

INVERSION OF THE ABEL–PRYM MAP FOR REAL CURVES WITH INVOLUTIONS

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ABSTRACT. Riemann vanishing theorem is a main ingredient of the conventional technique related to the Jacobi inversion problem. In the case of curves with a holomorphic involution, it has been exhaustively expounded in wellknown Fay’s Lectures on theta functions. The case of real algebraic curves with involution is presented with less completeness in the literature. We give a detailed presentation of that case, including real curves of non-separating type (with involution) not considered before with this relation. We obtain the Novikov–Veselov realness conditions in a different set-up.

Bibliography: 16 titles.

Key words: Abel–Prym transformation, Jacobi inversion problem, Prym theta function, real curve.

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1. INTRODUCTION

Let Σ be a genus g algebraic curve. It is classical that the Abel map establishes a birational (bimeromorphic on certain open dense subsets) correspondence between $\text{Sym}^g \Sigma$ and the Jacobian $Jac(\Sigma)$ of Σ . Also, it can be considered as a correspondence between equivalence classes of degree g divisors and points of $Jac(\Sigma)$. The inverse to the Abel map is given by the following Riemann theorem: the preimage of a point of the Jacobian (except for the points of a subvariety of codimension 1) is given by a zero

divisor of a certain auxiliary function (constructed using the Riemann θ -function) on the universal covering of Σ . The last theorem is referred to as the Riemann vanishing theorem below, and the above reversion procedure as a whole is referred to as the Jacobi inversion. Among other things, the Jacobi inversion is a powerful tool of the theory of integrable systems.

If Σ is endowed with a holomorphic involution (denoted by σ below) then Σ can be assigned with another Abelian variety called Prym variety (or Prymian), defined as the subset of σ -antiinvariant points in $Jac(\Sigma)$, and denoted by $Prym_\sigma(\Sigma)$ below. However there is a more primary object in relation to the Prymian, namely, its finite unramified covering called isoPrymian below ($isoPrym(\Sigma)$). The construction of $isoPrym(\Sigma)$ repeats the construction of $Jac(\Sigma)$ where the Riemann matrix is replaced with the Prym matrix. In particular, isoPrymian is always principally polarized. The corresponding analog of the Abel map using only σ -antiinvariant holomorphic differentials is called the Abel–Prym map (see Section 3.3 for definitions).

As we noted in [13], the Jacobi inversion problem should be modified if applied to the Abel–Prym map. It is shown below that it should be posed on the isoPrymian. The number of points returned by the Riemann vanishing theorem is twice as big as $\dim isoPrym(\Sigma)$ [4] but the divisor ζ formed by them satisfies the relation $\zeta + \sigma\zeta \sim D$ where D is the constant divisor. Thus in this case the Riemann vanishing theorem provides a less effective description of the preimage than in the classical case, in which the preimage is a divisor free of any relations. A more effective description is also possible in the case of two commuting involutions [15]. In the present paper we consider a general case. We show that at least a direct calculation of symmetric functions of the points in the divisor is possible which is a weakened approach to the Jacobi inversion problem due to Dubrovin [5] (going back to Riemann).

If Σ possesses an antiholomorphic involution, it is called a real curve. Jacobians of real curves have been first investigated in [7] with relation to real solutions to sin-Gordon equation. It was shown that the Abel map establishes a 1-to-1 correspondence between τ -invariant degree g divisors and the real part of the Jacobian $Jac(\Sigma)$. Investigation of Prymians of real curves was pioneered in [16] with relation to real solutions to the potential two-dimensional Schrödinger equation, and also presented in [9] in the case when σ has only two fixed points. A general result of [16] is that a certain shift of the real part of such Prymian is in a 1-to-1 correspondence with divisors of a certain degree on Σ satisfying the relations $\zeta + \sigma\zeta \sim D$ and $\tau\zeta = \zeta$. Let's stress that the above results are obtained by consideration of the classical Abel map, and the Riemann theta function. We reproduce them for the isoPrymian, the Abel–Prym map, and the Prym theta function. The precise statement is given by Theorem 4.3 which we refer to as the inversion theorem for real curves. Together with the Riemann vanishing theorem (Lemma 4.1), it implies that the inverse image of a certain real subvariety of isoPrymian under the Abel–Prym map is given by τ -invariant, or $\sigma\tau$ -invariant divisors ζ on Σ satisfying the relation $\zeta + \sigma\zeta = D$ where D is a constant divisor. Proof of the theorem is based on the study of symmetries of the Prym θ -function (Lemma 3.5) for both separating and non-separating real curves. It generalizes the results of [7, 8] on the Riemann θ -function of a real curve, and of [4] on the Prym θ -function of a real curve of separating type.

In Section 2, following [7, 8, 9], we give preliminaries on Jacobians and Riemann θ -functions of real curves. In Section 3 we introduce main notions related to Prym

varieties, study the realness properties of Prym matrices and symmetries of Prym θ -functions of real curves. In Section 4 we present our inversion theorem for real curves.

2. REAL CURVES, THEIR JACOBIANS AND θ -FUNCTIONS

This section contains a background information, and follows the lines of [7, 9, 10, 11]

2.1. Separating and non-separating curves. Topological type. Real bases. Let Σ be a compact algebraic curve over \mathbb{C} . If Σ possesses an antiholomorphic involution (antiinvolution for short) then it is referred to as a real curve. Let τ stay for such involution. Connected components of the set of fixed points of τ are called ovals. If $\Sigma \setminus \cup \text{ovals}$ is not connected, the pair (Σ, τ) is called a real curve of separating type (separating curve), otherwise it is called a non-separating curve. For separating curves, the number of connected components of $\Sigma \setminus \cup \text{ovals}$ is always equal to 2 [9].

Two real curves (Σ_1, τ_1) and (Σ_2, τ_2) are called topologically equivalent if there is a homeomorphism $\psi : \Sigma_1 \rightarrow \Sigma_2$ such that $\psi\tau_1 = \tau_2\psi$. Set $\varepsilon = 1$ for separating curves, and $\varepsilon = 0$ for non-separating curves. Let g stay for the genus of Σ , and k stay for the number of ovals. The set (g, k, ε) is called the topological type of the pair (Σ, τ) .

