

# $\mathbb{Z}_p^m$ -ACTIONS OF TYPE $(d; p, n)$

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ABSTRACT. A  $\mathbb{Z}_p^m$ -action of type  $(d; p, n)$ , where  $2 \leq d \leq m \leq n$  are integers, is a pair  $(S, N)$  where  $S$  is a  $d$ -dimensional compact complex manifold,  $N \cong \mathbb{Z}_p^m$  is a group of holomorphic automorphisms of  $S$  such that the quotient orbifold  $S/N$  is the  $d$ -dimensional projective space  $\mathbb{P}^d$  whose branch locus consists of  $n+1$  hyperplanes in general position, each one of branch order  $p$ .

If  $(d; p, n) \notin \{(2; 2, 5), (2; 4, 3)\}$  and  $d+1 \leq n$ , then we prove that: (i)  $N$  is a normal subgroup of  $\text{Aut}(S)$  and (ii) if  $(S, M)$  is a  $\mathbb{Z}_{\hat{p}}^{\hat{m}}$ -action of type  $(d; \hat{p}, \hat{n})$ , then  $M = N$ . If, moreover,  $d+1 \leq n \leq 2d-1$ , then we observe that  $S$  is not algebraically hyperbolic.

## 1. INTRODUCTION

Let  $S$  be a compact complex manifold of dimension  $d \geq 1$ . Its group  $\text{Aut}(S)$  of holomorphic automorphisms is known to be a complex Lie group [2] and there is a natural short exact sequence  $1 \rightarrow \text{Aut}^0(S) \rightarrow \text{Aut}(S) \rightarrow \text{Aut}(S)/\text{Aut}^0(S)$ , where  $\text{Aut}^0(S)$  denotes the connected component of the identity. Let  $N$  be a subgroup of  $\text{Aut}(S)$  which acts properly discontinuously on  $S$ ; so, we have associated the quotient orbifold  $S/N$ . We are interested in the following two natural questions:

- (1) May we decide, in terms of the structure of the quotient orbifold  $S/N$ , if  $N$  is a normal subgroup of  $\text{Aut}(S)$ ?
- (2) Let  $M$  be another properly discontinuous subgroup of  $\text{Aut}(S)$ , which is isomorphic as an abstract group to  $N$  and such that the quotient orbifolds  $S/N$  and  $S/M$  are homeomorphic. May we decide, in terms of the structure of the quotient orbifold, if  $N = M$ ?

In this paper, we investigate the above questions in a very particular class of manifolds. More precisely, we consider those pairs  $(S, N)$ , where  $N \cong \mathbb{Z}_p^m$ ,  $m \geq 1$  and  $p \geq 2$  are integers, and the quotient orbifold  $S/N$  is the  $d$ -dimensional projective space  $\mathbb{P}^d$  whose branch locus consists of  $n+1$  hyperplanes in general position, each one of branch order  $p$ . Let us recall that the hyperplanes are in general position if: (i) the intersection of every subcollection of  $1 \leq k \leq d$  hyperplanes has dimension  $d-k$ , and (ii) every subcollection of  $k \geq d+1$  hyperplanes has empty intersection. In this situation, we will say that  $(S, N)$  is a  $\mathbb{Z}_p^m$ -action of type  $(d; p, n)$ . Necessarily,  $d \leq m \leq n$ , and  $S$  is known to be projective, i.e., it may be holomorphically embedded in some projective space (and  $\text{Aut}(S)$  is a group of biregular automorphisms). If  $n = d$ , then  $S$  is isomorphic to  $\mathbb{P}^d$ . If  $n = m = d+1$ , then  $S$  is isomorphic to the Fermat hypersurface of degree  $p$ .

**Theorem 1.** *Let  $(S, N)$  is a  $\mathbb{Z}_p^m$ -action of type  $(d; p, n) \notin \{(2; 2, 5), (2; 4, 3)\}$  and  $3 \leq d+1 \leq n$ . Then (i)  $\text{Aut}(S)$  is finite, (ii)  $N$  is a normal subgroup of  $\text{Aut}(S)$ , and (iii) if  $(S, M)$  is a  $\mathbb{Z}_q^r$ -action, then  $M = N$ .*

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We should note that the facts (ii) and (iii), in the previous result, are not generally true for the case of curves (i.e,  $d = 1$ ).

Examples of compact complex manifolds, for which the group of holomorphic automorphisms is finite, are provided by the so-called algebraically hyperbolic manifolds [3]. In [5], Demailly observed that every compact complex Kobayashi hyperbolic manifold is algebraically hyperbolic. In the same paper, he conjectured the converse.

Now, if  $(S, N)$  is a  $\mathbb{Z}_p^m$ -action of type  $(d; p, n) \notin \{(2; 2, 5), (2; 4, 3)\}$ , where  $3 \leq d + 1 \leq n$ , then  $\text{Aut}(S)$  is finite. It seems natural to ask if  $S$  is algebraically hyperbolic. The next result is a negative answer in some cases.

**Theorem 2.** *Let  $(S, N)$  be a  $\mathbb{Z}_p^m$ -action of type  $(d; p, n) \notin \{(2; 2, 5), (2; 4, 3)\}$ , where  $3 \leq d + 1 \leq n$ . If either (i)  $n \leq 2d - 1$ , or (ii)  $n = 2d$  and  $p \in \{2, 3\}$ , or (iii)  $n = 2d + 1$  and  $p = 2$ , then  $S$  is not algebraically hyperbolic, in particular, not Kobayashi hyperbolic.*

A natural question is whether the exceptional cases provided in the above result are the only ones for which  $S$  is not algebraically hyperbolic.

**Notations:** Suppose  $Y \subset \mathbb{P}^k$  is a smooth irreducible projective complex algebraic variety of dimension  $d$ . In that case, we will denote by  $\text{Aut}(Y)$  its group of all holomorphic automorphisms and by  $\text{Lin}(Y)$  its group of linear automorphisms (that is, its automorphisms obtained as the restriction of a projective linear transformation of  $\mathbb{P}^k$ ).

## 2. GENERALIZED FERMAT VARIETIES

As noticed above, the maximal value of  $m$ , in the definition of  $\mathbb{Z}_p^m$ -action of type  $(d; p, n)$ , is  $m = n$ . Also, as observed in [10],  $n \geq d$ .

**2.1. The group  $H$ .** Let  $n \geq 1, p \geq 2$  be integers. Set  $\omega_p = e^{2\pi i/p}$ . Let us consider the linear automorphisms  $\varphi_1, \dots, \varphi_{n+1} \in \text{PGL}_{n+1}(\mathbb{C})$  of  $\mathbb{P}^n$ , defined by

$$\varphi_j([x_1 : \dots : x_j : \dots : x_{n+1}]) := [x_1 : \dots : \omega_p x_j : \dots : x_{n+1}].$$

Then  $\varphi_1 \circ \dots \circ \varphi_{n+1} = 1$  and  $H := \langle \varphi_1, \dots, \varphi_n \rangle \cong \mathbb{Z}_p^n$ . We say that  $\{\varphi_1, \dots, \varphi_{n+1}\}$  is a set of canonical generators of  $H$ .

Let us denote by  $\text{Aut}_g(H)$  the group of automorphisms of  $H \cong \mathbb{Z}_p^n$  which correspond to permutations of the set of canonical generators  $\{\varphi_1, \dots, \varphi_{n+1}\}$ . Note that  $\text{Aut}_g(H) = \langle \Psi_1, \Psi_2 \rangle \cong \mathfrak{S}_{n+1}$ , where

$$\Psi_1 : (\varphi_1, \dots, \varphi_{n+1}) \mapsto (\varphi_2, \varphi_1, \varphi_3, \dots, \varphi_{n+1}), \quad \Psi_2 : (\varphi_1, \dots, \varphi_{n+1}) \mapsto (\varphi_{n+1}, \varphi_1, \varphi_2, \dots, \varphi_n).$$

**2.2. Generalized Fermat pairs.** A generalized Fermat pair of type  $(d; k, n)$  is a  $\mathbb{Z}_p^n$ -action  $(X, H_X)$  of type  $(d; p, n)$ . We also say that  $X$  is a generalized Fermat variety of type  $(d; p, n)$ , and that  $H_X$  is a generalized Fermat group of type  $(d; p, n)$ .

If  $d = 1$ , then  $X$  is a closed Riemann surface uniformized by the derived subgroup of a Fuchsian group of signature  $(0; p, \frac{n+1}{p}, p)$ ; we also say that  $X$  is a generalized Fermat curve of type  $(p, n)$ .

**2.3. Case  $n = d$ .** In this case, we may assume (up to biholomorphisms) that  $X = \mathbb{P}^d$ . The group  $H$  is a generalized Fermat group of type  $(d; p, d)$ . This is not the unique generalized Fermat group of such type, but any other is  $\text{PGL}_{d+1}(\mathbb{C})$ -conjugated to  $H$ .

**2.4. Case  $n = d + 1$ .** In this case, (up to biholomorphisms) we may assume that  $X = F_p = \{x_1^p + \cdots + x_{d+2}^p = 0\} \subset \mathbb{P}^{d+1}$ , the Fermat hypersurface of degree  $p$ . The group  $H$  is a generalized Fermat group of type  $(d; p, d + 1)$ . If (i)  $d \geq 2$  and  $(d, p) \neq (2, 4)$ , or (ii)  $d = 1$  and  $p > 3$ , then  $H$  is the unique generalized Fermat group of type  $(d; p, d + 1)$ , and  $\text{Aut}(X) = H \rtimes \mathfrak{S}_{d+2}$ , where  $\mathfrak{S}_{d+2}$  is the subgroup of  $\text{PGL}_{d+2}(\mathbb{C})$  given by permutations of the coordinates.

