CLASS GROUPS OF IMAGINARY BIQUADRATIC FIELDS

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ABSTRACT. We present two distinct families of imaginary biquadratic fields, each of which contains infinitely many members, with each member having large class groups. Construction of the first family involves elliptic curves and their quadratic twists, whereas to find the other family, we use a combination of elliptic and hyperelliptic curves. Two main results are used, one from Soleng [13] and the other from Banerjee and Hoque [3].

1. INTRODUCTION

The class number of number fields is one of the most intriguing and fundamental questions associated with field extensions. It is well known that there exist infinitely many quadratic fields, each with a class number divisible by a particular integer. The Cohen-Lenstra heuristic [6] predicts a certain behavior of the asymptotics of the class number of infinitely many real quadratic fields. We refer to [10] for additional related work in this direction.

In this paper, we study the class group of imaginary biquadratic fields. Currently, there is only a handful of literature available relating to class groups of biquadratic fields. Sime in [17] and [18] studied the class group of real biquadratic fields, but very little is known about the class group of imaginary biquadratic fields, and to our knowledge, Athaide et.al.[1] is the only work.

In this direction, we should also mention that the works of Buell [5], Buell-Call [4], Griffin-Ono-Tsai [11], are very interesting and important from the analytic perspective of class groups and its relations with elliptic curves.

This work is motivated towards shedding some light on the class group of imaginary biquadratic fields, and these fields (that we construct) are kind of geometric in the sense that their origin is from elliptic and hyperelliptic curves. The novelty of this work lies that we are

1

extending the Soleng's technique over imaginary quadratic fields and use the result in [3] to produce element in the class groups of imaginary biquadratic fields.

Let us begin by briefly stating our plan to move from curves to imaginary biquadratic fields.

We start with an elliptic curve E defined by

$$(1.1) \qquad E : y^2 = x^3 + ax + b,$$

where $a, b \in \mathbb{Q}$. Taking a positive square-free integer d and twisting E over $\mathbb{Q}(\sqrt{-d})$, we shall get an elliptic curve,

$$(1.2) \qquad E_d : y^2 = x^3 + d^2ax + d^3b,$$

and it is defined over $\mathbb{Q}(\sqrt{-d})$ for E_d.

We fix the notation of the twisted elliptic curve as E_d over $\mathbb{Q}(\sqrt{-d}) = K$. In the sequel, $cl(K)$ denotes the class group of K.

Note that if $P = (x, y)$ is a point on E, then $\left((x + s), y \right)$ is a point on

$$(1.3) \qquad y^2 = x^3 - 3x^2s + (3s^2 + a)x + (-s^3 - as + b)$$

for any integer s. A rational point $p = (x, y) = \left(A/C^2, B/C^3 \right)$ (here A, B and C are integers) gives a proper ideal $(A, -kB + \sqrt{-s^3 - as + b})$ (where $k \in \mathbb{Z}$ and satisfies $kC^3 + s_1 A = 1$ for some $s_1 \in \mathbb{Z}$) in imaginary quadratic field $\mathbb{Q}(\sqrt{-s^3 - as + b})$ with discriminant $D = 4(-s^3 - as + b)$ which is not a perfect square.

Thus, taking a torsion point of E_d over $\mathbb{Q}$ and applying Soleng's result (Theorem 4.1, [13]), we can go from elliptic curves to class groups of imaginary quadratic fields. Then by going up/going down theorem ([7], Chapter 15, §, 3, Theorem 26), we prove the following result.

Theorem 1.1. *Let d be a square-free integer and let E_d be the quadratic twist of an elliptic curve E, defined over $\mathbb{Q}(\sqrt{-d})$ such that d^3b is not a square. Let $E_d(\mathbb{Q})$ has a primitive element of order $n \leq 16$. Then there are infinitely many biquadratic fields $\mathbb{Q}(\sqrt{-d}, \sqrt{-p^3 - d^2pa + d^3b})$ such that the class group of each of them contains an element of order n or $2n$ with $n \neq 2$, provided that $(-p^3 - d^2pa + d^3b)$ is not a perfect square.*

Here primitive torsion point of an elliptic curve is defined in the beginning of §2.

In the second part, we consider the hyperelliptic curve

$$(1.4) \qquad y^2 = k^2 - x^l$$

for large l and $k \in \mathbb{Z}$. Then, using the construction described in Theorem 5.1 (see[3]), we get an element of order l in the class group of $\mathbb{Q}(\sqrt{k^2 - p^l})$.

