

Zeros of Stern polynomials in the complex plane

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Abstract

The classical Stern sequence of positive integers was extended to a polynomial sequence $S_n(\lambda)$ by Klavžar et. al. by defining $S_0(\lambda) = 0$, $S_1(\lambda) = 1$, and

$$S_{2n}(\lambda) = \lambda S_n(\lambda), \quad S_{2n+1}(\lambda) = S_n(\lambda) + S_{n+1}(\lambda).$$

Dilcher et. al. conjectured that all roots of $S_n(\lambda)$ lie in the half-plane $\{\operatorname{Re} w < 1\}$. We make partial progress on this conjecture by proving that $\{|w - 2| \leq 1\} \subseteq \mathbb{C}$ does not contain any roots of $S_n(\lambda)$. Our proof uses the Parabola Theorem for convergence of complex continued fractions. As a corollary, we establish a conjecture of Ulas and Ulas by showing that $S_p(\lambda)$ is irreducible in $\mathbb{Z}[\lambda]$ whenever p is a positive prime.

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1 Introduction

The Stern sequence $(s_n)_{n \geq 0}$ of positive integers, named after Moritz Abraham Stern, is given by $s_0 = 0$, $s_1 = 1$, and

$$s_{2n} = s_n, \quad s_{2n+1} = s_n + s_{n+1}. \tag{1.1}$$

There is a vast body of literature (e.g. [2, 10, 12, 15] and the references therein) about the Stern sequence. Perhaps the most striking property of the Stern sequence is that every positive rational number appears exactly once in the sequence $(s_{n+1}/s_n)_{n \geq 1}$, giving an explicit enumeration of \mathbb{Q}^+ . Stern proved this fact 15 years before Cantor introduced the notion of a countable set!

In [8], Klavžar, Milutinović, and Petr define a polynomial analogue $S_n(\lambda)$ of the Stern sequence via $S_0(\lambda) = 0$, $S_1(\lambda) = 1$, and

$$S_{2n}(\lambda) = \lambda S_n(\lambda), \quad S_{2n+1}(\lambda) = S_n(\lambda) + S_{n+1}(\lambda). \quad (1.2)$$

The first 16 terms of the sequence $S_n(\lambda)$ are given in Table 1. This table suggests many patterns, several of which are recorded in Proposition 1.

n	$S_n(\lambda)$	n	$S_n(\lambda)$
1	1	9	$1 + 2\lambda + \lambda^2$
2	λ	10	$\lambda + 2\lambda^2$
3	$1 + \lambda$	11	$1 + 3\lambda + \lambda^2$
4	λ^2	12	$\lambda^2 + \lambda^3$
5	$1 + 2\lambda$	13	$1 + 2\lambda + 2\lambda^2$
6	$\lambda + \lambda^2$	14	$\lambda + \lambda^2 + \lambda^3$
7	$1 + \lambda + \lambda^2$	15	$1 + \lambda + \lambda^2 + \lambda^3$
8	λ^3	16	λ^4

Table 1: The first 16 terms of the sequence $S_n(\lambda)$.

While there are several papers which analyze the sequence $S_n(\lambda)$ (see [3, 4, 5, 17] and the references therein), not much is known about the zeros of Stern polynomials. Because $S_{2n}(\lambda) = \lambda S_n(\lambda)$ for all $n \geq 1$, it suffices to examine the roots of $S_n(\lambda)$ only when n is odd. Define

$$\mathcal{S} := \{z \in \mathbb{C} : S_n(z) = 0 \text{ for some odd } n \geq 1\}.$$

In [5, Theorem 2.1], Gawron proves that 0 , -1 , $-\frac{1}{2}$, and $-\frac{1}{3}$ are the only rational zeros of any Stern polynomial by showing, more generally, that any real number in the closed interval $[-\frac{1}{4}, \frac{1}{4}]$ cannot be in \mathcal{S} . His methods extend easily to show that any complex number z with $|z| \leq \frac{1}{4}$ cannot be in \mathcal{S} . For completeness, we record the proof here.

Theorem 1. If $z \in \mathcal{S}$, then $|z| > \frac{1}{4}$.

Proof. Let z be any complex number with $|z| \leq \frac{1}{4}$. Let $b_n := |S_n(z)|$ for each positive integer n . We show more strongly that

$$b_{2n+1} > \frac{1}{2} \max\{b_n, b_{n+1}\} > 0 \quad (1.3)$$

for all $n \geq 1$.

The proof of (1.3) proceeds by induction on n . The base case, $n = 1$, follows because

$$b_3 = |z + 1| \geq \frac{3}{4} > \frac{1}{2} = \frac{1}{2} \max\{1, |z|\} = \frac{1}{2} \max\{b_1, b_2\}.$$

There are two cases to consider. First assume $n = 2k$ is even. Then

$$\begin{aligned} b_{4k+1} &= |S_{4k+1}(z)| = |S_{2k+1}(z) + S_{2k}(z)| \\ &= |zS_k(z) + S_{2k+1}(z)| \geq b_{2k+1} - \frac{1}{4}b_k \\ &\geq b_{2k+1} - \frac{1}{2}b_{2k+1} = \frac{1}{2}b_{2k+1}. \end{aligned}$$

Moreover, $b_{2k+1} \geq \frac{1}{2}b_k$, and thus $\max\{b_{2k}, b_{2k+1}\} = b_{2k+1} > 0$. In this case, our inequality is proved.

Now assume $n = 2k + 1$ is odd. Then

$$\begin{aligned} b_{4k+3} &= |S_{4k+3}(z)| = |S_{2k+1}(z) + S_{2k+2}(z)| \\ &= |S_{2k+1}(z) + zS_{k+1}(z)| \geq b_{2k+1} - \frac{1}{4}b_{k+1} \\ &\geq b_{2k+1} - \frac{1}{2}b_{2k+1} = \frac{1}{2}b_{2k+1}. \end{aligned}$$

Moreover, $b_{2k+1} \geq \frac{1}{2}b_{k+1}$, and so in this case we also have $\max\{b_{2k}, b_{2k+1}\} = b_{2k+1} > 0$. We have exhausted both cases, completing the proof of Theorem 1. \square

In [3], Dilcher et. al. focus more specifically on the complex roots of $S_n(\lambda)$. Their paper makes the following conjecture.

Conjecture 1. All elements of \mathcal{S} lie in the half-plane $\{\operatorname{Re} w < 1\}$.

By generalizing the Enestrom-Kakeya theorem, they prove Conjecture 1 for several classes of positive integers n taking the form $2^n \pm k$, where k is fixed and $2^n \geq k$. These are the only two papers the author could find which discuss the complex zeros of $S_n(\lambda)$.

Figure 1 shows a snapshot of \mathcal{S} . One striking feature of this figure is the contrasting behavior of these roots within the half-planes $\{\operatorname{Re} w \geq 0\}$ and $\{\operatorname{Re} w < 0\}$. These differences present difficulties in fully characterizing the geometry of \mathcal{S} .

In this paper, we partially resolve Conjecture 1 by establishing the following result.

Theorem 2. All roots of $S_n(\lambda)$ lie outside the disk $\{|z - 2| \leq 1\} \subseteq \mathbb{C}$.

To prove Theorem 2, we use a continued fraction representation for a ratio of Stern polynomials (Theorem 3) independently discovered by Reznick [12] and Schinzel [14]. This allows us to use the Parabola Theorem (Theorem 4) to show that, for certain values of $z \in \mathbb{C}$, the denominators of these continued fractions can never be zero. Along the way, we establish inequalities in \mathbb{C} relating to the sums $1 + z + \dots + z^{n-1}$ which may be of independent interest. The most notable of these inequalities is Theorem 7, which proves a lower bound for this geometric series whenever $\operatorname{Re} z \geq 1$.

As a corollary, we obtain the following surprising fact, resolving a conjecture of Ulas and Ulas ([17]).

Corollary 1. For each prime number p , the Stern polynomial $S_p(\lambda)$ is irreducible in \mathbb{Q} .

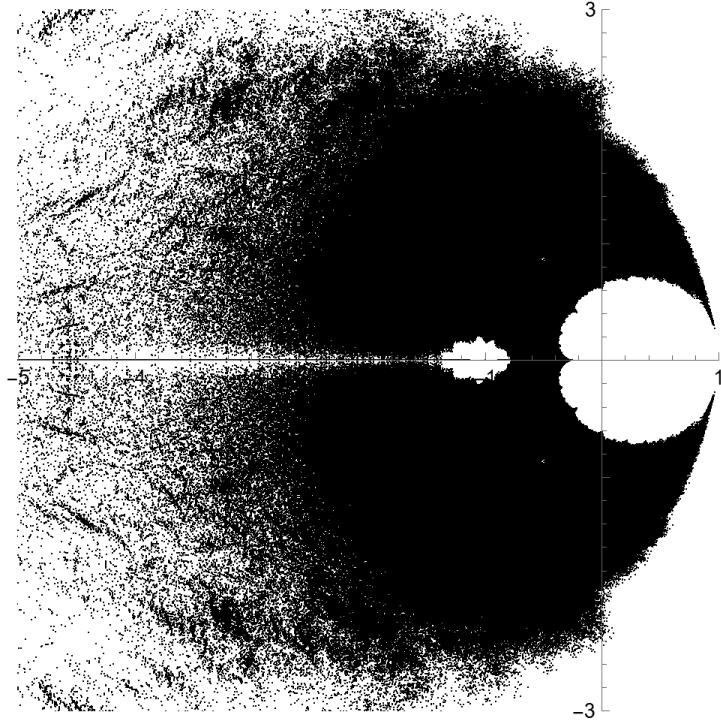


Figure 1: The zeros of $S_n(\lambda)$ in the range $\{a + bi : -4 \leq a \leq 1, |b| \leq 3\}$, where $1 \leq n < 2^{21}$ is odd.

Ulas and Ulas had verified this conjecture computationally for the first million primes p . Additionally, Schinzel in [13] proved Corollary 1 for all primes $p < 2017$ by using finite differences to bound the leading coefficient of any proper divisor of $S_n(\lambda)$. However, these previous attempts to prove the conjecture were algebraic in nature, whereas our proof depends on the analytic properties of \mathcal{S} .