**2.5. Case  $n \geq d + 2$ .** Next, we recall the algebraic models of  $(X, H_X)$  and the uniqueness results for generalized Fermat groups.

**2.5.1. The parameter space  $\Omega_{n,d}$ .** Assume  $d \geq 1$ , and  $n \geq d + 2$  are integers. If  $\Lambda = (\lambda_{i,j}) \in M_{(n-d-1) \times d}(\mathbb{C})$ , then we may consider the collection  $\mathcal{B}(\Lambda)$  consisting of the following  $(n + 1)$  hyperplane in  $\mathbb{P}^d$ :

$$\begin{aligned}\Sigma_j &= \{[y_1 : \cdots : y_{d+1}] \in \mathbb{P}^d : y_j = 0\}, \quad j = 1, \dots, d + 1, \\ \Sigma_{d+2} &= \{[y_1 : \cdots : y_{d+1}] \in \mathbb{P}^d : y_1 + \cdots + y_{d+1} = 0\}, \\ \Sigma_{d+2+j}(\Lambda) &= \{[y_1 : \cdots : y_{d+1}] \in \mathbb{P}^d : \lambda_{j,1}y_1 + \cdots + \lambda_{j,d}y_d + y_{d+j} = 0\}, \quad j = 1, \dots, n - d - 1.\end{aligned}$$

Let us denote by  $\Omega_{n,d} \subset M_{(n-d-1) \times d}(\mathbb{C})$  the subset consisting of those  $\Lambda$  such that the above collection is in general position. This space is a connected, open, and dense subset of  $M_{(n-d-1) \times d}(\mathbb{C}) \cong \mathbb{C}^{(n-d-1)d}$ .

**2.5.2. A family of algebraic varieties parametrized by  $\Omega_{n,d}$ .** If  $\Lambda = (\lambda_{i,j}) \in \Omega_{n,d}$ , then we may consider the following algebraic variety

$$(1) \quad X_n^p(\Lambda) := \left\{ \begin{array}{rcl} x_1^p + \cdots + x_d^p + x_{d+1}^p + x_{d+2}^p & = & 0 \\ \lambda_{1,1}x_1^p + \cdots + \lambda_{1,d}x_d^p + x_{d+1}^p + x_{d+3}^p & = & 0 \\ \vdots & \vdots & \vdots \\ \lambda_{n-d-1,1}x_1^p + \cdots + \lambda_{n-d-1,d}x_d^p + x_{d+1}^p + x_{n+1}^p & = & 0 \end{array} \right\} \subset \mathbb{P}^n.$$

**Remark 1.** The variety  $X_n^p(\Lambda)$  is an irreducible nonsingular complete intersection projective variety of dimension  $d$ . So, if  $d \geq 2$ , then  $X_n^d(\Lambda)$  is simply connected (this result is attributed to Lefschetz; see [8]).

The following facts can be deduced from the above algebraic model of  $X_n^p(\Lambda)$  and the form of the elements  $\varphi_i$ .

- (I)  $\mathbb{Z}_p^n \cong H < \text{Aut}(X_n^p(\Lambda)) < \text{PGL}_{n+1}(\mathbb{C})$ .
- (II)  $\varphi_1\varphi_2 \cdots \varphi_{n+1} = 1$ .
- (III) The only non-trivial elements of  $H$  with fixed set points being of maximal dimension  $d - 1$  are the non-trivial powers of the generators  $\varphi_1, \dots, \varphi_{n+1}$ . Moreover, for  $d \geq 2$ ,  $\text{Fix}(\varphi_j) := \{x_j = 0\} \cap X_n^p(\Lambda)$  is isomorphic to a generalized Fermat variety of type  $(d - 1; k, n - 1)$ .
- (IV)  $\pi : X_n^p(\Lambda) \rightarrow \mathbb{P}^d : [x_1 : \cdots : x_{n+1}] \mapsto [x_1^p : \cdots : x_{d+1}^p]$  is a Galois branched cover with deck group  $H$ , whose branch locus is the collection  $\mathcal{B}(\Lambda)$ . In particular,  $(X_n^p(\Lambda), H)$  is a generalized Fermat pair of type  $(d; p, n)$ .

**Remark 2.** As a consequence of Randell's isotopy theorem [17], for  $\Lambda_1, \Lambda_2 \in \Omega_{n,d}$ , there is an orientation-preserving homeomorphism  $f : \mathbb{P}^d \rightarrow \mathbb{P}^d$  carrying  $\mathcal{B}(\Lambda_1)$  onto  $\mathcal{B}(\Lambda_2)$ . This homeomorphism lifts to an orientation-preserving homeomorphism  $h : X_n^p(\Lambda_1) \rightarrow X_n^p(\Lambda_2)$  such that  $hHh^{-1} = H$ .

The following fact was obtained in [10], as a consequence of the results in [14, 15].

**Theorem 3** ([10]). (1) The linear group  $\text{Lin}(X_n^p(\Lambda))$  consists of matrices such that only an element in each row and column is non-zero. (2) If  $(d; p, n) \notin \{(2; 2, 5), (2; 4, 3)\}$ , then  $\text{Aut}(X_n^p(\Lambda)) = \text{Lin}(X_n^p(\Lambda))$ .

**2.5.3. Algebraic equations of all generalized Fermat varieties.** Let  $(X, H_X)$  be a generalized Fermat pair of type  $(d; p, n)$  and let  $\pi : X \rightarrow \mathbb{P}^d$  be a Galois branched cover, with deck group  $H_X$ , and whose branch locus consists of  $(n+1)$  hyperplanes  $B_1, \dots, B_{n+1}$  which are in general position. Let us consider any permutation  $\sigma \in \mathfrak{S}_{n+1}$ . There is a unique  $T \in \text{PGL}_{d+1}(\mathbb{C})$  such that  $T(B_{\sigma^{-1}(i)}) = \Sigma_i$ , for  $i = 1, \dots, d+2$ . As the  $T$ -image of these  $(n+1)$  hyperplanes are in general position, there is a unique  $\Lambda = \Lambda_\sigma \in \Omega_{n,d}$  such that  $T(B_{\sigma^{-1}(d+2+j)}) = \Sigma_{d+1+j}(\Lambda)$ , for  $j = 1, \dots, n-1-d$ .

**Remark 3.** The above construction of  $T_\sigma \in \text{PGL}_{d+1}(\mathbb{C})$ , for each  $\sigma \in \mathfrak{S}_{n+1}$ , induces a one-to-one homomorphism  $\Theta : \mathfrak{S}_{n+1} \rightarrow \text{Aut}(\Omega_{n,d})$ . We set  $\mathbb{G}_{n,d} = \Theta(\mathfrak{S}_{n+1}) \cong \mathfrak{S}_{n+1}$ .

**Theorem 4** ([7], [10]). If  $n \geq d+2$  and  $(X, H_X)$  is a generalized Fermat pair of type  $(d; p, n)$ , then there is some  $\Lambda \in \Omega_{n,d}$  and a biholomorphism  $\phi : X \rightarrow X_n^p(\Lambda)$  such that  $\phi H_X \phi^{-1} = H$ . Moreover,  $\Lambda_1, \Lambda_2 \in \Omega_{n,d}$  produce isomorphic pairs if and only if they belong to the same  $\mathbb{G}_{n,d}$ -orbit.

**Remark 4.** The above result, for  $d \geq 2$ , may be seen as a consequence of Pardini's classification of abelian branched covers [16], and that of maximal branched abelian covers [1]. The proof of the case  $d = 1$  in [7] was obtained from Fuchsian group theory.

**2.6. A simple remark on the cohomological information of generalized Fermat varieties.** The fact that  $X := X_n^p(\Lambda)$  is a complete intersection variety allows us to compute the cohomology groups of the twisting sheaf  $\mathcal{O}_X(r)$  in a relatively direct way, and in particular, to obtain the following.

**Proposition 1.** Let  $d \geq 2$ ,  $\Lambda \in \Omega_{n,d}$ ,  $n \geq d+1$ , and  $X := X_n^p(\Lambda)$ . Set  $r_1 = (n-d)p - n - 1$ . Then

(1) The plurigenera  $P_m(X)$  of  $X$  satisfies

$$P_m(X) = \frac{p^{n-d}((n-d)p - n - 1)^d}{d!} m^d + O(m^{d-1}).$$

(2) The arithmetic genus  $p_a(X)$  and the geometric genus  $p_g(X)$  are given by

$$p_a(X) = p_g(X) = \begin{cases} 0 & \text{if } r_1 < 0 \\ \binom{r_1+n}{n} & \text{if } 0 \leq r_1 < p \\ \sum_{j \in \Delta_{r_1}} \binom{r_1-j+d}{d} & \text{if } r_1 \geq p \end{cases}$$

(3) If  $(n-d)p - n - 1 = 0$ , then  $X$  is a Calabi-Yau variety.

(4) If  $d = 2$ , then  $X$  is a general type surface except for the rational varieties cases  $(p, n) \in \{(2, 3), (3, 3), (2, 4)\}$  and the K3 varieties  $(p, n) \in \{(4, 3), (2, 5)\}$ .

*Proof.* Let  $\mathbb{C}[x_1, \dots, x_m]_l$  be the homogeneous polynomials of degree  $l$ .