This gives us a family of infinitely many imaginary quadratic fields (there exist infinitely many primes p such that the expression $(k^2 - p^l)$ is negative and is not a square). Thus, by the construction of Soleng ([13]), $\mathbb{Q}(\sqrt{-q^3 - d^2qa + d^3b})$ has an element of order n, provided that the elliptic curve E given by (1.1) has a n-torsion (here $n \leq 16$ according to Mazur's theorem). Next, we consider the compositum of the fields

$$\mathbb{Q}(\sqrt{k^2 - p^l}) \text{ and } \mathbb{Q}(\sqrt{-q^3 - d^2qa + d^3b})$$

to obtain a biquadratic field L that has an element of order ln. This is possible only if

$$\mathbb{Q}(\sqrt{k^2 - p^l}) \cap \mathbb{Q}(\sqrt{-q^3 - d^2qa + d^3b}) = \mathbb{Q},$$

that is, they are linearly disjoint. The second result is the following.

Theorem 1.2. *The imaginary biquadratic field $\mathbb{Q}(\sqrt{k^2 - p^l}, \sqrt{-q^3 - d^2qa + d^3b})$ has an element of order ln in the class group, where $n \leq 16$ and $n \neq 2$.*

In §2, we shall briefly discuss Soleng's technique ([13]), which is the stepping stone of this article. Then in §3, we shall prove Theorem (1.1) followed by an example. Finally, in §4, we shall prove Theorem (1.2)

2. Soleng's technique

Let $P = (x, y)$ be a rational point on the elliptic curve

$$(2.1) \qquad\qquad Y^2 = X^3 + a_2X^2 + a_4X + a_6$$

defined over $\mathbb{Z}$. According to Soleng [13], P is a primitive element if $x, 2y$ and $x^2 + a_2x + a_4$ are pairwise coprime. By convention, the point at infinity is a primitive point.

The following Proposition ([13], Proposition 2.1, Page 3) confirms the group structure of these points on the curve.

Proposition 2.1. *The set of all primitive points in $E(\mathbb{Q})$ is a group.*

Let $P = (A/C^2, B/C^3)$ be a point on (2.1), then (A, B) is a point on the elliptic curve

$$Y^2 = X^3 + a_2C^2X^2 + a_4C^4X + a_6C^6.$$

Here $a_6 \neq 0$ and it is not a perfect square.

Let us consider the integral ideal given by

$$I = [A, -B + C^3\sqrt{a_6}].$$

Then I defines a proper integral ideal in the order of $\mathbb{Q}(\sqrt{a_6})$. The following homomorphism ([13], Theorem 2.1) gives the desired connection. Let a_6 be not a perfect square. Then we have the following theorem.

Theorem 2.2 (Theorem 2.1 [13]). *The homomorphism*

$$\Phi : E(\mathbb{Q})_{prim} \to cl(\mathbb{Q}[\sqrt{a_6}])$$

is given by

$$(A/C^2, B/C^3) \to (A, -kB + \sqrt{a_6})$$

where k is an integer satisfying

$$kC^3 + sA = 1$$

for some $s \in \mathbb{Z}$. Here, $cl(\mathbb{Q}[\sqrt{a_6}])$ denotes the class group of the field $\mathbb{Q}(\sqrt{a_6})$.

Therefore, if we start with an element of order n in $E(\mathbb{Q})$ which is a primitive element, then the image of this element under the homomorphism Φ is of order at most n. This homomorphism restricted on non-trivial torsion elements of the group $E(\mathbb{Q})_{prim}$ is non-trivial.

3. The connection between the class groups of biquadratic fields and elliptic curves

Soleng's main theorem [13], gives the connection between the rational points of the elliptic curves and the class group of certain number fields.

Now using Theorem (2.2), we shall prove Theorem (1.1).

First, we prove the following lemma.

Lemma 3.1. *Let E_d be the twist of the elliptic curve $E : y^2 = x^3 + ax + b$ defined over $\mathbb{Q}(\sqrt{-d})$, given by*

$$y^2 = x^3 + d^2 ax + d^3 b$$

Then the above homomorphism Φ is non-trivial from $(E_d)_{prim}$ to

$$\mathbb{Q}(\sqrt{-p^3 - d^2 p + d^3 b})$$

for infinitely many primes p (i.e. except for finitely many primes p) with loss of primitivity.

Proof. We need to verify that the hypothesis of Theorem 2.2 is satisfied for infinitely many primes p (except for finitely many primes with loss of primitivity). The equation of the twisted elliptic curve is

$$y^2 = x^3 + d^2 ax + d^3 b$$

If (x, y) is a point in this curve then

$$(x + p, y)$$

is a point on the curve given by the equation

$$y^2 = x^3 - 3px^2 + (3p^2 + d^2a)x + (-p^3 - d^2p + d^3b)$$

Let us call this curve as E_{1p}. By a result of[9], we know that the number

$$(-p^3 - d^2p + d^3b)$$

is not a perfect square for infinitely many primes p / or except for a finite set of primes. This can also be proven by using a celebrated result of Siegel [14], which states that, any algebraic curve with genus bigger than zero has finitely many integral points on it. Thus the elliptic curve

$$(3.1) \qquad\qquad y^2 = x^3 + d^2x + d^3b$$

has only finitely many integral solutions. So for any prime p for which the curve (3.1) has an non integral solution, the expression

$$(-p^3 - d^2p + d^3b)$$

will not a perfect square and these happens except finitely many primes p. Hence, in this way we can establish the positive density of the set of primes such that the above number is squarefree.

