
HYPERBOLICITY AND FUNDAMENTAL GROUPS OF COMPLEX QUASI-PROJECTIVE VARIETIES (I): MAXIMAL QUASI-ALBANESE DIMENSION BY NEVANLINNA THEORY

by

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Abstract. — This is the first part of a series of three papers. In this paper, we establish a Big Picard type theorem for holomorphic maps $f : Y \rightarrow X$, where Y is a ramified covering of the punctured disc \mathbb{D}^* with small ramification and X is a complex quasi-projective variety of log-general type and of maximal quasi-Albanese dimension. As a byproduct, we prove the generalized Green–Griffiths–Lang conjecture for such X . This paper summarizes the parts of the three-paper series that are based primarily on Nevanlinna theory.

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0. Introduction

0.1. Generalized Green-Griffiths-Lang conjecture. — The generalized version of the Green–Griffiths and Lang conjectures is the complex analogue of the Bombieri–Lang conjecture about the Zariski density of the set of rational points of an algebraic variety of general type. It is a fundamental problem in the study of the hyperbolicity of algebraic varieties (cf. [Lan91, I, 3.5] and [Lan91, VIII, Conj. 1.3]). To formulate this conjecture, we first introduce several notions of non-hyperbolicity loci, usually referred to as special subsets in various senses.

Definition 0.1 (Special subsets). — Let X be a smooth quasi-projective variety.

- (i) $\mathrm{Sp}_{\mathrm{sab}}(X) := \overline{\bigcup_f f(A_0)}^{\mathrm{Zar}}$, where f ranges over all non-constant rational maps $f : A \dashrightarrow X$ from all non-trivial semi-abelian varieties A to X such that f is regular $A_0 \rightarrow X$ on a Zariski open subset $A_0 \subset A$ whose complement $A \setminus A_0$ has codimension at least two;
- (ii) $\mathrm{Sp}_h(X) := \overline{\bigcup_f f(\mathbb{C})}^{\mathrm{Zar}}$, where f ranges over all non-constant holomorphic maps from \mathbb{C} to X ;

- (iii) $\text{Sp}_{\text{alg}}(X) := \overline{\bigcup_V V}^{\text{Zar}}$, where V ranges over all positive-dimensional closed subvarieties of X which are not of log general type;
- (iv) $\text{Sp}_p(X) := \overline{\bigcup_f f(\mathbb{D}^*)}^{\text{Zar}}$, where f ranges over all holomorphic maps from the punctured disk \mathbb{D}^* to X with essential singularity at the origin, i.e., f has no holomorphic extension $\bar{f} : \mathbb{D} \rightarrow \overline{X}$ to a projective compactification \overline{X} .

The *generalized Green-Griffiths-Lang conjecture* can be stated as follows.

Conjecture 0.2 (Generalized Green-Griffiths-Lang). — *Let X be a smooth quasi-projective variety. Then the following properties are equivalent:*

- (i) X is of log general type;
- (ii) X is strongly of general type, i.e., $\text{Sp}_{\text{alg}}(X) \subseteq X$;
- (iii) X is pseudo Picard hyperbolic, i.e., $\text{Sp}_p(X) \subseteq X$;
- (iv) X is pseudo Brody hyperbolic, i.e., $\text{Sp}_h(X) \subseteq X$;
- (v) $\text{Sp}_{\text{sab}}(X) \subseteq X$.

The first two sets $\text{Sp}_{\text{sab}}(X)$ and $\text{Sp}_h(X)$ are introduced by Lang for the compact case. He made the following two conjectures (cf. [Lan91, I, 3.5] and [Lan91, VIII, Conjecture 1.3]):

- $\text{Sp}_{\text{sab}}(X) \subseteq X$ if and only if X is of general type.
- $\text{Sp}_{\text{sab}}(X) = \text{Sp}_h(X)$.

The first assertion implicitly include the following third conjecture:

- $\text{Sp}_{\text{sab}}(X) = \text{Sp}_{\text{alg}}(X)$.

The original two conjectures imply the famous strong Green-Griffiths conjecture that varieties of (log) general type are pseudo Brody hyperbolic.

Let us discuss some known cases of the conjecture. Conjecture 0.2 is known to hold for curves. However, when $\dim X \geq 2$, there has been only few progress. We summarize below four classes of varieties for which Conjecture 0.2 has been established:

- Complex projective surfaces with big cotangent bundles, proved by McQuillan [McQ98], and later extended to quasi-projective surfaces with big logarithmic cotangent bundles by El Goul [EG03].
- Subvarieties of abelian varieties, by the classical theorem of Bloch–Ochiai–Kawamata, and more generally subvarieties of semi-abelian varieties, by Noguchi [Nog81].
- Projective varieties of maximal Albanese dimension, by Kawamata [Kaw81] and the third author [Yam15a].
- General hypersurfaces in projective space \mathbb{P}^n ($n \geq 3$) of sufficiently high degree, proved in [DMR10] (based on the strategy of Siu [Siu04]), with degree bounds subsequently improved in [Dar16, MT22, BK24, Cad24]; as well as complements of general hypersurfaces of high degree in \mathbb{P}^n ($n \geq 2$), proved in [Dar16, BD19], to mention only a few.

In this paper, we establish Conjecture 0.2 for quasi-projective varieties of maximal quasi-Albanese dimension (see Theorem C), thus generalizing the result of [Yam15a] into the non-compact setting. Here we say that a quasi-projective manifold X has maximal quasi-Albanese dimension if the quasi-Albanese map $\alpha : X \rightarrow A(X)$ is generically finite onto $\overline{\alpha(X)}$. By the universal property of the quasi-Albanese maps, this is equivalent to the existence of a morphism $a : X \rightarrow A$ to a semi-Abelian variety A such that $\dim X = \dim a(X)$.

0.2. Main results. — We consider a Riemann surface Y equipped with a proper surjective holomorphic map $\pi : Y \rightarrow \mathbb{C}_{>\delta}$, where $\mathbb{C}_{>\delta} := \{z \in \mathbb{C} \mid |z| > \delta\}$ for some positive constant $\delta > 0$. Our first result is a Big Picard type theorem for holomorphic maps $f : Y \rightarrow X$, where X is a smooth complex quasi-projective variety of log-general type and of maximal quasi-Albanese dimension. This theorem holds provided that $\pi : Y \rightarrow \mathbb{C}_{>\delta}$ has small ramifications compared with the growth order of f , i.e., $N_{\text{ram}}\pi(r) = O(\log r) + o(T_f(r))||$. The notations are from Nevanlinna theory; specifically, the symbol $||$ means that the stated estimate is valid as $r \rightarrow \infty$ with the possible exception of a set of finite Lebesgue measure. We will introduce other notations in § 3.

Theorem A. — *Let X be a smooth quasi-projective variety which is of log general type. Assume that there is a morphism $a : X \rightarrow A$ to a semi-Abelian variety A such that $\dim X = \dim a(X)$.*

Then there exists a proper Zariski closed set $\Xi \subsetneq X$ with the following property: Let $f : Y \rightarrow X$ be a holomorphic map such that $N_{\text{ram}}(\pi)(r) = O(\log r) + o(T_f(r))$ and that $f(Y) \not\subseteq \Xi$. Then f does not have essential singularity over ∞ , i.e., there exists an extension $\bar{f} : \bar{Y} \rightarrow \bar{X}$ of f , where \bar{Y} is a Riemann surface such that $\pi : Y \rightarrow \mathbb{C}_{>\delta}$ extends to a proper map $\bar{\pi} : \bar{Y} \rightarrow \mathbb{C}_{>\delta} \cup \{\infty\}$ and \bar{X} is a compactification of X .

Note that the identity map $\mathbb{C}_{>\delta} \rightarrow \mathbb{C}_{>\delta}$ trivially satisfies the small ramification condition in the theorem. Thus, we can apply Theorem A when Y is $\mathbb{C}_{>\delta}$ itself, which implies that X in the theorem is Picard hyperbolic. Nevertheless, for further applications it is essential to study Nevanlinna theory not only on $\mathbb{C}_{>\delta}$ itself, but also on its ramified coverings. This broader perspective is crucial for establishing the hyperbolicity of algebraic varieties. Such applications appear, for instance, in the recent work of Kebekus and Rousseau [KR24] on the hyperbolicity of Campana orbifolds (or C -pairs), as well as in Part (II) of this paper series, where we study the hyperbolicity of complex quasi-projective varieties carrying a big and semi-simple complex local system (see [CDY25, Theorems A, C & D]). In particular, for further applications in [CDY25], we will make use of the following consequence of Theorem A. See § 11 for more discussion about the condition stated in the theorem.

Theorem B (=Theorem 11.1). — *Let X be a smooth quasi-projective variety with a smooth projective compactification \bar{X} such that the boundary divisor $D = \bar{X} \setminus X$ is a simple normal crossing divisor. Assume that there exists a finite surjective morphism $p : \bar{\Sigma} \rightarrow \bar{X}$ from a normal projective variety $\bar{\Sigma}$ together with non-zero sections $\tau_1, \dots, \tau_l \in H^0(\bar{\Sigma}, p^*\Omega_{\bar{X}}(\log D))$ satisfying the following two conditions:*

- (i) *setting $\Sigma = p^{-1}(X)$, the variety Σ is of log-general type and admits a morphism $a : \Sigma \rightarrow A$ into a semi-abelian variety A with $\dim \Sigma = \dim a(\Sigma)$;*
- (ii) *defining*

$$R := \{s \in \bar{\Sigma} \mid \exists i \text{ such that } \tau_i(s) = 0\},$$

then $R \subsetneq \bar{\Sigma}$ is a proper Zariski closed subset, and $p : \bar{\Sigma} \rightarrow \bar{X}$ is étale outside R .

Then every holomorphic map $f : \mathbb{D}^ \rightarrow X$ with Zariski dense image has a holomorphic extension $\bar{f} : \mathbb{D} \rightarrow \bar{X}$.*

Next we state our result on Conjecture 0.2, which we prove as a consequence of Theorem A.

Theorem C. — *Let X be a smooth quasi-projective variety. Assume that there is a morphism $a : X \rightarrow A$ to a semi-Abelian variety A such that $\dim X = \dim a(X)$. Then the following properties are equivalent:*

- (a) *X is of log general type;*
- (b) *X is strongly of log general type;*
- (c) *X is pseudo Picard hyperbolic;*
- (d) *X is pseudo Brody hyperbolic;*
- (e) *$\text{Sp}_{\text{ab}}(X) \subsetneq X$.*

It is worth noting that the equivalence of (a) and (b) is a purely algebro-geometric statement. However our implication (a) \implies (b) involves the analytic statement (a) \implies (c) (namely, Theorem A) and is therefore not purely algebro-geometric. Theorem C also plays a crucial role in the second part of this paper series [CDY25].

The paper is organized as follows. We first prove some results on varieties with maximal quasi-Albanese dimension and logarithmic Kodaira dimension zero, and then deduce Theorem C from Theorem A. After introducing the necessary notations and estimates from Nevanlinna theory in § 3 and § 4, we present the proof of Theorem A in § 5–§ 10. The proof relies on the arguments of [NWY13] and [Yam15a]. In [NWY13], the non-compact setting is treated, while it only considers entire curves $\mathbb{C} \rightarrow X$ with Zariski dense images. In contrast, [Yam15a] handles holomorphic maps with non-Zariski dense images, but is restricted to the compact setting. Our situation requires us to bridge these two cases. In Remark 10.11, we briefly discuss the additional difficulties of our proof compared to the proofs in [NWY13] and [Yam15a], coming from the simultaneous consideration of non-compactness and non-Zariski dense images. Finally in § 11, we prove Theorem B using Theorem A.

This paper forms the first part of the long preprint on arXiv [CDY22], which has been split into three parts for journal submission. It corresponds to Sections 3 and 4 of [CDY22]. This paper treats the part of the series which is mainly based on Nevanlinna theory. In Part Two, we shall treat Sections 2 and 5–9, and in Part Three, Sections 10–12 of [CDY22].

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1. Varieties with logarithmic Kodaira dimension zero and maximal quasi-Albanese dimension

We first recall a fundamental result for semiabelian varieties.

Lemma 1.1 ([NW14, Propositions 5.6.21 & 5.6.22]). — *Let \mathcal{A} be a semi-abelian variety.*

- (a) *Let $X \subset \mathcal{A}$ be a closed subvariety. Then $\bar{\kappa}(X) \geq 0$ with equality if and only if X is a translate of a semi-abelian subvariety.*
- (b) *Let $Z \subset \mathcal{A}$ be a Zariski closed subset. Then $\bar{\kappa}(\mathcal{A} - Z) \geq 0$ with equality if and only if Z has no component of codimension 1.* \square

Next we prove two lemmas that will be used to discuss the structures on varieties with maximal quasi-Albanese dimension and logarithmic Kodaira dimension zero.

Lemma 1.2. — *Let $f : X \rightarrow Y$ be a dominant birational morphism of quasi-projective manifolds. Let $E \subset X$ be a Zariski closed subset such that $\overline{f(E)}$ has codimension at least two. Assume $\bar{\kappa}(Y) \geq 0$. Then $\bar{\kappa}(X - E) = \bar{\kappa}(X)$.*

Proof. — Let $\bar{f} : \bar{X} \rightarrow \bar{Y}$ be a proper birational morphism which extends $f : X \rightarrow Y$, where \bar{X} and \bar{Y} are smooth projective compactifications of X and Y such that $B = \bar{Y} - Y$ is a simple normal crossing divisor. Since the logarithmic Kodaira dimension is birationally invariant, we may further blow-up \bar{X} to assume that

– $\bar{E} + (\bar{X} - X)$ is a simple normal crossing divisor,

where $\bar{E} \subset \bar{X}$ is the closure of E . Then we note that $\bar{X} - X$ is also simple normal crossing. Write $\bar{X} - X = \bar{f}^{-1}(B) + D$, where D is a reduced divisor on \bar{X} and $\bar{f}^{-1}(B) = \text{supp } \bar{f}^*B$. Then we have $f^*K_{\bar{Y}}(B) + \bar{E} \leq K_{\bar{X}}(\bar{f}^{-1}(B))$. So we write $K_{\bar{X}}(\bar{f}^{-1}(B)) = f^*K_{\bar{Y}}(B) + \bar{E} + F$, where F is effective. Then we have

$$K_{\bar{X}}(\bar{f}^{-1}(B)) + D + \bar{E} = \bar{f}^*K_{\bar{Y}}(B) + D + 2\bar{E} + F$$

By the assumption $\bar{\kappa}(Y) \geq 0$, one has $nK_{\bar{Y}}(B) \geq 0$ for some positive integer $n > 0$. Therefore,

$$n(\bar{f}^*K_{\bar{Y}}(B) + D + 2\bar{E} + F) \leq 2n(\bar{f}^*K_{\bar{Y}}(B) + D + \bar{E} + F) = 2n(K_{\bar{X}}(\bar{f}^{-1}(B)) + D).$$

Thus $n(K_{\bar{X}}(\bar{f}^{-1}(B)) + D + \bar{E}) \leq 2n(K_{\bar{X}}(\bar{f}^{-1}(B)) + D)$. Recall that $\bar{f}^{-1}(B) + D + \bar{E}$ and $\bar{f}^{-1}(B) + D$ are both simple normal crossing divisors. It follows $\bar{\kappa}(X - E) \leq \bar{\kappa}(X)$. Hence $\bar{\kappa}(X - E) = \bar{\kappa}(X)$. \square

Lemma 1.3. — *Let $\alpha : X \rightarrow \mathcal{A}$ be a (possibly non-proper) birational morphism from a quasi-projective manifold X to a semi-abelian variety \mathcal{A} with $\bar{\kappa}(X) = 0$. Then there exists a Zariski closed subset $Z \subset \mathcal{A}$ of codimension at least two such that α is isomorphic over $\mathcal{A} \setminus Z$.*

Proof. — Since $\alpha : X \rightarrow \mathcal{A}$ is birational, we remove the exceptional locus of $X \rightarrow \mathcal{A}$ from X to get $X^\circ \subset X$ such that $\alpha : X^\circ \rightarrow \alpha(X^\circ)$ is an isomorphism. By $\bar{\kappa}(\mathcal{A}) = 0$, we may apply Lemma 1.2 to get $\bar{\kappa}(X^\circ) = \bar{\kappa}(X) = 0$. By Lemma 1.1, $\mathcal{A} - \alpha(X^\circ)$ has codimension at least two. \square

We establish several results concerning structures on varieties with maximal quasi-Albanese dimension and logarithmic Kodaira dimension zero.

Lemma 1.4. — *Let $\alpha : X \rightarrow \mathcal{A}$ be a (possibly non-proper) morphism from a quasi-projective manifold X to a semi-abelian variety \mathcal{A} with $\bar{\kappa}(X) = 0$. Assume that $\dim X = \dim \alpha(X)$. Then $\pi_1(X)$ is abelian.*

Proof. — *Step 1.* We may assume that α is dominant and birational. Consider the quasi-Stein factorisation $X \xrightarrow{h} Y \xrightarrow{k} \mathcal{A}$ of α , where h is birational (might not proper), and k is a finite morphism. This is constructed as follows. Let $X \subset X'$ be a partial smooth compactification of X such that α extends to a proper map $\alpha' : X' \rightarrow \mathcal{A}$. Let $X' \rightarrow Y \rightarrow \mathcal{A}$ be the Stein factorization

of α' , where $h' : X' \rightarrow Y$ have connected fibers and $k : Y \rightarrow \mathcal{A}$ is finite. By $\dim X' = \dim Y$, the map h' is birational. Let $h : X \rightarrow Y$ be the restriction of h' to X . Then h is birational.

Now since $\bar{\kappa}(X) = 0$, one has

$$0 = \bar{\kappa}(X) \geq \bar{\kappa}(Y) \geq \bar{\kappa}(k(Y)) \geq 0$$

where the last inequality follows from Lemma 1.1. Hence $\bar{\kappa}(Y) = \bar{\kappa}(k(Y)) = 0$. By Lemma 1.1 and Kawamata [Kaw81, Theorem 26], $k(Y)$ is a semi-abelian variety and k is finite étale. In conclusion, it now suffices to prove the proposition with \mathcal{A} replaced by Y . Hence in the following, we may assume that α is dominant and birational.

Step 2. We apply Lemma 1.3 to get the isomorphism $\alpha|_{X^\circ} : X^\circ \rightarrow \mathcal{A}^\circ$, where $\mathcal{A} - \mathcal{A}^\circ$ of codimension at least two. Let $\bar{\alpha} : \overline{X} \rightarrow \mathcal{A}$ be a proper birational morphism which extends $\alpha : X \rightarrow \mathcal{A}$ such that \overline{X} is smooth. One gets the following commutative diagram

$$\begin{array}{ccc} \pi_1(X^\circ) & \xrightarrow{\sim} & \pi_1(\mathcal{A}^\circ) \\ \downarrow & & \downarrow \cong \\ \pi_1(X) & & \\ \downarrow & & \downarrow \cong \\ \pi_1(\bar{\alpha}^{-1}(\mathcal{A})) & \xrightarrow{\sim} & \pi_1(\mathcal{A}) \end{array}$$

where the two rows are isomorphisms because they are induced by proper birational morphisms between smooth quasi-projective varieties. The map $\pi_1(X^\circ) \rightarrow \pi_1(X)$ is surjective since it is induced by the inclusion of a dense Zariski open subset. It follows that all the groups in the previous diagram are isomorphic, so $\pi_1(X) \cong \pi_1(\mathcal{A})$ is abelian. \square

Lemma 1.5. — *Let $\alpha : X \rightarrow \mathcal{A}$ be a (possibly non-proper) morphism from a quasi-projective manifold X to a semi-abelian variety \mathcal{A} with $\bar{\kappa}(X) = 0$. Assume that $\dim X = \dim \alpha(X)$ and $\dim X > 0$. Then X admits a Zariski dense entire curve $\mathbb{C} \rightarrow X$. In particular, $\text{Sp}_h(X) = X$.*

Proof. — As in the step 1 of the proof of Lemma 1.4, we may assume that $\alpha : X \rightarrow \mathcal{A}$ is birational. We apply Lemma 1.3 to get the isomorphism $\alpha|_{X^\circ} : X^\circ \rightarrow \mathcal{A}^\circ$, where $\mathcal{A} - \mathcal{A}^\circ$ of codimension at least two. Set $Z = \mathcal{A} - \mathcal{A}^\circ$. We shall show that $\mathcal{A}^\circ = \mathcal{A} - Z$ contains a Zariski dense entire curve. Let $\varphi : \mathbb{C} \rightarrow \mathcal{A}$ be a one parameter group such that $\varphi(\mathbb{C}) \subset \mathcal{A}$ is Zariski dense. We define $F : \mathbb{C} \times Z \rightarrow \mathcal{A}$ by $F(c, z) = z + \varphi(c)$. Then F is a holomorphic map. Let $Z_1 \subset Z_2 \subset \dots$ be a sequence of compact subsets of Z such that $\cup_n Z_n = Z$. Let $K_n = \{|z| \leq n\} \subset \mathbb{C}$. Then $F(K_n \times Z_n) \subset \mathcal{A}$ is a compact subset. Set $O_n = \mathcal{A} \setminus F(K_n \times Z_n)$. Then $O_n \subset \mathcal{A}$ is an open subset. Note that $\dim(\mathbb{C} \times Z) < \dim \mathcal{A}$. Hence O_n is a dense subset. Hence by the Baire category theorem, the intersection $\cap_n O_n$ is dense in \mathcal{A} . In particular, we may take $x \in \cap_n O_n$. We define $f : \mathbb{C} \rightarrow \mathcal{A}$ by $f(c) = x + \varphi(c)$. Then $f(\mathbb{C}) \cap Z = \emptyset$. Indeed suppose contrary $f(c) \in Z$. Then $F(-c, f(c)) = f(c) + \varphi(-c) = x$. Since $(-c, f(c)) \in K_n \times Z_n$ for sufficiently large n , we have $x \in F(K_n \times Z_n)$, so $x \notin O_n$. This contradicts the choice of x . \square

Lemma 1.6. — *Let $\alpha : X \rightarrow \mathcal{A}$ be a (possibly non-proper) morphism from a quasi-projective manifold X to a semi-abelian variety \mathcal{A} with $\bar{\kappa}(X) = 0$. Assume that $\dim X = \dim \alpha(X)$ and $\dim X > 0$. Then $\text{Sp}_{\text{sab}}(X) = X$.*

Proof. — As in the step 1 of the proof of Lemma 1.4, we may assume that $\alpha : X \rightarrow \mathcal{A}$ is birational. We apply Lemma 1.3 to get the isomorphism $\alpha|_{X^\circ} : X^\circ \rightarrow \mathcal{A}^\circ$, where $\mathcal{A} - \mathcal{A}^\circ$ of codimension at least two. Then we have the inverse $\mathcal{A}^\circ \rightarrow X$ of the isomorphism $X^\circ \rightarrow \mathcal{A}^\circ$. Thus $\text{Sp}_{\text{sab}}(X) = X$. \square

2. Proof of Theorem C from Theorem A

We now apply the results of § 1 to deduce Theorem C from Theorem A. We begin with some general facts about special subsets.

Lemma 2.1. — *Let X be a smooth quasi-projective variety. Then one has $\text{Sp}_{\text{sab}} \subseteq \text{Sp}_h \subseteq \text{Sp}_p$.*

Proof. — Let $A_0 \subset A$ be a Zariski open subset of a non-trivial semi-Abelian variety A such that the complement $A \setminus A_0$ has codimension at least two. By Lemma 1.1, we have $\bar{\kappa}(A_0) = 0$. Thus A_0 admits a Zariski dense entire curve $\mathbb{C} \rightarrow A_0$ by Lemma 1.5, which shows $\text{Sp}_{\text{sab}} \subseteq \text{Sp}_h$.

For any non-constant holomorphic map $f : \mathbb{C} \rightarrow X$, the holomorphic map

$$\begin{aligned} g : \mathbb{C}^* &\rightarrow X \\ z &\mapsto f(\exp(\frac{1}{z})) \end{aligned}$$

has essential singularity at 0. Note that the Zariski closure of $g(\mathbb{D}^*)$ coincides with the Zariski closure of f . It follows that $\text{Sp}_h \subseteq \text{Sp}_p$. \square

Proof of Theorem C. — (a) \implies (c). It follows from Theorem A directly.

(c) \implies (d). This follows from Lemma 2.1.

(d) \implies (e). This follows from Lemma 2.1.

(e) \implies (b). Let $E \subsetneq X$ be a proper Zariski closed subset such that $a|_{X \setminus E} : X \setminus E \rightarrow A$ is quasi-finite. Set $\Xi = \text{Sp}_{\text{sab}}(X) \cup E$. Under the hypothesis (e), we have $\Xi \subsetneq X$.

Now we prove that all closed subvarieties $V \subset X$ with $V \not\subset \Xi$ are of log-general type. Note that $\dim V = \dim a(V)$. Hence $\bar{\kappa}(V) \geq 0$ by Lemma 1.1. We prove that the logarithmic Iitaka fibration $V \dashrightarrow W$ is birational. So assume contrary that a very general fiber F is positive dimensional. We have $F \not\subset \Xi$, hence $F \not\subset \text{Sp}_{\text{sab}}(X)$, and the logarithmic Kodaira dimension satisfies $\bar{\kappa}(F) = 0$. Then $a|_F : F \rightarrow A$ satisfies $\dim a(F) = \dim F$. Hence by Lemma 1.6, we have $\text{Sp}_{\text{sab}} F = F$. This contradicts to $F \not\subset \text{Sp}_{\text{sab}}(X)$. Hence V is of log general type.

