

# Kummer-faithfulness over $p$ -adic fields

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## Abstract

The notion of a Kummer-faithful field, defined by Mochizuki, is expected as one of suitable base fields for anabelian geometry. In this paper, we study Kummer-faithfulness for algebraic extension fields of  $p$ -adic fields. We show that Kummer-faithfulness for such fields are deeply related with various finiteness properties on torsion points of (semi-)abelian varieties. For example, a Galois extension  $K$  of a  $p$ -adic field is Kummer-faithful with finite residue field if and only if, for any finite extension  $L$  of  $K$  and any abelian variety over  $L$ , its  $L$ -rational torsion subgroup is finite. In addition, we study Kummer-faithfulness for Lubin-Tate extension fields.

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## 1 Introduction

Anabelian geometry is an area of arithmetic geometry that studies how much information concerning the geometry of certain geometric objects, so-called “anabelian varieties”, can be reconstructed from various data associated with their arithmetic fundamental groups. The philosophy of anabelian geometry was first suggested by Grothendieck in *Esquisse d’un Programme* and *Brief an G. Faltings* (cf. [SL97]), and he proposed that anabelian geometry should be considered over fields that are finitely generated over their prime fields. Nowadays, Grothendieck’s original conjecture for hyperbolic curves over finitely generated

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fields over the prime field has been proved by Nakamura ([Nak90b], [Nak90a]), Tamagawa ([Tam97]), Mochizuki ([Moc96], [Moc07]) and Stix ([Sti02a], [Sti02b]).

However, led by Mochizuki, it has been revealed that anabelian geometry can be developed over a broader class of fields. *Kummer-faithful fields*, defined by Mochizuki [Moc15] and the main topic of this paper, form one such class. A perfect field  $K$  is Kummer-faithful if, for every finite extension  $L$  of  $K$  and every semi-abelian variety  $A$  over  $L$ , the Mordell-Weil group  $A(L)$  has a trivial divisible part; see Definition 2.2. (Precisely, in [Moc15], Kummer-faithful fields are assumed to be of characteristic zero. In [Hos17], Kummer-faithfulness are defined also for positive characteristic cases.) Kummer-faithfulness asserts the injectivity of the Kummer map associated with semi-abelian varieties; thus, roughly speaking, Kummer-faithfulness guarantees that “Kummer theory for semi-abelian varieties works effectively”. Typical examples of Kummer-faithful fields are finitely generated fields over  $\mathbb{Q}$ , which are related to the Grothendieck’s original setting. Moreover, *sub-p-adic fields* are Kummer-faithful (here, a field is sub- $p$ -adic if it is isomorphic to a finitely generated field over  $\mathbb{Q}_p$ ). As one of the important results related to the Grothendieck conjecture over Kummer-faithful fields, Hoshi proved in [Hos17] that a “point-theoretic” and “Galois-preserving” isomorphism between the étale fundamental groups of affine hyperbolic curves over Kummer-faithful fields arises from an isomorphism of schemes.

A study of various properties of Kummer-faithfulness, together with the construction of examples of Kummer-faithful fields, is important for understanding the range of fields over which anabelian geometry can be developed. Moreover, in recent years, studying these topics has become a research interest in its own right. Let  $k$  be a number field and  $K/k$  an algebraic extension. In [OT22, Corollary 2.15], Taguchi and the author studied Kummer-faithfulness of  $K$  in terms of ramification theory. We showed that  $K$  is Kummer-faithful if  $K/k$  is a Galois extension with “finite maximal ramification break everywhere” (cf. Definition 2.14 of *loc. cit.*). As a typical example, the field obtained by adjoining to  $\mathbb{Q}$  all  $\ell$ -th roots of unity for all prime  $\ell$  is Kummer-faithful. Ohtani [Oht22] and Asayama-Taguchi [AT25] studied Kummer-faithfulness for extremely large fields. Let  $G$  be the absolute Galois group of  $k$  and  $e$  a positive integer. For  $\sigma = (\sigma_1, \dots, \sigma_e) \in G^e$  denote by  $\bar{k}(\sigma)$  the fixed field of  $\sigma$  in  $\bar{k}$ , and by  $\bar{k}[\sigma]$  the maximal Galois subextension of  $k$  in  $\bar{k}(\sigma)$ . It is known that  $\bar{k}[\sigma]$  is Kummer-faithful for almost all  $\sigma \in G^e$  (in terms of the (normalized) Haar measure); see [Oht22, Corollary 1] for the case where  $e \geq 2$ , and [AT25, Theorem 5.3] for the general case. Moreover, if  $e \geq 2$ ,  $\bar{k}(\sigma)$  is Kummer-faithful for almost all  $\sigma \in G^e$  (cf. [AT25, Theorem 5.2]). The structure of the Mordell-Weil group of (semi-)abelian varieties over  $\bar{k}(\sigma)$  or  $\bar{k}[\sigma]$  satisfies interesting properties; see Section 3 of [AT25] for more information. On the other hand, Murotani showed in [Mur23a] that an algebraic extension field  $\mathbb{F}$  over  $\mathbb{F}_p$  is Kummer-faithful if and only if the absolute Galois group of  $\mathbb{F}$  is isomorphic to  $\hat{\mathbb{Z}}$ . It should be notable that he also studied Kummer-faithfulness for higher local fields (cf. [Mur23b]).

In this paper, we study Kummer-faithfulness for algebraic extension fields of  $p$ -adic fields (= finite extension fields of  $\mathbb{Q}_p$ ). Since sub- $p$ -adic fields are Kummer-faithful, we know that  $p$ -adic fields are Kummer-faithful (but this can be checked immediately from the main theorem of [Mat55]). If we restrict our attention to tamely ramified (or unramified) Galois extensions  $K$  of some  $p$ -adic field, we will see that Kummer-faithfulness has a simple interpretation; in fact, for such  $K$ ,  $K$  is Kummer-faithful if and only if  $K/\mathbb{Q}_p$  is quasi-finite (see Corollaries 3.4 and 3.7). Thus our main interest is Kummer-faithfulness

for algebraic extensions of  $p$ -adic fields with infinite wild ramification. For this, we focus on Lubin-Tate extension fields of  $p$ -adic fields. For a power  $q$  of  $p$ , we say that  $\alpha$  is a *Weil  $q$ -integer* if  $|\iota(\alpha)| = \sqrt{q}$  for every embedding  $\iota: \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$ . We denote by  $W(q)$  the set of Weil  $q$ -integers.

**Theorem 1.1** (A part of Theorem 4.2). *Let  $k$  be a  $p$ -adic field with residue field  $\mathbb{F}_q$  and  $\pi$  a uniformizer of  $k$ . Denote by  $k_\pi$  the Lubin-Tate extension field of  $k$  associated with  $\pi$ .*

- (1) *If  $k$  is a Galois extension of  $\mathbb{Q}_p$  and  $k_\pi$  is not Kummer-faithful, then either of the following holds.*
  - (a)  $q^{-1}\text{Nr}_{k/\mathbb{Q}_p}(\pi) \in \mu_{p-1}$ .
  - (b) *For some  $(r_\sigma)_{\sigma \in \Gamma_k}$  with  $r_\sigma \in \{0, 1\}$ , it holds  $\prod_{\sigma \in \Gamma_k} \sigma \pi^{r_\sigma} \in W(q)$ . Here,  $\Gamma_k$  is the set of  $\mathbb{Q}_p$ -algebra embeddings  $k \hookrightarrow \overline{\mathbb{Q}_p}$ .*
- (2) *If  $q^{-1}\text{Nr}_{k/\mathbb{Q}_p}(\pi) \in \mu_{p-1}$  or  $\text{Nr}_{k/\mathbb{Q}_p}(\pi) \in W(q)$ , then  $k_\pi$  is not Kummer-faithful.*

Applying Theorem 1.1 with  $k = \mathbb{Q}_p$ , we obtain

**Corollary 1.2.** *Assume  $k = \mathbb{Q}_p$ . Then the following are equivalent.*

- (i)  $k_\pi$  is Kummer-faithful.
- (ii)  $p^{-1}\pi \notin \mu_{p-1}$  and  $\pi \notin W(p)$ .

Similar results related to assertion (1) of Theorem 1.1 have already been studied in [Oze23, Theorem 1.1], and we will essentially follow the same arguments provided there in our proof. Assertion (2), which can be seen as a partial converse of (1), is not studied in *loc. cit.* In Section 4.2, we construct an example of non-Kummer faithful  $k_\pi$  such that the assumptions in Theorem 1.1 (2) do not hold but (b) in Theorem 1.1 (1) holds. The author has a slight hope that  $k_\pi$  is not Kummer-faithful if and only if either (a) or (b) in Theorem 1.1 (1) holds.

On the other hand, in the  $p$ -adic setting, thanks to the theory of Tate curves and non-archimedean rigid uniformization theorems, we can show that Kummer-faithfulness is equivalent to certain finiteness properties of torsion points of abelian varieties.

**Theorem 1.3** (Corollary of Theorem 3.9). *Let  $K$  be a Galois extension of a  $p$ -adic field.*

- (1) *The following are equivalent.*
  - (a)  $K$  is Kummer-faithful.
  - (b) *Any finite extension of  $K$  has only finitely many  $\ell$ -power roots of unity for every prime  $\ell$ , and the group  $A(L)[p^\infty]$  is finite for any finite extension  $L/K$  and any abelian variety  $A/L$  with good reduction.*
- (2) *The following are equivalent.*
  - (a)  $K$  is Kummer-faithful with finite residue field.
  - (b) *The torsion subgroup of  $A(L)$  is finite for any finite extension  $L/K$  and any abelian variety  $A/L$ .*

The finiteness properties such as (b) of Theorem 1.3 (2) have been studied as a standard problem in arithmetic theory of abelian varieties. It seems that many known results are for abelian varieties over number fields, but some results are also known for those over  $p$ -adic fields. It is a theorem of Imai [Ima75] that, if  $K = \mathbb{Q}_p(\mu_{p^\infty})$ , the torsion subgroup of  $A(L)$  is finite for any finite extension  $L/K$  and any abelian variety  $A/K$  with potential good reduction. This result is known to be essentially valuable in studies such as Iwasawa theory. Kubo and Taguchi [KT13] generalized Imai's theorem in the sense that Imai's theorem holds even after replacing  $\mathbb{Q}_p(\mu_{p^\infty})$  with the field  $k(k^{1/p^\infty})$  obtained by adjoining to a  $p$ -adic field  $k$  all  $p$ -power roots of all element of  $k$ . However, the fields  $K$  appearing here are not Kummer-faithful since Tate curves over  $\mathbb{Q}_p$  contain non-trivial divisible  $K$ -rational element (see also Corollary 1.2).

It would be very interesting to consider analogues of Theorem 1.3 over global fields; however, the author currently has no idea what such statements might look like.

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**Notation.** A *number field* is a finite extension of the field  $\mathbb{Q}$  of rational numbers. Let  $p$  be a rational prime. A  *$p$ -adic field* is a finite extension of the field  $\mathbb{Q}_p$  of  $p$ -adic numbers. For any field  $F$ , we fix a separable closure  $\overline{F}$  of  $F$  and we denote by  $G_F$  the absolute Galois group  $\text{Gal}(\overline{F}/F)$  of  $F$ .

## 2 General theory for Kummer-faithful fields

In this section, we recall the definition of Kummer-faithful fields (cf. [Moc15, Def. 1.5], [Hos17, Def. 1.2]) and study some standard properties for Kummer-faithfulness in general settings. Before starting the main part, we briefly recall the degree of (not necessary finite) algebraic extensions. A *supernatural number* is a formal product  $\mathbf{n} = \prod_\ell \ell^{n_\ell}$  where  $\ell$  runs over the set of primes and  $n_\ell \in \mathbb{Z} \cup \{\infty\}$ . Using the unique decomposition into prime powers, we can view any natural number as a supernatural number. The product of supernatural numbers are defined by the natural way, and also the greatest common divisors and the least common multiple of supernatural numbers. Let  $L$  be an algebraic extension of a perfect field  $K$ . We define the *extension degree*  $[L : K]$  of  $L/K$  by

$$[L : K] := \text{lcm}\{[K' : K] \mid K' \text{ is a finite extension of } K \text{ contained in } L\}$$

as a supernatural number. By Galois theory, we have a group theoretic interpretation of  $[L : K]$  as follows. Take any Galois extension  $\tilde{L}$  of  $K$  which contains  $L$ , and set  $G := \text{Gal}(\tilde{L}/K)$  and  $H := \text{Gal}(\tilde{L}/L)$ . Then we see  $[L : K] = (G : H)$ , where the right hand side is the index of profinite groups in the sense of Section 1.3 of [Ser97]. For algebraic extensions  $K \subset L \subset M$ , we have  $[M : K] = [M : L] \cdot [L : K]$ . If an algebraic extension  $L/K$  is of the form  $L = \bigcup_i K_i$  for some finite extensions  $K_i$  of  $K$ , one has  $[L : K] = \text{lcm}\{[K_i : K]\}_i$ .

