

UNIFORM IRREDUCIBILITY OF GALOIS ACTION ON THE ℓ -PRIMARY PART OF ABELIAN 3-FOLDS OF PICARD TYPE

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ABSTRACT. Half a century ago Manin showed that given a number field k and a rational prime ℓ , there exists a uniform bound for the order of cyclic ℓ -power isogenies between two non-CM elliptic curves over k . We generalize this to certain 2-dimensional families of abelian 3-folds with multiplication by an imaginary quadratic field.

Dedicated to the memory of Yuri Manin

INTRODUCTION

Given a prime number ℓ and a number field k , Manin showed in [18] that there exists an integer $r = r(\ell, k)$ such that for any non-CM elliptic curve E over k , $E[\ell^r] \simeq (\mathbf{Z}/\ell^r \mathbf{Z})^2$ does not contain a k -rational line, or equivalently that the image of the reduction modulo ℓ^r of its ℓ -adic Galois representation

$$\mathrm{Gal}_k = \mathrm{Gal}(\bar{k}/k) \longrightarrow \mathrm{Aut}_{\mathbf{Z}/\ell^r \mathbf{Z}}(E[\ell^r]) \simeq \mathrm{GL}(2, \mathbf{Z}/\ell^r \mathbf{Z})$$

is not contained in a Borel subgroup. Manin's original proof can be greatly simplified using Faltings' proof of Mordell's conjecture, which came later. In a series of papers Cadoret and Tamagawa established a definitive result regarding the uniform boundedness of the ℓ -primary torsion for 1-dimensional families of abelian varieties. In this paper we prove an analogous statement for certain 2-dimensional families of abelian 3-folds which we believe to be the first genuine result over a two-dimensional base.

Henceforth we fix an imaginary quadratic field M of odd fundamental discriminant $-D$ and denote by \mathcal{O}_M its ring of integers. An abelian 3-fold of *Picard type* over a field k containing M will always stand for a principally polarized abelian variety over k of dimension 3 having multiplication by \mathcal{O}_M defined over k . Its ℓ -adic Tate module $T_\ell A$ is free of rank 3 over $\mathcal{O} := \mathbf{Z}_\ell \otimes \mathcal{O}_M$ endowed with a continuous \mathcal{O} -linear action of Gal_k . By a line (resp. plane) in $T_\ell A$, we would mean a \mathcal{O} -submodule of rank 1 (resp. 2) which is a direct factor. More generally, given a positive integer r , a line (resp. plane) in $A[\ell^r]$ will always be assumed be the image, under the natural reduction map, of a line (resp. plane) in $T_\ell A$. Finally, by a full flag we would mean a tuple of a line sitting as direct factor in a plane. Lines (resp. planes) will be called k -rational if they are stable by Gal_k (but not necessarily point-wise fixed).

Our first main result addresses the semi-stable case.

Theorem A. *Given a number field k , a prime number ℓ inert in M and a finite set S of places of M , there exists an integer $r = r(\ell, k, S)$ such that for any non-CM abelian 3-fold A over k of Picard type which is semi-stable outside S , $A[\ell^r]$ does not contain a full k -rational flag.*

As in the case of elliptic curves, the conclusion of Theorem A asks the image of the attached Galois representation

$$\mathrm{Gal}_k \longrightarrow \mathrm{Aut}_{\mathcal{O}/\ell^r\mathcal{O}}(A[\ell^r]) \simeq \mathrm{GU}(3, \mathbf{Z}/\ell^r\mathbf{Z})$$

not to be contained in a Borel subgroup. Also, as in the case of elliptic curves, it is necessary to cast aside the CM abelian varieties, as their ℓ -adic representations are potentially reducible.

We next show how one can relax the semi-stability assumption by adding a tiny bit of level structure at D . Given a prime v of M above some $p \mid D$, the projective Gal_k -action on the \mathbf{F}_p -vector space $A[v]$ yields a homomorphism $\tilde{\rho}_{A,p} : \mathrm{Gal}_k \rightarrow \mathrm{PGL}_2(\mathbf{F}_p)$ (see (8)). Taking quotient by the unique index two subgroup $\mathrm{PSL}_2(\mathbf{F}_p)$ of $\mathrm{PGL}_2(\mathbf{F}_p)$ yields a canonical homomorphism $\varepsilon_{A,p} : \mathrm{Gal}_k \rightarrow \{\pm 1\}$ and we let $\varepsilon_{A,D} = \prod_{p \mid D} \varepsilon_{A,p} : \mathrm{Gal}_k \rightarrow \{\pm 1\}$.

Theorem B. *Given a number field k containing M and a prime number ℓ inert in M , there exists an integer $r = r(\ell, k)$ such that for any non-CM abelian 3-fold A over k of Picard type and such that $\varepsilon_{A,D}$ is trivial, $A[\ell^r]$ does not contain a full k -rational flag.*

Theorem B is the main result of this paper and implies Theorem A as follows. Let k' be the compositum of the (finitely many) quadratic extensions of k which are unramified outside S and the primes dividing D . Given any abelian 3-fold A as in Theorem A, we claim that $\varepsilon_{A,D}(\mathrm{Gal}_{k'})$ is trivial. Indeed, by a theorem of Grothendieck [14, Prop. 3.5] the semi-stability of A at $v \notin S$, $v \nmid D$ implies that the inertia subgroup of Gal_k at v acts unipotently on the D -adic Tate module of A , in particular its image by $\varepsilon_{A,D}$ is pro- D hence trivial (as D is odd). Therefore the base change of A to k' satisfies the additional assumption in Theorem B, implying that Theorem A holds with $r(\ell, k')$ from Theorem B.

For an individual abelian variety A , the conclusion of Theorem B is a consequence of the Mumford–Tate conjecture which is known for abelian 3-folds (see §2.3), so the important feature of the result is its uniformity. As abelian 3-folds of Picard type are parametrized by Shimura surfaces of Picard type, a natural way to proceed would be to show that the k -rational points are not Zariski dense in any of their connected components Y_Γ . Let us for the moment consider the simpler situation from our earlier paper [10] where the congruence subgroups Γ were neat. Our method there had two principal steps. The first step involved showing the existence of three linearly independent global holomorphic 1-forms on the toroidal compactification X_Γ (see §4.3 for an amended list of Γ to which our methods apply). By a theorem of Faltings concerning the associated Albanese variety this implies that the k -rational points on X_Γ are contained in a divisor Z , as predicted by a conjecture of Bombieri and Lang as X_Γ turns out to be of general

type. The second step consisted in applying a result of Nadel requiring Γ to be neat and the canonical divisor to be big (in his sense) to deduce that any curve C of genus ≤ 1 contained in X_Γ is in fact contained in the complement of Y_Γ . Consequently, every curve in Z meeting the open surface Y_Γ must be of genus ≥ 2 thus, by Faltings' proof of Mordell's conjecture for curves, $Y_\Gamma(k)$ is finite for any number field k .

Let us now say a few words about the techniques involved in the proof of Theorem B. As we are led to consider congruence subgroups of Iwahori type $\Gamma_0(\ell^r)$, which are *never neat* as they have torsion, both steps mentioned above encounter difficulty and we have to resort to new methods. We produce irregularity by constructing explicit endoscopic automorphic forms in certain non-generic representations π on the unitary group in 3 variables. It is here that the index 2 projective Galois image condition at D , suggested to us by Gross, is essential, as otherwise all of the Picard modular surfaces involved would have trivial Albanese and our approach would fail for global reasons. Making this strategy actually work yet requires to address some delicate representation theoretic questions to which a significant part of the paper is devoted and on which we will elaborate now.

By Rogawski's theory π is an element of an endoscopic Arthur packet parametrized by an anti-cyclotomic (more precisely, conjugate-symplectic) Hecke character λ of M . Theorem B imposes conditions so stringent so that λ must differ from Gross' minimally ramified 'canonical' characters by a finite order character χ only ramified at ℓ . The local Arthur packet at ℓ contains two representations, a supercuspidal $\pi_{c,\ell}$ and a non-tempered $\pi_{n,\ell}$, both non-generic. The difficulty of finding $\Gamma_0(\ell^r)$ -invariants in $\pi_{c,\ell}$ forces us to work with the global π_n , which is automorphic if, and only if, the root number $W(\lambda^3)$ is $+1$. For $D \equiv 3 \pmod{8}$ Gross' canonical characters work and a computation of matrix coefficients performed in §1.4 shows that the resulting $\pi_{n,\ell}$ has invariants even by the hyperspecial maximal compact subgroup.

When $D \equiv 7 \pmod{8}$ the canonical characters yield the wrong sign, leading us to consider λ 's which are tamely ramified at ℓ to switch the sign. It remains however to show that the non-tempered representations $\pi_{n,\ell}$ attached to such λ 's admit $\Gamma_0(\ell^r)$ -invariants for some r . For this we use a more involved argument, based on Jacquet modules and intertwining operators involving some precise averages of exponential sums, occupying the entirety section §1.5.

Once the irregularity of X_Γ has been shown to the at least 3 and the Bombieri–Lang conjecture established, one has to deal with the possible curves C of genus ≤ 1 contained in Y_Γ . Using our key Lemma 4.1 affirming that our Picard modular surfaces only admit a finite number of isolated singularities, we show that, after removing a finite number of points, C is endowed with an abelian family (see §2.2). This allows us apply the results of Cadoret and Tamagawa regarding the uniform boundedness of the Galois action on the Tate module of such 1-dimensional families.

Finally, each of the finitely many non-CM k -rational points are dealt with using the results on the Mumford–Tate conjecture for abelian 3-folds of Picard type recalled in §2.3.

As our Picard modular surfaces X_Γ have irregularity $q \geq 3$, the Kodaira–Spencer classification implies that they are either ruled of genus q , or elliptic, or else they are of general type. In the last case, which according to Holzapfel [16, §5.4] occurs for all odd $D \notin \{3, 7, 11, 19, 23, 31, 39, 47, 71\}$, we show that the Bombieri–Lang Conjecture holds, *i.e.*, that the k -rational points are not Zariski dense. Investigating small values of D , as suggested by Mazur, seems even more interesting. It is established in *loc. cit.* that for all $D \neq 71$ in the above list the level 1 Picard modular surfaces are rational and it would be natural to investigate the nature of their degree 2 Gross covers that we consider. A way to shed light on this question would be to find an explicit 2-parameter family of abelian 3-folds of Picard type to which our theorem applies.

It might be worthwhile remarking that we could have also considered the simpler case of the moduli of principally polarized abelian *surfaces* A over k with multiplication by \mathcal{O}_M , which will involve $U(1, 1)$. However, as $SU(1, 1) \simeq SL(2)$ this essentially reduces to the modular curve case. On the other hand, if we consider principally polarized abelian surfaces A with *real multiplication*, then the family is parametrized by a Hilbert modular surface which has trivial H^1 , thus our methods, which rely on the Albanese variety, do not lead to an establishment the Bombieri–Lang Conjecture.

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CONTENTS

1. Levels for endoscopic non-tempered representations of $U(3)$	5
1.1. The Bruhat-Tits tree of U_3	6
1.2. Review of L -parameters and A -packets	7
1.3. The Gross subgroup K''	9
1.4. Higher Iwahori invariants via matrix coefficients	11
1.5. Intertwining and an exponential sum	13
2. Galois representations for 3-folds of Picard type	15
2.1. Abelian 3-folds of Picard type and Tate modules	15
2.2. Shimura surfaces and families of abelian threefolds of Picard type	16
2.3. Étale fundamental groups and Mumford–Tate groups	18
2.4. Galois stable lattices and rationality	19
2.5. Lie images of Galois representations	20
3. Irregularity for Picard modular surfaces	22

3.1.	A lemma on surfaces with isolated singularities	22
3.2.	A formula for the irregularity	22
3.3.	Twists of canonical characters and root numbers	24
3.4.	The Bombieri–Lang Conjecture for Picard modular surfaces	25
4.	Uniform irreducibility of Galois images	27
4.1.	Complex reflexions, elliptic elements and abelian families	27
4.2.	Proof of Main Theorem	27
4.3.	Odd and ends	29

1. LEVELS FOR ENDOSCOPIC NON-TEMPERED REPRESENTATIONS OF $U(3)$

It goes back to the work of Casselman that admissible irreducible representations having non-zero Iwahori invariants are exactly those occurring as sub-quotients in parabolic inductions of unramified characters. Whereas the dimension of the invariants by the depth r Iwahori subgroup in the full induced representation grows as r goes to infinity, this might not always be the case for all its sub-quotients, as shown by the example of the trivial representation of GL_2 , realized as a quotient of a unramified principal series representation.

Another challenging question is to determine which sub-quotient of a parabolically induced unramified character picks up the invariants by a given maximal open compact subgroup. Whereas MacDonald’s formula for zonal spherical functions yields an answer in the case of a maximal hyperspecial subgroup, the general case appears to be an open question.

In this section we fully answer those two natural questions in the case of certain non-tempered endoscopic representations of U_3 attached to a quadratic extension E/\mathbf{Q}_p . It will be later applied in a global setting to $E = M_p$, where M is an imaginary quadratic field in which the prime p does not split.

In this section of our paper we will adopt local notations.

Let E be a quadratic field extension of \mathbf{Q}_p , \mathcal{O} be its ring of integers, \mathcal{P} its maximal ideal and ϖ a uniformizer. We assume that E is *not* a ramified extension of \mathbf{Q}_2 . Denote by $x \mapsto \bar{x}$ the automorphism of E induced by the non-trivial element of $\text{Gal}(E/\mathbf{Q}_p)$ and fix a generator ξ of the different \mathcal{D} of E/\mathbf{Q}_p such that $\bar{\xi} = -\xi$. We fix an additive character $\psi : \mathbf{Q}_p \rightarrow \mathbf{C}^\times$ of conductor 0, i.e. $\ker(\psi) = \mathbf{Z}_p$, and we consider the additive character ψ_E of E defined as $\psi_E(x) = \psi(\text{Tr}_{E/\mathbf{Q}_p}(x))$.

