

PRIME DETECTING QUASI-MODULAR FORMS IN HIGHER LEVEL

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ABSTRACT. In a previous work, the authors resolved a conjecture about the structure of prime-detecting quasi-modular forms by studying sign changes occurring in quasi-modular cusp forms. In this paper, we extend the considerations to prime-detecting quasi-modular forms of higher level, in particular describing the structure of the space of quasi-modular forms that detect primes in various arithmetic progressions. We also provide an “analytic” proof of the level one case.

1. INTRODUCTION

Recently Craig, van Ittersum and Ono [CvIO24], showed that the set of primes is “partition theoretic”, meaning that the set of primes can be described as the set of solutions certain Diophantine equations involving partition functions. In fact they showed that there is an infinite family of such partition-theoretic identities that “strongly detect” primes. To describe one of the simplest such examples, given an $a \in \mathbb{N}$, we define the MacMahon partition function

$$M_a(n) := \sum_{\substack{0 < s_1 < \dots < s_a \\ n = m_1 s_1 + \dots + m_a s_a}} m_1 \dots m_a.$$

Then one of the results of [CvIO24] states that an integer n is a prime if and only if

$$(n^2 - 3n + 2)M_1(n) = 8M_2(n).$$

As mentioned before, this is just one of an infinite family of such relations. Some more recent results are available in [Cra25, Gom25, KMS25].

Definition 1.1. A sequence of numbers $a(n)$ is said to *detect* a set $A \subseteq \mathbb{N}$ if $a(n) = 0$ whenever $n \in A$. We say that $a(n)$ *strongly detects* A if in addition, $a(n) \neq 0$ whenever $n \notin A$.

The existence of prime detecting partition identities arose within the larger context of quasi-modular forms whose n -th Fourier coefficient detects (or strongly detects) primes. In particular, define a subset Ω of the graded ring of (integer weight) quasi-modular forms (of full level) such that $f \in \Omega$ if and only if for $(q := e^{2\pi i\tau})$

$$f(\tau) = \sum_{n \geq 0} c_f(n) q^n,$$

we have $c_f(n)$ strongly detects the primes. Let \mathcal{E} denote the space of *quasi-modular Eisenstein series* (i.e., the vector space spanned by Eisenstein series and their derivatives). In [CvIO24, Theorem 2.3] the authors classify $\mathcal{E} \cap \Omega$, and propose the following conjecture.

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Conjecture 1.2. *With notation as above, $\Omega \subset \mathcal{E}$.*

In a recent work [KKL25], we prove this conjecture for quasi-modular forms of full level. In fact we deduce Conjecture 1.2 as a consequence of the following slightly stronger result. Let $\tilde{\Omega}$ be the set of all quasi-modular forms that detect primes. By this we mean that the Fourier coefficients of the quasi-modular form detect primes as in Definition 1.1.

Theorem 1.3. *With notation as above, $\tilde{\Omega} \subset \mathcal{E}$. In particular, $\Omega \subset \mathcal{E}$.*

The purpose of this note is to investigate the situation for quasi-modular forms of arbitrary level. For $N, M \in \mathbb{N}$ and $m \in \mathbb{Z}$ with $\gcd(m, M) = 1$, we let $\tilde{\Omega}_{m,M}(\Gamma_1(N))$ be the set of quasi-modular forms f of level N (i.e., modular on $\Gamma_1(N)$) for which $c_f(p) = 0$ for every $p \equiv m \pmod{M}$ with $p \nmid N$. Similarly, we let $\Omega_{m,M}(\Gamma_1(N))$ be those which are prime detecting in the sense that $f \in \Omega_{m,M}(\Gamma_1(N))$ if and only if

$$\{n \equiv m \pmod{M} : \gcd(n, N) = 1, c_f(n) = 0\} = \{p \equiv m \pmod{M} : p \text{ prime}, p \nmid N\}.$$

More generally we may define $\tilde{\Omega}_{m,M}(\Gamma_0(N), \chi)$ to be the set of all quasi-modular forms of level $\Gamma_0(N)$ and character χ which detect primes.

It turns out that a naive generalization of Theorem 1.3 to higher level is not true. Contrary to the results in [KKL25], a quasi-modular form that detects primes need not lie in $\mathcal{E}(\Gamma_0(N), \chi)$ itself. That is, it is no longer true that $\tilde{\Omega}_{m,M}(\Gamma_0(N), \chi) \subseteq \mathcal{E}(\Gamma_0(N), \chi)$, the set of quasi-modular Eisenstein series (of suitable level, character etc.). More precisely, the “cuspidal part” of f need not be zero. For example, if $E \in \mathcal{E}(\Gamma_0(N), \chi) \cap \tilde{\Omega}_{1,3}(\Gamma_0(N), \chi)$, then for any quasi-modular form f of level N and character χ , so is

$$E + f - f \otimes \chi_{-3},$$

where $f \otimes \chi$ denotes the quadratic twist $f \otimes \chi(\tau) = \sum_{n \geq 0} \chi(n) c_f(n) q^n$ of f and $\chi_D(n) := (\frac{D}{n})$ is the Kronecker character. However, *all* of the coefficients of $f - f \otimes \chi_{-3}$ vanish in the given arithmetic progression, so this function doesn’t affect the detection problem *within this arithmetic progression*.

To make matters more precise, we introduce the following sieving operator. For $m \in \mathbb{Z}$ and $M \in \mathbb{N}$, we define the *sieving operator* $S_{M,m}$ acting on quasi-modular forms by

$$f|S_{M,m}(\tau) := \sum_{n \equiv m \pmod{M}} c_f(n) q^n.$$

The sieving operator $S_{M,m}$ maps quasi-modular forms of level N to quasi-modular forms of level $\text{lcm}(N, M^2, MN)$ (for example, see [BHHK24, Lemma 2.2 (2)]).