**Definition 2.1.** Let  $L$  be an algebraic extension of a field  $K$  with  $[L : K] = \prod_\ell \ell^{n_\ell}$ . We say that  $L/K$  is *quasi-finite* if  $n_\ell$  is finite for any prime  $\ell$ .

Every finite extensions are clearly quasi-finite. For an algebraic extension  $\mathbb{E}$  of a finite field  $\mathbb{F}$  with  $[\mathbb{E} : \mathbb{F}] = \prod_\ell \ell^{n_\ell}$ , then one sees that  $\mathbb{E}/\mathbb{F}$  is quasi-finite if and only if  $G_{\mathbb{E}}$  is isomorphic to  $\hat{\mathbb{Z}}$ . If  $L$  is an unramified extension of a  $p$ -adic field  $K$ , then  $L/K$  is quasi-finite if and only if the residue field extension of  $L/K$  is quasi-finite.

Now let us recall the definition of Kummer-faithful fields and study some basic properties. Let  $M$  be a  $\mathbb{Z}$ -module and  $\ell$  a prime. We say that  $P \in M$  is *divisible* (resp.  $\ell$ -*divisible*) if, for any integer  $n > 0$ , there exists  $Q \in M$  such that  $P = nQ$  (resp.  $P = \ell^n Q$ ). We denote by  $M_{\text{div}}$  (resp.  $M_{\ell\text{-div}}$ ) the set of divisible (resp.  $\ell$ -divisible) elements of  $M$ , that is,

$$M_{\text{div}} = \bigcap_{n>0} nM, \quad M_{\ell\text{-div}} = \bigcap_{n>0} \ell^n M.$$

Note that  $P \in M$  is divisible if and only if it is  $\ell$ -divisible for all primes  $\ell$ . In fact, we have  $M_{\text{div}} = \bigcap_\ell M_{\ell\text{-div}}$ . If a  $\mathbb{Z}$ -module  $M$  is divisible (i.e.,  $M_{\text{div}} = M$ ), it is known (e.g., [Fuc15, Chapter 4, Theorem 3.1]) that  $M$  is isomorphic to  $\mathbb{Q}^I \oplus \left( \bigoplus_p (\mathbb{Z}[1/p]/\mathbb{Z})^{I_p} \right)$ , where  $p$  runs over the set of primes and  $I, I_p$  are some index sets.

**Definition 2.2.** A perfect field  $K$  is *Kummer-faithful* (resp. *AVKF*, resp. *torally Kummer-faithful*) if, for every finite extension  $L$  of  $K$  and every semi-abelian variety (resp. abelian variety, resp. torus)  $A$  over  $L$ , it holds that

$$A(L)_{\text{div}} = 0.$$

It is clear that any subfield of a Kummer-faithful field is also Kummer-faithful, and any finite extension of a Kummer-faithful field is also Kummer-faithful. As is remarked in [Moc15, Rem. 1.5.2], by considering the Weil restriction, one verifies immediately that one obtains an equivalent definition of Kummer-faithful, if, in Definition 2.2, one restricts  $L$  to be equal to  $K$ . The same statements holds if we replace “Kummer-faithful” with “AVKF” or “torally Kummer-faithful”.

It is shown by Mochizuki in Remark 1.5.4 of [Moc15] that any *sub-p-adic field* is Kummer-faithful. Here, recall that a field  $k$  is sub- $p$ -adic if there exists a prime  $p$  and a finitely generated field extension  $L$  of  $\mathbb{Q}_p$  such that  $k$  is isomorphic to a subfield of  $L$ . Let  $F^{\text{cyc}}$  be the maximal cyclotomic field of a number field  $F$ . Then it follows from the result of Ribet (and Proposition 2.6) that  $F^{\text{cyc}}$  is AVKF (however, it is not Kummer-faithful). Furthermore, if  $L$  is a quasi-finite Galois extension of  $F^{\text{cyc}}$ , then  $L$  is also AVKF (cf. [HMT25, Proposition 6.3 (i)]). Furthermore, it is shown by Murotani and the author in [MO25, Theorem 1.3] that the field  $F(F^{1/\infty})$  obtained by adjoining to  $F$  all roots of all elements of  $F$  is AVKF.

**Proposition 2.3.** A perfect field  $K$  is Kummer-faithful if and only if it is both torally Kummer-faithful and AVKF.

*Proof.* Suppose that  $K$  is both torally Kummer-faithful and AVKF. Let  $L$  be a finite extension of  $K$  and  $A$  a semi-abelian variety over  $L$ . There exists an exact sequence  $0 \rightarrow T \rightarrow A \rightarrow B \rightarrow 0$  of  $K$ -group schemes where  $T$  is a torus and  $B$  is an abelian variety, which induces an exact sequence  $0 \rightarrow T(L) \rightarrow A(L) \rightarrow B(L)$  of  $\mathbb{Z}$ -modules. Since  $K$  is both torally Kummer-faithful and AVKF, we know that both  $T(L)_{\text{div}}$  and  $B(L)_{\text{div}}$  are zero. Hence it follows from Lemma 2.4 below that  $A(L)_{\text{div}}$  is also zero.  $\square$

**Lemma 2.4.** *Let  $0 \rightarrow L \rightarrow M \rightarrow N$  be an exact sequence of  $\mathbb{Z}$ -modules. Assume that  $L_{\text{div}} = 0$ ,  $N_{\text{div}} = 0$  and the kernel of the  $n$ -th multiplication map on  $M$  is finite for any  $n > 0$ . Then we have  $M_{\text{div}} = 0$ .*

*Proof.* Take any  $x \in M_{\text{div}}$ . For any  $n > 0$ , we denote by  $X_n$  the set of all  $y \in M$  such that  $ny = x$ . Then  $\{X_n\}_{n>0}$  forms a projective system with transition maps  $f_{n,m}: X_m \rightarrow X_n$  given by  $f_{n,m}(z) = (m/n)z$  for  $n \mid m$ . Since each  $X_n$  is a non-empty finite set, the projective limit  $\varprojlim_n X_n$  is non-empty. Take any  $(x_n)_n \in \varprojlim_n X_n$  and denote by  $f: M \rightarrow N$  the map in the statement of the lemma. Since each  $x_n$  is a divisible element of  $M$ , we see  $f(x_n) \in N_{\text{div}}$ . This shows  $f(x_n) = 0$  and thus  $x_n \in L$ . Since  $x = nx_n \in L$  for any  $n$ , we find that  $x$  is a divisible element of  $L$ . This gives  $x = 0$  as desired.  $\square$

We study some relations between Kummer-faithfulness and finiteness of prime power torsion part of semi-abelian varieties.

**Definition 2.5.** (1) Let  $\ell$  be a prime. A perfect field  $K$  is  $\ell^\infty$ -semi-AV-tor-finite (resp.  $\ell^\infty$ -AV-tor-finite) if, for every finite extension  $L$  of  $K$  and every semi-abelian variety (resp. abelian variety)  $A$  over  $L$ , it holds that  $A(L)[\ell^\infty]$  is finite.  
(2) A perfect field  $K$  is locally semi-AV-tor-finite (resp. locally AV-tor-finite) if it is  $\ell^\infty$ -semi-AV-tor-finite (resp.  $\ell^\infty$ -AV-tor-finite) for every prime  $\ell$ .  
(3) A perfect field  $K$  is semi-AV-tor-finite (resp. AV-tor-finite) if, for every finite extension  $L$  of  $K$  and every semi-abelian variety (resp. abelian variety)  $A$  over  $L$ , it holds that  $A(L)_{\text{tor}}$  is finite.

It is helpful to the readers to refer [HMT25, Section 6] for various properties of Kummer-faithful fields, AVKF fields,  $\ell^\infty$ -AV-tor-finite fields and so on. Note that locally semi-AV-tor-finite is equivalent to  $\mathfrak{P}\text{times}^\infty$ -AV-tor-finite in the sense of [HMT25, Definition 6.1 (iv)]. Any sub- $p$ -adic field is semi-AV-tor-finite by Proposition 2.9 of [OT22]. The theorem of Ribet [KL81] shows that the maximal cyclotomic field  $F^{\text{cyc}}$  of a number field  $F$  is AV-tor-finite. Moreover, it is shown in [MO25, Corollary 1.2] that the field  $F((\mathcal{O}_F^\times)^{1/\infty})$  obtained by adjoining to  $F$  all roots of all units of the integer ring of  $F$  is also AV-tor-finite.

**Proposition 2.6.** *Let  $K$  be an algebraic extension of a field  $k$ . Let  $A$  be a semi-abelian variety (resp. abelian variety, resp. torus) over  $k$ . Consider the following conditions.*

- (a)  $A(K)_{\text{div}}$  is zero.
- (b)  $A(K)[\ell^\infty]$  is finite for any prime  $\ell$ .

*Then we have (a)  $\Rightarrow$  (b). If  $k$  is Kummer-faithful (resp. AVKF, resp. torally Kummer-faithful) and  $K$  is a Galois extension of  $k$ , then we have (a)  $\Leftrightarrow$  (b).*

*Proof.* The statement for Kummer-faithful fields follows from Proposition 2.4 of [OT22]. The arguments in *loc. cit.* proceed also for AVKF fields and torally Kummer-faithful fields.  $\square$

Here is an immediate consequence of the proposition above.

**Corollary 2.7.** (1) *A perfect field is locally semi-AV-tor-finite if it is Kummer-faithful.*  
(2) *For a Galois extension field of a Kummer-faithful field, it is locally semi-AV-tor-finite if and only if it is Kummer-faithful.*

Note that there exists a locally semi-AV-tor-finite field which is not Kummer-faithful. Let  $a > 1$  be a natural number and take a system  $(a_n)_{n>0}$  in  $\overline{\mathbb{Q}}$  such that  $a_1 = 1$  and  $a_{nm}^m = n$  for all  $n, m > 0$ . Denote by  $K$  the extension field of  $\mathbb{Q}$  obtained by adjoining all  $a_n$  for  $n > 0$ . Then,  $K$  is locally semi-AV-tor-finite but is not Kummer-faithful since  $K^\times$  contains a non-trivial divisible element  $a$ .

**Proposition 2.8.** *Let  $\square \in \{\text{semi-AV}, \text{AV}\}$  and  $\ell$  a prime. Let  $K$  be a perfect field and  $L$  a potentially prime-to- $\ell$  extension of  $K$ . If  $K$  is  $\ell^\infty$ - $\square$ -tor-finite, then  $L$  is also  $\ell^\infty$ - $\square$ -tor-finite.*

*In particular, any quasi-finite extension of a locally semi-AV-tor-finite field is also locally semi-AV-tor-finite.*

*Proof.* Let  $L'$  be a finite extension of  $L$  and  $A$  a (semi-)abelian variety over  $L'$ . Take any finite extension  $K'/K$  contained in  $L'$  such that  $A$  is defined over  $K'$ . Setting  $L'' := L'(K'(A[\ell]))$ , we know that  $L''/K'(A[\ell])$  is potentially prime-to- $\ell$  since so is  $L/K$ . Thus there exists a finite extension  $K''$  of  $K'(A[\ell])$  contained in  $L''$  such that  $L''/K''$  is prime-to- $\ell$ . Since  $K''(A[\ell^\infty])$  is a pro- $\ell$  extension of  $K''$  but  $L''$  is a prime-to- $\ell$  extension of  $K''$ , we see that the intersection  $L'' \cap K''(A[\ell^\infty])$  is equal to  $K''$ . Hence we obtain  $A(L')[\ell^\infty] \subset A(L'')[\ell^\infty] = A(K'')[\ell^\infty]$ . Since  $K$  is  $\ell^\infty$ -semi-AV-tor-finite, the finiteness of  $A(K'')[\ell^\infty]$  is assured, which gives the fact that  $A(L')[\ell^\infty]$  is finite.  $\square$

It may be helpful to write down the following implications:

$$\begin{array}{ccccc} \text{semi-AV-tor-finite} & \Longrightarrow & \text{locally semi-AV-tor-finite} & \Longrightarrow & \ell^\infty\text{-semi-AV-tor-finite} \\ \uparrow & & \uparrow (*) & & \\ \text{sub-}p\text{-adic} & \Longrightarrow & \text{KF} = \text{TKF and AVKF} & & \end{array}$$

Here, KF and TKF stand for Kummer-faithful and torally Kummer-faithful, respectively. The vertical arrow  $(*)$  is an equivalence relation for the class of Galois extension fields of Kummer-faithful fields. It follows from Theorem 3.9 (2) below that, for Galois extension fields of  $p$ -adic fields, we have an equivalence “semi-AV-tor-finite  $\Leftrightarrow$  KF with finite residue fields”.

