_1 \alpha_i : 2 \leq i \leq n \rangle + \langle \alpha_i \alpha_j \alpha_{1,1} : \text{for } 2 \leq i < j \leq n \rangle.$$

Similarly, for $k \geq 2$, we consider the ring

$$(56) \quad \mathcal{S}_k = \mathbb{C}[\alpha_{i,j} : 1 \leq i \leq n, 1 \leq j \leq k \text{ and } i \neq j] \otimes \mathbb{C}[\beta_i : 1 \leq i \leq k] \otimes \mathbb{C}[\beta_{i,s} : 1 \leq i \leq k \text{ and } 2 \leq s \leq u_i - 1],$$

and the ideal of \mathcal{S}_k given by

$$(57) \quad \begin{aligned} \mathcal{G}_k := & \langle \alpha_{i,j} \alpha_{j,r} : 1 \leq i \leq n, 1 \leq j, r \leq k \text{ and } i \neq j, j \neq r \rangle + \\ & \langle \alpha_{i,j} \beta_j - \alpha_{i,r} \beta_r : 1 \leq i \leq n, 1 \leq j < r \leq k \text{ and } i \neq j, i \neq r \rangle + \\ & \langle \alpha_{i,r} \alpha_{j,r} \beta_r : 1 \leq i \leq n, k+1 \leq j \leq n \text{ and } 1 \leq r \leq k, i \neq k, i \neq j \rangle. \end{aligned}$$

Lemma B.2. For $1 \leq k \leq n$, $\mathcal{S}_k / \mathcal{J}_k$ is isomorphic to $\mathcal{S}_k / \mathcal{G}_k$.

Proof. First, we give a simplified list of generators of J_k . First of all, using the generators of the form

$$(58) \quad \alpha_{i,j,u_j-1}\alpha_{j,r,u_r-1} - \alpha_{i,j,l},$$

we may reduce the last ideal in (26) to

$$(59) \quad \langle \alpha_{i,j,u_j-1}\alpha_{j,r,u_r-1} : \text{for } j \in [k], i \in [n] \setminus \{j\} \text{ and } r \in [k] \setminus \{j\} \rangle.$$

Similarly, the generators of the second ideal in (26) are reduced to

$$(60) \quad \langle A_i\alpha_{j,r,u_r-1} : \text{for } k+1 \leq j \leq n, i \in [n] \setminus \{j\} \text{ and } r \in [k] \rangle.$$

Using the generators (60) and the third ideal in (26), we get that the generators of the first ideal in (26) are redundant. Now, we conclude that \mathcal{G}_k can be written as

$$(61) \quad \begin{aligned} \mathcal{G}_k = & \langle A_i\alpha_{j,r,u_r-1} : \text{for } k+1 \leq j \leq n, i \in [n] \setminus \{j\} \text{ and } r \in [k] \rangle \\ & + \langle \alpha_{i,j,u_j-1}\alpha_{j,j,1} - A_i : \text{for } j \in [k] \text{ and } i \in [n] \setminus \{j\} \rangle \\ & + \langle \alpha_{i,j,u_j-1}A_j : \text{for } j \in [k] \text{ and } i \in [n] \setminus \{j\} \rangle \\ & + \langle \alpha_{i,j,u_j-1}\alpha_{j,j,l+1} - \alpha_{i,j,l} : \text{for } j \in [k], i \in [n] \setminus \{j\} \text{ and } l \in [u_j - 2] \rangle \\ & + \langle \alpha_{i,j,u_j-1}\alpha_{j,r,u_r-1} : \text{for } j \in [k], i \in [n] \setminus \{j\} \text{ and } r \in [k] \setminus \{j\} \rangle. \end{aligned}$$

Now, assume first that $k = 1$. In this case, the ideal \mathcal{G}_1 is

$$(62) \quad \begin{aligned} \mathcal{G}_1 = & \langle A_i\alpha_{j,1,u_1-1} : \text{for } 2 \leq j \leq n \text{ and } i \in [n] \setminus \{j\} \rangle \\ & + \langle \alpha_{i,1,u_1-1}\alpha_{1,1,1} - A_i : \text{for } 2 \leq i \leq n \rangle \\ & + \langle \alpha_{i,1,u_1-1}A_1 : \text{for } 2 \leq i \leq n \rangle \\ & + \langle \alpha_{i,1,u_1-1}\alpha_{1,1,l+1} - \alpha_{i,1,l} : \text{for } 2 \leq i \leq n \text{ and } l \in [u_1 - 2] \rangle. \end{aligned}$$