Theorem 2.1 ([9]). *Two real curves are topologically equivalent iff they have the same topological type.*

For more information on the structure of real curves we refer to [9], and the works quoted there.

Theorem 2.2 ([9]). *Let (Σ, τ) be a real curve of the type (g, k, ε) , and q is a real point in Σ . Then there exists a symplectic base $\{a_j, b_j | j = 1, \dots, g\}$ of cycles on Σ such that*

1° for $\varepsilon = 1$

$$\begin{cases} \tau(a_i) = a_i, \tau(b_i) = -b_i, & i = 1, \dots, k-1; \\ \tau(a_i) = a_{i+m}, \tau(b_i) = -b_{i+m}, & i = k, \dots, k+m-1 \end{cases}$$

where $m = \frac{1}{2}(g+1-k)$, and the oval containing q is homological to $\sum_{i=1}^{k-1} a_i$.

2° for $\varepsilon = 0$

$$\begin{cases} \tau(a_i) = a_i, & i = 1, \dots, g; \\ \tau(b_i) = -b_i, & i = 1, \dots, k-1; \\ \tau(b_i) = -b_i + a_i, & i = k, \dots, g, \end{cases}$$

and the oval containing q is homological to $\sum_{i=1}^g a_i$.

A base satisfying to the conditions of Theorem 2.2 is called a real base.

2.2. Realness properties of Riemann matrices of real curves. Let $\{a_j, b_j\}$ be a real base of cycles of a curve, $\{\omega_i\}$ be the normalized base of holomorphic differentials where the normalization conditions are of the form

$$(2.1) \quad \int_{a_j} \omega_i = 2\pi i \delta_{ij}.$$

Define the permutation t of indices $1, \dots, g$, such that $\tau(a_j) = a_{t(j)}$ by virtue of Theorem 2.2. For separating curves t writes as follows via cycles: $t = (1) \dots (k-1)(k, k+m) \dots (k+m-1, g)$. For non-separating curves t is trivial. Observe that $t^2 = t$.

Let $\omega = (\omega_1, \dots, \omega_g)^T$ be the column of normalized differentials, $t\omega = (\omega_{t(1)}, \dots, \omega_{t(g)})^T$, $A_i = \int_{a_i} \omega$, $B_i = \int_{b_i} \omega$ be the corresponding periods. Then $(A_1, \dots, A_g) = 2\pi i E$,

$(B_1, \dots, B_g) = B$, where E is the unit matrix, B is referred to as the matrix of periods. It is a symmetric matrix with negative defined real part.

Below we describe specific properties of the period matrix, and symmetries of the Riemann θ -function for real curves.

Lemma 2.3. 1° . $\tau^*\omega = -t\bar{\omega}$;

2° . For separating real curves $\tau^*\omega_i = \begin{cases} -\bar{\omega}_i, & i < k; \\ -\bar{\omega}_{i+m}, & k \leq i < k+m. \end{cases}$

3° . For non-separating curves $\tau^*\omega = -\bar{\omega}$.

Proof. By change of variables $\int_{a_j} \tau^*\omega_i = \int_{a_{t(j)}} \omega_i$. By symmetry of the a -period matrix (it is just diagonal) $\int_{a_{t(j)}} \omega_i = \int_{a_j} \omega_{t(i)}$. Besides, the matrix of a -periods is imaginary, hence $\int_{a_j} \omega_{t(i)} = -\int_{a_j} \overline{\omega_{t(i)}}$. Both $\tau^*\omega_i$ and $-\overline{\omega_{t(i)}}$ are antiholomorphic differentials, and they have the same a -periods. Hence $\tau^*\omega_i = -\overline{\omega_{t(i)}}$, $i = 1, \dots, g$, and 1° is proven. 2° and 3° immediately follow from 1° by definition of the permutation t . \square

Lemma 2.4. Let $\{a_j, b_j\}$ be a real base of cycles of a real curve of the type (g, k, ε) , then

1° . $\overline{B_j} = B_j$ for $j \leq k-1$;

2° . $\overline{B_j} = B_j - A_j$ for $\varepsilon = 0$, $j = k, \dots, g$;

3° . $\overline{B_j} = tB_{j+m}$ for $\varepsilon = 1$, $j = k, \dots, k+m-1$, where $m = \frac{1}{2}(g+1-k)$.

Remark 2.5. For separating curves, symmetries of B given by the cases 1° , 3° of the lemma are the same as claimed in [4, p.109]. They are also formulated in Lemma 8.2 [9], however without mentioning the permutation t in the 3d relation. For non-separating curves, the lemma was first formulated in [7] for hyperelliptic curves, and then in Lemma 8.2 [9] for general curves.

Proof of Lemma 2.4. The proof in the cases 1° and 3° immediately follows from the relation $\overline{B_j} = tB_{t(j)}$. Let's check the last. $\overline{B_j} = \overline{\int_{b_j} \omega} = \int_{b_j} \bar{\omega} = -t \int_{b_j} \tau^*\omega = -t \int_{\tau(b_j)} \omega = -t \int_{-b_{t(j)}} \omega = tB_{t(j)}$.

In the case 2° t is trivial. Similar to the cases 1° and 3° we get $\overline{B_j} = -\int_{\tau(b_j)} \omega$. We proceed as follows: $\int_{\tau(b_j)} \omega = \int_{-b_j+a_j} \omega = -B_j + A_j$. \square

Using the Abel transform we can transfer the antiinvolution τ to $Jac(\Sigma)$. Indeed, by the Riemann vanishing theorem we uniquely represent almost every $z \in Jac(\Sigma)$ as $z = \int^D \omega$ where D is a degree g divisor on Σ . Then we set by definition $\tau(z) = \int^D \tau^*\omega$. To prove that $\tau(z)$ is well-defined, we check that the period lattice is τ -invariant:

$$(2.2) \quad \tau(A_j) = A_{t(j)}, \quad \tau(B_j) = -B_{t(j)}$$

where the second is true modulo a -periods. Indeed, $\tau(A_j) = \int_{a_j} \tau^*\omega = -t \int_{a_j} \bar{\omega} = -t\overline{A_j} = tA_j = A_{t(j)}$ (here we used imaginary and symmetry of the matrix of a -periods). Similarly, $\tau(B_j) = \int_{b_j} \tau^*\omega = -t\overline{B_j} = -t(tB_{t(j)}) = -B_{t(j)}$ (since Lemma 2.4 implies that $\overline{B_j} \equiv tB_{t(j)} \pmod{A_j}$).