(b) \implies (a). This is obvious. \square

Therefore, Theorem A proves Conjecture 0.2 for quasi-projective varieties with maximal quasi-Albanese dimension. After the preparations in § 5–§ 9, we prove Theorem A in § 10.

3. Some notions in Nevanlinna theory

We shall recall some elements of Nevanlinna theory (cf. [NW14], [Yam15b]). Let $\delta > 0$ be a positive constant and let Y be a Riemann surface equipped with a proper surjective holomorphic map $\pi : Y \rightarrow \mathbb{C}_{>\delta}$. For $r > 2\delta$, define $Y(r) = \pi^{-1}(\mathbb{C}_{>2\delta}(r))$ where $\mathbb{C}_{>2\delta}(r) = \{z \in \mathbb{C} \mid 2\delta < |z| < r\}$. In the following, we always assume that $r > 2\delta$. The *ramification counting function* of the covering $\pi : Y \rightarrow \mathbb{C}_{>\delta}$ is defined by

$$N_{\text{ram } \pi}(r) := \frac{1}{\deg \pi} \int_{2\delta}^r \left[\sum_{y \in Y(t)} \text{ord}_y \text{ram } \pi \right] \frac{dt}{t},$$

where $\text{ram } \pi \subset Y$ is the ramification divisor of $\pi : Y \rightarrow \mathbb{C}_{>\delta}$.

Let X be a projective variety and let Z be a closed subscheme of X . Let $f : Y \rightarrow X$ be a holomorphic map such that $f(Y) \not\subset Z$. Since Y is one dimensional, the pull-back f^*Z is a divisor on Y . The counting function and truncated function are defined to be

$$\begin{aligned} N_f(r, Z) &:= \frac{1}{\deg \pi} \int_{2\delta}^r \left[\sum_{y \in Y(t)} \text{ord}_y f^*Z \right] \frac{dt}{t}, \\ N_f^{(k)}(r, Z) &:= \frac{1}{\deg \pi} \int_{2\delta}^r \left[\sum_{y \in Y(t)} \min \{k, \text{ord}_y f^*Z\} \right] \frac{dt}{t}, \end{aligned}$$

where $k \in \mathbb{Z}_{\geq 1}$ is a positive integer. We define the proximity function by

$$m_f(r, Z) := \frac{1}{\deg \pi} \int_{y \in \pi^{-1}(\{|z|=r\})} \lambda_Z(f(y)) \frac{d\arg \pi(y)}{2\pi},$$

where $\lambda_Z : X - \text{supp } Z \rightarrow \mathbb{R}_{\geq 0}$ is a Weil function for Z (cf. [Yam04, Prop 2.2.3]).

Let L be a line bundle on X . Let $f : Y \rightarrow X$ be a holomorphic map. We define the order function $T_f(r, L)$ as follows. First suppose that X is smooth. We equip with a smooth hermitian metric h_L , and let $c_1(L, h_L)$ be the curvature form of (L, h_L) . Set

$$T_f(r, L) := \frac{1}{\deg \pi} \int_{2\delta}^r \left[\int_{Y(t)} f^* c_1(L, h_L) \right] \frac{dt}{t}.$$

This definition is independent of the choice of the hermitian metric up to a function $O(\log r)$. For the general case, let $V \subset X$ be the Zariski closure of $f(Y)$ and let $V' \rightarrow V$ be a smooth model of V . This induces a morphism $p : V' \rightarrow X$. We have a natural lifting $f' : Y \rightarrow V'$ of $f : Y \rightarrow V$. Then we set

$$T_f(r, L) := T_{f'}(r, p^* L) + O(\log r).$$

This definition does not depend on the choice of $V' \rightarrow V$ up to a function $O(\log r)$.

Theorem 3.1 (First Main Theorem). — *Let X be a projective variety and let D be an effective Cartier divisor on X . Let $f : Y \rightarrow X$ be a holomorphic curve such that $f(Y) \not\subset D$. Then*

$$N_f(r, D) + m_f(r, D) = T_f(r, \mathcal{O}_X(D)) + O(\log r).$$

Proof. — Let $V \subset X$ be the Zariski closure of $f(Y)$ and let $p : V' \rightarrow V$ be a desingularization. Let $f' : Y \rightarrow V'$ be the canonical lifting of f . Replacing X, D and f by $V', p^* D$ and f' , respectively, we may assume that X is smooth.

Let $\|\cdot\|$ be a smooth Hermitian metric on $\mathcal{O}_X(D)$ and let s_D be the associated section for D . By the Poincaré-Lelong formula, we have

$$2dd^c \log(1/\|s_D \circ f(y)\|) = - \sum_{y \in Y} (\text{ord}_y f^* D) \delta_y + f^* c_1(\mathcal{O}_X(D), \|\cdot\|),$$

where δ_y is Dirac current supported on y . Integrating over $Y(t)$, we get

$$2 \int_{Y(t)} dd^c \log(1/\|s_D \circ f(y)\|) = - \sum_{y \in Y(t)} \text{ord}_y f^* D + \int_{Y(t)} f^* c_1(\mathcal{O}_X(D), \|\cdot\|)$$

Hence, we get

$$\begin{aligned} -N_f(r, D) + T_f(r, \mathcal{O}_X(D)) &= \lim_{\delta' \rightarrow \delta^+} \frac{2}{\deg \pi} \int_{2\delta'}^r \frac{dt}{t} \int_{\pi^{-1}(\mathbb{C}_{>2\delta'}(t))} dd^c \log \left(\frac{1}{\|s_D \circ f(z)\|} \right) \\ &= \lim_{\delta' \rightarrow \delta^+} \frac{2}{\deg \pi} \int_{2\delta'}^r \frac{dt}{t} \int_{\partial \pi^{-1}(\mathbb{C}_{>2\delta'}(t))} d^c \log \left(\frac{1}{\|s_D \circ f(z)\|} \right) \\ &= m_f(r, D) - m_f(2\delta, D) - C_f \log(r/2\delta), \end{aligned}$$

where we set

$$C_f = \lim_{\delta' \rightarrow \delta^+} \frac{2}{\deg \pi} \int_{\pi^{-1}\{|z|=2\delta'\}} d^c \log \left(\frac{1}{\|s_D \circ f(z)\|} \right),$$

which is a constant independent of r . \square

For any effective divisor $D_L \in |L|$ with $f(Y) \not\subset D_L$, the First Main theorem (cf. Theorem 3.1) implies the following Nevanlinna inequality:

$$(3.1) \quad N_f(r, D_L) \leq T_f(r, L) + O(\log r).$$

When L is an ample line bundle on X , then we have $T_f(r, L) + O(\log r) > 0$. If L' is another ample line bundle on X , then we have $T_f(r, L') = O(T_f(r, L)) + O(\log r)$. We write $T_f(r)$ for short instead of $T_f(r, L)$ when the order of magnitude as $r \rightarrow \infty$ is concerned.

Lemma 3.2. — *Let X be a smooth projective variety and let $f : Y \rightarrow X$ be a holomorphic map. Assume that $T_f(r) = O(\log r)$ and $N_{\text{ram}}(r) = O(\log r)$. Then f does not have essential singularity over ∞ .*

Here, we use the symbol $\|$ to denote that the stated estimate holds for $r \rightarrow \infty$ outside an exceptional set of finite Lebesgue measure.

Proof. — For $s > 2\delta$, we set $\nu(s) = \sum_{y \in Y(s)} \text{ord}_y \text{ram } \pi$. Then for $2\delta < s < r$, we have

$$(3.2) \quad \begin{aligned} N_{\text{ram } \pi}(r) - N_{\text{ram } \pi}(s) &= \frac{1}{\deg \pi} \int_s^r \left[\sum_{y \in Y(t)} \text{ord}_y \text{ram } \pi \right] \frac{dt}{t} \\ &\geq \frac{1}{\deg \pi} \int_s^r \nu(s) \frac{dt}{t} = \frac{\nu(s)}{\deg \pi} (\log r - \log s). \end{aligned}$$

Set $K = \lim_{r \rightarrow \infty} N_{\text{ram } \pi}(r)/\log r$. By $N_{\text{ram } \pi}(r) = O(\log r)$, we have $K < \infty$. By (3.2), we have $\nu(s) \leq K \deg \pi$. Since $s > 2\delta$ is arbitrary, the ramification divisor $\text{ram } \pi \subset Y$ consists of finite points. Thus we may take $s_0 > \delta$ such that $\pi^{-1}(\mathbb{C}_{>s_0}) \rightarrow \mathbb{C}_{>s_0}$ is unramified covering. Thus $\pi^{-1}(\mathbb{C}_{>s_0})$ is a disjoint union of punctured discs. Hence we may take an extension $\bar{\pi} : \bar{Y} \rightarrow \mathbb{C}_{>\delta} \cup \{\infty\}$ of the covering $\pi : Y \rightarrow \mathbb{C}_{>\delta}$.

In the following, we replace δ by s_0 and take a connected component of $\pi^{-1}(\mathbb{C}_{>s_0})$. Then we may assume that $\pi : Y \rightarrow \mathbb{C}_{>\delta}$ is unramified and Y is a punctured disc. Let ω be a smooth positive $(1, 1)$ -form on X . For $s > 2\delta$, we set $\alpha(s) = \int_{Y(s)} f^* \omega$. Then by the similar computation as in (3.2), the assumption $T_f(r) = O(\log r)$ yields that $\alpha(s)$ is bounded on $s > 2\delta$. Thus $\int_Y f^* \omega < \infty$. By Lemma 3.3 below, we conclude the proof. \square

The following lemma is well-known to the experts. We refer the readers to [CD21, Lemma 3.3] for a simpler proof based on Bishop's theorem.

Lemma 3.3. — *Let X be a compact Kähler manifold and let $f : \mathbb{D}^* \rightarrow X$ be a holomorphic map from the punctured disc \mathbb{D}^* . Let ω be a smooth Kähler form on X . Suppose $\int_{\mathbb{D}^*} f^* \omega < \infty$. Then f has holomorphic extension $\bar{f} : \mathbb{D} \rightarrow X$.* \square

We state the following lemma, which will be applied later.

Lemma 3.4. — *Let X be a smooth projective variety and let $f : Y \rightarrow X$ be a holomorphic map. If $\lim_{r \rightarrow \infty} T_f(r)/\log r < +\infty$, then $T_f(r) = O(\log r)$.*

Proof. — For $s > 2\delta$, we set $\alpha(s) = \int_{Y(s)} f^* \omega$, where ω is a smooth positive $(1, 1)$ -form on X . Then by the similar computation as in (3.2), the assumption $\lim_{r \rightarrow \infty} T_f(r)/\log r < +\infty$ yields that $\alpha(s)$ is bounded, i.e., there exists a positive constant $C > 0$ such that $\alpha(s) < C$ for all $s > 2\delta$. Hence for all $r > 2\delta$, we have

$$\frac{1}{\deg \pi} \int_{2\delta}^r \left[\int_{Y(t)} f^* \omega \right] \frac{dt}{t} < \frac{C}{\deg \pi} \log \frac{r}{2\delta}.$$

Thus we get $T_f(r) = O(\log r)$. \square

4. Lemma on logarithmic derivatives

Let $\pi_Y : Y \rightarrow \mathbb{C}_{>\delta}$ be a Riemann surface with a proper surjective holomorphic map. Set $D_Y = \pi_Y^*(\partial/\partial z)$. Then D_Y is a meromorphic vector field on Y . For a meromorphic function $f : Y \rightarrow \mathbb{P}^1$ on Y , we set $f' = D_Y(f)$. For $r > 2\delta$, we set as follows:

$$m(r, f) = \frac{1}{\deg \pi_Y} \int_{y \in \pi_Y^{-1}(\{|z|=r\})} \log \sqrt{1 + |f(y)|^2} \frac{d \arg \pi_Y(y)}{2\pi},$$

$$T(r, f) = \frac{1}{\deg \pi_Y} \int_{2\delta}^r \frac{dt}{t} \int_{Y(t)} f^* \omega_{\text{F.S.}},$$

where $\omega_{\text{F.S.}}$ is the Fubini-Study form:

$$\omega_{\text{F.S.}} = \frac{1}{(1 + |w|^2)^2} \frac{\sqrt{-1}}{2\pi} dw \wedge d\bar{w}.$$

Note that $m(r, f)$ is a proximity function function for $f : Y \rightarrow \mathbb{P}^1$ with respect to the divisor (∞) on \mathbb{P}^1 , and $T(r, f)$ is a order function with respect to the ample line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$. In this section, we provide a simple proof for Nevanlinna's lemma on logarithmic derivatives, Theorem 4.3.

Lemma 4.1. — *For $n \in \mathbb{Z}_{\geq 1}$, we have $T(r, f^n) = nT(r, f) + O(\log r)$.*

Proof. — Let $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be defined by $\varphi(w) = w^n$, then we have $f^n = \varphi \circ f$. By $\varphi^* \mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_{\mathbb{P}^1}(n)$, we obtain our lemma. \square

Lemma 4.2. — $m(r, f'/f) \leq 3T(r, f) + O(\log r) \parallel$

Proof. — Set

$$f^\# = \frac{|f'|}{1 + |f|^2}$$

and

$$m^\#(r, f) = \frac{1}{\deg \pi_Y} \int_{y \in \pi_Y^{-1}(\{|z|=r\})} \log \sqrt{1 + (f^\#(y))^2} \frac{d \arg \pi_Y(y)}{2\pi}.$$

We first show

$$(4.1) \quad m^\#(r, f) \leq T(r, f) + O(\log r) \parallel.$$

Indeed using convexity of \log , we have

$$m^\#(r, f) \leq \frac{1}{2} \log \left(1 + \frac{1}{\deg \pi_Y} \int_{y \in \pi_Y^{-1}(\{|z|=r\})} f^\#(y)^2 \frac{d \arg \pi_Y(y)}{2\pi} \right).$$

Using polar coordinate, we get

$$\int_{Y(r)} f^* \omega_{\text{F.S.}} = \frac{1}{\pi} \int_{2\delta}^r t dt \int_{y \in \pi_Y^{-1}(\{|z|=t\})} f^\#(y)^2 d \arg \pi_Y(y).$$

This shows

$$\frac{1}{2r} \frac{d}{dr} \left(r \frac{d}{dr} T(r, f) \right) = \frac{1}{\deg \pi_Y} \int_{y \in \pi_Y^{-1}(\{|z|=r\})} f^\#(y)^2 \frac{d \arg \pi_Y(y)}{2\pi}.$$

Hence using Borel's growth lemma [NW14, p. 13], we get

$$\begin{aligned} m^\#(r, f) &\leq \frac{1}{2} \log \left(1 + \frac{1}{2r} \frac{d}{dr} \left(r \frac{d}{dr} T(r, f) \right) \right) \\ &\leq \frac{1}{2} \log \left(1 + \frac{1}{2r} \left(r \frac{d}{dr} T(r, f) \right)^{1+\delta} \right) \parallel_\delta \\ &\leq \frac{1}{2} \log \left(1 + \frac{1}{2} r^\delta T(r, f)^{(1+\delta)^2} \right) \parallel_\delta \\ &\leq T(r, f) + O(\log r) \parallel. \end{aligned}$$

This shows our estimate (4.1). Here we take $\delta = 1$ in the final equation.

Now by

$$\begin{aligned} \log \sqrt{1 + \left| \frac{1}{f} \right|^2} + \log \sqrt{1 + |f'|^2} + \log \sqrt{1 + \left(\frac{|f'|}{1 + |f|^2} \right)^2} \\ = \log \sqrt{\left(|f| + \frac{1}{|f|} \right)^2 + \left| \frac{f'}{f} \right|^2} \geq \log \sqrt{1 + \left| \frac{f'}{f} \right|^2}, \end{aligned}$$

we get

$$m(r, f'/f) \leq m(r, 1/f) + m(r, f) + m^\#(r, f).$$

Using the first main theorem (cf. Theorem 3.1) and (4.1), we get Lemma 4.2. \square

We prove Nevanlinna's lemma on logarithmic derivatives (cf. [NW14, Lemma 1.2.2]) in the following form.

Theorem 4.3. — $m(r, f'/f) = o(T(r, f)) + O(\log r) \parallel$.

Several methods exist for proving Nevanlinna's lemma on logarithmic derivatives. The proof presented here is an application of the ramified covering method used in [Yam15b, Thm 4.8] to a simpler context.

Proof. — We first show that for every $\varepsilon > 0$, we have

$$(4.2) \quad m(r, f'/f) \leq \varepsilon T(r, f) + O_\varepsilon(\log r) \parallel.$$

Here O_ε indicates that the implicit constant in the Landau symbol O may depend on ε .

Indeed we take $n \in \mathbb{Z}_{\geq 1}$ so that $3/n < \varepsilon$. We set $f_n = \sqrt[n]{f}$. Then f_n is a multi-valued meromorphic function on Y . Let $\pi_{Y_n} : Y_n \rightarrow \mathbb{C}_{>\delta}$ be the Riemann surface for f_n . Both f and f_n are considered as meromorphic functions on Y_n . We have $f'/f = n f'_n/f_n$. By Lemma 4.1, we get

$$m(r, f'/f) = m(r, f'_n/f_n) + O(1) \leq 3T(r, f_n) + O(\log r) \parallel.$$

Hence by Lemma 4.2, we have

$$m(r, f'/f) \leq \frac{3}{n} T(r, f) + O(\log r) \parallel.$$

This shows (4.2).

Suppose that $\liminf_{r \rightarrow \infty} T(r, f)/\log r < +\infty$. Then by Lemma 3.4, we have $T(r, f) = O(\log r)$. Then by (4.2), we have $m(r, f'/f) = O(\log r)$, in particular $m(r, f'/f) = o(T(r, f)) + O(\log r) \parallel$.

Next we assume $\liminf_{r \rightarrow \infty} T(r, f)/\log r = +\infty$. Then $\log r = o(T(r, f))$. Hence by (4.2), we have

$$(4.3) \quad m(r, f'/f) \leq \varepsilon T(r, f) + o(T(r, f)) \parallel_\varepsilon$$

for all $\varepsilon > 0$. This implies $m(r, f'/f) = o(T(r, f)) \parallel$. Indeed we take a sequence $2\delta = r_0 < r_1 < r_2 < \dots$ with $r_n \rightarrow \infty$ as follows. By (4.3), we have $m(r, f'/f) \leq \frac{1}{n} T(r, f)$ for all $r > 2\delta$ outside some exceptional set $E_n \subset (2\delta, \infty)$ with $|E_n| < \infty$. We take r_n such that $|(r_n, \infty) \cap E_n| < 1/2^n$. We set $\varepsilon(r) = 1/n$ if $r_n \leq r < r_{n+1}$, and $\varepsilon(r) = 1$ if $r_0 < r < r_1$. Then $\varepsilon(r) \rightarrow 0$ if $r \rightarrow \infty$. Set $E = (r_0, r_1) \cup \bigcup ((r_n, r_{n+1}) \cap E_n)$. Then we have $m(r, f'/f) \leq \varepsilon(r) T(r, f)$ for all $r > 2\delta$ outside E , and $|E| < r_1 + 1$. Hence we get $m(r, f'/f) = o(T(r, f)) \parallel$, in particular we get $m(r, f'/f) = o(T(r, f)) + O(\log r) \parallel$. This concludes the proof of the theorem. \square

Let V be a smooth projective variety and let $D \subset V$ be a simple normal crossing divisor. Let $T(V; \log D)$ be the logarithmic tangent bundle. Set $\bar{T}(V; \log D) = P(T(V; \log D) \oplus \mathcal{O}_V)$, which is a smooth compactification of $T(V; \log D)$. Let $\partial T(V; \log D) \subset \bar{T}(V; \log D)$ be the boundary divisor. Let $f : Y \rightarrow V$ be a holomorphic map such that $f(Y) \not\subset D$. Then we get a derivative map $j_1(f) : Y \rightarrow \bar{T}(V; \log D)$. By Theorem 4.3, we obtain the following estimate

$$(4.4) \quad m_{j_1(f)}(r, \partial T(V; \log D)) = o(T(r, f)) + O(\log r) \parallel.$$

This estimate is a direct modification of [Yam04, Thm 5.1.7 (2)], whose proof applies to that of (4.4). The original proof uses [Yam04, Prop 5.1.2 (1)] together with [Yam04, Thm 2.5.1 (2)]. Our modification is to substitute [Yam04, Thm 2.5.1 (2)] with Theorem 4.3 and then follow the same line of reasoning. Here we note that we write $j_1(f)$ instead of " $j_1^{\log}(f)$ " as in [Yam04] for simplicity. See also [Voj00, Appendix A]. The same modification shows that the corresponding estimate for [Yam04, Thm 5.1.7 (1)] is also valid with the error term $S(r, f) = o(T(r, f)) + O(\log r) \parallel$ by the same manner using [Yam04, Prop 5.1.2 (2)] combined with Theorem 4.3. This will be used in the estimate (7.3).

Let us discuss an application of (4.4) to an estimate for logarithmic 1-forms as in [Nog81]. Let D be a simple normal crossing divisor on a projective manifold V as above. Let $f : Y \rightarrow V$ be a holomorphic map such that $f(Y) \not\subset D$. Let $\omega \in H^0(V, \Omega_V(\log D))$ be a logarithmic 1-form. Set $\xi = f^*\omega/\pi^*(dz)$. Then ξ is a meromorphic function on Y , which gives a holomorphic map $\xi : Y \rightarrow \mathbb{P}^1$.

We claim

$$(4.5) \quad m(r, \xi) = O(\log r) + o(T(r, f)) \parallel.$$

The proof goes as follows. The logarithmic 1-form ω defines a morphism $\varphi : T(V; \log D) \rightarrow \mathbb{A}^1$, which extends to a rational map $\bar{\varphi} : \bar{T}(V; \log D) \dashrightarrow \mathbb{P}^1$. By $f(Y) \not\subset D$, we get the lifting $j_1(f) : Y \rightarrow \bar{T}(V; \log D)$. Then we have $\xi = \bar{\varphi} \circ j_1(f)$. Hence by [Yam04, Prop 2.3.2 (6)], we have

$$m(r, \xi) = m_\xi(r, [\infty]) = O(m_{j_1(f)}(r, \partial T(V; \log D))) + O(1).$$

Thus by (4.4), we get (4.5).

It follows from the First Main theorem (cf. Theorem 3.1) that

$$N_\xi(r, [0]) + m_\xi(r, [0]) = N_\xi(r, [\infty]) + m_\xi(r, [\infty]) + O(\log r).$$

Hence by (4.5), we get

$$(4.6) \quad N_\xi(r, [0]) \leq N_\xi(r, [\infty]) + O(\log r) + o(T(r, f)).$$

Convention and notation. — In the rest of this paper, we use the following convention and notation. Let Σ be a quasi-projective variety and let $\bar{\Sigma}$ be a projective compactification. We denote by $f : Y \rightarrow \Sigma$ a holomorphic map $\bar{f} : Y \rightarrow \bar{\Sigma}$ such that $\bar{f}^{-1}(\Sigma) \neq \emptyset$. Moreover for a Zariski closed set $W \subset \Sigma$, we denote by $f(Y) \subset W$ if $\bar{f}(Y) \subset \bar{W}$, where $\bar{W} \subset \bar{\Sigma}$ is the Zariski closure. The choice of $\bar{\Sigma}$ is not canonical. Thus, we will use these notation to avoid mentioning the compactification explicitly.

Let A be a semi-abelian variety and let S be a projective variety. For $f : Y \rightarrow A \times S$, we set

$$\bar{N}_f^\partial(r) := \frac{1}{\deg \pi} \int_{2\delta}^r \text{card} \left(Y(t) \cap \bar{f}^{-1}(\partial A \times S) \right) \frac{dt}{t}.$$

This definition does not depend on the choice of \bar{A} . For $f : Y \rightarrow A \times S$, we define $f_A : Y \rightarrow A$ and $f_S : Y \rightarrow S$ by the compositions of f and the projections $A \times S \rightarrow A$ and $A \times S \rightarrow S$, respectively. See Remark 10.11 for the reason why we need to treat the situation above in the proof.

5. Preliminaries for the proof of Theorem A

Let V be a smooth algebraic variety. Let TV be the tangent bundle of V . Then we have $TV = \mathbf{Spec}(\text{Sym } \Omega_V^1)$. Set $\bar{TV} = P(TV \oplus \mathcal{O}_V)$, which is a smooth compactification of TV . Then we have $\bar{TV} = \mathbf{Proj}((\text{Sym } \Omega_V^1) \otimes_{\mathcal{O}_V} \mathcal{O}_V[\eta])$. Let $Z \subset V$ be a closed subscheme. We define a closed subscheme $Z^{(1)} \subset \bar{T}(V)$ as follows (cf. [Yam04, p. 38]): Let $U \subset V$ be an affine open subset. Let $(g_1, \dots, g_l) \subset \Gamma(U, \mathcal{O}_V)$ be the defining ideal of $Z \cap U$. Then $Z^{(1)} \cap \bar{T}(U)$ is defined by the homogeneous ideal

$$(g_1, \dots, g_l, dg_1 \otimes 1, \dots, dg_l \otimes 1) \subset \Gamma(U, (\text{Sym } \Omega_V^1) \otimes_{\mathcal{O}_V} \mathcal{O}_V[\eta]).$$

We glue $Z^{(1)} \cap \bar{T}(U)$ to define $Z^{(1)}$. If $Z \subset V$ is a closed immersion of a smooth algebraic variety, then $Z^{(1)} \cap T(V) = T(Z)$.