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2 Stern Polynomial Preliminaries

Notation. Write \mathbb{C} as usual for the field of complex numbers, and let $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ be the extended complex plane. For all positive integers n , we introduce the (non-standard) notation

$$(\lambda)_n := 1 + \lambda + \cdots + \lambda^{n-1} = \frac{\lambda^n - 1}{\lambda - 1} \in \mathbb{Z}[\lambda].$$

For continuity purposes, we set $(1)_n = n$. For any set $S \subseteq \mathbb{C}$, define the auxiliary sets S^+ and S^- via

$$S^+ := S \cap \{\operatorname{Im} w \geq 0\} \quad \text{and} \quad S^- := S \cap \{\operatorname{Im} w \leq 0\}.$$

Furthermore, for $U \subseteq \hat{\mathbb{C}}$, set $U^{-1} := \{w^{-1} : w \in U\} \subseteq \hat{\mathbb{C}}$. Finally, for all $z \in \mathbb{C}$ and $r \geq 0$, let $B_r(z)$ be the open Euclidean ball centered at z with radius r , and let $B_r[z]$ denote the closed Euclidean ball centered at z with radius r .

We begin by recording some basic properties of the Stern polynomials. All of these can be proven by induction.

Proposition 1. Let n be a nonnegative integer.

1. $S_n(0) = 0$ when n is even, and $S_n(0) = 1$ when n is odd.
2. $S_n(1) = s_n$, where s_n is the Stern sequence (1.1).
3. $S_n(2) = n$.
4. When $n = 2^r$ for a nonnegative integer r , $S_n(\lambda) = \lambda^r$.
5. When $n = 2^r - 1$ for a nonnegative integer r , $S_n(\lambda) = (\lambda)_r$.

One of the most fundamental properties of the Stern sequence is the recursion

$$s_{2^k m + r} = s_{2^k - r} s_m + s_r s_{m+1},$$

valid for $m \geq 0$ and $2^k > r \geq 0$. The Stern polynomials satisfy an analogous recursion.

Proposition 2 ([14, Lemma 1]). For all integers $m \geq 0$ and $2^k > r \geq 0$,

$$S_{2^k m + r}(\lambda) = S_{2^k - r}(\lambda) S_m(\lambda) + S_r(\lambda) S_{m+1}(\lambda).$$

Corollary 2. For positive integers a and b ,

$$S_{2^{a+b} - 2^b + 1}(\lambda) = (\lambda)_a(\lambda)_b + \lambda^a.$$

Proof. Write $2^{a+b} - 2^b + 1 = 2^b(2^a - 1) + 1$. Apply Proposition 2 with $k = b$, $m = 2^a - 1$, and $r = 1$ to obtain

$$\begin{aligned} S_{2^{a+b} - 2^b + 1}(\lambda) &= S_{2^b - 1}(\lambda) S_{2^a - 1}(\lambda) + S_1(\lambda) S_{2^a}(\lambda) \\ &= (\lambda)_b(\lambda)_a + \lambda^a. \end{aligned} \quad \square$$

Proposition 2 may be restated in a form that highlights its underlying structure. Note that any odd integer can be expressed in the form

$$[[a_1, \dots, a_t]] := 2^{a_1+\dots+a_t} - 2^{a_2+\dots+a_t} + \dots + (-1)^{t-1} 2^{a_t} + (-1)^t, \quad (2.1)$$

where a_1, \dots, a_t are positive integers. For example, $2^{a+b} - 2^b + 1 = [[a, b]]$, so Corollary 2 takes the form

$$S_{[[a,b]]}(\lambda) = (\lambda)_a(\lambda)_b + \lambda^a. \quad (2.2)$$

The notation (2.1) satisfies the recursive properties

$$[[a_1, \dots, a_{t-1}, a_t]] = 2^{a_1+\dots+a_t} - [[a_2, \dots, a_t]] \quad (2.3)$$

$$= 2^{a_t} [[a_1, \dots, a_{t-1}]] + (-1)^t. \quad (2.4)$$

Theorem 3 (Stern Polynomial Recursion). For all sequences of positive integers a_1, \dots, a_t ,

$$S_{[[a_1, \dots, a_t]]}(\lambda) = (\lambda)_{a_1} S_{[[a_2, \dots, a_t]]}(\lambda) + \lambda^{a_1} S_{[[a_3, \dots, a_t]]}(\lambda). \quad (2.5)$$

Proof. For simplicity let $r := a_1 + \dots + a_t$. Define

$$\begin{aligned} n &= [[a_1, \dots, a_t]], \\ n' &= [[a_2, \dots, a_t]], \quad \text{and} \\ n'' &= [[a_3, \dots, a_t]]. \end{aligned}$$

By repeated application of (2.3),

$$n = 2^r - n' = 2^r - (2^{r-a_1} - n'') = 2^{r-a_1}(2^{a_1} - 1) + n''.$$

Applying Proposition 2 with $k = r - a_1$, $m = 2^{a_1} - 1$, and $p = n''$ yields

$$\begin{aligned} S_n(\lambda) &= S_{2^{r-a_1}-n''}(\lambda)S_{2^{a_1}-1}(\lambda) + S_{n''}(\lambda)S_{2^a}(\lambda) \\ &= S_{n'}(\lambda)(\lambda)_{a_1} + S_{n''}(\lambda)\lambda^{a_1}. \end{aligned} \quad \square$$

Equation (2.5) bears similarities identities satisfied by continuants of continued fractions (see [7]). This suggests the Stern polynomials have a continued fraction representation in terms of the sequence a_i , which we now state and prove.

Corollary 3 ([14, Theorem 1]). Let a_1, \dots, a_t be positive integers, where $t \geq 2$. Then

$$\frac{S_{[[a_1, \dots, a_t]]}(\lambda)}{S_{[[a_2, \dots, a_t]]}(\lambda)} = (\lambda)_{a_1} + \cfrac{\lambda^{a_1}}{(\lambda)_{a_2} + \cfrac{\lambda^{a_2}}{\dots + \cfrac{\lambda^{a_{t-1}}}{(\lambda)_{a_{t-1}} + \cfrac{\lambda^{a_{t-1}}}{(\lambda)_{a_t}}}}. \quad (2.6)$$

Proof. Proceed by induction on t . The base case $t = 2$ follows from (2.2) and the calculation

$$\frac{S_{[[a,b]]}(\lambda)}{S_{[[b]]}(\lambda)} = \frac{(\lambda)_a(\lambda)_b + \lambda^a}{(\lambda)_b} = (\lambda)_a + \frac{\lambda^a}{(\lambda)_b}.$$

For the inductive step, apply Theorem 3. \square

Corollary 3 will be the workhorse for our analysis of the roots of $S_n(\lambda)$.

3 Continued Fractions and the Parabola Theorem

Recall that, for a complex number b_0 and any finite sequences of complex numbers a_1, \dots, a_t and b_1, \dots, b_t , the (generalized) continued fraction $b_0 + K(a_n|b_n)$ is the quotient

$$b_0 + K(a_n|b_n) := b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{\cdots + \cfrac{a_t}{b_t}}}.$$

While continued fractions are most often studied in number theory, there has been significant study toward the *convergence problem* for generalized continued fractions – that is, determining sufficient conditions on $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ for which the infinite continued fraction $b_0 + K(a_n|b_n)$ converges in \mathbb{C} . In this regard, one may also consider a continued fraction as a sequence of fractional linear transformations

$$S_n = s_0 \circ s_1 \circ \cdots \circ s_n, \quad (3.1)$$

where

$$s_0(w) := b_0 + w \quad \text{and} \quad s_n(w) := \cfrac{a_n}{b_n + w}.$$

This connection between continued fractions and fractional linear transformations was first discovered by Weyl [18]. While we will not need convergence of continued fractions, we will borrow some of the techniques and theorems from this field. For a more detailed investigation, consult [9, Chapter 3] and the references therein.

One way to analyze a generalized continued fraction is to find an **element set** and a **value set** for this fraction. We present the definitions for these sets in the special case where $b_j = 1$ for all j ; later, we will see that this presents no loss of generality.

Definition 1. A set $V \subset \hat{\mathbb{C}}$ is a **value set** for $K(a_n|1)$ if

$$\cfrac{a_n}{1+V} \subseteq V \text{ for } n = 1, 2, 3, \dots$$

Definition 2. For a given value set V , the set $E \subset \hat{\mathbb{C}}$ given by

$$E := \left\{ a \in \mathbb{C} : \cfrac{a}{1+V} \subseteq V \right\}$$

is called the **element set** corresponding to the value set V .

Observe that if $a_n \in E$ for all n , and $0 \in V$, then $K(a_n|1) = S_n(0) \in V$ for all $n \geq 0$. This allows us to bound $K(a_n|1)$ in $\hat{\mathbb{C}}$ using geometric arguments that are more precise than those found through just the Triangle Inequality.

One may hope that, given an element set E for a continued fraction, we may find a value set V corresponding to E . However, in most cases, this is a nearly impossible task. It is significantly easier to proceed in the opposite direction: fix a value set V and find an element set E corresponding to V . There are many theorems in the literature which take this form. We state the most relevant result for us below.

Theorem 4 (Parabola Theorem, [9, Theorem 3.43]). For fixed $|\alpha| < \frac{\pi}{2}$, let

$$V_\alpha := -\frac{1}{2} + e^{i\alpha} \overline{\mathbb{H}} = \{w \in \mathbb{C} : \operatorname{Re}(we^{-i\alpha}) \geq -\frac{1}{2} \cos \alpha\}$$

and

$$E_\alpha := \{a \in \mathbb{C} : |a| - \operatorname{Re}(ae^{-2i\alpha}) \leq \frac{1}{2} \cos^2 \alpha\}.$$

Then E_α is the element set for continued fractions $K(a_n|1)$ corresponding to the value set V_α .