From the generators of \mathcal{G}_1 , we see that the variables A_2, \dots, A_n and $\alpha_{i,1,l}$ for $2 \leq i \leq n$ and $l \in [u_1 - 2]$ are redundant. By eliminating these variables we get the ideal

$$\langle \alpha_{i,1,u_1-1}\alpha_{j,1,u_1-1}\alpha_{1,1,1} : \text{for } 2 \leq i < j \leq n \rangle + \langle \alpha_{i,1,u_1-1}A_1 : \text{for } 2 \leq i \leq n \rangle.$$

The proof follows from the fact that this ideal is the kernel of the map S_1/J_1 to S_1/\mathcal{G}_1 given by

$$\begin{aligned} A_1 & \longmapsto A_1, \\ \alpha_{i,1,u_1} & \longmapsto \alpha_i \quad \text{for } 2 \leq i \leq n, \\ \alpha_{1,1,l} & \longmapsto \alpha_{1,l} \quad \text{for } l \in [u_1 - 1]. \end{aligned}$$

Now assume that $k \geq 2$. Using the second type of generators in (61), we get that

$$\alpha_{i,j,u_j-1}A_j = \alpha_{i,j,u_j-1}\alpha_{j,r,u_r-1}\alpha_{r,r,1}$$

for $i \in [n]$, $j \in [k] \setminus \{i\}$, $r \in [k] \setminus \{j\}$. Note that such r exists since $k \geq 2$. Therefore, we deduce that the generators of the form $\alpha_{i,j,u_j-1}A_j$ are contained in the last ideal in (61). Similarly, using the second type of generators in (61), we get that

$$A_i\alpha_{j,r,u_r-1} = \alpha_{j,r,u_r-1}\alpha_{i,j,u_j-1}\alpha_{j,j,1}$$

for $k+1 \leq j \leq n$, $i \in [n] \setminus \{j\}$ and $r \in [k]$. We conclude that for $k \geq 2$, we have

$$(63) \quad \begin{aligned} \mathcal{G}_k = & \langle \alpha_{i,j,u_j-1}\alpha_{j,r,u_r-1}\alpha_{j,j,1} : \text{for } k+1 \leq j \leq n, i \in [n] \setminus \{j\} \text{ and } r \in [k] \rangle \\ & + \langle \alpha_{i,j,u_j-1}\alpha_{j,j,1} - A_i : \text{for } j \in [k] \text{ and } i \in [n] \setminus \{j\} \rangle \\ & + \langle \alpha_{i,j,u_j-1}\alpha_{j,j,l+1} - \alpha_{i,j,l} : \text{for } j \in [k], i \in [n] \setminus \{j\} \text{ and } l \in [u_j - 2] \rangle \\ & + \langle \alpha_{i,j,u_j-1}\alpha_{j,r,u_r-1} : \text{for } j \in [k], i \in [n] \setminus \{j\} \text{ and } r \in [k] \setminus \{j\} \rangle. \end{aligned}$$

Now, consider the map from \mathcal{S}_k to \mathcal{S}_k/J_k given by

$$\begin{aligned} \alpha_{i,j} & \longmapsto \alpha_{i,j,u_j-1}, \\ \beta_i & \longmapsto \alpha_{i,1,1}, \\ \beta_{i,l} & \longmapsto \alpha_{i,i,l}. \end{aligned}$$

This map is surjective and its kernel is exactly the ring \mathcal{G}_k

□

By Theorem 4.1 and Theorem B.2, the schematic structure $\mathcal{Hilb}^m(X_n)$ around $[J]$ can be studied through the analysis of the scheme $\text{Spec}(\mathcal{S}_k/\mathcal{G}_k)$ around the origin. Since the variables $\beta_{i,s}$ do not appear in the generators of the ideal \mathcal{G}_k . It is enough to study the quotient of

$$\mathcal{S}'_k := \mathbb{C}[\alpha_{i,j}, \beta_i : 1 \leq i \leq n, 1 \leq j \leq k \text{ and } i \neq j]$$

by \mathcal{G}_k . We now compute the primary decomposition of the ideal \mathcal{G}_k in the following technical lemmas. We start with the case $k = 1$.