When detecting primes in the arithmetic progression $m \pmod{M}$, we may obviously add an arbitrary $g|S_{M',m'}$ for any arithmetic progression $m' \pmod{M'}$ that does not intersect the arithmetic progression $m \pmod{M}$. More generally, we may consider all the forms in the kernel of the sieving operator $S_{M,m}$. Thus $\ker(S_{M,m})$ sits naturally inside $\tilde{\Omega}_{m,M}$ as a subspace. Also, in view of Remark 3 below, it is also necessary to sieve the “old-forms” away from f with the family $\{S_{N,n}\}_{(n,N)=1}$ of operators. Our main theorem asserts that, once this is done, what remains is Eisenstein.

Theorem 1.4. *For $M, N \in \mathbb{N}$ and $m \in \mathbb{Z}$ with $\gcd(m, M) = 1$, let $\mathcal{V}_{m,M}$ be the subset of $n \in \{0, \dots, N-1\}$ for which $(N\mathbb{Z} + n) \cap (M\mathbb{Z} + m)$ is non-empty¹ and $\gcd(n, N) = 1$. Then for*

¹ $(N\mathbb{Z} + n) \cap (M\mathbb{Z} + m) \neq \emptyset$ iff $m \equiv n \pmod{(M, N)}$. The “only if” part is clear, while for the “if” part, we have the solution $x = m - (m - n)\alpha M / (M, N) = n + (m - n)\beta N / (M, N)$ where $\alpha M + \beta N = (M, N)$.

a quasi-modular form f of level N , we have $f \in \tilde{\Omega}_{m,M}(\Gamma_1(N))$ if and only if $f \in \tilde{\Omega}_{n,N}(\Gamma_1(N))$ for every $n \in \mathcal{V}_{m,M}$.

Moreover, all of the coefficients of elements of $\tilde{\Omega}_{m,M}(\Gamma_1(N))$ from the arithmetic progression $n \equiv m \pmod{M}$ relatively prime to N come from coefficients of Eisenstein series in the sense that

$$\bigoplus_{n \in \mathcal{V}_{m,M}} \tilde{\Omega}_{m,M}(\Gamma_1(N)) |S_{M,m}|S_{N,n} = \bigoplus_{n \in \mathcal{V}_{m,M}} \mathcal{E}(\Gamma_1(N)) \cap \tilde{\Omega}_{n,N}(\Gamma_1(N)) |S_{M,m}|S_{N,n}.$$

In particular, we have

$$\bigoplus_{n \in \mathcal{V}_{m,M}} \Omega_{m,M}(\Gamma_1(N)) |S_{M,m}|S_{N,n} = \bigoplus_{n \in \mathcal{V}_{m,M}} \mathcal{E}(\Gamma_1(N)) \cap \Omega_{n,N}(\Gamma_1(N)) |S_{M,m}|S_{N,n}.$$

Remark 1. Before we move forward, we mention that a Galois theoretic proof of Conjecture 1.2 was recently obtained in [vIMOS25]. Their proof relies on an extension of the fundamental lemma of Ono-Skinner [OS98]. Their proof rests on showing that the Fourier coefficients of cusp forms vary erratically in congruence classes, while the proof of Theorem 1.3 rests on showing that the signs of the Fourier coefficients of cusp forms vary erratically. With regard to forms of higher level, the proof of [vIMOS25] seems to follow through but for an important caveat. The forms considered should not have complex multiplication (in the sense of Ribet [Rib85]). Our proof does not require this restriction and works in the most general setting. The key input here is provided by the prime number theorems² for L -functions and Rankin-Selberg L -functions attached to quasi-modular forms (these are easily seen to be shifts of the corresponding L functions attached to the original holomorphic modular forms and enjoy all of their analytic properties). These results rest on the analytic properties of the associated L -functions and therefore ultimately on the theory of newforms, irrespective of whether or not the forms possess complex multiplication. Thus our results are slightly more general than what can be deduced using Galois theoretic considerations.

Remark 2. A weaker statement (than Theorem 1.4) along the lines of Theorem 1.3 reads that, for any $n \in \mathcal{V}_{m,M}$,

$$\tilde{\Omega}_{m,M}(\Gamma_1(N)) |S_{M,m}|S_{N,n} \subset \mathcal{E}(\Gamma_1(N)) |S_{M,m}|S_{N,n}. \quad (1.1)$$

Remark 3. The condition $\gcd(n, N) = 1$ in the theorem is necessary. There is a natural operator V_d defined by

$$f|V_d(\tau) = f(d\tau) = \sum_{n \geq 0} c_f(n)q^{dn}.$$

This operator sends quasi-modular forms of level N to those of level Nd . By applying the operators V_p to forms of level $N' \mid N$, we may artificially force $c_f(p) = 0$ for $p \mid N$. For example, if f is a quasi-modular form of level N and g is a quasi-modular form of level $\frac{N}{p}$ with $c_g(1) \neq 0$, then the Fourier coefficients of

$$f - \frac{c_f(p)}{c_g(1)} g|V_p$$

²For an L -function $L(s) = \sum_{n \geq 1} a_n n^{-s}$ (initially defined for $\Re(s) > 1$), the prime number theorem refers to the asymptotic formula of the form

$$\sum_{p \leq x} a_p = \frac{cx}{\log x} + o(x),$$

where the constant c may be zero.

vanish precisely at the prime p and any p' for which $c_f(p') = 0$.

Remark 4. The sieving operators $S_{M,m}$ and $S_{N,n}$ are a commuting family of projections in the sense that $S_{N,n} \circ S_{M,m} = S_{M,m} \circ S_{N,n} = S_{(n+N\mathbb{Z}) \cap (m+M\mathbb{Z})}$. Recall that, if non-empty, by the Chinese remainder theorem, $(n+N\mathbb{Z}) \cap (m+M\mathbb{Z})$ defines an arithmetic progression modulo $\text{lcm}(N, M)$. If $(n+N\mathbb{Z})$ does not intersect with $(m+M\mathbb{Z})$, then by $S_{(n+N\mathbb{Z}) \cap (m+M\mathbb{Z})}$, we denote the zero operator (which annihilates every quasi-modular form). Therefore the direct sum over $\bigoplus_{n \in \mathcal{V}_{m,M}}$ in Theorem 1.4 maybe replaced with $\bigoplus_{\substack{1 \leq n \leq N \\ \gcd(n, N)=1}}$.