The region V_α is a half-plane whose boundary is a line intersecting the real axis at $z = -\frac{1}{2}$. In particular, because $|\alpha| < \frac{\pi}{2}$, V_α contains the half-line $[-\frac{1}{2}, \infty)$. The boundary of E_α is a parabola with focus at the origin and vertex at $-\frac{1}{4}e^{2i\alpha} \cos^2 \alpha$. This parabola intersects the real axis at $z = -\frac{1}{4}$.

Remark 1. Some sources include ∞ in the half-plane V_α . However, because $-1 \notin V_\alpha$, we know *a priori* that $S_n(w) = \frac{a_n}{1+S_{n-1}(w)}$ never equals $\infty \in \hat{\mathbb{C}}$. We choose to ignore ∞ to make the presentation cleaner.

We now tie the continued fraction theory back to Stern polynomials. Recall that Corollary 3 expresses the ratio of two Stern polynomials as a continued fraction with elements in \mathbb{C} . This form on its own is difficult to work with. We instead note (e.g. [9, Corollary 2.15]) that any continued fraction $K(a_n|b_n)$ with $b_j \neq 0$ is equivalent to the continued fraction $K(c_n|1)$, where $c_j = \frac{a_j}{b_j b_{j+1}}$. As an explicit example,

$$\frac{S_{[[2,3,5]]}(\lambda)}{S_{[[3,5]]}(\lambda)} = (\lambda)_2 + \frac{\lambda^2}{(\lambda)_3 + \frac{\lambda^3}{(\lambda)_5}} = (\lambda)_2 \left[1 + \frac{\frac{\lambda^2}{(\lambda)_2(\lambda)_3}}{1 + \frac{\lambda^3}{(\lambda)_3(\lambda)_5}} \right]. \quad (3.2)$$

Remark 2. While continued fractions of the form $K(1|d_n)$ are more common, the coefficients d_n are significantly messier than the coefficients c_n (see [9, Corollary 2.15]). We opt to use the less-common form $K(c_n|1)$ to simplify our analysis.

Equation (3.2) suggests that the ratios $\frac{z^a}{(z)_a(z)_b}$, where a and b are any positive integers, may play some importance. For this reason, we define $z_{a,b} := \frac{z^a}{(z)_a(z)_b}$ and

$$\mathcal{A}_z := \{z_{a,b} : (a, b) \in \mathbb{N}^2\}.$$

Suppose $\mathcal{A}_z \subseteq E_\alpha$ for some angle α with $|\alpha| < \frac{\pi}{2}$. Then V_α is a value set for the continued fraction (2.6) at $\lambda = z$. In particular, ∞ is not a possible value for this fraction, so $S_n(z) \neq 0$ for any z .

Throughout the rest of the paper, we use $\alpha = \frac{\pi}{12}$ in our applications of Theorem 4. In this case, Theorem 4 reduces to the following result.

Theorem 5. Let E denote the parabola

$$\begin{aligned} E := E_{\pi/12} &= \left\{ a \in \mathbb{C} : |a| - \operatorname{Re}(ae^{-\pi i/6}) \leq \frac{2 + \sqrt{3}}{8} \right\} \\ &= \left\{ x + yi \in \mathbb{C} : \sqrt{x^2 + y^2} \leq \frac{\sqrt{3}}{2}x + \frac{1}{2}y + \frac{2 + \sqrt{3}}{8} \right\}. \end{aligned}$$

Suppose $z \in \mathbb{C}$ is a complex number such that $z_{a,b} \in E$ for all $(a,b) \in \mathbb{N}^2$. Then z is not a root of any Stern polynomial.

The remainder of this paper is dedicated to showing $z_{a,b} \in E$ whenever z lies in the set

$$\mathcal{B} := B_1[2] = \{|w - 2| \leq 1\},$$

thus proving Theorem 5. See Figure 2 for an example.

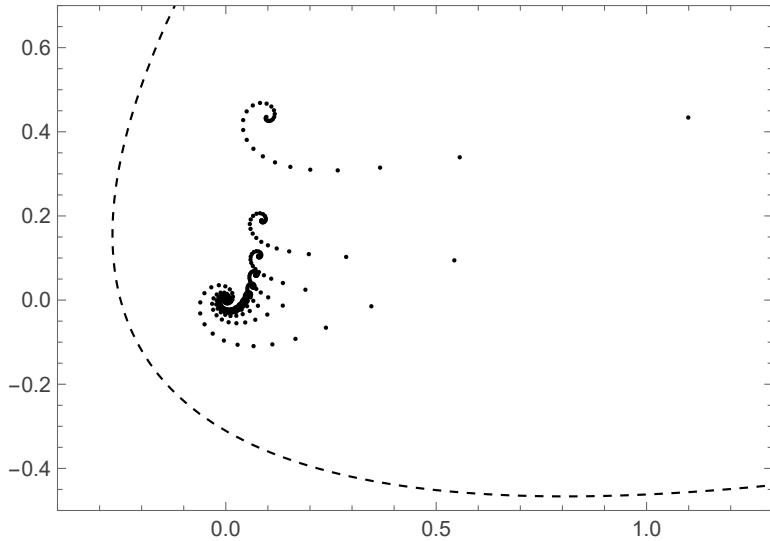


Figure 2: The set \mathcal{A}_z , where $z = (2 - \cos \frac{\pi}{7}) + i \sin \frac{\pi}{7} \in \partial \mathcal{B}$, enclosed in $E_{\pi/12}$ (dashed).

Let us see how Theorem 5 proves Theorem 1.

Proof (Theorem 1). Suppose P is a factor of some Stern polynomial with $\deg P = d$, and factor it over \mathbb{C} :

$$P(\lambda) = 1 + c_1\lambda + \cdots + c_d\lambda^d = c_d \prod_{k=1}^d (\lambda - \mu_k).$$

Then $P(2) = c_d \prod_{k=1}^d (2 - \mu_k)$. Because $c_d \geq 1$ and $|2 - \mu_k| > 1$ by Theorem 5, it follows that $P(2) > 1$. Thus, if $S_p(\lambda) = P(\lambda)Q(\lambda)$ for non-constant P and Q in $\mathbb{Z}[\lambda]$, then

$$p = S_p(2) = P(2)Q(2).$$

This implies $P(2) = 1$ or $Q(2) = 1$, which is impossible. \square

4 A Potpourri of Complex Inequalities

This chapter is the main content of the paper. We prove several inequalities in \mathbb{C} needed for the main theorem. Some results tie directly into the Stern polynomials $S_n(\lambda)$, while others are independent results which seem interesting in their own rights.

4.1 Preliminaries

Here we list some inequalities which are not directly related to complex numbers but which will be used several times in the following three sections.

Our first inequality is a lower bound on $\sin x$. While it is not as tight as the Taylor inequality $\sin x \geq x - \frac{x^3}{6}$ for small x , it exhibits a second equality case at $x = \pi$.

Proposition 3. For all $x \in \mathbb{R}$,

$$\sin x \geq x - \frac{x^2}{\pi}. \quad (4.1)$$

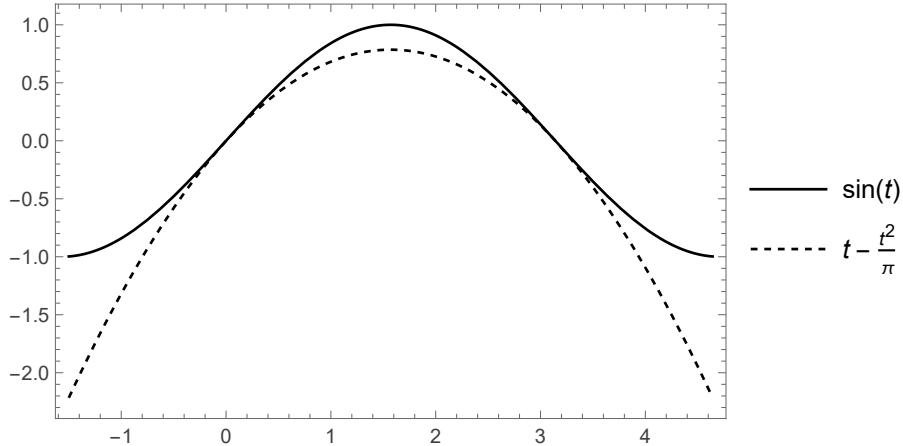


Figure 3: A plot of the inequality in Proposition 3.

Proof. There are two cases to consider.

- First suppose $x \in [0, \pi]$. Because both $\sin x$ and $x - \frac{x^2}{\pi}$ are symmetric about the axis $x = \frac{\pi}{2}$, it suffices to prove the inequality for $x \in [0, \frac{\pi}{2}]$. Within this smaller interval, cosine is concave down with $\cos 0 = 1$ and $\cos \frac{\pi}{2} = 0$, so

$$\cos x \geq 1 - \frac{2x}{\pi}. \quad (4.2)$$

Integrating both sides of (4.2) yields (4.1) in this case.

- Now suppose $x \notin [0, \pi]$. By symmetry, it suffices to prove the inequality for $x \geq \pi$. In this interval, $1 - \frac{2x}{\pi} \leq -1 \leq \cos x$. It follows that (4.2), and thus (4.1), holds as well.

□

We also need a variation of Bernoulli's Inequality which lower bounds $(1+x)^r$ by its second-order Taylor expansion. If $r \geq 2$ is a positive integer, then certainly

$$(1+x)^r = \sum_{j=0}^r \binom{r}{j} x^j \geq 1 + rx + \binom{r}{2} x^2 \quad \text{whenever } x \geq 0. \quad (4.3)$$

However, this argument does not work for general $r \in \mathbb{R}$, because the binomial coefficients $\binom{r}{j}$ may be negative. Nevertheless, (4.3) still holds if r is real. We present a proof using Taylor expansions. For a generalization of this inequality, consult [6].