Lemma B.3. *The primary decomposition of the ideal \mathcal{G}_1 is given by the ideals*

- $\langle A_1, \alpha_{1,1} \rangle$.
- $\langle \alpha_2, \dots, \alpha_n \rangle$.
- $\langle A_1, \alpha_j : 2 \leq j \leq n \text{ and } j \neq i \rangle$ for $2 \leq i \leq n$.

Proof. Assume first that $A_1 \neq 0$. Then, from (55), we deduce that $\alpha_2 = \dots = \alpha_n = 0$ and we get the component $\langle \alpha_2, \dots, \alpha_n \rangle$. On the contrary, suppose that $A_1 = 0$, and assume that $\alpha_{1,1} \neq 0$. Then, from (55) we derive that $\alpha_i \alpha_j = 0$ for every $2 \leq i < j \leq n$. This leads to the $n - 1$ components of the form $\langle A_1, \alpha_j : 2 \leq j \leq n \text{ and } j \neq i \rangle$ for $2 \leq i \leq n$. Finally, assume that $A_1 = \alpha_{1,1} = 0$. Then, every generator of \mathcal{G}_1 vanishes. Hence, we conclude that $\langle A_1, \alpha_{1,1} \rangle$ is the last irreducible component of \mathcal{G}_1 . \square

Now we continue with the case $k \geq 2$. To compute the primary decomposition in this case, we need the following lemmas:

Lemma B.4. *For $S \subseteq [k]$ and $S \neq [n]$, the primary decomposition of the ideal*

$$(64) \quad \langle \alpha_{i,j} \alpha_{j,r} : j, r \in S, i \in [n], i \neq j, j \neq r \rangle$$

is given by the ideals

$$\mathcal{G}_{S,T} = \langle \alpha_{r,s} : r, s \in S \setminus T, r \neq s \rangle + \langle \alpha_{i,j} : \text{for } j \in T \text{ and } i \in [n] \setminus \{j\} \rangle$$

for every $T \subsetneq S$. For $S = [n]$, the primary decomposition of (64) is given by the ideals $\mathcal{G}_{S,T}$ for $T \subsetneq S$ and $T \neq \emptyset$.

Proof. Assume first that for every $j \in S$ there exists $i_j \in [n] \setminus \{j\}$ such that $\alpha_{i_j,j} \neq 0$. Note that this condition defines an open subset that we denote by $U_{S,\emptyset}$. From $\alpha_{i_j,j} \alpha_{j,r} = 0$, we deduce that $\alpha_{j,r} = 0$ for every $j, r \in S$ and $r \neq j$. In particular, we get that the restriction of (64) to $U_{S,\emptyset}$ is $\mathcal{G}_{S,\emptyset}$. In particular, since $U_{S,\emptyset}$ is open, we deduce that $\mathcal{G}_{S,\emptyset}$ is part of the primary decomposition

On the contrary, assume now that there exists $j_1 \in S$ such that $\alpha_{i,j_1} = 0$ for all $i \in [n] \setminus \{j_1\}$. We distinguish two cases. First, we assume that for every $j \in S \setminus \{j_1\}$ there exists $i_j \in [n] \setminus \{j\}$ such that $\alpha_{i_j,j} \neq 0$. We denote the set defined by these constraints by $U_{S,\{j_1\}}$. As before, we get that $\alpha_{j,r} = 0$ for every $j \in S \setminus \{j_1\}$ and $r \in S \setminus \{j\}$. Therefore, the restriction of (64) to these constraints is $\mathcal{G}_{S,\{j_1\}}$. Since the ideal $\mathcal{G}_{S,\emptyset}$ is not contained in $\mathcal{G}_{S,\{j_1\}}$ and $U_{S,\{j_1\}}$ is an open subset in the complement of $U_{S,\emptyset}$, we deduce that $\mathcal{G}_{S,\{j_1\}}$ is in the primary decomposition.