The other thing we need to check is that if (x, y) is a primitive point on E_d, then $(x + p, y)$ is a primitive point on E_{1p}. As (x, y) is a primitive point, we know that $x, 2y, x^2 + d^2a$ are pairwise coprime. Thus, it follows that $x + p, 2y, (x + p)^2 + d^2a$ are pairwise coprime for infinitely many primes p. Hence $(x + p, y)$ defines a primitive point on E_{1p} for each such prime p, and then we can follow Soleng's construction of Φ. $\qquad\square$

Now using Theorem (2.2) and the above lemma, we shall prove Theorem (1.1).

Proof. Let us take an elliptic curve E defined over $\mathbb{Q}$ given by the equation (1.1) and we consider its twist over the field $\mathbb{Q}(\sqrt{-d})$, denoted by E_d, which is given by the equation (1.2). This quadratic twist is defined over $\mathbb{Q}(\sqrt{-d})$. Now, if we consider the elliptic curve E_d and a non-trivial torsion α on E_d, then the specialization of the element α at the point $-p$, denoted by α_p is in the class group of $F = \mathbb{Q}(\sqrt{-p^3 - d^2p + d^3b})$ by the main theorem of Soleng [13] and by 3.1. Then by the going up/going down theorem for integral extensions

(see [7], Chapter 15, §3, Theorem 26), we have an ideal β_p in $cl(FK)$ such that $N_{FK/K}(\beta_p) = \alpha_p \neq 0$, and then we have

$$j_{FK/K}N_{FK/K}(\beta_p) = 2\beta_p \ .$$

Here $j_{FK/K}, N_{FK/K}$ denote the transfer and the Norm homomorphism at the level of class groups of number fields.

Lemma 3.2. *Suppose α_p is of order $n \neq 2$, then we have β_p is of order n or $2n$.*

Proof. This follows from the identity

$$N_{FK/K}(\beta_p) = \alpha_p$$

and

$$j_{FK/K}N_{FK/K}(\beta_p) = 2\beta_p \ .$$

Therefore, if $n\alpha_p = 0$ we have

$$N_{FK/K}(n\beta_p) = 0$$

and hence we have

$$j_{FK/K}N_{FK/K}(n\beta_p) = 2n\beta_p = 0$$

So the order of β_p divides $2n$. Now suppose that the order is less than $2n$. From the above identity it follows that $n\beta_p$ can be zero if it is not zero then $2n\beta_p = 0$. So, in the case when $n\beta_p \neq 0$ we have $2n\beta_p = 0$. If the order of β_p is less than $2n$ in this case, we have the order divides $2n$ but not n. Hence the order is 2. Then $2\alpha_p = 0$ but n is the order of α_p which is not equal to 2. Therefore we achieve a contradiction.

In the other case, $n\beta_p = 0$, so the order of β_p is less than n, say m. Then by the equality:

$$N_{FK/K}(\beta_p) = \alpha_p$$

we have

$$m\alpha_p = 0$$

which is not possible as the order of α_p is n and $m < n$. Therefore the lemma is proven.

□

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Now we shall give one, example to illustrate Theorem (1.1)

Example 3.3. *Let us consider the elliptic curve E given by the equation*

$$E : \ y^2 = x^3 + 16$$

over the rational numbers and its twist over

$$E_d : \ \mathbb{Q}(\sqrt{-3}) \ .$$

The equation of the twist E_{-3} is given by

$$y^2 = x^3 - 432 \ .$$

  Here, $E_{-3}(\mathbb{Q})_{tors} \cong \mathbb{Z}/3$.
  If we specialize the torsion point in $E_{-3}(\mathbb{Q})$ generating $\mathbb{Z}/3$, on the class group of $L = \mathbb{Q}(\sqrt{-3}, \sqrt{-p^3 - 432})$ for a prime p, we get a nontrivial element in the class group of L.
  If we call this element α, then by the previous theorem (1.1), the order of α is 3 or 6.

Such examples of elliptic curves can be found in the LMFDB database [12], and for each of them, we can make computations related to Soleng's result and produce elements in the class groups of imaginary quadratic fields.