Let G be the unique quasi-split unitary group in 3 variables relative to the extension E/\mathbf{Q}_p . It can be realized as the automorphisms of E^3 preserving the hermitian pairing

$$\langle x, y \rangle = \bar{x}_1 y_3 + \xi \bar{x}_2 y_2 - \bar{x}_3 y_1.$$

1.1. The Bruhat-Tits tree of U_3 . As G has rank 1, its Bruhat-Tits building is a tree. We will first describe its standard apartment. The relative roots of G are obtained by decomposing the adjoint action on the Lie algebra of the maximal \mathbf{Q}_p -split torus $T_0 = \{\text{diag}(a, 1, a^{-1}) | a \in \mathbf{Q}_p^\times\}$ of G . The positive elements of the associated root system Φ are $\{\zeta, 2\zeta\}$. Let $h : \mathbf{G}_m \rightarrow T_0 \subset G$ be the generator of the co-character lattice $X_*(T_0) \simeq \mathbf{Z}$ such that $\langle \zeta, h \rangle = 1$. Then the co-root sub-lattice is generated by $\zeta^\vee = 2h$, so that we have the standard normalization $\langle \zeta, \zeta^\vee \rangle = 2$. According to [32, §1.15] the affine roots are $\{\pm\zeta + \mathbf{Z}\} \cup \{\pm 2\zeta + \mathbf{Z}\}$ if E is unramified, and $\{\pm\zeta + \frac{1}{2}\mathbf{Z}\} \cup \{\pm 2\zeta + \mathbf{Z} + \frac{1}{2}\}$ if E is ramified; note that $\delta = 0$ in *loc. cit.* as E is not a ramified extension of \mathbf{Q}_2 . The apartment associated to T_0 is $\mathbf{R}h$ and its walls are the vanishing sets of these (affine) roots, hence they are given by $\frac{1}{2}\mathbf{Z}h = \mathbf{Z}h \cup \frac{1}{2}\mathbf{Z}h$, resp. $\frac{1}{4}\mathbf{Z}h = (\frac{1}{2}\mathbf{Z} + \frac{1}{4})h \cup \frac{1}{2}\mathbf{Z}h$, if E is unramified, resp. ramified. Given an \mathcal{O} -lattice \mathcal{L} in E^3 we define its dual as

$$\mathcal{L}^\perp = \text{Hom}_{\mathcal{O}}(\mathcal{L}, \mathcal{D}^{-1}) = \{x \in E^3 | \langle x, \mathcal{L} \rangle \subset \mathcal{D}^{-1}\}.$$

Lemma 1.1. *There are two conjugacy classes of maximal compact subgroups in G , those which are stabilizers of self-dual lattices, and those which are stabilizers of almost self-dual lattices, i.e., lattices \mathcal{L} such that $\mathcal{L} \subsetneq \mathcal{L}^\perp \subsetneq \varpi^{-1}\mathcal{L}$. They are all special, and the hyperspecial ones are those stabilizing a self-dual lattice when E is unramified.*

Proof. A conjugacy class of maximal compact subgroups can be represented by a wall in the standard apartment. By definition, a wall is hyperspecial if for every $\zeta' \in \Phi$ there exists an affine root with gradient ζ' vanishing on that wall. Since $(\frac{1}{2}\mathbf{Z} + \frac{1}{4})h \cap \frac{1}{2}\mathbf{Z}h = \emptyset$ this only can happen when E is unramified, in which case the hyperspecial walls are $\mathbf{Z}h \cap \frac{1}{2}\mathbf{Z}h = \mathbf{Z}h$. All walls are special, as elements of Φ are rational multiple of one another. \square

We now give an explicit description of the maximal compact subgroups corresponding to the walls of a chamber in the standard apartment.

The standard maximal compact subgroup K° of G is defined as the stabilizer of the self-dual lattice $\mathcal{L}^\circ = \mathcal{O} \oplus \xi^{-1}\mathcal{O} \oplus \xi^{-1}\mathcal{O}$. It is hyperspecial if and only if E is unramified, and

$$K^\circ = \begin{pmatrix} \mathcal{O} & \xi\mathcal{O} & \xi\mathcal{O} \\ \xi^{-1}\mathcal{O} & \mathcal{O} & \mathcal{O} \\ \xi^{-1}\mathcal{O} & \mathcal{O} & \mathcal{O} \end{pmatrix} \cap G.$$

The reductive quotient \overline{G}° is given by $U_3(\mathbf{F}_p)$ if E is unramified, and by $O_3(\mathbf{F}_p)$ if E is ramified.

The other standard maximal compact subgroup K' of G , defined as the stabilizer of the almost self-dual lattice $\mathcal{L}' = \mathcal{O} \oplus \xi^{-1}\mathcal{O} \oplus \xi^{-1}\mathcal{P}$, is given by

$$K' = \begin{pmatrix} \mathcal{O} & \xi\mathcal{O} & \xi\mathcal{P}^{-1} \\ \xi^{-1}\mathcal{P} & \mathcal{O}^\times & \mathcal{O} \\ \xi^{-1}\mathcal{P} & \mathcal{P} & \mathcal{O} \end{pmatrix} \cap G.$$

One has $\mathcal{L}'^\perp = \mathcal{P}^{-1} \oplus \xi^{-1}\mathcal{O} \oplus \xi^{-1}\mathcal{O}$ and K' acts on $\varpi^{-1}\mathcal{L}'/\mathcal{L}'^\perp \simeq \mathcal{O}/\mathcal{P}$ via its middle coefficient. The reductive quotient \overline{G}' is isomorphic to $\mathrm{U}_{1,1}(\mathbf{F}_p) \times \mathrm{U}_1(\mathbf{F}_p)$ if E is unramified, and to $\pm \mathrm{SL}_2(\mathbf{F}_p) \times \{\pm 1\}$, if E is ramified.

The standard Iwahori subgroup of G is defined as $I = K^\circ \cap K' = \begin{pmatrix} \mathcal{O}^\times & \xi\mathcal{O} & \xi\mathcal{O} \\ \xi^{-1}\mathcal{P} & \mathcal{O}^\times & \mathcal{O} \\ \xi^{-1}\mathcal{P} & \mathcal{P} & \mathcal{O}^\times \end{pmatrix} \cap G$.

Finally let $I_{2,1} = K^\circ \cap \gamma K^\circ \gamma^{-1} = \begin{pmatrix} \mathcal{O}^\times & \xi\mathcal{O} & \xi\mathcal{O} \\ \xi^{-1}\mathcal{P} & \mathcal{O}^\times & \mathcal{O} \\ \xi^{-1}\mathcal{P}^2 & \mathcal{P} & \mathcal{O}^\times \end{pmatrix} \cap G$, where $\gamma = \begin{pmatrix} \varpi^{-1} & & \\ & 1 & \\ & & \varpi \end{pmatrix}$.

The standard apartment in the Bruhat-Tits tree of G is as follows:

$$\begin{array}{ccccccc} & & & I_{2,1} & & & \\ & & & \text{-----} & & & \\ \text{-----} & \gamma^{-1}K'\gamma & \text{-----} & K^\circ & \xrightarrow{I} & K' & \text{-----} & \gamma K^\circ \gamma^{-1} & \text{-----} \end{array}$$

If E is unramified, then $G \supset K' \supset I \supset I_{2,1} \supset I_2$, where $I_r = \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} & \mathcal{O} \\ p^r\mathcal{O} & \mathcal{O}^\times & \mathcal{O} \\ p^r\mathcal{O} & p^r\mathcal{O} & \mathcal{O}^\times \end{pmatrix} \cap G$.

1.2. Review of L -parameters and A -packets. For any integer $n \geq 1$ there are exactly two (up to isomorphism) n -dimensional hermitian spaces over E , depending on the image of the discriminant in $\mathbf{Q}_p^\times/\mathrm{N}_{E/\mathbf{Q}_p}(E^\times)$, and the corresponding unitary groups $\mathrm{U}(n)$ are isomorphic if and only if n is odd. When $n = 2$, by analogy with the Archimedean case, we will denote by $\mathrm{U}(1,1)$ the quasi-split form and by $\mathrm{U}(2)$ the compact one.

The L -group (of the quasi-split form) of $\mathrm{U}(n)$ is given by $\mathrm{GL}_n(\mathbf{C}) \rtimes W_{\mathbf{Q}_p}$ with the Weil group $W_{\mathbf{Q}_p}$ acting on $\mathrm{GL}_n(\mathbf{C})$ through its quotient $\mathrm{Gal}(E/\mathbf{Q}_p)$ whose non-trivial element sends g to $w_n {}^t g^{-1} w_n^{-1}$, where w_n denotes the anti-diagonal matrix $(1, -1, 1, \dots, (-1)^{n-1})$. By definition, an L -parameter for $\mathrm{U}(n)$ is a homomorphism $W_{\mathbf{Q}_p} \times \mathrm{SL}_2(\mathbf{C}) \rightarrow {}^L G$, but as one knows (see [13, §3]) it is equivalent to ask for its restriction

$$\phi : W_E \times \mathrm{SL}_2(\mathbf{C}) \rightarrow \mathrm{GL}_n(\mathbf{C}),$$

to be conjugate $(-1)^n$ -dual, *i.e.*, *conjugate-orthogonal* if n is odd and *conjugate-symplectic* if n is even. Recall that ϕ is conjugate-self-dual if $\bar{\phi} \simeq \phi^\vee$, or equivalently, if the induced representation $\mathrm{Ind}_{W_E}^{W_{\mathbf{Q}_p}}(\phi)$ is self-dual. Furthermore, ϕ is conjugate-orthogonal, resp. conjugate-symplectic, if it preserves a non-degenerate symmetric, resp. skew-symmetric, bilinear form. Note that while Schur's Lemma implies that any irreducible self-dual (or conjugate-self-dual) parameter has a well defined sign, this need not be always the case for reducible parameters.

For $n = 1$, a character of E^\times is conjugate-orthogonal (resp. conjugate-symplectic) if its restriction to \mathbf{Q}_p^\times is trivial (resp. is the quadratic character attached to E/\mathbf{Q}_p). For $n \in \mathbf{Z}_{\geq 0}$, the n -th symmetric power of the standard 2-dimensional representation St of $\mathrm{SL}(2, \mathbf{C})$, with W_E acting trivially, is conjugate-symplectic if n is odd and conjugate-orthogonal if n is even.

The base change $\nu_E(z) = \nu(z/\bar{z})$ of a character ν of E^1 is conjugate-orthogonal and conversely any conjugate-orthogonal character of E^\times is obtained in that way. For λ a conjugate-symplectic character of E^\times , an example of key relevance to us is the conjugate-orthogonal representation

$$(\lambda \otimes \text{St}) \oplus \nu_E : W_E \times \text{SL}_2(\mathbf{C}) \longrightarrow \text{GL}_3(\mathbf{C}).$$

It yields an L -parameter $\phi_{\lambda,\nu}$ of G , coming from an L -parameter of the (unique) cuspidal endoscopic subgroup $H = \text{U}(1,1) \times \text{U}(1)$ of G . The cardinality of the corresponding L -packet $\Pi_L(\phi_{\lambda,\nu})$ is given by the order of the centralizer in ${}^L G^0$ modulo the center which turns out to be 2. More precisely, $\Pi_L(\phi_{\lambda,\nu})$ contains two discrete series representations π_2 and π_c of $\text{U}(3)$, exactly one of them, namely π_c , being supercuspidal (see [25, Chap. 12.2] where this L -packet is denoted $\Pi_L(\text{St}_H(\xi))$). There is another endoscopic L -packet for G consisting of a single non-tempered representation π_n whose the L -parameter is given by

$$\lambda| \cdot |_E^{1/2} \oplus \lambda| \cdot |_E^{-1/2} \oplus \nu_E : W_E \times \text{SL}_2(\mathbf{C}) \longrightarrow \text{GL}_3(\mathbf{C}).$$

Rogawski's theory [25, 26] describes the automorphic representations contributing to the H^1 of Shimura surfaces of Picard type in terms global Arthur packets (see [10, §3.1] for a summary). The corresponding local Arthur packet at p has 2 elements $\Pi(\lambda, \nu) = \{\pi_n, \pi_c\}$ (see [25, §12.3.3], where π_c is denoted π^s), and the restriction to W_E of its A -parameter is given by

$$(\lambda \otimes \mathbf{1} \otimes \text{St}) \oplus \nu_E : W_E \times \text{SL}_2(\mathbf{C}) \times \text{SL}_2(\mathbf{C}) \longrightarrow \text{GL}_3(\mathbf{C}),$$

while the A -parameter of π_2 is given by $(\lambda \otimes \text{St} \otimes \mathbf{1}) \oplus \nu_E$.

Crucial for us would be the description π_n and π_2 as the Jordan–Hölder constituents of a principal series representation π . Indeed, by [26, §1], π_n is the Langlands quotient of the (unitarily normalized) parabolic induction of the character

$$(1) \quad \mu(\bar{\alpha}, \beta, \alpha^{-1}) = \lambda(\bar{\alpha})\nu(\beta)|\alpha|_E^{1/2},$$

with π_2 being the unique non-zero irreducible sub-representation. Moreover, the extension

$$(2) \quad 0 \rightarrow \pi_2 \rightarrow \pi = \text{Ind}_B^G(\mu) \xrightarrow{\text{pr}} \pi_n \rightarrow 0$$

does not split, and the sub and quotient are switched when μ is replaced by $\mu^w(\bar{\alpha}, \beta, \alpha^{-1}) = \lambda(\bar{\alpha})\nu(\beta)|\alpha|_E^{-1/2}$. The Jacquet functor is exact and it sends π_2 , resp. π_n , to $\mu\delta^{1/2}$, resp. $\mu^w\delta^{1/2}$, where w is the non-trivial element of the Weyl group of G , and $\delta(\bar{\alpha}, \beta, \alpha^{-1}) = |\alpha|_E^2$ is the modulus character. Fixing a non-degenerate character of the unipotent radical N of B , one knows by Rodier [24, Thm. 2] that the image of $\text{Ind}_B^G(\mu)$ by a twisted version of the Jacquet functor, singling out generic representations, is a line. The latter being also exact, this implies that exactly one amongst π_2 and π_n is generic. Since π_n is non-generic (see [25, p.174]), this implies that π_2 is generic.

As π_n is non-tempered, the subspace π_2 consists of $f \in \pi$ such that for all $f^\vee \in \pi^\vee$ the matrix coefficient $g \mapsto \langle g \cdot f, f^\vee \rangle$ belongs to $L^2(G)$. Conversely the following lemma holds.