Secondly, we assume the contrary. In other words, we assume that there exists $j_2 \in S \setminus \{j_1\}$ such that $\alpha_{i,j_2} = 0$ for all $i \in [n] \setminus \{j_2\}$. We distinguish again two cases. First, we assume that for every $j \in S \setminus \{j_1, j_2\}$ there exists $i_j \in [n] \setminus \{j\}$ such that $\alpha_{i_j,j} \neq 0$. Arguing as before, we get that (64) restricted to these constraints is $\mathcal{G}_{S,\{j_1,j_2\}}$ and it forms part of the primary decomposition. Secondly, we assume that there exists $j_3 \in S \setminus \{j_1, j_2\}$ such that $\alpha_{i,j_3} = 0$ for all $i \in [n] \setminus \{j_3\}$. Recursively, applying these restrictions we get that $\mathcal{G}_{S,T}$ for $T \subsetneq S$ appears in the primary decomposition of (64). Note that if $T = S$, the ideal $\mathcal{G}_{S,S}$ is generated by all $\alpha_{i,j}$ and therefore it is not in the primary decomposition.

Finally, for $S = [n]$, we get the same primary decomposition with one distinction. In this case, $\mathcal{G}_{[n],\emptyset}$ is also generated by all $\alpha_{i,j}$. Therefore, it does not appear in the primary decomposition. \square

Lemma B.5. *For $k \geq 2$, the ideal*

$$(65) \quad I_k := \langle a_1 b_1 - a_i b_i : 2 \leq i \leq k \rangle$$

in $\mathbb{C}[a_1, \dots, a_k, b_1, \dots, b_k]$ is toric of Krull dimension $k + 1$.

Proof. It suffices to prove that the binomial ideal $I_k = \langle a_1 b_1 - a_i b_i : 2 \leq i \leq k \rangle$ is prime of Krull dimension $k + 1$, we will do it by induction on k . For $k = 2$, I_k is the ideal of the cone over Segre variety in \mathbb{P}^3 , and therefore, it is prime of Krull dimension 3. Now, assume that $I_{k'}$ is prime for all $k' < k$. Assume first that $a_1 \neq 0$, then $b_1 = \frac{a_2 b_2}{a_1}$. After eliminating the variable b_1 , we get the ideal $\langle a_2 b_2 - a_i b_i : 3 \leq i \leq n \rangle$ which by induction is prime of dimension k . Therefore, in the open subset $a_1 \neq 0$ we get a unique reduced irreducible component of dimension $k + 1$. Assume now that $a_1 = 0$. Then, $a_i b_i = 0$ for every $2 \leq i \leq k$. The irreducible components of the corresponding variety are given by the equations $a_1 = a_i = b_j = 0$ for $i \in S_1$ and $j \in S_2$ with $S_1 \sqcup S_2 = \{2, \dots, k\}$. All these components lie in the boundary of the component obtained by assuming $a_1 \neq 0$. \square

We now give a description of the projective toric variety defined by the ideal in Theorem B.5.

Proposition B.6. *The polytope P_k associated to the toric ideal (65) is*

$$(66) \quad P_k = \text{Conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_i, \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_i : \text{for } i \in \{3, \dots, k\}).$$

The facets of P_k are the convex hull of the vertices

$$V_0 \cup \{\mathbf{e}_i : i \in S_1\} \cup \{\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_i : i \in S_2\},$$

for $S_1 \sqcup S_2 = \{3, \dots, k\}$ and $V_0 = \{\mathbf{0}, \mathbf{e}_1\}, \{\mathbf{0}, \mathbf{e}_2\}, \{\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2\}$ or $\{\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2\}$. In particular, the facets of P_k are simplices.