Let S be a projective variety, and let $W \subset A \times S$ be a Zariski closed set, where A is a semi-Abelian variety. Let $q : A \times S \rightarrow S$ be the second projection and let $q_W : W \rightarrow S$ be the restriction of q on W . Given $f : Y \rightarrow A \times S$, we denote by $\Sigma \subset S$ the Zariski closure of $f_S(Y) \subset S$. We define $e(f, W)$ to be the dimension of the generic fiber of $q_W^{-1}(\Sigma) \rightarrow \Sigma$. We set $e(f, W) = -1$ if the generic fiber of $q_W^{-1}(\Sigma) \rightarrow \Sigma$ is an empty set.

Now assume that $W \subset A \times S$ is irreducible and $\overline{q(W)} = S$. We also assume that $\dim \text{St}(W) = 0$, where the action $A \curvearrowright A \times S$ is defined by $(a, s) \mapsto (a + \alpha, s)$ for $\alpha \in A$. Since W and S are integral, and q_W is dominant, there exists a non-empty Zariski open subset $W^\circ \subset W$ such that q_W is a smooth morphism over W° . Let $S^\circ \subset S$ be a non-empty Zariski open subset such that every $s \in S^\circ$ satisfies $q_W^{-1}(s) \neq \emptyset$ and:

- (1) every irreducible component of $q_W^{-1}(s)$ has non-trivial intersection with W° , and
- (2) the stabilizer of every irreducible component of $q_W^{-1}(s)$ is 0-dimensional.

To justify the second condition, we argue as follows. We note that the assertion is local on S . Therefore, we may shrink S to ensure that any property holding generically on S holds over all of S . Let \bar{A} be a projective, equivariant compactification and let $\bar{W} \subset \bar{A} \times S$ be the closure. By the generic reduced theorem, the fibers \bar{W}_s are reduced for general $s \in S$ (cf. [Sta22, Tag 054Z, Tag 020I, Tag 0578]). Hence by shrinking S , we may assume that all fibers are reduced. Let $\bar{W}' \rightarrow \bar{W}$ be a desingularization, let $\bar{W}' \rightarrow S^\dagger \xrightarrow{\sim} S$ be the Stein factorization and let $V \subset \bar{A} \times S^\dagger$ be the image of the induced map $\bar{W}' \rightarrow \bar{A} \times S^\dagger$. Note that general fibers of $\bar{W}' \rightarrow S^\dagger$ are smooth and connected, hence irreducible. Therefore by shrinking S , all fibers of $V \rightarrow S^\dagger$ are irreducible, and are also reduced by the generic reduced theorem. Now we have a natural surjection $v : V \rightarrow \bar{W}$. Hence by

$\dim \text{St}(W) = 0$, we have $\dim \text{St}(V) = 0$. Thus by [Yam23, Lemma 11.12], we have $\dim \text{St}(V_y) = 0$ for general fibers V_y , thus for all $y \in S^\dagger$ by shrinking S . Then for any $s \in S$ and any irreducible component of \overline{W}_s , there exists $y \in S^\dagger$ such that $u(y) = s$ and V_y is isomorphic to that component. This confirms the validity of the second assertion.

Assume S is smooth and projective. We note

$$(5.1) \quad \overline{T}(A \times S) = A \times S',$$

where $S' = \overline{\text{Lie}(A) \times TS}$. We may define $e(j_1 f, W^{(1)})$ from $j_1 f : Y \rightarrow A \times S'$ and $W^{(1)} \subset A \times S'$.

Lemma 5.1. — Assume that S is smooth and projective, and that $W \subset A \times S$ is irreducible with $\overline{q(W)} = S$. Let $f : Y \rightarrow A \times S$ satisfies $e(j_1 f, W^{(1)}) = e(f, W)$ and $f_S(Y) \not\subset S - S^o$. Then we have

$$T_{f_A}(r) = O(T_{f_S}(r)) + O(\log r).$$

Remark 5.2. — In the statement of the lemma, by $T_{f_A}(r)$ we mean $T_{f_A}(r, L)$ for some ample line bundle on a compactification \overline{A} . So this notion has ambiguity, but we have $T_{f_A}(r, L') = O(T_{f_A}(r, L)) + O(\log r)$ for other choices. Hence the term $O(T_{f_A}(r)) + O(\log r)$ has fixed meaning. Similar for $O(T_{f_S}(r)) + O(\log r)$.

Proof of Lemma 5.1. — We define $\mathcal{Z}(W) \subset A \times S \times S$ by

$$\mathcal{Z}(W) = \{(a, s_1, s_2) \in A \times S \times S; \dim(q_W^{-1}(s_1) \cap (a + q_W^{-1}(s_2))) \geq \dim W - \dim S\},$$

where we consider $q_W^{-1}(s_1)$ and $q_W^{-1}(s_2)$ as Zariski closed subsets of A . For $s \in S$, let $\mathcal{Z}_s(W) \subset A \times S$ be the fiber of the composite map $\mathcal{Z}(W) \hookrightarrow A \times S \times S \rightarrow S$ over $s \in S$, where the second map $A \times S \times S \rightarrow S$ is the third projection.

Claim 5.3. — Let $s \in S^o$. Let $g : \mathbb{D} \rightarrow A \times S$ be a holomorphic map such that $e(j_1 g, W^{(1)}) = e(g, W)$ with $g(0) = (a, s) \in A \times S^o$. Then $g(\mathbb{D}) \subset \mathcal{Z}_s(W) + a$.

Proof. — This is proved in the sublemma of [Yam15a, Lemma 3] when A is compact. Our argument proceeds along the same lines as that proof.

By $g \circ g(0) = s \in S^o$, we may take a non-empty open subset $\Omega \subset \mathbb{D}$ such that $g \circ g(\Omega) \subset S^o$. Hence the condition (1) of S^o implies

$$e(g, W) = \dim W - \dim S.$$

Let $q'_{W^{(1)}} : W^{(1)} \rightarrow S'$ be the restriction of the second projection $q' : A \times S' \rightarrow S'$ to $W^{(1)}$. By shrinking Ω if necessary, the assumption $e(j_1 g, W^{(1)}) = e(g, W)$ implies that

$$\dim(q'_{W^{(1)}})^{-1}(q' \circ j_1 g(z)) = \dim W - \dim S$$

for all $z \in \Omega$. Indeed denoting by $\Sigma \subset S'$ the Zariski closure of $q' \circ j_1 g(\mathbb{D}) \subset S'$, the set $\{s \in \Sigma; \dim(q'_{W^{(1)}})^{-1}(s) = \dim W - \dim S\}$ is constructible (cf. [Sta22, Tag 05F9]). Hence we may shrink Ω as desired above.

Now for each $z \in \Omega$, we may take an irreducible component I_z of $(q'_{W^{(1)}})^{-1}(q' \circ j_1 g(z))$ such that

$$(5.2) \quad \dim I_z = \dim W - \dim S.$$

Since $(q'_{W^{(1)}})^{-1}(q' \circ j_1 g(z)) \subset q_W^{-1}(q \circ g(z))$ as Zariski closed subsets of A , we conclude that I_z is an irreducible component of $q_W^{-1}(q \circ g(z))$. Hence the condition (1) of S^o implies

$$(5.3) \quad I_z \cap W^o \neq \emptyset.$$

Define

$$\psi : A \times \mathbb{D} \rightarrow A \times S, \quad \psi(b, z) = (b, q \circ g(z))$$

and set

$$W_{\mathbb{D}}^o := \psi^{-1}(W^o).$$

On W^o , the map q_W is smooth, hence a submersion. Therefore $W_{\mathbb{D}}^o$ is a (possibly non-connected) smooth complex manifold, and the induced map $q_{W_{\mathbb{D}}^o} : W_{\mathbb{D}}^o \rightarrow \mathbb{D}$ is a submersion. Hence every connected component of $W_{\mathbb{D}}^o$ has dimension $\dim W - \dim S + 1$.

Define an automorphism $\tau : A \times \mathbb{D} \rightarrow A \times \mathbb{D}$ by

$$\tau(b, z) = (b - a + p \circ g(z), z).$$

Let \mathcal{X} be the holomorphic vector field on $A \times \mathbb{D}$ given by

$$\mathcal{X} := \tau_*(0, \partial/\partial z),$$

where $(0, \partial/\partial z)$ is the standard vector field on $A \times \mathbb{D}$ and τ_* denotes the push-forward. For each $b \in A$ set

$$\tau_b : \mathbb{D} \rightarrow A \times \mathbb{D}, \quad \tau_b(z) = \tau(b, z).$$

Each map τ_b is an integral curve of the vector field \mathcal{X} .

We shall show that there exists a connected component $\tilde{W}_{\mathbb{D}}^o$ of $W_{\mathbb{D}}^o$ such that \mathcal{X} is tangent to $\tilde{W}_{\mathbb{D}}^o$ at every point of $\tilde{W}_{\mathbb{D}}^o$. We specify $\tilde{W}_{\mathbb{D}}^o$ as follows. Set

$$\mathcal{W} := \{x \in W^{(1)}; \dim_x(q'_{W^{(1)}})^{-1}(q'_{W^{(1)}}(x))) \geq \dim W - \dim S\}.$$

Then $\mathcal{W} \subset W^{(1)}$ is Zariski closed. Put

$$V := \{(b, z) \in W_{\mathbb{D}}^o; \psi_*(\mathcal{X}_{(b, z)}) \in \mathcal{W}\},$$

where $\psi_* : T(A \times \mathbb{D}) \rightarrow T(A \times S)$ is the push-forward of ψ . Then $V \subset W_{\mathbb{D}}^o$ is an analytic subset. Note that $\psi_*(\mathcal{X}_{(b, z)})$ is a translate of $j_1(g)(z) \in T_{g(z)}(A \times S)$. Thus taking into account (5.2), we get $I_z \cap W^o \subset \varphi^{-1}(z) \cap V$ for $z \in \Omega$, where $\varphi : A \times \mathbb{D} \rightarrow \mathbb{D}$ is the projection. In particular, by (5.3), we have

$$\Omega \subset \varphi(V),$$

and by the definition of V ,

$$\dim_x(\varphi^{-1}(z) \cap V) = \dim W - \dim S$$

for every $z \in \Omega$ and $x \in \varphi^{-1}(z) \cap V$. Since $V \subset W_{\mathbb{D}}^o$ is an analytic subset, these two conditions imply that V contains a connected component of $W_{\mathbb{D}}^o$. We denote this component by $\tilde{W}_{\mathbb{D}}^o$.

Next we show that \mathcal{X} is tangent to $\tilde{W}_{\mathbb{D}}^o$ at every point of $\tilde{W}_{\mathbb{D}}^o$. Since $q_W|_{W^o} : W^o \rightarrow S$ is submersion, we have the identity of subbundles of $T(A \times \mathbb{D})$:

$$TW_{\mathbb{D}}^o = (\psi_*)^{-1}(TW^o).$$

Therefore it is enough to show

$$(5.4) \quad \psi_*(\mathcal{X}_{(b, z)}) \in W^{(1)}$$

for every $(b, z) \in \tilde{W}_{\mathbb{D}}^o$. However by the definition of V , this holds for all points of V , in particular holds for the subset $\tilde{W}_{\mathbb{D}}^o \subset V$. Therefore \mathcal{X} is tangent to $\tilde{W}_{\mathbb{D}}^o$ at every point of $\tilde{W}_{\mathbb{D}}^o$.

Now we take $\zeta \in \Omega$, so that $\varphi^{-1}(\zeta) \cap \tilde{W}_{\mathbb{D}}^o \neq \emptyset$. For each $w \in \varphi^{-1}(\zeta) \cap \tilde{W}_{\mathbb{D}}^o$, we set $c = w + a - p \circ g(\zeta)$. Then $\tau_c(\zeta) = (w, \zeta)$. Since $\tau_c : \mathbb{D} \rightarrow A \times \mathbb{D}$ is an integral curve of \mathcal{X} and \mathcal{X} is tangent to $\tilde{W}_{\mathbb{D}}^o$ at every point of $\tilde{W}_{\mathbb{D}}^o$, we have $\tau_c(U) \subset \tilde{W}_{\mathbb{D}}^o$ for some neighbourhood $U \subset \mathbb{D}$ of ζ . Put $W_{\mathbb{D}} := \psi^{-1}(W)$, then $W_{\mathbb{D}} \subset A \times \mathbb{D}$ is an analytic set, and $W_{\mathbb{D}}^o \subset W_{\mathbb{D}}$ is an open subset. Therefore we have $\tau_c(\mathbb{D}) \subset W_{\mathbb{D}}$. Thus for every $z \in \mathbb{D}$ and every $w \in \varphi^{-1}(\zeta) \cap \tilde{W}_{\mathbb{D}}^o$, we have $\tau_c(z) \in \varphi^{-1}(z) \cap W_{\mathbb{D}}$. Equivalently,

$$(p \circ g(z) - p \circ g(\zeta)) + (\varphi^{-1}(\zeta) \cap \tilde{W}_{\mathbb{D}}^o) \subset \varphi^{-1}(z) \cap W_{\mathbb{D}}.$$

Considering this relation for $z = 0$ and arbitrary z , we obtain

$$-p \circ g(\zeta) + (\varphi^{-1}(\zeta) \cap \tilde{W}_{\mathbb{D}}^o) \subset \{-p \circ g(0) + (\varphi^{-1}(0) \cap W_{\mathbb{D}})\} \cap \{-p \circ g(z) + (\varphi^{-1}(z) \cap W_{\mathbb{D}})\}$$

for all $z \in \mathbb{D}$. This implies

$$(p \circ g(z) - p \circ g(0), q \circ g(z), q \circ g(0)) \in \mathcal{Z}(W)$$

for all $z \in \mathbb{D}$, which exactly means $g(\mathbb{D}) \in \mathcal{Z}_s(W) + a$. This completes the proof of Claim 5.3. \square

Now we return to the proof of Lemma 5.1. We remark that $\mathcal{Z}_s(W) \subset A \times S$ is a constructible subset for $s \in S$. To see this, we consider the composite map

$$\phi : W \times W_s \hookrightarrow (A \times S) \times A \rightarrow A \times S,$$

where the second map is defined by

$$(A \times S) \times A \ni ((a, t), a') \mapsto (a - a', t) \in A \times S.$$

Then $\mathcal{Z}_s(W)$ is given by

$$\mathcal{Z}_s(W) = \{x \in A \times S \mid \dim \phi^{-1}(x) \geq \dim W - \dim S\}.$$

Hence, $\mathcal{Z}_s(W) \subset A \times S$ is a constructible subset (cf. [Sta22, Tag 05F9]).

Now we take $y \in Y$ such that $f_S(y) \in S^o$ and $f(y) = (a, s) \in A \times S^o$. By the second condition of the definition of S^o , the stabilizer of every irreducible component of $q_W^{-1}(s)$ is 0-dimensional. Therefore, the restriction of the natural map

$$q|_{\mathcal{Z}_s(W)} : \mathcal{Z}_s(W) \rightarrow S$$

over S^o is quasi-finite. Let $E \subset S$ be the Zariski closure of $f_S(Y)$ and let \bar{A} be a projective compactification of A . By Claim 5.3, we have

$$f(Y) \subset \overline{q^{-1}(E) \cap (\mathcal{Z}_s(W) + a)} \subset \bar{A} \times S.$$

Hence there exists an irreducible component Z of $\overline{q^{-1}(E) \cap (\mathcal{Z}_s(W) + a)}$ such that $f(Y) \subset Z$. Since $q^{-1}(E) \cap (\mathcal{Z}_s(W) + a) \subset A \times S$ is a constructible subset, it contains a dense Zariski open subset of Z . Hence the projection $\mu : Z \rightarrow S$ is generically-finite onto its image $\mu(Z) = E$. Thus $\dim Z = \dim E$, and $f(Y) \subset Z$ is Zariski dense. Then by [Yam15a, Lemma 4], we have $T_{f_A}(r) = O(T_{f_S}(r)) + O(\log r)$. \square

We recall $S' = \overline{\text{Lie}A \times TS}$ from (5.1), so that $\bar{T}(A \times S) = A \times S'$.

Lemma 5.4. — For $f : Y \rightarrow A \times S$, we have

$$T_{(j_1 f)_{S'}}(r) = O(T_{f_S}(r) + N_{\text{ram } \pi_Y}(r) + \bar{N}_f^\partial(r)) + O(\log r) + o(T(r, f_A)) \parallel,$$

where $j_1(f)_{S'} : Y \rightarrow S'$ is the composition of $j_1(f) : Y \rightarrow A \times S'$ and the second projection $A \times S' \rightarrow S'$.

Proof. — (cf. [Yam15a, Lemma 2]) Let \bar{A} be an equivariant compactification, which is smooth and projective. Set $D = (\partial A) \times S$. Then D is a simple normal crossing divisor on $\bar{A} \times S$. Let $T(\bar{A} \times S; \log D)$ be the logarithmic tangent bundle. We set $\bar{T}(\bar{A} \times S; \log D) = P(T(\bar{A} \times S; \log D) \oplus \mathcal{O}_{\bar{A} \times S})$, which is a smooth compactification of $T(\bar{A} \times S; \log D)$. We set $E = \bar{T}(\bar{A} \times S; \log D) - T(\bar{A} \times S; \log D)$, which is a divisor on $\bar{T}(\bar{A} \times S; \log D)$. By $T(A \times S) \subset \bar{T}(\bar{A} \times S; \log D)$, we have $j_1 f : Y \rightarrow \bar{T}(\bar{A} \times S; \log D)$. By [Yam04, (2.4.8)] and (4.4), we have

$$T_{j_1 f}(r, E) \leq N_{\text{ram } \pi}(r) + \bar{N}_f^\partial(r) + O(\log r) + o(T_f(r)) \parallel.$$

Note that $T(\bar{A}; \log \partial A) = \bar{A} \times \text{Lie}A$ (cf. [NW14, Prop 5.4.3]). Hence we have $\bar{A} \times S' = \bar{T}(\bar{A} \times S; \log D)$ and $S' = \mathbf{Proj}((\text{Sym } \Omega_S^1) \otimes_{\mathcal{O}_S} \mathcal{O}_S[\eta, dz_1, \dots, dz_{\dim A}])$, where $\{dz_1, \dots, dz_{\dim A}\} \subset H^0(\bar{A}, \Omega_{\bar{A}}^1(\log \partial A))$ is a basis. Let $F \subset S'$ be the divisor defined by $\eta = 0$. Then we have $p^* F = E$, where $p : \bar{A} \times S' \rightarrow S'$ is the second projection. Hence we have

$$T_{(j_1 f)_{S'}}(r, F) \leq N_{\text{ram } \pi_Y}(r) + \bar{N}_f^\partial(r) + O(\log r) + o(T(r, f_A)) \parallel.$$

By $\text{Pic}(S') = \text{Pic}(S) \oplus \mathbb{Z}[F]$, we obtain our lemma. \square

6. Refinement of log Bloch-Ochiai Theorem

Let A be a semi-Abelian variety. Let $\mathcal{S}_0(A)$ be the set of all semi-abelian subvarieties of A . Let $W \subset A \times S$ be a Zariski closed subset. For $B \in \mathcal{S}_0(A)$, we set

$$(6.1) \quad W^B = \{x \in W; x + B \subset W\}.$$

Then $W^B = \cap_{b \in B} (b + W) \subset W$ is a Zariski closed subset. When $B = \{0\}$, we have $W^{\{0\}} = W$.

Proposition 6.1. — Let S be a projective variety. Let $W \subset A \times S$ be an irreducible Zariski closed subset. Then there exists a finite subset $P \subset \mathcal{S}_0(A)$ such that for every $f : Y \rightarrow W$, there exists $B \in P$ such that $f(Y) \subset W^B$ and

$$(6.2) \quad T_{q_B \circ f_A}(r) = O(N_{\text{ram } \pi}(r) + T_{f_S}(r) + \bar{N}_f^\partial(r)) + O(\log r) + o(T_f(r)) \parallel,$$

where $q_B : A \rightarrow A/B$ is the quotient.

Proof. — Given $f : Y \rightarrow W$, we have $e(f, W) \neq -1$, hence $e(f, W) \in \{0, 1, \dots, \dim A\}$. Proposition 6.1 is reduced to the following claim by setting $k = \dim A$.

Claim 6.2. — Let $(k, d) \in \mathbb{Z}_{\geq 0}^2$. Let S be a projective variety such that $\dim S = d$, and let A be a semi-Abelian variety. Let $W \subset A \times S$ be an irreducible Zariski closed subset. Then there exists a finite subset $P \subset \mathcal{S}_0(A)$ such that for every $f : Y \rightarrow W$ with $e(f, W) \leq k$, there exists $B \in P$ such that $f(Y) \subset W^B$ and (6.2).

We prove this claim by induction on the pair of integers $(k, d) \in \mathbb{Z}_{\geq 0}^2$, with respect to the lexicographical order on $\mathbb{Z}_{\geq 0}^2$. We therefore prove the claim for the pair (k, d) by assuming that it holds for all pairs (k', d') such that $(k', d') < (k, d)$.

Let $q : A \times S \rightarrow S$ be the second projection. Replacing S by the Zariski closure of $q(W)$, we may assume that $q(W) \subset S$ is dominant.

Step 1. In the preceding proof, an exceptional locus $V \subsetneq S$ naturally arises, which leads us to distinguish between two cases: $f_S(Y) \subset V$ or not. As a preliminary step, we treat the first case by fixing a proper Zariski closed subset $V \subsetneq S$ and assuming $f_S(Y) \subset V$. Let V_1, \dots, V_l be the irreducible components of V . For $j = 1, \dots, l$, we set $W_j = W \cap (A \times V_j)$. Let $W_j^1, \dots, W_j^{t_j}$ be the irreducible components of W_j . By $\dim V_j < d$, the induction hypothesis applies: There exists a finite subset $P_{V_j, W_j^i} \subset \mathcal{S}_0(A)$ such that if $f(Y) \subset W_j^i$ and $e(f, W_j^i) \leq k$, then there exists $B \in P_{V_j, W_j^i}$ such that $f(Y) \subset (W_j^i)^B$ and (6.2). Set $P_V = \cup_j \cup_i P_{V_j, W_j^i}$. Thus if $f_S(Y) \subset V$ and $e(f, W) \leq k$, then there exists $B \in P_V$ such that $f(Y) \subset W^B$ and (6.2).

Step 2. We consider the case that S is smooth and $\text{St}^0(W) = \{0\}$. Let $f : Y \rightarrow W$ satisfy $e(f, W) \leq k$. Suppose $f_S(Y) \subset S \setminus S^\circ$, then by the above consideration (Step 1), there exists $B \in P_{S \setminus S^\circ}$ such that $f(Y) \subset W^B$ and (6.2). So we consider the case $f_S(Y) \not\subset S \setminus S^\circ$. We consider the first jet $j_1 f : Y \rightarrow \overline{T}(A \times S) = A \times (\overline{\text{Lie}A \times TS})$. Set $S' = \overline{\text{Lie}A \times TS}$. If $e(j_1 f, W^{(1)}) = e(f, W)$, then Lemma 5.1 yields

$$T_{f_A}(r) = O(T_{f_S}(r)) + O(\log r).$$

This shows (6.2) for $B = \{0\}$, and obviously $f(Y) \subset W = W^{\{0\}}$. If $e(j_1 f, W^{(1)}) < e(f, W) \leq k$, then $e(j_1 f, W^{(1)}) \leq k - 1$. Thus the induction hypothesis yields a finite set $P' \subset \mathcal{S}_0(A)$ such that there exists $B \in P'$ such that $j_1 f(Y) \subset (W^{(1)})^B$ and

$$T_{q_B \circ f_A}(r) = O(N_{\text{ram}} \pi(r) + \bar{N}_f^\partial(r) + T_{(j_1 f)_{S'}}(r)) + O(\log r) + o(T_f(r))||.$$

By Lemma 5.4, we have

$$T_{(j_1 f)_{S'}}(r) = O(N_{\text{ram}} \pi(r) + \bar{N}_f^\partial(r) + T_{f_S}(r)) + O(\log r) + o(T_{f_A}(r))||.$$

Hence we have $f(Y) \subset W^B$ and (6.2). We set $P = P' \cup P_{S \setminus S^\circ} \cup \{0\}$. This concludes the proof of the claim if S is smooth and $\text{St}^0(W) = \{0\}$.

Step 3. We remove the assumption that S is smooth. Let $\tilde{S} \rightarrow S$ be a smooth modification which is an isomorphism outside a proper Zariski closed set $E \subsetneq S$. If $f_S(Y) \subset E$, then we apply Step 1: there exists $B \in P_E$ such that $f(Y) \subset W^B$ and (6.2). If $f_S(Y) \not\subset E$, then there exists a unique lift $f : Y \rightarrow A \times \tilde{S}$. Let $\tilde{W} \subset A \times \tilde{S}$ be the proper transform of W . Then by the consideration above (Step 2), there exists $P' \subset \mathcal{S}_0(A)$ such that $f(Y) \subset \tilde{W}^B$ and (6.2) for some $B \in P'$. We set $P = P' \cup P_E$ to conclude the proof of the claim when $\text{St}^0(W) = \{0\}$.