Lemma 1.2. *Let $f \in \pi$. If $g \mapsto \langle g \cdot f, f^\vee \rangle$ belongs to $L^2(G)$ for some $0 \neq f^\vee \in \pi^\vee$, then $f \in \pi_2$.*

Proof. The dual of (2) is given by

$$0 \rightarrow \pi_n^\vee \rightarrow \pi^\vee = \text{Ind}_B^G(\mu^{-1}) \rightarrow \pi_2^\vee \rightarrow 0,$$

and the irreducibility of π_2 and π_n implies that $\pi_n^\vee = \{f^\vee \in \pi^\vee \mid \langle \pi_2, f^\vee \rangle = 0\}$. As $f^\vee \neq 0$, its G -span contains π_n^\vee , implying that the matrix coefficient $g \mapsto \langle g \cdot f, f^\vee \rangle$ belongs to $L^2(G)$ for all $f^\vee \in \pi_n^\vee$. One deduces that

$$g \mapsto \langle g \cdot f, f^\vee \rangle = \langle \text{pr}(g \cdot f), f^\vee \rangle = \langle g \cdot \text{pr}(f), f^\vee \rangle \in L^2(G)$$

As the irreducible π_n is not a discrete series representation, this implies $\text{pr}(f) = 0$, i.e. $f \in \pi_2$. \square

We will be mostly interested in the following A -packets having trivial central characters:

$$(3) \quad \Pi(\lambda) = \Pi(\lambda, \lambda_{|E^1}^{-1}).$$

1.3. The Gross subgroup K'' . In this subsection, E is ramified (hence p is odd). Then $\mathcal{O}/\mathcal{P} = \mathbf{F}_p$ and $\mathcal{P} = \xi \cdot \mathcal{O}$. As $|\text{PGL}_2(\mathbf{F}_p)| = |\text{SL}_2(\mathbf{F}_p)|$ all vertices in the tree of G have valence $p^3 + 1$. The map from K° to its reductive quotient \overline{G}° is obtained by reducing $\begin{pmatrix} \xi/2 & & \\ & 1 & \\ & & 1 \end{pmatrix}^{-1} K^\circ \begin{pmatrix} \xi/2 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ modulo \mathcal{P} and a direct computation shows that \overline{G}° is isomorphic to the orthogonal group $\text{O}_3(\mathbf{F}_p)$ with respect to the quadratic form represented by $\begin{pmatrix} & & 1 \\ & 2 & \\ 1 & & \end{pmatrix}$.

Note that $\text{O}_3(\mathbf{F}_p) = \pm \text{SO}_3(\mathbf{F}_p)$ and the adjoint action on matrices $\begin{pmatrix} y & x \\ z & -y \end{pmatrix}$ preserving the determinant $-(y^2 + xz)$ allows us to identify $\text{PGL}_2(\mathbf{F}_p)$ and $\text{SO}_3(\mathbf{F}_p)$ as follows:

$$(4) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{1}{ad - bc} \begin{pmatrix} a^2 & 2ab & -b^2 \\ ac & ad + bc & -bd \\ -c^2 & -2cd & d^2 \end{pmatrix}.$$

It follows from that description that $\text{SO}_3(\mathbf{F}_p)$ is generated by the set

$$\left\{ \begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix}, \begin{pmatrix} a & & \\ & 1 & \\ & & a^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ -c^2 & -2c & 1 \end{pmatrix} \mid a \in \mathbf{F}_p^\times, c \in \mathbf{F}_p \right\}.$$

Definition 1.3. Let K'' be the index 2 subgroup of K° defined as the inverse image of the subgroup of $\text{O}_3(\mathbf{F}_p)$ generated by -1 and the image of $\text{PSL}_2(\mathbf{F}_p)$. Let $I'' = K'' \cap K' \subset I$.

We recall that E/\mathbf{Q}_p is a ramified quadratic extension with p odd. A conjugate-symplectic character λ of E^\times is necessarily ramified and its restriction to \mathbf{Z}_p^\times is given by its unique quadratic character. If λ is tamely ramified, then its restriction to \mathcal{O}^\times is also given by its unique quadratic character, and the equation $\lambda(\xi)^2 = \lambda(-\xi\bar{\xi}) = \lambda(-1) = (-1)^{(p-1)/2}$ shows that there are precisely two such characters.

Interested in determining a level for an element of the A -packet $\Pi(\lambda)$ considered in (3), we are indebted to B. Gross for generously sharing a suggestion that led to the following proposition.

Proposition 1.4. *Let λ be a tamely ramified conjugate-symplectic character of E^\times and let π_n be the non-tempered member of the A -packet $\Pi(\lambda)$. Then $\dim \pi_n^{K''} = \dim \pi_2^{I''} = 1$.*

Proof. As $\mu|_{(T \cap I'')} = \mu|_{(T \cap I'')}^w = \mathbf{1}$, applying the Jacquet functor to the exact sequence of admissible G -representations (2) allows one to see that both π_2 and π_n have non-trivial I'' -invariants. Moreover, as $|B \backslash G / I''| = |(B \cap K'') \backslash K'' / I''| = 2$, both $\pi_2^{I''}$ and $\pi_n^{I''}$ must be 1-dimensional. By Iwasawa decomposition, the restriction of $\text{Ind}_B^G(\mu)$ to K'' is given by $\text{Ind}_{B \cap K''}^{K''}(\mu)$, hence the line $\text{Ind}_B^G(\mu)^{K''}$ admits a basis f uniquely characterized by $f|_{K''} = \mathbf{1}_{K''}$. It follows that $\dim \pi_n^{K''} + \dim \pi_2^{K''} = 1$ and we will show that $\pi_2^{K''} = \{0\}$.

The line $(\text{Ind}_B^G(\mu^{-1}))^{K''}$ admits a basis f^\vee uniquely characterized by

$$\langle v, f^\vee \rangle = \frac{1}{\sqrt{\text{vol}(K'')}} \int_{K''} v(k) dk.$$

By Lemma 1.2 one has $f \notin \pi_2$ if, and only if, $(g \mapsto \langle g \cdot f, f^\vee \rangle) \notin L^2(K'' \backslash G / K'')$.

Recall $\gamma = \begin{pmatrix} -\xi^{-1} & \\ & 1 \\ & & \xi \end{pmatrix}$ and let $\eta = \begin{pmatrix} \bar{u} & & \\ & 1 & \\ & & u^{-1} \end{pmatrix}$ where $u \in \mathcal{O}^\times$ is a fixed non-square element.

As $K^\circ = K'' \amalg \eta K''$, Cartan decomposition for the special maximal compact K° yields:

$$G = \coprod_{n \geq 0} (K'' \gamma^n K'') \amalg (K'' \gamma^n \eta K'').$$

Since $\eta \cdot f = -f$ one deduces that $\langle \gamma^n \eta \cdot f, f^\vee \rangle = -\langle \gamma^n \cdot f, f^\vee \rangle$ and checking that $f \notin \pi_2$ amounts to proving the divergence of the numerical sequence with general term

$$\text{vol}(K'' \gamma^n K'') |\langle \gamma^n \cdot f, f^\vee \rangle|^2 = [K'' : (K'' \cap \gamma^n K'' \gamma^{-n})] \left| \int_{K''} f(k \gamma^n) dk \right|^2.$$

By Iwahori decomposition one has $[I'' : (K'' \cap \gamma^n K'' \gamma^{-n})] = p^{2n-1}$ for all $n \geq 1$. As $|(\mu \delta^{1/2})(\gamma)| = p^{3/2}$ we are led to establish the divergence of the sequence with general term

$$\Phi_n = p^{5n/2} \cdot \left| \int_{K''} f(\gamma^{-n} k \gamma^n) dk \right|.$$

In view of the inequality $p > \sqrt{p} + 1$ for $p \geq 3$, this will follow from the next lemma. \square

Lemma 1.5. *For all $n \geq 1$ one has $\left| \int_{K'' \backslash K_{2n}''} f(\gamma^{-n} k \gamma^n) dk \right| \leq \text{vol}(I'') \cdot (\sqrt{p} + 1) p^{-2n}$ and $\left| \int_{K_{2n}''} f(\gamma^{-n} k \gamma^n) dk \right| = \text{vol}(I'') \cdot p^{1-2n}$.*

Proof. The last row of an element $k \in K''$ is given by $(0, 0, 1) \cdot k = (\xi^{-1} \cdot c_1(k), c_2(k), c_3(k))$ with $(c_1(k), c_2(k), c_3(k)) \in \mathcal{O}^3 \backslash (\xi \mathcal{O})^3$. For $j \geq 0$ we let

$$K_j'' = \{k \in K'' \mid c_1(k) \in \xi^j \mathcal{O}\} \text{ and } K_j''^\times = K_j'' \backslash K_{j+1}''.$$

Note that $K_0'' = K''$ and $K_1'' = I''$. We use the partition $K'' \setminus K_{2n}'' = K_0'' \amalg K_1'' \amalg \cdots \amalg K_{2n-1}''$ to compute the first integral and $K_{2n}'' = K_{2n}'' \amalg K_{2n+1}''$ for the second.

For $j \geq 1$, using Iwahori decomposition, one finds that $[I'' : K_j''] = [I'' \cap N^- : K_j'' \cap N^-] = p^{j-1}$.

Given any $k \in K_j''^\times$ ($0 \leq j \leq 2n$), using the Iwasawa decomposition $G = \gamma^\mathbf{Z} N K^\circ = \gamma^\mathbf{Z} N K'$, one finds that $\gamma^{-j} k \gamma^n \in N K^\circ$, hence $|f(\gamma^{-n} k \gamma^n)| \leq |\mu(\gamma^{j-2n})| = p^{\frac{3}{2}j-3n}$. Therefore

$$\begin{aligned} \left| \int_{K'' \setminus K_{2n}''} f(\gamma^{-n} k \gamma^n) dk \right| &\leq \text{vol}(K'' \setminus I'') \cdot p^{-3n} + \text{vol}(I'') \sum_{j=1}^{2n-1} p^{\frac{1}{2}j-3n} (p-1) = \\ &= \text{vol}(I'') \left(p^{\frac{1}{2}-2n} + p^{-2n} - p^{1-3n} - p^{\frac{1}{2}-3n} + ([K'' : I''] - 1) p^{-3n} \right), \end{aligned}$$

proving the desired inequality, as $[K'' : I''] = p+1$ (obtained by going to the reductive quotient).

Since $\text{vol}(I'') \cdot p^{1-2n} = \text{vol}(K_{2n}'')$, in order to complete the proof of the lemma, it suffices to show that $f(\gamma^{-n} k \gamma^n)$ is constant on $k \in K_{2n}''$. This is evident for $k \in K_{2n+1}''$, as then k and $\gamma^{-n} k \gamma^n$ both belong to I'' and share same determinant and lower right coefficient c_3 , implying that $f(\gamma^{-n} k \gamma^n) = f(k)$. Miraculously, as one can see from (4), this remains true for $k \in K_{2n}'' \setminus K_{2n+1}''$ as well, *i.e.* even though $\gamma^{-n} k \gamma^n \in K^\circ \setminus I$, the fact that $c_3(\gamma^{-n} k \gamma^n) = c_3(k)$ still implies that $\gamma^{-n} k \gamma^n \in K''$. \square

1.4. Higher Iwahori invariants via matrix coefficients. In this subsection we assume that E/\mathbf{Q}_p is unramified. The unique unramified Arthur packet is given by $\Pi(\lambda_0) = \Pi(\lambda_0, \mathbf{1})$, where λ_0 is the unique quadratic unramified character of E^\times .

It follows from Iwasawa decomposition that the corresponding $\text{Ind}_B^G(\mu_0)$ has one dimensional invariants by any given maximal open compact subgroup K of G . The following proposition states, depending on K , whether the K -invariant line belongs to π_2 or maps non-trivially to π_n . We recall that K° and K' are the two standard maximal compact subgroups, K° being the hyperspecial one, and that the standard Iwahori subgroup I equals $K^\circ \cap K'$.

Proposition 1.6. *One has $\dim \pi_2^{K'} = \dim \pi_n^{K^\circ} = 1$, if μ is unramified, and $\pi_2^I = \pi_n^I = \{0\}$, otherwise.*

Proof. Applying the Jacquet functor to the exact sequence (2) allows one to see that both π_2 and π_n have non-trivial I -invariants if μ is unramified, and none, otherwise. Assuming henceforth that $\mu = \mu_0$ is unramified, we observe that both π_2^I and π_n^I must be 1-dimensional. Moreover, as by Cartan decomposition G is generated by K and γ , hence by K° and K' , it follows that necessarily one amongst π_2^I and π_n^I is fixed by K° , while the other one is fixed by K' .

By Iwasawa decomposition, the restriction of $\pi = \text{Ind}_B^G(\mu_0)$ to K is given by $\text{Ind}_{B \cap K}^K(\mu_0)$, hence the line π^K admits a basis f_K characterized by asking its restriction to K to be $\mathbf{1}_K$. Moreover, the line $(\pi^\vee)^K$ admits a basis f_K^\vee characterized by

$$\langle f, f_K^\vee \rangle = \frac{1}{\sqrt{\text{vol}(K)}} \int_K f(k) dk.$$

The remainder of the proof consists in computing the bi- K -invariant function $g \mapsto \langle g \cdot f_K, f_K^\vee \rangle$ and checking whether it belongs or not to $L^2(K \backslash G/K)$. Using Cartan decomposition $G = \coprod_{n \geq 0} K \gamma^n K$ this amounts to checking whether $L^2(\mathbf{Z}_{\geq 0})$ contains the numerical sequence

$$\sqrt{\text{vol}(K \gamma^n K)} \langle \gamma^n \cdot f_K, f_K^\vee \rangle = \sqrt{[K : (K \cap \gamma^{-n} K \gamma^n)]} \int_K f_K(k \gamma^n) dk.$$

Using Iwahori decomposition for all $n \geq 1$ we have $[K : (K \cap \gamma^{-n} K \gamma^n)]/[K : I] = p^{4n-3}$ (resp. p^{4n-1}), where $K = K^\circ$ (resp. K'). Since $(\mu \delta^{1/2})(\gamma) = -p^3$ we have $f_K \in \pi_2$ if, and only if,

$$(5) \quad (\Phi_n^K)_n \in L^2(\mathbf{Z}_{\geq 0}), \text{ where } \Phi_n^K = p^{5n} \cdot \int_K f_K(\gamma^{-n} k \gamma^n) dk.$$

The proof of Proposition is then completed by the following Lemma. □

Lemma 1.7. *The quantity $p^n \cdot \Phi_n^{K'}$ is independent of $n \geq 1$, in particular $(\Phi_n^{K'})_n \in L^2(\mathbf{Z}_{\geq 0})$.*

Proof. The last row of an element $k \in K'$ is given by $(0, 0, 1) \cdot k = (p \cdot c_1(k), p \cdot c_2(k), c_3(k))$ with $c_2(k) \in \mathcal{O}$ and $(c_1(k), c_3(k)) \in (\mathcal{O} \times \mathcal{O}) \setminus (\mathcal{P} \times \mathcal{P})$. For $j \geq 0$ we let

$$K'_j = \left\{ k \in K' \mid c_1(k) \in \mathcal{P}^j \right\} \text{ and } K_j'^{\times} = K'_j \setminus K'_{j+1}.$$

To compute the above integral we use the partition $K' = K_0'^{\times} \coprod K_1'^{\times} \coprod \cdots \coprod K_{2n-1}'^{\times} \coprod K_{2n}'$.