### 3 Kummer-faithfulness in $p$ -adic settings

In this section, we study Kummer-faithfulness for algebraic extensions  $K$  of  $\mathbb{Q}_p$ . Throughout this section, we denote by  $\mathbb{F}_K$  the residue field of  $K$ . First we should point out that, by the existence of Tate curves, one can check that Kummer-faithfulness is equivalent to AVKF property in this situation.

**Proposition 3.1.** *Let  $K$  be an algebraic extension of  $\mathbb{Q}_p$ .*

- (1) *For a prime  $\ell$ ,  $K$  is  $\ell^\infty$ -semi-AV-tor-finite if and only if it is  $\ell^\infty$ -AV-tor-finite.*
- (2)  *$K$  is Kummer-faithful if and only if  $K$  is AVKF.*

*Proof.* Let  $E_{/\mathbb{Q}_p}$  be the Tate curve associated with uniformizing element  $p$ . To show (1), it suffices to prove that  $\mu_{\ell^\infty}(L)$  is finite for any finite extension  $L$  of  $K$  under the assumption

that  $K$  is  $\ell^\infty$ -AV-tor-finite, but this follows immediately from the existence of an injection  $\mu_{\ell^\infty}(L) \hookrightarrow E(L)[\ell^\infty]$ . To show (2), it suffices to prove that the divisible part  $(L^\times)_{\text{div}}$  of  $L^\times$  is trivial for any finite extension  $L$  of  $K$  under the assumption that  $K$  is AVKF, but this also follows immediately from Lemma 2.4 and an exact sequence  $0 \rightarrow p^{\mathbb{Z}} \rightarrow L^\times \rightarrow E(L)$  of abelian groups.  $\square$

### 3.1 Unramified extensions and tamely ramified extensions

In this section, we give criteria of Kummer-faithfulness for unramified, or tamely ramified, extension fields of some  $p$ -adic fields. In addition, we show that Kummer-faithful fields that are Galois extensions of  $p$ -adic fields admit a decomposition of quasi-finite extension fields and (possibly of infinite degree)  $p$ -power extensions.

Recall that  $\mathbb{F}_K$  is the residue field of an algebraic extension field  $K$  of  $\mathbb{Q}_p$ . We say that a field  $K$  is *stably  $\mu_{\ell^\infty}$ -finite* if  $\mu_{\ell^\infty}(L)$  is finite for any finite extension  $L$  of  $K$ .

**Lemma 3.2.** *Let  $K$  be an algebraic extension of  $\mathbb{Q}_p$ . Denote by  $G_{\mathbb{F}_K}^\ell$  the maximal pro- $\ell$  quotient of  $G_{\mathbb{F}_K}$  for any prime  $\ell$ .*

- (1) *Assume  $\ell \neq p$ . Then  $K$  is stably  $\mu_{\ell^\infty}$ -finite if and only if  $G_{\mathbb{F}_K}^\ell \simeq \mathbb{Z}_\ell$ .*
- (2) *If  $K$  is  $p^\infty$ -semi-AV-tor-finite, then  $G_{\mathbb{F}_K}^p \simeq \mathbb{Z}_p$ .*

*Proof.* (1) We may suppose that  $K$  contains  $\mu_\ell$ . Then the maximal pro- $\ell$  extension of  $\mathbb{F}_K$  is  $\mathbb{F}_K(\mu_{\ell^\infty})$ , and  $G_{\mathbb{F}_K}^\ell \simeq \text{Gal}(\mathbb{F}_p(\mu_{\ell^\infty})/\mathbb{F}_K \cap \mathbb{F}_p(\mu_{\ell^\infty})) \subset \text{Gal}(\mathbb{F}_p(\mu_{\ell^\infty})/\mathbb{F}_p(\mu_\ell)) \simeq \mathbb{Z}_\ell$ . Thus we have the following equivalent relations:  $G_{\mathbb{F}_K}^\ell \simeq \mathbb{Z}_\ell \Leftrightarrow \#\mu_{\ell^\infty}(\mathbb{F}_K) < \infty \Leftrightarrow \#\mu_{\ell^\infty}(K) < \infty$ . The result immediately follows.

(2) Assume  $G_{\mathbb{F}_K}^p \not\simeq \mathbb{Z}_p$ , that is,  $G_{\mathbb{F}_K}^p$  is trivial. Take a CM elliptic curve  $E$  defined over a  $p$ -adic subfield  $k$  of  $K$  with the properties that  $E$  has good ordinary reduction over  $k$  and every endomorphism of  $E$  is defined over  $k$ . Let  $V_p(E)$  and  $V_p(\bar{E})$  be the  $p$ -adic Tate module of  $E$  and that of the reduction  $\bar{E}$  of  $E$ , respectively. With a suitable choice of a basis of  $V_p(E)$ , the representation  $\rho: G_k \rightarrow GL_{\mathbb{Q}_p}(V_p(E)) \simeq GL_2(\mathbb{Q}_p)$  is of the form

$$\rho = \begin{pmatrix} \chi_p \varepsilon^{-1} & u \\ 0 & \varepsilon \end{pmatrix},$$

where  $\chi_p$  is the  $p$ -adic cyclotomic character,  $\varepsilon$  is an unramified character defined by the  $G_k$ -action on  $V_p(\bar{E})$  and  $u$  is a continuous map. Since  $G_{\mathbb{F}_K}^p$  is trivial, we find that an open subgroup of  $G_{\mathbb{F}_K}$  acts on  $\bar{E}[p^\infty]$  trivially. Replacing  $K$  by a finite extension, we may assume that  $\bar{E}[p^\infty]$  is defined over  $\mathbb{F}_K$ . This implies  $G_K \subset \ker \varepsilon$ . Since  $\rho$  has an abelian image, we have  $\rho(\sigma)\rho(\tau) = \rho(\tau)\rho(\sigma)$  for any  $\sigma, \tau \in G_k$ , which gives  $(\varepsilon^{-1}(\tau)\chi_p(\tau) - \varepsilon(\tau))u(\sigma) = (\varepsilon^{-1}(\sigma)\chi_p(\sigma) - \varepsilon(\sigma))u(\tau)$ . In particular, for  $\sigma, \tau \in G_K$ , we have

$$(\chi_p(\tau) - 1)u(\sigma) = (\chi_p(\sigma) - 1)u(\tau). \tag{3.1}$$

Take an element  $\tau_0 \in G_K \setminus \ker \chi_p$  (such an element exists since  $p^\infty$ -semi-AV-tor-finiteness of  $K$  in particular implies  $K$  is stably  $\mu_{p^\infty}$ -finite). By (3.1), we have  $u(\sigma) = \frac{u(\tau_0)(\chi_p(\sigma)-1)}{\chi_p(\tau_0)-1}$  for any  $\sigma \in G_K$ . This shows that  $u = c(\chi_p - 1)$  on  $G_K$  for some  $c \in \mathbb{Q}_p$ . We see that the vector  $\mathbf{x} = {}^t(-c, 1)$  satisfies  $\rho(\sigma)\mathbf{x} = \mathbf{x}$  for any  $\sigma \in G_K$ . Hence  $V_p(E)^{G_K}$  is not zero, that is,  $E(K)[p^\infty]$  is infinite. This contradicts the assumption that  $K$  is  $p^\infty$ -semi-AV-tor-finite.  $\square$

**Proposition 3.3.** *Let  $K$  be an algebraic extension of  $\mathbb{Q}_p$ .*

- (1) *If  $K$  is Kummer-faithful, then  $\mathbb{F}_K$  is quasi-finite, that is,  $G_{\mathbb{F}_K} \simeq \hat{\mathbb{Z}}$ .*
- (2) *If  $K$  is AV-tor-finite, then  $\mathbb{F}_K$  is finite.*

*Proof.* (1) Since Kummer-faithful fields over  $p$ -adic fields are stably  $\mu_{\ell^\infty}$ -finite for any prime  $\ell$  and  $p^\infty$ -semi-AV-tor-finite, the result follows immediately from Lemma 3.2.

(2) Assume that  $\mathbb{F}_K$  is infinite. There exists an increasing extensions  $\mathbb{F}_{p^{m_1}} \subset \mathbb{F}_{p^{m_2}} \subset \dots$  of subfields of  $\mathbb{F}_K$  with  $m_1 < m_2 < \dots$ . This in particular implies that  $\mathbb{F}_K$  contains all the  $(p^{m_i} - 1)$ -th roots of unity for all  $i$ , and the same holds for  $K$ . Hence, for the Tate curve  $E_{/\mathbb{Q}_p}$  associated with any choice of uniformizing element,  $E(K)$  contains infinitely many torsion points but this contradicts the assumption that  $K$  is AV-tor-finite.  $\square$

**Corollary 3.4.** *Let  $K$  be an unramified extension of some  $p$ -adic field. Then, the following are equivalent.*

- (a)  *$K$  is Kummer-faithful.*
- (b)  *$K/\mathbb{Q}_p$  is quasi-finite.*
- (c)  *$\mathbb{F}_K$  is Kummer-faithful.*

*Proof.* It is shown by Murotani [Mur23a, Theorem B] that  $\mathbb{F}_K$  is Kummer-faithful if and only if  $G_{\mathbb{F}_K} \simeq \hat{\mathbb{Z}}$ . By assumption on  $K$ , this is equivalent to say that  $K/\mathbb{Q}_p$  is quasi-finite. Thus the result immediately follows from Propositions and 2.6, 2.8 and 3.3.  $\square$

**Remark 3.5.** Let  $L$  be the completion of an algebraic extension, with finite ramification, of some  $p$ -adic field. Then, it follows from [Mur23a, Proposition 3.7] that, if the residue field of  $L$  is Kummer-faithful, then  $L$  is Kummer-faithful (see also [Tsu23, Proposition 1.4]). Thus the equivalent conditions (a), (b) and (c) in Corollary 3.4 are also equivalent to the following condition:

- (d) *The completion of  $K$  is Kummer-faithful.*

Next we consider Kummer-faithfulness for tamely ramified extensions.

**Lemma 3.6.** *Let  $K$  be an algebraic extension of a  $p$ -adic field  $k$ . Let  $M$  (resp.  $N$ ) be the maximal unramified (resp. maximal tamely ramified) extension of  $k$  contained in  $K$ .*

- (1) *If  $K$  is Kummer-faithful, then  $M/k$  is quasi-finite.*
- (2) *If  $K$  is torally Kummer-faithful and  $N$  is a Galois extension of  $M$ , then  $N/M$  is quasi-finite.*
- (3) *If  $K$  is Kummer-faithful and is a Galois extension of  $k$ , then  $N/k$  is quasi-finite.*

*Proof.* The assertion (1) follows from Corollary 3.4, and (3) follows from (1) and (2). Thus it suffices to show (2). Assume that there exists a prime  $\ell$  such that  $v_\ell([N : M]) = \infty$ . Note that  $\ell$  is not equal to  $p$  since  $N/M$  is totally tamely ramified. There exists an infinite set of finite Galois subextensions  $\{M_i\}_{i \in \mathbb{M}}$  in  $N/M$  with the property that  $v_\ell(e_i) < v_\ell(e_{i+1})$  where  $e_i := [M_i : M]$ . Since  $M_i$  is a Galois extension of  $M$ , it follows from [Lan94, Chapter II, §5, Proposition 12] that  $M_i$  contains  $e_i$ -th roots of unity. Since  $\lim_{i \rightarrow \infty} v_\ell(e_i) = \infty$ , it follows that  $N$  contains all  $\ell$ -power roots of unity. This contradicts the assumption that  $K$  is torally Kummer-faithful.  $\square$

**Corollary 3.7.** *Let  $K$  be a Galois extension of some  $p$ -adic fields. Then, the following are equivalent.*

- (a)  $K/\mathbb{Q}_p$  is tame and  $K$  is Kummer-faithful.
- (b)  $K/\mathbb{Q}_p$  is quasi-finite.
- (c)  $K/\mathbb{Q}_p$  is tame,  $K$  is totally Kummer-faithful and  $\mathbb{F}_K$  is Kummer-faithful.

*Proof.* (a)  $\Rightarrow$  (b) follows from Lemma 3.6 (3). (b)  $\Rightarrow$  (a) follows from Propositions 2.6 and 2.8. (a), (b)  $\Rightarrow$  (c) follows from Corollary 3.4. (c)  $\Rightarrow$  (b) follows from Corollary 3.4 and Lemma 3.6 (2).  $\square$

**Proposition 3.8.** *Let  $K$  be a Galois extension of a  $p$ -adic field. Then the following are equivalent.*

- (a)  $K$  is Kummer-faithful.
- (b)  $K$  has a decomposition  $K = MN$ . Here,
  - $M$  is Kummer-faithful with (possibly infinite)  $p$ -power degree over a  $p$ -adic field, and
  - $N$  is a quasi-finite Galois extension of a  $p$ -adic field.