Remark 5. We also note that $S_{N,n} \circ S_{N,n} = S_{N,n}$ (and similarly for $S_{M,m}$ etc.). In particular, $S_{M,m} \circ S_{N,n}$ is a projection into the images of the respective sieving operators. Every element in $\tilde{\Omega}_{m,M}$ can be written *uniquely* as a sum of quasi modular forms coming from $\tilde{\Omega}_{m,M} \cap \ker(S_{M,m} \circ S_{N,n})$ and $\tilde{\Omega}_{m,M}|S_{M,m}|S_{N,n}$. Our main theorem now asserts that the second component is Eisenstein.

In light of Theorem 1.3, the space $\tilde{\Omega}$ naturally breaks into two components; the first one arising from the kernel of the sieving operator, and the second arising from quasi-modular Eisenstein series. The sieving operator $S_{m,M}$ is a projection operator and its kernel is quite large (for example given a cusp form f , $f - f|S_{m,M} \in \ker(S_{m,M})$). Thus we direct our attention to the Eisenstein series part of $\tilde{\Omega}_{m,M}$. We define the Eisenstein series following Sections 4.5 and 4.6 of [DS05]. Let χ, ψ be Dirichlet characters, primitive of level N_1, N_2 respectively. Let $k \geq 2$ be an integer. Suppose that $\chi(-1)\psi(-1) = (-1)^k$. Suppose

$$E_{k,\chi,\psi}(\tau) = \delta(\chi)L(1-k, \psi) + 2 \sum_{n=1}^{\infty} \sigma_{k-1}^{\chi,\psi}(n)q^n, \quad (1.2)$$

where as before $q = e^{2\pi i\tau}$ and where

$$\sigma_{k-1}^{\chi,\psi}(n) = \sum_{d|n} \chi\left(\frac{n}{d}\right) \psi(d)d^{k-1} \quad (1.3)$$

is the weighted divisor function. The constant $\delta(\psi)$ equals 1 if $\psi \equiv 1$ and is zero otherwise. If N is the least common multiple of N_1, N_2 so that $\chi\psi$ is a primitive Dirichlet character modulo N , $E_{k,\chi,\psi}$ is modular of weight k and level $\Gamma_0(N)$ with nebentype character $\chi\psi$. Finally, we denote by E_2 , the usual non-holomorphic Eisenstein series of weight 2, with Fourier expansion given by

$$E_2(\tau) := 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n.$$

Ultimately, we want to describe a basis for this space³. To this end, we make the following definition. Let $M \in \mathbb{N}$ be fixed and choose m coprime to M . For every pair of integers $\ell \geq 2$ and $k \geq 2$, and two Dirichlet characters χ, ψ modulo M define

$$H_{k,\ell,\chi,\psi} := \overline{\chi(m)}D^{\ell-1}E_{k,\psi,\chi} - \psi(m)E_{\ell,\overline{\psi},\overline{\chi}}$$

where we set $E_{0,\chi,\psi} \equiv 0$. Here D is the familiar differential operator, defined as

$$D := \frac{1}{2\pi i} \frac{d}{d\tau}.$$

³Similar to the H_k 's defined in [CvIO24].

Theorem 1.5. *The space*

$$\bigoplus_{n \in \mathcal{V}_{m,M}} \tilde{\Omega}_{m,M}(\Gamma_1(N)) |S_{M,m}|S_{N,n}$$

is spanned by (the image under the sieving operator of)

$$\bigcup_{K=2}^{\infty} \{H_{k_1, \ell_1, \chi_1, \psi_1} - H_{k_2, \ell_2, \chi_2, \psi_2} \mid k_1 + \ell_1 = k_2 + \ell_2 = K\}.$$

We also have an analogue of [KKL25, Theorem 1.3] and may be proved similarly.

Theorem 1.6. *Suppose that $f \in \mathcal{E}(\Gamma_1(N))$. There exists an integer r , such that if there exists primes $\{p_1, \dots, p_r\}$ all congruent to m mod M such that $c_f(p_i) = 0$ for $i = 1, 2, \dots, r$, then $f \in \tilde{\Omega}_{m,M}$.*

We conclude this paper by providing, in an appendix, a purely analytic proof of some of results for the level one case. In particular, we give an equivalent condition (see Theorem A.1 below) in terms of certain ratios of the Riemann zeta function for the linear combination of certain divisor functions to detect primes. This can be used to easily produce examples of such identities which in turn can be translated to prime-detecting identities involving partition functions. As the notation and results leading up to the proof of Theorem A.1 are self-contained and somewhat independent of the rest of the paper, we have given the details in a separate section.

The paper is organized as follows. In Section 2 we show that a quasimodular cusp form not in the kernel of the sieving operator exhibits infinitely many sign changes at the prime Fourier coefficients. In Section 3, we show that the corresponding Eisenstein series exhibit at most finitely many sign changes and conclude the proof of Theorem 1.4. In Sections 4 and 5 we prove Theorems 1.5 and 1.6 respectively.

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2. SIGN CHANGES OF QUASI-MODULAR CUSP FORMS

We let $\mathcal{S}(\Gamma_1(N))$ denote the space of quasi-modular cusp forms of level N (i.e., the space spanned by cusp forms and their derivatives), and omit $\Gamma_1(N)$ in the notation if $N = 1$ (i.e., $\mathcal{S} := \mathcal{S}(\Gamma_1(1))$).

In this section, we show that Fourier coefficients of quasi-modular cusp forms (of arbitrary level) exhibit sign changes. First we recall the main sign change lemma (for the full level case) from [KKL25].