First, we compute the volume of K'_j , for $j \geq 1$. Using Iwahori decomposition one finds that:

$$[K' : K'_j] = \frac{[K' : I]}{[K'_j : I \cap K'_j]} [I : I \cap K'_j] = \frac{[K' : I]}{[K' \cap N : I \cap N]} [I \cap N^- : K'_j \cap N^-] = c_0 \cdot p^{j+2[\frac{j}{2}]}.$$

Next we observe that by Iwasawa decomposition, for all $k \in K'_{2n}$ one has $c_2(k) \in \mathcal{P}^n$, i.e. $\gamma^{-n} k \gamma^n \in N \cdot K'$, and therefore $f_K(\gamma^{-n} k \gamma^n) = 1$.

Using again Iwasawa decomposition, one checks that for $0 \leq j \leq 2n-1$ and for every $k \in K_j'^{\times}$ one has $p^{n-j} c_2(k) \in \mathcal{O}^\times$, hence $\gamma^{-n} k \gamma^n \in \gamma^{j-2n} N \cdot K'$ and $f_K(\gamma^{-n} k \gamma^n) = (-p^3)^{j-2n}$.

$$\begin{aligned} \text{Therefore } \frac{1}{\text{vol}(K')} \int_{K'} f_K(\gamma^{-n} k \gamma^n) dk &= \frac{1}{[K' : K_{2n}']} + p^{-6n} \sum_{j=0}^{2n-1} (-1)^j p^{3j} \frac{1}{[K' : K_j'^{\times}]} = \\ &= c_0 \cdot p^{-6n} \left(p^{2n} + c_0^{-1} - \sum_{i=1}^n p^{6i-3} (p^{-4i+3} - p^{-4i}) + \sum_{i=0}^{n-1} p^{6i} (p^{-4i} - p^{-4i-1}) \right) = p^{-6n} (1 + c_0). \quad \square \end{aligned}$$

Remark 1.8. Alternatively, one could use MacDonald's formula for zonal spherical functions to see that π_2 does not admit non-zero vectors fixed by the hyperspecial maximal open compact subgroup K° . Indeed, [15, §5.5] allows to express $\Phi_n^{K^\circ}$ from (5), up to a non-zero constant, as

$$\Gamma_\mu \cdot \mu(\gamma^{-n}) + \Gamma_{\mu^w} \cdot \mu^w(\gamma^{-n}), \text{ where } \Gamma_\nu = \frac{1 - p^2 \cdot \nu(\gamma^{-2})}{1 - \nu(\gamma^{-2})} \cdot \frac{1 - p^2 \cdot \nu(\gamma^{-1})}{1 - \nu(\gamma^{-1})},$$

where the two factors in Γ_ν correspond respectively to the positive roots ζ and 2ζ of G (see §1.1). As $\mu(\gamma^{-1}) = \mu^w(\gamma) = -p^{-1}$, one has $\Gamma_\mu = 0 \neq \Gamma_{\mu^w}$, hence the sequence $(\Phi_n^{K^\circ})_{n \geq 0}$ is not L^2 . This also shows, in passing, that π_n is not square integrable.

As a consequence we obtain the following lower bound, in the unramified case.

Corollary 1.9. *For $r \geq 1$ and for $\pi \in \Pi(\lambda_0)$, one has $\dim(\pi^{I_{2r}}) \geq r + 1$.*

Proof. By Proposition 1.6 for all $r \in \mathbf{Z}$, π_2 contains a (unique) line fixed by $\gamma^r K' \gamma^{-r}$, having $\gamma^r \cdot f$ as basis, and moreover, the stabilizer in G of that line is $\gamma^r K' \gamma^{-r}$. We claim that the vectors $f, \gamma \cdot f, \dots, \gamma^r \cdot f \in \pi_2$ are linearly independent. Indeed, if f was a linear combination of the remaining r vectors then it would be fixed by $\cap_{1 \leq j \leq r} \gamma^j K' \gamma^{-j}$ of G which is *not* contained in K' . As any of these $(r + 1)$ vectors is fixed by I_{2r} , the claim follows for π_2 .

Arguing the exact same way, using K° instead of K' , proves the statement for π_n . \square

While using the unramified A -packet $\Pi(\lambda_0)$ would be sufficient in our global applications when $D \equiv 3 \pmod{8}$, the case of discriminants $D \equiv 7 \pmod{8}$ would require the use of certain tamely ramified A -packets $\Pi(\lambda)$ and providing explicit levels for them is the object of the next subsection.

1.5. Intertwining and an exponential sum. Our arithmetic applications will require to show existence of non-zero I_r -invariants, for some $r \in \mathbf{Z}_{\geq 1}$, in certain ramified A -packets. This is delicate because of the lack of new-vector theory for non-generic representations (see Remark 1.12). Also Casselman's result asserting that π^{I_r} surjects onto $\pi_N^{T \cap I_r}$ is inconclusive here as the latter vanishes, contrarily to [10] where the open compact is a pro- p -Iwahori subgroup of a sufficiently deep level (see §4.3 where these results are discussed). We will instead resort to explicit methods to prove in Proposition 1.10 that π_n has non-zero invariants K_T , which contains a conjugate of I_3 . It is relatively straightforward to determine all such vectors f in the full induced representation but it becomes a thorny issue to find a non-square integrable matrix coefficient $\langle g \cdot f, f^\vee \rangle$. By another result of Casselman, matrix coefficients can be expressed in terms of the corresponding ones in the Jacquet module, given here by an explicit scalar product on $\mathbf{C} \cdot \mu \oplus \mathbf{C} \cdot \mu^w$. Making this actually work requires non-vanishing under the Jacquet functor which, once verified, leads directly to the result we seek. The computation of the Jacquet functor is first reduced to a precise statement about the intertwining operator at the level of finite reductive groups. It involves showing non-vanishing of some explicit exponential sums, bringing out the arithmetic nature of the problem. Although these sums seem extremely hard to be computed individually, we manage to conclude by evaluating an average corresponding to the trace of finite intertwining.

Theorem 1.10. *Assume that p odd and E/\mathbf{Q}_p unramified. Let λ be a character of E^\times sending p to -1 and whose restriction to \mathcal{O}^\times equals χ_E , where $\chi : \mathcal{O}^1 \rightarrow \mathbf{F}_{p^2}^1 \rightarrow \mathbf{C}^\times$ is a (non-trivial) tamely ramified character. Letting π_n denote the non-tempered representation of the Arthur packet $\Pi(\lambda)$, one has $\dim \pi_n^{K_T} = 1$ where $K_T = \begin{pmatrix} \mathcal{O}^\times & p\mathcal{O} & p\mathcal{O} \\ p\mathcal{O} & \mathcal{O}^\times & p\mathcal{O} \\ p\mathcal{O} & p\mathcal{O} & \mathcal{O}^\times \end{pmatrix} \cap G$.*

Lemma 1.11. *One has*

$$B \backslash G / K_T = \overline{B} \backslash \overline{G} / \overline{T} = \left\{ \mathbf{1}, w, \overline{[0, 1]} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \xi & 0 & 1 \end{pmatrix}, \sigma_y = \overline{[1, y]} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \xi y - \frac{1}{2} & 1 & 1 \end{pmatrix} \mid y \in \mathbf{F}_p \right\}.$$

Moreover, for any non-trivial χ , the K_T -invariants in $\text{Ind}_B^G(\mu)$ are supported by the double cosets of $\{\sigma_y, y \in \mathbf{F}_p\}$ and, in addition if χ is quadratic, by the double coset of $\overline{[0, 1]}$.

Proof. Mackey's Theorem and Frobenius Reciprocity yield that the dimension of K_T -invariants in $\text{Ind}_B^G(\mu)$ equals the number of cosets $[\sigma] \in B \backslash G / K_T$ such that μ has trivial restriction to $B \cap \sigma K_T \sigma^{-1} \supset Z$. As μ is ramified this is never the case for $\sigma = \mathbf{1}$, nor for $\sigma = w$, while $\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \xi & 0 & 1 \end{pmatrix}$ works if and only if χ is quadratic.

If $b = \begin{pmatrix} \bar{\alpha} & * & * \\ 0 & 1 & * \\ 0 & 0 & \alpha^{-1} \end{pmatrix} \in \sigma K_T \sigma^{-1}$, for some $y \in \mathbf{F}_p$, performing a matrix multiplication shows that

$$\sigma_y^{-1} b \sigma_y \equiv \begin{pmatrix} * & * & * \\ (\bar{\alpha} - 1) & * & * \\ ? & (\alpha^{-1} - 1) & * \end{pmatrix} \pmod{p} \in K_T,$$

hence $\alpha \equiv 1 \pmod{p}$, ensuring the triviality of μ on $B \cap \sigma_y K_T \sigma_y^{-1}$.

In summary, the dimension of $(\text{Ind}_B^G(\mu))^{K_T}$ is $p + 1$ for χ quadratic, and p otherwise. \square

Remark 1.12. Recall that π_n is the Langlands quotient of $\text{Ind}_B^G(\mu)$ whose other Jordan-Hölder constituent is π_2 . As taking invariants by an open compact subgroup is an exact functor in the category of admissible representations, Lemma 1.11 implies that

$$\dim \pi_n^{K_T} + \dim \pi_2^{K_T} = p(1).$$

To show that $\pi_n^{K_T} \neq \{0\}$ one could try computing $\dim \pi_2^{K_T}$ as π_2 is a generic discrete series. Unfortunately Miyauchi's theory [19, 20] of conductors for $\text{U}(3)$ -representations with respect to the paramodular groups $K_r = \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} & p^{-r}\mathcal{O} \\ p^r\mathcal{O} & \mathcal{O}^\times & \mathcal{O} \\ p^r\mathcal{O} & p^r\mathcal{O} & \mathcal{O}^\times \end{pmatrix} \cap G$ predicts that the ramified π_n and π_c have no level, *i.e.* they have no invariants by K_r for any r , while the level of π_2 is given by its conductor. When χ is the quadratic character, the L -parameter $(\lambda \otimes \text{St}) \oplus \mathbf{1}$ has conductor 2, therefore the generic member π_2 in this L -packet has one dimensional invariants by K_2 , hence also by $\gamma^{-1} K_2 \gamma = \begin{pmatrix} \mathcal{O}^\times & p\mathcal{O} & \mathcal{O} \\ p\mathcal{O} & \mathcal{O}^\times & p\mathcal{O} \\ \mathcal{O} & p\mathcal{O} & \mathcal{O}^\times \end{pmatrix} \cap G \supset K_T$. For other ramified, χ 's π_2 has an invariant line by K_3 which has same volume as K_T but is not conjugated to it, thus non-settling the non-vanishing of $\pi_2^{K_T}$ let alone computing its dimension.

Recall the Jacquet functor given by

$$\text{Ind}_B^G(\mu) \longrightarrow \mathbf{C} \cdot \mu \oplus \mathbf{C} \cdot \mu^w, \quad f \mapsto (f(1), (Mf)(1))$$

where the standard intertwining operator $M : \text{Ind}_B^G(\mu) \rightarrow \text{Ind}_B^G(\mu^w)$ is defined via analytic continuation, as follows. For $s \in \mathbf{C}$, letting $\mu_s = \mu \cdot \delta^{s/2}$, the intertwining operator

$$M_s : \text{Ind}_B^G(\mu_s) \rightarrow \text{Ind}_B^G(\mu_s^w), \quad f \mapsto \int_N f(wn \cdot) dn$$

is absolutely convergent for $\Re(s) \gg 0$ and G -equivariant. Moreover, for any section $f_s \in \text{Ind}_B^G(\mu_s)$ such that for all $g \in G$ the function $f_s(g)$ is analytic in $s \in \mathbf{C}$, the function $(M_s(f_s))(g)$, a priori only defined for $\Re(s) \gg 0$, is a rational function in p^{-s} , hence extends to a meromorphic function on all of \mathbf{C} with only possibly a finite number of poles independent of f and of g . In fact, it continues as an intertwining operator, *i.e.*

$$(M_s(f_s))(gg') = (M_s(f_s(\cdot g')))(g), \text{ for all } g, g' \in G.$$

We refer to [1, §1] for more detail and proofs, and we will only use that

$$M(f_0) = \lim_{s \rightarrow 0} M_s(f_s)$$

computing explicitly the right hand side as a rational function in p^{-s} , simultaneously justifying that M_s does not have a pole at $s = 0$.

The Lemma 1.11 implies the existence of a non-zero element $f_y \in (\text{Ind}_B^G(\mu))^{K_T}$ supported on $B\sigma_y K_T$, which we normalize by $f_y(\sigma_y) = 1$. Consider a K° -flat section $f_{y,s}$ passing thru $f_{y,0} = f_y$, and computing $M_s(f_{y,s})$ for $\Re(s) \gg 0$.

Lemma 1.13. *For all $y, y' \in \mathbf{F}_p$, we have:*

$$M_s(f_{y',s})(\sigma_y) = \chi(-1)p^{-1}(1-p^{-1})(1-\mu_s(\gamma))^{-1}.$$

Thus, to complete the proof, it suffices to show that $\pi_n^{I_2} \neq \{0\}$ or equivalently find $y \in \mathbf{F}_p^\times$ such that $f_y \notin \pi_2$.

Although we only solve this question in the case of a unitary group in 3 variables, we feel that it deserves to be studied in greater generality for its own sake.

2. GALOIS REPRESENTATIONS FOR 3-FOLDS OF PICARD TYPE

From this point onwards, we will use global notations. The local results of the previous section can be applied to the completion E of M at any prime number which does not split in that field.

We denote by \mathbf{A}_f the ring of finite adeles of \mathbf{Q} , so that $\mathbf{A} = \mathbf{A}_f \times \mathbf{R}$.

2.1. Abelian 3-folds of Picard type and Tate modules. Let k be any field containing M . Consider an abelian 3-fold A/k together with an injection $\iota^0 : M \hookrightarrow \text{End}^0(A/k) = \text{End}(A/k) \otimes \mathbf{Q}$, or equivalently with an injection ι of an order of M into $\text{End}(A/k)$, the most important for us case being when ι^0 comes from $\iota : \mathcal{O}_M \hookrightarrow \text{End}(A/k)$.

The action of M splits the 3-dimensional k -vector space $\text{Lie}(A/k)$ in a direct sum of two sub-spaces: one on which the actions of M and k agree, and one on which they differ by the

complex conjugation. We say that A is of truly of Picard type if the pair of dimensions of these spaces, called the signature, equals $(2, 1)$.

A polarization on A is an isogeny $\theta : A \rightarrow A^\vee$, where A^\vee denotes the dual abelian variety. By positivity, since k is a field, the Rosati involution induced by θ on $\iota(\mathcal{O})$ is given by the complex conjugation (see [21, §21]). A polarization is called principal, if it is an isomorphism, and can always be acquired over a finite extension of k .