Step 4. Finally we remove the assumption $\text{St}^0(W) = \{0\}$. Suppose that $\text{St}^0(W) \neq \{0\}$. Set $C = \text{St}^0(W)$. Let $f' : Y \rightarrow W/C \subset (A/C) \times S$ be the induced map. Then we have $e(f', W/C) < e(f, W)$. Hence by the induction hypothesis, there exists $P' \subset \mathcal{S}_0(A/C)$ such that $f'(Y) \subset (W/C)^{B'}$ and (6.2) for some $B' \in P'$. We define P to be the set of all B such that $C \subset B$ and $B/C \in P'$. This concludes the proof of the claim. Thus the proof of Proposition 6.1 is completed. \square

Corollary 6.3. — In Proposition 6.1, we may take P so that $\text{St}^0(W) \subset B$ for all $B \in P$. Moreover there exists a proper Zariski closed set $\Xi \subsetneq W$ such that for every $f : Y \rightarrow W$, either one of the followings holds:

1. $f(Y) \subset \Xi$.

2. $T_{q_{\text{St}^0(W)} \circ f_A}(r) = O(N_{\text{ram}} \pi(r) + \bar{N}_f^\partial(r) + T_{f_S}(r)) + O(\log r) + o(T_f(r))||$, where $q_{\text{St}^0(W)} : A \rightarrow A/\text{St}^0(W)$ is the quotient map

Proof. — We apply Proposition 6.1 for $W/\text{St}^0(W) \subset (A/\text{St}^0(W)) \times S$ to get $P_0 \subset \mathcal{S}_0(A/\text{St}^0(W))$. Then we define $P \subset \mathcal{S}_0(A)$ as the set of all B such that $\text{St}^0(W) \subset B$ and $B/\text{St}^0(W) \in P_0$. We set $P' = P \setminus \{\text{St}^0(W)\}$ and $\Xi = \cup_{B \in P'} W^B$. Then Ξ is a proper Zariski closed set. \square

7. Second main theorem type estimate with weak truncation

Let A be a semi-Abelian variety. Let $\mathcal{S}(A)$ be the set of all positive dimensional semi-abelian subvarieties of A . Hence $\mathcal{S}(A) = \mathcal{S}_0(A) \setminus \{\{0\}\}$.

Proposition 7.1. — *Let \bar{A} be an equivariant compactification of A such that \bar{A} is smooth and projective. Let $W \subsetneq \bar{A} \times S$ be a closed subscheme, where S is a projective variety. Then there exist a finite subset $P \subset \mathcal{S}(A) \setminus \{A\}$ and a positive integer $\rho \in \mathbb{Z}_{>0}$ with the following property: Let $f : Y \rightarrow A \times S$ satisfies $f(Y) \not\subset \text{supp } W$ and $f_S(Y) \not\subset p(\text{Sp}_A W)$, where $\text{Sp}_A W = \cap_{a \in A} (a + W) \subset \bar{A} \times S$ and $p : \bar{A} \times S \rightarrow S$ is the second projection. Then either one of the followings are true:*

1. *There exists $B \in P$ such that*

$$T_{q \circ f_A}(r) = O(T_{f_S}(r) + N_{\text{ram}} \pi(r) + \bar{N}_f^\partial(r)) + O(\log r) + o(T_f(r))||,$$

where $q : A \rightarrow A/B$ is the quotient map.

2. *The following estimate holds:*

$$(7.1) \quad m_f(r, W) + N_f(r, W) - N_f^{(\rho)}(r, W) = O(N_{\text{ram}} \pi(r) + \bar{N}_f^\partial(r) + T_{f_S}(r)) + O(\log r) + o(T_f(r))||.$$

Proof. — For $W \subset \bar{A} \times S$, we set $W_o = W \cap (A \times S)$. Given $f : Y \rightarrow A \times S$, we have $e(f, W_o) \in \{-1, 0, \dots, \dim A\}$. Proposition 7.1 is reduced to the following claim by setting $k = \dim A$.

Claim 7.2. — *Let $(m, k, d) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq -1} \times \mathbb{Z}_{\geq 0}$. Let A be a semi-Abelian variety such that $\dim A = m$ and let S be a projective variety such that $\dim S = d$. Let \bar{A} and $W \subsetneq \bar{A} \times S$ be the same as in Proposition 7.1. Then there exist a finite subset $P \subset \mathcal{S}(A) \setminus \{A\}$ and a positive integer $\rho \in \mathbb{Z}_{>0}$ with the following property: Let $f : Y \rightarrow A \times S$ satisfies $f(Y) \not\subset \text{supp } W$, $f_S(Y) \not\subset p(\text{Sp}_A W)$, and $e(f, W_o) \leq k$. Then either one of the assertions of Proposition 7.1 is true.*

We prove this claim by the induction on the triple $(m, k, d) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq -1} \times \mathbb{Z}_{\geq 0}$ with the dictionary order. We therefore prove the claim for the triple (m, k, d) by assuming that it holds for all triples (m', k', d') such that $(m', k', d') < (m, k, d)$.

Step 1. As in the step 1 of the proof of Proposition 6.1, we fix a proper Zariski closed subset $V \subsetneq S$ and assume $f_S(Y) \subset V$. Let V_1, \dots, V_l be the irreducible components of V . For $j = 1, \dots, l$, we set $W_j = W_o \cap (A \times V_j)$. By $\dim V_j < d$, the induction hypothesis applies: There exist a finite subset $P_j \subset \mathcal{S}(A) \setminus \{A\}$ and $\rho_j \in \mathbb{Z}_{\geq 1}$ such that if $f : Y \rightarrow A \times S$ satisfies $f_S(Y) \subset V_j$ and $e(f, W_j) \leq k$ in addition to $f(Y) \not\subset W_j$ and $f_S(Y) \not\subset p(\text{Sp}_A W_j)$, either of the two assertions of Proposition 7.1 holds, with P and ρ replaced by P_j and ρ_j , respectively. Set $P_V = \cup_j P_j$ and $\rho_V = \max_j \rho_j$. Then if $f_S(Y) \subset V$ and $e(f, W_o) \leq k$ in addition to $f(Y) \not\subset W$ and $f_S(Y) \not\subset p(\text{Sp}_A W)$, either of the two assertions of Proposition 7.1 holds, with P and ρ replaced by P_V and ρ_V , respectively.

Step 2. We assume that the following three conditions are satisfied.

1. $q(W) = S$, where $q : \bar{A} \times S \rightarrow S$ is the second projection,
2. W is reduced, and irreducible,
3. S is smooth.

This step is separated into several cases.

Step 2-1. $W_o \neq \emptyset$. Set $C = \text{St}^0(W_o)$. We define $S^o \subset S$ from $W_o/C \subset (A/C) \times S$ (cf. Lemma 5.1). We consider $f : Y \rightarrow A \times S$ with $e(f, W_o) = k$ such that $f(Y) \not\subset W$ and $f_S(Y) \not\subset p(\text{Sp}_A W)$. If $f_S(Y) \subset S \setminus S_o$, then by the above consideration (Step 1), either of the two

assertions of Proposition 7.1 holds, with P and ρ replaced by $P_{S \setminus S_o}$ and $\rho_{S \setminus S_o}$, respectively. So we assume that $f_S(Y) \not\subset S \setminus S_o$. We consider the first jet $j_1 f : Y \rightarrow A \times S'$, where $S' = \overline{\text{Lie}(A)} \times TS$.

Step 2-1-1. $e(j_1 f, W_o^{(1)}) = e(f, W_o)$, where $W_o^{(1)} \subset A \times S'$. Note that $C = \text{St}^0(W_o^{(1)})$. Let $f' : Y \rightarrow (A/C) \times S$ be the composition of f and the projection $A \times S \rightarrow (A/C) \times S$. Then we have $e(f, W_o) = e(f', W_o/C) + \dim C$ and $e(j_1 f, W_o^{(1)}) = e(j_1 f', (W_o/C)^{(1)}) + \dim C$. Hence $e(f', W_o/C) = e(j_1 f', (W_o/C)^{(1)})$. Thus Lemma 5.1 yields that

$$(7.2) \quad T_{q_C \circ f_A}(r) = O(T_{f_S}(r)) + O(\log r),$$

where $q_C : A \rightarrow A/C$ is the quotient. Since we are assuming that $W \rightarrow S$ is dominant, we have $C \neq A$, for otherwise $f(Y) \subset W$. Thus if $\dim C > 0$, the estimate (7.2) shows the first assertion of Proposition 7.1, provided $C \in P$. If $\dim C = 0$, then (7.2) yields that $T_{f_A}(r) = O(T_{f_S}(r)) + O(\log r)$. Hence $m_f(r, W) + N_f(r, W) = O(T_{f_S}(r)) + O(\log r)$. This is stronger than the second assertion of Proposition 7.1. Thus assuming $C \in P$ if $C \neq \{0\}$, we conclude that one of the assertions of Proposition 7.1 holds.

Step 2-1-2. $e(j_1 f, W_o^{(1)}) < e(f, W_o)$. Then the induction hypothesis yields $P' \subset \mathcal{S}(A) \setminus \{A\}$ and ρ' such that either the first assertion of Proposition 7.1 for $P = P'$ or the estimate

$$\begin{aligned} m_{j_1 f}(r, W_{\log}^{(1)}) + N_{j_1 f}(r, W_{\log}^{(1)}) - N_{j_1 f}^{(\rho')}(r, W_{\log}^{(1)}) \\ = O(N_{\text{ram } \pi}(r) + \bar{N}_f^\partial(r) + T_{j_1 f_{S'}}(r)) + O(\log r) + o(T_f(r)) \end{aligned}$$

holds. Here $W_{\log}^{(1)} \subset \overline{A} \times S' = \overline{T}(\overline{A} \times S; \log \partial(A \times S))$ is defined in [Yam04, Section 5] so that $W_{\log}^{(1)} \cap (A \times S') = W_o^{(1)}$. By the same argument for the proof of [Yam15a, Lemma 5], using [Yam04, Thm 5.1.7], we have

$$\begin{aligned} (7.3) \quad m_f(r, W) + N_f(r, W) - N_f^{(\rho'+1)}(r, W) \\ \leq m_{j_1 f}(r, W_{\log}^{(1)}) + N_{j_1 f}(r, W_{\log}^{(1)}) - N_{j_1 f}^{(\rho')}(r, W_{\log}^{(1)}) + N_{\text{ram } \pi}(r) + o(T_f(r)). \end{aligned}$$

By Lemma 5.4, we have

$$T_{(j_1 f)_{S'}}(r) = O(T_{f_S}(r) + N_{\text{ram } \pi}(r) + \bar{N}_f^\partial(r)) + O(\log r) + o(T_f(r)).$$

Thus we get

$$m_f(r, W) + N_f(r, W) - N_f^{(\rho'+1)}(r, W) = O(T_{f_S}(r) + N_{\text{ram } \pi}(r) + \bar{N}_f^\partial(r)) + O(\log r) + o(T_f(r)).$$

Therefore, combining this with Case 2-1-1, we complete the proof of the induction step for Case 2-1. Here we set $P = (P_{S \setminus S_o} \cup P' \cup \{C\}) \setminus \{0\}$ and $\rho = \rho_{S \setminus S_o} + \rho' + 1$.

Step 2-2. $W_o = \emptyset$. In this case, we have $W \subset \partial A \times S$ and $e(f, W_o) = -1$. Let $I \subset A$ be the isotropy group for W and let $D \subset \partial A$ be the irreducible component of ∂A such that $W \subset D \times S$. Then D is an equivariant compactification of A/I . By [Yam23, Lem A.11], there exist an A -invariant Zariski open set $U \subset \overline{A}$ and an equivariant map $\psi : U \rightarrow D$ such that $D \subset U$ and ψ is an isomorphism over $D \subset U$. We define $g : Y \rightarrow (A/I) \times S \subset D \times S$ by $g = (\psi \circ f_A, f_S)$. By $\text{Sp}_A(W) = \text{Sp}_{A/I}(W)$, we have $g_S(Y) \not\subset p(\text{Sp}_{A/I}W)$. Suppose $g(Y) \not\subset W$. Note that $\dim(A/I) < \dim A$. Hence by the induction hypothesis, there exist $P' \subset \mathcal{S}(A/I) \setminus \{A/I\}$ and $\rho' \in \mathbb{Z}_{>0}$, which are independent of the choice of f , such that either the first assertion of Proposition 7.1 or the following estimate

$$m_g(r, W) + N_g(r, W) - N_g^{(\rho')}(r, W) = O(N_{\text{ram } \pi}(r) + \bar{N}_g^\partial(r) + T_{f_S}(r)) + O(\log r) + o(T_f(r))$$

holds. By $\bar{N}_g^\partial(r) \leq \bar{N}_f^\partial(r)$, $m_g(r, W) \leq m_f(r, W)$ and $\text{ord}_y f^* W \leq \text{ord}_y g^* W$ for all $y \in Y$, this estimate implies the second assertion of Proposition 7.1 for $\rho = \rho'$. If $g(Y) \subset W$, then we apply Proposition 6.1 to get $P'' \subset \mathcal{S}_0(A/I)$, which is independent of the choice of f . Then there exists $B \in P''$ such that

$$T_{q \circ g_{A/I}}(r) = O(T_{f_S}(r) + N_{\text{ram } \pi}(r) + \bar{N}_f^\partial(r)) + O(\log r) + o(T_f(r)),$$

where $q : A/I \rightarrow (A/I)/B$ is the quotient. We note $B \neq A/I$, for otherwise we have $g(Y) \subset W^{A/I}$, which contradicts to $g_S(Y) \not\subset p(\text{Sp}_{A/I}W)$. We define P to be the set of all B such that $I \subset B$ and

$B/I \in P' \cup (P'' \setminus \{A/I\})$. This concludes the proof of the induction step for our case $W_o = \emptyset$. Thus we have completed the induction step under the three assumptions of Step 2.

Step 3. We remove the three assumptions on W and S made in Step 2. First we remove the first assumption. Suppose $q(W) \subsetneq S$, where $q : \overline{A} \times S \rightarrow S$ is the second projection. If $f_S(Y) \not\subseteq q(W)$, then we have

$$m_f(r, W) + N_f(r, W) = O(T_{f_S}(r)) + O(\log r).$$

This is stronger than the second assertion of Proposition 7.1. Thus we may consider the case $f_S(Y) \subset q(W)$. We replace S by $q(W)$. Then, by the induction hypothesis, we get our claim. Hence the first assumption is removed.

Next we remove the second assumption. Let W_1, \dots, W_n be the irreducible components of $\text{supp } W$. Then for each $j \in \{1, \dots, n\}$, by the argument above (Step 2), there exist a finite subset $P_j \subset \mathcal{S}(A) \setminus \{A\}$ and a positive integer ρ_j such that for every $f : Y \rightarrow A \times S$ with $e(f, W_o) \leq k$, either the first assertion of Proposition 7.1 for $P = P_j$ or the following estimate holds:

$$(7.4) \quad m_f(r, W_j) + N_f(r, W_j) - N_f^{(\rho_j)}(r, W_j) = O(T_{f_S}(r) + N_{\text{ram } \pi_Y}(r) + \bar{N}_f^\partial(r)) + O(\log r) + o(T_f(r)).$$

There is a positive integer l such that

$$(\mathcal{I}_{W_1} \cdots \mathcal{I}_{W_n})^l \subset \mathcal{I}_W,$$

where $\mathcal{I}_W \subset \mathcal{O}_{A \times S}^-$ (resp. \mathcal{I}_{W_i}) is the defining ideal sheaf of W (resp. W_i). Then we have

$$(7.5) \quad m_f(r, W) \leq l \sum_{i=1}^n m_f(r, W_i).$$

For $y \in Y$, we have

$$\text{ord}_y f^* W \leq l \sum_{i=1}^n \text{ord}_y f^* W_i.$$

Hence setting $\tilde{\rho} = \rho_1 + \dots + \rho_n$, we get

$$\begin{aligned} \max\{0, \text{ord}_y f^* W - l\tilde{\rho}\} &\leq \max \left\{ 0, l \sum_{i=1}^n (\text{ord}_y f^* W_i - \rho_i) \right\} \\ &\leq l \sum_{i=1}^n \max \{0, (\text{ord}_y f^* W_i - \rho_i)\}. \end{aligned}$$

Hence we get

$$(7.6) \quad N_f(r, W) - N_f^{(l\tilde{\rho})}(r, W) \leq l \sum_{i=1}^n (N_f(r, W_i) - N_f^{(\rho_i)}(r, W_i)).$$

By (7.4), (7.5), (7.6), we get

$$m_f(r, W) + N_f(r, W) - N_f^{(l\tilde{\rho})}(r, W) \leq O(T_{f_S}(r) + N_{\text{ram } \pi_Y}(r) + \bar{N}_f^\partial(r)) + O(\log r) + o(T_f(r)).$$

Thus we have removed the assumption that W is irreducible and reduced. Here we set $\rho = l\tilde{\rho}$ and $P = P_1 \cup \dots \cup P_n$.

The assumption that S is smooth is removed similarly as in the proof of Proposition 6.1. This completes the induction step of the proof of the claim. Thus the claim is proved. The proof of Proposition 7.1 is completed. \square

8. Intersection with higher codimensional subvarieties

Lemma 8.1. — Let L be an ample line bundle on \overline{A} , where \overline{A} is a smooth equivariant compactification. Let $Z \subset A \times S$ be a closed subscheme whose codimension is greater than one, where S is a projective variety. Let $\varepsilon > 0$. Then there exist a finite subset $P \subset S(A) \setminus \{A\}$ and a proper Zariski closed subset $E \subsetneq S$ with the following property: Let $f : Y \rightarrow A \times S$ satisfies $f(Y) \not\subset \text{supp } Z$. Then either one of the followings is true:

1. $N_f^{(1)}(r, Z) \leq \varepsilon T_{f_A}(r, L) + O_\varepsilon(N_{\text{ram}} \pi(r) + \bar{N}_f^\partial(r) + T_{f_S}(r)) + O(\log r) + o(T_f(r))||_\varepsilon$.
2. $f_S(Y) \subset E$.
3. There exists $B \in P$ such that

$$T_{q_B \circ f_A}(r) = O(N_{\text{ram}} \pi(r) + \bar{N}_f^\partial(r) + T_{f_S}(r)) + O(\log r) + o(T_f(r))||,$$

where $q_B : A \rightarrow A/B$ is the quotient map.

Proof. — We first prove this lemma under the additional three assumptions:

- $q(Z) \subset S$ is dense, where $q : A \times S \rightarrow S$ is the second projection,
- Z is irreducible,
- S is smooth.

We use higher jet spaces. Let $J_l(A \times S)$ be the l -th jet space. We have the natural splitting $J_l(A \times S) = A \times (J_l(A \times S)/A)$ induced from the splitting $TA = A \times \text{Lie}A$. There exists a partial compactification $J_l(A \times S) \subset \bar{J}_l(A \times S)$ such that the natural map $\bar{J}_l(A \times S) \rightarrow A \times S$ is projective and $\bar{J}_l(A \times S)$ is \mathbb{Q} -factorial ([Yam04, 2.4]). We have the natural splitting $\bar{J}_l(A \times S) = A \times (\bar{J}_l(A \times S)/A)$. When $l = 1$, this reduces to (5.1).

Claim 8.2. — There exist a sequence of positive integers $n(1), n(2), n(3), \dots$ and an ample line bundle L_o on \overline{A} with the following conditions:

- (1) $\frac{n(l)}{l} \rightarrow 0$ when $l \rightarrow \infty$.
- (2) For $l \geq 1$, let $S_l \subset J_l(A \times S)/A$ be an irreducible Zariski closed subset whose image under the projection $J_l(A \times S)/A \rightarrow S$ is Zariski dense. Let $\overline{S}_l \subset \bar{J}_l(A \times S)/A$ be the compactification of S_l in $\bar{J}_l(A \times S)/A$. Then there exists an effective Cartier divisor $F_l \subset \overline{A} \times \overline{S}_l$ with the following properties:

- (a) F_l corresponds to a global section of

$$H^0(\overline{A} \times \overline{S}_l, p_l^* L_o^{\otimes n(l)} \otimes q_l^* M_l),$$

where $p_l : \overline{A} \times \overline{S}_l \rightarrow \overline{A}$ is the first projection, $q_l : \overline{A} \times \overline{S}_l \rightarrow \overline{S}_l$ is the second projection, and M_l is an ample line bundle on \overline{S}_l .

- (b) Let $f : \mathbb{D} \rightarrow A \times S$ be a holomorphic map such that $j_l(f)(\mathbb{D}) \subset A \times S_l$ and $j_l(f)(\mathbb{D}) \not\subset F_l$. Assume $f(0) \in Z$, then $\text{ord}_0 j_l(f)^* F_l \geq l + 1$.

Proof. — When A is compact, this is [Yam15a, Prop 6]. The same proof as in [Yam15a, Prop 6] can be adapted to our setting, with the only difference being that it requires the compactification of several objects. For the sake of completeness we give the proof of the claim in Remark 8.4. See also [NW14, Lemma 6.5.42]. \square

Now we are given $\varepsilon > 0$. Let $\mu > 0$ be a positive integer such that $L^{\otimes \mu} \otimes L_o^{-1}$ is ample. We take $l \in \mathbb{N}$ such that

$$(8.1) \quad n(l)/(l + 1) < \varepsilon/\mu.$$

Let \mathcal{V} be the set of all irreducible Zariski closed subsets of $J_l(A \times S)/A$. Let $\mathcal{W} \subset \mathcal{V}$ be the subset of \mathcal{V} which consists of $W \in \mathcal{V}$ whose image under the projection $J_l(A \times S)/A \rightarrow S$ is Zariski dense in S . We define the sequence $\mathcal{V}_1, \mathcal{V}_2, \dots$ of subsets of \mathcal{V} and the sequence $\mathcal{W}_1, \mathcal{W}_2, \dots$ of subsets of \mathcal{W} by the following inductive rule. Put $\mathcal{V}_1 = \mathcal{W}_1 = \{J_l(A \times S)/A\}$. For each $W \in \mathcal{W}_i$, let $F_W \subset A \times W$ be the divisor defined as the restriction of the divisor in Claim 8.2 to $A \times W$. Set $(F_W)^A = A \times W'$, where $W' \subset W$ is a proper Zariski closed subset. Put

$$\mathcal{V}_{i+1} = \bigcup_{W \in \mathcal{W}_i} \{V \in \mathcal{V}; V \text{ is an irreducible component of } W'\},$$

$$\mathcal{W}_{i+1} = \mathcal{V}_{i+1} \cap \mathcal{W}.$$

Since the number of the irreducible components of W' is finite, each \mathcal{V}_i is a finite set. Since $\dim W' < \dim W$, we have $\mathcal{V}_i = \mathcal{W}_i = \emptyset$ for $i \geq \dim(J_l(A \times S)/A) + 2$. We apply Proposition 6.1 for $F_W \subset A \times W$ to get $P_W \subset \mathcal{S}_0(A)$. (More precisely, we apply Proposition 6.1 for each irreducible component of F_W .) Set

$$P = \bigcup_i \bigcup_{W_i \in \mathcal{W}_i} P_{W_i} \setminus \{\{0\}, A\}.$$

Then $P \subset \mathcal{S}(A) - \{A\}$ is a finite subset. For $V \in \mathcal{V}_i \setminus \mathcal{W}_i$, let $S_V \subset S$ be the Zariski closure of the image of V under the projection $J_l(A \times S)/A \rightarrow S$. Then by the construction of \mathcal{W}_i , we have $S_V \neq S$. Set

$$E = \bigcup_i \bigcup_{V \in \mathcal{V}_i \setminus \mathcal{W}_i} S_V.$$

Then $E \subset S$ is a proper Zariski closed subset.

Now let $f : Y \rightarrow A \times S$ with $f(Y) \not\subseteq Z \cup q^{-1}(E)$. Let $q' : J_l(A \times S) \rightarrow J_l(A \times S)/A$ be the projection under the splitting $J_l(A \times S) = A \times (J_l(A \times S)/A)$. We may take $W_i \in \mathcal{W}_i$ such that $q' \circ j_l(f)(Y) \subset W_i$ but $q' \circ j_l(f)(Y) \not\subset W_{i+1}$ for all $W_{i+1} \in \mathcal{W}_{i+1}$. Then we have

$$(8.2) \quad j_l(f)(Y) \not\subseteq (F_{W_i})^A.$$

We consider the three possible cases.

Case 1. $j_l(f)(Y) \not\subseteq F_{W_i}$. We remark that $j_l(f)$ hits the boundary of $\bar{J}_l(A \times S)$ only at the ramification points of $\pi_Y : Y \rightarrow \mathbb{C}$ and the points in $\bar{f}^{-1}(\partial A \times S)$, where $\bar{f} : Y \rightarrow \overline{A} \times S$ is the extension of f . Hence, we have

$$(l+1)N_f^{(1)}(r, Z) \leq N_{j_l(f)}(r, F_{W_i}) + (l+1)N_{\text{ram } \pi_Y}(r) + (l+1)\bar{N}_f^\partial(r).$$

We recall that F_{W_i} corresponds to $H^0(\overline{A} \times W_i, p_l^* L_o^{\otimes n(l)} \otimes q_l^* M_l)$, where M_l is an ample line bundle on W_i and $q_l : A \times W_i \rightarrow W_i$ is the second projection. Hence by the Nevanlinna inequality (cf. (3.1)), we have

$$N_{j_l(f)}(r, F_{W_i}) \leq n(l)T_f(r, p_l^* L_o) + T_{j_l(f)}(r, q_l^* M_l) + O(\log r).$$

By the similar argument for the proof of Lemma 5.4, we get

$$(8.3) \quad T_{j_l(f)}(r, q_l^* M_l) = O(T_{f_S}(r) + N_{\text{ram } \pi_Y}(r) + \bar{N}_f^\partial(r)) + O(\log r) + o(T_f(r)) ||.$$

Hence we obtain

$$N_f^{(1)}(r, Z) \leq \varepsilon T_{f_A}(r, L) + O(T_{f_S}(r) + N_{\text{ram } \pi_Y}(r) + \bar{N}_f^\partial(r)) + O(\log r) + o(T_f(r)) ||.$$

Case 2. $j_l(f)(Y) \subset F_{W_i}$. We apply Proposition 6.1 to get $B \in P_{W_i}$. By $j_l f(Y) \not\subseteq (F_{W_i})^A$, we have $B \neq A$.