Proposition 4. For all $x \geq 0$ and real $r \geq 2$,

$$(1+x)^r \geq 1 + rx + \binom{r}{2} x^2.$$

Proof. Let $f(t) := (1+t)^r$. Observe that $f'''(t) = r(r-1)(r-2)(1+t)^{r-3}$ is nonnegative on $(0, \infty)$. It follows from Taylor's Theorem that

$$(1+x)^r = 1 + rx + \binom{r}{2} x^2 + \int_0^x \frac{f'''(t)}{6} (x-t)^2 dt \geq 1 + rx + \binom{r}{2} x^2. \quad \square$$

4.2 Bounding Geometric Series on $\{\operatorname{Re} w \geq 1\}$

The main technical difficulty in our analysis of the roots of $S_n(\lambda)$ comes from estimating $(z)_n$. This is significantly more difficult than it seems, especially for complex numbers z of the form $1+it$ where t is small. For example, it is tempting to write

$$|(z)_n| = \left| \frac{z^n - 1}{z - 1} \right| \geq \frac{|z|^n - 1}{|z - 1|} = \frac{(1+t^2)^{n/2} - 1}{t}. \quad (4.4)$$

However, the right hand side of (4.4) is $\frac{n}{2}t + \mathcal{O}(t^2)$ as $t \rightarrow 0$, whereas $|(z)_n|$ tends to n as $z \rightarrow 1$.

There seem to be few attempts to strengthen (4.4) in the literature. The most relevant result we could find is Problem 4795 in the American Mathematical Monthly, reproduced below. A proof appears in [11].

Theorem 6. For all $z \in \mathbb{C}$ with $\operatorname{Re} z \geq 1$, and for all $a \geq 1$,

$$|(z)_a| = |1 + z + \cdots + z^{a-1}| \geq |z|^{a-1}.$$

This estimate will be useful later in this paper, but it still is not tight for $z \approx 1$. We now focus on finding a lower bound for $|(z)_n|$ that is tight near 1.

We first show that it suffices to lower bound $|(z)_n|$ on the line $\operatorname{Re} w = 1$.

Lemma 1. Let $n \geq 1$ be an integer, and suppose x, x_0 , and y are real numbers with $x \geq x_0 \geq 1$. Then

$$|(x + yi)_n| \geq |(x_0 + yi)_n|.$$

Proof. Both sides equal 1 when $n = 1$, so assume $n \geq 2$. Let $z = x + yi$ and $z_0 := x_0 + yi$ for ease of readability. Observe that

$$|(z)_n| = |1 + z + z^2 + \cdots + z^{n-1}| = \prod_{j=1}^{n-1} |z - e^{2\pi ij/n}|$$

and

$$|(z_0)_n| = |1 + z_0 + z_0^2 + \cdots + z_0^{n-1}| = \prod_{j=1}^{n-1} |z_0 - e^{2\pi ij/n}|.$$

For each j , because $\operatorname{Re}(e^{2\pi ij/n}) \leq 1$ and $x \geq x_0$, we have $|z - e^{2\pi ij/n}| \geq |z_0 - e^{2\pi ij/n}|$. Multiplying all $n - 1$ such inequalities together yields

$$|1 + z + z^2 + \cdots + z^{n-1}| \geq |1 + z_0 + z_0^2 + \cdots + z_0^{n-1}|. \quad \square$$

We handle small values of n numerically. Observe that $(1 + it)_n$ is a polynomial in t , so

$$\begin{aligned} |(1 + it)_n|^2 &= (1 + it)_n(1 - it)_n = \frac{(1 + it)^n - 1}{it} \cdot \frac{(1 - it)^n - 1}{-it} \\ &= \frac{(1 + t^2)^n - 2 \operatorname{Re}(1 + it)^n + 1}{t^2} \end{aligned}$$

can be written in the form $P_n(t^2)$ for some polynomial P_n . This means we may manually compute P_n for small n and use standard calculus techniques to numerically bound their minima. These computations are shown in Table 2 for $1 \leq n \leq 14$.

Unfortunately, these explicit polynomials are hard to analyze for large n . (In particular, observe that the sequence of minima is not monotonic, instead peaking at $n = 9$.) This means we need a fundamentally different approach to bound P_n in general.

The key estimate for us is the following inequality.

Theorem 7. For all real numbers t and all positive integers n ,

$$\left| \left(1 + \frac{it}{n}\right)^n - 1 \right| \geq |e^{it} - 1| = 2 \sin \frac{t}{2}. \quad (4.5)$$

Carlo Beenakker conjectured Theorem 7 in response to a question by the author on MathOverflow [16], and Terence Tao later confirmed this conjecture in the same thread. To make this paper self-contained, and to record the proof of this inequality in print, we replicate Tao's argument below with some modifications in exposition.

Proof (Tao, [16]). The proof proceeds in four steps. The first step rewrites (4.5) into polar coordinate form to obtain the equivalent inequality (4.8). The next three steps prove (4.8) by performing casework on different ranges of t .

n	$P_n(x)$	$\sqrt{\min_{x \geq 0} P_n(x)}$
1	1	1
2	$x + 4$	2
3	$x^2 + 3x + 9$	3
4	$x^3 + 4x^2 + 4x + 16$	4
5	$x^4 + 5x^3 + 10x^2 + 25$	5
6	$x^5 + 6x^4 + 15x^3 + 22x^2 - 15x + 36$	5.8206
7	$x^6 + 7x^5 + 21x^4 + 35x^3 + 49x^2 - 49x + 49$	6.3003
8	$x^7 + 8x^6 + 28x^5 + 56x^4 + 68x^3 + 112x^2 - 112x + 64$	6.5136
9	$x^8 + 9x^7 + 36x^6 + 84x^5 + 126x^4 + 108x^3 + 252x^2 - 216x + 81$	6.5474
10	$x^9 + 10x^8 + 45x^7 + 120x^6 + 210x^5 + 254x^4 + 120x^3 + 540x^2 - 375x + 100$	6.4727
11	$x^{10} + 11x^9 + 55x^8 + 165x^7 + 330x^6 + 462x^5 + 484x^4 + 1089x^2 - 605x + 121$	6.3388
12	$x^{11} + 12x^{10} + 66x^9 + 220x^8 + 495x^7 + 792x^6 + 922x^5 + 924x^4 - 495x^3 + 2068x^2 - 924x + 144$	6.1771
13	$x^{12} + 13x^{11} + 78x^{10} + 286x^9 + 715x^8 + 1287x^7 + 1716x^6 + 1690x^5 + 1859x^4 - 1859x^3 + 3718x^2 - 1352x + 169$	6.0096
14	$x^{13} + 14x^{12} + 91x^{11} + 364x^{10} + 1001x^9 + 2002x^8 + 3003x^7 + 3434x^6 + 2821x^5 + 4004x^4 - 5005x^3 + 6370x^2 - 1911x + 196$	5.8386

Table 2: Polynomials $P_n(x)$ as well as their minima on $[0, \infty)$.

Step 1: Setup. For $n = 1$, observe that

$$|e^{it} - 1| = 2 \left| \sin \frac{t}{2} \right| \leq 2 \cdot \frac{|t|}{2} = |t|. \quad (4.6)$$

For $n = 2$, Theorem 7 holds by the calculation

$$\left| \left(1 + \frac{it}{2}\right)^2 - 1 \right| = \sqrt{t^2 + \left(\frac{t^2}{4}\right)^2} \geq |t| \geq |e^{it} - 1|,$$

where the last inequality is due to (4.6). (This shows that, interestingly, the left hand side of (4.5) is *not* always decreasing in n , despite converging to $|e^{it} - 1|$ in the limit as $n \rightarrow \infty$.) In what follows, assume $n \geq 3$.

Without loss of generality assume $t \geq 0$. We may take advantage of the estimate $(1 + \frac{it}{n})^n \approx e^{it}$ by writing $(1 + \frac{it}{n})^n = r e^{i(t-\varepsilon)}$, where

$$r := \left(1 + \frac{t^2}{n^2}\right)^{n/2} \quad \text{and} \quad \varepsilon := t - n \arctan \frac{t}{n}.$$

We expect that ε should be relatively small. Indeed, upon integrating the inequalities $1 - x^2 \leq \frac{1}{1+x^2} \leq 1$ (true for all x) we have $x - \frac{x^3}{3} \leq \arctan x \leq x$ for all $x \geq 0$. It follows that

$$0 \leq \varepsilon \leq \frac{t^3}{3n^2}. \quad (4.7)$$

To finish the setup, observe that squaring both sides of (4.5) yields the equivalent inequality

$$r^2 - 2r \cos(t - \varepsilon) + 1 \geq 2 - 2 \cos t.$$

This rearranges to

$$(r - 1)^2 + 2(r - 1)(1 - \cos(t - \varepsilon)) \geq 2(\cos(t - \varepsilon) - \cos t). \quad (4.8)$$

The rest of the proof will establish (4.8) by splitting into cases based on the value of t .

Step 2: $t > \frac{8}{3}$. In this region, $r - 1$ is large enough that we expect the first term $(r - 1)^2$ to dominate on its own. By Bernoulli's Inequality, we make the estimate

$$r - 1 = \left(1 + \frac{t^2}{n^2}\right)^{n/2} - 1 \geq \frac{n}{2} \cdot \frac{t^2}{n^2} = \frac{t^2}{2n}.$$

Furthermore, applying the Mean Value Theorem to $x \mapsto \cos x$ yields

$$\cos(t - \varepsilon) - \cos t \leq \varepsilon \leq \frac{t^3}{3n^2}.$$

It follows that $(r - 1)^2 \geq 2(\cos(t - \varepsilon) - \cos t)$ when

$$\left(\frac{t^2}{2n}\right)^2 \geq 2 \cdot \frac{t^3}{3n^2},$$

or, equivalently, when $t > \frac{8}{3}$.