To define a level structure on A we need to consider its Tate module. Given a place v of k , the v -adic Tate module $T_v A = \varprojlim_r A[v^r]$ of A is free of rank 3 over \mathcal{O}_v . Denote $V_v A = M_v \otimes_{\mathcal{O}_v} T_v A$. One also considers the adelic Tate module

$$V_f A = \mathbf{Q} \otimes_{\mathbf{Z}} \varprojlim_n A[n],$$

which is free of rank 3 over $\mathbf{A}_{M,f}$. Given a polarization $\theta : A \rightarrow A^\vee$, the Weil pairing endows $V_f A$ with a non-degenerate skew-hermitian form, *i.e.*, a non-degenerate alternating pairing

$$\langle \cdot, \cdot \rangle_A : V_f A \times V_f A \rightarrow \mathbf{A}_f$$

such that $\langle a \cdot v, v' \rangle_A = \langle v, \bar{a} \cdot v' \rangle_A$ for all $a \in M$. If θ is principal, then $\langle \cdot, \cdot \rangle_A$ is a perfect pairing.

2.2. Shimura surfaces and families of abelian threefolds of Picard type. Let $(V, \langle \cdot, \cdot \rangle)$ be a 3-dimensional (non-degenerate) hermitian space over M . The corresponding unitary similitude group $\tilde{G} = \text{GU}(V)$ is a reductive group over \mathbf{Q} such that for any \mathbf{Q} -algebra R one has:

$$\tilde{G}(R) = \{g \in \text{GL}(V \otimes_{\mathbf{Q}} R) \mid \forall v, v' \in V \otimes_{\mathbf{Q}} R, \langle g(v), g(v') \rangle = \nu(g) \langle v, v' \rangle\},$$

where $\nu : \text{GU}(V) \rightarrow \mathbf{G}_{m,\mathbf{Q}}$ is a homomorphism whose kernel is the unitary group $G = \text{U}(V)$.

Note that any hermitian form in 3 variables over a non-archimedean local field is isotropic, hence $\tilde{G}(\mathbf{Q}_p)$ is unique up to isomorphism, while at infinity $\langle \cdot, \cdot \rangle$ is uniquely determined by its signature, hence there are only two possibilities for $\tilde{G}(\mathbf{R})$ (as opposite signatures define isomorphic groups). Hasse's Principle applied to the semi-simple simply connected derived group $\tilde{G}^1 = \text{SU}(V)$ implies then that, up to an isomorphism, there exists a unique quasi-split unitary group, denoted $\text{GU}_{2,1}$, and a unique definite unitary group, denoted $\text{GU}_{3,0}$.

We will now define the Shimura variety for the unitary similitude group $\tilde{G} = \text{GU}_{2,1}$ represented by the matrix $\begin{pmatrix} & & 1 \\ & \sqrt{-D} & \\ -1 & & \end{pmatrix}$. The one for $\text{GU}_{1,2}$ is its complex conjugate. The homomorphism of \mathbf{R} -algebraic groups

$$\tilde{h} : \text{Res}_{\mathbf{R}}^{\mathbf{C}} \mathbf{G}_{m,\mathbf{R}} \rightarrow \tilde{G}_{\mathbf{R}}, z \mapsto \begin{pmatrix} \Re(z) & 0 & \Im(z) \\ 0 & z & 0 \\ -\Im(z) & 0 & \Re(z) \end{pmatrix}$$

satisfy the Shimura datum axioms for \tilde{G} , hence for any open compact subgroups \tilde{K} of $\tilde{G}(\mathbf{A}_f)$ one can consider the Shimura surface

$$Y_{\tilde{K}}(\mathbf{C}) = \tilde{G}(\mathbf{Q}) \backslash \left(\mathcal{H} \times G(\mathbf{A}_f) / \tilde{K} \right),$$

where $\mathcal{H} \simeq \tilde{G}(\mathbf{R}) / \tilde{K}_\infty$ is identified with the $\tilde{G}(\mathbf{R})$ -conjugacy classes of \tilde{h} . By a fundamental result of Shimura $Y_{\tilde{K}}$ admits a canonical model over the reflex field M . As we will see, the connected components of $Y_{\tilde{K}}$ are Picard modular surfaces, justifying the terminology.

For \tilde{G} anisotropic, can analogously define Shimura sets which are finite and therefore will not alter the uniformity of our results in §4.

The Shimura surfaces of Picard type are coarse moduli spaces of abelian 3-folds of Picard type. Namely, $Y_{\tilde{K}}(\mathbf{C})$ is in bijection with isogeny classes of $(A, \iota^0, \theta, \eta \circ \tilde{K})$, where (A, ι^0, θ) is a polarized abelian variety of Picard type over \mathbf{C} , and $\eta : \mathbf{A}_f \otimes_{\mathbf{Q}} V \xrightarrow{\sim} V_f A$ is an isomorphism sending $\langle \cdot, \cdot \rangle_A$ to a \mathbf{A}_f^\times -multiple of $\langle \cdot, \cdot \rangle_V$. Note that the usual \mathbf{Q}^\times -multiple condition is automatically satisfied as we are in the type C case (**provide reference**). When \tilde{K}° is the standard maximal open compact subgroup of $G(\mathbf{A}_f)$, the points of $Y_{\tilde{K}^\circ}(k)$ correspond to isomorphism classes of principally polarized abelian 3-folds over k having multiplication by \mathcal{O}_M .

Henceforth, we will only consider abelian 3-folds which are principally polarized and admit multiplication by \mathcal{O}_M , and we will refer to them simply as being of *Picard type*.

It must be noted that, even though each point of $Y_{\tilde{K}}(\mathbf{C})$ is associated to an abelian 3-fold of Picard type, there does not exist such a family over the entire $Y_{\tilde{K}}(\mathbf{C})$ unless there is no point with extra automorphisms, in which case $Y_{\tilde{K}}(\mathbf{C})$ would be a fine moduli space. In our cases of interest \tilde{K} is not neat, and therefore $[S_{\tilde{K}}]$ is not a fine moduli space.

However, given an open compact subgroup \tilde{K} , we claim that there is an abelian family of Picard type A over any open subset U of $Y_{\tilde{K}}$ which contains no point with a non-trivial stabilizer. By [17, §2.3.4], the moduli stack $S_{\tilde{K}}$ associated to this problem is an algebraic stack (for the étale topology), locally of finite type over the base which we may take to $\text{Spec}(M)$. Moreover, by [17, §A.7.5], there is a canonical surjective morphism ϕ from $S_{\tilde{K}}$ to the associated coarse moduli space $[S_{\tilde{K}}]$, which in our notations is $Y_{\tilde{K}}$. By [17, §7], $[S_{\tilde{K}}]$ is an algebraic space and even a quasi-projective scheme. Moreover, by a general property of moduli stacks (see [22, Chap. 7]), ϕ is an isomorphism over the locus U where there is *no non-trivial automorphism*, by which we mean it has no infinitesimal automorphism; analytically, this corresponds to points of $Y_{\tilde{K}}(\mathbf{C})$ having no non-trivial stabilizers. Now U is a priori an open subscheme of $[S_{\tilde{K}}]$, but since it is where ϕ is an isomorphism, we get a canonical open $j : U \rightarrow S_{\tilde{K}}$. This map tautologically yields the desired family $f : A \rightarrow U$ of abelian varieties of Picard type, whose existence is essential for our proof of the main results.

In analogy with Gross' index 2 subgroups of maximal compacts K_p° at ramified primes introduced in §1.3, we consider the open compact subgroup

$$(6) \quad \tilde{K}'' = \tilde{K}_D'' \prod_{p \nmid D} \tilde{K}_p^\circ \subset \tilde{G}(\mathbf{A}_f),$$

where \tilde{K}_D'' is defined as the kernel of the composed homomorphism

$$(7) \quad \prod_{p \mid D} \tilde{K}_p^\circ \twoheadrightarrow \prod_{p \mid D} \tilde{K}_p^\circ / \tilde{K}_p'' = \prod_{p \mid D} \{\pm 1\} \xrightarrow{\Pi} \{\pm 1\}.$$

Let (A, ι, θ) be a principally polarized abelian 3-fold of Picard type over k . For v the prime of M above $p \mid D$, the action of the absolute Galois group Gal_k on $A[v]$ factors through $\text{GO}(3, \mathbf{F}_p)$. Using the exceptional isomorphism $\text{PGO}(3, \mathbf{F}_p) \xrightarrow{\sim} \text{SO}(3, \mathbf{F}_p) \xrightarrow{\sim} \text{PGL}_2(\mathbf{F}_p)$, one defines its projectivization

$$(8) \quad \tilde{\rho}_{A,p} : \text{Gal}_k \rightarrow \text{PGL}_2(\mathbf{F}_p).$$

Taking quotient by the unique index two subgroup $\text{PSL}_2(\mathbf{F}_p)$ of $\text{PGL}_2(\mathbf{F}_p)$ yields a canonical homomorphism $\varepsilon_{A,p} : \text{Gal}_k \rightarrow \{\pm 1\}$ and we let $\varepsilon_{A,D} = \prod_{p \mid D} \varepsilon_{A,p} : \text{Gal}_k \rightarrow \{\pm 1\}$.

Note that, for any open compact subgroup \tilde{K} , A defines a k -rational point in $Y_{\tilde{K}}$ if, and only if, the Galois representation on the adelic Tate module has image in \tilde{K} . Hence a point in $Y_{\tilde{K}''}(k)$ corresponds precisely to an abelian 3-fold A over k of Picard type having trivial $\varepsilon_{A,D}$.

2.3. Étale fundamental groups and Mumford–Tate groups. Let k be a number field containing M over which the connected component of $Y_{\tilde{K}}$ are defined, and fix a connected component Y of $Y_{\tilde{K}} \times_M k$ and a smooth open U of Y endowed with an abelian scheme $f : A \rightarrow U$ of Picard type. Denote by η the generic point of the smooth surface U . Fixing a closed geometric point \bar{x} of U the étale fundamental group sits in the middle of a short exact sequence

$$(9) \quad 1 \rightarrow \Pi_1(U_{\bar{k}}, \bar{x}) \rightarrow \Pi_1(U, \bar{x}) \rightarrow \text{Gal}_k = \Pi_1(\{x\}, \bar{x}) \rightarrow 1.$$

The morphism $f : A \rightarrow U$ being proper and smooth, one can consider the étale sheaf $R^1 f_* \mathbf{Z}_\ell$ on U . As U is geometrically connected we have $\Pi_1(U, \bar{x}) \simeq \Pi_1(U, \bar{\eta})$ and the latter acts on

$$(R^1 f_* \mathbf{Z}_\ell)_{\bar{\eta}} = H^1(A_{\bar{\eta}}, \mathbf{Z}_\ell) = (T_\ell A_\eta)^\vee,$$

yielding a continuous representation

$$\text{Gal}(\bar{\eta}/\eta) \twoheadrightarrow \Pi_1(U, \bar{x}) \xrightarrow{\rho_{U,\ell}} \text{Aut}_{\mathbf{Z}_\ell}(T_\ell A_\eta).$$

Any closed point $x \in U(k)$ yields a section $s_x : \text{Gal}_k \rightarrow \Pi_1(U, \bar{x})$ of (9) allowing one to consider

$$\rho_{x,\ell} = \rho_{U,\ell} \circ s_x : \text{Gal}_k \rightarrow \text{Aut}_{\mathbf{Z}_\ell}(T_\ell A_\eta).$$

Finally for any closed curve $C \subset U$ defined over k , there is a natural map $\Pi_1(C, \bar{x}) \rightarrow \Pi_1(U, \bar{x})$ whose composition with $\rho_{U,\ell}$ is denoted $\rho_{C,\ell}$. As $f : A \rightarrow U$ is of Picard type, for any $x \in C(k)$

$$\Gamma_x = \text{im}(\rho_{x,\ell}) \subset \Gamma_C = \text{im}(\rho_{C,\ell}) \subset \Gamma_U = \text{im}(\rho_{U,\ell}) \subset K_\ell^\circ.$$

By a series of results of Cadoret–Tamagawa (see [3, 4]), the set C_ρ of all $x \in C(k)$ for which Γ_x is not open in Γ_C is finite and for all $x \in C(k) \setminus C_\rho$ the index $[\Gamma_C : \Gamma_x]$ is uniformly bounded.

The Mumford–Tate group $\text{MT}(A)$ of a polarized abelian variety A over \mathbf{C} is the smallest connected reductive subgroup of $\text{GL}(H_1(A, \mathbf{Q}))$ over \mathbf{Q} , whose \mathbf{R} -points contain the associated \mathbf{R} -morphism $h : \mathbf{C}^* \rightarrow \text{GL}(H_1(A(\mathbf{C}), \mathbf{R}))$ coming from the Hodge decomposition. If we assume further that A is defined over a number field $k \subset \mathbf{C}$ finitely generated over \mathbf{Q} , then the image Γ_ℓ of Gal_k acting on $(T_\ell A)$ is an ℓ -adic Lie group. By a theorem of Deligne [8, Chap. I.2], we have $\text{Lie}(\Gamma_\ell)_{\mathbf{Q}_\ell} \subset \text{Lie}(\text{MT}(A) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell)$ and the Mumford–Tate conjecture, known for abelian varieties of dimension at most 3, asserts that they are equal (see *e.g.* [5]).

As the Mumford–Tate group of the (generic point of the) universal family $f : A \rightarrow U$ is given by $\tilde{G} = \text{GU}_{2,1}$, it follows from the above discussion that the Mumford–Tate group of any abelian 3-fold of Picard type is a reductive subgroup of \tilde{G} . We have the following trichotomy.

Lemma 2.1. *Let \mathfrak{g} be a reductive Lie subalgebra of $\mathfrak{gu}(3, k)$ defined over a characteristic 0 field k . If $\mathfrak{g}' \subset \mathfrak{su}(3, k)$ denotes the semi-simple part of \mathfrak{g} , exactly one of the following holds:*

- (i) $\mathfrak{g}' = \{0\}$, *i.e.* \mathfrak{g} is abelian,
- (ii) \mathfrak{g}' is a form of $\mathfrak{sl}(2, k)$,
- (iii) $\mathfrak{g}' = \mathfrak{su}(3, k)$.