If  $K$  is Kummer-faithful and is abelian extension of a  $p$ -adic field, then we can choose  $N$  above so that it is an unramified quasi-finite extension of a  $p$ -adic field.

*Proof.* (b)  $\Rightarrow$  (a) follows from Proposition 2.8. We show (a)  $\Rightarrow$  (b). Assume that  $K$  is Kummer-faithful. Let  $k$  be a  $p$ -adic subfield of  $K$  so that  $K$  is a Galois extension of  $k$ . Let  $N$  be the maximal tamely ramified extension of  $k$  contained in  $K$ , which is a Galois extension of  $k$  since so is  $K$ . By Lemma 3.6 (3), replacing  $k$  by a finite subextension in  $K$ , we may suppose that  $N/k$  is prime-to- $p$ . Then, it follows from Lemma 5 of [Iwa55] that the exact sequence  $1 \rightarrow \text{Gal}(K/N) \rightarrow \text{Gal}(K/k) \rightarrow \text{Gal}(N/k) \rightarrow 1$  splits, that is, there exists an algebraic extension  $M$  of  $k$  contained in  $K$  such that  $K = MN$  and  $M \cap N = k$ . Since  $K/N$  is a pro- $p$  extension, we see that the supernatural number  $[M : k]$  is a power of  $p$ . This finishes a proof of (a)  $\Rightarrow$  (b). Finally we note that, under the assumption that  $K$  is abelian over a  $p$ -adic field  $k$ , the tame ramification index of  $K/k$  is finite.  $\square$

In view of the above proposition, the essential difficulty in constructing a Kummer-faithful field over a  $p$ -adic field lies in the problem of whether an (almost) pro- $p$  algebraic extension which is Kummer-faithful can be constructed. Later we will study some criterion of Kummer-faithfulness for Lubin-Tate extensions of  $p$ -adic fields.

### 3.2 Finiteness of torsion points

The purpose of this section is to give some equivalent conditions of Kummer-faithfulness in terms of finiteness of torsion points.

**Theorem 3.9.** *Let  $\mathbb{Q}_p \subset k \subset K$  be algebraic extensions.*

- (1) *Assume that  $k$  is Kummer-faithful and  $K$  is a Galois extension of  $k$ . Then, the following are equivalent.*

- (a)  $K$  is Kummer-faithful.
  - (b)  $K$  is stably  $\mu_{\ell^\infty}$ -finite for every prime  $\ell$ , and the group  $A(L)[p^\infty]$  is finite for any finite extension  $L/K$  and any abelian variety  $A/L$  with good reduction.
- (2) Assume that  $k$  is both AV-tor-finite and Kummer-faithful, and also assume that  $K$  is a Galois extension of  $k$ . Then the following are equivalent.
- (a)  $K$  is Kummer-faithful with finite residue field.
  - (b)  $K$  is semi-AV-tor-finite.
  - (c)  $K$  is AV-tor-finite.

We note that the implications  $(a) \Rightarrow (b)$  in Theorem 3.9 (1) and  $(b) \Rightarrow (c)$  in Theorem 3.9 (2) clearly hold for *every* algebraic extension field  $K$  of  $\mathbb{Q}_p$ . If we remove the assumption “Galois”, then  $(a) \Rightarrow (b)$ ,  $(c)$  in Theorem 3.9 (2) does not hold (although the author does not know about the converse implication), see Remark 3.11 (2).

First we show Theorem 3.9 (1); this is an immediate consequence of Proposition 2.7, Proposition 3.2 and Lemma 3.10 below.

**Lemma 3.10.** *Let  $\mathbb{Q}_p \subset k \subset K$  be algebraic extensions. Let  $\ell$  be a prime. Assume that  $k$  is  $\ell^\infty$ -semi-AV-tor-finite and  $K$  is a Galois extension of  $k$ . Consider the following conditions.*

- (a)  $K$  is  $\ell^\infty$ -semi-AV-tor-finite.
- (b)  $K$  is stably  $\mu_{\ell^\infty}$ -finite, and the group  $A(L)[\ell^\infty]$  is finite for any finite extension  $L/K$  and any abelian variety  $A/L$  with good reduction.
- (c)  $K$  is stably  $\mu_{\ell^\infty}$ -finite.

Then, the following hold.

- (1) We have  $(a) \Leftrightarrow (b) \Rightarrow (c)$ .
- (2) If  $\ell$  is not equal to  $p$ , then we have  $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ .

*Proof.* The equivalence  $(a) \Leftrightarrow (b)$  follows from essentially the same proof as that of Corollary 2.4 (1) of [Oze23]. (We should give two remarks about this. At first, in *loc cit.*,  $k$  is assumed to be  $p$ -adic fields. However, the same argument proceeds by using  $\ell^\infty$ -semi-AV-tor-finiteness instead of Matuck’s theorem. Secondly, we remark that the key for the proof is a non-archimedean rigid uniformization theorem. We will use a similar method in the proof of Theorem 3.9 (2) below.) It suffices to show  $(c) \Rightarrow (b)$  if  $\ell \neq p$ . Let  $\ell \neq p$  and assume that  $K$  is stably  $\mu_{\ell^\infty}$ -finite. Let  $L$  be a finite extension of  $K$  and  $A$  an abelian variety over  $L$  with good reduction. Let us denote by  $\bar{A}$  the reduction of  $A$ . Since  $\ell$  is prime to  $p$ , the reduction map induces a bijection between  $A(L)[\ell^\infty]$  and  $\bar{A}(\mathbb{F}_L)[\ell^\infty]$ . On the other hand, it follows from (c) that  $\mathbb{F}_L$  is a potential prime-to- $\ell$  extension over a finite field. Thus  $\bar{A}(\mathbb{F}_L)[\ell^\infty]$  is defined over a finite field. In particular, the group  $\bar{A}(\mathbb{F}_L)[\ell^\infty]$ , and thus also  $A(L)[\ell^\infty]$ , must be finite.  $\square$

Next we give a proof of Theorem 3.9 (2).

*Proof of Theorem 3.9 (2).* The implication  $(b) \Rightarrow (c)$  is clear, and  $(c) \Rightarrow (a)$  is a result of Propositions 2.6 and 3.3. We show  $(a) \Rightarrow (b)$ . Assume the condition (a). Let  $L/K$  be a finite extension and  $X/L$  a semi-abelian variety. The goal is to show that the torsion subgroup of  $X(L)$  is finite. We will prove this in four steps depending on the situation of  $X$ .

**Step 1.** Consider the case where  $X = T$  is a torus. Replacing  $L$  by a finite extension, we may assume that the torus  $T$  splits over  $L$ . Since  $L$  is Kummer-faithful,  $T(L)[\ell^\infty]$  is finite for every prime  $\ell$ . Moreover, one can check that  $T(L)[\ell]$  is trivial for almost all  $\ell$ : If not, we have  $\mu_\ell \subset L$  for infinitely many primes  $\ell$  but this contradicts the fact that  $L$  has a finite residue field. Hence  $T(L)_{\text{tor}}$  is finite.

**Step 2.** Consider the case where  $X = A$  is an abelian variety with potential good reduction. Replacing  $L$  by a finite extension, we may suppose that  $A$  has good reduction over  $L$ . Since  $L$  is AVKF,  $A(L)[\ell^\infty]$  is finite for every prime  $\ell$ . Furthermore, since the reduction map from the prime-to- $p$  part  $A(L)_{p'}$  of  $A(L)_{\text{tor}}$  is injective and the residue field of  $L$  is finite, we see that  $A(L)_{p'}$  is finite. Hence  $A(L)_{\text{tor}}$  is finite.

**Step 3.** Consider the case where  $X = A$  is an abelian variety (without reduction hypothesis). Let  $g$  be the dimension of  $A$ . Take a  $p$ -adic subfield  $K_0$  in  $L$  so that  $A$  is defined over  $K_0$ . Applying a non-archimedean rigid uniformization theorem to  $A_{/K_0}$  ([Ray71], [BL91] and [BX96]), we find that there exist the following data, which is called a rigid uniformization of  $A$  (cf. [BX96, Definition 1.1 and Theorem 1.2]):

- (i)  $S$  is a semi-abelian variety of dimension  $g$  fits into an exact sequence of  $K_0$ -group schemes  $0 \rightarrow T \rightarrow S \rightarrow B \rightarrow 0$  where  $T$  is a torus of rank  $m$  and  $B$  is an abelian variety which has potential good reduction,
- (ii) a closed immersion of rigid  $K_0$ -groups  $N^{\text{an}} \hookrightarrow S^{\text{an}}$  where  $N$  is a group scheme which is isomorphic to  $\mathbb{Z}^{\oplus m}$  after a finite base extension. Here, the subscript “an” is the GAGA functor, and
- (iii) a faithfully flat morphism  $S^{\text{an}} \rightarrow A^{\text{an}}$  of rigid  $K_0$ -groups with kernel  $N^{\text{an}}$ .

We have exact sequences

$$0 \rightarrow N(\overline{K}) \rightarrow S(\overline{K}) \rightarrow A(\overline{K}) \rightarrow 0 \quad \text{and} \quad 0 \rightarrow T(\overline{K}) \rightarrow S(\overline{K}) \rightarrow B(\overline{K}) \rightarrow 0$$

of  $G_{K_0}$ -modules. For the proof, replacing  $L$  and  $K_0$  by finite extensions, we may assume that  $L$  is a Galois extension of  $k$  and  $N$  is constant over the base field  $K_0$ . We set  $k_0 := kK_0$ . Note that  $L$  is a Galois extension of  $k_0$ . The exact sequence  $0 \rightarrow T(\overline{K}) \rightarrow S(\overline{K}) \rightarrow B(\overline{K}) \rightarrow 0$  of  $G_{k_0}$ -modules induces an exact sequence  $0 \rightarrow T(L)_{\text{tor}} \rightarrow S(L)_{\text{tor}} \rightarrow B(L)_{\text{tor}}$  of  $G_{k_0}$ -modules. By Step 1 and Step 2, it follows that  $T(L)_{\text{tor}}$  and  $B(L)_{\text{tor}}$  are finite, which shows that  $S(L)_{\text{tor}}$  is also finite. Since  $k$  is AV-tor-finite, there exists an integer  $M > 0$  such that both  $S(L)_{\text{tor}} \subset S(\overline{K})[M]$  and  $A(k_0)_{\text{tor}} \subset A(\overline{K})[M]$  hold. Let  $n > 0$  be any integer divided by  $M^2$ . Since  $L$  is a Galois extension of  $k_0$ , there exists an exact sequence  $0 \rightarrow S(L)[n] \rightarrow A(L)[n] \rightarrow N(\overline{K})/nN(\overline{K})$  of  $G_{k_0}$ -modules. Let  $P$  be an element of  $A(L)[n]$ . Since  $G_{k_0}$  acts on  $N(\overline{K})$  trivial, we know that  $\sigma P - P \in S(L)[n] \subset S(\overline{K})[M]$  for any  $\sigma \in G_{k_0}$ . Thus we find  $MP \in A(k_0)_{\text{tor}} \subset A(\overline{K})[M]$ , which implies  $A(L)[n] \subset A(\overline{K})[M^2]$ . Since this relation holds for any  $n$  divided by  $M^2$ , we obtain that  $A(L)_{\text{tor}}$  is contained in  $A(\overline{K})[M^2]$ , which must be finite.

**Step 4.** Finally, we consider the case where  $X$  is a semi-abelian variety. We have an exact sequence

$$0 \rightarrow T \rightarrow X \rightarrow A \rightarrow 0 \quad (3.2)$$

of group schemes over  $K$ , where  $T$  is a torus and  $A$  is an abelian variety. This induces an exact sequence  $0 \rightarrow T(L)_{\text{tor}} \rightarrow X(L)_{\text{tor}} \rightarrow A(L)_{\text{tor}}$  of abelian groups. Since  $T(L)_{\text{tor}}$  and  $A(L)_{\text{tor}}$  are finite by Step 1 and Step 3, we conclude that  $X(L)_{\text{tor}}$  is finite as desired.  $\square$

**Remark 3.11.** (1) Let  $\mathbb{Q}_p \subset k \subset K$  be as in Theorem 3.9 (2). The equivalent conditions (a), (b) and (c) in the theorem are also equivalent to the following condition (d):

- (d) For every finite extension  $L$  of  $K$  and every commutative algebraic group  $G$  over  $L$ , it holds that  $G(L)_{\text{tor}}$  is finite.