Observe that

$$(2.3) \quad \tau(z) = -t\bar{z}, \quad z \in Jac(\Sigma).$$

Indeed, $\tau(z) = \int^D \tau^* \omega = -t \overline{\int^D \omega} = -t\bar{z}$.

Next, we define an \mathbb{R} -linear involution $\tilde{\tau} : \mathbb{C}^g \rightarrow \mathbb{C}^g$ by

$$(2.4) \quad \tilde{\tau}(z) = -t\bar{z}, \quad z \in \mathbb{C}^g.$$

Hence

$$(2.5) \quad \tilde{\tau}z = -t\bar{z} = -(\bar{z}_1, \dots, \bar{z}_{k-1}, \bar{z}_{k+m}, \dots, \bar{z}_g, \bar{z}_k, \dots, \bar{z}_{k+m-1})^T.$$

for separating curves, and

$$(2.6) \quad \tilde{\tau}z = -\bar{z}.$$

for non-separating curves.

Lemma 2.6.

- 1°. $\tilde{\tau}A_j = A_{t(j)}$, $\tilde{\tau}B_j = -B_{t(j)}$, $\varepsilon = 1$, $j = 1, \dots, g$ or $\varepsilon = 0$, $j = 1, \dots, k-1$;
 $\tilde{\tau}A_j = A_j$, $\tilde{\tau}B_j = -B_j + A_j$, $\varepsilon = 0$, $j = k, \dots, g$;
- 2°. $\tilde{\tau}A_j = A_j$, $\tilde{\tau}B_j = -B_j$, $j = 1, \dots, k-1$;
 $\tilde{\tau}A_j = A_j$, $\tilde{\tau}B_j = -B_j + A_j$, $\varepsilon = 0$, $j = k, \dots, g$;
 $\tilde{\tau}A_j = A_{j+m}$, $\tilde{\tau}B_j = -B_{j+m}$, $\varepsilon = 1$, $j = k, \dots, k+m-1$;
 $\tilde{\tau}A_j = A_{j-m}$, $\tilde{\tau}B_j = -B_{j-m}$, $\varepsilon = 1$, $j = k+m, \dots, g$

where $m = \frac{1}{2}(g+1-k)$.

3°. The following diagram is commutative:

$$\begin{array}{ccc} \mathbb{C}^g & \xrightarrow{\tilde{\tau}} & \mathbb{C}^g \\ pr \downarrow & & pr \downarrow \\ Jac(\Sigma) & \xrightarrow{\tau} & Jac(\Sigma) \end{array}$$

where pr is the natural projection.

Proof. 1°. By (2.4) we have

$$\begin{aligned} \tilde{\tau}(A_j) &= -t\overline{A_j} = -t(-A_j) = tA_j = A_{t(j)} \text{ in all cases;} \\ \tilde{\tau}(B_j) &= -t\overline{B_j} = -t(tB_{t(j)}) = -B_{t(j)} \text{ except for } \varepsilon = 0, j = k, \dots, g; \\ \tilde{\tau}(B_j) &= -\overline{B_j} \text{ (by (2.6))} = -B_j + A_j \text{ for } \varepsilon = 0, j = k, \dots, g. \end{aligned}$$

In the last two lines we made use of Lemma 2.4.

Claim 2° follows from 1° by definition of t .

By claim 1° it follows that the fundamental lattice of the Jacobian is $\tilde{\tau}$ -invariant. Then (2.3) and (2.4) prove 3°. \square

Remark 2.7. $-\tilde{\tau}$ coincides with the $\tau_{\mathbb{R}}$ introduced in [9] except for the case 2°, $\varepsilon = 0$ of Lemma 2.6.

2.3. Riemann θ -function of a real curve. The Riemann theta function is defined by

$$(2.7) \quad \theta(z, B) = \sum_{N \in \mathbb{Z}^g} \exp\left(\frac{1}{2}N^T B N + N^T z\right).$$

Lemma 2.8 ([8]). *The Riemann theta function possesses the following symmetries:*

- 1° $\theta(t\bar{z}) = \overline{\theta(z)}$ for separating curves;

2° $\overline{\theta(z)} = \theta(\bar{z} + \lambda)$ for non-separating curves, where

$$(2.8) \quad \lambda = \pi i(0, \dots, 0, 1, \dots, 1)^T \quad (\text{units for } j \geq k).$$

Proof. 1°. [4, proof of Prop. 6.1]. By (2.6) (and for the reason that θ is even in z) $\theta(tz) = \sum_{N \in \mathbb{Z}^g} \exp(\frac{1}{2}N^T B N + N^T(t\bar{z}))$. Since $t^2 = 1$ and $t^T = t$ we have $N^T B N = (tN)^T(tBt)(tN)$. By Lemma 2.4 $\bar{B} = tBt$ (see the proof of Lemma 2.4). By setting $M = tN$ we get $\theta(\tilde{\tau}z) = \sum_{M \in \mathbb{Z}^g} \exp(\frac{1}{2}M^T \bar{B} M + M^T \bar{z}) = \overline{\theta(z)}$.

2°. $\theta(\bar{z} + \lambda) = \sum_{N \in \mathbb{Z}^g} \exp(\frac{1}{2}N^T B N + N^T \lambda + N^T \bar{z})$. We want to show that

$$(2.9) \quad \frac{1}{2}N^T B N + N^T \lambda \equiv \frac{1}{2}N^T (B - \tilde{A}) N \pmod{2\pi i \mathbb{Z}}, \quad \forall N \in \mathbb{Z}^g$$

where $\tilde{A} = (0, \dots, 0, A_k, \dots, A_g)$, $A_j = 2\pi i(\delta_j^i)^{i=1, \dots, g}$, $j = 1, \dots, g$ are columns of the matrix of a -periods. According to Lemma 2.4, 2° $B - \tilde{A} = \bar{B}$, which proves the required symmetry.