Proof. If $\mathfrak{g}'_{\mathbf{C}} = \{0\}$, then $\mathfrak{g}' = \{0\}$, whereas if $\mathfrak{g}'_{\mathbf{C}} = \mathfrak{sl}(3, \mathbf{C})$, then $\mathfrak{g}' = \mathfrak{su}(3, k)$ for dimension reasons. In the remaining cases, using the well known fact that any proper non-zero semi-simple Lie subalgebra of $\mathfrak{sl}(3, \mathbf{C})$ is isomorphic to $\mathfrak{sl}(2, \mathbf{C})$, we deduce that \mathfrak{g}' is a form of $\mathfrak{sl}(2, k)$. \square

If A is (potentially) of CM type (resp. admits (potentially) a non-trivial CM quotient), then its Mumford–Tate group is of the first (resp. second) type. It is natural to ask whether Theorem B could be further refined for abelian 3-folds without non-trivial CM quotients.

2.4. Galois stable lattices and rationality. Suppose henceforth that k is a number field. Let (A, ι, θ) be a polarized abelian 3-fold of Picard type over k , and let $\eta : \mathbf{A}_f \otimes_{\mathbf{Q}} V \xrightarrow{\sim} V_f A$ be an isomorphism sending $\langle \cdot, \cdot \rangle_A$ to a \mathbf{A}_f^\times -multiple of $\langle \cdot, \cdot \rangle_V$. As $\text{MT}(A, \iota, \theta) \subset \tilde{G}$ by Deligne [8, Cor. 6.2], the action of Gal_k on the adelic Tate module $V_f A$, together with the choice of η , yields a continuous homomorphism:

$$\rho_{A,f} : \text{Gal}_k \longrightarrow \tilde{G}(\mathbf{A}_f).$$

Moreover, the point $(A, \iota, \theta, \eta \circ \tilde{K})$ on $Y_{\tilde{K}}(\mathbf{C})$ is defined over k if, and only if, $\rho_{A,f}(\text{Gal}_k) \subset \tilde{K}$.

Given any prime number ℓ , the resulting continuous homomorphism:

$$\rho_\ell = \rho_{A,\ell} : \text{Gal}_k \longrightarrow \tilde{G}(\mathbf{Q}_\ell),$$

has compact image, which is thus necessarily contained in some maximal compact \tilde{K}_ℓ of $\tilde{G}(\mathbf{Q}_\ell)$.

We will denote by $\bar{\rho}_\ell$ the composition of ρ_ℓ with the natural surjection of \tilde{K}_ℓ onto its reductive quotient \bar{G}_ℓ . Then $\bar{\rho}_\ell$ acts on $\mathcal{L}_\ell \otimes_{\mathbf{Z}_\ell} \mathbf{F}_\ell$, where \mathcal{L}_ℓ is a $\mathcal{O} \otimes \mathbf{Z}_\ell$ -lattice whose stabilizer in $\tilde{G}(\mathbf{Q}_\ell)$ is \tilde{K}_ℓ . While in general $\bar{\rho}_\ell$ depends on the choice of \tilde{K}_ℓ , or equivalently on the choice of a Gal_k -stable lattice \mathcal{L}_ℓ , it follows from a Theorem of Brauer and Nesbitt that its semi-simplification is independent of these choices.

If ℓ splits in M then $G(\mathbf{Q}_\ell) = \text{GL}_3(\mathbf{Q}_\ell)$ and $\bar{\rho}_\ell : \text{Gal}_k \longrightarrow \widetilde{\text{GL}}_3(\mathbf{F}_\ell)$ is absolutely irreducible if, and only if, there exists a unique, up to homothecy, $\rho_\ell(\text{Gal}_k)$ -stable \mathbf{Z}_ℓ -lattice \mathcal{L}_ℓ .

If ℓ does not split in M , then $G(\mathbf{Q}_\ell)$ has rank 1 and every edge of the corresponding Bruhat-Tits tree links a vertex with reductive quotient \bar{G}_ℓ° to a vertex with reductive quotient \bar{G}'_ℓ (see §1.1). As $\rho_\ell(\text{Gal}_k)$ acts on the tree by isometries, if it fixes any two (or more) vertices, then it necessarily fixes an edge, hence its image in the reductive quotient of any fixed vertex would be reducible. Conversely, since no irreducible subgroup of \bar{G}_ℓ° or of \bar{G}'_ℓ does fix an adjacent vertices, one can characterize the representations ρ_ℓ fixing a unique vertex as follows.

Note that θ yields an integral pairing on $H_1(A(\mathbf{C}), \mathbf{Z})$ inducing a pairing on $T_\ell A \simeq H_1(A(\mathbf{C}), \mathbf{Z}_\ell)$ for each ℓ . If ℓ does not divide the degree of θ , then $T_\ell A$ is self-dual, and one can chose $\tilde{K}_\ell \simeq \tilde{K}_\ell^\circ$.

Lemma 2.2. *Let (A, ι, θ) be a polarized abelian 3-fold of Picard type over a number field k .*

- (i) *$\bar{\rho}_\ell$ is absolutely irreducible if, and only if, there exists a unique, up to homothecy, $\rho_\ell(\text{Gal}_k)$ -stable lattice. The latter is necessarily self-dual and \tilde{K}_ℓ is conjugated to \tilde{K}_ℓ° .*
- (ii) *Suppose ℓ be a prime that does not split in M . Then $\bar{\rho}_\ell(\text{Gal}_k)$ is an irreducible subgroup of \bar{G}'_ℓ if, and only if, there exists a unique, up to homothecy, pair of $\rho_\ell(\text{Gal}_k)$ -stable lattices. The latter are almost self-dual and \tilde{K}_ℓ is conjugated to \tilde{K}'_ℓ .*
- (iii) *If θ is principal, then one can chose $\tilde{K} \simeq \tilde{K}^\circ$, i.e., (A, ι, θ) defines a k -rational point on $Y_{\tilde{K}^\circ}$.*

2.5. Lie images of Galois representations. The Lie algebra $\mathfrak{h} \subset \mathfrak{gu}(3, \mathbf{Z}_\ell)$ of the image of $\rho_{A,\ell}$ is algebraic and $\mathfrak{h}_{\mathbf{Q}_\ell} = \mathbf{Q}_\ell \otimes_{\mathbf{Z}_\ell} \mathfrak{g}_{\mathbf{Z}_\ell}$ is reductive by Faltings [11, Thm. 3].

Serre [29, C.3.7] has defined an integral model of the Mumford–Tate group and refined Deligne’s theorem to show that its Lie algebra $\mathfrak{h}_{\mathbf{Z}_\ell}$ contains $\mathfrak{g}_{\mathbf{Z}_\ell}$ as a subgroup. The integral Mumford–Tate conjecture, known for abelian varieties of dimension ≤ 3 , asserts that the image is an open subgroup.

Theorem 2.3. *Let A be an abelian 3-fold of Picard type defined over a field k . Then $\rho_{A,\ell}$ is potentially reducible if, and only if, A has a non-trivial CM quotient.*

Proof. Suppose that $\mathfrak{g}_{\mathbf{Q}_\ell}$ is a proper subalgebra of $\mathfrak{gu}(3, \mathbf{Q}_\ell)$. It is, by Proposition 2.1, either abelian or contains $\mathfrak{su}(3, \mathbf{Q}_\ell)$ or its semisimple part is a form of $\mathfrak{sl}(2, \mathbf{Q}_\ell)$. In the second case, as it contains homotheties by Bogomolov [2], it must contain $\mathfrak{su}(3, \mathbf{Q}_\ell) \oplus \mathbf{Q}_\ell$. In each of the remaining cases, after extending scalars to $\overline{\mathbf{Q}_\ell}$, $\rho_{A,\ell}$ is potentially reducible, *i.e.*, after possibly replacing k by a finite extension, $\rho_{A,\ell}$ contains (as a direct factor by Faltings) a character $\chi_\ell : k^\times \backslash \mathbf{A}_k^\times \rightarrow \overline{\mathbf{Q}_\ell}^\times$. As a sub-representation of $\rho_{A,\ell}$, χ_ℓ is unramified outside a finite set of places and its restriction decomposition groups at places above ℓ is Hodge-Tate with weights belonging to $\{0, -1\}$. In addition it is pure of weight -1 . By Minkowski's proof of the Dirichlet unit theorem, χ_ℓ corresponds to an algebraic Hecke character $\chi : k^\times \backslash \mathbf{A}_k^\times \rightarrow \mathbf{C}^\times$ whose infinity component is necessarily of the form $N_{\Phi'} \circ N_{k/L'}$ where $N_{\Phi'}$ is the partial norm given by a CM type Φ' for a CM field $L' \subset k$. By [31, Lem. 2] replacing (L', Φ') by its double reflex yields the same infinite component, hence we may and do assume that (L', Φ') is a primitive, *i.e.*, coincides with the reflex of a CM field L endowed with a CM type Φ . Further replacing k by a finite (abelian) extension one can assume that χ_f takes values in L^\times . By Casselman (see [31, Thm. 6]), there exists an abelian variety B defined over $k \supset L'$ which is CM of type (L, Φ) and such that $\rho_{B,\ell} = \chi_\ell$, hence

$$\mathrm{Hom}_{\mathrm{Gal}_k}(\rho_{A,\ell}, \rho_{B,\ell}) \neq \{0\}.$$

By Faltings one deduces that $\mathrm{Hom}_k(A, B) \neq \{0\}$, hence A contains a non-trivial CM quotient A' . One can assume that $A' \neq A$, hence there exists an abelian variety A'' which is not of CM type and such that A is isogenous to $A' \times A''$, *i.e.*, $V_{A,\ell} = V_{A',\ell} \oplus V_{A'',\ell}$. Furthermore since $\mathrm{Hom}_k(A', A'') = \{0\}$, one can show that the isogeny is \mathcal{O} -linear. Hence A'' admits multiplication by \mathcal{O} and since it is not of CM type, it has dimension 2, from which one deduces that $\mathfrak{h}_{\mathbf{Z}_\ell} = \mathfrak{g}(\mathfrak{u}(2) \times \mathfrak{u}(1), \mathbf{Z}_\ell)$.

Then either $\rho_{A'',\ell}$ is Lie surjective, in which case $\mathfrak{g}_{\mathbf{Z}_\ell} = \mathfrak{h}_{\mathbf{Z}_\ell}$, or else $\rho_{A'',\ell}$ is reducible which by repeating the above argument would contradict A'' not being of CM type. \square

Lemma 2.4. *Let \tilde{K} be an open compact subgroup of $\tilde{G}(\mathbf{A}_f)$ and let $x \in Y_{\tilde{K}}(k)$ be such that $\mathrm{MT}(x) \neq \tilde{G}$. Then x belongs to a special subvariety defined over k .*

Proof. By the classification in §2.3, $\tilde{H} = \mathrm{MT}(x)$ is either isomorphic to a form of $\mathbf{G}(\mathrm{U}(2) \times \mathrm{U}(1))$ or a torus. Moreover by a theorem of Deligne [8], x belongs to the image of the canonical morphism of Shimura varieties $Y_{\tilde{H} \cap \tilde{K}}^{\tilde{H}} \rightarrow Y_{\tilde{K}}$. It remains to see that $Y_{\tilde{H} \cap \tilde{K}}^{\tilde{H}}$ is defined over k . This is clear if \tilde{H} is a torus, as then $Y_{\tilde{H} \cap \tilde{K}}^{\tilde{H}}$ is a finite set of points which are Galois conjugates. In the remaining case $Y_{\tilde{H} \cap \tilde{K}}^{\tilde{H}}$ is a Shimura curve and for any $\sigma \in \mathrm{Gal}_k$ a theorem of Kazhdan ensures that $\sigma(Y_{\tilde{H} \cap \tilde{K}}^{\tilde{H}})$ is also a Shimura subvariety of Y_K containing x . If $\sigma(Y_{\tilde{H} \cap \tilde{K}}^{\tilde{H}}) \neq Y_{\tilde{H} \cap \tilde{K}}^{\tilde{H}}$ then x would belong a smaller Shimura subvariety, namely $\sigma(Y_{\tilde{H} \cap \tilde{K}}^{\tilde{H}}) \cap Y_{\tilde{H} \cap \tilde{K}}^{\tilde{H}}$, contradicting the fact that $\tilde{H} = \mathrm{MT}(A)$. \square

3. IRREGULARITY FOR PICARD MODULAR SURFACES

We denote by $q(X)$ the irregularity of a projective algebraic surface X over \mathbf{C} , given by the dimension of $H^1(X, \mathcal{O}_X)$. If X is smooth and projective then $q(X) = \dim H^0(X, \Omega_X^1)$.

3.1. A lemma on surfaces with isolated singularities. Let X be a projective irreducible algebraic surface over \mathbf{C} with isolated singularities, *i.e.*, such that there exists a smooth open $j : U \hookrightarrow X$ whose complement $Z = X \setminus U$ consists of finitely many closed points. There exists a smooth resolution

$$\phi : \tilde{X} \rightarrow X$$

such that $\phi^{-1}(Z)$ is a divisor with normal crossings with ϕ restricting to an isomorphism from $\phi^{-1}(U)$ onto U . Thus we get an injection $\tilde{j} : U \hookrightarrow \tilde{X}$ such that $j = \phi \circ \tilde{j}$, and we denote by

$$\tilde{j}^* : H^1(\tilde{X}, \mathbf{Q}) \rightarrow H^1(U, \mathbf{Q})$$

the pullback homomorphism on cohomology. By [6, Thm.3.2.5(iii)] we know that \tilde{j}^* is a homomorphism of mixed Hodge structure, with $H^1(\tilde{X}, \mathbf{Q})$ being pure of weight 1.

Lemma 3.1. *The map \tilde{j}^* is an isomorphism, in particular $H^1(U, \mathbf{Q})$ is a pure weight 1 Hodge structure and $q(X) = \dim H^0(U, \Omega_U^1)$.*

Proof. Let $IH^\bullet(X, \mathbf{Q})$ denote the middle intersection cohomology of X . Since

$$\dim(X) - 1 > 0 = \dim(Z)$$

by [9, Thm.5.4.12] $j^* : IH^1(X, \mathbf{Q}) \rightarrow IH^1(U, \mathbf{Q})$ is an isomorphism, while \tilde{j}^* is injective. Moreover by Cor.5.4.11 and Prop.5.4.4 in *loc.cit.* $\phi^* : IH^1(X, \mathbf{Q}) \rightarrow IH^1(\tilde{X}, \mathbf{Q}) = H^1(\tilde{X}, \mathbf{Q})$ is an embedding, while $IH^1(U, \mathbf{Q}) = H^1(U, \mathbf{Q})$. This is summarized in the following commutative diagram:

$$\begin{array}{ccc} IH^1(X, \mathbf{Q}) & \xrightarrow[\sim]{j^*} & IH^1(U, \mathbf{Q}) \\ \phi^* \downarrow & & \parallel \\ H^1(\tilde{X}, \mathbf{Q}) & \xrightarrow{\tilde{j}^*} & H^1(U, \mathbf{Q}) \end{array}$$

It immediately follows that \tilde{j}^* is an isomorphism and $q(\tilde{X}) = \dim H^0(\tilde{X}, \Omega_{\tilde{X}}^1) = \dim H^0(U, \Omega_U^1)$. Finally $q(X) = q(\tilde{X})$ as the irregularity is a birational invariant. \square

3.2. A formula for the irregularity. Let $z \mapsto \bar{z}$ be the non-trivial automorphism of M/\mathbf{Q} . Put $M^1 = \{z \in M^\times \mid z\bar{z} = 1\}$, which we will view as an algebraic torus over \mathbf{Q} and denote by \mathbf{A}_M^1 its adelic points.