In fact, we can check the equivalence between (d) and the others by almost the same method as the proof of the theorem; we should use Chevalley's decomposition of (possibly non-connected) commutative algebraic groups (cf. [Bri17, Theorem 2.9]) instead of (3.2) in the argument of Step 4.

(2) We can not remove the assumption "Galois" from the statement of Theorem 3.9 (2). Let  $\mathfrak{n} = \prod_{\ell \text{ prime}} \ell$  be a supernatural number and take a compatible system  $(p_k)_{k|\mathfrak{n}}$  of  $(p^k - 1)$ -th roots  $p_k$  of  $p$ . Let  $K$  be the extension field  $\mathbb{Q}_p(p_k; k \mid \mathfrak{n})$  of  $\mathbb{Q}_p$  obtained by adjoining all  $p_k$  for all  $k \mid \mathfrak{n}$ . We see that  $K$  is totally ramified over  $\mathbb{Q}_p$  and the Galois closure  $\hat{K}$  of  $K$  over  $\mathbb{Q}_p$  is quasi-finite over  $\mathbb{Q}_p$ . The field  $\hat{K}$  is locally semi-AV-tor-finite by Proposition 2.8. In particular,  $\hat{K}$  is Kummer faithful by Proposition 2.7 (2) and thus so is  $K$ . On the other hand,  $K$  is not AV-tor-finite since there are infinitely many  $K$ -rational torsion points of the Tate curve with uniformizing parameter  $p$ .

## 4 Kummer-faithfulness of Lubin-Tate extensions

The aim of this section is to give some criteria on Kummer-faithfulness of Lubin-Tate extensions of  $p$ -adic fields. Throughout this section, we fix an algebraic closure  $\bar{\mathbb{Q}}_p$  of  $\mathbb{Q}_p$ . Unless otherwise specified, all algebraic extension fields of  $\mathbb{Q}_p$  appearing in this section are subfields of our fixed  $\bar{\mathbb{Q}}_p$ . Let  $k$  be a  $p$ -adic field and  $\pi$  a uniformizer. Let  $F_\pi$  be the Lubin-Tate formal group over the integer ring of  $k$  associated with  $\pi$ . Let  $k_\pi$  be the extension of  $k$  obtained by adjoining all  $\pi$ -power torsion points of  $F_\pi$ , which is called the Lubin-Tate extension of  $k$  associated with  $\pi$ . Local class field theory asserts that  $k_\pi$  is a maximal totally ramified abelian extension of  $k$ , and the composite of  $k_\pi$  and the maximal unramified extension field  $k^{\text{ur}}$  of  $k$  coincides with the maximal abelian extension field  $k^{\text{ab}}$  of  $k$ . We should remark that the following conditions on the Lubin-Tate extension field  $k_\pi$  are equivalent by Theorem 3.9 (2):

- (a))  $k_\pi$  is Kummer-faithful.
- (b)  $k_\pi$  is semi-AV-tor-finite.
- (c)  $k_\pi$  is AV-tor-finite.

We are interested in determining which Lubin-Tate extension fields  $k_\pi$  satisfy the three equivalent conditions mentioned above. Before the main part of this section, we note that it is easy to check whether  $k_\pi$  is torally Kummer-faithfulness or not.

**Proposition 4.1.**  $k_\pi$  is torally Kummer-faithful if and only if  $q^{-1}\text{Nr}_{k/\mathbb{Q}_p}(\pi) \notin \mu_{p-1}$ .

*Proof.* One verifies immediately that  $k_\pi$  is torally Kummer-faithful if and only if it is stably  $\mu_{p^\infty}$ -finite by Proposition 2.6. The latter condition is equivalent to  $q^{-1}\text{Nr}_{k/\mathbb{Q}_p}(\pi) \notin \mu_{p-1}$  by local class field theory.  $\square$

#### 4.1 Criterion for Lubin-Tate extensions

In this section, we give a proof of Theorem 1.1 in the Introduction. In fact, we prove more precise criterion for Kummer-faithfulness of Lubin-Tate extension fields. To simplify statements, we use the following notation. Let  $k$  be a  $p$ -adic field and  $\pi$  a uniformizer of  $k$ . For a tuple  $(r_\sigma)_{\sigma \in \Gamma_k}$  with  $r_\sigma \in \{0, 1\}$ , we often set

$$\hat{\pi} := \prod_{\sigma \in \Gamma_k} \sigma \pi^{r_\sigma}$$

(of course this notation depends on the choices of  $\pi$  and  $(r_\sigma)_{\sigma \in \Gamma_k}$ ). We recall that  $W(q)$  is the set of Weil  $q$ -integers.

**Theorem 4.2.** *Let  $k$  be a  $p$ -adic field with residue cardinality  $q = p^f$  and  $\pi$  a uniformizer of  $k$ . We denote by  $k_\pi$  the Lubin-Tate extension field of  $k$  associated with  $\pi$ .*

- (1) *If  $k$  is a Galois extension of  $\mathbb{Q}_p$  and  $k_\pi$  is not Kummer-faithful, then either of the following holds.*
  - (a)  $q^{-1}\text{Nr}_{k/\mathbb{Q}_p}(\pi) \in \mu_{p-1}$ .
  - (b) *For some  $(r_\sigma)_{\sigma \in \Gamma_k}$  with  $r_\sigma \in \{0, 1\}$ , it holds  $\hat{\pi} \in W(q)$ .*
- (2) *If  $q^{-1}\text{Nr}_{k/\mathbb{Q}_p}(\pi) \in \mu_{p-1}$  or  $\text{Nr}_{k/\mathbb{Q}_p}(\pi) \in W(q)$ , then  $k_\pi$  is not Kummer-faithful.*
- (3) *Assume that  $k$  is a Galois extension of  $\mathbb{Q}_p$ , and also assume that all the following conditions hold.*
  - (i) *For some  $(r_\sigma)_{\sigma \in \Gamma_k}$  with  $r_\sigma \in \{0, 1\}$ , it holds  $\hat{\pi} \in W(q)$ .*
  - (ii)  $\pi^r \in \mathbb{Q}_p$  for some integer  $r \geq 1$ .
  - (iii) *Let  $\hat{\pi}$  be as in (i).*
    1.  *$f$  divides  $\text{ord}_v(\hat{\pi})f_v$  for any finite place  $v$  of  $\mathbb{Q}(\hat{\pi})$  above  $p$ . Here,  $\text{ord}_v$  is the valuation associated with  $v$  normalized by  $\text{ord}_v(\mathbb{Q}(\hat{\pi})^\times) = \mathbb{Z}$  and  $f_v$  is the residue degree of  $\mathbb{Q}(\hat{\pi})/\mathbb{Q}$  at  $v$ .*
    2. *Let  $f(X) \in \mathbb{Q}[X]$  be the minimal polynomial of  $\hat{\pi}$  over  $\mathbb{Q}$ . Then, the degree of any irreducible factor of  $f(X)$  in  $\mathbb{Q}_p[X]$  is not of the form  $n \cdot [\mathbb{Q}_p(\hat{\pi}) : \mathbb{Q}_p]$  for any integer  $n \geq 2$ .*

*Then,  $k_\pi$  is not Kummer-faithful.*

The aim of this section is to prove the above theorem. Our argument relies heavily on both Honda–Tate theory and  $p$ -adic Hodge theory. We begin by briefly reviewing the aspects of these theories that are essential for our proof. Let  $A$  be a simple abelian variety over  $\mathbb{F}_q$  of dimension  $g > 0$  (here, “simple” stands for not “ $\bar{\mathbb{F}}_q$ -simple” but “ $\mathbb{F}_q$ -simple”). Put  $D = \text{End}_{\mathbb{F}_q}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$  and denote by  $\pi_A \in D$  the geometric Frobenius endomorphism

of  $A_{/\mathbb{F}_q}$ . Weil showed that  $\pi_A$  is a Weil  $q$ -integer. The degree  $2g$  characteristic polynomial  $f_{\bar{A}/\mathbb{F}_q} \in \mathbb{Z}[T]$  of  $\pi_A$  has a single monic irreducible factor over  $\mathbb{Q}$ , and  $f_{\bar{A}/\mathbb{F}_q}$  coincides with the characteristic polynomial the action of the arithmetic  $q$ -Frobenius element of  $G_{\mathbb{F}_q}$  on the  $\ell$ -adic Tate module  $V_\ell(A)$  of  $A$  for any prime  $\ell \neq p$ .

**Theorem 4.3.** *Let  $\mathbb{F}$  be the finite field of order  $q$ .*

- (1) *The assignment  $A \mapsto \pi_A$  defines a bijection from the set of isogeny classes of simple abelian varieties over  $\mathbb{F}$  to the set of  $G_{\mathbb{Q}}$ -conjugacy classes of Weil  $q$ -integers.*
- (2) *Let  $A$  be an abelian variety over  $\mathbb{F}$  which is  $\mathbb{F}$ -isogenous to a power of an  $\mathbb{F}$ -simple abelian variety. Then there exist a finite extension  $\mathbb{F}'/\mathbb{F}$  and a  $p$ -adic field  $k'$  with residue field  $\mathbb{F}'$  such that  $A$  is  $\mathbb{F}'$ -isogenous to the reduction of a CM abelian variety with good reduction over  $k'$ .*

The first assertion is a central result of Honda–Tate theory, while the second is commonly referred to as the "Honda–Tate CM lifting theorem". For more details, we refer the reader to Section 1.6 of [CCO14].

Next, we summarize some fundamental facts about  $p$ -adic Hodge theory. For an introduction to the basic notions of  $p$ -adic Hodge theory, it is helpful for the reader to refer [Fon94a] and [Fon94b]. Let  $K$  be a  $p$ -adic field. Below we often use the following notations. Let  $B_{\text{cris}}$  be the Fontaine's  $p$ -adic period ring and set  $D_{\text{cris}}^K(V) := (B_{\text{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K}$  for any  $\mathbb{Q}_p$ -representation  $V$  of  $G_K$ . Let us denote by  $K_0$  the maximal unramified subextension of  $K/\mathbb{Q}_p$ . Since  $B_{\text{cris}}^{G_K} = K_0$ ,  $D_{\text{cris}}^K(V)$  is a  $K_0$ -vector space. Denote by  $\varphi_{K_0}$  the arithmetic Frobenius map of  $K_0$ , that is, the (unique) lift of the  $p$ -th power map on the residue field of  $K_0$ . This extends to  $B_{\text{cris}}$ , and moreover this extended Frobenius map induces a  $K_0$ -semi-linear endomorphism  $\varphi$  on  $D_{\text{cris}}^K(V)$ , which is also called Frobenius endomorphism. Note that the  $f_K$ -th iterate  $\varphi^{f_K}$  of  $\varphi$  on  $D_{\text{cris}}^K(V)$  is  $K_0$ -linear where  $f_K$  is the residue degree of  $K/\mathbb{Q}_p$ . It is known that  $D_{\text{cris}}^K(V)$  is of finite dimensional over  $K_0$  with  $\dim_{K_0} D_{\text{cris}}^K(V) \leq \dim_{\mathbb{Q}_p} V$ . We say that  $V$  is crystalline if the equality  $\dim_{K_0} D_{\text{cris}}^K(V) = \dim_{\mathbb{Q}_p} V$  holds. Let us recall locally algebraic theory for crystalline characters; see [Ser98, Appendix of Chapter III] and [Con11, Appendix B] for more precise information. Let  $F$  be a  $p$ -adic field and  $\psi: G_K \rightarrow F^\times$  a continuous character. We denote by  $F(\psi)$  the  $\mathbb{Q}_p$ -representation of  $G_K$  underlying a 1-dimensional  $F$ -vector space endowed with an  $F$ -linear action by  $G_K$  via  $\psi$ . We say that  $\psi$  is crystalline if  $F(\psi)$  is crystalline. Suppose that  $\psi$  is crystalline. For any  $\sigma \in \Gamma_F$ , let  $\chi_{\sigma F}: I_{\sigma F} \rightarrow \sigma F^\times$  be the restriction to the inertia  $I_{\sigma F}$  of the Lubin-Tate character associated with any choice of uniformizer of  $\sigma F$  (it depends on the choice of a uniformizer of  $\sigma F$ , but its restriction to the inertia subgroup does not). Assume that  $K$  contains the Galois closure of  $F/\mathbb{Q}_p$ . Then, we have

$$\psi = \prod_{\sigma \in \Gamma_F} \sigma^{-1} \circ \chi_{\sigma F}^{h_\sigma} \tag{4.1}$$

on the inertia  $I_K$  for some integer  $h_\sigma$  where  $\Gamma_F := \text{Hom}_{\mathbb{Q}_p}(F, \overline{\mathbb{Q}}_p)$ . Note that  $\{h_\sigma \mid \sigma \in \Gamma_F\}$  is the (multi-)set of Hodge-Tate weights of  $F(\psi)$ , that is,  $C \otimes_{\mathbb{Q}_p} F(\psi) \simeq \bigoplus_{\sigma \in \Gamma_F} C(h_\sigma)$  where  $C$  is the completion of  $\overline{\mathbb{Q}}_p$ .