(a) We first proceed to describe the cohomology groups of the twisting sheaf  $\mathcal{O}_X(r)$ ,  $r \in \mathbb{Z}$ .

(a1) Let  $\Delta_r := \{(j_1, \dots, j_{n-d}) \in \mathbb{Z}^{n-d} : 0 \leq j_i \leq p-1, 0 \leq i \leq n-d, \text{ and } \bar{j} := j_1 + j_2 + \dots + j_{n-d} \leq r\}$ . Then

$$H^0(X, \mathcal{O}_X(r)) := \begin{cases} 0 & \text{if } r < 0 \\ \mathbb{C}[x_1, \dots, x_{n+1}]_r & \text{if } 0 \leq r < p \\ \bigoplus_{j \in \Delta_r} Q_j & \text{if } r \geq p \end{cases}$$

where  $Q_j := \mathbb{C}[x_1, \dots, x_{d+1}]_{(r-\bar{j})} x_{d+2}^{j_1} x_{d+3}^{j_2} \cdots x_{n+1}^{j_{n-d}}$ ,  $j := (j_1, \dots, j_{n-d})$ .

- (a2) By Grothendieck's vanishing theorem,  $H^i(X, \mathcal{O}_X(r)) = 0$  for  $i > d$ , and  $r \in \mathbb{Z}$ ,
- (a3) and, as  $X$  is a complete intersection variety,  $H^i(X, \mathcal{O}_X(r)) = 0$  for  $0 < i < d$ , and  $r \in \mathbb{Z}$  (see page 231 of [9]).
- (a4) Finally, using the Serre duality,  $H^d(X, \mathcal{O}_X(r)) \cong H^0(X, \mathcal{O}_X(r_1 - r))$ .  
Remember that  $\omega_X \cong \mathcal{O}_X(r_1)$  (see page 188 of [9]).
- (b) With the former, we can calculate the plurigenus of  $X$

$$P_m(X) = \dim_{\mathbb{C}} H^0(X, \omega_X^{\otimes m}) = \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(r_m))$$

where  $r_m := mr_1 = m((n-d)p - n - 1)$ .

- (b1) If  $(n-d)p - n - 1 < 0$ , we obtain that  $P_m(X) = 0$ . This implies that the Kodaira dimension of  $X$  is  $\kappa(X) = -\infty$ .
- (b2) If  $(n-d)p - n - 1 = 0$ , we obtain that  $P_m(X) = 1$ . This implies that the Kodaira dimension of  $X$  is  $\kappa(X) = 0$ .
- (b3) If  $(n-d)p - n - 1 > 0$ , the canonical sheaf is very ample and

$$P_m(X) = \begin{cases} \binom{r_m+n}{n} & \text{if } 0 \leq r_m < p \\ \sum_{j \in \Delta_{r_m}} \binom{r_m-\bar{j}+d}{d} & \text{if } r_m \geq p \end{cases}$$

In particular, if  $r_m \geq \max\{p, (n-d)(p-1)\}$ , we obtain the assertion (1).

This implies that the Kodaira dimension of  $X$  is  $\kappa(X) = d$ .

- (c) The former also permits us to determine the arithmetic genus and geometric genus of  $X$ . As seen from the above,  $p_a(X) = p_g(X) = \dim_{\mathbb{C}} H^d(X, \mathcal{O}_X) = \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(r_1))$ , so, we obtain assertion (2).

□

**2.6.1. Uniqueness of generalized Fermat groups.** If  $n = d$ , then the generalized Fermat group is not unique (but it is unique up to conjugation).

**Theorem 5** ([11]). *If  $d = 1$  and  $(n-1)(p-1) > 2$ , then a generalized Fermat curve of type  $(p, n)$  has a unique generalized Fermat group.*

**Theorem 6** ([10]). *Let  $d \geq 2$  and  $(X, H_X)$  be a generalized Fermat pair of type  $(d; p, n) \notin \{(2; 2, 5), (2; 4, 3)\}$ . If  $\hat{H}$  is a generalized Fermat group of  $X$  of some type  $(d; \hat{p}, \hat{n})$ , then  $\hat{H} = H_X$ .*

*Proof.* We may assume  $X = X_n^p(\Lambda)$ , for some  $\Lambda \in \Omega_{n,d}$  and  $H_X = H$ .

Let  $\psi \in \hat{H}$  be an element whose fixed point locus has dimension  $d-1$  (i.e., a canonical generator for  $\hat{H}$ ). By Theorem 3,  $\psi \in \text{Lin}(X)$  corresponds to a matrix such that only an element in each row and column is non-zero. If such a matrix is not diagonal, then its locus of fixed points in  $\mathbb{P}^n$  is a linear subspace of codimension at least two; so  $\text{Fix}(\psi) \cap X$  cannot have dimension  $d-1$ , a contradiction. So,

$$\psi([x_1 : \cdots : x_{n+1}]) = [\alpha_1 x_1 : \cdots : \alpha_{n+1} x_{n+1}].$$

If  $[x_1 : \cdots : x_{n+1}] \in X$ , then as  $\psi \in \text{Aut}(X)$ , it follows that

$$(2) \quad \left\{ \begin{array}{lcl} \alpha_1^p x_1^p + \cdots + \alpha_d^p x_d^p + \alpha_{d+1}^p x_{d+1}^p + \alpha_{d+2}^p x_{d+2}^p & = & 0 \\ \lambda_{1,1} \alpha_1^p x_1^p + \cdots + \lambda_{1,d} \alpha_d^p x_d^p + \alpha_{d+1}^p x_{d+1}^p + \alpha_{d+3}^p x_{d+3}^p & = & 0 \\ \vdots & & \vdots \\ \lambda_{n-d-1,1} \alpha_1^p x_1^p + \cdots + \lambda_{n-d-1,d} \alpha_d^p x_d^p + \alpha_{d+1}^p x_{d+1}^p + \alpha_{n+1}^p x_{n+1}^p & = & 0 \end{array} \right\} \subset \mathbb{P}^n.$$

Since  $x_1^p + \cdots + x_d^p + x_{d+1}^p + x_{d+2}^p = 0$ , we may observe that  $\alpha_1^p = \cdots = \alpha_{d+1}^p = \alpha_{d+2}^p$ .

Since, for  $i = 1, \dots, n-d-1$ ,  $\lambda_{i,1}x_1^p + \dots + \lambda_{i,d}x_d^p + x_{d+1}^p + x_{d+2+i}^p = 0$ , we also observe that  $\alpha_1^p = \dots = \alpha_{d+1}^p = \alpha_{d+2+i}^p$ .

All of the above asserts that  $\psi \in H$  and that it has a  $(d-1)$ -dimensional locus of fixed points. So,  $\psi$  is a non-trivial power of one of the canonical generators of  $H$ .

The above asserts that  $\hat{H} \leq H$ . Now, by interchanging the roles of  $\hat{H}$  and  $H$  in the above, we also obtain that  $H \leq \hat{H}$ .  $\square$

**Remark 5.** The two exceptional cases  $(d; p, n) \in \{(2; 2, 5), (2; 4, 3)\}$  correspond to the only K3-surfaces among generalized Fermat surfaces. They have infinite group of holomorphic automorphisms, the corresponding linear subgroup has infinite index and it is non-normal. Anyway, inside the linear subgroup of automorphisms there is a unique generalized Fermat group.

**2.7. Automorphisms of generalized Fermat varieties.** As a consequence of Theorem 6, is the following fact, which together with Theorem 3 below, might be used to explicitly compute the full group of automorphismsm of a generalized Fermat variety.

**Corollary 1.** *Let  $d \geq 2$ ,  $p \geq 2$ ,  $n \geq d+1$  be integers and  $(d; p, n) \notin \{(2; 2, 5), (2; 4, 3)\}$ . Let  $(X, H)$  be a generalized Fermat pair of type  $(d; p, n)$ . If  $G_0$  is the  $\mathrm{PGL}_{d+1}(\mathbb{C})$ -stabilizer of the  $n+1$  branch hyperplanes of  $X/H = \mathbb{P}^d$ , then  $|\mathrm{Aut}(X)| = |G_0|p^n$  and, if the order of  $G_0$  is relatively prime with  $p$ , then  $\mathrm{Aut}(X) \cong H \rtimes G_0$ .*

*Proof.* We know that  $X$  admits a unique generalized Fermat group  $H$  of type  $(d; p, n)$ . Let  $\pi : X \rightarrow \mathbb{P}^d$  be a Galois branched covering, with  $H$  as its desk group, and let  $\{L_1, \dots, L_{n+1}\}$  be its set of branch hyperplanes. Let  $G_0$  be the  $\mathrm{PGL}_{d+1}(\mathbb{C})$ -stabilizer of these  $n+1$  branch hyperplanes. As  $H$  is a normal subgroup of  $\mathrm{Aut}(X)$ , it follows the existence of a homomorphism  $\theta : \mathrm{Aut}(X) \rightarrow G_0$ , with kernel  $H$ . As  $X$  is a universal branched cover, every element  $Q$  of  $G_0$  lifts to a holomorphic automorphism  $\tilde{Q}$  of  $X$ . Then there is a short exact sequence  $1 \rightarrow H \rightarrow \mathrm{Aut}(X) \xrightarrow{\rho} G_0 \rightarrow 1$ . In particular,  $|\mathrm{Aut}(X)| = |G_0|p^n$ . Also, by the Schur-Zassenhaus theorem [6], in the case that the order of  $G_0$  is relatively prime with  $p$ , then  $\mathrm{Aut}(X) \cong H \rtimes G_0$ .  $\square$