Lemma 2.1. *Suppose that $F \in \mathcal{S}$ has a Fourier expansion*

$$F(\tau) = \sum_{n \geq 1} c_F(n) q^n$$

with $c_F(n) \in \mathbb{R}$. If $F \neq 0$, then the sequence $c_F(p)$, running over p prime, has infinitely many sign changes.

In higher levels, we have the following lemma.

Lemma 2.2. *Suppose that $F \in \mathcal{S}(\Gamma_1(N))$ has a Fourier expansion*

$$F(\tau) = \sum_{n \geq 1} c_F(n) q^n$$

with $c_F(n) \in \mathbb{R}$. If $F \neq 0$, then either $c_F(n) = 0$ for all n satisfying $(n, N) = 1$ or $\{c_F(p)\}$ has infinitely many sign changes, as p runs through the primes.

Proof. Suppose that $F \neq 0$. We may express a quasi-modular cusp form F as the linear combination of Hecke eigenforms and their derivatives;

$$F = \sum_f \sum_j A_f(j) f^{(j)}$$

where $A_f(j) \in \mathbb{C}$ and $f^{(j)} = D^j f$ is the normalized j -th derivative of f . We break the sum over f based on the image of the V map. Observe from [Li75, Lemma 4] that it suffices to restrict ourselves to those d which divide N . Hence, we write

$$F = \sum_{d|N} G_d$$

where

$$G_d = \sum_f \sum_j A_f(j) f^{(j)} = \sum_{n=1}^{\infty} c_{G_d}(n) q^{nd}$$

where the f sum is restricted to those Hecke eigenforms in the image of V_d operator.

From construction, it follows that $c_F(dp) = \sum_{\ell|d} c_{G_\ell}(dp/\ell)$ for almost all primes p . In particular, $c_F(p) = c_{G_1}(p)$ for almost all primes p as $c_{G_d}(p) = 0$ for every $d > 1$ and any prime $p > N$. If $G_1 = 0$, then $c_F(n) = 0$ for all $(n, N) = 1$, because for all $1 < d|N$, G_d contributes a coefficient of 0 to the term q^n whenever $(n, N) = 1$. Therefore the lemma would follow if we prove the infinite of sign changes in $c_{G_1}(p)$. This reduces the proof of the lemma to the case of $d = 1$.

When $d = 1$, we may without loss of generality suppose that $F = G_1$. In this case, the proof is very similar to the proof of Lemma 2.1. We briefly sketch the arguments for the sake of completeness and refer the reader to [KKL25] for more details.

After some rearrangement, we may write

$$c_F(p) = \sum_f \sum_j A_f(j) p^j a_f(p) = \sum_f P_f(p) a_f(p)$$

for some polynomials $P_f(x) \in \mathbb{C}[x]$. For every f , we let the weight of f to be $k_f \geq 4$ and the degree of P_f to be $j_f \geq 0$. Denote the leading coefficient of P_f as simply A_f . From the Ramanujan bound for a_f [Del74, Del80], we know

$$P_f(p) a_f(p) = A_f p^{j_f} a_f(p) + O\left(p^{j_f + \frac{k_f - 1}{2} - 1}\right),$$

where the implied constant depends at most on F .

Set $\alpha_0 := \max_f \left\{ \alpha_f := j_f + \frac{k_f+1}{2} \right\}$. Summing over the primes, and using the prime number theorem in this setting (see [IK04, Theorem 5.13]) gives us

$$\sum_{p \leq x} c_F(p) = o\left(\frac{x^{\alpha_0}}{\log(x)}\right). \quad (2.1)$$

Similarly,

$$\begin{aligned} |c_F(p)|^2 &= \sum_{f,g} P_f(p) \overline{P_g(p)} a_f(p) \overline{a_g(p)} \\ &= \sum_f |P_f(p)|^2 |a_f(p)|^2 + \sum_{f \neq g} P_f(p) \overline{P_g(p)} a_f(p) \overline{a_g(p)}. \end{aligned}$$

Appealing to Rankin-Selberg theory and the Selberg orthogonality conjecture [LY05], the sum over p of $|c_F(p)|^2$ is dominated by the diagonal term. In particular the leading order of the asymptotic is obtained from the forms of the largest weight. Plugging this all in we get,

$$\sum_{p \leq x} |c_F(p)|^2 \gg_F \frac{x^{\beta_0}}{\log(x)}. \quad (2.2)$$

where $\beta_0 = 2\alpha_0 - 1$. It follows from Deligne's bound that for all prime p ,

$$|c_F(p)| \leq \sum_f |P_f(p) a_f(p)| \leq \sum_f \|P_f\| p^{j_f} |a_f(p)| \leq C_F p^{\alpha_0-1}$$

where $\|P\| = \sum_{r=0}^m |A_r|$ if $P(x) = \sum_{r=0}^m A_r x^r \in \mathbb{C}[x]$ and $C_F > 0$ is a constant. This yields

$$\sum_{p \leq y} |c_F(p)|^2 = O\left(\frac{y^{2\alpha_0-1}}{\log(y)}\right).$$

With these ingredients we may now deduce the infinitude of sign changes as in [KKL25]. This completes the proof. \square

Corollary 2.3. *Suppose that $f \in \mathcal{S}(\Gamma_1(N))$ for some $N \in \mathbb{N}$ and*

$$f(\tau) = \sum_{l \geq 1} c_f(l) q^l$$

with $c_f(l) \in \mathbb{R}$. If $M \in \mathbb{N}$ and $m \in \mathbb{Z}$ with $\gcd(m, M) = 1$, then for any $n \in \mathbb{Z}$ with $\gcd(n, N) = 1$, either $f \in \ker(S_{M,m} \circ S_{N,n})$ or $\{c_f(p)\}_{p \equiv m \pmod{M}, p \equiv n \pmod{N}}$ has infinitely many sign changes.