Here we recall a remarkable observation of Conrad for crystalline characters. We denote by  $\underline{K}^\times$  the Weil restriction  $\text{Res}_{K/\mathbb{Q}_p}(\mathbb{G}_m)$ . This is an algebraic torus such that, for a  $\mathbb{Q}_p$ -algebra  $R$ , the  $R$ -valued points  $\underline{K}^\times(R)$  of  $\underline{K}^\times$  is  $\mathbb{G}_m(R \otimes_{\mathbb{Q}_p} K)$ . If  $\psi: G_K \rightarrow F^\times$

is a continuous character, we may regard  $\psi$  as a character of  $\text{Gal}(K^{\text{ab}}/K)$  where  $K^{\text{ab}}$  is the maximal abelian extension of  $K$ . We denote by  $\psi_K: K^\times \rightarrow F^\times$  the composite of the reciprocity map  $K^\times \rightarrow \text{Gal}(K^{\text{ab}}/K)$  of local class field theory and  $\psi: \text{Gal}(K^{\text{ab}}/K) \rightarrow F^\times$ . For example, if  $\chi: G_K \rightarrow \mathbb{Q}_p^\times$  is the  $p$ -adic cyclotomic character, we have  $\chi_K(x) = (\text{Nr}_{K/\mathbb{Q}_p}(x)|\text{Nr}_{K/\mathbb{Q}_p}|_p)^{-1}$  for  $x \in K^\times$  where  $|\cdot|_p$  is the  $p$ -adic absolute value normalized by  $|p|_p = p^{-1}$ . The following result is shown in Proposition B.4 of [Con11].

**Proposition 4.4.** *Let  $\psi: G_K \rightarrow F^\times$  be a continuous character.*

- (1)  *$F(\psi)$  is crystalline if and only if there exists a (necessarily unique)  $\mathbb{Q}_p$ -homomorphism  $\psi_{\text{alg},K}: \underline{K}^\times \rightarrow \underline{F}^\times$  such that  $\psi_K$  and  $\psi_{\text{alg},K}$  (on  $\mathbb{Q}_p$ -points) coincides on  $\mathcal{O}_K^\times \subset \underline{K}^\times(\mathbb{Q}_p)$ .*
- (2) *Assume that  $F(\psi)$  is crystalline and let  $\psi_{\text{alg},K}$  be as in (1). Then,  $D_{\text{cris}}^K(F(\psi^{-1}))$  is free of rank 1 over  $K_0 \otimes_{\mathbb{Q}_p} F$  and its  $K_0$ -linear endomorphism  $\varphi^{f_K}$  is given by the action of the product*

$$\psi_K(\pi_K) \cdot \psi_{\text{alg},K}^{-1}(\pi_K) \in F^\times,$$

*where  $\pi_K$  is any uniformizer of  $K$  and  $f_K$  is the residue degree of  $K/\mathbb{Q}_p$ . Note that this product is independent of the choice of  $\pi_K$ .*

Motivated by the above proposition, for a crystalline character  $\psi: G_K \rightarrow F^\times$  of  $G_K$ , we set

$$\Phi_K(\psi) := \psi_K(\pi_K) \cdot \psi_{\text{alg},K}^{-1}(\pi_K) \in F^\times.$$

This is invariant under the coefficient field extension  $F'/F$ . For a finite extension  $K'/K$ , we set  $\Phi_{K'}(\psi) := \Phi_{K'}(\psi|_{G_{K'}}) \in F^\times$ . For example, one verifies  $\Phi_K(\chi_\pi) = \pi^{f_{K/k}}$  where  $\chi_\pi: G_k \rightarrow k^\times$  is the Lubin-Tate character of  $k$  associated with a uniformizer  $\pi$  and  $K$  is a finite extension of  $k$  with residue degree  $f_{K/k}$  (see also Lemma 4.5 (2) below). Note that the character  $\psi_{\text{alg},K}$  on  $\mathbb{Q}_p$ -points coincides with  $\prod_{\sigma \in \Gamma_F} \sigma^{-1} \circ \text{Nr}_{K/\sigma F}^{-h_\sigma}$  if  $K$  contains the Galois closure of  $F/\mathbb{Q}_p$  and  $\psi$  is of the form (4.1). By Proposition 4.4 (2), one can verify that the roots of the characteristic polynomial of the  $\varphi^{f_K}$ -action on the  $K_0$ -vector space  $D_{\text{cris}}^K(F(\psi^{-1}))$  are the  $\mathbb{Q}_p$ -conjugates of  $\Phi_K(\psi)$ , that is,

$$\det(T - \varphi^{f_K} | D_{\text{cris}}^K(F(\psi^{-1}))) = \prod_{\sigma \in \Gamma_F} (T - \sigma \Phi_K(\psi)).$$

We summarize some basic properties of  $\Phi_K(\psi)$ .

**Lemma 4.5.** *Let  $\psi: G_K \rightarrow F^\times$  be a crystalline character and  $\{h_\sigma\}_{\sigma \in \Gamma_F}$  the (multi-)set of Hodge-Tate weights of  $F(\psi)$ .*

- (0) *The  $\varphi^{f_K}$ -action on the free  $K_0 \otimes_{\mathbb{Q}_p} F$ -module  $D_{\text{cris}}^K(F(\psi^{-1}))$  of rank 1 is given  $\Phi_K(\psi)$ .*
- (1) *For a crystalline character  $\psi': G_K \rightarrow F^\times$ , we have  $\Phi_K(\psi\psi') = \Phi_K(\psi)\Phi_K(\psi')$ .*
- (2) *For a finite extension  $K'/K$  with residue degree  $f_{K'/K}$ , we have  $\Phi_{K'}(\psi) = \Phi_K(\psi)^{f_{K'/K}}$ .*
- (3) *For  $\sigma \in \Gamma_F$ , we have  $\Phi_K(\sigma\psi) = \sigma\Phi_K(\psi)$ .*
- (4)  *$\Phi_K(\psi) \in \mathcal{O}_F^\times$  if and only if  $\sum_{\sigma \in \Gamma_F} h_\sigma = 0$ .*

- (5) Assume either “ $h_\sigma \geq 0$  for all  $\sigma$ ” or “ $h_\sigma \leq 0$  for all  $\sigma$ ”. If  $\Phi_K(\psi) = 1$ , then  $\psi = 1$ .
- (6) Let  $\pi_F$  be a uniformizer of  $F$  with the property that  $\pi_F^r \in \mathbb{Q}_p$  for some  $r > 0$  (such  $\pi_F$  exists if  $F/\mathbb{Q}_p$  is tamely ramified). If  $K \supset F$ ,  $F$  is a Galois extension of  $\mathbb{Q}_p$  and  $\Phi_K(\psi) = 1$ , then we have

$$\psi = \prod_{\sigma \in \Gamma_F} \sigma^{-1} \circ \chi_{\pi_F}^{h_\sigma}$$

on an open subgroup of  $G_K$ . Moreover,

- (i)  $\psi: G_K \rightarrow F^\times$  extends to  $G_F$ , and
- (ii) there exists a character  $\epsilon: G_F \rightarrow \overline{\mathbb{Q}}_p^\times$  such that  $\epsilon^r = 1$  and  $\psi = \epsilon \cdot \prod_{\sigma \in \Gamma_F} \sigma^{-1} \circ \chi_{\pi_F}^{h_\sigma}$  on  $G_F$ .

*Proof.* (0) is Proposition 4.4 (2). (1) and (3) are clear from the definition of  $\Phi_K$ . We consider (2). By local class field theory,  $\psi_{K'}: (K')^\times \rightarrow F^\times$  is the composite of the norm map  $\text{Nr}_{K'/K}: K'^\times \rightarrow K^\times$  and  $\psi_K: K^\times \rightarrow F^\times$ , and also  $\psi_{\text{alg}, K'}: \underline{K'}^\times \rightarrow \underline{F}^\times$  is the composite of the norm map  $\text{Nr}_{K'/K}: \underline{K'}^\times \rightarrow \underline{K}^\times$  and  $\psi_{\text{alg}, K}: \underline{K}^\times \rightarrow \underline{F}^\times$ . Take uniformizers  $\pi_K$  and  $\pi_{K'}$  of  $K$  and  $K'$ , respectively. Writing  $f = f_{K'/K}$  and  $\text{Nr}_{K'/K}(\pi_{K'}) = \pi_K^f u$  for some  $u \in \mathcal{O}_K^\times$ , we have  $\Phi_{K'}(\psi) = \psi_{K'}(\pi_{K'}) \cdot \psi_{\text{alg}, K'}^{-1}(\pi_{K'}) = \psi_K(\pi_K^f u) \cdot \psi_{\text{alg}, K}^{-1}(\pi_K^f u) = (\psi_K(\pi_K) \cdot \psi_{\text{alg}, K}^{-1}(\pi_K))^f = \Phi_K(\psi)^f$ . This shows (2). Next we show (4). By (2), we may assume that  $K$  contains the Galois closure of  $F/\mathbb{Q}_p$ . Thus  $\psi$  is of the form (4.1) on  $I_K$ . Note that  $\Phi_K(\psi) \in \mathcal{O}_F^\times$  if and only if  $\psi_{\text{alg}, K}(\pi_K) \in \mathcal{O}_F^\times$  since  $\psi_K$  has values in  $\mathcal{O}_F^\times$ . Since  $\psi_{\text{alg}, K}(\pi_K)$  coincides with  $\prod_{\sigma \in \Gamma_M} \sigma^{-1} \text{Nr}_{K/\sigma F}(\pi_K)^{-h_\sigma}$ , we see that  $\psi_{\text{alg}, K}(\pi_K)$  is a  $p$ -adic unit if and only if  $\sum_{\sigma \in \Gamma_F} h_\sigma = 0$ , which shows (4). We show (5). It follows from the assumption and (4) that  $h_\sigma = 0$  for all  $\sigma$ . Thus  $\psi_{\text{alg}, K}$  is trivial and then  $\psi_K$  is trivial on  $\mathcal{O}_K^\times$ . Furthermore,  $\Phi_K(\psi) = 1$  implies  $\psi_K(\pi_K) = 1$ . Hence we have  $\psi_K = 1$  on  $K^\times$ , equivalently,  $\psi = 1$ . Finally, we show (6). By  $\Phi_K(\psi) = 1$ , we have

$$\psi_K(x) = \prod_{\sigma \in \Gamma_F} \sigma^{-1} \circ \text{Nr}_{K/F}(x)^{-h_\sigma} \tag{4.2}$$

if  $x$  is any uniformizer  $\pi_K$  of  $K$ . It follows from the definition of  $\{h_\sigma\}_{\sigma \in \Gamma_F}$  that (4.2) also holds for any  $x \in \mathcal{O}_K^\times$ . Hence the equality (4.2) holds for every  $x \in K^\times$ . Thus  $\psi: G_K \rightarrow F^\times$  extends to  $G_F$  so that

$$\psi_F(x) = \prod_{\sigma \in \Gamma_F} \sigma^{-1} x^{-h_\sigma}$$

for  $x \in F^\times$ . Taking  $\pi_F$  as in the statement of (6), by (4), we see

$$\psi_F(\pi_F^r) = \prod_{\sigma \in \Gamma_F} (\sigma^{-1} \pi_F^r)^{-h_\sigma} = \prod_{\sigma \in \Gamma_F} \pi_F^{-rh_\sigma} = 1.$$

Hence we have  $\psi^r = \left( \prod_{\sigma \in \Gamma_F} \sigma^{-1} \circ \chi_{\pi_F}^{h_\sigma} \right)^r$  on  $G_F$ . Now the result immediately follows.  $\square$

Now we are ready to prove Theorem 4.2.