**Corollary 2.** *Let  $d \geq 2$  and  $p \geq 2$  be integers. If  $G_0$  be a finite subgroup of  $\mathrm{PGL}_{d+1}(\mathbb{C})$ , then there exists a generalized Fermat pair  $(X, H)$  of type  $(d; p, n)$ , for some  $n \geq d+1$ , such that  $\mathrm{Aut}(X/H) \cong G_0$ . In fact, for  $|G_0| \leq d+1$  we may assume  $n = d+1$  and, for  $|G_0| \geq d+2$ , we may assume  $n = |G_0|-1$ .*

*Proof.* If  $|G_0| \leq d+1$ , then take  $n = d+1$  and note that for the classical Fermat hypersurface  $F_p \subset \mathbb{P}^n$  of degree  $p$  one has that  $\mathrm{Aut}(F_p)/H$  contains the permutation group of  $d+1$  letters. Let us assume  $|G_0| \geq d+2$ . The linear group  $G_0$  induces a linear action on the space  $\mathbb{P}_{\text{hyper}}^d$  of hyperplanes of  $\mathbb{P}^d$ . As  $G_0$  is finite, we may find (generically) a point  $q \in \mathbb{P}_{\text{hyper}}^d$  whose  $G_0$ -orbit is a generic set of points. Such an orbit determines a collection of  $|G_0|$  lines in general position in  $\mathbb{P}^d$ . Let us observe that, by the generic choice, we may even assume the above set of points to have  $\mathrm{PGL}_{d+1}(\mathbb{C})$ -stabilizer exactly  $G_0$ , so the same situation for our collection of hyperplanes. Now, the results follow from Corollary 1.  $\square$

**2.8. Fixed points of elements of  $H$ .** Let us consider a generalized Fermat pair  $(X_p^n(\Lambda), H)$  of type  $(d; p, n)$ , where  $d \geq 2$ , and let  $\pi : X_p^n(\Lambda) \rightarrow \mathbb{P}^d$  be as previously defined in Section 2.5.2. The branch locus of  $\pi$  is the collection  $\mathcal{B}(\Lambda)$ , the union of the following  $n+1$  hyperplanes (in general position)

$$\Sigma_1, \dots, \Sigma_{d+2}, \Sigma_{d+3} = \Sigma_{d+3}(\Lambda), \dots, \Sigma_{n+1} = \Sigma_{n+1}(\Lambda).$$

Next, we describe those elements of  $H$  acting with fixed points on  $X_n^p(\Lambda)$ .

**Proposition 2.** *Let  $\varphi \in H$  be different from the identity. Then  $\varphi$  has fixed points on  $X_n^p(\Lambda)$  if and only if there exist  $1 \leq j \leq d$ ,  $1 \leq i_1 < \dots < i_j \leq n+1$ , and  $1 \leq m_{i_1}, \dots, m_{i_j} \leq p-1$ , such that  $\varphi := \varphi_{i_1}^{m_{i_1}} \circ \dots \circ \varphi_{i_j}^{m_{i_j}}$ .*

*Proof.* Let  $p \in X_n^p(\Lambda)$  be a fixed point of  $\varphi$ . Then  $\pi(p) \in \mathcal{B}(\Lambda)$ . Let  $1 \leq i_1 < \dots < i_j \leq n+1$  a maximal collection of indices so that  $p \in \Sigma_{i_1} \cap \dots \cap \Sigma_{i_j}$ . As the hyperplanes  $\Sigma_j$  are in general position, necessarily  $j \leq d$ . Now, the previous asserts that  $p \in \text{Fix}(\varphi_{i_1}) \cap \dots \cap \text{Fix}(\varphi_{i_j})$ , so  $\varphi \in \langle \varphi_{i_1}, \dots, \varphi_{i_j} \rangle$ . The converse is clear.  $\square$

**Remark 6.** Let  $d \geq 2$ ,  $n \geq d+1$ ,  $p \geq 2$ ,  $\Lambda \in \Omega_{n,d}$ ,  $X_n^p(\Lambda)$ . Let us consider an element  $\varphi \in H$ , different from the identity, acting with fixed points on  $X_n^p(\Lambda)$ . As seen above, we can write  $\varphi := \varphi_1^{m_1} \circ \dots \circ \varphi_{n+1}^{m_{n+1}} \in H$ , where there are  $1 \leq j \leq d$  and  $1 \leq i_1 < \dots < i_j \leq n+1$  such that (i)  $m_i = 0$  if and only if  $i \notin \{i_1, \dots, i_j\}$  and (ii)  $m_{i_1}, \dots, m_{i_j} \in \{1, \dots, p-1\}$ . For each  $l \in \{0, 1, \dots, p-1\}$ , set

$$L_l(\varphi) := \{j \in \{1, \dots, n+1\} : m_j = l\},$$

and the (possibly empty) algebraic sets

$$\widetilde{F}_l(\varphi) = \{[x_1 : \dots : x_{n+1}] \in \mathbb{P}^n : x_i = 0, \forall i \notin L_l(\varphi)\}, F_l(\varphi) := \widetilde{F}_l(\varphi) \cap X_n^p(\Lambda).$$

The locus of fixed points of  $\varphi$  in  $\mathbb{P}^n$  is the disjoint union of the algebraic sets  $\widetilde{F}_l(\varphi)$ .

Note that each  $\widetilde{F}_l(\varphi)$  is: (i) just a point if  $\#L_l(\varphi) = 1$ , and (ii) a projective linear space of dimension  $\#L_l(\varphi) - 1$  if  $\#L_l(\varphi) > 1$ . The locus of fixed points of  $\varphi$  on  $X_n^p(\Lambda)$  is then given as the disjoint union of the sets  $F_l(\varphi) = \widetilde{F}_l(\varphi) \cap X_n^p(\Lambda)$ . But on  $X_n^p(\Lambda)$  we cannot have points  $[x_1 : \dots : x_{n+1}]$  with at least  $d+1$  coordinates equal to zero. This fact asserts that for  $\#L_l(\varphi) \leq n-d$  one has that  $F_l(\varphi) = \emptyset$ . Also, for  $\#L_l(\varphi) \geq n+1-d$ , we obtain that  $F_l(\varphi) \neq \emptyset$  is a generalized Fermat variety of dimension  $\#L_l(\varphi) + d - n - 1$ .

In particular, its number of (non-empty) connected components (if non-empty) equals the number of exponents  $l$  appearing in  $\varphi$  at least  $n+1-d$  times.

**Example 1.** Let  $d \geq 2$ ,  $n \geq d+1$ ,  $p \geq 2$ ,  $\Lambda \in \Omega_{n,d}$ ,  $X := X_n^p(\Lambda)$ .

- (1) If  $p = 2$ , and  $\varphi \in H \cong \mathbb{Z}_2^n$ , different from the identity. In this case, we have only two sets to consider, say  $\#L_0(\varphi)$  and  $\#L_1(\varphi)$ , satisfying that  $\#L_0(\varphi) + \#L_1(\varphi) = n+1$ . By Proposition 6,  $\varphi$  has no fixed points on  $X_n^2(\Lambda)$  if and only if

$$\#L_0(\varphi), \#L_1(\varphi) \leq n-d.$$

Since,  $n+1 = \#L_0(\varphi) + \#L_1(\varphi) \leq (n-d) + (n-d)$ , necessarily  $n \geq 2d+1$ . In other words, if  $n \leq 2d$ , then  $H$  does not have non-trivial elements acting freely.

- (2) If  $d = 2$ , and  $\varphi \in H$ , different from the identity. By Proposition 2,  $\text{Fix}(\varphi) \neq \emptyset$  if and only if there exists some  $l \in \{0, 1, \dots, p-1\}$  such that  $\#L_l(\varphi) \geq n-1$ . In other words, if and only if  $\varphi$  is one of the following elements:  $\varphi_i^l$  or  $\varphi_j^s \circ \varphi_k^r$ , where  $l, r, s \in \{1, \dots, p-1\}$ , and  $i, j, k \in \{1, \dots, n+1\}$  with  $j \neq k$ .
- (3) Let us assume  $p \geq 2$  is a prime integer. Let  $K \cong \mathbb{Z}_p^{n-r}$  be a subgroup of  $H$  acting freely on  $X$ . Let  $F_j \subset X$ ,  $j = 1, \dots, n+1$ , be the locus of fixed points of the canonical generator  $\varphi_j$ . As  $H$  is an abelian group, each  $F_j$  is invariant under  $K$  and acts freely on it. Let  $S = X/K$  (which is a compact complex manifold of dimension  $d$ ) and  $X_j = F_j/K$  (a connected complex submanifold of  $S$ ). The  $(n+1)$  connected sets  $X_j$  are the locus of fixed points of the induced holomorphic automorphism by  $\varphi_j$ . As each two different  $F_i$  and  $F_j$  always intersect transversely, it follows that the same happens for  $X_i$  and  $X_j$ . As the locus of fixed points of (finite) holomorphic automorphisms is

smooth, it follows that different  $X_i$  and  $X_j$  are the fixed points of different cyclic groups of  $N = H/K \cong \mathbb{Z}_p^r$ . This in particular asserts that  $n+1 \leq (p^r - 1)/(p-1)$ . So, for instance, the cases (i)  $r=1$  and (ii)  $r=2$  and  $p=2$ , are impossible (note that this is in contrast to the case  $p=2$  and  $d=1$ , where these subgroups exist and are related to hyperelliptic Riemann surfaces).