Proof. We find a monomial parametrization of the projective toric variety defined by (65). Assume that $a_1 = 1$ and $a_i \neq 0$ for $i \neq 1$. Then, $b_1 = a_2 b_2$ and $b_i = b_1 a_i^{-1} = a_2 b_2 a_i^{-1}$ for $3 \leq i \leq k$. Fixing the coordinates of \mathbb{P}^{2k-1} as $[a_1, \dots, a_k, b_1, \dots, b_k]$, the monomial parametrization is given by

$$\begin{aligned} (\mathbb{C}^*)^k &\longrightarrow \mathbb{P}^{2k-1} \\ t = (t_1, \dots, t_k) &\longmapsto [1, t_2, \dots, t_k, t_1 t_2, t_1, t_1 t_2 t_3^{-1}, \dots, t_1 t_2 t_k^{-1}]. \end{aligned}$$

where $a_1 = 1$, $a_i = t_i$ for $2 \leq i \leq k$ and $b_2 = t_1$. The description of P_k by (66) follows from the exponents of this monomial map.

To find the faces of the polytope P_k , we minimize the scalar product $\langle \mathbf{u}, - \rangle$ by a vector $\mathbf{u} = (u_1, \dots, u_k) \neq \mathbf{0}$ over P_k . Let V be the set of vertices among the ones in (66) where the minimum of $\langle \mathbf{u}, - \rangle$ is achieved. Assume first that such minimum is 0. In other words, $\mathbf{0}$ is contained in V . This implies that $u_i \geq 0$ for every $i \in [k]$. First, we claim that \mathbf{e}_1 and \mathbf{e}_2 are not contained simultaneously in V . Indeed, assume that $\mathbf{e}_1, \mathbf{e}_2 \in V$. Then, we have that $u_1 = u_2 = 0$. Therefore, $\langle \mathbf{u}, \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_i \rangle = -u_i \leq 0$ for $3 \leq i \leq k$. Since the minimum is 0, we deduce $u_i = 0$ for $3 \leq i \leq k$ and $\mathbf{u} = \mathbf{0}$. Therefore, \mathbf{e}_1 and \mathbf{e}_2 are not contained simultaneously in V . Now, $\mathbf{e}_1 + \mathbf{e}_2$ is not contained in V . Indeed, if $\mathbf{e}_1 + \mathbf{e}_2 \in V$, then $\langle \mathbf{u}, \mathbf{e}_1 + \mathbf{e}_2 \rangle = u_1 + u_2 = 0$. Since $u_1, u_2 \geq 0$, we get that $u_1 = u_2 = 0$, and hence, $\mathbf{e}_1, \mathbf{e}_2 \in V$. We conclude that $\mathbf{e}_1 + \mathbf{e}_2$ is not contained in V . Similarly, we claim that \mathbf{e}_i and $\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_i$ for $3 \leq i \leq k$ are not contained simultaneously in V . Assume that $\mathbf{e}_i, \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_i \in V$ for $3 \leq i \leq k$. Then, $u_i = 0$ and $u_1 + u_2 - u_i = u_1 + u_2 = 0$. This implies that the minimum is also achieved at $\mathbf{e}_1 + \mathbf{e}_2$, and thence, $\mathbf{e}_1 + \mathbf{e}_2 \in V$.

Now assume that the minimum of $\langle \mathbf{u}, - \rangle$ is obtained at a facet of P_k . This implies that V must contain at least k vertices. Since \mathbf{e}_i and $\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_i$ are not simultaneously contained in V , we deduce that among the vertices $\{\mathbf{e}_i, \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_i : 3 \leq i \leq k\}$ only $k - 2$ can be simultaneously in V . Among the other 4 vertices, only $\mathbf{0}$ and \mathbf{e}_1 or $\mathbf{0}$ and \mathbf{e}_2 may be simultaneously contained in V . We conclude that V must be of the form

$$V = V_0 \cup \{\mathbf{e}_i : i \in S_1\} \cup \{\mathbf{e}_i : i \in S_2\},$$

for $S_1 \sqcup S_2 = \{3, \dots, n\}$ and $V_0 = \{\mathbf{0}, \mathbf{e}_1\}$ or $\{\mathbf{0}, \mathbf{e}_2\}$.