Step 3: $\frac{\pi}{2} < t \leq \frac{8}{3}$. Within this range, we instead show that the second term, namely $2(r - 1)(1 - \cos(t - \varepsilon))$, dominates. Here, $\cos t \leq 0$ and

$$1 - \cos(t - \varepsilon) \geq 1 - \varepsilon \geq 1 - \frac{t^3}{3n^2}.$$

It follows that (4.8) will be satisfied (just using the second term) if

$$2 \cdot \frac{n}{2} \cdot \frac{t^2}{n^2} \left(1 - \frac{t^3}{3n^2}\right) \geq 2 \frac{t^3}{3n^2},$$

which simplifies to

$$\frac{2}{3}t + \frac{t^3}{3n^2} \leq n.$$

It remains to check this inequality for $n \geq 3$ and $t \leq \frac{8}{3}$. This follows from the computation

$$\frac{2}{3}t + \frac{t^3}{3n^2} \leq \frac{2}{3} \cdot \frac{8}{3} + \frac{(8/3)^3}{3 \cdot 3^2} = \frac{1808}{729} < 3.$$

Step 4: $0 \leq t \leq \frac{\pi}{2}$. We finish with the most delicate case of the proof. Rewrite (4.8) slightly as

$$(r-1)^2 + 2(r-1)(1-\cos t) \geq 2r(\cos(t-\varepsilon) - \cos t). \quad (4.9)$$

We do this to take advantage of concavity of cosine in this region. Indeed, because t and $t-\varepsilon = n \arctan \frac{t}{n}$ are both in the interval $[0, \frac{\pi}{2}]$,

$$\cos(t-\varepsilon) - \cos t \leq \varepsilon \sin t.$$

This yields a crucial extra power of t in the limit as $t \rightarrow 0^+$.

Now observe that

$$\frac{1-\cos t}{\sin t} = \tan \frac{t}{2} \geq \frac{t}{2}.$$

Thus, after dividing through by $\sin t$, it suffices to establish the bound

$$2(r-1)\frac{t}{2} \geq 2r\varepsilon$$

(again taking only the second term in the left hand side of (4.9)). From our previous inequalities it would suffice to show that

$$2 \cdot \frac{n}{2} \cdot \frac{t^2}{n^2} \cdot \frac{t}{2} \geq 2r \frac{t^3}{3n^2},$$

or $r \leq \frac{3}{4}n$. Magically, all powers of t have canceled out.

To finish this case, we make the bound

$$r \leq \left(1 + \frac{(\pi/2)^2}{n^2}\right)^{n/2} \leq \exp\left(\frac{(\pi/2)^2 n}{n^2} \frac{n}{2}\right) = \exp\left(\frac{\pi^2}{8n}\right).$$

It remains to show $\exp(\frac{\pi^2}{8n}) \leq \frac{3}{4}n$ for all $n \geq 3$. Indeed, the left-hand side is decreasing in n while the right-hand side is increasing, and at $n = 3$ the inequality becomes

$$\exp\left(\frac{\pi^2}{24}\right) \leq \frac{9}{4}.$$

A quick calculation reveals

$$\exp\left(\frac{\pi^2}{24}\right) < \exp\left(\frac{1}{2}\right) = \sqrt{e} < 2 < \frac{9}{4},$$

so $\exp(\frac{\pi^2}{8n}) \leq \frac{3}{4}n$ holds for all $n \geq 3$. □

4.3 Bounding Geometric Series on \mathcal{B}

Recall that \mathcal{B} is the disk $\{|w-2| \leq 1\} \subseteq \mathbb{C}$. While Theorem 7 is strong in its own right, it is not quite sufficient to show Theorem 1. We opt to establish a separate lower bound for $|(w)_n|$ when $w \in \mathcal{B}$. This bound is not the strongest possible (see the remark before Proposition 8), but it is sufficient for our purposes.

Theorem 8. Let n be a positive integer, and suppose $z \in \mathcal{B}$. Then $|(z)_n| \geq \min(n, \frac{11}{2})$.

For all $n \in \mathbb{N}$ and $t \in (0, n\pi]$, let

$$w_{n,t} := \overline{2 - e^{it/n}} = \left(2 - \cos \frac{t}{n}\right) + i \sin \frac{t}{n}.$$

Just as in the proof of Theorem 7, we aim to take advantage of the approximation $w_{n,t} \approx e^{it}$ for large n . The next few propositions will establish several lower bounds for $|(w_{n,t})_n|$.

Proposition 5. For integer $n \geq 1$ and real $t \geq 0$, we have

$$\left| \frac{w_{n,t}^n - 1}{w_{n,t} - 1} \right| \geq n \left(1 - \frac{t}{2\pi}\right).$$

Proof. By Lemma 1, because $\operatorname{Im} w_{n,t} = \sin \frac{t}{n}$ and $\operatorname{Re} w_{n,t} \geq 1$, we know

$$\left| \frac{w_{n,t}^n - 1}{w_{n,t} - 1} \right| \geq \left| \frac{(1 + i \sin \frac{t}{n})^n - 1}{(1 + i \sin \frac{t}{n}) - 1} \right| = \frac{|(1 + i \sin \frac{t}{n})^n - 1|}{\sin \frac{t}{n}}.$$

Now write $\sin \frac{t}{n} = \frac{1}{n} \cdot n \sin \frac{t}{n}$ and apply Lemma 7 to obtain

$$\begin{aligned} \left| \left(1 + i \sin \frac{t}{n}\right)^n - 1 \right| &= \left| \left(1 + \frac{i}{n} \cdot n \sin \frac{t}{n}\right)^n - 1 \right| \\ &\geq \left| e^{in \sin \frac{t}{n}} - 1 \right| = 2 \sin \left(\frac{n}{2} \sin \frac{t}{n} \right). \end{aligned}$$

Finally, using the inequalities $\frac{\sin x}{x} \geq 1 - \frac{x}{\pi}$ and $\sin x \leq x$, valid for $x > 0$, we obtain

$$\begin{aligned} \frac{|(1 + i \sin \frac{t}{n})^n - 1|}{\sin \frac{t}{n}} &\geq \frac{2 \sin(\frac{n}{2} \sin \frac{t}{n})}{\sin \frac{t}{n}} = n \cdot \frac{\sin(\frac{n}{2} \sin \frac{t}{n})}{\frac{n}{2} \sin \frac{t}{n}} \\ &\geq n \left(1 - \frac{\frac{n}{2} \sin \frac{t}{n}}{\pi}\right) \geq n \left(1 - \frac{t}{2\pi}\right). \end{aligned} \quad \square$$

The second bound concerns values of t away from 0.

Proposition 6. For all $n \geq 4$ and $t \geq 0$, we have

$$\left| \frac{w_{n,t}^n - 1}{w_{n,t} - 1} \right| \geq 2n \sin \frac{t}{2n}.$$

Proof. Because $w_{n,t} = (2 - \cos \frac{t}{n}) + i \sin \frac{t}{n}$, compute

$$\begin{aligned} |w_n|^2 &= \left(2 - \cos \frac{t}{n}\right)^2 + \sin^2 \frac{t}{n} \\ &= 5 - 4 \cos \frac{t}{n} = 1 + 8 \sin^2 \frac{t}{2n}. \end{aligned}$$

It follows by the Triangle Inequality and $\sin x \geq x - \frac{x^3}{6}$ that

$$\left| \frac{w_{n,t}^n - 1}{w_{n,t} - 1} \right| \geq \frac{|w_{n,t}|^n - 1}{|w_{n,t} - 1|} = \frac{(1 + 8 \sin^2 \frac{t}{2n})^{n/2} - 1}{2 \sin \frac{t}{2n}}.$$

Finally, Bernoulli's Inequality yields

$$\frac{(1 + 8 \sin^2 \frac{t}{2n})^{n/2} - 1}{2 \sin \frac{t}{2n}} \geq \frac{(1 + 4n \sin^2 \frac{t}{2n}) - 1}{2 \sin \frac{t}{2n}} = 2n \sin \frac{t}{2n}. \quad \square$$

Our third inequality is a slight tightening of Proposition 6 as $t \rightarrow 0$.

Proposition 7. For all n and $t \in [0, 2\pi]$, we have

$$\left| \frac{w_{n,t}^n - 1}{w_{n,t} - 1} \right| \geq t + t^3 \left(\frac{1}{2n} - \frac{25}{24n^2} \right) - t^5 \left(\frac{1}{8n^3} - \frac{1}{16n^4} \right). \quad (4.10)$$

The right hand side of (4.10) is the fifth-order Taylor approximation of the left hand side as a function of t . The simpler inequality $|w_{n,t}^n - 1| \geq t|w_{n,t} - 1|$ is easier to prove, but the resulting bounds are not strong enough to prove Theorem 8 without quadrupling the number of base cases.

Proof. As before, derive the inequality

$$\left| \frac{w_{n,t}^n - 1}{w_{n,t} - 1} \right| \geq \frac{(1 + 8 \sin^2 \frac{t}{2n})^{n/2} - 1}{2 \sin \frac{t}{2n}}.$$

This time, we use Proposition 4 in conjunction with $\sin x \geq x - \frac{x^3}{6}$ to obtain

$$\begin{aligned} \frac{(1 + 8 \sin^2 \frac{t}{2n})^{n/2} - 1}{2 \sin \frac{t}{2n}} &\geq \frac{(1 + 4n \sin^2 \frac{t}{2n} + 8n(n-2) \sin^4 \frac{t}{2n}) - 1}{2 \sin \frac{t}{2n}} \\ &= 2n \sin \frac{t}{2n} + 4n(n-2) \sin^3 \frac{t}{2n} \\ &\geq 2n \left(\frac{t}{2n} - \frac{t^3}{48n^3} \right) + 4n(n-2) \left(\frac{t}{2n} - \frac{t^3}{48n^3} \right)^3 \\ &= t - \frac{t^3}{24n^2} + \frac{(n-2)t^3}{2n^2} \left(1 - \frac{t^2}{24n^2} \right)^3. \end{aligned}$$

Finally, Bernoulli's Inequality allows us to bound the nonlinear terms:

$$\begin{aligned} t - \frac{t^3}{24n^2} + \frac{(n-2)t^3}{2n^2} \left(1 - \frac{t^2}{24n^2} \right)^3 &\geq t - \frac{t^3}{24n^2} + \frac{(n-2)t^3}{2n^2} \left(1 - \frac{t^2}{8n^2} \right) \\ &= t + t^3 \left(\frac{1}{2n} - \frac{25}{24n^2} \right) - t^5 \left(\frac{1}{8n^3} - \frac{1}{16n^4} \right). \quad \square \end{aligned}$$

We are now able to prove Theorem 8.

Proof. Table 2 shows Theorem 8 for $n \leq 14$, so assume $n \geq 15$.