- (4) Let  $n=p=3$  and  $d=2$ . In this case,  $X$  is just the Fermat hypersurface  $\{x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0\} \subset \mathbb{P}^3$ . If  $\varphi = \varphi_1\varphi_2\varphi_3^2$ , then  $(m_1, m_2, m_3, m_4) = (1, 1, 2, 0)$  and  $L_0(\varphi) = \{4\}$ ,  $L_1(\varphi) = \{1, 2\}$ ,  $L_2(\varphi) = \{3\}$ . The locus of fixed points (in  $\mathbb{P}^3$ ) of  $\varphi$  is given by

$$\begin{aligned} \widetilde{F}_0(\varphi) \cup \widetilde{F}_1(\varphi) \cup \widetilde{F}_2(\varphi) = \\ \{[0 : 0 : 0 : 1]\} \cup \{[x_1 : x_2 : 0 : 0] \in \mathbb{P}^3\} \cup \{[0 : 0 : 1 : 0]\}. \end{aligned}$$

As the cardinalities of  $L_0(\varphi)$  and  $L_2(\varphi)$  are at most equal to  $n-d$ , these two do not introduce fixed points of  $\varphi$  on  $X$  (this can be seen also directly). The set  $L_1(\varphi)$  has cardinality  $2 \geq n-d+1$ , so it produces a zero-dimensional set of fixed points consisting of the three points  $[1 : -1 : 0]$ ,  $[1 : \omega_6 : 0]$  and  $[1 : \omega_6^{-1} : 0]$ , where  $\omega_6 = e^{\pi i/3}$ .

- (5) Let us consider the case  $n=d+1$ , that is,  $X$  is the Fermat hypersurface of degree  $p$ . Let us consider an element  $\varphi \in H$ , different from the identity. Let us write

$$\varphi = \varphi_1^{m_1} \circ \cdots \circ \varphi_{d+1}^{m_{d+1}}, \quad 0 \leq m_i \leq p-1.$$

By Proposition 2, for  $\varphi$  to act freely on  $X$ , necessarily  $1 \leq m_i \leq p-1$ . Since  $\varphi_1 \circ \cdots \circ \varphi_{d+2} = 1$ , we also have that, for every  $i \in \{1, \dots, d+1\}$ ,

$$\varphi = \varphi_1^{m_1-m_i} \circ \cdots \circ \varphi_{i-1}^{m_{i-1}-m_i} \circ \varphi_{i+1}^{m_{i+1}-m_i} \cdots \circ \varphi_{d+1}^{m_{d+1}-m_i} \circ \varphi_{d+2}^{-m_i}.$$

So, for  $\varphi$  to acts freely, we must also have that  $m_j - m_i \not\equiv 0 \pmod{p}$ , for every  $i \neq j$ .

These conditions ensure that the existence of such  $\varphi$  obligates for  $p \geq d+2$ . Now, if  $p \geq d+2$ , then we may consider  $m_i = i$ , for  $i = 1, \dots, d+1$ , and set  $K = \langle \varphi \rangle \cong \mathbb{Z}_p$ . Then,  $(S = X/K, N = H/K)$  is a  $\mathbb{Z}_p^d$ -action of type  $(d; p; d+1)$ .

### 3. $\mathbb{Z}_p^m$ -ACTIONS OF TYPE $(d; p, n)$ , $d \geq 2$

In this section, we assume  $d \geq 2$ .

**3.1.  $\mathbb{Z}_p^m$ -actions as quotients of generalized Fermat varieties.** Let us consider a  $\mathbb{Z}_p^m$ -action  $(S, N)$  of type  $(d; p, n)$ , and let  $A = \text{Aut}(S)$  be the group of holomorphic automorphisms of  $S$ .

Let us consider a Galois branched cover  $\pi_N : S \rightarrow \mathbb{P}^d$  with deck group  $N \cong \mathbb{Z}_p^m$  and whose branch locus consists of  $(n+1)$  hyperplanes in general position. Up to postcomposition with a suitable element of  $\text{PGL}_{d+1}(\mathbb{C})$ , we may assume this  $(n+1)$  hyperplanes to be given by the collection  $\mathcal{B}(\Lambda)$ , for a suitable  $\Lambda \in \Omega_{n,d}$ .

As generalized Fermat varieties of type  $(d; p, n)$  are universal (branched) covers of orbifolds with underlying space  $\mathbb{P}^d$  and branch locus consisting of  $(n+1)$  hyperplanes in general position (each one of cone order  $p$ ), we may observe the following fact.

**Theorem 7.** *There is a subgroup  $\mathbb{Z}_p^{n-1} \cong K \triangleleft H$ , acting freely on  $X_n^p(\Lambda)$ , and a biholomorphism  $\phi : S \rightarrow X_n^p(\Lambda)/K$  such that  $\phi N \phi^{-1} = H/K$ . In particular, (i)  $m \leq n$ , and (ii) if  $m = n$ , then  $K = \{1\}$ .*

As a consequence of the above, we will assume (and this will be in what follows) that  $m \leq n-1$ .

Let us denote by  $\pi_K : X_p^n(\Lambda) \rightarrow S$  a Galois covering with deck group  $K$ . The fact that  $X_p^n(\Lambda)$  is simply connected ensures that  $A$  lifts, under  $\pi_K$ , to a group  $Q$  of biholomorphisms of  $X_p^n(\Lambda)$ , i.e., there is a short exact sequence

$$(3) \quad 1 \rightarrow K \rightarrow Q \xrightarrow{\rho} A \rightarrow 1,$$

where  $\pi_K \circ \psi = \rho(\psi) \circ \pi_K$ .

As  $H/K = N \leq A$ , it follows that  $H \leq Q$ . So, if  $(d; p, n) \notin \{(2; 2, 5), (2; 4, 3)\}$ , then the uniqueness of  $H$  ensures that  $H \triangleleft Q$ , i.e.,  $N \triangleleft A$ . In particular, the above short exact sequence determines (i) a short exact sequence

$$(4) \quad 1 \rightarrow N \rightarrow A \xrightarrow{\theta} L \rightarrow 1,$$

where  $\pi_N \circ \psi = \theta(\psi) \circ \pi_N$ ,  $L = A/N = Q/H$  is a subgroup of the  $\mathrm{PGL}_{d+1}$ -stabilizer of the configuration  $\mathcal{B}(\Lambda)$ , and (ii) a short exact sequence

$$(5) \quad 1 \rightarrow H \rightarrow Q \xrightarrow{\eta} L \rightarrow 1,$$

where  $\pi \circ \psi = \eta(\psi) \circ \pi$ .

In particular, if  $(p, |L|) = 1$ , then (by the Schur-Zassenhaus theorem),  $Q \cong H \rtimes L$  and  $A \cong K \rtimes L$ .

We have proved the following.

**Theorem 8.** *Let  $(S, N)$  be a  $\mathbb{Z}_p^m$ -action  $(S, N)$  of type  $(d; p, n) \notin \{(2; 2, 5), (2; 4, 3)\}$  and  $d \geq 2$ . Then*

- (1)  $N \triangleleft \mathrm{Aut}(S)$ .
- (2) *Let  $\pi : S \rightarrow \mathbb{P}^d$  be a Galois branched cover with deck group  $N$  and with branch locus  $\mathcal{B}$  being a collection of  $n + 1$  hyperplanes in general position. Then, there is a short exact sequence*

$$(6) \quad 1 \rightarrow N \rightarrow \mathrm{Aut}(S) \xrightarrow{\theta} L \rightarrow 1,$$

where  $\pi \circ \psi = \theta(\psi) \circ \pi$ , and  $L$  is a subgroup of the  $\mathrm{PGL}_{d+1}$ -stabilizer of  $\mathcal{B}$ .

**3.2. Uniqueness.** As already noticed, a generalized Fermat variety of type  $(d; p, n) \notin \{(2; 2, 5), (2; 4, 3)\}$  admits a unique generalized Fermat group. The following result states a similar uniqueness result for  $\mathbb{Z}_p^m$ -action  $(S, N)$  of type  $(d; p, n) \notin \{(2; 2, 5), (2; 4, 3)\}$  and  $d \geq 2$ .

**Theorem 9.** *Let  $d \geq 2$  and  $(S, N)$  be a  $\mathbb{Z}_p^m$ -action  $(S, N)$  of type  $(d; p, n) \notin \{(2; 2, 5), (2; 4, 3)\}$ . If  $(S, M)$  is a  $\mathbb{Z}_q^r$ -action of type  $(d; q, s)$ , then  $M = N$ .*

*Proof.* Assume  $S = X_n^p(\Lambda)/K$ . Let  $\hat{\psi} \in M \cong \mathbb{Z}_q^r$  be such that its locus of fixed points has dimension  $d - 1$ . Let us consider a lifting  $\psi \in \mathrm{Aut}(X_n^p(\Lambda))$  of  $\hat{\psi}$ . We may take  $\psi$  so that its locus of fixed points has dimension  $d - 1$ , so  $\psi \in H$  is a non-trivial power of some canonical generator. So,  $M \leq N$ . Now, by looking at the equations for  $H$  and  $X_n^p$ , we may observe that the only subgroup  $L$  of  $N$ , for which  $(S, L)$  is a  $\mathbb{Z}_p^m$ -action, is for  $L = N$ .  $\square$

#### 4. FREELY ACTING SUBGROUPS OF $H$

As previously seen, if  $(S, N)$  is a  $\mathbb{Z}_p^m$ -action of type  $(d; p, n)$ , then  $(S, N)$  is biholomorphically equivalent to  $(X_n^p(\Lambda)/K, H/K)$ , where  $\Lambda \in \Omega_{n,d}$  and  $K$  is a subgroup of  $H$  acting freely on  $X_n^p(\Lambda)$  such that  $H/K \cong \mathbb{Z}_p^m$ . The freely acting condition for  $K$  is, by Proposition 2, independent of the choice of  $\Lambda$ .