*Proof of Theorem 4.2.* (1) The assertion is an immediate consequence of [Oze23, Theorem 1.2]. However, we include a proof here for the sake of completeness (in fact, arguments will be simpler since we only need to consider abelian varieties here). Assume that  $k$  is a Galois extension of  $\mathbb{Q}_p$  and  $k_\pi$  is not Kummer-faithful. By Theorem 3.9, we know that (i)  $k_\pi$  is not stably  $\mu_{p^\infty}$ -finite, or, (ii) the group  $A(L)[p^\infty]$  is infinite for some finite extension  $L/k_\pi$  and some abelian variety  $A/L$  with good reduction. In the former case (i), we have  $q^{-1}\text{Nr}_{k/\mathbb{Q}_p}(\pi) \in \mu_{p-1}$ . In the rest of the proof we assume the latter case (ii) and let  $L/k_\pi$  and  $A/L$  be as in (ii). Take a finite subextension  $K/k$  in  $L/k$  so that  $L = Kk_\pi$ ,  $A$  is defined over  $K$  and  $A$  has good reduction over  $K$ . Since  $L$  is a Galois extension of  $K$  and the group  $A(L)[p^\infty]$  is infinite,  $V := V_p(A)^{G_L}$  is a non-zero  $G_K$ -stable submodule of the  $p$ -adic Tate module  $V_p(A)$  of  $A$ . Since the  $G_K$ -action on  $V$  factors through an abelian quotient, by taking a  $p$ -adic field  $F$  large enough, we have an isomorphism

$$(V \otimes_{\mathbb{Q}_p} F)^{\text{ss}} \simeq F(\psi_1) \oplus F(\psi_2) \oplus \cdots \oplus F(\psi_t)$$

of  $F[G_K]$ -modules for some continuous characters  $\psi_i : G_K \rightarrow F^\times$ . Here, “ss” stands for the semi-simplification as a  $F[G_K]$ -module. Each  $\psi_i$  is crystalline since  $A$  has good reduction (see [Fon82, Section 6]; see also [CI99, Theorem 1]). Furthermore, each  $\psi_i$  factors through  $\text{Gal}(L/K)$ . Replacing  $L, K$  and  $F$  by finite extensions, by Lemma 2.5 of [Oze23], we find that  $\psi_i = \prod_{\sigma \in \Gamma_k} \sigma \circ \chi_\pi^{r_{i,\sigma}}$  on  $G_K$  for some  $r_{i,\sigma} \in \{0, 1\}$  (here, we need the assumption that  $k$  is a Galois extension of  $\mathbb{Q}_p$  to apply the lemma). Thus

$$\Phi_K(\psi_i) = \prod_{\sigma \in \Gamma_k} \sigma \Phi_K(\chi_\pi)^{r_{i,\sigma}} = \left( \prod_{\sigma \in \Gamma_k} \sigma \pi^{r_{i,\sigma}} \right)^{f_{K/k}}. \quad (4.3)$$

Now we set

$$f(T) := \det(T - \varphi^{f_K} \mid D_{\text{cris}}^K(V^\vee))$$

where  $f_K$  is the residue degree of  $K/\mathbb{Q}_p$  and “ $\vee$ ” stands for the dual of  $\mathbb{Q}_p[G_K]$ -modules. We have

$$f(T)^{[F:\mathbb{Q}_p]} = \det(T - \varphi^{f_K} \mid D_{\text{cris}}^K(V^\vee \otimes_{\mathbb{Q}_p} F)) = \prod_{i=1}^t \prod_{\sigma \in \Gamma_F} (T - \sigma \Phi_K(\psi_i))$$

by Proposition 4.4 (2). The polynomial  $f(T)$  is a divisor of  $f_A(T) := \det(T - \varphi^{f_K} \mid D_{\text{cris}}^K(V_p(A)^\vee))$  and it follows from  $p$ -adic Hodge theory (cf. [Fal89] and [CLS98]) that

$$f_A(T) = \det(T - \varphi^{f_K} \mid D_{\text{cris}}^K(H_{\text{ét}}^1(A_{\bar{K}}, \mathbb{Q}_p))) = \det(T - \text{Frob}_K^{-1} \mid H_{\text{ét}}^1(A_{\bar{K}}, \mathbb{Q}_\ell)) \quad (4.4)$$

for any prime  $\ell \neq p$ , where  $\text{Frob}_K$  stands for the arithmetic Frobenius of  $K$ . By the Weil conjecture, we conclude that each  $\Phi_K(\psi_i)$  is a Weil  $q_K$ -integer. It follows from (4.3) that  $\prod_{\sigma \in \Gamma_k} \sigma \pi^{r_{i,\sigma}}$  is a Weil  $q$ -integer.

(2) If  $q^{-1}\text{Nr}_{k/\mathbb{Q}_p}(\pi)$  is a root of unity, then  $k_\pi$  is not torally-Kummer faithful by Proposition 4.1 and thus it is not Kummer-faithful. We consider the case where  $\text{Nr}_{k/\mathbb{Q}_p}(\pi) \in W(q)$ . It suffices to show that there exists a  $p$ -adic field  $K$  and an abelian variety defined over  $K$  which has infinitely many  $Kk_\pi$ -rational  $p$ -power torsion points. Let  $\bar{A}$  be the simple abelian variety defined over  $\mathbb{F}_q$  (uniquely determined up to  $\mathbb{F}_q$ -isogeny) which corresponds to the Weil  $q$ -number  $\text{Nr}_{k/\mathbb{Q}_p}(\pi)$  via Honda-Tate theory (cf. Theorem 4.3 (1)). (Note that

$\bar{A}$  is ordinary by the first Theorem of [WM71, Section III].) Moreover, by Honda-Tate lifting theorem (cf. Theorem 4.3 (2)), there exists a finite extension  $K/k$  and a CM abelian variety  $B$  over  $K$  with good reduction such that  $\bar{A}$  is isogenous to the reduction  $\bar{B}$  of  $B$  over the residue field of  $K$ . Put  $g = \dim B$  and denote by  $L$  the CM field of  $B$  (so there exists an embedding from  $L$  into  $\text{End}_K(B) \otimes_{\mathbb{Z}} \mathbb{Q}$ ). Let  $\prod_i L_i$  denote the decomposition of  $L \otimes_{\mathbb{Q}} \mathbb{Q}_p$  into a finite product of finite extensions of  $\mathbb{Q}_p$  (note that, a priori, each factor field  $L_i$  does not live in the fixed algebraically closed field  $\bar{\mathbb{Q}}_p$ ). Since the  $G_K$ -action on  $V_p(B)$  commutes with  $L \otimes_{\mathbb{Q}} \mathbb{Q}_p$ -action, the above decomposition gives the decomposition  $V_p(B) \simeq \bigoplus_i V_i$  of  $G_K$ -modules with the property that each  $V_i$  is naturally equipped with a structure of one dimensional  $L_i$ -representation of  $G_K$ . By choosing a  $\mathbb{Q}_p$ -algebra embedding from  $L_i$  into our fixed  $\bar{\mathbb{Q}}_p$  for each  $i$ , we may regard  $L_i$  as a subfield of  $\bar{\mathbb{Q}}_p$ . Denote by  $\psi_i: G_K \rightarrow L_i^{\times}$  the character obtained by the  $L_i$ -linear  $G_K$ -action on  $V_i$ . Each  $\psi_i$  is crystalline since  $B$  has good reduction (note that the  $L_i$ -representation  $L_i(\psi_i)$  considered as a  $\mathbb{Q}_p$ -representation is isomorphic to  $V_i$ , which is a  $\mathbb{Q}_p$ -subrepresentation of  $V_p(B)$ ). It holds that

$$\det(T - \varphi^{f_K} \mid D_{\text{cris}}^K(V_i^{\vee})) = \prod_{\tau \in \Gamma_{L_i}} (T - \Phi_K(\tau \circ \psi_i)) \quad (4.5)$$

by Proposition 4.4 (2) and Lemma 4.5 (3). Now we set

$$f_B(T) := \det(T - \varphi^{f_K} \mid D_{\text{cris}}^K(V_p(B)^{\vee})).$$

Note that  $f_B(T)$  is equal to the characteristic polynomial  $\det(T - \text{Frob}_K \mid V_{\ell}(B)) = \det(T - \text{Frob}_K \mid V_{\ell}(\bar{B}))$  for any prime  $\ell \neq p$  (see (4.4)), which also coincides with the characteristic polynomial  $\det(T - \text{Frob}_K \mid V_{\ell}(\bar{A}))$ . Hence  $\text{Nr}_{k/\mathbb{Q}_p}(\pi)^{f_{K/k}}$  is a root of  $f_B(T)$  by the choice of  $\bar{A}$ . Since  $f_B(T)$  is the product of the polynomial (4.5) over all  $i$ , we have an equality  $\text{Nr}_{k/\mathbb{Q}_p}(\pi)^{f_{K/k}} = \Phi_K(\tau \circ \psi_i)$  for some  $i$  and some  $\tau \in \Gamma_{L_i}$ . Since  $\text{Nr}_{k/\mathbb{Q}_p}(\pi)$  is an element of  $\mathbb{Q}_p$ , we have  $\text{Nr}_{k/\mathbb{Q}_p}(\pi)^{f_{K/k}} = \Phi_K(\psi_i)$  by Lemma 4.5 (3). Note that  $\text{Nr}_{k/\mathbb{Q}_p}(\pi)^{f_{K/k}} = (\Phi_k(\text{Nr}_{k/\mathbb{Q}_p} \circ \chi_{\pi}))^{f_{K/k}} = \Phi_K(\text{Nr}_{k/\mathbb{Q}_p} \circ \chi_{\pi})$  since  $\Phi_k(\chi_{\pi}) = \pi$ . Hence we obtain

$$\Phi_K(\psi_i^{-1} \cdot (\text{Nr}_{k/\mathbb{Q}_p} \circ \chi_{\pi})) = 1. \quad (4.6)$$

On the other hand, replacing  $K$  by a finite extension so that  $K$  contains the Galois closure of  $L_i/\mathbb{Q}_p$ , we find  $\psi_i = \prod_{\sigma \in \Gamma_{L_i}} \sigma^{-1} \circ \chi_{\sigma L_i}^{r_{\sigma}}$  for some  $r_{\sigma} \in \{0, 1\}$  on  $I_K$ . Furthermore, we find  $\text{Nr}_{k/\mathbb{Q}_p} \circ \chi_{\pi} = \prod_{\sigma \in \Gamma_{L_i}} \sigma^{-1} \circ \chi_{\sigma L_i}$  as characters from  $I_K$  to  $L_i^{\times}$ . Then we have an equality  $\psi_i^{-1} \cdot (\text{Nr}_{k/\mathbb{Q}_p} \circ \chi_{\pi}) = \prod_{\sigma \in \Gamma_{L_i}} \sigma^{-1} \circ \chi_{\sigma L_i}^{1-r_{\sigma}}$  on  $I_K$  and this in particular implies that all the Hodge-Tate weights of  $L_i(\psi_i^{-1} \cdot (\text{Nr}_{k/\mathbb{Q}_p} \circ \chi_{\pi}))$  are non-negative. By (4.6) and Lemma 4.5 (5), we obtain  $\psi_i = \text{Nr}_{k/\mathbb{Q}_p} \circ \chi_{\pi}$  on  $G_K$ . Thus the Galois group  $G_{Kk_{\pi}}$  acts on  $V_i$  trivial. This in particular implies that  $V_p(B)^{G_{Kk_{\pi}}}$  is not zero, which is equivalent to say that  $B(Kk_{\pi})[p^{\infty}]$  is infinite. Therefore,  $k_{\pi}$  is not Kummer-faithful.

(3) First we consider the case where  $\hat{\pi}^2 = q$ . Calculating the  $p$ -adic valuation on both sides, we have  $2 \sum_{\sigma \in \Gamma_k} r_{\sigma} = [k : \mathbb{Q}_p]$ . On the other hand, taking the norm, we also have  $\text{Nr}_{k/\mathbb{Q}_p}(\hat{\pi})^2 = q^{[k : \mathbb{Q}_p]}$ . Since  $k$  is a Galois extension of  $\mathbb{Q}_p$ , we have  $\text{Nr}_{k/\mathbb{Q}_p}(\hat{\pi})^2 = \text{Nr}_{k/\mathbb{Q}_p}(\pi)^{2 \sum_{\sigma \in \Gamma_k} r_{\sigma}}$ . Hence we see that  $q^{-1} \text{Nr}_{k/\mathbb{Q}_p}(\pi)$  is a root of unity, and hence  $k_{\pi}$  is not Kummer-faithful by (2).