Case 2-1. $B = \{0\}$. Then by Proposition 6.1, we have

$$T_{f_A}(r) = O(N_{\text{ram } \pi}(r) + \bar{N}_f^\partial(r) + T_{j_l f_{W_i}}(r)) + O(\log r) + o(T_f(r)) ||.$$

Thus by (8.3), we have

$$T_f(r) = O(T_{f_S}(r) + N_{\text{ram } \pi_Y}(r) + \bar{N}_f^\partial(r)) + O(\log r) + o(T_f(r)) ||.$$

Hence

$$N_f(r, Z) = O(T_{f_S}(r) + N_{\text{ram } \pi_Y}(r) + \bar{N}_f^\partial(r)) + O(\log r) + o(T_f(r)) ||.$$

Case 2-2. $B \neq \{0\}$. Then $B \in P$. By Proposition 6.1, we have

$$T_{q_B \circ f_A}(r) = O(N_{\text{ram } \pi}(r) + \bar{N}_f^\partial(r) + T_{j_l f_{W_i}}(r)) + O(\log r) + o(T_f(r)) ||.$$

Thus by (8.3), we have

$$T_{q_B \circ f_A}(r) = O(N_{\text{ram } \pi}(r) + \bar{N}_f^\partial(r) + T_{f_S}(r)) + O(\log r) + o(T_f(r)) ||.$$

We combine the three cases above to conclude the proof of the lemma (Lemma 8.1) under the three assumptions above.

We remove the three assumptions. First suppose $q(Z) \subsetneq S$. We set $E = \overline{q(Z)}$. Then $E \subsetneq S$ is a Zariski closed set. If $f_S(Y) \not\subset E$, then we have $m_f(r, Z) + N_f(r, Z) = O(T_{f_S}(r)) + O(\log r)$.

This is stronger than the first assertion in Lemma 8.1. Thus we have removed the assumption that $q(Z) \subset S$ is dense.

Next we remove the assumption that Z is irreducible. Let $Z = Z_1 \cup \dots \cup Z_k$ be the irreducible decomposition. We apply Lemma 8.1 for Z_i and ε/k to get $P_i \subset \mathcal{S}(A) \setminus \{A\}$ and $E_i \subsetneq S$. We set $P = \cup_i P_i$ and $E = \cup_i E_i$. Then $P \subset \mathcal{S}(A) \setminus \{A\}$ is a finite subset and $E \subsetneq S$ is a proper Zariski closed subset. If $f : Y \rightarrow A \times S$ does not satisfy the second and the third assertions in Lemma 8.1, then we have

$$N_f^{(1)}(r, Z_i) \leq (\varepsilon/k)T_{f_A}(r, L) + O_\varepsilon(N_{\text{ram}} \pi(r) + \bar{N}_f^\partial(r) + T_{f_S}(r)) + O(\log r) + o(T_f(r))||_\varepsilon.$$

By $N_f^{(1)}(r, Z) \leq \sum_{i=1}^k N_f^{(1)}(r, Z_i)$, we have removed the assumption that Z is irreducible.

Finally we remove the assumption that S is smooth. Let $E_o \subset S$ be the singular locus of S and let $\tau : S' \rightarrow S$ be a smooth modification, which is isomorphic outside E_o . Let $Z' \subset A \times S'$ be the Zariski closure of $Z \cap (A \times (S \setminus E_o))$. Then the codimension of $Z' \subset A \times S'$ is greater than one. We apply Lemma 8.1 for Z' to get $P \subset \mathcal{S}(A) \setminus \{A\}$ and $E' \subsetneq S'$. We set $E = E_o \cup \tau(E')$. If $f : Y \rightarrow A \times S$ does not satisfy the second and the third assertions in Lemma 8.1, then we have a lift $f' : Y \rightarrow A \times S'$ and

$$N_{f'}^{(1)}(r, Z') \leq \varepsilon T_{f_A}(r, L) + O_\varepsilon(N_{\text{ram}} \pi(r) + \bar{N}_f^\partial(r) + T_{f_S}(r)) + O(\log r) + o(T_f(r))||_\varepsilon.$$

By

$$N_f^{(1)}(r, Z) \leq N_{f'}^{(1)}(r, Z') + N_{f_S}^{(1)}(r, E_o) \leq N_{f'}^{(1)}(r, Z') + O(T_{f_S}(r)) + O(\log r),$$

we have removed the assumption that S is smooth. The proof of Lemma 8.1 is completed. \square

Let $Z \subset A \times S$ be a closed subscheme. For $f : Y \rightarrow A \times S$, we recall $e(f, Z)$ from § 5.

Proposition 8.3. — *Let $Z \subset A \times S$ be a closed subscheme, where S is a projective variety. Let L be an ample line bundle on \overline{A} , where \overline{A} is a smooth equivariant compactification. Let $\varepsilon > 0$. Then there exists a finite subset $P \subset \mathcal{S}(A) \setminus \{A\}$ with the following property: Let $f : Y \rightarrow A \times S$ satisfies $f(Y) \not\subseteq Z$. Then either one of the followings is true:*

1. $N_f^{(1)}(r, Z) \leq \varepsilon T_{f_A}(r, L) + O_\varepsilon(N_{\text{ram}} \pi(r) + \bar{N}_f^\partial(r) + T_{f_S}(r)) + O(\log r) + o(T_f(r))||_\varepsilon$.
2. $e(f, Z) \geq \dim A - 1$.
3. There exists $B \in P$ such that

$$T_{q_B \circ f_A}(r) = O(N_{\text{ram}} \pi(r) + \bar{N}_f^\partial(r) + T_{f_S}(r)) + O(\log r) + o(T_f(r))||,$$

where $q_B : A \rightarrow A/B$ is the quotient map.

Proof. — We prove by Noether induction on S . So we assume that the the assertion is true for all irreducible $V \subsetneq S$ and $Z|_{A \times V} \subset A \times V$. We are given $\varepsilon > 0$.

We set

$$\Psi(A, Z) = \{x \in A \times S; \dim((x + A) \cap Z) \geq \dim A - 1\}.$$

Note that $\Psi(A, Z)/A \subset S$ is a constructible set.

First we consider the case that $\Psi(A, Z)/A \subset S$ is Zariski dense. Then there exists a non-empty Zariski open set $U \subset S$ such that $U \subset \Psi(A, Z)/A$. Set $V = S \setminus U$. Then $V \subsetneq S$ is a proper Zariski closed subset. If $f_S(Y) \not\subseteq V$, then we have $e(f, Z) \geq \dim A - 1$. Hence the second assertion of Proposition 8.3 is valid. If $f_S(Y) \subset V$, then the induction hypothesis yields a finite subset $P' \subset \mathcal{S}(A) \setminus \{A\}$ so that one of the three assertions in Proposition 8.3 is valid. Hence our proposition is proved by setting $P = P'$.

Next we consider the case $V_o = \overline{\Psi(A, Z)/A} \subsetneq S$. Let $Z' \subset A \times S$ be the Zariski closure of $Z \cap (A \times (S \setminus V_o))$. Then the codimension of $Z' \subset A \times S$ is greater than one. We apply Lemma 8.1 for $Z' \subset A \times S$ to get $E \subsetneq S$ and $P' \subset \mathcal{S}(A) \setminus \{A\}$. Then $E \subsetneq S$. Let V_1, \dots, V_l be the irreducible components of $E \cup V_o$. By the induction hypothesis applied for each $V_i \subsetneq S$, we get $P_i \subset \mathcal{S}(A) \setminus \{A\}$. We set $P = P' \cup \cup_i P_i$. Now let $f : Y \rightarrow A \times S$ satisfies $f(Y) \not\subseteq Z$. Assume that the second and the third assertions of Proposition 8.3 are not valid. If $f_S(Y) \subset V_i$ for some V_i , then by the induction hypothesis, we get the first assertion of Proposition 8.3. Hence we assume that $f_S(Y) \not\subseteq V_i$ for all V_i . Then we have $f_S(Y) \not\subseteq E$. By Lemma 8.1, we get

$$N_f^{(1)}(r, Z') \leq \varepsilon T_{f_A}(r, L) + O_\varepsilon(N_{\text{ram}} \pi(r) + \bar{N}_f^\partial(r) + T_{f_S}(r)) + O(\log r) + o(T_f(r))||_\varepsilon.$$

By

$$N_f^{(1)}(r, Z) \leq N_f^{(1)}(r, Z') + N_{fs}^{(1)}(r, V_o) \leq N_f^{(1)}(r, Z') + O(T_{fs}(r)) + O(\log r),$$

we get the first assertion of Proposition 8.3 for the case that $f_S(Y) \notin V_i$ for all V_i . Thus the proof is completed. \square

Remark 8.4. — For the sake of completeness, we provide a proof of Claim 8.2. The proof is identical to that of [Yam15a, Prop 6], except for the compactifications of several objects. Before we start the proof, we note that the statement of [Yam15a, Cor 3] also holds for semi-abelian varieties A , not just abelian varieties as stated in the original paper. This is because the proof of [Yam15a, Cor 3] relies only on the natural splitting $J_l(A \times S) = A \times (J_l(A \times S)/A)$ induced from the splitting $TA = A \times \text{Lie}A$, a property that also holds for semi-abelian varieties.

Proof of Claim 8.2. — First we construct the sequence of positive integers $n(1), n(2), n(3), \dots$ and the ample line bundle L on A . Let $\bar{Z} \subset \bar{A} \times S$ be the schematic closure. Let $r : \bar{Z} \rightarrow S$ be the composition of the morphisms

$$(8.4) \quad \bar{Z} \hookrightarrow \bar{A} \times S \xrightarrow{q} S,$$

where q is the second projection and $\bar{Z} \hookrightarrow \bar{A} \times S$ is the closed immersion. By the generic flatness theorem, there exists a non-empty Zariski open subset $S^o \subset S$ such that the restriction of the family $r : \bar{Z} \rightarrow S$ over S^o is a flat family.

Consider the pull back of the sequence of morphisms (8.4) by the second projection $q : A \times S \rightarrow S$:

$$A \times \bar{Z} \xrightarrow{s_1} A \times \bar{A} \times S \xrightarrow{s_2} A \times S,$$

where s_2 maps as

$$A \times \bar{A} \times S \ni (a, a', s) \mapsto (a, s) \in A \times S.$$

Put $s = s_2 \circ s_1 : A \times \bar{Z} \rightarrow A \times S$.

Let L be an ample line bundle on \bar{A} . (Later, we replace L by a suitable translate.) We define $\phi : A \times \bar{A} \times S \rightarrow \bar{A}$ by

$$A \times \bar{A} \times S \ni (a, a', s) \mapsto a + a' \in \bar{A}.$$

Let L^\dagger be the line bundle on $A \times \bar{Z}$ which is the pull back of L by the composition of morphisms

$$A \times \bar{Z} \xrightarrow{s_1} A \times \bar{A} \times S \xrightarrow{\phi} \bar{A}.$$

Since the restriction of $s : A \times \bar{Z} \rightarrow A \times S$ to $A \times S^o$ is a flat family, the semicontinuity theorem [Har77, p.288] implies that there exists a Zariski open subset $U_n \subset A \times S^o$ ($n > 0$) such that $H^0((A \times \bar{Z})_P, L_P^{\dagger \otimes n})$ are all the same dimensional \mathbb{C} vector spaces for $P \in U_n$; put this number G_n . Here $(A \times \bar{Z})_P$ denotes the fiber of the morphism $s : A \times \bar{Z} \rightarrow A \times S$ over $P \in A \times S$, and $L_P^{\dagger \otimes n}$ is the induced line bundle. Since the intersection $\cap_n U_n$ is non-empty, we may take a point $(a, w) \in \cap_n U_n$. Replacing L by the pull back by the morphism

$$\bar{A} \ni x \mapsto x + a \in \bar{A}$$

we may assume that

$$(8.5) \quad \{0\} \times S \subset A \times S \text{ and } \cap_n U_n \text{ have non-trivial intersection.}$$

Since $s : A \times \bar{Z} \rightarrow A \times S$ is flat over $A \times S^o$ and $r : \bar{Z} \rightarrow S$ is dominant, we have

$$\dim(A \times \bar{Z})_P = \dim(A \times Z) - \dim(A \times S)$$

for all $P \in \cap_n U_n \subset A \times S^o$. Since $\text{codim}(Z, A \times S) \geq 2$, we have

$$\dim(A \times \bar{Z})_P \leq \dim A - 2.$$

Hence there exist a positive integer n_0 and positive constants C_1, C_2 such that

$$G_n < C_1 n^{\dim A - 2}, \quad \dim_{\mathbb{C}} H^0(\bar{A}, L^{\otimes n}) > C_2 n^{\dim A}$$

for $n > n_0$. Hence for a positive integer l , we may take a positive integer $n(l)$ such that

$$(8.6) \quad \dim_{\mathbb{C}} H^0(\bar{A}, L^{\otimes n(l)}) > (l + 1)G_{n(l)}$$

and

$$\lim_{l \rightarrow \infty} \frac{n(l)}{l} \rightarrow 0.$$

For instance, $n(l) \sim l^{2/3}$.

Now let $S_l \subset J_l(A \times S)/A$ be an irreducible Zariski closed subset whose image under the projection $J_l(A \times S)/A \rightarrow S$ is Zariski dense. Let $\mathcal{T}_l \subset J_l(A \times S) \times S$ by the closed subscheme considered in [Yam15a, Corollary 3]. Let $\mathcal{S}_l \subset A \times S_l \times S$ be the closed subscheme obtained by the pull-back of \mathcal{T}_l by the closed immersion

$$A \times S_l \times S \hookrightarrow J_l(A \times S) \times S.$$

Note that the under lying topological spaces of \mathcal{S}_l and S_l are the same. We consider the following commutative diagram (8.7) which is obtained by the base change of (8.4) with a sequence of morphisms

$$\mathcal{S}_l \hookrightarrow A \times S_l \times S \rightarrow A \times S \rightarrow S.$$

Here $A \times S_l \times S \rightarrow A \times S$ is the product of the first and third projections.

$$(8.7) \quad \begin{array}{ccccccc} \mathcal{Z}_l & \xrightarrow{v'} & A \times S_l \times \overline{Z} & \longrightarrow & A \times \overline{Z} & \longrightarrow & \overline{Z} \\ \downarrow u_1 & & \downarrow t_1 & & \downarrow s_1 & & \downarrow \\ \cdot & \longrightarrow & A \times S_l \times \overline{A} \times S & \longrightarrow & A \times \overline{A} \times S & \longrightarrow & \overline{A} \times S \\ \downarrow u_2 & & \downarrow t_2 & & \downarrow s_2 & & \downarrow q \\ \mathcal{S}_l & \xrightarrow{v} & A \times S_l \times S & \longrightarrow & A \times S & \longrightarrow & S \end{array}$$

Let \mathcal{L}^\dagger be a line bundle on \mathcal{Z}_l obtained by the pull back of L^\dagger by the morphisms in the above diagram (8.7). Let $\mathcal{S}_{l,n} \subset \mathcal{S}_l$ be a Zariski open set of \mathcal{S}_l obtained by the inverse image of $U_n \subset A \times S$. Since the image of S_l under the projection $J_l(A \times S)/A \rightarrow S$ is Zariski dense and we are assuming (8.5), $\mathcal{S}_{l,n}$ is non-empty.

Since $\dim_{\mathbb{C}} H^0((A \times \overline{Z})_P, L_P^{\dagger \otimes n}) = G_n$ for all $P \in U_n$, the direct image sheaf $s_* L^{\dagger \otimes n}$ is a locally free sheaf of rank G_n on U_n and the natural map

$$s_* L^{\dagger \otimes n} \otimes \mathbb{C}(P) \rightarrow H^0((A \times \overline{Z})_P, L_P^{\dagger \otimes n})$$

is an isomorphism for $P \in U_n$. This follows by the Theorem of Grauert [Har77, p.288] since U_n is reduced and irreducible. Let $u : \mathcal{Z}_l \rightarrow \mathcal{S}_l$ be the morphism obtained by the composition $u = u_2 \circ u_1$, where u_1 and u_2 are the morphisms in (8.7). Then the natural map

$$u_* \mathcal{L}^{\dagger \otimes n} \otimes \mathbb{C}(P) \rightarrow H^0((\mathcal{Z}_l)_P, \mathcal{L}_P^{\dagger \otimes n})$$

is also surjective, so isomorphism on $P \in \mathcal{S}_{l,n}$. This follows by the Theorem of Cohomology and Base Change [Har77, p.290]. Hence $u_* \mathcal{L}^{\dagger \otimes n}$ is locally generated by G_n elements as $\mathcal{O}_{\mathcal{S}_l}$ module on $\mathcal{S}_{l,n} \subset \mathcal{S}_l$. Let $\pi : \mathcal{S}_l \rightarrow S_l$ be the composition of the morphisms

$$\mathcal{S}_l \xrightarrow{v} A \times S_l \times S \xrightarrow{\text{2nd proj}} S_l,$$

and let $S_{l,n} = \pi(\mathcal{S}_{l,n})$ be a Zariski open set (note that the under lying topological spaces of \mathcal{S}_l and S_l are the same). By [Yam15a, Corollary 3], π is a finite morphism and the direct image sheaf $\pi_* \mathcal{O}_{\mathcal{S}_l}$ is locally generated by $(l+1)$ elements as an \mathcal{O}_{S_l} module on $S_l \cap J_l^{\text{reg}}(A \times S)/A$. Hence $(\pi \circ u)_* \mathcal{L}^{\dagger \otimes n}$ is locally generated by $(l+1)G_n$ elements as an \mathcal{O}_{S_l} module on $S_{l,n} \cap J_l^{\text{reg}}(A \times S)/A$.

Now look at the following commutative diagram

$$\begin{array}{ccccccc} \mathcal{Z}_l & \xrightarrow{t_1 \circ v'} & A \times S_l \times \overline{A} \times S & \xrightarrow{\psi} & \overline{A} \times S_l & \xrightarrow{p_l|_{S_l}} & \overline{A} \\ \downarrow \pi \circ u & & \downarrow \text{2nd proj} & & \downarrow q_l|_{S_l} & & \\ S_l & \xlongequal{\quad} & S_l & \xlongequal{\quad} & S_l & \xlongequal{\quad} & S_l \end{array}$$

where ψ is the morphism

$$\psi : A \times S_l \times \overline{A} \times S \ni (a, s, a', s') \mapsto (a + a', s) \in \overline{A} \times S_l,$$

and $p_l|_{S_l}$ (resp. $q_l|_{S_l}$) is the restriction of p_l (resp. q_l) over $\overline{A} \times S_l$. Since $(p_l|_{S_l} \circ \psi \circ t_1 \circ v')^* L = \mathcal{L}^\dagger$, we have a natural morphism

$$(8.8) \quad (q_l|_{S_l})_* (p_l|_{S_l})^* L^{\otimes n} = H^0(\overline{A}, L^{\otimes n}) \otimes_{\mathbb{C}} \mathcal{O}_{S_l} \rightarrow (\pi \circ u)_* \mathcal{L}^{\dagger \otimes n}.$$

Let \mathcal{F} be the kernel of (8.8) for $n = n(l)$. Then by (8.6), we have $\mathcal{F} \neq 0$. Let

$$\mathcal{F}' \subset (q_l)_* p_l^* L^{\otimes n} = H^0(\overline{A}, L^{\otimes n}) \otimes_{\mathbb{C}} \mathcal{O}_{\overline{S}_l}$$

be a coherent sheaf on \overline{S}_l which is an extension of \mathcal{F} (cf. [2, II Ex. 5.15]). By tensoring a sufficiently ample line bundle M_l on \overline{S}_l with \mathcal{F}' , we may assume that $H^0(\overline{S}_l, \mathcal{F}' \otimes M_l) \neq 0$. Since we have

$$\begin{aligned} H^0(\overline{S}_l, \mathcal{F}' \otimes M_l) &\subset H^0(\overline{S}_l, ((q_l)_* p_l^* L^{\otimes n}) \otimes M_l) \\ &= H^0(\overline{S}_l, (q_l)_* (p_l^* L^{\otimes n} \otimes q_l^* M_l)) \\ &= H^0(\overline{A} \times \overline{S}_l, p_l^* L^{\otimes n} \otimes q_l^* M_l), \end{aligned}$$

we may take a divisor $F_l \subset \overline{A} \times \overline{S}_l$ which is defined by a non-zero global section of $H^0(\overline{S}_l, \mathcal{F}' \otimes M_l)$. Let $F_l|_{S_l}$ be the restriction of F_l on $\overline{A} \times S_l$. Then we have

$$\mathcal{Z}_l \subset \psi^*(F_l|_{S_l})$$

as closed subschemes of $A \times S_l \times \overline{A} \times S$.

Now let $f : \mathbb{D} \rightarrow A \times S$ be a holomorphic map such that $j_l(f)(\mathbb{D}) \subset A \times S_l$, $j_l(f)(\mathbb{D}) \not\subset F_l$, and $f(0) = (a, s) \in Z$. We define a holomorphic map $\tilde{f} : \mathbb{D} \rightarrow A \times S_l \times \overline{A} \times S$ by

$$\tilde{f}(z) = (p \circ f(z) - a, q_l \circ j_l(f)(z), a, s),$$

where $p : A \times S \rightarrow A$ is the first projection. Then we have

$$\tilde{f}(\mathbb{D}) \subset A \times S_l \times \overline{A}, \quad \tilde{f}(0) \in \text{supp } \mathcal{Z}_l, \quad \psi \circ \tilde{f} = j_l(f).$$

Since v' is the base change of v in (8.7), we have

$$\text{ord}_0 \tilde{f}^* \mathcal{Z}_l = \text{ord}_0 (t_2 \circ \tilde{f})^* \mathcal{S}_l = \text{ord}_0 (j_l(f) - a)^* (\mathcal{T}_l)_s.$$

Hence by [Yam15a, Lemma 6 (2)], we have

$$\text{ord}_0 \tilde{f}^* \mathcal{Z}_l \geq l + 1.$$

Hence we have

$$\text{ord}_0 j_l(f)^* F_l = \text{ord}_0 \tilde{f}^* (\psi^*(F_l|_{S_l})) \geq \text{ord}_0 \tilde{f}^* \mathcal{Z}_l \geq l + 1.$$

This complete the proof of Claim 8.2. \square

9. Second main theorem type estimate with truncation level one

In this subsection, we assume that S is smooth and projective. In particular, all divisors on $A \times S$ are Cartier divisors.

To prove the main result of this section, Proposition 9.2, we start from the following lemma.

Lemma 9.1. — *Let $D \subset A \times S$ be a reduced divisor. Let L be an ample line bundle on \overline{A} , where \overline{A} is a smooth equivariant compactification. Let $\varepsilon > 0$. Then there exist a finite subset $P \subset \mathcal{S}(A) \setminus \{A\}$ and a proper Zariski closed subset $E \subsetneq S$ with the following property: Let $f : Y \rightarrow A \times S$ satisfies $f(Y) \not\subset D$. Then either one of the followings is true:*

1. $N_f^{(2)}(r, D) - N_f^{(1)}(r, D) \leq \varepsilon T_{f_A}(r, L) + O_\varepsilon(N_{\text{ram}} \pi(r) + \bar{N}_f^\partial(r) + T_{f_S}(r)) + O(\log r) + o(T_f(r))||_\varepsilon$.
2. $f(Y) \subset A \times E$.
3. There exists $B \in P$ such that

$$T_{q_B \circ f_A}(r) = O(N_{\text{ram}} \pi(r) + \bar{N}_f^\partial(r) + T_{f_S}(r)) + O(\log r) + o(T_f(r))||,$$

where $q_B : A \rightarrow A/B$ is the quotient map.

Proof. — We prove in the following three cases.

Case 1. D is irreducible and $\text{St}^0(D) = \{0\}$. We recall a Zariski open set $S^o \subset S$ (cf. Lemma 5.1). By replacing S^o by a smaller non-empty Zariski open set, we may assume that $D_s \subset A$ is a reduced divisor for all $s \in S^o$. Set $E = S \setminus S^o$. Set $S' = \overline{(\text{Lie}A \times TS)}$. We apply Proposition 8.3 for $D^{(1)} \subset A \times S'$ and $\varepsilon > 0$ to get a finite subset $P \subset \mathcal{S}(A) \setminus \{A\}$.