Relation (2.9) obviously descends to the following relation:

$$(2.10) \quad N^T \lambda + \frac{1}{2}N^T \tilde{A} N \equiv 0 \pmod{2\pi i \mathbb{Z}}, \quad \forall N \in \mathbb{Z}^g.$$

Observe that $\tilde{A} = 2\text{diag}(\lambda)$. Hence $\frac{1}{2}N^T \tilde{A} N = \sum_{k=1}^g \lambda_k n_k^2$ where $N = \sum_{k=1}^g n_k e_k$, and $N^T \lambda = \sum_{k=1}^g \lambda_k n_k$. Then we have

$$(2.11) \quad N^T \lambda + \frac{1}{2}N^T \tilde{A} N = \sum_{k=1}^g \lambda_k n_k (n_k + 1) \in 2\pi i \mathbb{Z}$$

because $\lambda_k \in \pi i \mathbb{Z}$, and $n_k(n_k + 1) \in 2\mathbb{Z}$ for every $k = 1, \dots, g$. □

Remark 2.9. In [8] claim 1° is given in the form which in our notation reads as $\theta(\tilde{\tau}z) = \overline{\theta(z)}$. It immediately follows from our form of the claim due to relation (2.3) and to the fact that θ is even in z .

Remark 2.10. In [8], λ is given without any factor πi , like in [7], by mistake, because the normalizations of basis differentials (the expressions for θ , resp.) are different in [7] and [8].

3. REAL CURVES WITH INVOLUTIONS, THEIR PRYM VARIETIES, AND θ -FUNCTIONS

By a real curve with an involution we mean a compact algebraic curve Σ over \mathbb{C} endowed with two commuting involutions σ and τ where the first is holomorphic, while the second is antiholomorphic. We will refer to τ as to antiinvolution. Let $g = g(\Sigma)$ denote the genus of Σ , $g_\sigma = g(\Sigma/\sigma)$.

3.1. Prym matrix for a general curve with an involution. Let the degree of the branching divisor of the natural covering $\Sigma \rightarrow \Sigma/\sigma$ be equal to $2n$. According to [4], there exists a base of cycles a_i, b_i ($i = 1, \dots, g_\sigma$), a_i, b_i ($i = g_\sigma + 1, \dots, h = g_\sigma + n - 1$), a_{i+h}, b_{i+h} ($i = 1, \dots, g_\sigma$) on Σ , where the first and the third groups of cycles are pulled back from Σ_σ . Consider the permutation induced by σ on indices of the base: $s = (1, h + 1) \dots (g_\sigma, h + g_\sigma)(g_\sigma + 1) \dots (h)$. Then the following relations hold:

$$(3.1) \quad \sigma(a_j) + a_{s(j)} = \sigma(b_j) + b_{s(j)} = 0, \quad j = 1, \dots, g.$$

Let $\{w_i | i = 1, \dots, g\}$ be the dual base of normalized holomorphic differentials on Σ , $w = (w_1, \dots, w_g)^T$. Then

$$(3.2) \quad \sigma^* w = -s w$$

where having been written on the left, s denotes the matrix of the linear transformation permuting coordinates according to the permutation s . Indeed, $\int_{a_j} \sigma^* w = \int_{\sigma(a_j)} w$ by change of variables, $\int_{\sigma(a_j)} w = \int_{-a_{s(j)}} w = -\int_{a_{s(j)}} w$ by (3.1), and $\int_{a_{s(j)}} w = \int_{a_j} s w$ by symmetry of the matrix of a -periods.

Relations (3.2) can be also written in the form

$$(3.3) \quad \sigma^* w_i = \begin{cases} -w_{i+h}, & i = 1, \dots, g_\sigma; \\ -w_i, & i = g_\sigma + 1, \dots, h; \\ -w_{i-h}, & i = h + 1, \dots, g, \end{cases}$$

Following Fay [4] we introduce $\alpha, \beta = 1, \dots, g_\sigma$ and $\alpha' = \alpha + h, \beta' = \beta + h$, and with this convention it is assumed that $i, j = g_\sigma + 1, \dots, h$. Then $s = (\alpha, \alpha')(i)$, and

$$(3.4) \quad \begin{aligned} \sigma(a_\alpha) + a_{\alpha'} &= \sigma(b_\alpha) + b_{\alpha'} = \sigma(a_i) + a_i = 0; \\ \sigma^* w_\alpha &= -w_{\alpha'}, \quad \sigma^* w_i = -w_i. \end{aligned}$$

Differentials $\{\omega_i = w_i + w_{i+h} | i = 1, \dots, g_\sigma\}$ and $\{\omega_i = w_i | i = g_\sigma + 1, \dots, h\}$ form a base of Prym differentials on Σ . This base is normalized in a sense that $\oint_{a_j} \omega_k = 2\pi i \delta_{kj}$, $k, j = 1, \dots, h$. The Riemann matrix of a Prym variety is defined as the matrix $\Pi = (\Pi_{ij})_{i,j=1,\dots,h}$ where

$$(3.5) \quad \Pi_{ij} = \oint_{b_j} \omega_i \quad (j = 1, \dots, g_\sigma); \quad \Pi_{ij} = \frac{1}{2} \oint_{b_j} \omega_i \quad (j = g_\sigma + 1, \dots, h).$$

In the Fay notation it looks as

$$(3.6) \quad \Pi = \left(\begin{array}{c|c} \Pi_{\alpha\beta} & \Pi_{\alpha j} \\ \hline \Pi_{i\beta} & \Pi_{ij} \end{array} \right) = \left(\begin{array}{c|c} \int_{b_\beta} \omega_\alpha & \frac{1}{2} \int_{b_j} \omega_\alpha \\ \hline \int_{b_\beta} \omega_i & \frac{1}{2} \int_{b_j} \omega_i \end{array} \right)$$

(cf. [4, Eq. (92)]). The expression of the Riemann–Prym matrix via the Riemann matrix is given by

Lemma 3.1. 1°. $\Pi_{\alpha\beta} = B_{\alpha\beta} + B_{\alpha'\beta} = \Pi_{\beta\alpha}$;
 2°. $\Pi_{\alpha j} = \frac{1}{2}(B_{\alpha j} + B_{\alpha'j}) = \Pi_{j\alpha}$;
 3°. $\Pi_{ij} = \frac{1}{2}B_{ij} = \Pi_{ji}$.