First suppose $t \geq 2\pi$. Here, use Proposition 6 to write

$$|(w_{n,t})_n| \geq 2n \sin \frac{t}{2n} \geq 2n \sin \frac{\pi}{n}.$$

The right hand side is increasing as a function of n and evaluates to $12 \sin \frac{\pi}{6} = 6$ for $n = 6$. It follows that $|(w_{n,t})_n| \geq 6$ for $n \geq 6$, ergo for $n \geq 15$.

We may now assume $t \in [0, 2\pi]$. By Propositions 5 and 6, we know

$$|(w_{n,t})_n| \geq \max \left\{ n \left(1 - \frac{t}{2\pi} \right), t + t^3 \left(\frac{1}{2n} - \frac{25}{24n^2} \right) - t^5 \left(\frac{1}{8n^3} - \frac{1}{16n^4} \right) \right\}. \quad (4.11)$$

To make the analysis simpler, we bound the t^5 term in (4.11). Because $t \in [0, 2\pi]$,

$$\begin{aligned} t^5 \left(\frac{1}{8n^3} - \frac{1}{16n^4} \right) &= \frac{t^3}{n^2} \cdot t^2 \left(\frac{1}{8n} - \frac{1}{16n^2} \right) \\ &\leq \frac{t^3}{n^2} \cdot (2\pi)^2 \left(\frac{1}{8 \cdot 15} - \frac{1}{16 \cdot 15^2} \right) = \frac{29\pi^2}{900} \frac{t^3}{n^2} < \frac{t^3}{3n^2}. \end{aligned}$$

Combining this with Propositions 5 and 6, it follows that

$$\begin{aligned} |(w_{n,t})_n| &\geq \max \left\{ n \left(1 - \frac{t}{2\pi} \right), t + t^3 \left(\frac{1}{2n} - \frac{25}{24n^2} \right) - \frac{t^3}{3n^2} \right\} \\ &= \max \left\{ n \left(1 - \frac{t}{2\pi} \right), t + t^3 \left(\frac{1}{2n} - \frac{11}{8n^2} \right) \right\}. \end{aligned} \quad (4.12)$$

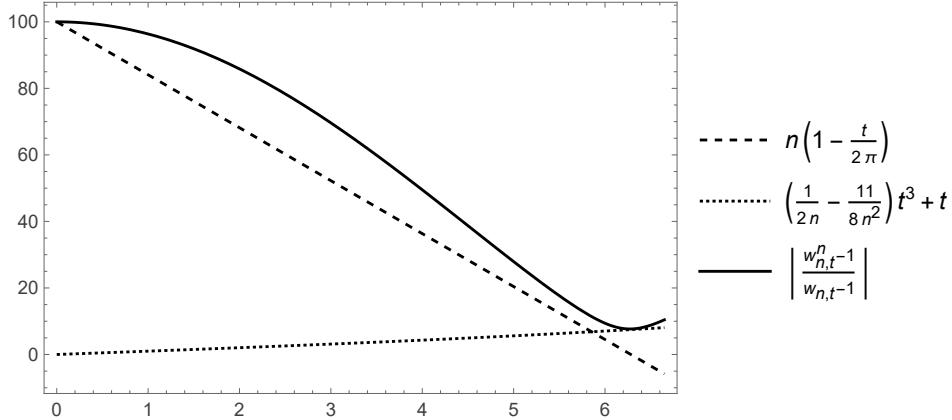


Figure 4: Plot comparing $|(w_{n,t})_n|$ to the two expressions in (4.12) when $n = 100$. Note the near-equality case around $t = 2\pi$.

Now let

$$M := 2\pi \left(1 - \frac{5.5}{n} \right) = \pi \left(2 - \frac{11}{n} \right).$$

There are two cases to consider. First, suppose $t \leq M$. Then

$$n \left(1 - \frac{t}{2\pi}\right) \geq n \left(1 - \frac{M}{2\pi}\right) = \frac{11}{2}.$$

Now suppose $t \geq M$. Because $n \geq 15$, the coefficient $\frac{1}{2n} - \frac{11}{8n^2}$ is positive, so

$$t + t^3 \left(\frac{1}{2n} - \frac{11}{8n^2}\right) \geq \pi \left(2 - \frac{11}{n}\right) + \pi^3 \left(2 - \frac{11}{n}\right)^3 \left(\frac{1}{2n} - \frac{11}{8n^2}\right). \quad (4.13)$$

To estimate (4.13), let $x = \frac{1}{n}$, so it suffices to analyze the polynomial

$$P(x) := \pi(2 - 11x) + \pi^3(2 - 11x)^3 \left(\frac{1}{2}x - \frac{11}{8}x^2\right).$$

The polynomial P has exactly one critical point in the interval $[0, \frac{1}{15}]$, occurring at $x = x_0 \approx 0.024$. In particular, P is increasing on $[0, x_0]$ and decreasing on $[x_0, \frac{1}{15}]$. Compute

$$P(0) = 2\pi \quad \text{and} \quad P\left(\frac{1}{15}\right) = \frac{19\pi}{15} + \frac{336091\pi^3}{6075000} \approx 5.695 > \frac{11}{2}.$$

It follows that $P(x) > \frac{11}{2}$ for all $x \in [0, \frac{1}{15}]$, and thus $|w_{n,t}^n - 1| \geq M|w_{n,t} - 1|$ for all $t \in [0, 2\pi]$ and $n \geq 15$. \square

Remark 3.

1. In the above proof, the maximum value of $P(x)$ in the interval $[0, \frac{1}{15}]$ is $P(x_0) \approx 7.2723$.
2. We conjecture the stronger bound $|(z)_n| \geq \min(n, 2\pi)$. The methods above cannot extend to this tougher inequality without extensive computer assistance.

Finally, we will need two more special cases. Our previous estimates are not *quite* strong enough to obtain these inequalities, so we need to prove them separately.

Proposition 8. Suppose $z \in \mathcal{B}$. Then

$$|(z)_5| = |1 + z + z^2 + z^3 + z^4| \geq 5|z|, \quad (4.14)$$

$$|(z)_6| = |1 + z + z^2 + z^3 + z^4 + z^5| \geq 6|z|. \quad (4.15)$$

Proof. We first prove (4.14). Observe that the function $f(z) = \frac{z}{(z)_5}$ is analytic in \mathcal{B} , so by the Maximum Modulus Principle we may assume $z \in \partial\mathcal{B}$, i.e. $|z - 2| = 1$. Write $z = re^{i\theta}$. Rewrite the given inequality to $|z^5 - 1| \geq 5|z||z - 1|$ and square both sides; this yields the equivalent inequality

$$r^{10} - 2r^5 \cos(5\theta) + 1 \geq 25r^2(r^2 - 2r \cos \theta + 1). \quad (4.16)$$

Observe that $|z - 2| = 1$ implies $r^2 - 4r \cos \theta + 4 = 1$, or $\cos \theta = \frac{r^2+3}{4r}$. Under this substitution, the right hand side of (4.16) becomes

$$25r^2 \left[r^2 - 2r \cdot \frac{r^2+3}{4r} + 1 \right] = \frac{25}{2}(r^4 - r^2).$$

Analogously, the left hand side of (4.16) becomes

$$\begin{aligned}
r^{10} - 2r^5 \cos(5\theta) + 1 &= r^{10} - 2r^5 (16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta) + 1 \\
&= r^{10} - 2r^5 \left[16 \left(\frac{r^2+3}{4r} \right)^5 - 20 \left(\frac{r^2+3}{4r} \right)^3 + 5 \left(\frac{r^2+3}{4r} \right) \right] + 1 \\
&= r^{10} - 32 \left(\frac{r^2+3}{4} \right)^5 - 20r^2 \left(\frac{r^2+3}{4} \right)^3 + 5r^4 \left(\frac{r^2+3}{4} \right) + 1 \\
&= \frac{1}{32} (31r^{10} + 5r^8 + 10r^6 + 30r^4 + 135r^2 - 211).
\end{aligned}$$

It follows that (4.16) is equivalent to the inequality

$$\frac{1}{32} (31r^{10} + 5r^8 + 10r^6 + 30r^4 + 135r^2 - 211) \geq \frac{25}{2} (r^4 - r^2). \quad (4.17)$$

But (4.17) follows from the unexpected factorization

$$\text{LHS} - \text{RHS} = \frac{(r^2 - 1)^3 (31r^4 + 98r^2 + 211)}{32} \geq 0.$$

This proves (4.14).

The proof of (4.15) is similar, so we only sketch the details. By the Maximum Modulus Principle we may assume $|z - 2| = 1$. Squaring both sides yields the equivalent inequality

$$r^{12} - 2r^6 \cos(6\theta) + 1 \geq 36r^2(r^2 - 2r \cos \theta + 1). \quad (4.18)$$

Substituting $\cos \theta = \frac{r^2+3}{4r}$ and expanding yields the inequality

$$\frac{1}{64} (63r^{12} + 6r^{10} + 9r^8 + 20r^6 + 81r^4 + 486r^2 - 665) \geq 18(r^4 - r^2).$$

Here, we have the factorization

$$\text{LHS} - \text{RHS} = \frac{(r^2 - 1)^2 (63r^8 + 132r^6 + 210r^4 + 308r^2 - 665)}{64}. \quad (4.19)$$

The polynomial $g(x) := 63x^8 + 132x^6 + 210x^4 + 308x^2 - 665$ is increasing on $[0, \infty)$ and $g(1) = 48$. Because $r \in [1, 3]$, we know $g(r) \geq 0$, and thus the right hand side of (4.19) is always nonnegative. \square

Remark 4. One might suspect that $|(z)_n| \geq n|z|$ more generally whenever $z \in \mathcal{B}$. However, numerical evidence suggests that the value $\min_{z \in \mathcal{B}} \left| \frac{(z)_n}{z} \right|$ approaches 2π as $n \rightarrow \infty$. It seems that this inequality is true *only* for $n = 5, 6$, and 7 , but we do not have a proof.

4.4 Set Inclusion Inequalities

This section focuses on showing that certain sets in \mathbb{C} are subsets of $E = E_{\pi/12}$, where we use the definition of E_α given in Theorem 4.