Let $f : Y \rightarrow A \times S$ satisfies $f(Y) \not\subseteq D$. We assume that the assertions 2 and 3 of the statement do not occur. Then $e(f, D) = \dim A - 1$. If $e(j_1 f, D^{(1)}) = \dim A - 1$, then by Lemma 5.1, we have $T_f(r) = O(T_{f_S}(r)) + O(\log r)$. Hence

$$m_f(r, D) + N_f(r, D) = O(T_{f_S}(r)) + O(\log r).$$

This is stronger than the assertion 1. Thus in the following, we assume $e(j_1 f, D^{(1)}) < \dim A - 1$. Then by Proposition 8.3, we get

$$N_{j_1 f}^{(1)}(r, D^{(1)}) \leq \varepsilon T_{f_A}(r, L) + O_\varepsilon(N_{\text{ram} \pi}(r) + \bar{N}_f^\partial(r) + T_{(j_1 f)_{S'}}(r)) + O(\log r) + o(T_f(r))||_\varepsilon.$$

By Lemma 5.4 and

$$N_f^{(2)}(r, D) - N_f^{(1)}(r, D) \leq N_{j_1 f}^{(1)}(r, D^{(1)}) + N_{\text{ram} \pi}(r) + \bar{N}_f^\partial(r),$$

we conclude the proof in the case 1.

Case 2. D is irreducible, but $\text{St}^0(D)$ is general. We set $B = \text{St}^0(D)$. We apply the case 1 above for $A' = A/B$, $D' = D/B$ to get $P' \subset \mathcal{S}(A') \setminus \{A'\}$ and $E \subsetneq S$. We define $P \subset \mathcal{S}(A) \setminus \{A\}$ by $C \in P$ if and only if $B \subset C$ and $C/B \in P'$.

Case 3. We treat the general case. Let $D = D_1 + \dots + D_k$ be the irreducible decomposition. Set $Z_{ij} = D_i \cap D_j$. Then $Z_{ij} \subset A \times S$ has codimension greater than one. We apply Proposition 8.3 for Z_{ij} to get $P_{ij} \subset \mathcal{S}(A) \setminus \{A\}$ and $E_{ij} \subsetneq S$ such that $\overline{\Psi(A, Z_{ij})} = A \times E_{ij}$. We apply the case 2 above for each D_i to get $P_i \subset \mathcal{S}(A) \setminus \{A\}$ and $E_i \subsetneq S$. Then we set $P = \cup_i P_i \cup \cup_{i,j} P_{ij}$ and $E = \cup_i E_i \cup \cup_{i,j} E_{ij}$. \square

Proposition 9.2. — Let $D \subset A \times S$ be a reduced divisor. Let \overline{A} be a smooth equivariant compactification such that $p(\text{Sp}_A \overline{D}) \subsetneq S$ is a proper Zariski closed set, where $p : \overline{A} \times S \rightarrow S$ is the second projection and $\overline{D} \subset \overline{A} \times S$ is the Zariski closure. Let L be an ample line bundle on \overline{A} . Let $\varepsilon > 0$. Then there exist a finite subset $P \subset \mathcal{S}(A) \setminus \{A\}$, and a proper Zariski closed subset $E \subsetneq S$ such that for every $f : Y \rightarrow A \times S$ with $f(Y) \not\subseteq D$, either one of the followings holds:

1. There exists $B \in P$ such that

$$T_{q_B \circ f_A}(r) = O(N_{\text{ram} \pi}(r) + \bar{N}_f^\partial(r) + T_{f_S}(r)) + O(\log r) + o(T_f(r))||,$$

where $q_B : A \rightarrow A/B$ is the quotient map.

2. $f(Y) \subset A \times E$.

- 3.

$$\begin{aligned} m_f(r, \overline{D}) + N_f(r, \overline{D}) &\leq N_f^{(1)}(r, \overline{D}) + \varepsilon T_{f_A}(r, L) \\ &\quad + O_\varepsilon(N_{\text{ram} \pi}(r) + \bar{N}_f^\partial(r) + T_{f_S}(r)) + O(\log r)||. \end{aligned}$$

Proof. — We apply Proposition 7.1 for $\overline{D} \subset \overline{A} \times S$ to get a finite subset $P_1 \subset \mathcal{S}(A) \setminus \{A\}$ and $\rho \in \mathbb{Z}_{>0}$. We apply Lemma 9.1 for $D \subset A \times S$ and $\varepsilon' = \varepsilon/(\rho - 1)$ to get a finite subset $P_2 \subset \mathcal{S}(A) \setminus \{A\}$ and $E' \subsetneq S$. We set $P = P_1 \cup P_2$ and $E = p(\text{Sp}_A \overline{D}) \cup E'$.

Now let $f : Y \rightarrow A \times S$ satisfies $f(Y) \not\subseteq D$. We assume that the assertions 1 and 2 of our lemma do not hold. By Proposition 7.1, we get

$$m_f(r, D) + N_f(r, D) - N_f^{(\rho)}(r, D) = O(N_{\text{ram} \pi}(r) + \bar{N}_f^\partial(r) + T_{f_S}(r)) + O(\log r) + o(T_f(r))||.$$

We apply Lemma 9.1 to get

$$N_f^{(2)}(r, D) - N_f^{(1)}(r, D) \leq \varepsilon' T_{f_A}(r, L) + O(N_{\text{ram} \pi}(r) + \bar{N}_f^\partial(r) + T_{f_S}(r)) + O(\log r) + o(T_f(r))||.$$

By $N_f^{(\rho)}(r, D) - N_f^{(1)}(r, D) \leq (\rho - 1)(N_f^{(2)}(r, D) - N_f^{(1)}(r, D))$, we get

$$(9.1) \quad m_f(r, D) + N_f(r, D) - N_f^{(1)}(r, D) \leq (\rho - 1)\varepsilon' T_{f_A}(r, L) \\ + O(N_{\text{ram}}\pi(r) + \bar{N}_f^{\partial}(r) + T_{f_S}(r)) + O(\log r) + o(T_f(r)).$$

This completes the proof. \square

10. Proof of Theorem A

In this section, we complete the proof of Theorem A by proving a stronger result Theorem 10.7. Let $B \subset A$ be a semi-abelian subvariety and let \bar{A} be an equivariant compactification.

Definition 10.1. — We say that \bar{A} is compatible with B if there exists an equivariant compactification \bar{A}/B for which the quotient map $q_B : A \rightarrow A/B$ extends to a morphism $\bar{q}_B : \bar{A} \rightarrow \bar{A}/B$.

Let $W \subset A \times S$ be a B -invariant closed subvariety. Then we have $W/B \subset (A/B) \times S$. If \bar{A} is compatible with B , then \bar{q}_B yields $p : \bar{A} \times S \rightarrow (\bar{A}/B) \times S$.

Lemma 10.2. — Let $W \subset A \times S$ be a B -invariant closed subvariety. Let $D \subset W$ be a reduced divisor. Then there exists a smooth, projective, equivariant compactification \bar{A} which is compatible with B and $p(\text{Sp}_B \bar{D}) \subsetneq \bar{W}/B$, where $\bar{D} \subset \bar{A} \times S$ is the Zariski closure.

Proof. — We consider $W \rightarrow W/B$. We take the schematic generic point $\eta \in W/B$. Let $D_\eta \subset W_\eta$ be the schematic generic fiber. Let $\bar{\eta} : \text{Spec } \overline{\mathbb{C}(\eta)} \rightarrow W/B$ be the geometric point. Then $W_{\bar{\eta}} = B \times_{\mathbb{C}} \overline{\mathbb{C}(\eta)}$ is a semi-abelian variety over $\overline{\mathbb{C}(\eta)}$. By [Yam23, Lem 12.7], there exists a smooth equivariant compactification $\bar{W}_{\bar{\eta}}$ such that $\bar{D}_{\bar{\eta}} \subset \bar{W}_{\bar{\eta}}$ satisfies

$$(10.1) \quad \text{Sp}_{W_{\bar{\eta}}} \bar{D}_{\bar{\eta}} = \emptyset.$$

We remark that there exists a smooth equivariant compactification \bar{B} such that $\bar{W}_{\bar{\eta}}$ is obtained by the base change of \bar{B} . Namely

$$\bar{W}_{\bar{\eta}} = \bar{B} \times_{\mathbb{C}} \overline{\mathbb{C}(\eta)}.$$

To see this, we consider the canonical extension $0 \rightarrow T_B \rightarrow B \rightarrow B_0 \rightarrow 0$, where T_B is an algebraic torus and B_0 is an abelian variety. By [Yam23, Lem A.6], there exists a smooth equivariant compactification $(T_B)_{\bar{\eta}}$ such that $\bar{W}_{\bar{\eta}} = ((T_B)_{\bar{\eta}} \times W_{\bar{\eta}})/(T_B)_{\bar{\eta}}$, where $(T_B)_{\bar{\eta}} = T_B \times_{\mathbb{C}} \overline{\mathbb{C}(\eta)}$. By Sumihiro's theorem, $(T_B)_{\bar{\eta}}$ is a torus embedding associated to some complete fan (cf. [Yam23, Sec. A.2]). Hence there exists a smooth equivariant compactification \bar{T}_B such that $(T_B)_{\bar{\eta}} = \bar{T}_B \times_{\mathbb{C}} \overline{\mathbb{C}(\eta)}$. Let $\bar{B} = (\bar{T}_B \times B)/T_B$. Then $\bar{B} \times_{\mathbb{C}} \overline{\mathbb{C}(\eta)} = \bar{W}_{\bar{\eta}}$ as desired. By [Yam23, Lem A.8], we may assume moreover that \bar{B} is projective by replacing \bar{B} by a modification $\hat{B} \rightarrow \bar{B}$, for the property (10.1) remains true under this modification. Then \bar{T}_B is projective.

Next we shall construct an equivariant compactification \bar{A} which is compatible with B and the general fibers of $\bar{A} \rightarrow \bar{A}/B$ are \bar{B} constructed above. We have the canonical extensions $0 \rightarrow T_B \rightarrow B \rightarrow B_0 \rightarrow 0$, $0 \rightarrow T_A \rightarrow A \rightarrow A_0 \rightarrow 0$ and $0 \rightarrow T_{A/B} \rightarrow A/B \rightarrow A_0/B_0 \rightarrow 0$. Then we have $0 \rightarrow T_B \rightarrow T_A \rightarrow T_{A/B} \rightarrow 0$. We may take a section $T_{A/B} \rightarrow T_A$ so that $T_A = T_B \times T_{A/B}$. Set $A' = A/T_{A/B}$. Then $B \subset A'$ and $A'/B = A_0/B_0$. The B -torsor $A \rightarrow A/B$ is the pull-back of $A' \rightarrow A'/B$ by $A/B \rightarrow A'/B$.

Let \bar{B} be the compactification above so that $\bar{B} = (\bar{T}_B \times B)/T_B$. Set $\bar{A}' = (\bar{T}_B \times A')/T_B$. Then $\bar{A}' = (\bar{B} \times A')/B$. Then the general fibers of $\bar{A}' \rightarrow A'/B$ are \bar{B} . Let $\bar{T}_{A/B}$ be a smooth projective equivariant compactification. Set $\bar{A} = ((\bar{T}_B \times \bar{T}_{A/B}) \times A)/(T_B \times T_{A/B})$ and $\bar{A}/B = (\bar{T}_{A/B} \times (A/B))/T_{A/B}$. Then \bar{A} is compatible with B and the general fibers of $\bar{A} \rightarrow \bar{A}/B$ are \bar{B} . Moreover \bar{A} and \bar{A}/B are smooth and projective by the proof of [Yam23, Lem A.8].

Now let $\bar{D} \subset \bar{W}$ be the compactification of D . Then we claim

$$(10.2) \quad (\bar{D})_{\bar{\eta}} = \bar{D}_{\bar{\eta}}$$

in $(\bar{W})_{\bar{\eta}}$. To see this, we note that the induced map $\bar{W}_\eta \rightarrow \bar{W}$ is homeomorphism onto its image (cf. [Sta22, Tag 01K1]). Hence $(\bar{D})_\eta = \bar{D}_{\bar{\eta}}$. Note that $(\bar{W})_{\bar{\eta}} \rightarrow (\bar{W})_\eta$ is a closed morphism

(cf. [Sta22, Tag 01WM]). Hence $(\overline{D_\eta})_{\bar{\eta}} = \overline{D_{\bar{\eta}}}$. Combining these two observations, we have $(\overline{D})_{\bar{\eta}} = (\overline{D_\eta})_{\bar{\eta}} = \overline{D_{\bar{\eta}}}$. This shows (10.2). Since $\mathrm{Sp}_B \overline{D} \subset \overline{W}$ is B -invariant, $(\mathrm{Sp}_B \overline{D})_{\bar{\eta}} \subset (\overline{W})_{\bar{\eta}}$ is $B_{\bar{\eta}}$ -invariant. Moreover $(\mathrm{Sp}_B \overline{D})_{\bar{\eta}} \subset (\overline{D})_{\bar{\eta}}$. Hence $(\mathrm{Sp}_B \overline{D})_{\bar{\eta}} \subset \mathrm{Sp}_{B_{\bar{\eta}}}((\overline{D})_{\bar{\eta}})$. Combining this with (10.1) and (10.2), we have $(\mathrm{Sp}_B \overline{D})_{\bar{\eta}} \subset \mathrm{Sp}_{B_{\bar{\eta}}} \overline{D_{\bar{\eta}}} = \emptyset$. Hence $p(\mathrm{Sp}_B \overline{D}) \subset \overline{W/B}$ is a proper Zariski closed set. \square

Let X be a quasi-projective variety. Let $D \subset X$ be a reduced (Weil) divisor. Let $E \subset D$ be a subset. Let $f : Y \rightarrow X$ be a holomorphic map such that $f(Y) \not\subset D$. We set

$$\overline{N}_f(r, E) = \frac{1}{\deg \pi} \int_{2\delta}^r \mathrm{card}(Y(t) \cap f^{-1}E) \frac{dt}{t}.$$

When D is a Cartier divisor, then we have $\overline{N}_f(r, D) = N_f^{(1)}(r, D)$.

Let S be a projective variety and let $B \subset A$ be a semi-abelian subvariety. We denote by I_B the set of all holomorphic maps $f : Y \rightarrow A \times S$ such that the following three estimates hold:

- $T_{q_B \circ f_A}(r) = O(\log r) + o(T_f(r))||$, where $q_B : A \rightarrow A/B$ is the quotient,
- $N_{\mathrm{ram} \pi}(r) = O(\log r) + o(T_f(r))||$,
- $T_{f_S}(r) = O(\log r) + o(T_f(r))||$.

Remark 10.3. — If $f \in I_{\{0\}}$, then $f : Y \rightarrow A \times S$ does not have essential singularity over ∞ . Indeed, $f \in I_{\{0\}}$ yields $T_{f_A}(r) = O(\log r) + o(T_f(r))||$ and $T_{f_S}(r) = O(\log r) + o(T_f(r))||$. Hence $T_f(r) = O(\log r)||$, thus $N_{\mathrm{ram} \pi}(r) = O(\log r)||$. These two estimates implies that f does not have essential singularity over ∞ (cf. Lemma 3.2).

For a proper birational morphism $\varphi : V \rightarrow W$, we denote by $\mathrm{Ex}(\varphi) \subset V$ the exceptional locus of φ . If $f : Y \rightarrow W$ is a holomorphic map such that $f(Y) \not\subset \varphi(\mathrm{Ex}(\varphi))$, then there exists a unique lift $g : Y \rightarrow V$ of f .

The following lemma brings together the results from § 6 to § 9 and is essential for the applications that follow.

Lemma 10.4. — Let $B \subset A$ be a semi-abelian subvariety, and let $W \subset A \times S$ be a B -invariant closed subvariety, where S is projective. Let $D \subset W$ be a reduced Weil divisor. Then the following objects exist with the specified property described later:

- a smooth projective equivariant compactification $A \subset \overline{A}$,
- a proper birational morphism $\varphi : V \rightarrow \overline{W}$ with V smooth, such that $\varphi^{-1}(\overline{D} \cup \partial W) \subset V$ is a simple normal crossing divisor.

The satisfied property: Let $Z \subset W$ be a Zariski closed subset with $\mathrm{codim}(Z, W) \geq 2$ and $Z \subset D$, let L be an ample line bundle on V , and let $\varepsilon > 0$. Then there exist

- a finite subset $P \subset \mathcal{S}(B) \setminus \{B\}$, and
- a proper Zariski closed subset $\Phi \subsetneq W$ with $\varphi(\mathrm{Ex}(\varphi)) \subset \overline{\Phi}$,

such that for every holomorphic map $f : Y \rightarrow W$ satisfying $f(Y) \not\subset D \cup \Phi$ and $f \in I_B$, one of the following holds:

1. There exists $C \in P$ such that $f \in I_C$.
2. Let $f' : Y \rightarrow V$ be the lift of f . Then

$$T_{f'}(r, K_V(\varphi^{-1}(\overline{D} \cup \partial W))) \leq \overline{N}_f(r, D \setminus Z) + \varepsilon T_{f'}(r, L) + O(\log r) + o(T_{f'}(r))||.$$

Our main technical issue in the proof is that W cannot be decomposed into $B \times (W/B)$, this obstructs the application of the previous results to $f : Y \rightarrow W$. We handle this issue by considering a generically finite surjective map $\sigma : \Sigma \rightarrow \overline{W/B}$ which gives a rational map $\overline{B} \times \Sigma \dashrightarrow \overline{W}$. We can then apply our previous results to the lift $Y' \rightarrow \overline{B} \times \Sigma$ and translate them to obtain the estimates for $f : Y \rightarrow W$. A careful argument is required to ensure that these strategies work correctly. This involves constructing several additional objects not mentioned above. See also Remark 10.11 for additional discussion.

Proof. — We divide the proof into several steps.

Step 1. The construction of \overline{A} and $\varphi : V \rightarrow \overline{W}$. By Lemma 10.2, we may take an equivariant compactification \overline{A} such that

- \overline{A} is smooth and projective,

- \overline{A} is compatible with B , and
- $p(\mathrm{Sp}_B \overline{D}) \subsetneq \overline{W/B}$, where $p : \overline{W} \rightarrow \overline{W/B}$ is induced from $\overline{A} \rightarrow \overline{A/B}$.

Let $U \subset W/B$ be the smooth locus of W/B . Then $p^{-1}(U) \subset \overline{W}$ is smooth. We take a smooth modification $\varphi_o : V_o \rightarrow \overline{W}$ which is isomorphism over $p^{-1}(U)$. We apply the embedded resolution of Szabó [Sza94] for $\varphi_o^{-1}(\overline{D} + \partial W) \subset V_o$ to get a smooth modification $\tau : V \rightarrow V_o$ such that $\widetilde{D} = (\varphi_o \circ \tau)^{-1}(\overline{D} + \partial W)$ is simple normal crossing and τ is an isomorphism outside the locus where $\varphi_o^{-1}(\overline{D} + \partial W)$ is not simple normal crossing. Let $\varphi : V \rightarrow \overline{W}$ be the composite of $\tau : V \rightarrow V_o$ and $\varphi_o : V_o \rightarrow \overline{W}$. We have

$$(10.3) \quad K_V(\varphi^{-1}(\overline{D} \cup \partial W)) \leq \tau^* K_{V_o}(\varphi_o^{-1}(\overline{D} + \partial W)) + E$$

for some effective divisor $E \subset V$ such that $\tau(E) \subset V_o$ is contained in the locus where $\varphi_o^{-1}(\overline{D} + \partial W)$ is not simple normal crossing. Since $p^{-1}(U) \cap \partial W$ is simple normal crossing, we have $p^{-1}(U) \cap \tau(E) \subset \overline{D}$.

Step 2. The construction of Σ . Let $\Psi : B \times \overline{W} \rightarrow \overline{W}$ be the B -action. We may take a smooth subvariety $S' \subset W$ such that the induced map $S' \rightarrow U$ is étale. Then Ψ yields an étale map

$$\psi : B \times S' \rightarrow p^{-1}(U) \cap W.$$

Let $\overline{B} \subset \overline{A}$ be the compactification. Then \overline{B} is an equivariant compactification. The map ψ extends to a map

$$\bar{\psi} : \overline{B} \times S' \rightarrow p^{-1}(U).$$

Since \overline{A} is compatible with B , $\bar{\psi}$ is étale. We take a smooth compactification $S' \subset \Sigma$ such that the maps $S' \hookrightarrow W$ and $S' \rightarrow W/B$ extend to $\Sigma \rightarrow \overline{W}$ and $\sigma : \Sigma \rightarrow \overline{W/B}$. Then $\bar{\psi} : \overline{B} \times S' \rightarrow p^{-1}(U) \subset V_o$ induces a rational map $\widehat{\psi} : \overline{B} \times \Sigma \dashrightarrow V_o$.

$$\begin{array}{ccccc} & & \bar{\psi} & & \\ & \swarrow & & \searrow & \\ \overline{B} \times S' & \xhookrightarrow{\quad} & \overline{B} \times \Sigma & \dashrightarrow & \overline{W} \xleftarrow{\varphi_o} V_o \xleftarrow{\tau} V \\ \downarrow & & \downarrow q & & \downarrow p \\ S' & \xhookrightarrow{\quad} & \Sigma & \xrightarrow{\sigma} & \overline{W/B} \end{array}$$

We take a closed subscheme $I \subset \overline{B} \times \Sigma$ such that $q(\mathrm{supp} I) \subset \Sigma \setminus S'$ and the rational map $\widehat{\psi} : \overline{B} \times \Sigma \dashrightarrow V_o$ extends to a morphism

$$\tilde{\psi} : \mathrm{Bl}_I(\overline{B} \times \Sigma) \rightarrow V_o.$$

Step 3. The construction of P and Φ from the given Z, L , and $\varepsilon > 0$. Let $Z \subset W$ be a Zariski closed subset such that $\mathrm{codim}(Z, W) \geq 2$ and $Z \subset D$. We take a closed subscheme $\widehat{Z} \subset V_o$ such that

- $\mathrm{supp} \widehat{Z} = \overline{p^{-1}(U) \cap (\tau(E) \cup \varphi_o^{-1}(Z))}$,
- $(p \circ \varphi)^{-1}(U) \cap E \subset (p \circ \varphi)^{-1}(U) \cap \tau^* \widehat{Z}$, as closed subschemes of $(p \circ \varphi)^{-1}(U)$.

Then we have $\mathrm{supp}(\widehat{Z}) \subset \varphi_o^{-1}(\overline{D})$ and $\mathrm{codim}(\widehat{Z}, V_o) \geq 2$.

Let L be an ample line bundle on V . Let L_{V_o} be an ample line bundle on V_o . Then, there exists a positive constant $c_1 > 0$ such that for every $g : Y \rightarrow V$, we have the estimate:

$$(10.4) \quad T_{\tau \circ g}(r, L_{V_o}) \leq c_1 T_g(r, L) + O(\log r).$$

Let $L_{\overline{B}}$ be an ample line bundle on \overline{B} . Let $\kappa : \mathrm{Bl}_I(\overline{B} \times \Sigma) \rightarrow \overline{B}$ be the composite of $\mathrm{Bl}_I(\overline{B} \times \Sigma) \rightarrow \overline{B} \times \Sigma$ and the first projection $\overline{B} \times \Sigma \rightarrow \overline{B}$. Then, there exists a positive constant $c_2 > 0$ such that for every $g : Y \rightarrow V_o$ with $g(Y) \not\subset \tilde{\psi}(\mathrm{Bl}_I(\overline{B} \times \Sigma) \setminus (\overline{B} \times S'))$, we have the estimate (cf. [Yam15a, Lemma 1]):

$$(10.5) \quad T_{g'}(r, \kappa^* L_{\overline{B}}) \leq c_2 T_g(r, L_{V_o}) + O(\log r),$$

where $g' : Y \rightarrow \mathrm{Bl}_I(\overline{B} \times \Sigma)$ is a lift of g .

Let $D' \subset \overline{B} \times \Sigma$ be the reduced divisor defined by the Zariski closure of $\bar{\psi}^{-1}(\overline{D} \cap p^{-1}(U)) \subset \overline{B} \times S'$. Then we have

$$q(\mathrm{Sp}_B D') \not\subseteq \Sigma.$$

Let $Z' \subset \overline{B} \times \Sigma$ be the schematic closure of $\bar{\psi}^*(\widehat{Z} \cap p^{-1}(U)) \subset \overline{B} \times S'$. Then $\mathrm{codim}(Z', \overline{B} \times \Sigma) \geq 2$ and $\mathrm{supp} Z' \subset D'$. Hence we have $q(\mathrm{Sp}_B Z') \not\subseteq \Sigma$. We apply Proposition 7.1 for $Z' \subset \overline{B} \times \Sigma$ to get $P_0 \subset \mathcal{S}(B) \setminus \{B\}$ and $\rho \in \mathbb{Z}_{>0}$.

Let $\varepsilon > 0$. We set $\varepsilon' = \frac{\varepsilon}{c_1 c_2 (\rho+2)}$. We may apply Proposition 9.2 for $D' \subset \overline{B} \times \Sigma$, $L_{\overline{B}}$ and ε' to get $P_1 \subset \mathcal{S}(B) \setminus \{B\}$ and $E_1 \not\subseteq \Sigma$. We apply Lemma 8.1 for $Z' \subset \overline{B} \times \Sigma$, $L_{\overline{B}}$ and ε' to get $P_2 \subset \mathcal{S}(B) \setminus \{B\}$ and $E_2 \not\subseteq \Sigma$. We take a non-empty Zariski open set $U' \subset U$ such that $S' \rightarrow U$ is finite over U' . We set

$$\Phi = (\varphi(\mathrm{Ex}(\varphi)) \cup p^{-1}(\overline{W/B} \setminus U') \cup p^{-1}(\sigma(E_1 \cup E_2 \cup q(\mathrm{Sp}_B Z')))) \cap W.$$

We set $P = P_0 \cup P_1 \cup P_2$.