Proof. The form $F := f|S_{M,m}|S_{N,n} \in \mathcal{S}(\Gamma_1(L))$ for some $L|M^4N^2$. By Lemma 2.2, either $c_F(l) = 0$ for all but finitely many $(l, L) = 1$ or $\{c_F(p)\}$ has infinitely many sign changes. As the Fourier coefficients are supported on (a subset of) $(m + M\mathbb{Z}) \cap (n + N\mathbb{Z})$, all integers in the support are coprime with L . The desired result follows. \square

3. VANISHING AT PRIMES IN ARITHMETIC PROGRESSIONS

We begin with a lemma similar to [KKL25, Lemma 4.1].

Lemma 3.1. *Suppose that $f \in \mathcal{E}(\Gamma_1(N))$ with real Fourier coefficients. Then for every arithmetic progression $m \pmod{N}$ with $\gcd(m, N) = 1$ there exists $\varepsilon_{m,N} \in \{-1, 0, 1\}$ for which every sufficiently large prime $p \equiv m \pmod{N}$*

$$\operatorname{sgn}(c_f(p)) = \varepsilon_{m,N}.$$

Proof. The basis elements of $\mathcal{E}(\Gamma_1(N))$ are given by $D^\ell E_{k,\chi,\psi}|V_\delta$, where χ and ψ are Dirichlet characters modulo N . Suppose $f \in \mathcal{E}(\Gamma_1(N))$. Let us denote the Fourier coefficient of f as $c_f(n)$. For $n = p$ prime, expressing f as a linear combination of the basis above, and using (1.3), we see that the p -th coefficient can be written as a polynomial

$$c_f(p) = \sum_r \beta_r(\chi(p), \psi(p)) p^r,$$

where the coefficients $\beta_r(\chi(p), \psi(p)) \in \mathbb{C}$ only depend on f , $\chi(p)$ and $\psi(p)$. Since χ and ψ are characters modulo N , for $p \equiv m \pmod{N}$ we have $\chi(p) = \chi(m)$ and $\psi(p) = \psi(m)$, so

$$c_f(p) = \sum_r \beta_r(\chi(m), \psi(m)) p^r$$

is a polynomial in p whose coefficients only depend on f and m . If this polynomial vanishes identically, then we may take $\varepsilon_{m,N} := 0$, and otherwise we may choose r_0 largest so that $\beta_{r_0}(\chi(m), \psi(m)) \neq 0$, in which case we may choose

$$\varepsilon_{m,N} := \operatorname{sgn}(\beta_{r_0}(\chi(m), \psi(m))) \in \{\pm 1\}.$$

The fact that $\beta_{r_0}(\chi(m), \psi(m)) \in \mathbb{R}$ follows from the assumption that the Fourier coefficients of f are real. \square

Theorem 1.4 now follows by an argument similar to the proof of Theorem 1.3.

Proof of Theorem 1.4. Suppose that $f \in \tilde{\Omega}_{m,M}(\Gamma_1(N))$. We split

$$f = f_E + f_S$$

where $f_E \in \mathcal{E}(\Gamma_1(N))$ and $f_S \in \mathcal{S}(\Gamma_1(N))$. As in the proof of Theorem 1.3 (see [KKL25]), we may isolate the real and imaginary parts of f_E, f_S and deal with them separately. For brevity, we shall suppose that the Fourier coefficients of f_E and f_S are real valued and move forward. For any $n \in \mathcal{V}_{m,M}$, Lemma 3.1 gives us $\varepsilon_{n,N}$ for which

$$\operatorname{sgn}(c_{f_E}(p)) = \varepsilon_{n,N}$$

for sufficiently large $p \equiv n \pmod{N}$. Since $(N\mathbb{Z} + n) \cap (M\mathbb{Z} + m)$ is non-trivial by assumption, there exist infinitely many p in this intersection (since $(N\mathbb{Z} + n) \cap (M\mathbb{Z} + m)$ defines an arithmetic progression and since $(n, N) = (m, M) = 1$), and for such sufficiently large p we have $c_{f_S}(p) = -c_{f_E}(p)$, implying that

$$\operatorname{sgn}(c_{f_S}(p)) = -\operatorname{sgn}(c_{f_E}(p)) = -\varepsilon_{n,N}.$$

However, by Corollary 2.3, $\{c_{f_S}(p)\}_{p \in (N\mathbb{Z} + n) \cap (M\mathbb{Z} + m)}$ has infinitely many sign changes unless $f_S|S_{M,m}|S_{N,n} = 0$. Therefore,

$$\sum_{n \in \mathcal{V}_{m,M}} f_S|S_{M,m}|S_{N,n} = 0.$$

But then

$$\sum_{n \in \mathcal{V}_{m,M}} f|S_{M,m}|S_{N,n} = \sum_{n \in \mathcal{V}_{m,M}} f_E|S_{M,m}|S_{N,n}.$$

Since the vanishing of the coefficients of $f_E|S_{M,m}$ in arithmetic progressions are determined by the vanishing of the $\varepsilon_{n,N}$, and these only depend on $n \pmod{N}$ (or, equivalently, by the polynomials from the proof of Lemma 3.1 vanishing identically), we see that the p -th coefficient of $f_E|S_{M,m}$ vanishes at every $p \equiv m \pmod{M}$ if and only if the p -th coefficient of f_E vanishes at every $p \equiv n \pmod{N}$ for every $n \in \mathcal{V}_{m,M}$. \square