Assume now that the minimum of $\langle \mathbf{u}, - \rangle$ is strictly negative. Then, $\mathbf{0} \notin V$. As before, if \mathbf{e}_1 and \mathbf{e}_2 are simultaneously contained in V , then the minimum is $u_1 = u_2 < 0$. This is a contradiction since $\langle \mathbf{u}, \mathbf{e}_1 + \mathbf{e}_2 \rangle = u_1 + u_2 < u_1$. Therefore, \mathbf{e}_1 and \mathbf{e}_2 are not simultaneously contained in V . Similarly, assume that \mathbf{e}_i and $\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_i$ are simultaneously contained in V . Then, the minimum would be $u_i = u_1 + u_2 - u_i$. In particular, $\langle \mathbf{u}, \mathbf{e}_1 + \mathbf{e}_2 \rangle u_1 + u_2 = 2u_i < u_i$, which is a contradiction since the minimum is u_i . In this case, we conclude that V must be of the form

$$V = V_0 \cup \{\mathbf{e}_i : i \in S_1\} \cup \{\mathbf{e}_i : i \in S_2\},$$

for $S_1 \sqcup S_2 = \{3, \dots, n\}$ and $V_0 = \{\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2\}$ or $\{\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2\}$. Finally, the facets of P_n are simplices since they have dimension $k - 1$ and they are the convex hull of k vertices. \square

Lemma B.7. *For $n \geq 2$, the polytope P_k is normal. In particular, the affine cone of $\mathbb{V}(I_k)$ is a normal affine variety.*

Proof. We use the fact that a polytope admitting a unimodular simplicial subdivision is normal, and we show that P_k admits such a decomposition by induction on k . For $k = 2$, $P_2 = \text{Conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2)$, which is normal. In this case, the subdivision is given by the simplices $\text{Conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2)$ and $\text{Conv}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2)$. Assume now that P_{k-1} admits a unimodular simplicial subdivision \mathcal{P}_{k-1} and let Δ be a simplex in the subdivision. Since P_{k-1} is contained in P_k , we consider the simplex Δ^+ obtained by taking the convex hull of Δ and \mathbf{e}_n . Similarly, we consider the simplex Δ^- obtained by taking the convex hull of Δ and $\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_n$. We claim that the $\mathcal{P}_k := \{\Delta^+, \Delta^- : \Delta \in \mathcal{P}_{k-1}\}$ is a unimodular simplicial subdivision of P_k . Indeed, by construction the normalized volume of Δ^+ and Δ^- is one. Therefore, \mathcal{P}_k is unimodular and simplicial. Now, P_k is the union of the polytopes

$$P_k^+ := \text{Conv}(P_{k-1}, \mathbf{e}_n) \text{ and } P_k^- := \text{Conv}(P_{k-1}, \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_n).$$

The polytopes P_k^+ and P_k^- are subdivided by Δ^+ and Δ^- respectively for $\Delta \in \mathcal{P}_{k-1}$. Therefore, \mathcal{P}_k is a unimodular simplicial subdivision of P_k , and hence, P_k is normal. \square

Lemma B.8. *The primary decomposition of the ideal \mathcal{G}_k for $k \geq 2$ is given by the ideals*

- For every $i \in [n]$,

$$\mathcal{Q}_i := \langle \alpha_{i,r}\beta_r - \alpha_{i,s}\beta_s : r, s \in [k], r, s \neq i \rangle + \langle \alpha_{r,s} : r \in [n] \setminus \{i\}, s \in [k], r \neq s \rangle.$$

- For every $S \subseteq [k]$ with $1 \leq |S| \leq \min\{k, n - 2\}$

$$\mathcal{G}_S := \langle \beta_j : j \in S \rangle + \langle \alpha_{j,r} : j \in S, r \in [k], j \neq r \rangle + \langle \alpha_{r,s} : r \in [n] \setminus S, s \in [k] \setminus S, r \neq s \rangle.$$