Step 4. The estimates for the lift $Y' \rightarrow \overline{B} \times \Sigma$. Now let $f : Y \rightarrow W$ be a holomorphic map such that $f(Y) \not\subset D \cup \Phi$ and $f \in I_B$. We assume that the first assertion of Lemma 10.4 does not valid. By $p \circ f(Y) \not\subset \overline{W/B} \setminus U'$, we may take a lift

$$g : Y' \rightarrow \overline{B} \times \Sigma$$

such that

$$g^{-1}(\partial B \times \Sigma) \subset g_\Sigma^{-1}(\Sigma \setminus S').$$

Hence we have

$$\begin{aligned} \bar{N}_g^\partial(r) &\leq N_{g_\Sigma}(r, \Sigma \setminus S') = O(\log r) + o(T_f(r))||, \\ T_{g_\Sigma}(r) &= O(\log r) + o(T_f(r))||. \end{aligned}$$

Note that the covering $\mu : Y' \rightarrow Y$ is unramified outside $g_\Sigma^{-1}(\Sigma \setminus S')$. Hence we have

$$N_{\mathrm{ram}} \pi'(r) = O(\log r) + o(T_f(r))||,$$

where $\pi' : Y' \rightarrow \mathbb{C}_{>\delta}$ is the composite $\pi' = \pi \circ \mu$.

Hence by Proposition 9.2, we get

$$m_g(r, D') + N_g(r, D') - \bar{N}_g(r, D') \leq \varepsilon' T_{g_B}(r, L_{\overline{B}}) + O(\log r) + o(T_f(r))||.$$

By Proposition 7.1

$$m_g(r, Z') + N_g(r, Z') \leq N_g^{(\rho)}(r, Z') + O(\log r) + o(T_f(r))||.$$

By Lemma 8.1, we get

$$N_g^{(1)}(r, Z') \leq \varepsilon' T_{g_B}(r, L_{\overline{B}}) + O(\log r) + o(T_f(r))||.$$

Hence

$$m_g(r, Z') + N_g(r, Z') \leq \rho \varepsilon' T_{g_B}(r, L_{\overline{B}}) + O(\log r) + o(T_f(r))||.$$

Note that $K_{\overline{B} \times \Sigma}(\partial(B \times \Sigma)) = q^* K_\Sigma$. Hence

$$\begin{aligned} (10.6) \quad m_g(r, Z') + N_g(r, Z') + T_g(r, K_{\overline{B} \times \Sigma}(D' + \partial(B \times \Sigma))) &\leq \bar{N}_g(r, D' \setminus Z') \\ &\quad + (\rho + 2) \varepsilon' T_{g_B}(r, L_{\overline{B}}) + O(\log r) + o(T_f(r))||. \end{aligned}$$

Step 5. Translating the estimate (10.6) to the original map $f : Y \rightarrow W$. We continue to consider $f : Y \rightarrow W$ from the previous step. Since $\bar{\psi} : \overline{B} \times S' \rightarrow p^{-1}(U)$ is étale, we have

$$\bar{\psi}^* K_{V_o}(\varphi_o^{-1}(\overline{D} + \partial W)) = K_{\overline{B} \times \Sigma}(D' + \partial(B \times \Sigma))|_{\overline{B} \times S'}.$$

Let $f' : Y \rightarrow V$ be the lift of $f : Y \rightarrow W$. Then $\tau \circ f' : Y \rightarrow V_o$ is the lift of f . By [Yam06, Lemma 3.1] we have

$$(10.7) \quad T_g(r, K_{\overline{B} \times \Sigma}(D' + \partial(B \times \Sigma))) = T_{\tau \circ f'}(r, K_{V_o}(\varphi_o^{-1}(\overline{D} + \partial W))) + O(\log r) + o(T_f(r))||.$$

Similarly we have (cf. [Yam15a, Lemma 1])

$$(10.8) \quad m_g(r, Z') + N_g(r, Z') = m_{\tau \circ f'}(r, \widehat{Z}) + N_{\tau \circ f'}(r, \widehat{Z}) + O(\log r) + o(T_f(r))||,$$

$$(10.9) \quad \bar{N}_g(r, D' \setminus Z') \leq \bar{N}_f(r, D \setminus Z) + O(\log r) + o(T_f(r))||.$$

Hence combining (10.6)–(10.9), we get

$$(10.10) \quad m_{\tau \circ f'}(r, \widehat{Z}) + N_{\tau \circ f'}(r, \widehat{Z}) + T_{\tau \circ f'}(r, K_{V_o}(\varphi_o^{-1}(\overline{D} + \partial W))) \leq \overline{N}_f(r, D \setminus Z) \\ + (\rho + 2)\varepsilon' T_{g_B}(r, L_{\overline{B}}) + O(\log r) + o(T_f(r))||.$$

By the choice of $\widehat{Z} \subset V_o$, we have $(p \circ \varphi)^{-1}(U) \cap E \subset (p \circ \varphi)^{-1}(U) \cap \tau^*\widehat{Z}$. Hence

$$m_{f'}(r, E) + N_{f'}(r, E) \leq m_{\tau \circ f'}(r, \widehat{Z}) + N_{\tau \circ f'}(r, \widehat{Z}) + O(\log r) + o(T_f(r))||.$$

Hence by (10.3), we get

$$(10.11) \quad T_{f'}(r, K_V(\varphi^{-1}(\overline{D} \cup \partial W))) \leq \overline{N}_f(r, D \setminus Z) \\ + (\rho + 2)\varepsilon' T_{g_B}(r, L_{\overline{B}}) + O(\log r) + o(T_f(r))||.$$

By (10.4) and (10.5), we have $T_{g_B}(r, L_{\overline{B}}) \leq c_1 c_2 T_{f'}(r, L)$. Thus we get the second assertion of Lemma 10.4. The proof is completed. \square

Let V be a \mathbb{Q} -factorial, projective variety. Let $F \subset V$ be an effective Weil divisor. Then there exists a positive integer k such that kF is a Cartier divisor. Let $g : Y \rightarrow V$ be a holomorphic map. Then we set

$$T_g(r, F) = \frac{1}{k} T_g(r, \mathcal{O}_V(kF)) + O(\log r).$$

This definition does not depend on the choice of k .

The following lemma generalizes Lemma 10.4 by considering a finite map $X \rightarrow A \times S$ instead of a closed subvariety $W \subset A \times S$. This lemma is a consequence of Lemma 10.4.

Lemma 10.5. — *Let $B \subset A$ be a semi-abelian subvariety. Let X be a normal variety with a finite map $a : X \rightarrow A \times S$, where S is a projective variety. Let $D \subset X$ be a reduced Weil divisor. Then the following objects exist with the specified property described later:*

- a compactification \overline{X} ,
- a proper birational morphism $\varphi : \overline{X}' \rightarrow \overline{X}$, where \overline{X}' is \mathbb{Q} -factorial.

The satisfied property: Let $Z \subset X$ be a Zariski closed set such that $\text{codim}(Z, X) \geq 2$ and $Z \subset D$. Let L be a big line bundle on \overline{X}' and let $\varepsilon > 0$. Then there exist

- a finite subset $P \subset \mathcal{S}(B) \setminus \{B\}$,
- a proper Zariski closed set $\Xi \subsetneq X$ with $\varphi(\text{Ex}(\varphi)) \subset \Xi$

such that for every holomorphic map $f : Y \rightarrow X$ with $f(Y) \not\subset D \cup \Xi$ and $a \circ f \in I_B$, one of the following holds:

1. *There exists $C \in P$ such that $a \circ f \in I_C$.*
2. *Let $H \subset \overline{X}'$ be a reduced Weil divisor defined by $\varphi^{-1}(\overline{D \cup X_{\text{sing}} \cup \partial X}) = H \cup \Gamma$, where $\Gamma \subset \overline{X}'$ is a Zariski closed subset of codimension greater than one. Let $g : Y \rightarrow \overline{X}'$ be the lift of f . Then*

$$T_g(r, K_{\overline{X}'}(H)) \leq \overline{N}_f(r, D \setminus Z) + \varepsilon T_g(r, L) + O(\log r) + o(T_g(r))||.$$

Proof. — We set $W = a(X)$. Since a is finite, $W \subset A \times S$ is a Zariski closed set. We divide the proof into several steps.

Step 1. We first consider the case that $B \not\subset \text{St}^0(W)$. We apply Corollary 6.3 to get $\Xi \subset W$. We set $B' = B \cap \text{St}^0(W)$. Then $B' \subsetneq B$. We set $P = \{B'\}$. Now let $f : Y \rightarrow X$ satisfies $f(Y) \not\subset D \cup \Xi$ and $a \circ f \in I_B$. By Corollary 6.3 we have $a \circ f \in I_{\text{St}^0(W)}$. Hence $a \circ f \in I_{B'}$. This shows Lemma 10.5, provided $B \not\subset \text{St}^0(W)$. In the following, we consider the case $B \subset \text{St}^0(W)$, which allows us to apply Lemma 10.4.

Step 2. We find \overline{X} and \overline{X}' . Let $R \subset W$ be a reduced Weil divisor such that $a : X \rightarrow W$ is étale outside R . We take a reduced Weil divisor $D' \subset W$ such that $a(D \cup X_{\text{sing}}) \cup R \subset D'$. By Lemma 10.4, there exist a compactification $W \subset \overline{W}$ and a proper birational morphism $\psi : V \rightarrow \overline{W}$, where V is smooth and $\psi^{-1}(\overline{D'}) \cup \partial W \subset V$ is a simple normal crossing divisor. Let $\overline{X} \rightarrow \overline{W}$ be obtained from $X \rightarrow W$ by the normalization. Let $p : \overline{X}' \rightarrow V$ be obtained from the base change and normalization.

$$\begin{array}{ccccc}
X & \hookrightarrow & \overline{X} & \xleftarrow{\varphi} & \overline{X}' \\
a \downarrow & & \downarrow & & \downarrow p \\
W & \hookrightarrow & \overline{W} & \xleftarrow{\psi} & V
\end{array}$$

Note that $p : \overline{X}' \rightarrow V$ is unramified outside $\tilde{D} = \psi^{-1}(\overline{D}' \cup \partial W)$. Since \tilde{D} is simple normal crossing, [NWY13, Lemma 2] yields that \overline{X}' is \mathbb{Q} -factorial.

Step 3. We find P and Ξ from the given Z, L and ε . Let $Z \subset X$ be a Zariski closed set such that $\text{codim}(Z, X) \geq 2$ and $Z \subset D$. We set

$$Z' = a(Z \cup X_{\text{sing}}) \cup (W \cap (\psi \circ p)(\overline{X}'_{\text{sing}})).$$

Then $\text{codim}(Z', W) \geq 2$ and $Z' \subset D'$. Let L be a big line bundle on \overline{X}' . Let L_V be an ample line bundle on V . Then there exists a positive integer $l \in \mathbb{Z}_{\geq 1}$ such that $p^*L_V(E) = L^{\otimes l}$ for some effective divisor $E \subset \overline{X}'$. Let $\varepsilon > 0$. We set $\varepsilon' = \varepsilon/l$.

Now by Lemma 10.4 for $Z' \subset D', L_V$ and ε' , we get a finite subset $P \subset \mathcal{S}(B) \setminus \{B\}$ and a proper Zariski closed subset $\Phi \subsetneq W$. We set

$$\Xi = a^{-1}(D' \cup \Phi) \cup (X \cap \varphi(E)) \cup (X \cap \varphi(\text{Ex}(\varphi))).$$

Step 4. Let $f : Y \rightarrow X$ be a holomorphic map such that $a \circ f \in I_B$ and $f(Y) \not\subset D \cup \Xi$. Assume that the first assertion of Lemma 10.5 does not valid. Let $g : Y \rightarrow \overline{X}'$ be the lift of f . Then $p \circ g : Y \rightarrow V$ is the lift of $a \circ f : Y \rightarrow W$. Hence by Lemma 10.4, we get

$$(10.12) \quad T_{p \circ g}(r, K_V(\tilde{D})) \leq \overline{N}_{a \circ f}(r, D' \setminus Z') + \varepsilon' T_{p \circ g}(r, L_V) + O(\log r) + o(T_{p \circ g}(r))||$$

We define a reduced divisor F on \overline{X}' by $p^{-1}(\tilde{D}) = F + H$. By the ramification formula, we have

$$K_{\overline{X}'}(F + H) = p^*K_V(\tilde{D}).$$

Thus we have

$$T_g(r, K_{\overline{X}'}(F + H)) = T_{p \circ g}(r, K_V(\tilde{D})) + O(\log r).$$

Combining this estimate with (10.12) and $p^*L_V(E) = L^{\otimes l}$, we get

$$(10.13) \quad T_g(r, K_{\overline{X}'}(F + H)) \leq \overline{N}_{a \circ f}(r, D' \setminus Z') + \varepsilon T_g(r, L) + O(\log r) + o(T_g(r))||$$

We estimate the right hand side of (10.13). We have

$$(10.14) \quad \overline{N}_{a \circ f}(r, D' \setminus Z') \leq \overline{N}_g(r, F \setminus \overline{X}'_{\text{sing}}) + \overline{N}_g(r, H \setminus \varphi^{-1}(Z \cup X_{\text{sing}}))$$

We set H' by $H = H' + \varphi^{-1}(\partial X + \overline{D})$. Then $\varphi(H') \subset X_{\text{sing}}$. Hence by $\overline{N}_f(r, \partial X) = 0$, we have

$$(10.15) \quad \overline{N}_g(r, H \setminus \varphi^{-1}(Z \cup X_{\text{sing}})) \leq \overline{N}_f(r, D \setminus Z).$$

Since \overline{X}' is \mathbb{Q} -factorial, we may take a positive integer k such that kF is a Cartier divisor. Since $F \cap (\overline{X}' \setminus \overline{X}'_{\text{sing}})$ is a Cartier divisor on $\overline{X}' \setminus \overline{X}'_{\text{sing}}$, we have

$$k \text{ord}_z g^*F = \text{ord}_z g^*(kF)$$

for $z \in g^{-1}(\overline{X}' \setminus \overline{X}'_{\text{sing}})$, and hence

$$k \min\{1, \text{ord}_z g^*F\} \leq \text{ord}_z g^*(kF).$$

Thus we get

$$k \overline{N}_g(r, F \setminus \overline{X}'_{\text{sing}}) \leq N_g(r, kF).$$

By the Nevanlinna inequality (cf. (3.1)), we have

$$N_g(r, kF) \leq T_g(r, kF) + O(\log r).$$

Hence

$$\overline{N}_g(r, F \setminus \overline{X}'_{\text{sing}}) \leq T_g(r, F) + O(\log r).$$

Combining this with (10.14) and (10.15), we get

$$\overline{N}_{a \circ f}(r, D' \setminus Z') \leq \overline{N}_f(r, D \setminus Z) + T_g(r, F) + O(\log r).$$

Combining this estimate with (10.13), we get the second assertion of Lemma 10.5. This concludes the proof. \square

Proposition 10.6. — *Let Σ be a smooth quasi-projective variety which is of log general type. Assume that there is a morphism $a : \Sigma \rightarrow A \times S$ such that $\dim \Sigma = \dim a(\Sigma)$, where S is a projective variety. Let $B \subset A$ be a semi-abelian subvariety. Then there exist a finite subset $P \subset \mathcal{S}(B) \setminus \{B\}$ and a proper Zariski closed set $\Phi \subsetneq \Sigma$ with the following property: Let $f : Y \rightarrow \Sigma$ be a holomorphic map such that $a \circ f \in I_B$ and $f(Y) \not\subset \Phi$. Then either one of the followings holds:*

1. *There exists $C \in P$ such that $a \circ f \in I_C$.*
2. *f does not have essential singularity over ∞ .*

Proof. — Let $W \subset A \times S$ be the Zariski closure of $a(\Sigma)$. Let $\pi : X \rightarrow W$ be the normalization with respect to a and let $\varphi : \Sigma \rightarrow X$ be the induced map. Then φ is birational. Let $\bar{\Sigma}$ be a smooth partial compactification such that φ extends to a projective morphism $\bar{\varphi} : \bar{\Sigma} \rightarrow X$ and $\bar{\Sigma} \setminus \Sigma$ is a divisor on $\bar{\Sigma}$. Since $\bar{\varphi}$ is birational and X is normal, there exists a Zariski closed subset $Z \subset X$ whose codimension is greater than one such that $\bar{\varphi} : \bar{\Sigma} \rightarrow X$ is an isomorphism over $X \setminus Z$. In particular $X \setminus Z$ is smooth. Let D be the Zariski closure of $(X \setminus Z) \cap \bar{\varphi}(\bar{\Sigma} \setminus \Sigma)$ in X . Then D is a reduced divisor on X and $X \setminus (Z \cup D)$ is of log-general type. Thus by [NWY13, Lemma 4], $X \setminus (X_{\text{sing}} \cup D)$ is of log-general type. Note that $(D \cap Z) \cup X_{\text{sing}} \subset Z$.

We apply Lemma 10.5 to get \bar{X} and $\psi : \bar{X}' \rightarrow \bar{X}$. We define a reduced divisor $H \subset \bar{X}'$ to be $\psi^{-1}(\overline{D \cup X_{\text{sing}} \cup \partial X}) = H \cup \Gamma$, where $\Gamma \subset \bar{X}'$ is a Zariski closed subset of codimension greater than one. Since $X \setminus (D \cup X_{\text{sing}})$ is of log-general type, we deduce that $\psi^{-1}(X \setminus (D \cup X_{\text{sing}})) = \bar{X}' \setminus (H \cup \Gamma)$ is also of log-general type. Thus by [NWY13, Lemma 3], $K_{\bar{X}'}(H)$ is big.

By Kodaira's lemma, there exist an effective divisor $E \subset \bar{X}'$ and a positive integer $l \in \mathbb{Z}_{\geq 1}$ such that $lK_{\bar{X}'}(H) - E$ is ample. Hence if $g : Y \rightarrow \bar{X}'$ satisfies $g(Y) \not\subset E$, then

$$T_g(r) = O(T_g(r, K_{\bar{X}'}(H))) + O(\log r).$$

By Lemma 10.5 applied to $Z \cap D$, $L = K_{\bar{X}'}(H)$ and $\varepsilon = 1/2$, we get $P \subset \mathcal{S}(B) \setminus \{B\}$ and $\Xi \subsetneq X$. We set $\Phi = \varphi^{-1}(\Xi \cup \psi(E))$. Let $f : Y \rightarrow \Sigma$ be a holomorphic map such that $a \circ f \in I_B$ and $f(Y) \not\subset \Phi$. Then $\varphi \circ f(Y) \not\subset \Xi$. Suppose that the first assertion of Proposition 10.6 is not valid. Then by Lemma 10.5, we get

$$T_g(r, K_{\bar{X}'}(H)) \leq \bar{N}_{\varphi \circ f}(r, D \setminus Z) + \frac{1}{2} T_g(r, K_{\bar{X}'}(H)) + O(\log r) + o(T_g(r))||,$$

where $g : Y \rightarrow \bar{X}'$ is the lift of $\varphi \circ f : Y \rightarrow X$. We have

$$\bar{N}_{\varphi \circ f}(r, D) = \bar{N}_{\varphi \circ f}(r, D \cap Z).$$

Hence we get

$$T_g(r, K_{\bar{X}'}(H)) = O(\log r) + o(T_g(r))||.$$

Thus $T_{\varphi \circ f}(r) = O(\log r)||$. This shows $T_{a \circ f}(r) = O(\log r)||$. Hence $a \circ f \in I_{\{0\}}$. Hence by Remark 10.3, f does not have essential singularity over ∞ . \square

The following theorem implies Theorem A, when S is a single point.

Theorem 10.7. — *Let X be a smooth quasi-projective variety which is of log general type. Assume that there is a morphism $a : X \rightarrow A \times S$ such that $\dim X = \dim a(X)$, where S is a projective variety. Then there exists a proper Zariski closed set $\Xi \subsetneq X$ with the following property: Let $f : Y \rightarrow X$ be a holomorphic map with the following three properties:*

- (a) $T_{(a \circ f)_S}(r) = O(\log r) + o(T_f(r))||$,
- (b) $N_{\text{ram } \pi}(r) = O(\log r) + o(T_f(r))||$,
- (c) $f(Y) \not\subset \Xi$.

Then f does not have essential singularity over ∞ .

Proof. — We take a Zariski closed set $E \subset X$ such that if $g : Y \rightarrow X$ satisfies $g(Y) \not\subset E$, then $T_g(r) = O(T_{a \circ g}(r)) + O(\log r)$. We take a sequence $\mathcal{P}_i \subset \mathcal{S}(A)$ of finite sets as follows. Set $\mathcal{P}_1 = \{A\}$. Given \mathcal{P}_i , we define \mathcal{P}_{i+1} as follows. For each $B \in \mathcal{P}_i$, we apply Proposition 10.6 to get

$\Phi_B \subsetneq X$ and $P_B \subset S(B) \setminus \{B\}$. We set $\mathcal{P}_{i+1} = \cup_{B \in \mathcal{P}_i} P_B$ and $\Xi_{i+1} = \cup_{B \in \mathcal{P}_i} \Phi_B$. Set $\Xi = E \cup \cup_i \Xi_i$. Then $\Xi \subsetneq X$ is a proper Zariski closed set.

Let $f : Y \rightarrow X$ satisfy the three properties in Theorem 10.7. Then $f \in I_A$. We take i which is maximal among the property that there exists $B \in \mathcal{P}_i$ such that $f \in I_B$. Then by $f(Y) \not\subset \Phi_B \subset \Xi$, Proposition 10.6 implies that f does not have essential singularity over ∞ . \square

We state and prove a more general result than that used in the implications of Theorem C: (a) \implies (c) and (e) \implies (b). This result will be used in the proof of [CDY25, Theorem C].

Corollary 10.8. — Let X be a smooth quasi-projective variety and let $a : X \rightarrow A \times S^\circ$ be a morphism such that $\dim X = \dim a(X)$, where S° is a smooth quasi-projective variety (S° can be a point). Write $b : X \rightarrow S^\circ$ as the composition of a with the projection map $A \times S^\circ \rightarrow S^\circ$. Assume that b is dominant.

- (i) Suppose S° is pseudo Picard hyperbolic. If X is of log general type, then X is pseudo Picard hyperbolic.
- (ii) Suppose $\text{Sp}_{\text{alg}}(S^\circ) \subsetneq S^\circ$. If $\text{Sp}_{\text{sab}}(X) \subsetneq X$, then $\text{Sp}_{\text{alg}}(X) \subsetneq X$.

Proof. — Let S be a smooth projective variety that compactifies S° .

Proof of Corollary 10.8.(i): Since S° is pseudo Picard hyperbolic, there exists a proper Zariski closed subset $Z \subset S^\circ$ such that each holomorphic map $g : \mathbb{D}^* \rightarrow S^\circ$ with $g(\mathbb{D}^*) \not\subset Z$ has no essential singularity at ∞ . By applying Theorem 10.7 to $X \rightarrow A \times S$, we obtain a proper Zariski closed set $\Xi \subsetneq X$ that satisfies the property given in Theorem 10.7. Set $\Sigma := b^{-1}(Z) \cup \Xi$. Since b is dominant, Σ is a proper Zariski closed subset of X . Then for any holomorphic map $f : \mathbb{D}^* \rightarrow X$ with $f(\mathbb{D}^*) \not\subset \Sigma$, we have $b \circ f(\mathbb{D}^*) \not\subset Z$ and $f(\mathbb{D}^*) \not\subset \Xi$. It follows that $T_{b \circ f}(r) = O(\log r)$. Hence f verifies Properties Item (a) and Item (c) in Theorem 10.7. Note that Property Item (b) in Theorem 10.7 is automatically satisfied as $N_{\text{ram}}(\pi) = 0$. We can now apply Theorem 10.7 to conclude that f does not have an essential singularity over ∞ . Therefore, X is pseudo Picard hyperbolic.

Proof of Corollary 10.8.(ii): Since $\dim X = \dim a(X)$, there is a proper Zariski closed subset $Y \subsetneq X$ such that the restriction of $a|_{X \setminus Y} : X \setminus Y \rightarrow A \times S^\circ$ is quasi-finite. Let $\Xi := b^{-1}(\text{Sp}_{\text{alg}}(S^\circ)) \cup \text{Sp}_{\text{sab}}(X) \cup Y$. Then by the assumptions $\text{Sp}_{\text{alg}}(S^\circ) \subsetneq S^\circ$ and $\text{Sp}_{\text{sab}}(X) \subsetneq X$, we have $\Xi \subsetneq X$.

Let V be a closed subvariety of X that is not contained in Ξ . Then we have

$$(10.16) \quad \text{Sp}_{\text{sab}}(V) \subsetneq V.$$

In the following, we shall prove that V is of log general type to conclude $\text{Sp}_{\text{alg}}(X) \subset \Xi$.