4. A SPANNING SET FOR $\tilde{\Omega}_{m,M}$

In this section, we prove Theorem 1.5. Before the proof, it is convenient to first quickly verify that for any integer K , $H_{k_1,\ell_1,\psi_1,\chi_1} - H_{k_2,\ell_2,\psi_2,\chi_2} \in \tilde{\Omega}_{m,M}$, where $k_1 + \ell_1 = k_2 + \ell_2 = K$. With k_1, ℓ_1, k_2, ℓ_2 fixed as above, for ease of notation, let $H := H_{k_1,\ell_1,\psi_1,\chi_1} - H_{k_2,\ell_2,\psi_2,\chi_2}$. For a prime $p \equiv m \pmod{M}$, we may calculate the p -th Fourier coefficient of H as

$$\begin{aligned} a_H(p) &= \overline{\chi_1(m)} \left(p^{\ell_1-1} (\chi_1(m)p^{k_1-1} + \psi_1(m)) \right) - \psi_1(m) \overline{(\chi_1(m)p^{\ell_1-1} + \psi_1(m))} \\ &\quad - \overline{\chi_2(m)} \left(p^{\ell_2-1} (\chi_2(m)p^{k_2-1} + \psi_2(m)) \right) + \psi_2(m) \overline{(\chi_2(m)p^{\ell_2-1} + \psi_2(m))} = 0. \end{aligned}$$

Now we move to the proof of Theorem 1.5. Suppose that f is a quasi-modular form detecting primes congruent to m modulo M and lies in $\bigoplus_{n \in \mathcal{V}_{m,M}} \tilde{\Omega}_{m,M}(\Gamma_1(N))|S_{M,m}|S_{N,n}$. From Theorem 1.4, f is spanned by quasi-modular Eisenstein series. From Proposition 20⁴ of [Zag08], f is a linear combination of the derivatives of the level N Eisenstein series and E_2 . Suppose that

$$f = \sum_{i=1}^t \alpha_i D^{\ell_i} E_{k_i,\chi_i,\psi_i}|S_{M,m}|S_{N,n} + \alpha_{t+1} D^{\ell_{t+1}} E_2|S_{M,m}|S_{N,n}.$$

for some Dirichlet characters χ_i, ψ_i . For ease of notation, we shall denote $E_{k_i,\chi_i,\psi_i}|S_{M,m}|S_{N,n}$ as simply E_i .

Suppose without loss of generality that $i = 1, \dots, r$ are the indices for which $k_i + \ell_i = K$. For E_2 , we may consider the characters χ_{t+1}, ψ_{t+1} as trivial characters. We write

$$f = \sum_{i=1}^r \alpha_i D^{\ell_i} E_i + g$$

for some quasi-modular form g . If we define K_g analogous to K_f above, then we observe that $K_g < K_f$. For a prime $p \equiv m \pmod{M}$, we have

$$\begin{aligned} a_f(p) &= \sum_{i=1}^r \alpha_i p^{\ell_i} \left(\chi_i(m)p^{k_i-1} + \psi_i(m) \right) + a_g(p) \\ &= p^{K_f-1} \sum_{i=1}^r \alpha_i \chi_i(m) + \sum_{i=1}^r \alpha_i \psi_i(m) p^{\ell_i} + a_g(p). \end{aligned}$$

In particular we have

$$a_f(p) = p^{K-1} \sum_{i=1}^r \alpha_i \chi_i(m) + O(p^{K-2})$$

⁴In Zagier's notation, $\phi = E_2$.

since $k_i > 1$. As f detects primes on that arithmetic progression, considering $p \rightarrow \infty$, this forces

$$\sum_{i=1}^r \alpha_i \chi_i(m) = 0.$$

Let $\{e_i\}$ denote the standard basis of \mathbb{C}^r . The orthogonal complement of $(\chi_1(m), \chi_2(m), \dots, \chi_r(m))$ in \mathbb{C}^r is spanned by $\{v_j := \overline{\chi_1(m)}e_1 - \overline{\chi_j(m)}e_j\}$ for $2 \leq j \leq r$. Hence there exist constants β_j 's such that $(\alpha_1, \dots, \alpha_r) = \beta_2 v_2 + \dots + \beta_r v_r$.

Therefore, on rewriting the above equation, we have

$$f = \sum_{j=2}^r \beta_j \left(\overline{\chi_1(m)} D^{\ell_1} E_1 - \overline{\chi_j(m)} D^{\ell_j} E_j \right) + g.$$

We define the quasi modular form h such that

$$\begin{aligned} f &= \sum_{j=2}^r \beta_j \left(\overline{\chi_1(m)} D^{\ell_1} E_1 - \psi_1(m) E_{\ell_1, \overline{\chi_1}, \overline{\psi_1}} + \psi_j(m) E_{\ell_j, \overline{\chi_j}, \overline{\psi_j}} - \overline{\chi_j(m)} D^{\ell_j} E_j \right) + h \\ &= \sum_{j=2}^r \beta_j (H_{k_1, \ell_1, \chi_1, \psi_1} - H_{k_j, \ell_j, \chi_j, \psi_j}) + h \end{aligned}$$

Since we have already shown that $H_{k_1, \ell_1, \chi_1, \psi_1} - H_{k_j, \ell_j, \chi_j, \psi_j} \in \tilde{\Omega}_{m, M}$ earlier in the proof, we observe that $h \in \tilde{\Omega}_{m, M}$ and $K_h < K_f$. If K_f were equal to 2, then $h \equiv 0$, since the space of quasi-modular Eisenstein series are spanned by forms of weight 2 or larger. Thus the theorem follows by induction on K_f .

5. FINITE CHECKS FOR PRIME DETECTION

The proof is a ready adaptation of the proof of [KKL25, Theorem 1.3] Suppose that

$$f = \sum_{\ell, k} \alpha_{\ell, k} D^{\ell} G_{k, \chi, \psi} |V_{\delta}.$$

Choose r to be the maximum of $\ell + k$ for which $\alpha_{\ell, k} \neq 0$. From the proof of Lemma 3.1, we may write

$$c_f(p) = \sum_r \beta_r (\chi(m), \psi(m)) p^r,$$

whenever $p \equiv m \pmod{M}$ is a prime. Now, if $c_f(p_i) = 0$ for $1 \leq i \leq r+1$, then we obtain the system of equations

$$\begin{pmatrix} 1 & p_1 & p_1^2 & \dots & p_1^r \\ 1 & p_2 & p_2^2 & \dots & p_2^r \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & p_{r+1} & p_{r+1}^2 & \dots & p_{r+1}^r \end{pmatrix} \begin{pmatrix} \beta_0(\chi(m), \psi(m)) \\ \beta_1(\chi(m), \psi(m)) \\ \vdots \\ \beta_r(\chi(m), \psi(m)) \end{pmatrix} = 0.$$

This system, being a Vandermonde system is uniquely solvable and hence $\beta_0(\chi(m), \psi(m)) = \beta_1(\chi(m), \psi(m)) = \dots = \beta_r(\chi(m), \psi(m)) = 0$. Hence the claim follows.