Proof. 1°. By definition of $\Pi_{\alpha\beta}$ and symmetry of B we have $\Pi_{\alpha\beta} = \int_{b_\beta} (w_\alpha + w_{\alpha'}) = B_{\alpha\beta} + B_{\alpha'\beta} = B_{\beta\alpha} + B_{\beta\alpha'}$. Next, $B_{\beta\alpha'} = B_{\beta'\alpha}$. Indeed, by definition, relations (3.4) and change of variables $B_{\beta\alpha'} = \int_{b_{\alpha'}} w_\beta = -\int_{\sigma(b_\alpha)} w_\beta = -\int_{b_\alpha} \sigma^* w_\beta = \int_{b_\alpha} w_{\beta'} = B_{\beta'\alpha}$. Now $\Pi_{\alpha\beta} = B_{\beta\alpha} + B_{\beta'\alpha} = \Pi_{\beta\alpha}$.

2°. By definition $\Pi_{\alpha j} = \frac{1}{2}(B_{\alpha j} + B_{\alpha'j})$, $\Pi_{j\alpha} = B_{j\alpha}$. But $B_{\alpha j} = \int_{b_j} w_\alpha = \int_{-b_j} \sigma^* w_\alpha = \int_{b_j} w_{\alpha'} = B_{\alpha'j}$. Hence $\Pi_{\alpha j} = B_{\alpha j} = B_{j\alpha} = \Pi_{j\alpha}$.

3°. The proof is similar to the two above ones. □

3.2. Symmetries of the Prym matrix for a real curve with an involution. As above (Section 2.2), let t denote the permutation of indices induced by τ . Since σ and τ commute, there exists a real base of cycles satisfying conditions (3.4).

Lemma 3.2. *For separating real curves with an involution, the matrix Π satisfies to the following equivalent conditions:*

- 1°. $\overline{\Pi_{pq}} = \Pi_{t(p),t(q)}$, $p, q = 1, \dots, h$;
- 2°. $\overline{\Pi} = t\Pi t$ where t is the matrix giving the permutation t of coordinates: $t_{pq} = \delta_{p,t(q)}$;
- 3°. $\overline{\Pi_q} = t\Pi_{t(q)}$ where Π_q is the q -th column of Π .

Proof. For separating curves, Lemma 2.4 can be summarized as follows: $\overline{B_{pq}} = B_{t(p),t(q)}$, $p, q = 1, \dots, g$. Making use of that we prove the claim 1° of the present lemma for every block of the matrix (3.6), separately.

Indeed, by (3.6) $\Pi_{\alpha\beta} = B_{\alpha\beta} + B_{\alpha'\beta}$, hence $\overline{\Pi_{\alpha\beta}} = \overline{B_{\alpha\beta}} + \overline{B_{\alpha'\beta}} = B_{t(\alpha),t(\beta)} + B_{t(\alpha'),t(\beta)} = B_{t(\alpha),t(\beta)} + B_{t(\alpha'),t(\beta)} = \Pi_{t(\alpha),t(\beta)}$ ($t(\alpha') = t(\alpha)'$ by commutativity of σ and τ). Similarly

$$\begin{aligned}\overline{\Pi_{i\beta}} &= \overline{B_{i\beta}} = B_{t(i),t(\beta)} = \Pi_{t(i),t(\beta)}, \\ \overline{\Pi_{\alpha j}} &= \frac{1}{2}\overline{B_{\alpha j}} = \frac{1}{2}B_{t(\alpha),t(j)} = \Pi_{t(\alpha),t(j)}, \\ \overline{\Pi_{ij}} &= \frac{1}{2}\overline{B_{ij}} = \frac{1}{2}B_{t(i),t(j)} = \Pi_{t(i),t(j)}.\end{aligned}$$

This completes the proof of the claim 1°. The claims 2° and 3° are obviously equivalent to the claim 1°. \square

Now assume the curve to be non-separating. Assume a_1, \dots, a_{r_0} and $a_{g_\sigma+1}, \dots, a_{g_\sigma+r_1}$ ($r_0 \leq g_\sigma$, $r_1 \leq n-1$) are ovals, and the other a -cycles are not.

Lemma 3.3. *Let $\{a_j, b_j\}$ be a σ -invariant real base of cycles of a non-separating real curve, then*

- 1°. $\overline{\Pi_\beta} = \Pi_\beta$ for $\beta = 1, \dots, r_0$ (i.e. for β corresponding to ovals permutable by σ);
- 2°. $\overline{\Pi_j} = \Pi_j$ for $j = g_\sigma + 1, \dots, g_\sigma + r_1$ (i.e. for j corresponding to σ -invariant ovals);
- 3°. $\overline{\Pi_\beta} = \Pi_\beta - A_\beta$ for $\beta = r_0 + 1, \dots, g_\sigma$ (i.e. for non-ovals permutable by σ);
- 4°. $\overline{\Pi_j} = \Pi_j - \frac{1}{2}A_j$ for $j = g_\sigma + r_1 + 1, \dots, h$ (i.e. for σ -invariant non-ovals).

Proof. 1°. For Π_β , $\beta \leq r_0 \leq g_\sigma$ we have

$$\begin{aligned}\Pi_{\alpha\beta} &= B_{\alpha\beta} + B_{\alpha'\beta} \quad \text{by Lemma 3.1;} \\ \Pi_{i\beta} &= B_{i\beta} \quad \text{by (3.6).}\end{aligned}$$

Since number β corresponds to an oval, we obtain by Lemma 2.4 that $B_{\alpha\beta} = \overline{B_{\alpha\beta}}$, $B_{\alpha'\beta} = \overline{B_{\alpha'\beta}}$, $B_{i\beta} = \overline{B_{i\beta}}$. Hence $\overline{\Pi_\beta} = \Pi_\beta$.

2°. For Π_j , $g_\sigma + 1 \leq j \leq g_\sigma + r_1$ we have the same but the coefficient $\frac{1}{2}$ on the right hand sides of the above relations (Lemma 3.1).