We first establish two preliminary facts about elements of \mathcal{B} . Because nonreal zeros of $S_n(\lambda)$ come in conjugate pairs, it suffices to examine $z \in \mathcal{B}^+$.

Proposition 9. Let $z \in \mathcal{B}^+$. Then $0 \leq \arg z \leq \frac{\pi}{6}$ and

$$|\operatorname{Re} z^{-4}| \leq |\operatorname{Re} z^{-2}|. \quad (4.20)$$

Proof. The lower bound $0 \leq \arg z$ is clear because z lies in the first quadrant. For the upper bound, observe that the line $y = \frac{1}{\sqrt{3}}x$ is tangent to the circle $(x - 2)^2 + y^2 = 1$, so $\arg z \leq \arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6}$.

To prove (4.20), we claim that

$$|\cos 4\theta| \leq |\cos 2\theta| \quad \text{whenever } \theta \in [0, \frac{\pi}{6}]. \quad (4.21)$$

To prove this, note that the function $h(x) = |2x^2 - 1|$ is decreasing for $x \in [\frac{1}{2}, \frac{\sqrt{2}}{2}]$ and increasing for $x \in [\frac{\sqrt{2}}{2}, 1]$. Furthermore, $h(\frac{1}{2}) = \frac{1}{2}$ and $h(1) = 1$. It follows that

$$|2x^2 - 1| \leq |x| \quad \text{for all } x \in [\frac{1}{2}, 1],$$

which is equivalent to (4.21) under the substitution $x = \cos 2\theta$.

This implies (4.20), as

$$|\operatorname{Re} z^{-4}| = |z|^{-4} |\cos 4\theta| \leq |z|^{-2} |\cos 2\theta| = |\operatorname{Re} z^{-2}|. \quad \square$$

We now turn to the main results of this section.

Proposition 10. The following sets in \mathbb{C} are subsets of E :

- | | |
|--|---|
| (a) $\mathcal{E}_a := B_{0.23}(0),$
(c) $\mathcal{E}_c := B_{1/5}(0) + B_1(2)^+,$
(e) $\mathcal{E}_e := B_{0.35}(0.25).$ | (b) $\mathcal{E}_b := B_{7/4}(\frac{3}{2})^+,$
(d) $\mathcal{E}_d := \{\operatorname{Re} w \geq 1.15\}^{-1} = B_{0.575}(0.575),$ |
|--|---|

See Figure 5 for plots of \mathcal{E}_a through \mathcal{E}_e . The upshot of this proposition is that it will allow us to show that a complex number w is in E by instead showing it lies in one of the sets \mathcal{E}_a through \mathcal{E}_e . These new subsets appear more naturally in our calculations.

Proof. It is known (e.g. Chapter 3 Exercise 26 of [9]) that E_α can be written in the form

$$\left\{ re^{i(\theta+2\alpha)} : 0 \leq r \leq \frac{\frac{1}{2}\cos^2 \alpha}{1 - \cos \theta} \right\}.$$

In our case, $\alpha = \frac{\pi}{12}$, so

$$E_{\pi/12} = \left\{ re^{i(\theta+\pi/6)} : 0 \leq r \leq \frac{\lambda}{1 - \cos \theta} \right\} \quad \text{for } \lambda = \frac{2 + \sqrt{3}}{8}.$$

We now proceed with the bounding.

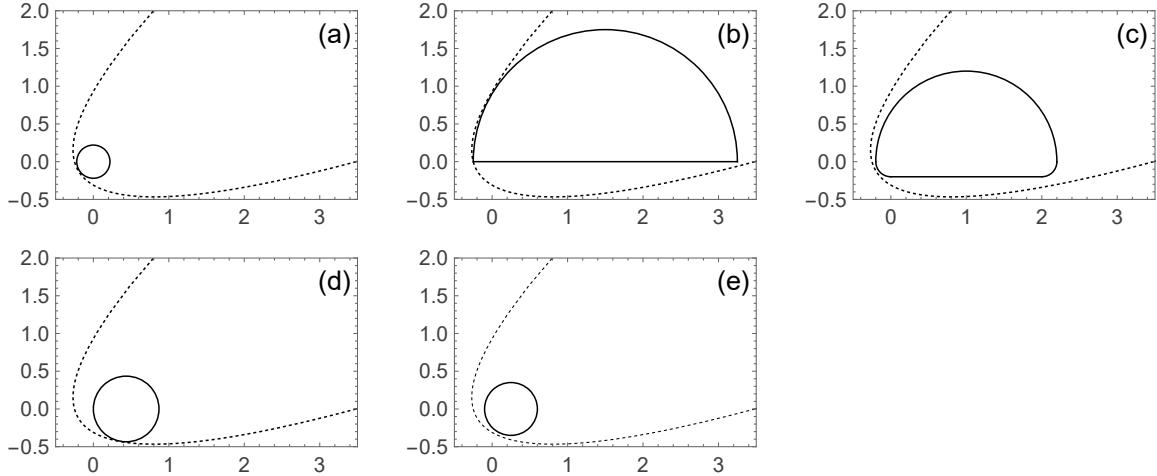


Figure 5: Plots of each of the five sets in Proposition 10 (solid) compared with E (dashed). Some of the bounds are quite tight.

For part (a), observe that $1 - \cos \theta \geq 2$, so along ∂E the minimum value of r is $\frac{1}{2}\lambda > 0.233$. This implies $\mathcal{E}_a \subseteq E$.

For part (b), again let z lie on ∂E . The restriction $\text{Im } z \geq 0$ implies $\theta \in [-\frac{\pi}{6}, \frac{5\pi}{6}]$. Compute

$$\begin{aligned} |z - \frac{3}{2}|^2 &= r^2 - 3r \cos\left(\theta + \frac{\pi}{6}\right) + \frac{9}{4} \\ &= \frac{\lambda^2}{(1 - \cos \theta)^2} - \frac{3\lambda \cos(\theta + \frac{\pi}{6})}{1 - \cos \theta} + \frac{9}{4} =: g(\theta). \end{aligned}$$

It remains to analyze the behavior of g . Although g is increasing on $[-\frac{\pi}{6}, 0]$, there are two critical points in $(0, \frac{5\pi}{6}]$: one at $\theta \approx 1.11731$ and one at $\theta \approx 1.75855$. Checking these points, as well as the endpoints, yields that the minimum of g on $[-\frac{\pi}{6}, \frac{5\pi}{6}]$ is $g(\frac{5\pi}{6}) = \frac{49}{16}$. Thus $|w - \frac{3}{2}| \geq \frac{7}{4}$ and $\mathcal{E}_b \subseteq E$.

For part (c), we remark that

$$\mathcal{E}_c \subset C_1 \cup C_2 \cup C_3,$$

where

$$C_1 = B_{6/5}(1)^+, \quad C_2 = B_{1/5}(0), \quad \text{and} \quad C_3 = [0, \frac{11}{5}] \times [-\frac{1}{5}, 0].$$

(See Figure 5(c).) By part (a), $C_1 \subseteq \mathcal{E}_a \subseteq E$, and by part (b), $C_2 \subseteq \mathcal{E}_b \subseteq E$. To show $C_3 \subseteq E$, observe that E is the convex hull of the four complex numbers $0, -\frac{1}{5}i, \frac{11}{5},$ and $\frac{11}{5} - \frac{1}{5}i$. Each of these four points lies in E by simple computation, and E is a convex set. It follows that $C_3 \subseteq E$, and so $\mathcal{E}_c \subseteq E$.

For part (d), observe that

$$E^{-1} = \left\{ re^{i(\theta - \pi/6)} : r \geq \frac{1}{\lambda}(1 - \cos \theta) \right\}.$$

Thus, writing $w = re^{i(\theta-\pi/6)} \in \partial E$ in polar coordinates,

$$\operatorname{Re} w = \frac{1}{\lambda}(1 - \cos \theta) \cos(\theta - \frac{\pi}{6}). \quad (4.22)$$

The maximum value of (4.22) on $[0, 2\pi]$ is $1.13861\dots < 1.14$, occurring when $\theta = \frac{4\pi}{9}$. It follows that $\{\operatorname{Re} w \geq 1.15\} \subseteq E^{-1}$, so $\mathcal{E}_d \subseteq E$.

Finally, for part (e), we proceed as we did in our proof of (b). In this case, our goal is to minimize the quantity

$$\begin{aligned} |z - \frac{1}{4}|^2 &= r^2 - \frac{1}{2}r \cos \left(\theta + \frac{\pi}{6} \right) + \frac{1}{16} \\ &= \frac{\lambda^2}{(1 - \cos \theta)^2} - \frac{\lambda \cos(\theta + \frac{\pi}{6})}{2(1 - \cos \theta)} + \frac{1}{16} =: h(\theta). \end{aligned}$$

Within the interval $[0, 2\pi]$, h has exactly one local minimum at $\theta =: \theta_0 \approx 4.46993$. It follows that $h(\theta) \geq h(\theta_0) > 0.15$, and so $|z - \frac{1}{4}| > \sqrt{0.15} \geq 0.387 > 0.35$. In turn, $\mathcal{E}_e \subseteq E$. \square

5 Proof of Theorem 5

We are finally ready to prove Theorem 5.

Proof (Theorem 5). Our proof will involve splitting the ordered pairs $(a, b) \in \mathbb{N}^2$ into several cases and analyzing each case independently. In some cases, we will analyze the original ratio $z_{a,b} = \frac{z^a}{(z)_a(z)_b}$, while in others we instead examine its reciprocal $z_{a,b}^{-1} = \frac{(z)_a(z)_b}{z^a}$. We shall refer to these expressions by $(*)$ and $(*^{-1})$, respectively.