In the rest of the proof, we assume  $\hat{\pi}^2 \neq q$ . Then  $\mathbb{Q}(\hat{\pi})$  can not be embedded into  $\mathbb{R}$ . Moreover,  $\mathbb{Q}(\hat{\pi})$  must be a CM field since it is a totally imaginary quadratic extension

of the totally real number field  $\mathbb{Q}(\hat{\pi} + q/\hat{\pi})$ . The proof below is based on the method used in the proof of (2) but we need more careful treatments. The notable difference here is that we apply a CM lifting theorem due to Chai-Conrad-Oort [CCO14], which refines the Honda-Tate lifting theorem. Let  $\bar{A}$  be the simple abelian variety defined over  $\mathbb{F}_q$  which corresponds to the Weil  $q$ -number  $\hat{\pi}$  via Honda-Tate theory. We put  $g = \dim \bar{A}$ ,  $D = \text{End}_{\mathbb{F}_q}(\bar{A}) \otimes_{\mathbb{Z}} \mathbb{Q}$  and  $Z = \mathbb{Q}(\hat{\pi})$ . Since  $\hat{\pi}$  is (a  $\mathbb{Q}$ -conjugation of) the  $q$ -Frobenius map of  $\bar{A}$ ,  $Z$  is a subfield of  $D$ . It is a theorem of Tate (cf. Corollary 1.6.2.2 (3) of [CCO14]) that  $Z$  is a center of  $D$ ,  $D$  is a division algebra over  $Z$  of degree  $d^2$  for some integer  $d > 0$ , and  $2g = d \cdot [Z : \mathbb{Q}]$ . Tate moreover showed that there exists a maximal subfield  $L$  in  $D$  of degree  $d$  over  $Z$  which is a CM field. Now we claim that  $L = Z$ . For any field  $F$ , we denote by  $\text{Br}(F)$  the Brauer group of  $F$ . Let  $[D] \in \text{Br}(Z)$  be the class of  $D$ . We denote by  $\text{inv}_v: \text{Br}(Z_v) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$  the local invariant map of  $\text{Br}(Z_v)$  for any finite place  $v$  of  $Z$ . Here,  $Z_v$  is the completion of  $Z$  at  $v$ . It is known (cf. Corollary 1.6.2.2 (3) of [CCO14]) that  $\text{inv}_v([D]) = 0$  if  $v$  is not above  $p$  and  $\text{inv}_v([D]) = \frac{\text{ord}_v(\hat{\pi})f_v}{f} \pmod{\mathbb{Z}}$  if  $v$  is above  $p$ . By the assumption 1 of (iii) and the fact that  $Z$  has no real infinite place, we find that  $[D]$  is trivial; in fact, the local invariant maps induces an injection  $\text{Br}(Z) \hookrightarrow \bigoplus_v \text{Br}(Z_v)$  where  $v$  runs through all finite places of  $Z$ . On the other hand, the order of  $[D]$  in  $\text{Br}(Z)$  coincides with the square root  $d$  of  $[D : Z]$  by Theorem 1.2.4.4 of [CCO14]. Thus we have  $d = 1$  and the claim follows. Consequently, we obtain the fact that  $\bar{A}$  has complex multiplication over  $\mathbb{F}_q$  by the CM field  $L = Z$ .

Let  $\mathbb{Q}_q$  be the unramified extension of  $\mathbb{Q}_p$  of degree  $f$ . By Theorem 4.1.1 of [CCO14] (and the construction of “ $D$ ” in page 246–247 of *loc. cit.*), there exists a finite totally ramified extension  $K/\mathbb{Q}_q$  and an abelian variety  $B$  over  $K$  with good reduction such that  $\bar{A}$  is  $\mathbb{F}_q$ -isogenous to the reduction of  $B$ . Moreover, we can take  $B$  so that  $B$  has complex multiplication over  $K$  by the same chosen CM field as  $\bar{A}$ ; thus, we may suppose that  $B$  has complex multiplication by  $L$ . Let  $\prod_i L_i$  denote the decomposition of  $L \otimes_{\mathbb{Q}} \mathbb{Q}_p$  into a finite product of finite extensions of  $\mathbb{Q}_p$  and let  $V_p(B) \simeq \bigoplus_i V_i$  be the corresponding decomposition of  $G_K$ -modules. Each  $V_i$  is naturally equipped with a structure of one dimensional  $L_i$ -representation of  $G_K$ . By choosing a  $\mathbb{Q}_p$ -algebra embedding from  $L_i$  into our fixed  $\overline{\mathbb{Q}}_p$  for each  $i$ , we may regard  $L_i$  as a subfield of  $\overline{\mathbb{Q}}_p$ . We denote by  $\psi_i: G_K \rightarrow L_i^\times$  the character obtained by the  $G_K$ -action on  $V_i$ . By a similar argument of the proof of (2), we know that the set of the roots of  $\det(T - \text{Frob}_K | V_\ell(\bar{A}))$  is  $\{\Phi_K(\tau \circ \psi_i) \mid \tau \in \Gamma_{L_i}, i\}$ . Furthermore,  $\hat{\pi}$  is also a root of this polynomial since  $K$  is totally ramified over  $\mathbb{Q}_q$ . Thus we obtain

$$\Phi_K(\tau \circ \psi_i) = \hat{\pi} \tag{4.7}$$

for some  $i$  and some  $\mathbb{Q}_p$ -algebra embedding  $\tau: L_i \hookrightarrow \overline{\mathbb{Q}}_p$ . Since the left hand side of (4.7) is contained in  $\tau L_i$ , we have  $\tau L_i \supset \mathbb{Q}_p(\hat{\pi})$ . In particular, we have  $[L_i : \mathbb{Q}_p] = n \cdot [\mathbb{Q}_p(\hat{\pi}) : \mathbb{Q}_p]$  for some integer  $n \geq 1$ . Since  $[L_i : \mathbb{Q}_p]$  coincides with the degree of some irreducible factor of  $f(X)$  in  $\mathbb{Q}_p[X]$ , it follows from the assumption 2 of (iii) that  $n = 1$ , that is,

$$\tau L_i = \mathbb{Q}_p(\hat{\pi}). \tag{4.8}$$

Note that (4.7) implies  $\Phi_K(\tau \circ \psi_i) = \Phi_k \left( \prod_{\sigma \in \Gamma_k} \sigma \circ \chi_\pi^{r_\sigma} \right)$ . Taking  $f_{Kk}$ -th power of this equality, we obtain  $\Phi_{Kk}((\tau \circ \psi_i)^{f_K}) = \Phi_{Kk} \left( \left( \prod_{\sigma \in \Gamma_k} \sigma \circ \chi_\pi^{r_\sigma} \right)^{f_k} \right)$ . Here we remark that, by (4.8), we may consider  $\tau \circ \psi_i$  as a crystalline character of  $G_K$  with values in  $k^\times$ .

Combining this with the assumption that  $k$  is a Galois extension of  $\mathbb{Q}_p$  and the assumption (ii), it follows from Lemma 4.5 (6) that we have

$$(\tau \circ \psi_i)^{f_K} = \prod_{\sigma \in \Gamma_k} \sigma \circ \chi_\pi^{m_\sigma}$$

on  $G_{K'}$  for some finite extension  $K'$  of  $Kk$  and some integers  $m_\sigma$ . Thus the character  $\tau \circ \psi_i$  restricted to  $G_{K'k_\pi}$  has values in the set of  $f_K$ -th roots of unity, and hence  $\psi_i$  is trivial on  $G_{K''k_\pi}$  for some finite extension  $K''$  of  $K'$ , which in particular implies that  $B(K''k_\pi)[p^\infty]$  is infinite. Therefore, we conclude that  $k_\pi$  is not Kummer-faithful.  $\square$

Theorem 4.2 naturally leads us to consider the following question.

**Question.** Let  $k$  be a Galois extension of  $\mathbb{Q}_p$ . Are the following conditions equivalent?

- (i)  $k_\pi$  is not Kummer-faithful.
- (ii) Either of the following holds:
  - (a)  $q^{-1}\text{Nr}_{k/\mathbb{Q}_p}(\pi) \in \mu_{p-1}$ .
  - (b) For some  $(r_\sigma)_{\sigma \in \Gamma_k}$  with  $r_\sigma \in \{0, 1\}$ , it holds  $\hat{\pi} := \prod_{\sigma \in \Gamma_k} \sigma \pi^{r_\sigma} \in W(q)$ .

The implications (i)  $\Rightarrow$  (ii) and (ii-a)  $\Rightarrow$  (i) were shown in Theorem 4.2 (1) and Proposition 4.1, respectively. Thus the remaining problem is to determine whether (ii-b) always implies (i). At the moment, the author does not have an answer for this problem. Theorem 4.2 (2) gives a partial answers for this; if  $\hat{\pi}$  is the norm  $\text{Nr}_{k/\mathbb{Q}_p}(\pi)$ , then  $\hat{\pi} \in W(q)$  implies (i). It is natural to ask whether there actually exists an example in which  $k_\pi$  is not Kummer-faithful, even though  $q^{-1}\text{Nr}_{k/\mathbb{Q}_p}(\pi) \notin \mu_{p-1}$ ,  $\hat{\pi} \neq \text{Nr}_{k/\mathbb{Q}_p}(\pi)$  and  $\hat{\pi} \in W(q)$ . By applying Theorem 4.2 (3), we construct such examples in the next section.

## 4.2 Examples of non-Kummer-faithful Lubin-Tate extension fields

Here we give an example of a non-Kummer-faithful Lubin-Tate extension field by applying Theorem 4.2 (3). Let  $r$  be a divisor of  $p - 1$ . Let  $F_1, F_2, \dots, F_r$  be imaginary quadratic fields such that  $p$  splits completely by principal ideals for each  $F_i$  and  $\text{Gal}(F/\mathbb{Q}) \simeq \prod_{i=1}^r \text{Gal}(F_i/\mathbb{Q})$ , where  $F$  is the composite field of  $F_1, F_2, \dots, F_r$ . Let  $\mathfrak{p}$  be a finite place of  $F$  above  $p$ . Denote by  $\mathfrak{p}_i$  the finite place of  $F_i$  below  $\mathfrak{p}$  and take a generator  $\omega_i$  of  $\mathfrak{p}_i$ . Let  $\omega_i^c$  be the complex conjugate of  $\omega_i$ ; we have  $\omega_i^c = \omega_i^{-1}p$ . We set

$$\pi_0 := \omega_1 \cdot \prod_{i=2}^r \omega_i^c \quad \text{and} \quad \pi := \pi_0^{1/r}.$$

We fix an embedding  $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$  with respect to  $\mathfrak{p}$ . With this embedding,  $\omega_1$  is a uniformizer of  $\mathbb{Q}_p$  and  $\omega_2^c, \dots, \omega_r^c$  are  $p$ -adic units. Set  $k := \mathbb{Q}_p(\pi)$ . Then  $k$  is a totally ramified extension of  $\mathbb{Q}_p$  of degree  $r$  and  $\pi$  is its uniformizer. Since  $r$  divides  $p - 1$ ,  $k$  is a Galois extension of  $\mathbb{Q}_p$ . The minimal polynomial  $f(X)$  of  $\pi$  over  $\mathbb{Q}$  is a divisor of  $\prod_{i=1}^r \prod_{\sigma_i \in \{1, c\}} (X^r - \omega_1^{\sigma_1} \cdots \omega_r^{\sigma_r})$  and each  $\omega_1^{\sigma_1} \cdots \omega_r^{\sigma_r}$  is of the form  $p^a \omega_1^{\pm 1} \cdots \omega_r^{\pm 1}$ . Hence any irreducible factor in  $\mathbb{Q}_p[X]$  of  $f(X)$  is a divisor of some  $X^r - \omega_1^{\sigma_1} \cdots \omega_r^{\sigma_r}$ . By Theorem 4.2 (3), we obtain that  $k_\pi$  is not Kummer-faithful.

**Proposition 4.6.** *Let  $\mathbb{Q}_p \subset k_0 \subset k$  be finite extensions. Let  $\pi_0$  and  $\pi$  be uniformizers of  $k_0$  and  $k$ , respectively. Denote by  $k_{0,\pi_0}$  and  $k_\pi$  the Lubin-Tate extensions of  $k_0$  and  $k$  associated with  $\pi_0$  and  $\pi$ , respectively. Let  $f$  be the residue degree of  $k/k_0$ . If  $\pi_0^{-f} \text{Nr}_{k/k_0}(\pi)$  is a root of unity and  $k_{0,\pi_0}$  is not Kummer-faithful, then  $k_\pi$  is not Kummer-faithful.*

*Proof.* The result follows from local class field theory; if  $\pi_0^{-f} \text{Nr}_{k/k_0}(\pi)$  is a root of unity, then a finite extension of  $k_\pi$  contains  $k_{0,\pi_0}$ .  $\square$

Let  $k_0$  be a  $p$ -adic field and  $\pi_0$  a uniformizer of  $k_0$  so that  $k_{0,\pi_0}$  is not Kummer-faithful. If we choose  $k$  and  $\pi$  by one of the following manner, it follows from Proposition 4.6 that  $k_\pi$  is not Kummer-faithful.

- (i) Let  $\pi$  be a  $n$ -th root of unity of  $\pi_0$  for an integer  $n > 0$  and set  $k := k_0(\pi)$ .
- (ii) Let  $k$  be a finite unramified extension of  $k_0$  and take a uniformizer  $\pi$  of  $k$  such that  $\pi_0^{-f} \text{Nr}_{k/k_0}(\pi) = 1$ . (The existence of such  $\pi$  is assured by, for example, [Ser68, Chapter V, §.2, Corollary of Proposition 3].)

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