Let us denote by  $\mathcal{F}(d; p, n, m)$  the collection of the subgroups  $K$  of  $H$  such that:

- (1)  $H/K \cong \mathbb{Z}_p^m$ , and
- (2)  $K$  does not contain those  $\varphi_{i_1}^{l_1} \varphi_{i_2}^{l_2} \cdots \varphi_{i_j}^{l_j}$ , where  $1 \leq j \leq d$ ,  $l_j \in \{1, \dots, p-1\}$  and  $1 \leq i_1 < \dots < i_j \leq n+1$ .

Observe that this collection is invariant under the action of  $\text{Aut}_g(H)$ .

**Lemma 1.** *If  $d \geq 2$  and  $\mathcal{F}(d; p, n, m) \neq \emptyset$ , then  $d \leq m$ . Moreover, if  $m = d = 2$ , then  $p \geq 4$ .*

*Proof.* Let  $\theta : H \rightarrow \mathbb{Z}_p^m$  be a surjective homomorphism such that  $\ker(\theta) = K \in \mathcal{F}(d; p, n, m)$ . Let us set  $\theta(\varphi_j) = \phi_j$ . As  $\text{Aut}_g(H)$  keeps invariant  $\mathcal{F}(d; p, n, m)$ , up to precomposition of  $\theta$  by a suitable element of  $\text{Aut}_g(H)$ , we may assume that  $\theta(H) = \langle \phi_1, \dots, \phi_m \rangle$ .

As  $\varphi_1 \circ \cdots \circ \varphi_{n+1} = 1$ , we may observe that

$$K = \langle \varphi_1^{l_{m+1,1}} \circ \cdots \circ \varphi_m^{l_{m+1,m}} \varphi_{m+1}^{-1}, \dots, \varphi_1^{l_{n,1}} \circ \cdots \circ \varphi_m^{l_{n,m}} \varphi_n^{-1} \rangle.$$

So, if  $m < d$ , then  $K$  has elements of  $H$  acting with fixed points, a contradiction.

Let us now assume  $m = d = 2$ ,  $p \in \{2, 3\}$ , and that there is a surjective homomorphism  $\theta : H \rightarrow \mathbb{Z}_p^2$  such that  $\varphi_k, \varphi_i \varphi_j^l \notin K = \ker(\theta)$ , for  $l \in \{1, \dots, p-1\}$ . In particular,  $\langle \theta(\varphi_1) = \phi_1, \theta(\varphi_2) = \phi_2 \rangle = \mathbb{Z}_p^2$ . For  $j = 3, \dots, n+1$ ,  $\theta(\varphi_j) = \phi_1^{r_j} \phi_2^{s_j}$ , where  $r_j, s_j \in \{0, \dots, p-1\}$ . Since  $\varphi_j, \varphi_1 \varphi_j, \varphi_2 \varphi_j, \varphi_1 \varphi_j^{p-1}, \varphi_2 \varphi_j^{p-1} \notin K$ , then  $r_j = s_j \in \{1, 2\}$ . But, in this situation  $\varphi_3 \varphi_4$  or  $\varphi_3 \varphi_4^2 \in K$ , a contradiction.  $\square$

**4.0.1. Description of elements of  $\mathcal{F}(2; p, n, m)$ .** Let  $K \in \mathcal{F}(2; p, n, m)$ . By the definition of  $\mathcal{F}(2; p, n, m)$ ,  $K$  does not contain those non-trivial elements of the form  $\varphi_k, \varphi_i \varphi_j^l$ , where  $1 \leq k \leq n+1$ ,  $1 \leq i < j \leq n+1$ , and  $l \in \{1, \dots, p-1\}$ .

Let us consider a surjective homomorphism  $\theta_1 : H \rightarrow \mathbb{Z}_p^m$  whose kernel is  $K$ . There is a subset (not unique) of indices  $1 = i_1 < i_2 < \cdots < i_m \leq n+1$  such that  $\langle \phi_1 = \theta_1(\varphi_{i_1}), \dots, \phi_m = \theta_1(\varphi_{i_m}) \rangle = \mathbb{Z}_p^m$ . Let  $\Phi \in \text{Aut}_g(H)$  be such that  $\Phi^{-1}(\varphi_j) = \varphi_{i_j}$ , for  $j = 1, \dots, m$ . Then  $\Phi(K) \in \mathcal{F}(2; p, n, m)$  is the kernel of the surjective homomorphism  $\theta = \theta_1 \circ \Phi^{-1} : H \rightarrow \mathbb{Z}_p^m$ . Note that

$$\begin{aligned} \theta(\varphi_j) &= \phi_j, \quad j = 1, \dots, m, \\ \theta(\varphi_i) &= \phi_1^{r_{i,1}} \cdots \phi_m^{r_{i,m}}, \quad i = m+1, \dots, n+1, \end{aligned}$$

where the tuples  $(r_{i,1}, \dots, r_{i,m}) \in \{0, 1, \dots, p-1\}^m$  satisfy the following properties.

- (1)  $(\varphi_1 \cdots \varphi_{n+1} = 1)$   
 $1 + r_{m+1,i} + r_{m+2,i} + \cdots + r_{n+1,i} \equiv 0 \pmod{p}, \quad i = 1, \dots, m.$
- (2)  $(\varphi_i \notin K, \text{ for } i = m+1, \dots, n+1)$   
 $(r_{i,1}, \dots, r_{i,m}) \neq (0, \dots, 0), \quad i = m+1, \dots, n+1.$
- (3)  $(\varphi_k \varphi_i^l \notin K, \text{ for } k = 1, \dots, m, i = m+1, \dots, n+1, \text{ and } l = 1, \dots, p-1)$   
 $(r_{i,1}, \dots, r_{i,m}) \text{ cannot have } (m-1) \text{ of its coordinates equal to zero, for } i = m+1, \dots, n+1.$
- (4)  $(\varphi_i \varphi_j^l \notin K, \text{ for } m+1 \leq i < j \leq n+1, \text{ and } l = 1, \dots, p-1)$   
 $(r_{i,1} + lr_{j,1}, \dots, r_{i,m} + lr_{j,m}) \not\equiv (0, \dots, 0) \pmod{p}, \quad m+1 \leq i < j \leq n+1, \quad l = 1, \dots, p-1.$

In this case,

$$\Phi(K) = \langle \varphi_1^{r_{m+1,1}} \cdots \varphi_m^{r_{m+1,m}} \varphi_{m+1}^{-1}, \dots, \varphi_1^{r_{n,1}} \cdots \varphi_m^{r_{n,m}} \varphi_n^{-1} \rangle.$$

Summarizing the above is the following.

**Theorem 10.** Up to  $\text{Aut}_g(H)$ , the elements of  $\mathcal{F}(2; p, n, m)$  are given by the following normalized ones

$$K = \langle \varphi_1^{r_{m+1,1}} \cdots \varphi_m^{r_{m+1,m}} \varphi_{m+1}^{-1}, \dots, \varphi_1^{r_{n,1}} \cdots \varphi_m^{r_{n,m}} \varphi_n^{-1} \rangle,$$

where the exponents  $r_{i,j} \in \{0, 1, \dots, p-1\}$  satisfy the conditions (1)-(4) as described above.

4.0.2. The case  $d = p = 2$ . As already noticed in Lemma 1, in this case  $m \geq 3$ . In the following, we observe that, for  $m = 3$ , necessarily  $n = 6$ .

**Proposition 3.**

- (1)  $\mathcal{F}(2; 2, n, 3) \neq \emptyset$  if and only if  $n = 6$ . Moreover,  $\mathcal{F}(2; 2, 6, 3)/\text{Aut}_g(H)$  has exactly one element, this one represented by the group  $K = \langle \varphi_1 \varphi_2 \varphi_4, \varphi_1 \varphi_3 \varphi_5, \varphi_2 \varphi_3 \varphi_6 \rangle$ .
- (2)  $\mathcal{F}(2; 2, n, n-1) \neq \emptyset$ , for  $n \geq 5$ .
- (3)  $\mathcal{F}(2; 2, n, n-2) \neq \emptyset$ , for  $n \geq 6$ .
- (4)  $\mathcal{F}(2; 2, (m-1)(m+2)/2, m) \neq \emptyset$ , for  $m \geq 4$  even.
- (5)  $\mathcal{F}(2; 2, m(m+1)/2, m) \neq \emptyset$ , for  $m \geq 3$  odd.