- **$b = 1, a \leq 4$:** Begin with $(*^{-1})$. The quotient expands to

$$\frac{(z)_a}{z^a} = \frac{1 + z + \dots + z^{a-1}}{z^a} = \frac{1}{z} + \frac{1}{z^2} + \dots + \frac{1}{z^a}.$$

Note that $\operatorname{Im} z^{-j} \leq 0$ for $1 \leq j \leq a$. Furthermore, $\operatorname{Re} z^{-j} \geq 0$ for $j = 1, 2, 3$, and $\operatorname{Re} z^{-4} \geq \operatorname{Re} z^{-2}$ by part 2 of Proposition 9. It follows that

$$\operatorname{Re} \frac{(z)_a}{z^a} \geq \operatorname{Re} \frac{1}{z} \geq \frac{1}{3}.$$

Combining both parts yields $\frac{z^a}{(z)_a} \in B_{3/2}(\frac{3}{2})^+ \subseteq \mathcal{E}_b \subseteq E$.

- **$2 \leq b \leq 4, a \leq 4$:** Begin with $(*^{-1})$. We perform similar computations to the previous part, but we can no longer assert that $\frac{(z)_b(z)_a}{z^a}$ lies in the lower half plane. Write

$$\begin{aligned} \frac{(z)_b(z)_a}{z^a} &= (1 + z + \dots + z^{b-1}) \left(\frac{1}{z} + \dots + \frac{1}{z^a} \right) \\ &= \sum_{\substack{0 \leq j \leq b-1 \\ 1 \leq k \leq a}} z^{j-k} = \sum_{m=-a}^{b-2} c_m z^m, \end{aligned}$$

where

$$c_m = \#\{(j, k) \in [0, a] \times [1, b] : j - k = m\}.$$

The sequence $(c_m)_m$ is weakly increasing between $m = -b$ and $m = -1$, so $c_{-4} \leq c_{-2}$. (These coefficients may be zero.) This means

$$\operatorname{Re}(c_{-4}z^{-4} + c_{-2}z^{-2}) \geq c_{-2} \operatorname{Re} z^{-2} - c_{-4} |\operatorname{Re} z^{-4}| \geq 0.$$

It follows that

$$\begin{aligned} \operatorname{Re} \frac{(z)_b(z)_a}{z^a} &= \sum_{m=-a}^{b-2} c_m \operatorname{Re} z^m \\ &\geq c_0 + c_{-1} \operatorname{Re} z^{-1} + \operatorname{Re}(c_{-2}z^{-2} + c_{-4}z^{-4}) \\ &\geq \frac{4}{3} > 1.15. \end{aligned}$$

Thus $\frac{z^a}{(z)_a(z)_b} \in \mathcal{E}_d \subseteq E$.

- **$b = 1, a \geq 5$:** Begin with (*). Write

$$\frac{z^a}{(z)_a} = \frac{z^a - 1 + 1}{(z)_a} = z - 1 + \frac{1}{(z)_a}.$$

Because $z - 1 \in B_1(2)^+$ and $|(z)_a| \geq 5$, we deduce

$$\frac{z^a}{(z)_a} \subseteq B_{1/5}(0) + B_1(2)^+ = \mathcal{E}_c \subseteq E.$$

- **$b = 2, a \geq 5$:** Begin with (*). Write

$$\frac{z^a}{(z)_a(1+z)} = \frac{z-1}{z+1} + \frac{1}{(z)_a(z+1)} \in B_{1/10}\left(\frac{z-1}{z+1}\right).$$

The function $\eta(z) := \frac{z-1}{z+1}$ is a Möbius transformation with $\eta(1) = 0$ and $\eta(3) = \frac{1}{2}$. This means η sends the ball $B_1(2)$ to the ball $B_{1/4}(\frac{1}{4})$. It follows from Proposition 9 that

$$B_{1/10}\left(\frac{z-1}{z+1}\right) \subseteq B_{0.35}(0.25) = \mathcal{E}_e \subseteq E.$$

- **$b \in \{3, 4\}, a \geq 5$:** Begin with (*). Recall that $|(z)_a| \geq 5$ and $|(z)_b| \geq 3$ by Theorem 8. Write

$$\frac{z^a}{(z)_a(z)_b} = \frac{z-1}{(z)_b} + \frac{1}{(z)_a(z)_b} \in B_{1/15}\left(\frac{z-1}{(z)_b}\right).$$

To bound the location of $\frac{z-1}{(z)_b}$, observe that its reciprocal is

$$\begin{aligned} \frac{(z)_b}{z-1} &= \frac{z^{b-1} + \cdots + z + 1}{z-1} \\ &= z^{b-2} + \cdots + (b-2)z + (b-1) + \frac{b}{z-1}. \end{aligned} \tag{5.1}$$

Because $b \leq 4$, Proposition 9 implies that all terms in (5.1) have nonnegative real part, and furthermore $z \in \mathcal{B}$ implies $\operatorname{Re} \frac{1}{z-1} \geq \frac{1}{2}$. It follows that

$$\operatorname{Re} \left(\frac{(z)_b}{z-1} \right) \geq (b-2) + (b-1) + \frac{1}{2}b = \frac{5}{2}b - 3 \geq \frac{9}{2}.$$

This means that $\frac{z-1}{(z)_b} \in B_{1/9}(\frac{1}{9})$, and so

$$B_{1/15} \left(\frac{z-1}{(z)_b} \right) \subseteq B_{8/45} \left(\frac{1}{9} \right) \subseteq \mathcal{E}_e \subseteq E.$$

- $b \in \{5, 6\}$: Begin with (*). By Theorem 6 and Lemma 8,

$$\left| \frac{z^a}{(z)_a(z)_b} \right| = \left| \frac{z^{a-1}}{(z)_a} \cdot \frac{z}{(z)_b} \right| \leq \left| \frac{z}{(z)_b} \right| \leq \frac{1}{5}.$$

Thus $\frac{z^a}{(z)_a(z)_b} \in \mathcal{E}_a \subseteq E$.

- $b \geq 7$: Begin with $(*)^{-1}$. We are finally able to use our more general inequalities. Observe that

$$\left| \frac{(z)_b}{z} \right| = \left| \frac{1}{z} + (z)_{b-1} \right| \geq |(z)_{b-1}| - \frac{1}{|z|} \geq 4.5,$$

so $|(z)_b| < 0.233$. It follows that $\left| \frac{z^a}{(z)_a(z)_b} \right| \leq \left| \frac{z}{(z)_b} \right| < 0.233$ and $\frac{z^a}{(z)_a(z)_b} \in \mathcal{E}_a \subseteq E$.

We have *finally* covered all ordered pairs $(a, b) \in \mathbb{N}^2$, thus proving Theorem 5. \square

6 Comments and Future Work

Because the sequence $S_n(\lambda)$ is relatively new, several unresolved questions regarding their properties and zero distributions persist. These questions touch many areas of mathematics, including continued fraction theory and combinatorics.

While Theorem 4 was sufficient to show \mathcal{B} has no Stern zeros, it cannot be used to show that all Stern zeros have real part less than 1. There exist complex numbers z with $\operatorname{Re} z \geq 1$ for which the set \mathcal{A}_z is not a subset of any parabolic region E_α . Figure 6 illustrates $z = 1 + 2.5i$ as an example. Any attempts to prove this stronger conjecture will need more general results from continued fraction theory.

Stronger estimates for $|(z)_m|$ would also be desirable. For example, it may be possible to strengthen Lemma 1 to a statement of the form

$$|(x + yi)_n| \geq C|(x_0 + yi)_n|,$$

where $C = C_{n,y,x,x_0}$ is a positive number greater than 1. We hope that this stronger statement may lead to cleaner proofs and further generalizations of Theorem 8.

Finally, let us comment on a remark from the introduction of this paper. In Section 1, we mentioned the stark contrast in the behavior of \mathcal{S} between the two half-planes $\{\operatorname{Re} w > 0\}$ and $\{\operatorname{Re} w < 0\}$. Recent papers have aimed to make these differences explicit. For example, while we conjecture that the set $\{\operatorname{Re} w : w \in \mathcal{S}\}$ is bounded above, it is *not* bounded below.

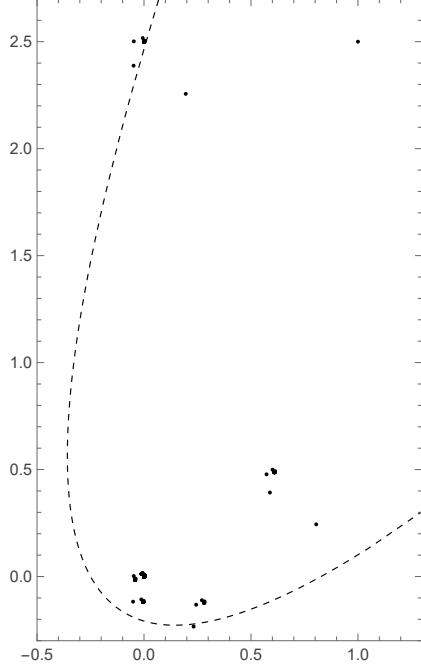


Figure 6: The set $\mathcal{A}_{1+2.5i}$ (solid) compared with the near-miss region $E_{0.16\pi}$ (dashed).

Theorem 9 ([3, Proposition 3.3]). The roots of $S_{\alpha_n}(\lambda)$, where $\alpha_n = \frac{1}{3}(2^n - (-1)^n)$, are negative real numbers of the form

$$-\frac{1}{4} \sec^2 \left(\frac{j\pi}{n} \right), \quad j = 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor.$$

Consequently, the interval $(-\infty, -\frac{1}{4}]$ is dense in \mathcal{S} .

This leads into a related question: is the set of *imaginary* parts bounded? Surprisingly, the answer is also “no”.

Theorem 10. The polynomials $S_{t_n}(\lambda)$, where

$$t_n = \frac{4^{n+1} + (-2)^n + 1}{3},$$

have roots $z_n \in \mathbb{C}$ satisfying $\text{Im } z_n \rightarrow \infty$ as $n \rightarrow \infty$.

The proof begins by finding a closed form for $S_{t_n}(\lambda)$ (involving the principal square root in \mathbb{C}), then uses Rouché’s Theorem along with careful asymptotics to show there exist roots with arbitrarily large imaginary part. The interested reader may consult the author’s doctoral thesis [1] for the details. The author believes $\text{Re } z_n \asymp (\text{Im } z_n)^2$, but has not worked out the details.

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