3°. For $\alpha \leq g_\sigma$ it follows from $\Pi_{\alpha\beta} = B_{\alpha\beta} + B_{\alpha'\beta}$ and Lemma 2.4, 2°. Observe that $A_{\alpha'\beta} = 2\pi i \delta_{\alpha'\beta} = 0$ since $\alpha \leq g_\sigma$, $\alpha' > h \geq g_\sigma$. Hence $\overline{\Pi_{\alpha\beta}} - A_{\alpha\beta} = B_{\alpha\beta} - A_{\alpha\beta} + B_{\alpha'\beta} - A_{\alpha'\beta}$. According to Lemma 2.4, 2° the last is equal to $\overline{B_{\alpha\beta}} + \overline{B_{\alpha'\beta}} = \overline{\Pi_{\alpha\beta}}$.

For $i > g_\sigma$, $\Pi_{i\beta} = B_{i\beta}$ (like in the point 1°), hence $\overline{\Pi_{i\beta}} - A_{i\beta} = B_{i\beta} - A_{i\beta} = \overline{B_{i\beta}} = \overline{\Pi_{i\beta}}$.

4°. For $\alpha \leq g_\sigma$ the claim from $\Pi_{\alpha j} = \frac{1}{2}(B_{\alpha j} + B_{\alpha'j})$ and Lemma 2.4, 2°. In this case $A_{\alpha j} = A_{\alpha'j} = 0$ since $\alpha \leq g_\sigma < j \leq h < \alpha'$. Hence $\overline{\Pi_{\alpha j}} - \frac{1}{2}A_{\alpha j} = \frac{1}{2}(B_{\alpha j} - A_{\alpha j} + B_{\alpha'j} -$

$A_{\alpha'j}$). According to Lemma 2.4, 2° the last is equal to $\frac{1}{2}(\overline{B_{\alpha j}} + \overline{B_{\alpha'j}}) = \overline{\Pi_{\alpha j}}$. In fact, we have obtained $\overline{\Pi_{\alpha j}} = \Pi_{\alpha j}$ in this case.

For $i > g_\sigma$ the claim follows from $\Pi_{ij} = \frac{1}{2}B_{ij}$ (Lemma 3.1, 3°). Indeed, $\Pi_{ij} - \frac{1}{2}A_{ij} = \frac{1}{2}(B_{ij} - A_{ij}) = \frac{1}{2}\overline{B_{ij}} = \overline{\Pi_{ij}}$. \square

Corollary 3.4. *All entries of Π are real numbers except for diagonal entries Π_{jj} with $j = r_0 + 1, \dots, g_\sigma$ and $j = g_\sigma + r_1 + 1, \dots, h$ (for Jacobians, except for the entries of the right lower corner).*

3.3. Prym theta function, isoPrym variety and the Abel–Prym map. A Prym theta function is a Riemann θ function with the Riemann matrix Π :

$$(3.7) \quad \theta(z, \Pi) = \sum_{N \in \mathbb{Z}^h} \exp\left(\frac{1}{2}N^T \Pi N + N^T z\right), \quad z \in \mathbb{C}^h.$$

It corresponds to the principally polarized Abelian variety $P_0 = \mathbb{C}^h / \mathbb{Z}(2\pi i E, \Pi)$ which is a finite unramified covering of the Prym variety (thus isogeneous to the Prymian). Below, P_0 is referred to as isoPrymian. The map $\mathcal{A} : \Sigma \rightarrow P_0$:

$$(3.8) \quad \mathcal{A}(\gamma) = \left(\int_{\gamma_0}^{\gamma} \omega \right) \pmod{\mathbb{Z}(2\pi i E, \Pi)}$$

where $\omega = (\omega_1, \dots, \omega_h)^T$ (equivalently, $\omega = (\omega_\alpha, \omega_j)^T$) is referred to as the Abel–Prym map.

Lemma 3.5. *The Prym θ function of a real curve possesses the following symmetries:*

1° $\overline{\theta(z)} = \theta(t\bar{z})$ for separating curves;

2° For non-separating curves, let $\lambda = (\lambda_1, \dots, \lambda_h)^T$ where

$$(3.9) \quad \lambda_k = \begin{cases} \pi i, & k = r_0 + 1, \dots, g_\sigma; \\ \frac{1}{2}\pi i, & k = g_\sigma + r_1 + 1, \dots, h; \\ 0 & \text{otherwise.} \end{cases}$$

If all σ -invariant basis cycles are ovals (i.e. $g_\sigma + r_1 = h$) then

$$\overline{\theta(z)} = \theta(\bar{z} + \lambda).$$

Proof. 1°. Due to Lemma 3.2, 2° the proof of the claim 1° of the present lemma is the same as the proof of Lemma 2.8, 1°.

2°. Similar to Lemma 2.8, we can summarize Lemma 3.3 as $\overline{\Pi} = \Pi - \tilde{A}$ where $\tilde{A} = 2\text{diag}(\lambda)$. Again, the desired symmetry reduces to the relation

$$(3.10) \quad N^T \lambda + \frac{1}{2}N^T \tilde{A} N \equiv 0 \pmod{2\pi i \mathbb{Z}}, \quad \forall N \in \mathbb{Z}^g,$$

and

$$(3.11) \quad N^T \lambda + \frac{1}{2}N^T \tilde{A} N = \sum_{k=1}^g \lambda_k n_k (n_k + 1).$$

However, it follows from (3.9) that for $k = g_\sigma + r_1 + 1, \dots, h$ (i.e. for σ -invariant non-ovals) we only can claim that $\lambda_k n_k (n_k + 1) \in \pi i \mathbb{Z}$ while for the remainder of values of k we have $\lambda_k n_k (n_k + 1) \in 2\pi i \mathbb{Z}$. Hence in absence of such non-ovals the desired symmetry takes place. The requirement that all σ -invariant basis cycles are ovals is exactly equivalent to the absence of σ -invariant non-ovals among the basis cycles. \square

4. THE ABEL–PRYM MAP, AND INVERSION THEOREM

4.1. Riemann vanishing theorem. Assume Σ to be endowed with a holomorphic involution σ . Let $F_z(P) = \theta(\int_q^P \omega - z)$ ($P \in \Sigma$, $z \in \mathbb{C}^h$) where θ is the Prym θ -function. Thus $F_z(P)$ is defined up to a θ -multiplier depending on the integration path from q to P . Hence the zero divisor of $F_z(P)$ is well defined. We denote it by ζ . As above, let A stay for the Abel–Prym map, and $\varepsilon = \text{diag}(\varepsilon_1, \dots, \varepsilon_h)$ where $\varepsilon_j = 1$ for $j = 1, \dots, g_\sigma$, $\varepsilon_j = 2$ for $j = g_\sigma + 1, \dots, h$. Let $\sum_{i=1}^{2n} Q_i$ be the branch divisor of the natural covering $\Sigma \rightarrow \Sigma_\sigma$, K_σ be the canonical divisor on Σ_σ . With this notation, Corollary 5.6 [4] combined with Eq.(108) [4] reads as