APPENDIX A. AN ANALYTIC PROOF FOR THE LEVEL ONE CASE

As in [CvIO24], polynomial expressions involving Macmahon partition functions can be rewritten as polynomial equations involving various divisor functions. Thus the study of partition identities that detect primes is in principle a study of “divisor function” identities that detect primes. In this spirit we consider the following general situation.

For $1 \leq i \leq r$, we choose polynomials $P_i(x) \in \mathbb{Q}[x]$. We also choose and fix non-negative integers $\{k_i\}_{i=1}^r$. We define the function

$$a(n) := \sum_{i=1}^r P_i(n) \sigma_{k_i}(n) = \sum_{j=1}^t A_j n^{\ell_j} \sigma_{k_j}(n), \quad (\text{A.1})$$

for some $A_j \in \mathbb{Q}$ and (not necessarily distinct) non-negative integers ℓ_j . We observe that

$$W(s) := \sum_{j=1}^t A_j \zeta(s - \ell_j) \zeta(s - \ell_j - k_j) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}. \quad (\text{A.2})$$

Associate to W , two integers R_W and S_W defined by $R_W := \max_j \{\ell_j, \ell_j + k_j\} = \max_j \{\ell_j + k_j\}$ and $S_W := \sum |A_j|$, where the sum is over all indices j such that $\ell_j + k_j = R_W$. We also define the closely related function

$$Z_W(s) := \prod_{j=1}^t (\zeta(s - \ell_j) \zeta(s - \ell_j - k_j))^{A_j} = \sum_{n=1}^{\infty} \frac{b(n)}{n^s} \quad (\text{A.3})$$

for $\Re(s) \gg 1$. In order to state the main theorem, we introduce the following notation. For a quadruple of integers $\mathbf{m} = (m_1, m_2, m_3, m_4) \in \mathbb{Z}^4$, we define

$$\begin{aligned} W_{\mathbf{m}}(s) &:= \zeta(s - m_1) \zeta(s - m_3) + \zeta(s - m_2) \zeta(s - m_4) \\ &\quad - \zeta(s - m_1) \zeta(s - m_4) - \zeta(s - m_2) \zeta(s - m_3) =: \sum_{n=1}^{\infty} \frac{a_{\mathbf{m}}(n)}{n^s}. \end{aligned} \quad (\text{A.4})$$

Theorem A.1. *Let notation be as above and fix a W as in (A.2). Then the following are equivalent.*

- (1) $a(p)$ vanishes for all primes p ,
- (2) $Z_W(s) \equiv 1$,
- (3) There exist integers $\{c_{\mathbf{m}}\}_{\mathbf{m} \in \mathbb{Z}^4}$, at most finitely many of them non-zero, such that $W(s) = \sum_{\mathbf{m} \in \mathbb{Z}^4} c_{\mathbf{m}} W_{\mathbf{m}}(s)$,
- (4) There exist R_W distinct primes $\{p_1, \dots, p_{R_W}\}$ such that $a(p_i) = 0$ for $1 \leq i \leq R_W$.

Proof. The equivalence of (1) and (4) is the content of [KKL25, Theorem 1.3]. So we shall prove that (1), (2) and (3) are equivalent to one another.

Let us prove that (1) implies (2). Clearing out denominators in the definition of $a(n)$ and by taking a suitable power of $Z_W(s)$ we may without loss of generality assume that all the A_j 's are integers. After some rearrangement, we may write

$$Z_W(s) = \prod_{j=1}^u \zeta(s - m_j)^{B_j}$$

where, $m_1 < m_2 < \dots < m_u$. Let $M := \{m_j : 1 \leq j \leq u\}$. To prove (2), it suffices to show that $B_j = 0$ for all $j = 1, \dots, u$. From definition, observe that $b(n)$ is a multiplicative function. In other words, Z_W has an Euler product expansion. We have

$$\begin{aligned} \log Z_W(s) &= \sum_p b(p)p^{-s} + \dots \\ &= \sum_p \left(\sum_{m_j \in M} B_j p^{m_j} \right) p^{-s} + \dots \end{aligned}$$

On the other hand, we observe that $a(p) = b(p)$ for every prime p . This implies that $B_j = 0$ for every $1 \leq j \leq u$. This proves (2).

Let us assume (2) and prove (3). Henceforth we shall adopt the notation of (A.2). In that notation, we shall suppose that $A_j \neq 0$ for every $1 \leq j \leq t$. For simplicity, we shall also suppose that $(\ell_i, k_i) \neq (\ell_j, k_j)$ if $i \neq j$. We shall proceed by successive reductions, first on S_W and then on R_W , and so on. If $W \equiv 0$, then we may choose $c_{\mathbf{m}} = 0$ for all \mathbf{m} and we are done. So we suppose otherwise, that is $W \neq 0$. In particular, since $Z_W \equiv 1$ by assumption, we have $S_W \geq 2$ and, since all ℓ_j and k_j are non-negative, $R_W \geq 0$.