4. Proof of Theorem (1.2)

Proof. In the proof of Theorem (1.2), we shall use the techniques of [3] to prove that $\mathbb{Q}(\sqrt{k^2 - p^l})$ has an element of order l in the class group. Also, by Soleng's technique, $\mathbb{Q}(\sqrt{-q^3 - qd^2a + d^3b})$ has an element of order n where $n \leq 16$ in the class group. Then the compositum has an element of order ln in the class group, provided the intersection of the individual fields in the compositum is $\mathbb{Q}$. This is equivalent to $\mathbb{Q}(\sqrt{k^2 - p^l})$ and $\mathbb{Q}(\sqrt{-q^3 - qd^2a + d^3b})$ are linearly disjoint field extensions $\mathbb{Q}$. This amounts to showing :

Lemma 4.1. *The fundamental discriminants satisfy*

$$gcd(k^2 - p^l, -q^3 - qd^2a + d^3b) = 1$$

for infinitely many primes p, q, such that both of the above numbers are squarefree and congruent to 1 modulo 4.

Proof. The claim mentioned above is true because if for infinitely many primes p, q the above numbers have a common divisor, then we have

$$d' | k^2 - p^l$$

$$d' | - q^3 - qd^2a + d^3b$$

and $d' > 1$. On the other hand, we have

$$-sp^l + t(-q^3) = 1$$

for some integers s, t. So,

$$d'|s(k^2 - p^l) + t(-q^3 - qd^2a + d^3b)$$

hence we have

$$d'|1 + sk^2 + td^3b - tqd^2a$$

So we have

$$1 + sk^2 + td^3b - tqd^2a = md'$$

So, it follows that d' is not a common factor of $k^2, d^3b - qd^2a$. Since one of the above numbers is k^2 which is fixed, then the number of common factors of the two above mentioned numbers is fixed and finite. Hence, the result follows.

$\square$

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References

[1] Elizabeth Athaide, Emma Cardwell and Christina Thompson, *Class Number Formulas for Certain Biquadratic Fields*, Hardy Ramanujan Journal, **46** (2023), 63-89.

[2] Antoniewicz, A., *On a family of elliptic curves*, Universitatis Iagellonicae Acta Mathematica, **1285** (2005), 21–32.

[3] K. Banerjee and A. Hoque, *Chow groups, pull back and class groups*, Monatsh Math (2024).`doi:10.1007/s00605-024-02008-3`.

[4] Buell, D., Call, G., *Class pairings and isogenies on elliptic curves*, Journal of Number Theory, **167** (2016), 31-73.

[5] Buell, D., *Elliptic Curves and Class Groups of Quadratic Fields*, Journal of LMS, **2** (1977), no. 15, 19-25.

[6] Cohen, H., Lenstra, H.(Jr), *Heuristics on class groups of number fields*, Number Theory Noordwijkerhout, 1984, Lecture notes in Mathematics, 33-62.

[7] Dummit, D.S., Foote, R., *Abstract Algebra*, 3rd Edition, Wiley , 2004.

[8] Darmon, H., Merel, L., *Winding quotients and some variants of Fermat's Last Theorem*, J. Reine Angew. Math., **490** (1997), 81-100.

[9] Hooley, C., *Application of Sieve Methods to the Theory of of Numbers*, Cambridge Univ. Press, London, New York, 1976.

[10] Fouvry, É., Klüners, J., *Cohen–Lenstra Heuristics of Quadratic Number Fields.* In: Hess, F., Pauli, S., Pohst, M. (eds) Algorithmic Number Theory. ANTS 2006. Lecture Notes in Computer Science, **4076**, Springer, Berlin, Heidelberg. $https : //doi.org/10.1007/11792086_4$.

[11] Griffin, M., Ono, Ken, Tsai, Wei-Lun, (2021), *Quadratic twists of elliptic curves and class numbers*, Journal of Number Theory, **227** (2021), 1-29.

[12] https://www.lmfdb.org/EllipticCurve/Q/.

[13] R.Soleng *Homomorphisms From the Group of Rational Points On Elliptic Curves to Class Groups of Quadratic Number Fields*, Journal of Number Theory, **46** (1994), no. 2, 214-229.

[14] C. L. Siegel, Uber einige Anwendungen Diophantischer Approximationen, Abh. Preuss. Akad. Wiss. Phys. Math. Kl. **1** (1929), 1-70; Ges. Abh., Band **1**, 209–266.

[15] J.H. Silverman and J.T. Tate, *Rational points on Elliptic Curves*, 2nd edition, Springer, 2015.

[16] J.H. Silverman, *The arithmetic of Elliptic Curves*, 2nd edition, Springer, 2009.

[17] P. Sime, *On the ideal class groups of real biquadratic fields*, Transactions of AMS, **347** (1995), no. 12, 4855-4876.

[18] P. Sime, *Hilbert Class Fields of Real Biquadratic Fields*, Journal of Number Theory, **50** (1995), no. 1, 154-166.

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