For \tilde{K} an open compact subgroup of $\tilde{G}(\mathbf{A}_f)$ we recall the Shimura variety of Picard type defined as the adelic quotient

$$(10) \quad Y_{\tilde{K}} = \tilde{G}(\mathbf{Q}) \backslash \tilde{G}(\mathbf{A}) / \tilde{K} \tilde{K}_\infty.$$

Let $G^1 = \mathrm{SU}(V)$ be the derived group of \tilde{G} . As G^1 is simply connected and G_∞^1 is not compact, the Strong Approximation Theorem (see [23, Thm. 7.12]) implies that $G^1(\mathbf{Q})$ is dense in $G^1(\mathbf{A}_f)$. It follows that the determinant map defines an isomorphism between the group of connected components $\pi_0(Y_{\tilde{K}})$ and the idele class group $\mathbf{A}_M^\times / M^\times \det(\tilde{K}) M_\infty^\times$. Further, Shimura's theory of canonical models implies that the connected components of $Y_{\tilde{K}}$ are all Galois conjugates, hence share the same irregularity, and the same is true for the Shimura variety $Y_K = G(\mathbf{Q}) \backslash G(\mathbf{A}) / KK_\infty$ for G , where $K = \tilde{K} \cap G(\mathbf{A}_f)$. Letting

$$(11) \quad \Gamma = \tilde{G}(\mathbf{Q}) \cap \tilde{K} \tilde{G}(\mathbf{R}),$$

it follows from $\nu(\Gamma) \subset \mathbf{Q}^\times \cap \hat{\mathbf{Z}}^\times \hat{\mathbf{R}}_+^\times = \{1\}$ that both $Y_{\tilde{K}}$ and Y_K share the same connected component of identity given by $Y_\Gamma = \Gamma \backslash \mathcal{H}$ (see [10, (8)]). One should be careful to observe that the natural dominant map $Y_{K^1} \rightarrow Y_\Gamma$, where Y_{K^1} is the Shimura variety of level $K^1 = K \cap G^1(\mathbf{A}_f)$ for G^1 is an isomorphism precisely when, either $\det(\Gamma) = \{1\}$, or $-1 \in \Gamma$.

Proposition 3.2. *The irregularity of any connected component of the minimal compactification $Y_{\tilde{K}}^*$ of $Y_{\tilde{K}}$ is given by the formula*

$$(12) \quad q(Y_\Gamma^*) = \sum_{(\lambda, \nu) \in \Xi / \pi_0(\widehat{Y_K})} \sum_{\pi_f \in \Pi_f(\lambda, \nu)} \dim(\pi_f^K) \frac{1 + W(\lambda \nu_M)(-1)^{s(\pi_f)}}{2}, \text{ where}$$

- Ξ is the set of pairs (λ, ν) of a unitary Hecke character λ of M whose restriction to \mathbf{Q} is $(\frac{\cdot}{D})$, and of a unitary character ν of \mathbf{A}_M^1 / M^1 , such that

$$\lambda_\infty(z) = \frac{\bar{z}}{|z|}, \text{ for all } z \in M_\infty^\times \simeq \mathbf{C}^\times, \text{ and } \nu_\infty(z) = z, \text{ for all } z \in M_\infty^1,$$

- $\Pi_f(\lambda, \nu)$ is the finite part of a global Arthur packet for G (see §1.2),
- $W(\lambda \nu_M) \in \{\pm 1\}$ is the global root number, where $\nu_M(z) = \nu(\bar{z}/z)$ for $z \in \mathbf{A}_M^\times$,
- $s(\pi_f)$ the number of finite places v at which $\pi_v \simeq \pi_c(\lambda_v, \nu_v)$, and
- $\mu \in \pi_0(\widehat{Y_K})$ acts freely on Ξ by sending (λ, ν) to $(\lambda \mu_M^{-1}, \nu \mu)$.

Proof. We show that Y_Γ^* admits only isolated singularities and we first observe that the complement of Y_Γ in Y_Γ^* consists of finitely many points, the cusps. A singular point of Y_Γ which is not an elliptic point, is necessarily a fixed point of a single complex reflexion (an order 2 element in Γ fixing a hyperbolic plane). Although the universal cover $\mathcal{H} \rightarrow Y_\Gamma$ is not étale at such a point, this is still a smooth point on the quotient, as locally in the analytic geometry the complex reflexion sends (τ, z) to $(\tau, -z)$ (see [16, §4.5] for more details and additional material). Thus there exists a smooth open U_K of the normal projective surface Y_K^* whose complement consists of finitely many closed points. Lemma 3.1 applied component-wise to U_K yields $q(Y_K^*) = \dim H^0(U_K, \Omega_{U_K}^1)$. Let K' be any normal finite index torsion free subgroup of

K , *e.g.* the intersection with the principal congruence subgroup of level 3 (see [10, Lem. 1.4]). By Koecher's Principle, as $Y_{K'} \setminus U_{K'}$ has codimension at least 2 in $Y_{K'}$, we have

$$\dim H^0(U_K, \Omega_{U_K}^1) = \dim H^0(U_{K'}, \Omega_{U_{K'}}^1)^{K/K'} = \dim H^0(Y_{K'}, \Omega_{Y_{K'}}^1)^{K/K'},$$

where $U_{K'}$ is the inverse image of U_K under the natural projection $Y_{K'} \rightarrow Y_K$. Taking invariants by the finite group K/K' in Rogawski's formula [10, (14)] for $q(Y_{K'}^*) = \dim H^0(Y_{K'}, \Omega_{Y_{K'}}^1)$ yields

$$\dim H^0(U_K, \Omega_{U_K}^1) = \sum_{(\lambda, \nu) \in \Xi} \sum_{\pi_f \in \Pi(\lambda_f, \nu_f)} \dim(\pi_f^K) \frac{1 + W(\lambda\nu_M)(-1)^{s(\pi_f)}}{2}.$$

One should note a misprint in *loc. cit.* where one should read $(1 + W(\lambda\nu_M)(-1)^{s(\pi_f)})$ instead of $(W(\lambda\nu_M) + (-1)^{s(\pi_f)})$. The presence of this root number translates the fact that for $\pi_f \in \Pi(\lambda_f, \nu_f)$ and π_∞ the unique non-tempered holomorphic representation in the local Arthur packet $\Pi(\lambda_\infty, \nu_\infty)$, $\pi = \pi_f \otimes \pi_\infty$ is automorphic if, and only if, $W(\lambda\nu_M) = (-1)^{s(\pi_f)}$. Both $\dim(\pi_f^K)$ and $1 + W(\lambda\nu_M)(-1)^{s(\pi_f)}$ being preserved by the action of $\pi_0(\widehat{Y_K})$, one deduces the desired formula for $q(Y_K^*)$ as in [10, (15)]. \square

3.3. Twists of canonical characters and root numbers. Hecke characters $(\lambda, \nu) \in \Xi$ whose local components at each finite place have ‘minimal’ ramification are intimately related to the canonical characters studied by Gross and Rohrlich. They play a pivotal role in our production of automorphic forms contributing to the irregularity of the Picard modular surfaces of low level. We will now briefly recall some of their properties under the assumption that $D > 3$ is odd. Consider the character $\lambda_\infty(z) = \bar{z} \cdot |z|^{-1}$ of $M_\infty^\times \simeq \mathbf{C}^\times$ and let $\lambda_f : \widehat{\mathcal{O}}_M^\times \rightarrow \mathbf{C}^\times$ be a continuous character whose restriction to $\mathcal{O}_{M,p}^\times$ is given by the unique quadratic character

$$\mathcal{O}_{M,p}^\times \rightarrow \mathbf{F}_p^\times \xrightarrow{\left(\frac{\cdot}{p}\right)} \{\pm 1\},$$

for all p dividing D , and is trivial otherwise. As $\left(\frac{-1}{D}\right) = -1$, it follows that λ_∞ and λ_f agree on $\mathcal{O}_M^\times = \{\pm 1\}$. The finiteness of the class group $\mathcal{C}_M = \mathbf{A}_M^\times / M^\times \widehat{\mathcal{O}}_M^\times M_\infty^\times$ guarantees that the resulting character of $M^\times \widehat{\mathcal{O}}_M^\times M_\infty^\times$ extends to a character λ of \mathbf{A}_M^\times and clearly two such extensions must differ by a class character. As by construction the restriction of λ_f to $\widehat{\mathbf{Z}}^\times$ agrees with the quadratic Dirichlet character $\left(\frac{\cdot}{D}\right)$ viewed as a character of $\mathbf{A}^\times / \mathbf{Q}^\times \text{Nm}(\mathbf{A}_M^\times) = \text{Gal}(M/\mathbf{Q})$, and $\mathbf{A}^\times = \mathbf{Q}^\times \widehat{\mathbf{Z}}^\times \mathbf{R}_+^\times$, it follows that the restriction of λ to \mathbf{A}^\times equals $\left(\frac{\cdot}{D}\right)$ (*i.e.* λ is conjugate-symplectic). Such a character is called a canonical character and we will denote it by λ_c , remembering that it is only unique up to a multiplication by a character of \mathcal{C}_M . The root number $W(\lambda_c^3) = -W(\lambda_c) = \left(\frac{-2}{D}\right)$ is 1 if, and only if, $D \equiv 3 \pmod{8}$ (see [27]).

Assume henceforth that $\det(K) = \widehat{\mathcal{O}}_M^1$, so that $\pi_0(Y_K) = \mathcal{C}_M^1$. Assuming further that $\widehat{\mathcal{O}}_M^1$ embeds centrally into K , the central character $\omega = \nu \cdot \lambda|_{M^1}$ of any π contributing to (12) has to

be everywhere unramified, *i.e.*,

$$(13) \quad q(Y_\Gamma^*) = \frac{1}{|\mathcal{C}\ell_M^1|} \sum_{\chi \in \Xi^1, \omega \in \widehat{\mathcal{C}\ell_M^1}} \sum_{\pi_f \in \Pi_f(\lambda_c \chi_M, \lambda_{c|M^1}^{-1} \chi^2 \omega)} \dim(\pi_f^K) \frac{1 + W(\lambda_c^3 \chi_M^3)(-1)^{s(\pi_f)}}{2},$$

where Ξ^1 denotes the set of finite order characters of \mathbf{A}_M^1/M^1 (see [27, (3)] for the fact that multiplication by a class character does not change the root number).

If 3 does not divide $|\mathcal{C}\ell_M^1|$, then the action of $\mu \in \widehat{\mathcal{C}\ell_M^1} = \mathbf{A}_M^1/M^1 \widehat{\mathcal{O}}_M^1 M_\infty^1$ sending (χ, ω) to $(\chi\mu^{-1}, \omega\mu^3)$ allows to twist out the central character and obtain the simpler formula:

$$(14) \quad q(Y_\Gamma^*) = \sum_{\chi \in \Xi^1} \sum_{\pi_f \in \Pi_f(\lambda_c \chi_M)} \dim(\pi_f^K) \frac{1 + W(\lambda_c^3 \chi_M^3)(-1)^{s(\pi_f)}}{2},$$

where $\Pi_f(\lambda)$ is a short-hand for $\Pi_f(\lambda, \lambda_{|M^1}^{-1})$. Proving this formula in general amounts to showing that $\dim(\pi_f^K)$ remains unchanged when multiplying λ or ν by class characters, which we will later check in all cases of interest.

Successfully applying (14) requires to understand how root numbers behave under twisting. As we are interested in creating irregularity at level $\Gamma_0''(\ell^r)$, we focus on characters χ which are only ramified at the fixed inert prime ℓ .

Lemma 3.3. *Assuming that χ_M^3 has Artin conductor ℓ^a , we have*

$$W(\lambda_c^3 \chi_M^3) = (-1)^a \chi_\ell(-1) W(\lambda_c^3).$$

Proof. Using the factorization of root numbers $W = \prod_v W_v$, it suffices to prove that

$$(15) \quad W_\ell(\lambda_c^3 \chi_M^3) = (-1)^a \chi_\ell(-1) W_\ell(\lambda_c^3), \text{ and}$$

$$(16) \quad W_v(\lambda_c^3 \chi_M^3) = W_v(\lambda_c^3 \chi_M^3), \text{ for all } v \neq \ell,$$

where the local factors are defined using the standard additive character $\psi_M = \psi_{\mathbf{Q}} \circ \text{Tr}_{M/\mathbf{Q}}$. Applying [28, Prop. 3] to both $\lambda_{c,\ell}^3$ and $\lambda_{c,\ell}^3 \chi_{M,\ell}^3$ yields (15). As $\chi_{M,\infty} = 1$, it suffices to check (16) for v finite. Moreover, the characters $\lambda_{c,v}^3$ and $\chi_{M,v}$ are unramified for $v \nmid \ell D$, hence both sides of (16) are 1. Finally, for v dividing D , $\chi_{M,v}$ is unramified, $\lambda_{c,v}^3$ is tamely ramified, while the additive character ψ_E has conductor 1, implying by [7, (5.5.1)] that $W_v(\lambda_c^3 \chi_M^3)$ and $W_v(\lambda_c^3)$ differ by $\chi_{M,v}^3(\xi^{1+1 \cdot 1}) = (\chi_v(-1))^2 = 1$. \square

3.4. The Bombieri–Lang Conjecture for Picard modular surfaces. In this part we follow the general strategy of [10] by adapting it to the case where the level is not neat, the main point being to show that the irregularity of the Picard modular surfaces under consideration is at least 3. This requires different techniques also providing a new proof to the cases treated in *loc. cit.*

We recall the index 2 subgroup \tilde{K}'' of the maximal open compact subgroup \tilde{K}° of $\tilde{G}(\mathbf{A}_f)$ introduced in (6). Given $r \in \mathbf{Z}_{\geq 1}$ and a prime ℓ inert in M , we let $\tilde{K}_0''(\ell^r)$ be the subgroup of

\tilde{K}'' whose component at ℓ is the depth r Iwahori subgroup I_r and we let

$$K_0''(\ell^r) = \tilde{K}_0''(\ell^r) \cap G(\mathbf{A}_f) \quad \text{and} \quad \Gamma_0''(\ell^r) = G(\mathbf{Q}) \cap K_0''(\ell^r) \cdot G(\mathbf{R}).$$

Consider an automorphic representation $\pi \in \Pi(\lambda_c \chi_M)$ having non-zero $K_0''(\ell^r)$ -invariants. By Propositions 1.4 and 1.6, for all finite place $v \neq \ell$, we have $\pi_v = \pi_{n,v}$ with χ_v unramified, and conversely it follows from (7) that the line $\bigotimes_{p|D} \pi_{n,p}^{K_D''}$ is fixed by K_D'' . By (14) that π contributes to $q(Y_{\Gamma_0''(\ell^r)}^*)$ only when $\pi_\ell = \pi_{n,\ell}$ and $W(\lambda_c^3 \chi_M^3) = 1$, or $\pi_\ell = \pi_{c,\ell}$ and $W(\lambda_c^3 \chi_M^3) = -1$. As the choice of a character $\chi_\ell : \mathbf{Z}_{\ell^2}^1 \rightarrow \mathbf{C}^\times$ uniquely determines, up to a class character, a global character χ_M unramified outside ℓ , the irregularity formula becomes:

$$(17) \quad \frac{1}{h} \cdot q(Y_{\Gamma_0''(\ell^r)}^*) = \sum_{W(\lambda_c^3 \chi_M^3)=-1} \dim(\pi_{c,\ell}^{I_r}) + \sum_{W(\lambda_c^3 \chi_M^3)=1} \dim(\pi_{n,\ell}^{I_r}),$$

and it can be rendered even more explicit using Lemma 3.3.