*Proof.* Part (1): we may check by direct inspection that  $\mathcal{F}(2; 2, 4, 3) = \mathcal{F}(2; 2, 5, 3) = \emptyset$ . Assume  $\mathcal{F}(2; 2, n, 3) \neq \emptyset$ , where  $n \geq 6$ . Up to  $\text{Aut}_g(H)$ , there is a surjective homomorphism  $\theta : H \rightarrow \mathbb{Z}_2^3 = \langle \phi_1, \phi_2, \phi_3 \rangle$ , where  $\phi_j = \theta(\varphi_j)$ , for  $j = 1, 2, 3$ , and  $\varphi_k, \varphi_i \varphi_j \notin K = \ker(\theta)$ , where  $1 \leq k \leq n+1$ , and  $1 \leq i < j \leq n+1$ . Let us write, for  $j = 4, \dots, n+1$ ,  $\theta(\varphi_j) = \phi_1^{r_j} \phi_2^{s_j} \phi_3^{t_j}$ , where  $r_j, s_j, t_j \in \{0, 1\}$ . The condition that  $\varphi_j \notin K$  is equivalent to have that  $(r_j, s_j, t_j) \neq (0, 0, 0)$ . The condition that  $\varphi_i \varphi_j \notin K$ , for  $i \in \{1, 2, 3\}$  and  $j \in \{4, \dots, n+1\}$ , is equivalent to have that  $(r_j, s_j, t_j) \neq (1, 0, 0), (0, 1, 0), (0, 0, 1)$ . In particular,  $(r_j, s_j, t_j) \in \{(1, 1, 1), (1, 1, 0), (0, 1, 1), (1, 0, 1)\}$ . The condition that  $\varphi_i \varphi_j \notin K$ , for  $4 \leq i < j \leq n+1$  is equivalent to have that for different indices  $4 \leq i < j \leq n+1$ ,  $(r_i, s_i, t_i) \neq (r_j, s_j, t_j)$ . This ensures that  $n = 6$  and that, up to  $\text{Aut}_g(H)$ , we may choose  $(r_4, s_4, t_4) = (1, 1, 0)$ ,  $(r_5, s_5, t_5) = (1, 0, 1)$ ,  $(r_6, s_6, t_6) = (0, 1, 1)$ , and  $(r_7, s_7, t_7) = (1, 1, 1)$ .

Part (2): just consider the surjective homomorphism  $\theta : H \rightarrow \mathbb{Z}_2^{n-1} = \langle \phi_1, \dots, \phi_{n-1} \rangle$ , defined by  $\theta(\varphi_k) = \phi_k$ ,  $k = 1, \dots, n-1$ ,  $\theta(\varphi_n) = \phi_{i_1} \cdots \phi_{i_{l_1}}$ , and  $\theta(\varphi_{n+1}) = \phi_{j_1} \cdots \phi_{j_{l_2}}$ , where  $\{i_1, \dots, i_{l_1}\}$  and  $\{j_1, \dots, j_{l_2}\}$  is a disjoint partition of  $\{1, \dots, n-1\}$ , with  $l_1, l_2 \geq 2$ .

Part (3): just consider the surjective homomorphism  $\theta : H \rightarrow \mathbb{Z}_2^{n-2} = \langle \phi_1, \dots, \phi_{n-2} \rangle$ , defined by  $\theta(\varphi_k) = \phi_k$ ,  $k = 1, \dots, n-2$ ,  $\theta(\varphi_{n-1}) = \phi_{i_1} \cdots \phi_{i_{l_1}}$ ,  $\theta(\varphi_n) = \phi_{j_1} \cdots \phi_{j_{l_2}}$  and  $\theta(\varphi_{n+1}) = \phi_{k_1} \cdots \phi_{k_{l_3}}$ , where  $\{i_1, \dots, i_{l_1}\}$ ,  $\{j_1, \dots, j_{l_2}\}$ , and  $\{k_1, \dots, k_{l_3}\}$  is a disjoint partition of  $\{1, \dots, n-2\}$ , with  $l_j \geq 2$ .

Part (4): just consider the surjective homomorphism  $\theta : H \rightarrow \mathbb{Z}_2^m = \langle \phi_1, \dots, \phi_m \rangle$ , defined by  $\theta(\varphi_k) = \phi_k$ ,  $k = 1, \dots, m$ , and  $\{a_{m+1}, \dots, n+1\}$  are sent to  $\{\phi_1 \phi_2, \dots, \phi_{m-1} \phi_m\}$  bijectively.

Part (5): just consider the surjective homomorphism  $\theta : H \rightarrow \mathbb{Z}_2^m = \langle \phi_1, \dots, \phi_m \rangle$ , defined by  $\theta(\varphi_k) = \phi_k$ ,  $k = 1, \dots, m$ , and  $\{a_{m+1}, \dots, n\}$  are sent to  $\{\phi_1 \phi_2, \dots, \phi_{m-1} \phi_m\}$  bijectively, and  $\theta(\varphi_{n+1}) = \phi_1 \cdots \phi_m$ .  $\square$

**Example 2.** By Proposition 3, for the type  $\mathcal{F}(2; 2, 6, 3)/\text{Aut}_g(H)$  has cardinality one. A representative is

$$K = \langle \varphi_1 \varphi_2 \varphi_4, \varphi_1 \varphi_3 \varphi_5, \varphi_2 \varphi_3 \varphi_6 \rangle.$$

This provides the 6-dimensional family

$$\left\{ \left( S_\Lambda = X_6^2(\Lambda)/K, N_\Lambda = H/K \right) : \Lambda \in \Omega_{6,2} \right\}$$

of  $\mathbb{Z}_2^3$ -actions of type  $(2; 2, 6, 3)$ , all of them topologically conjugated. Below, we proceed to compute algebraic equations for these pairs  $(S_\Lambda, N_\Lambda)$ .

Let us first consider the affine model  $X(\Lambda) \subset \mathbb{C}^6$  of  $X_6^2(\Lambda)$  by taking  $x_7 = 1$ . In this affine model,  $K$  is generated by the linear transformations

$$\begin{aligned}\eta_1(x_1, \dots, x_6) &= (-x_1, -x_2, x_3, -x_4, x_5, x_6), \\ \eta_2(x_1, \dots, x_6) &= (-x_1, x_2, -x_3, x_4, -x_5, x_6), \\ \eta_3(x_1, \dots, x_6) &= (x_1, -x_2, -x_3, x_4, x_5, -x_6).\end{aligned}$$

A set of generators for the invariants  $\mathbb{C}[x_1, \dots, x_6]^K$  is

$$\begin{aligned}u_1 &= x_1^2, u_2 = x_2^2, u_3 = x_3^2, u_4 = x_4^2, u_5 = x_5^2, u_6 = x_6^2, u_7 = x_1 x_2 x_3, u_8 = x_1 x_4 x_5, \\ u_9 &= x_2 x_4 x_6, u_{10} = x_3 x_5 x_6, u_{11} = x_1 x_2 x_5 x_6, u_{12} = x_1 x_3 x_4 x_6, u_{13} = x_2 x_3 x_4 x_5.\end{aligned}$$

So, if we consider the map  $\Phi : \mathbb{C}^6 \rightarrow \mathbb{C}^{13}$ , defined by  $\Phi(x_1, \dots, x_6) = (u_1, \dots, u_{13})$ , then  $\Phi(X(\Lambda))$  is isomorphic to the affine model of  $S_\Lambda$ . The image (affine) surface  $\Phi(X(\Lambda))$  is defined by the following equalities

$$\begin{aligned}u_6 u_{13} &= u_9 u_{10}, u_5 u_{12} = u_8 u_{10}, u_1 u_2 u_3 = u_7^2, u_5 u_6 u_7 = u_{10} u_{11}, u_4 u_{11} = u_8 u_9, u_1 u_2 u_5 u_6 = u_{11}^2, \\ u_4 u_6 u_7 &= u_9 u_{12}, u_1 u_2 u_{10} = u_7 u_{11}, u_4 u_5 u_7 = u_8 u_{13}, u_3 u_{11} = u_7 u_{10}, u_1 u_3 u_4 u_6 = u_{12}^2, u_3 u_6 u_8 = u_{10} u_{12}, \\ u_3 u_5 u_9 &= u_{10} u_{13}, u_3 u_5 u_6 = u_{10}^2, u_3 u_8 u_9 = u_{12} u_{13}, u_2 u_{12} = u_7 u_9, u_1 u_3 u_9 = u_7 u_{12}, u_2 u_6 u_8 = u_9 u_{11}, \\ u_2 u_8 u_{10} &= u_{11} u_{13}, u_2 u_4 u_{10} = u_9 u_{13}, u_2 u_4 u_6 = u_9^2, u_1 u_4 u_5 = u_8^2, u_2 u_3 u_8 = u_7 u_{13}, u_2 u_3 u_4 u_5 = u_{13}^2, \\ u_1 u_{13} &= u_7 u_8, u_1 u_4 u_{10} = u_8 u_{12}, u_1 u_5 u_9 = u_8 u_{11}, u_1 u_9 u_{10} = u_{11} u_{12} \\ u_4 &= -u_1 - u_2 - u_3, u_5 = -\lambda_{1,1} u_1 - \lambda_{1,2} u_2 - u_3, u_6 = -\lambda_{2,1} u_1 - \lambda_{2,2} u_2 - u_3, u_3 = -\lambda_{3,1} u_1 - \lambda_{2,2} u_2 - 1.\end{aligned}$$

In this model, the group  $N = \langle \phi_1, \phi_2, \phi_3 \rangle$  is given by:

$$\begin{aligned}\phi_1 : &\begin{cases} u_i \mapsto -u_i, & i = 7, 8, 11, 12 \\ u_j \mapsto u_j, & \text{otherwise} \end{cases} \\ \phi_2 : &\begin{cases} u_i \mapsto -u_i, & i = 7, 9, 11, 13 \\ u_j \mapsto u_j, & \text{otherwise} \end{cases} \\ \phi_3 : &\begin{cases} u_i \mapsto -u_i, & i = 7, 10, 12, 13 \\ u_j \mapsto u_j, & \text{otherwise} \end{cases}\end{aligned}$$

**4.1. On topologically equivalence.** Two  $\mathbb{Z}_p^m$ -actions  $(S_1, N_1)$  and  $(S_2, N_2)$ , both of type  $(d; p, n)$ , are topologically equivalent if there is an orientation-preserving homeomorphism  $F : S_1 \rightarrow S_2$  such that  $F N_1 F^{-1} = N_2$ . Assume that  $S_j = X_n^p(\Lambda_j)/K_j$ , and  $N_j = H/K_j$ , where  $\Lambda_j \in \Omega_{n,d}$  and  $K_j \in \mathcal{F}(d; p, n, m)$ . Then, as  $X_n^p(\Lambda_j)$  are universal covers,  $F$  lifts to an orientation-preserving homeomorphism  $\tilde{F} : X_n^p(\Lambda_1) \rightarrow X_n^p(\Lambda_2)$  such that  $\tilde{F} K_1 \tilde{F}^{-1} = K_2$ . The homomorphism  $\tilde{F}$  induces, by the conjugation action, an element  $\Phi \in \text{Aut}_g(H)$ , which satisfies that  $\Phi(K_1) = K_2$ . We have obtained the following fact.