Without loss of generality, suppose that $\ell_t + k_t = R_W$. Define $m_4 = \ell_t + k_t$. Suppose also without loss of generality that $A_t > 0$. Since $Z_W(s) \equiv 1$, there exists at least one $j < t$ such that $\ell_j + k_j = \ell_t + k_t$ and $A_j < 0$. Since we have chosen the tuples (ℓ_i, k_i) 's to be distinct it follows that $\ell_j \neq \ell_t$ and $k_j \neq k_t$. Relabeling indices if necessary we may suppose that $k_t > k_j$. In particular $k_t \neq 0$. Choose and fix such a j . We set $m_2 = \ell_j$ and $m_1 = m_3 = \ell_t$. Define $\tilde{W}(s) := W(s) + W_{\mathbf{m}}(s)$. We observe that $Z_{\tilde{W}}(s) = Z_W(s) \equiv 1$ by assumption. We crucially observe that $R_{\tilde{W}} \leq R_W$ and if $R_{\tilde{W}} = R_W$, then $S_{\tilde{W}} \leq S_W - 2 < S_W$. Furthermore, $R_{\tilde{W}} < R_W$ if $S_W = 2$. If $\tilde{W} \neq 0$, repeating the above process for \tilde{W} , we may “peel off” one $W_{\mathbf{m}}$ at a time from W .

To complete the proof, we need to make sure that this process terminates in finitely many steps. To see this, first observe that R_W is non-increasing in this process. Second, at each step, at least one of R_W or S_W is strictly decreasing. Moreover, whenever R_W is non-decreasing, the parameter S_W is strictly decreasing, ultimately forcing R_W to decrease after finitely many steps. Even though S_W grows occasionally⁵, it is at most finite, at each step, and hence eventually goes down to zero, which in turn decreases R_W , keeping the reduction argument going. Continuing this process, we eventually end up with $R_W = 0$ and $S_W = 2$; but then $W = \zeta^2(s) - \zeta^2(s) = 0$ (since $Z_W \equiv 1$). Following this procedure, we may write W as a integral linear combination of $W_{\mathbf{m}}$'s proving (3).

Finally let us suppose (3) and prove (1). It suffices to verify that for every $\mathbf{m} \in \mathbb{Z}^4$, and for every prime p , we have $a_{\mathbf{m}}(p) = 0$. We shall give the proof assuming $m_1 < m_2 < m_3 < m_4$, the other cases being treated similarly. Given $\mathbf{m} = (m_1, m_2, m_3, m_4) \in \mathbb{Z}^4$, by direct computation, we have that

$$\begin{aligned} a_{\mathbf{m}}(p) &= p^{m_1} \sigma_{m_3-m_1}(p) + p^{m_2} \sigma_{m_4-m_2}(p) - p^{m_1} \sigma_{m_4-m_1}(p) - p^{m_2} \sigma_{m_3-m_2}(p) \\ &= p^{m_3} + p^{m_1} + p^{m_4} + p^{m_2} - p^{m_4} - p^{m_1} - p^{m_3} - p^{m_2} = 0. \end{aligned} \quad (\text{A.5})$$

This completes the proof of the theorem. \square

⁵When $S_W = 2$, there are exactly two choices for j such that the maximum R_W is attained. We cancel them out by adding the corresponding $W_{\mathbf{m}}$ and obtain \tilde{W} . In this step, $R_{\tilde{W}} < R_W$, but $S_{\tilde{W}}$ now counts the sum of coefficients of the pairs (ℓ_j, k_j) such that $\ell_j + k_j = R_{\tilde{W}}$ and not R_W . Thus $S_{\tilde{W}} > S_W$ (and in fact this might be considerably larger).

Remark 6. It is natural to want to extend this proof to forms of higher level, but this does not seem to be straightforward. When considering Eisenstein series of higher level, the associated Dirichlet series involves products of Dirichlet L -functions. More precisely, the analogue of Z_W (say $Z_{W,N}$, for level N) in this situation is no longer a ratio of shifts of the Riemann zeta function, but of L -functions associated to Dirichlet characters. The pole of $\zeta(s)$ at the point $s = 1$ was used to pinpoint the rightmost singularity of $\log(Z_W(s))$. But, as it is well known that the L -function associated to non-principal Dirichlet characters have neither zeros nor poles on the boundary of absolute convergence, we run into trouble when looking for the rightmost singularity of $\log(Z_{W,N}(s))$, unless a principal character appears in the decomposition. Furthermore, GRH predicts that $\log(Z_{W,N}(s))$ should not have any poles in the vertical strip of width $1/2$ to the left of region of convergence if there is no principal character. A workaround to this obstacle seems to require new ideas.

Along with the vanishing at the primes, it is interesting to investigate when $a(n)$ is non-negative. For a general W as above, the answer depends on the of $c_{\mathbf{m}}$. For $W_{\mathbf{m}}$ however, we have the following precise result.

Lemma A.2. *Let $\mathbf{m} = (m_1, m_2, m_3, m_4)$ be given. Then $\text{sgn}(a_{\mathbf{m}}(n)) = \text{sgn}(m_2 - m_1)\text{sgn}(m_4 - m_3)$ for every composite number n .*

Proof. We need only generalize the calculation in (A.5). Suppose n is a composite number. From definition,

$$\begin{aligned} a_{\mathbf{m}}(n) &= n^{m_1}\sigma_{m_3-m_1}(n) + n^{m_2}\sigma_{m_4-m_2}(n) - n^{m_1}\sigma_{m_4-m_1}(n) - n^{m_2}\sigma_{m_3-m_2}(n) \\ &= \sum_{d|n} (n^{m_1}d^{m_3-m_1} + n^{m_2}d^{m_4-m_2} - n^{m_1}d^{m_4-m_1} - n^{m_2}d^{m_3-m_2}) \\ &= \sum_{d|n} \left(\left(\frac{n}{d}\right)^{m_2} - \left(\frac{n}{d}\right)^{m_1} \right) (d^{m_4} - d^{m_3}). \end{aligned}$$

The terms corresponding to $d = 1$ and $d = n$ vanish. The remaining terms all non-zero and have the same sign which is $\text{sgn}(m_2 - m_1)\text{sgn}(m_4 - m_3)$. The lemma follows from here. \square

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