The study of the conductor of the super-cuspidal non-generic representation $\pi_{c,\ell}$ appears to be very delicate even in the depth 0 case (*i.e.* trivial χ), where some preliminary computations suggest that it does not have K_T -invariants. Consequently we will use the lower bound on the irregularity corresponding to the contribution of π which are the everywhere non-tempered.

Proposition 3.4. *We recall that $D > 3$ is odd and let h be the class number of $M = \mathbf{Q}(\sqrt{-D})$.*

- (i) *If $D \equiv 3 \pmod{8}$ then $q(Y_{\Gamma_0''(\ell)}^*) = q(Y_{\Gamma''}^*) = h$ and $q(Y_{\Gamma_0''(\ell^{2r})}^*) \geq (r+1)h$, for $r \in \mathbf{Z}_{\geq 1}$.*
- (ii) *If $D \equiv 7 \pmod{8}$ then $q(Y_{\Gamma_0''(27)}^*) \geq h$ and $q(Y_{\Gamma_0''(\ell^3)}^*) \geq 3h$, for $\ell \geq 7$.*

Proof. The proof is based on the inequality $(Y_{\Gamma_0''(\ell^r)}^*) = h \cdot \sum_{W(\lambda_c^3 \chi_M^3)=1} \dim(\pi_{n,\ell}^{I_r})$ resulting from (17) and the results on root numbers in §3.3.

(i) As $W(\lambda_c^3) = 1$ we can take $\chi = \mathbf{1}$. The claim follows from Proposition 1.6 and Corollary 1.9.

(ii) As $W(\lambda_c^3) = -1$ we use here a tamely ramified χ_ℓ to switch the sign. By Lemma 3.3, in order to have $W(\lambda_c^3 \chi_M^3) = -W(\lambda_c^3) = 1$, we need χ_ℓ^3 to be non-trivial and $\chi_\ell(-1) = 1$. For $\ell \geq 3$, there are precisely $\frac{\ell-1}{2}$ choices for χ_ℓ , if $3 \nmid (\ell+1)$, and $\frac{\ell-5}{2}$ choices, if $3 \mid (\ell+1)$. In particular, there are at least 3 choices for all $\ell \geq 7$. \square

The following results from Faltings [12] (see [10, §3.2] for detail).

Theorem 3.5. *Let \tilde{K} an open compact subgroup of $\tilde{G}(\mathbf{A}_f)$ and let $\Gamma = \tilde{G}(\mathbf{Q}) \cap \tilde{K}\tilde{G}(\mathbf{R})$. If $q(Y_\Gamma^*) \geq 3$, then $Y_{\tilde{K}}^*$ satisfies the Bombieri–Lang Conjecture, *i.e.*, any number field k the k -rational points of $Y_{\tilde{K}}^*$ are not Zariski dense.*

Corollary 3.6. *If $D \equiv 3 \pmod{8}$, then the Bombieri–Lang Conjecture holds for $Y_{\tilde{K}''(\ell^4)}^*$, and even for $Y_{\tilde{K}''}^*$, when $h \geq 3$. If $D \equiv 7 \pmod{8}$, then the Bombieri–Lang Conjecture holds for $Y_{\tilde{K}''(\ell^3)}^*$ for $\ell \geq 7$, and for $Y_{\tilde{K}''(37)}^*$.*

As 2 splits in M for $D \equiv 7 \pmod{8}$, we only exclude $\ell = 5$ when $D \equiv 7$ or $23 \pmod{40}$.

4. UNIFORM IRREDUCIBILITY OF GALOIS IMAGES

4.1. Complex reflexions, elliptic elements and abelian families. In this subsection we prove that, after enlarging k , there exists a natural family of abelian 3-folds of Picard type over $Y_{\tilde{K}''} \times_M k$ minus a finite number of k -rational elliptic points. This will be crucially used in the proof of Theorem B in the next subsection. Fix a geometrically connected component Y'' of $Y_{\tilde{K}''} \times_M k$. We have $Y''(\mathbf{C}) = \Gamma'' \backslash \mathcal{H}$, where $\Gamma'' = \tilde{G}(\mathbf{Q}) \cap g_f \tilde{K}'' g_f^{-1} \tilde{G}(\mathbf{R})$ for some $g_f \in G(\mathbf{A}_f)$. We will first show that Gross' level structure \tilde{K}''_D introduced in (7) prevents Γ from containing complex reflexions. Then we will classify the elliptic elements in Γ'' and deduce the sought for abelian scheme.

Recall that non-scalar element γ of the discrete subgroup $\Gamma \subset \tilde{G}(\mathbf{R})$ (*i.e.* a non-trivial element in $\bar{\Gamma} = \Gamma/(\Gamma \cap M^\times)$) has a fixed point in \mathcal{H} if, and only if, γ has finite order (this is because the stabilizers in $\tilde{G}(\mathbf{R})$ of points in \mathcal{H} are maximal compact subgroups). Such a γ is called elliptic if it only fixes an isolated point in \mathcal{H} , otherwise it is called a complex reflexions.

Lemma 4.1. *The group Γ'' does not contain any complex reflexions.*

Proof. Consider a complex reflexion $\gamma \in \tilde{G}(\mathbf{Q}) \cap g_f \tilde{K}'' g_f^{-1} \tilde{G}(\mathbf{R})$ as an endomorphism of the Hermitian space M^3 having signature $(2, 1)$. As eigenspaces for γ are mutually orthogonal, at most one such eigenspaces could contain a negative line (corresponding to a point in \mathcal{H}). This if γ fixes more that one point of \mathcal{H} , it necessarily fixes a hyperbolic line in \mathcal{H} . The corresponding endomorphism of M^3 has an eigenplane and an orthogonal eigenline (both M -rational), forcing the eigenvalues to be in $\mathcal{O}_M^\times = \{\pm 1\}$ (as $D > 4$) and not all equal. It follows, that for any $p \mid D$, the image of $g_f^{-1} \gamma g_f \in \tilde{K}''$ into the projectivization of the reductive quotient of \tilde{K}''_p belongs to the image under the adjoint isomorphism $\mathrm{PGL}_2(\mathbf{F}_p) \xrightarrow{\sim} \mathrm{PGO}(3, \mathbf{F}_p)$ of an element represented by a matrix having both eigenvalues 1 and -1 . In particular its image in $\mathrm{PGL}_2(\mathbf{F}_p)/\mathrm{PSL}_2(\mathbf{F}_p) = \{\pm 1\}$ equals $\left(\frac{-1}{p}\right)$. As $\prod_{p \mid D} \left(\frac{-1}{p}\right) = \left(\frac{-1}{D}\right) = -1$ it follows that $g_f^{-1} \gamma g_f \notin \tilde{K}''$ (see (6)), *i.e.* $\gamma \notin \Gamma''$. \square

Lemma 4.2. *The set of fixed points in $Y''(\mathbf{C})$ is finite and defined over a finite extension of M .*

Proof. One proceeds by explicitly determining the elliptic elements of Γ'' . \square

After possibly enlarging k we may assume that the set Z from Lemma 4.2 is defined over k and we denote by U the complementary open in Y'' . Following the discussion in §2.2 the open U lifts to an open in the corresponding algebraic moduli stack, as such it is naturally endowed with a family $f : A \rightarrow U$ of abelian 3-folds of Picard type.

4.2. Proof of Main Theorem. At different stages of the proof we will remove finite set and deal with them in the last step. By §2.1 and §2.4 an abelian variety A as in the Theorem B defines a k -rational point on $Y_{\tilde{K}''}$, where \tilde{K}'' is defined in (6). If $A[\ell^r]$ admits a full k -rational

flag (or equivalently, a k -rational isotropic line), we claim that A defines a k -rational point on $Y_{\tilde{K}_0''(\ell^r)}$, where $\tilde{K}_0''(\ell^r) \subset \tilde{K}''$ is the subgroup whose component at ℓ consists of elements whose reduction modulo ℓ^r belong to the standard Borel subgroup $\tilde{B}(\mathbf{Z}/\ell^r\mathbf{Z})$ of $\tilde{G}(\mathbf{Z}/\ell^r\mathbf{Z})$. Indeed, by assumption one knows that there is some Borel subgroup containing the image of Gal_k acting on $A[\ell^r]$. However $\tilde{G}(\mathbf{Q}_\ell)$ acts transitively on isotropic lines (because isometries between hermitian subspaces always extend), hence all Borel subgroups are conjugated by $\tilde{G}(\mathbf{Q}_\ell)$, and in fact by $\tilde{K}_\ell^\circ = \tilde{G}(\mathbf{Z}_\ell)$ (using Iwasawa decomposition). As \tilde{K}'' is a normal subgroup of \tilde{K}° we deduce that the Galois image is contained in $\tilde{K}'' \cap \tilde{K}_0(\ell^r) = \tilde{K}_0''(\ell^r)$, proving the claim.

By Corollary 3.6, $Y_{\tilde{K}_0''(\ell^4)}^*$ satisfies the Strong Bombieri–Lang Conjecture. In particular all its k -rational points lie in a subvariety Z defined over k which is a finite union of points and curves.

Let us now take one of the (finitely many) geometrically connected curve C in Z , and after removing finitely many of its points (which would not affect the wanted result), we may assume that C is contained in the smooth open U from §4.1. In particular, there exists a family $f : A \rightarrow C$ of abelian 3-folds of Picard type.

Consider image $\Gamma_C \subset \tilde{K}_\ell^\circ = \tilde{G}(\mathbf{Z}_\ell)$ of the étale fundamental group acting on the ℓ -adic Tate module of the generic fiber of the family. By Cartan’s theorem (see [30, LG5.42]), Γ_C is an ℓ -adic Lie group hence admits a Lie algebra \mathfrak{h} . By Bogomolov [2] the Lie algebra $\mathfrak{h}_{\mathbf{Q}_\ell}$ is the Lie algebra of the Zariski closure of Γ_C in $\tilde{G}(\mathbf{Q}_\ell)$, the latter being furthermore reductive over \mathbf{Q}_ℓ by Faltings [11, Thm. 3]. By the Mumford–Tate Conjecture, which is known for abelian 3-folds (see *e.g.* [5]), we know that $\mathfrak{h}_{\mathbf{Q}_\ell}$ is the Lie algebra of the Mumford–Tate group $\text{MT}(A) \otimes \mathbf{Q}_\ell$. As C has positive dimension, it has to contain non-CM points, whose Mumford–Tate group is not abelian. By Lemma 2.1, the Lie subalgebra $\mathfrak{h}_{\mathbf{Q}_\ell} \subset \mathfrak{su}(3, \mathbf{Q}_\ell)$ contains a form of $\mathfrak{sl}(2, \mathbf{Q}_\ell)$. By [4, Thm. 1.1] applied to the abelian family $f : A \rightarrow C$ there exist $B > 0$ such that for all $x \in C(k)$ outside a finite set C_ρ we have

$$(18) \quad [\Gamma_C : \Gamma_x] \leq B,$$

where $\Gamma_x = \rho_{A_x, \ell}(\text{Gal}_k)$ with A_x the abelian 3-fold of Picard type corresponding to x .

Lemma 4.3. *There exists $r = r(C) \in \mathbf{Z}$ such that $\Gamma_x \supset \exp(\mathfrak{su}(2, \ell^r \mathbf{Z}_\ell))$ for all $x \in C(k) \setminus C_\rho$.*

Proof. We fix an exponential map on $\mathfrak{su}(2, \mathbf{Q}_\ell)$ so that $\Gamma_C \supset \exp(\mathfrak{su}(2, \mathbf{Z}_\ell))$. Using that a subgroup of index at most B contains a normal subgroup of index at most $B!$, (18) implies that Γ_x contains $B! \cdot \exp(\mathfrak{su}(2, \mathbf{Z}_\ell)) = \exp(\mathfrak{su}(2, \ell^r \mathbf{Z}_\ell))$, where r is the ℓ -adic valuation of $B!$. \square

It follows that $r \geq r(C)$ and for all A as above corresponding to a k -rational point on $C \setminus C_\rho$, $A[\ell^r]$ does not admit a full k -rational flag. Finally, a direct application of Theorem 2.3 to the finitely many remaining k -rational points yields an integer r such that all k -rational points in $Y_{\tilde{K}_0''(\ell^r)}$ are of CM type, proving the Theorem.

4.3. Odd and ends. In a previous paper, we mistakenly switched π_c and π_2 (which are in the same L -packet) in the proof of Proposition 3.8 in [10] which also had consequences for the assertions of Theorem 0.3 in *loc. cit.*. Providing I_T -invariants for its supercuspidal member π_c seem to be a thorny issue. Even in the simplest case of $\pi_c \in \Pi(\lambda_0)$ which can be shown to be of depth 0, *i.e.* coming from representation theory of finite groups, preliminary computation via Deligne–Lusztig theory suggest that it does not have K_T -invariants. This leaves us with no choice but to use π_n .

The methods of this paper based on ramified π_n allow us to find a new proof of part (ii) of Theorem 0.3 in *loc. cit.*, in fact yielding an even stronger result. The case (i) will be discussed elsewhere.

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