Lemma 4.1. *Either $F_z(P) \equiv 0$ or*

- 1° $\deg \zeta = 2h$;
- 2° $A(\zeta) = \tilde{z}$ where $\tilde{z} = \varepsilon z + \Delta$, $\Delta = \sum_{i=1}^{2n} A(Q_i) + \pi^*(\frac{1}{2}K_\sigma)$;
- 3° $\pi_*\zeta$ is the divisor of zeroes of a differential on Σ_σ with at most simple poles at $\pi_*(\sum_{i=1}^{2n} Q_i)$: $\pi_*\zeta \sim \sum_{i=1}^{2n} \pi_*(Q_i) + K_\sigma$.

For an analytic proof see [15]. The form of Δ assumes that the base point is σ -invariant, otherwise Δ slightly modifies [4].

Lemma 4.1 is an analog of the Riemann vanishing theorem for curves with an involution. Unlike the Riemann theorem itself, Lemma 4.1, in particular its claim 1°, leads to a description of the inverse image of the Abel–Prym map as of an h -dimensional subvariety of a $2h$ -dimensional variety of divisors. The subvariety is distinguished by the equations

$$(4.1) \quad A(\zeta) + A(\sigma\zeta) = 2\Delta.$$

Indeed, $\sigma\zeta$ is obviously the divisor of zeroes of $\theta(A(\sigma P) - z)$. If the base point of the transform A has been chosen to be σ -invariant, then $\theta(A(\sigma P) - z) = \theta(-A(P) - z) = \theta(A(P) + z)$. Hence by Lemma 4.1, 2° $A(\sigma\zeta) = -\varepsilon z + \Delta$. This implies (4.1).

4.2. θ -function formula for symmetric functions of zeroes of $F_{\varepsilon^{-1}(z-\Delta)}(P)$. In general, any effective procedure of finding out the divisor ζ is unknown. However, there exists an effective (avoiding any direct solution of the equation $\theta(A(P) - \varepsilon^{-1}(z - \Delta)) = 0$) procedure of calculating symmetric functions of the points of the divisor ζ . Here we present it following the lines of [15], with modifications due to the difference between the particular case considered there, and the general case studied here. In turn, the approach in [15] develops the one proposed in [5], and finally goes back to Riemann.

Let $z \in \text{isoPrym}(\Sigma)$, $\mathcal{A}^{-1}(z) = \zeta$, and $|\zeta| = \{P_1, \dots, P_{2h}\}$ is the support of ζ . Symmetric functions of P_1, \dots, P_{2h} are well-defined functions of z . It is our goal to find out a theta function formulae for a certain set of such functions.

For any meromorphic function f on Σ let $\sigma_f(z) = \sum_{P \in \zeta} f(P)$. Below, we assume that Σ is a branch covering of the Riemann sphere, and f has no pole except those over infinity. Let $\pi : \Sigma \rightarrow \mathbb{P}^1$ be the covering map. Then we have the following relation close to the relation due to Dubrovin ([6, Eq. (11.23)], [5, Eq. (2.4.29)]) (see the proof in [15]):

$$(4.2) \quad \sigma_f(z) = c - \sum_{q \in \pi^{-1}(\infty)} \text{res}_q f d \ln F_{\varepsilon^{-1}(z-\Delta)},$$

where c is constant in z .

Let $x_i = \pi(P_i)$, $i = 1, \dots, 2h$. We take $f(P) = x^k$ where $x = \pi(P)$. Denote σ_f by σ_k , then

$$(4.3) \quad \sigma_k(z) = x_1^k + \dots + x_{2h}^k,$$

i.e. $\sigma_k(z)$ is the k th Newton polynomial in x_1, \dots, x_{2h} . Exactly as in [15] we obtain

$$(4.4) \quad \sigma_k(z) = c - \sum_{q \in \pi^{-1}(\infty)} \sum_{i=1}^h \sum_{1 \leq |j| \leq 2k-1} \mathfrak{x}_{qik}^j D^j \partial_i \ln \theta(\mathcal{A}(q) - \epsilon^{-1}(z - \Delta))$$

where $j = (j_1, \dots, j_{h_1})$, $|j| = j_1 + \dots + j_{h_1}$,

$$(4.5) \quad D^j = \frac{1}{j_1! \dots j_h!} \frac{\partial^{|j|}}{\partial z_1^{j_1} \dots \partial z_h^{j_h}}, \quad \mathfrak{x}_{qik}^j = \sum_{l_{qi} + \sum_{s=1}^h \sum_{p=1}^{j_s} l_{qsp} = 2k-1} \varphi_{qi}^{(l_{qi})} \prod_{s=1}^h \prod_{p=1}^{j_s} \frac{\varphi_{qsp}^{(l_{qsp}-1)}}{l_{qsp}},$$

l_{qs} and $\varphi_{qs}^{(l_{qs})}$ are defined from the relation $\mathcal{A}_s(P) = \mathcal{A}_s(q) + \sum_{l_{qs} \geq 1} \frac{\varphi_{qs}^{(l_{qs})}}{l_{qs}} z_q^{l_{qs}}$, $P = P(z_q)$, z_q is a local parameter at the point q .

The functions $\sigma_k(z)$, $k = 1, \dots, 2h$ give a full set of symmetric functions of x -coordinates of the points in $A^{-1}(z)$. They determine x_1, \dots, x_{2h} up to a permutation. With this relation, observe that, for hyperelliptic curves, there exists an alternative to the Riemann vanishing theorem [2]. Namely, the x_i 's can be calculated as zeroes of an algebraic equation with coefficients given by means of the Weierstraß \wp -functions on the curve in that case. The following statement shows that knowledge of $\sigma_k(z)$, $k = 1, \dots, 2h$ gives the solution of the Jacobi problem of the same degree of effectiveness in the case of an arbitrary curve with involution.