Proof. We stratify the affine space given by \mathcal{S}_k through the subsets U_S for $S \subseteq [k]$

$$U_S = \{\beta_j = 0 : j \in S\} \cap \{\beta_j \neq 0 : j \notin S\}$$

and we check the irreducible components of $\mathbb{V}(\mathcal{G}_k)$ restricted to each U_S . We first focus on the case U_\emptyset . In other words, $S = \emptyset$ and we assume that $\beta_j \neq 0$ for every $j \in [k]$. Under this assumption, the ideal \mathcal{G}_k becomes

$$\begin{aligned} & \langle \alpha_{i,j}\alpha_{j,r} : 1 \leq i \leq n, 1 \leq j, r \leq k \text{ and } i \neq j, j \neq r \rangle + \\ & \langle \alpha_{i,j}\beta_j - \alpha_{i,r}\beta_r : 1 \leq i \leq n, 1 \leq j < r \leq k \text{ and } i \neq j, i \neq r \rangle + \\ & \langle \alpha_{i,r}\alpha_{j,r} : 1 \leq i \leq n, k+1 \leq j \leq n \text{ and } 1 \leq r \leq k, i \neq k, i \neq j \rangle. \end{aligned}$$

Without loss of generality, we may further assume that $\alpha_{i_0,j_0} \neq 0$ for some $i_0 \in [n], j_0 \in [k]$ with $i_0 \neq j_0$. From the generators of \mathcal{G}_k , we deduce that $\alpha_{j_0,r} = 0$ for $r \in [k] \setminus \{j_0\}$. For every $r \in [k] \setminus \{j_0\}$ we also have that

$$\alpha_{i_0,j_0}\beta_{j_0} = \alpha_{i_0,r}\beta_r.$$

Since $\alpha_{i_0,j_0}\beta_{j_0}, \beta_r \neq 0$, we deduce that $\alpha_{i_0,r} \neq 0$ for every $r \in [k]$. In particular, we obtain that $\alpha_{r,s} = 0$ for $r, s \in [k]$ with $r \neq i_0, s$. Similarly, from the generators of \mathcal{G}_k , we also get that $\alpha_{i,j_0} = 0$ for $i \geq k+1$ and $i \neq i_0$. Thus, we get that

$$0 = \alpha_{i,j_0}\beta_{j_0} = \alpha_{i,r}\beta_r.$$

for $r \in [k], i \geq k+1$ and $i \neq 0$. Since $\beta_r \neq 0$, we deduce that $\alpha_{i,r} = 0$ for $r \in [k], i \geq k+1$ and $i \neq i_0$. From all these conditions, we deduce that the restriction of \mathcal{G}_k to the open subsets $U_\emptyset \cap \{\alpha_{i_0,j_0}\}$ is \mathcal{K}_{i_0} which is prime by Theorem B.5.

Next, we compute the primary components of \mathcal{G}_k in U_S with $|S| = 1$. In other words, assume that there exists $j_0 \in [k]$ such that $\beta_{j_0} = 0$ and $\beta_j \neq 0$ for $j \in [k] \setminus \{j_0\}$. Since $k \geq 2$, we get that for every $j \in [k] \setminus \{j_0\}$ and $i \in [n] \setminus \{j, j_0\}$

$$0 = \alpha_{i,j}\beta_j - \alpha_{i,j_0}\beta_{j_0} = \alpha_{i,j}\beta_j.$$

Since $j \neq j_0$, we deduce that $\alpha_{i,j} = 0$ for $j \in [k] \setminus \{j_0\}$ and $i \in [n] \setminus \{j, j_0\}$. In particular, the restriction of \mathcal{G}_k to $U_{\{j_0\}}$ is given by

$$(67) \quad \begin{aligned} & \langle \beta_{j_0} \rangle + \langle \alpha_{i,j_0}\alpha_{j_0,r} : r \in [k] \setminus \{j_0\} \text{ and } i \in [n] \setminus \{j_0\} \rangle + \langle \alpha_{j_0,j}\beta_j - \alpha_{j_0,r}\beta_r : r, j \in [k] \setminus \{j_0\} \rangle + \\ & \langle \alpha_{i,j} : i \in [n] \setminus \{j_0\}, j \in [k] \setminus \{j_0\}, i \neq j \rangle. \end{aligned}$$

We distinguish two cases:

- Assume first that $\alpha_{j_0,r} \neq 0$ for every $r \in [k] \setminus \{j_0\}$. From (67) we deduce that $\alpha_{i,j_0} = 0$ for every $i \neq j_0$. One may check that under this assumption, \mathcal{G}_k becomes the ideal $\mathcal{Q}_{j_0} + \langle \beta_{j_0} \rangle$. Therefore, it does not lead to a new component of the primary decomposition.
- On the contrary, assume that there exists $r \in [k] \setminus \{j_0\}$ with $\alpha_{j_0,r_0} = 0$. Then, we get that

$$0 = \alpha_{j_0,r}\beta_r - \alpha_{j_0,r_0}\beta_{r_0} = \alpha_{j_0,r}\beta_r$$

for every $r \in [k] \setminus j_0$. Since $\beta_r \neq 0$, we deduce that $\alpha_{j_0,r} = 0$ for every $r \in [k] \setminus \{j_0\}$. In particular, in this case, \mathcal{G}_k becomes

$$\langle \beta_{j_0} \rangle + \langle \alpha_{j_0,r} : r \in [k] \setminus \{j_0\} \rangle + \langle \alpha_{i,j} : i \in [n] \setminus \{j_0\}, j \in [k] \setminus \{j_0\}, i \neq j \rangle = \mathcal{G}_{\{j_0\}}.$$

Since $\mathcal{G}_{\{j_0\}}$ does not contain any of the \mathcal{Q}_i , it forms part of the irreducible decomposition of \mathcal{G}_k .

Next, we apply induction on the cardinality of $S \subseteq [k]$ with $|S| \leq n - 2$. Let $2 \leq a \leq \min\{k, n - 2\}$ and assume that \mathcal{G}_S is part of the primary decomposition of \mathcal{G}_k for every $S \subseteq [k]$ with $|S| < a$. Let $S \subseteq [k]$ with $|S| = a$ and restrict \mathcal{G}_k to U_S . In other words, assume that $\beta_i = 0$ for $i \in S$ and $\beta_i \neq 0$ for $i \notin S$. We show that the only ideal in the primary decomposition of \mathcal{G}_k appearing in U_S is \mathcal{G}_S . For $r \in [k] \setminus S$, $j \in S$ and $i \in [n] \setminus \{j, r\}$ we get

$$0 = \alpha_{i,r}\beta_r - \alpha_{i,j}\beta_j = \alpha_{i,r}\beta_r.$$

Since $r \notin S$, $\beta_r \neq 0$ and we get $\alpha_{i,r} = 0$ for every $r \in [k] \setminus S$ and $i \in [n] \setminus \{r\}$. Therefore, the restriction of \mathcal{G}_k to U_S is

$$(68) \quad \langle \beta_j : j \in S \rangle + \langle \alpha_{i,r} : r \in [k] \setminus S, i \in [n] \setminus \{j\} \rangle + \langle \alpha_{i,j}\alpha_{j,r} : j, r \in S, i \in [n], i \neq j \text{ and } j \neq r \rangle.$$

By Theorem B.4, the primary decomposition of (68) is given by the ideals

$$(69) \quad \langle \beta_j : j \in S \rangle + \langle \alpha_{i,r} : r \in [k] \setminus S, i \in [n] \setminus \{j\} \rangle + \mathcal{G}_{S,T}$$

for $T \subsetneq S$. One may check that for $T \neq \emptyset$, we have that $\mathcal{G}_{S,T}$ contains the ideal $\mathcal{G}_{S \setminus T}$. In particular, the ideal (69) does not appear in the primary decomposition for $T \neq \emptyset$. For $T = \emptyset$, we obtain that $\mathcal{G}_S = \mathcal{G}_{S,\emptyset}$. Moreover, \mathcal{G}_S does not contain and it is not contained in any of the ideals \mathcal{Q}_i and $\mathcal{G}_{S'}$ for $S' \subseteq [k]$ with $|S'| < a$. We conclude that \mathcal{G}_S is in the primary decomposition of \mathcal{G}_k .

It remains to show that \mathcal{G}_S does not appear in the primary decomposition for $|S| = n - 1, n$. For $|S| = n - 1$, we denote by i_S the only integer in $[n] \setminus S$. Then, we get that \mathcal{G}_S contains \mathcal{Q}_{i_S} . Therefore, it does not appear in the primary decomposition. \square

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