**Proposition 4.** *If  $K_1, K_2 \in \mathcal{F}(d; p, n, m)$  determine topologically equivalent  $\mathbb{Z}_p^m$ -actions of type  $(d; p, n)$ , then there exists some  $\Phi \in \text{Aut}_g(H)$  such that  $K_2 = \Phi(K_1)$ .*

Now, assume that we have  $K_1, K_2 \in \mathcal{F}(d; p, n, m)$  such that there is some  $\Phi \in \text{Aut}_g(H)$  satisfying  $K_2 = \Phi(K_1)$ . Is such  $\Phi$  induced by an orientation-preserving homeomorphism? If this is the case, then the above result will state that the number of topologically equivalent  $\mathbb{Z}_p^m$ -actions of type  $(d; p, n)$  is equal to the cardinality of  $\mathcal{F}(d; p, n, m)/\text{Aut}_g(H)$ . This is true for  $d = 1$  [12], but it is not clear for  $d \geq 2$ .

## 5. ON HYPERBOLICITY OF $\mathbb{Z}_p^m$ -ACTIONS

Let  $S$  be a compact complex manifold of dimension  $d \geq 2$ . The manifold  $S$  is Kobayashi hyperbolic if its Kobayashi pseudometric is non-degenerate. In [4], Brody observed that  $S$  is Kobayashi hyperbolic if and only if there is no non-constant holomorphic map  $f : \mathbb{C} \rightarrow S$ .

Assume that  $S$  is a projective variety. In [5], Demailly introduced an algebraic analogue for hyperbolicity. More precisely,  $S$  is called algebraically hyperbolic if there exists a positive constant  $A$  such that the degree of any curve of genus  $g$  on  $S$  is bounded from above by  $A(g-1)$ . In the same paper, Demailly proved that Kobayashi hyperbolicity implies algebraically hyperbolicity. By the definition, an algebraically hyperbolic manifold does not contain genus  $g \in \{0, 1\}$  curves.

In [3], Bogomolov, Kamenova, and Verbitsky proved that, if  $S$  is algebraically hyperbolic, then  $\text{Aut}(S)$  is finite (for the Kobayashi hyperbolic case, this was proved by Kobayashi in [13]).

Let us consider a  $\mathbb{Z}_p^m$ -action  $(S, N)$  of type  $(d; p, n)$ , where  $n \geq d + 1$ .

**5.1. Case  $m = n$  and  $(d; p, n) \in \{(2; 4, 3), (2; 2, 5)\}$ .** If  $(d; k, n) = (2; 4, 3)$ , then  $S$  corresponds to the classical Fermat hypersurface of degree 4 in  $\mathbb{P}^3$  for which  $\text{Lin}(S) \cong \mathbb{Z}_4^3 \rtimes \mathbb{S}_4$  and  $\text{Aut}(S)$  infinite; so  $S$  is not algebraically hyperbolic. If  $(d; k, n) = (2; 2, 5)$ , then  $\text{Lin}(S)$  is a finite extension of  $\mathbb{Z}_2^5$  (generically a trivial extension) and  $\text{Aut}(S)$  is infinite by results due to Shioda and Inose in [18, Thm 5] (in [19] Vinberg computed it for a particular case). So, again, these surfaces are not algebraically hyperbolic.

**5.2. Case  $m = n$  and  $(d; p, n) \notin \{(2; 4, 3), (2; 2, 5)\}$ .** Let us now assume that  $(d; p, n) \notin \{(2; 4, 3), (2; 2, 5)\}$ , where  $n \geq d + 1$ . In this case, we know that  $S$  is a compact projective complex manifold of dimension  $d$  with  $\text{Aut}(S)$  finite. We wonder if, in these cases,  $S$  is or is not algebraically hyperbolic.

**5.3. Case  $d + 1 \leq m \leq n \leq 2d - 1$ .** In the next result, we observe that, for  $n \leq 2d - 1$ ,  $S$  cannot be algebraically hyperbolic.

**Theorem 11.** *If  $(S, N)$  is a  $\mathbb{Z}_p^m$ -action of type  $(d; p, n)$ , where  $3 \leq d + 1 \leq n$ . Then, in the following situations,  $S$  is not algebraically hyperbolic.*

- (1)  $n \leq 2d - 1$ .
- (2)  $n = 2d$  and  $p \in \{2, 3\}$ .
- (3)  $n = 2d + 1$  and  $p = 2$ .

*Proof.* Let  $\pi_N : S \rightarrow \mathbb{P}^d$  be a Galois branched covering with deck group  $N$ , whose branch locus is given by the collection  $\mathcal{B}$ , consisting of the  $n + 1$  hyperplanes  $\Sigma_1, \dots, \Sigma_{n+1}$ , that are in general position. By the general position condition, the intersection of the planes  $\Sigma_1, \dots, \Sigma_d$  consists of a unique point  $\alpha$ .

(1) Let us first consider the case  $n \leq 2d - 1$ . Now, let us consider the intersection of the  $n + 1 - d$  hyperplanes  $\Sigma_{d+1}, \dots, \Sigma_{n+1}$ , which is non-empty since  $n + 1 - d \leq d$ . Again, by the general position condition, we can find a point  $\beta$  in that intersection that does not belong to  $\Sigma_j$ , for  $j = 1, \dots, d$ . Let  $L \subset \mathbb{P}^d$  the line connecting  $\alpha$  with  $\beta$ . We observe that  $L \cap \mathcal{B}(\Lambda) = \{\alpha, \beta\}$ . Set  $L^* = L \setminus \{\alpha, \beta\} \cong \mathbb{C} \setminus \{0\}$ . Let  $\hat{L}$  be any connected component of  $\pi_N^{-1}(L^*)$ , which is a Riemann surface that finitely covers  $L^*$ . In this way, inside  $S$  we have a genus zero curve (by adding the two missing points to  $\hat{L}$ ), so  $S$  cannot be algebraically hyperbolic.

(2) Let us now assume that  $n = 2d$ . We proceed similarly as in the previous case, but in this case, we consider the intersection of the  $d$  hyperplanes  $\Sigma_{d+1}, \dots, \Sigma_{2d}$ ; which is a point  $\beta$ . We consider the line  $L \subset \mathbb{P}^d$  connecting  $\alpha$  and  $\beta$ . In this case,  $L$  intersects  $\Sigma_{2d+1}$  in a third point  $\gamma$ . Set  $L^* = L \setminus \{\alpha, \beta, \gamma\} \cong \mathbb{C} \setminus \{0, 1\}$ . Let  $\hat{L}$  be any connected component of  $\pi_N^{-1}(L^*)$ , which is a punctured Riemann surface. Moreover,  $\pi_N : \hat{L} \rightarrow L^*$  is a finite abelian cover of degree  $p^2$ . By adding the missing punctures to  $\hat{L}$ , we obtain a closed Riemann surface  $W$  such that  $\pi_N : W \rightarrow L$  is an abelian covering, with three branch values, each of order  $p$ . By the Riemann-Hurwitz formula, if  $p \in \{2, 3\}$ , then  $W$  has genus 0 or 1. So,  $S$  cannot be algebraically hyperbolic.

(3) The argument is similar to that in case (2), except that in this case  $L$  intersects the branch locus of  $\pi_N$  in four points. So, we will have an abelian covering  $W \rightarrow L$ , branched at four points, each of order 2. This again ensures that  $W$  has genus one.  $\square$

**Example 3.** Let us consider a generalized Fermat variety  $X = X_4^2(\Lambda)$  of type  $(2; 2, 4)$ ; so  $n = 2d$  and we are in case (2) of the previous result. In this case, the locus of fixed points  $F_1 \subset X$  of  $\varphi_1$  has genus one, in particular,  $X$  is not algebraically hyperbolic.

**Question 1.** Let  $(S, N)$  be a  $\mathbb{Z}_p^m$ -action of type  $(d; p, n)$ , where  $d \geq 2$ ,  $n \geq 2d$  and, if  $n = 2d$ , then  $p \geq 4$ , and if  $n = 2d + 1$ , then  $p \geq 3$ . When is  $S$  algebraically hyperbolic?

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