Theorem 4.2. *Calculating $\sigma_k(z)$, $k = 1, \dots, 2h$ reduces the problem of finding out the divisor ζ to resolving a degree $2h$ algebraic equation.*

Indeed, we pass from the Newton polynomials to elementary symmetric functions in (4.3), and then find out x_1, \dots, x_{2h} as the roots of the corresponding algebraic equation.

4.3. Inversion theorem for real curves. It is a purpose of this section to show that for curves possessing a real structure τ , the inverse image of certain real subvarieties of the isoPrymian (actually being shifts of varieties of fixed points of symmetries of the Prym theta function) can be given a more effective description as the variety of τ -invariant (or $\sigma\tau$ -invariant) degree $2h$ divisors (automatically satisfying (4.1) by Lemma 4.1). Observe that both τ and $\sigma\tau$ are antiholomorphic involutions on Σ , and both of them are involved into the description. The following theorem we refer to as the inversion theorem for real curves.

Theorem 4.3. *Assume, $F_{\epsilon^{-1}(z-\Delta)}(P)$ is not identical 0. Then for separating curves*

- 1°. *if $z + t\bar{z} = \Delta + t\bar{\Delta}$ then $\zeta = A^{-1}(z)$ is τ -invariant;*
- 2°. *if $z - t\bar{z} = \Delta - t\bar{\Delta}$ then $\zeta = A^{-1}(z)$ is $\sigma\tau$ -invariant;*

and for non-separating curves

- 3°. *if $z - \bar{z} = \Delta - \bar{\Delta} - \epsilon\lambda$ then $\zeta = A^{-1}(z)$ is $\sigma\tau$ -invariant.*

Proof. By Lemma 4.1 the zero divisor of $F_{\epsilon^{-1}(z-\Delta)}(P)$, and $A^{-1}(z)$ are the same. Consider first the case of separation curves.

1°. Let's find out when the zero divisor of $F_{\varepsilon^{-1}(z-\Delta)}(P)$ is τ -invariant. It takes place if $F_{\varepsilon^{-1}(z-\Delta)}(\tau(P)) = \overline{F_{\varepsilon^{-1}(z-\Delta)}(P)}$ (up to a θ -multiplier depending on the integration path). By Lemma 2.3, 1° $A(\tau(P)) = -t\overline{A(P)}$. Hence $F_{\varepsilon^{-1}(z-\Delta)}(\tau(P)) = \theta(-t\overline{A(P)} - \varepsilon^{-1}z + \varepsilon^{-1}\Delta) = \theta(t\overline{A(P)} + \varepsilon^{-1}z - \varepsilon^{-1}\Delta)$ (the last because θ is an even function). By Lemma 3.5, 1°

$$\overline{F_{\varepsilon^{-1}(z-\Delta)}(P)} = \overline{\theta(A(P) - \varepsilon^{-1}z + \varepsilon^{-1}\Delta)} = \theta(t\overline{A(P)} - \varepsilon^{-1}\overline{z} + \varepsilon^{-1}\overline{\Delta}).$$

$F_{\varepsilon^{-1}(z-\Delta)}(\tau(P)) = \overline{F_{\varepsilon^{-1}(z-\Delta)}(P)}$ provided $t\overline{A(P)} + \varepsilon^{-1}z - \varepsilon^{-1}\Delta = t(\overline{A(P)} - \varepsilon^{-1}\overline{z} + \varepsilon^{-1}\overline{\Delta})$. Since ε commutes with t , the last relation implies the claim 1° of the lemma.

2°. By Lemma 3.5, 1°, and due to the fact that ω in (3.8) is σ -antiinvariant, we have $A(\sigma\tau(P)) = t\overline{A(P)}$. Hence $F_{\varepsilon^{-1}(z-\Delta)}(\sigma\tau(P)) = \theta(t\overline{A(P)} - \varepsilon^{-1}z + \varepsilon^{-1}\Delta)$. By Lemma 3.5, 1°

$$\overline{F_{\varepsilon^{-1}(z-\Delta)}(P)} = \overline{\theta(A(P) - \varepsilon^{-1}z + \varepsilon^{-1}\Delta)} = \theta(t\overline{A(P)} - \varepsilon^{-1}\overline{z} + \varepsilon^{-1}\overline{\Delta}).$$

$F_{\varepsilon^{-1}(z-\Delta)}(\sigma\tau(P)) = \overline{F_{\varepsilon^{-1}(z-\Delta)}(P)}$ provided $t\overline{A(P)} - \varepsilon^{-1}z + \varepsilon^{-1}\Delta = t(\overline{A(P)} - \varepsilon^{-1}\overline{z} + \varepsilon^{-1}\overline{\Delta})$. Since ε commutes with t , the last relation implies the claim 2° of the lemma.

3°. Non-separation curves.

By Lemma 2.3, 3°, and due to the fact that ω in (3.8) is σ -antiinvariant, $A(\sigma\tau(P)) = \overline{A(P)}$. Hence $F_{\varepsilon^{-1}(z-\Delta)}(\sigma\tau(P)) = \theta(\overline{A(P)} - \varepsilon^{-1}z + \varepsilon^{-1}\Delta)$. By Lemma 3.5, 2°

$$\overline{F_{\varepsilon^{-1}(z-\Delta)}(P)} = \overline{\theta(A(P) - \varepsilon^{-1}z + \varepsilon^{-1}\Delta)} = \theta(\overline{A(P)} - \varepsilon^{-1}\overline{z} + \varepsilon^{-1}\overline{\Delta} + \lambda).$$

$F_{\varepsilon^{-1}(z-\Delta)}(\sigma\tau(P)) = \overline{F_{\varepsilon^{-1}(\varepsilon^{-1}z - \varepsilon^{-1}\Delta)}(P)}$ provided $\overline{A(P)} - \varepsilon^{-1}z + \varepsilon^{-1}\Delta = \overline{A(P)} - \varepsilon^{-1}\overline{z} + \varepsilon^{-1}\overline{\Delta} + \lambda$. The last relation implies the claim 3° of the lemma. \square

Remark 4.4. For non-separating curves the case when $\zeta = A^{-1}(z)$ is τ -invariant does not exist, for the reason that it descends to the relation $z + \overline{z} = \Delta + \overline{\Delta} - \lambda$ which is contradictory because λ is imaginary, non-zero.

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