We first show $\bar{\kappa}(V) \geq 0$. Note that for a general fiber F of $b|_V : V \rightarrow S^\circ$, we have $\dim F = \dim c(F)$, where $c : X \rightarrow A$ is the composition of a with the projection map $A \times S^\circ \rightarrow A$. By Lemma 1.1, it follows that $\bar{\kappa}(c(F)) \geq 0$, and hence $\bar{\kappa}(F) \geq 0$. Also note that $b(V)$ is not contained in $\text{Sp}_{\text{alg}}(S^\circ)$, and thus $\overline{b(V)}$ is of log general type. We use Fujino's addition formula for logarithmic Kodaira dimensions [Fuj17, Theorem 1.9] to conclude that

$$\bar{\kappa}(V) \geq \bar{\kappa}(F) + \bar{\kappa}(\overline{b(V)}) \geq 0.$$

Hence we have proved $\bar{\kappa}(V) \geq 0$. We may consider the logarithmic Iitaka fibration of V .

Next we show $\bar{\kappa}(V) = \dim V$. After replacing V with a birational modification, we can assume that V is smooth and that the logarithmic Iitaka fibration $j : V \rightarrow J(V)$ is regular. Assume contrary that V is not of log general type. Note that for a very general fiber F of j , the followings hold:

- (a) $\dim F > 0$;
- (b) F is smooth;
- (c) $\bar{\kappa}(F) = 0$;
- (d) $b(F) \not\subset \text{Sp}_{\text{alg}}(S^\circ)$;
- (e) $F \not\subset Y$.

By Item (d), we have $\bar{\kappa}(\overline{b(F)}) = \dim b(F) \geq 0$. Item (e) implies that for a general fiber Y of $b|_F : F \rightarrow S^\circ$, we have $\dim Y = \dim c(Y)$. Using Lemma 1.1, we have $\bar{\kappa}(\overline{c(Y)}) \geq 0$, and thus $\bar{\kappa}(Y) \geq 0$. Using [Fuj17, Theorem 1.9] again, we can conclude that

$$\bar{\kappa}(F) \geq \bar{\kappa}(Y) + \bar{\kappa}(\overline{b(F)}) \geq 0.$$

Item (c) implies that $\bar{\kappa}(Y) = 0$ and $\bar{\kappa}(\overline{b(F)}) = 0$. Hence $b(F)$ is a point. This implies that $\dim F = \dim c(F)$. Combining with Items (a) and (c), Lemma 1.6 yields $\text{Sp}_{\text{sab}}(F) = F$. Since F is a very general fiber of $j : V \rightarrow J(V)$, we get $\text{Sp}_{\text{sab}}(V) = V$. This contradicts to (10.16). Thus we have proved that V is of log general type, hence $\text{Sp}_{\text{alg}}(X) \subset \Xi$. \square

The following proposition finds applications in the third part of this paper series.

Proposition 10.9. — *Let $D \subset A$ be a reduced divisor such that $\text{St}(D) = \{a \in A; a + D = D\}$ is finite. Let $Z \subset D$ be a Zariski closed subset such that $\text{codim}(Z, A) \geq 2$. Then there exists a proper Zariski closed subset $\Xi \subsetneq A$ with the following property: Let $f : Y \rightarrow A$ be a holomorphic map with the following three properties:*

- (a) $N_{\text{ram}} \pi(r) = O(\log r) + o(T_f(r))||,$
- (b) $f(Y) \not\subset \Xi \cup D,$
- (c) $\text{ord}_y f^* D \geq 2$ for all $y \in f^{-1}(D \setminus Z)$.

Then f does not have essential singularity over ∞ .

Proof. — For a semi-abelian variety $B \subset A$, we denote by J_B the set of all holomorphic maps $f : Y \rightarrow A$ such that

- $f \in I_B$,
- $f(Y) \not\subset D$, and
- $\text{ord}_y f^* D \geq 2$ for all $y \in f^{-1}(D \setminus Z)$.

Given a semi-abelian variety $B \subset A$, we first prove the following claim.

Claim 10.10. — *There exist a finite subset $P_B \subset S(B) \setminus \{B\}$ and a proper Zariski closed set $\Phi_B \subsetneq A$ with the following property: Let $f : Y \rightarrow A$ be a holomorphic map such that $f \in J_B$ and $f(Y) \not\subset \Xi_B$. Then either one of the followings holds:*

- (a) *There exists $C \in P_B$ such that $f \in J_C$.*
- (b) *f does not have essential singularity over ∞ .*

Proof of Claim 10.10. — Since $\text{St}(D)$ is finite, $A \setminus D$ is of log-general type. We apply Lemma 10.4 to get a smooth projective equivariant compactification \overline{A} and a proper birational morphism $\varphi : V \rightarrow \overline{A}$, where V is smooth and $\varphi^{-1}(\overline{D} \cup \partial A)$ is a simple normal crossing divisor. Since $A \setminus D$ is of log-general type, we deduce that $\varphi^{-1}(A \setminus D) = V \setminus \varphi^{-1}(\overline{D} \cup \partial A)$ is also of log-general type. Thus $K_V(\varphi^{-1}(\overline{D} \cup \partial A))$ is big.

By Kodaira's lemma, there exist an effective divisor $E \subsetneq \overline{A}'$ and a positive integer $l \in \mathbb{Z}_{\geq 1}$ such that $L = lK_V(\varphi^{-1}(\overline{D} \cup \partial A)) - E$ is ample. Hence if $g : Y \rightarrow V$ satisfies $g(Y) \not\subset E$, then

$$T_g(r) = O(T_g(r, K_V(\varphi^{-1}(\overline{D} \cup \partial A)))) + O(\log r).$$

By Lemma 10.4 applied to $(Z \cup \varphi(\text{Ex}(\varphi))) \cap D$, L and $\varepsilon = 1/3l$, we get a finite subset $P_B \subset S(B) \setminus \{B\}$ and a proper Zariski closed set $\Phi_B \subsetneq A$ with $\varphi(\text{Ex}(\varphi)) \subset \overline{\Phi_B}$. We set $\Xi_B = \Phi_B \cup \varphi(E)$. Let $f : Y \rightarrow A$ be a holomorphic map such that $f \in J_B$ and $f(Y) \not\subset \Xi_B$. Then $f(Y) \not\subset D \cup \Xi_B$ and $f \in I_B$. Suppose that the first assertion of Claim 10.10 is not valid. Then $f \notin I_C$ for all $C \in P_B$. By Lemma 10.4, we get

$$T_g(r, K_V(\varphi^{-1}(\overline{D} \cup \partial A))) \leq \overline{N}_f(r, D \setminus (Z \cup \varphi(\text{Ex}(\varphi)))) + \frac{1}{3l} T_g(r, L) + O(\log r) + o(T_g(r))||,$$

where $g : Y \rightarrow V$ is the lift of $f : Y \rightarrow A$. By $g(Y) \not\subset E$, we have

$$T_g(r, L) \leq lT_g(r, K_V(\varphi^{-1}(\overline{D} \cup \partial A))) + O(\log r).$$

Hence we get

$$\frac{2}{3} T_g(r, K_V(\varphi^{-1}(\overline{D} \cup \partial A))) \leq \overline{N}_f(r, D \setminus (Z \cup \varphi(\text{Ex}(\varphi)))) + O(\log r) + o(T_g(r))||.$$

We estimate the first term of the right hand side. Since \overline{A} is smooth and equivariant, ∂A is a simple normal crossing divisor. Hence we may decompose as $\varphi^{-1}(\overline{D} \cup \partial A) = H + F$ so that $F = \varphi^{-1}(\partial A)$. Then $H = \varphi^{-1}(D)$. The induced map $V \setminus (F \cup \text{Ex}(\varphi)) \rightarrow A \setminus \varphi(\text{Ex}(\varphi))$ is isomorphic. Hence by $\text{ord}_y f^* D \geq 2$ for all $y \in f^{-1}(D \setminus Z)$, we have

$$2\overline{N}_f(r, D \setminus (Z \cup \varphi(\text{Ex}(\varphi)))) \leq T_g(r, H).$$

By $\bar{\kappa}(V \setminus F) = 0$ and $g(Y) \not\subset \text{Ex}(\varphi)$, we have $T_g(r, K_V(F)) + O(\log r) > 0$. Hence

$$2\overline{N}_f(r, D \setminus (Z \cup \varphi(\text{Ex}(\varphi)))) \leq T_g(r, K_V(H + F)) + O(\log r).$$

Hence we get

$$T_g(r, K_V(H + F)) = O(\log r) + o(T_g(r)).$$

Thus $T_g(r) = O(\log r)$. This shows $T_f(r) = O(\log r)$. Hence $f \in I_{\{0\}}$. Hence by Remark 10.3, f does not have essential singularity over ∞ . \square

We take a sequence $\mathcal{P}_i \subset \mathcal{S}(A)$ of finite sets as follows. Set $\mathcal{P}_1 = \{A\}$. Given \mathcal{P}_i , we define \mathcal{P}_{i+1} as follows. For each $B \in \mathcal{P}_i$, we apply Claim 10.10 to get $\Xi_B \subsetneq A$ and $P_B \subset \mathcal{S}(B) \setminus \{B\}$. We set $\mathcal{P}_{i+1} = \cup_{B \in \mathcal{P}_i} P_B$ and $\Xi_{i+1} = \cup_{B \in \mathcal{P}_i} \Xi_B$. Set $\Xi = \cup_i \Xi_i$. Then $\Xi \subsetneq A$ is a proper Zariski closed set.

Let $f : Y \rightarrow A$ satisfy the three properties in Proposition 10.9. Then $f \in J_A$. We take i which is maximal among the property that there exists $B \in \mathcal{P}_i$ such that $f \in J_B$. Then by $f(Y) \not\subset \Xi_B \subset \Xi$, Claim 10.10 implies that f does not have essential singularity over ∞ . \square

Remark 10.11. — The proof of Theorem A is based on the arguments in [Yam15a] and [NWY13]. A major difficulty in treating the non-compact case, compared with the compact case in [Yam15a], is the lack of the Poincaré reducibility theorem. To overcome this problem, we use a more general type of "cover" than an étale cover. In [NWY08], Noguchi, Winkelmann, and the third author utilized a "flat cover" (cf. the commutative diagram just after [NWY08, (5.9)]). An important issue with using this flat cover is to construct a lift of holomorphic maps onto it, such that the order function of the lift is bounded by that of the original map (cf. [NWY08, Lemma 5.8]). The authors in [NWY08] only considered holomorphic maps from the complex plane \mathbb{C} . Thanks to the simple connectedness of \mathbb{C} , such a lift could be constructed easily. In contrast, we consider holomorphic maps from the covering space $\pi : Y \rightarrow \mathbb{C}_{>\delta}$, which is not simply connected. Therefore, in this paper, we choose a different cover $\sigma : \Sigma \rightarrow \overline{W/B}$ in the proof of Lemma 10.4. After the base change by this cover, we obtain the lift $g : Y' \rightarrow \overline{B} \times \Sigma$ of $f : Y \rightarrow W$ from the covering space $Y' \rightarrow Y$. We then apply the results of § 5–§ 9 to this lift g . We remark that the map $g_B : Y' \rightarrow \overline{B}$ may hit the boundary ∂B . This is the reason why we treat the situation $f : Y \rightarrow A \times S$ in § 5–§ 9. We also remark that even if we restrict the statement of Theorem A to $f : \mathbb{D}^* \rightarrow X$, we still need to consider a ramified covering $Y' \rightarrow \mathbb{D}^*$ in the proof to obtain the map $g : Y' \rightarrow \overline{B} \times \Sigma$ mentioned above.

11. Proof of Theorem B

We consider another application of Theorem A, whose proof requires a ramified covering of $\mathbb{C}_{>\delta}$. The following theorem is used in Part II of this series. Although it appears only implicitly in [CDY22, §6] and is not stated explicitly, we find it convenient to formulate it here as follows, since the original work [CDY22] was divided into separate papers. See also [Yam10, §3].

Theorem 11.1. — Let X be a smooth quasi-projective variety with a smooth projective compactification \overline{X} such that the boundary divisor $D = \overline{X} \setminus X$ is a simple normal crossing divisor. Assume that there exists a finite surjective morphism $p : \overline{\Sigma} \rightarrow \overline{X}$ from a normal projective variety $\overline{\Sigma}$ together with non-zero sections $\tau_1, \dots, \tau_l \in H^0(\overline{\Sigma}, p^*\Omega_{\overline{X}}(\log D))$ satisfying the following two conditions:

- (i) setting $\Sigma = p^{-1}(X)$, the variety Σ is of log-general type and admits a morphism $a : \Sigma \rightarrow A$ into a semi-abelian variety A with $\dim \Sigma = \dim a(\Sigma)$;
- (ii) defining

$$R := \{s \in \overline{\Sigma} \mid \exists i \text{ such that } \tau_i(s) = 0\},$$

then $R \subsetneq \overline{\Sigma}$ is a proper Zariski closed subset, and $p : \overline{\Sigma} \rightarrow \overline{X}$ is étale outside R .

Then every holomorphic map $f : \mathbb{D}^* \rightarrow X$ with Zariski dense image has a holomorphic extension $\tilde{f} : \mathbb{D} \rightarrow \overline{X}$.

In the second condition (ii), we observe that $p^*\Omega_{\overline{X}}(\log D)$ is a locally free sheaf of rank $\dim X$ on $\overline{\Sigma}$. At a point $s \in \overline{\Sigma}$, we then consider $\tau_i(s) \in (p^*\Omega_{\overline{X}}(\log D))_s \otimes \mathbb{C}$, where $(p^*\Omega_{\overline{X}}(\log D))_s$ denotes the stalk at $s \in \overline{\Sigma}$. Thus the set $\{s \in \overline{\Sigma} ; \tau_i(s) = 0\}$ is a Zariski closed subset of $\overline{\Sigma}$, since it coincides with the zero locus of a section τ_i of the corresponding vector bundle on $\overline{\Sigma}$.

We shall see in Part II of our series that a covering $\bar{\Sigma} \rightarrow \bar{X}$ with the property of Theorem 11.1 exists provided there is a representation $\varrho : \pi_1(X) \rightarrow G(K)$ into an almost simple algebraic group G defined over a p -adic field K , where ϱ is big, unbounded, and Zariski dense. This covering is realized by a spectral cover constructed from the ϱ -equivariant pluriharmonic map $\bar{X} \rightarrow \Delta(G)$ from the universal cover \bar{X} of X to the Bruhat–Tits building $\Delta(G)$ of G . This pluriharmonic map was developed by the second author and co-authors [BDDM22], and it will be employed in Part II of our series to construct the corresponding spectral cover.

To prove Theorem 11.1 we start with the following lemma.

Lemma 11.2. — *Let X be a smooth quasi-projective variety with a smooth projective compactification \bar{X} such that the boundary divisor $D = \bar{X} \setminus X$ is a simple normal crossing divisor. Let $\omega \in H^0(\bar{X}, \Omega_{\bar{X}}(\log D))$ be a nonzero section, and let $u : Z \rightarrow X$ be a morphism from a smooth quasi-projective variety Z such that $u^*\omega \in H^0(Z, \Omega_Z)$ vanishes. Then, for every holomorphic map $g : Y \rightarrow X$ with Zariski dense image and $g^*\omega = 0$, we have $\bar{N}_g(r, u(Z)) = o(T_g(r)) + O(\log r)$.*

Proof. — Let $\alpha : X \rightarrow A_X$ be the quasi-Albanese map. Then there exists a non-zero holomorphic one-form $\tilde{\omega}$ on A_X such that $\alpha^*\tilde{\omega} = \omega$. Let $B \subset A_X$ be the largest semi-abelian subvariety of A_X on which the pull-back of the holomorphic one form $\tilde{\omega}$ vanishes. Set $A' = A_X/B$. Let $q : A_X \rightarrow A'$ be the quotient map, and let $\beta : X \rightarrow A'$ be the composition of $\alpha : X \rightarrow A_X$ with $q : A_X \rightarrow A'$.

We claim that $\beta(u(Z))$ is a point. Indeed, we have the following commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{\alpha} & A_X & \xrightarrow{q} & A_X/B = A' \\ u \uparrow & & j \uparrow & & \\ Z & \xrightarrow{a} & A_Z & & \end{array}$$

Here $a : Z \rightarrow A_Z$ denotes the quasi-Albanese map, and $j : A_Z \rightarrow A_X$ is the induced morphism of algebraic groups. By the definition of B , the image of $j(A_Z)$ is contained in B . Therefore, $\beta(u(Z))$ is a single point, as claimed.

Let $\beta(u(Z)) = \{a\}$ for some $a \in A'$. Let $\phi : S \rightarrow A'$ be the normalization of the Zariski closure $\overline{\beta(X)} \subset A'$. Since X is smooth, β factors ϕ as follows:

$$X \xrightarrow{\psi} S \xrightarrow{\phi} A'.$$

Since ϕ is finite and $\beta(u(Z)) = \{a\}$, there is a point $s \in S$ so that $\psi(u(Z)) = \{s\}$.

There exists a non-zero logarithmic one-form ω' on A' such that $\tilde{\omega} = q^*\omega'$. Then

$$(11.1) \quad \beta^*\omega' = \omega.$$

Since ω' is a linear logarithmic one-form on A' , we have $d\omega' = 0$. Hence, by the Poincaré lemma, in some analytic neighborhood U of a in A' , there exists a holomorphic function $h \in \mathcal{O}_{A'}^{\text{an}}(U)$ such that $dh = \omega'$ on U and $h(a) = 0$. By (11.1), β^*dh is not identically zero. Consequently, $h \circ \phi \in \mathcal{O}_S^{\text{an}}(\phi^{-1}(U))$ is a non-constant holomorphic function such that $h \circ \phi(s) = 0$.

Let $\mathcal{O}_{S,s}^{\text{an}}$ be the stalk at s in the sense of analytic spaces, and let $\mathfrak{m}_s \subset \mathcal{O}_{S,s}^{\text{an}}$ denote the maximal ideal. Since S is normal, $\mathcal{O}_{S,s}^{\text{an}}$ is an integral domain. We have verified that $h \circ \phi \in \mathcal{O}_{S,s}^{\text{an}}$ is nonzero. Moreover, since $h \circ \phi(s) = 0$, we have $h \circ \phi \in \mathfrak{m}_s$. Hence we have

$$(11.2) \quad \dim \mathcal{O}_{S,s}^{\text{an}}/(h \circ \phi) = \dim S - 1.$$

For every $n \in \mathbb{Z}_{>0}$, consider the zero-dimensional subscheme on S defined by $V_n := \text{Spec } \mathcal{O}_{S,s}^{\text{an}}/((h \circ \phi) + \mathfrak{m}_s^n)$. Then V_n is supported at s .

We take a projective compactification \bar{S} for S . We fix an ample line bundle M on \bar{S} . Consider the following short exact sequence

$$0 \rightarrow M^k \otimes \mathcal{I}_{V_n} \rightarrow M^k \rightarrow M^k \otimes \mathcal{O}_{V_n} \rightarrow 0,$$

which yields a short exact sequence

$$0 \rightarrow H^0(\bar{S}, M^k \otimes \mathcal{I}_{V_n}) \rightarrow H^0(\bar{S}, M^k) \rightarrow H^0(\bar{S}, M^k \otimes \mathcal{O}_{V_n})$$

By (11.2), we have

$$h^0(\bar{S}, M^k \otimes \mathcal{O}_{V_n}) = h^0(V_n, \mathcal{O}_{V_n}) = O(n^{\dim S - 1})$$

as $n \rightarrow \infty$. By Riemann-Roch theorem, there are positive constants $c > 0$ and $\kappa > 0$ such that $h^0(\bar{S}, M^k) > ck^{\dim S}$ for $k > \kappa$. If we take $k_n \sim n^{1 - \frac{1}{2\dim S}}$ when $n \rightarrow \infty$, it follows that

$$h^0(\bar{S}, M^{k_n}) > h^0(V_n, \mathcal{O}_{V_n})$$

when $n \gg 1$. Hence there are nonzero sections

$$\sigma_n \in H^0(\bar{S}, M^{k_n} \otimes \mathcal{I}_{V_n})$$

when $n \gg 1$. Let $D_n \subset \bar{S}$ be the corresponding divisor. Then we have $V_n \subset D_n$ as closed subschemes on \bar{S} .

We now consider $\psi \circ g : Y \rightarrow S$ which is a holomorphic map with Zariski dense image. Hence $\psi \circ g(Y) \not\subset D_n$. By Theorem 3.1, we have

$$N_{\psi \circ g}(r, V_n) \leq N_{\psi \circ g}(r, D_n) \leq T_{\psi \circ g}(r, M^{k_n}) + O(\log r) = k_n T_{\psi \circ g}(r, M) + O(\log r).$$

For every $y \in Y$ so that $g(y) \in u(Z)$, it follows that $\psi \circ g(y) = s$. Since $g^* \omega = 0$, it follows that

$$(11.3) \quad (\beta \circ g)^* \omega' = g^* \omega = 0.$$

Hence $(h \circ \phi) \circ (\psi \circ g)$ is constant on the connected component of $(\beta \circ g)^{-1}(U)$ containing y which is thus zero. Therefor $\psi \circ g : U \rightarrow S$ factors the zero locus $(h \circ \phi = 0)$. This implies that $\text{ord}_y(\psi \circ g)^* V_n \geq n$, hence

$$n\bar{N}_g(r, u(Z)) \leq N_{\psi \circ g}(r, V_n).$$

In conclusion,

$$\bar{N}_g(r, u(Z)) \leq \frac{k_n}{n} T_{\psi \circ g}(r, M) + O(\log r).$$

On the other hand, $T_{\psi \circ g}(r, M) \leq cT_g(r, L) + O(\log r)$ for some constant $c > 0$ since order functions decrease under rational map $\bar{X} \dashrightarrow \bar{S}$ induced by ψ . By $k_n/n \rightarrow 0$ as $n \rightarrow \infty$, we get

$$(11.4) \quad \bar{N}_g(r, u(Z)) \leq \varepsilon T_g(r, L) + O_\varepsilon(\log r)$$

for every $\varepsilon > 0$.

Suppose that $\lim_{r \rightarrow \infty} T_g(r, L)/\log r < +\infty$. Then by Lemma 3.4, we have $T_g(r, L) = O(\log r)$. Then by (11.4), we have $\bar{N}_g(r, u(Z)) = O(\log r)$, in particular

$$(11.5) \quad \bar{N}_g(r, u(Z)) = o(T_g(r, L)) + O(\log r).$$

Next we assume $\lim_{r \rightarrow \infty} T_g(r, L)/\log r = +\infty$. Then $\log r = o(T_g(r, L))$. Then by (11.4), we have $\bar{N}_g(r, u(Z)) \leq \varepsilon T_g(r, L) + o(T_g(r, L))$ for all $\varepsilon > 0$. Hence we get $\bar{N}_g(r, u(Z)) = o(T_g(r, L))$, in particular we get (11.5). The proof is completed. \square

Proof of Theorem 11.1. — Consider a resolution of singularities $\mu : S \rightarrow \Sigma$ and a projective compactification \bar{S} of S with $E := \bar{S} \setminus S$ a simple normal crossing divisor. We may assume that $\mu : S \rightarrow \Sigma$ extends to $\mu : \bar{S} \rightarrow \bar{\Sigma}$. Let $q : \bar{S} \rightarrow \bar{X}$ be the composite of $\mu : \bar{S} \rightarrow \bar{\Sigma}$ and $p : \bar{\Sigma} \rightarrow \bar{X}$. Set $\omega_i = \mu^* \tau_i$ as an element of $H^0(\bar{S}, q^* \Omega_{\bar{X}}(\log D))$. Set $Z_i := \{s \in \bar{S}; \omega_i(s) = 0\}$ for $i \in \{1, \dots, l\}$. Then

$$(11.6) \quad \mu^{-1}(R) \subset \bigcup_i Z_i$$

as Zariski closed subsets in \bar{S} .

Consider a holomorphic map $f : \mathbb{C}_{>\delta} \rightarrow X$ with Zariski dense image. The generically finite proper surjective morphism $q : S \rightarrow X$ induces a proper surjective holomorphic map $\pi : Y \rightarrow \mathbb{C}_{>\delta}$ from a Riemann surface Y to $\mathbb{C}_{>\delta}$ and a holomorphic map $g : Y \rightarrow S$ satisfying the following commutative diagram:

$$(11.7) \quad \begin{array}{ccc} Y & \xrightarrow{g} & S \\ \downarrow \pi & & \downarrow q \\ \mathbb{C}_{>\delta} & \xrightarrow{f} & X \end{array}$$

Note that π is unramified outside $(\mu \circ g)^{-1}(R)$. By (11.6), it follows that

$$(11.8) \quad \text{supp}(\text{ram } \pi) \subset \bigcup_i g^{-1}(Z_i).$$

We claim

$$(11.9) \quad N_{\text{ram}\pi}(r) = o(T_g(r)) + O(\log r)\|.$$

We prove this. By (11.8), we have

$$N_{\text{ram}\pi}(r) \leq \deg \pi \cdot \overline{N}_g(r, \bigcup_i Z_i).$$

It then suffices to prove that

$$\overline{N}_g(r, Z_i) = o(T_g(r, L)) + O(\log r)\|$$

for every Z_i . By (11.7), we have a morphism of sheaves $g^* q^* \Omega_X \rightarrow \pi^* \Omega_{\mathbb{C}_{>\delta}}$. Since $\omega_i \in H^0(S, q^* \Omega_X)$, we may consider $g^* \omega_i$ as a holomorphic section of $\pi^* \Omega_{\mathbb{C}_{>\delta}}^1$.

Case 1. $g^* \omega_i \neq 0$. Write

$$\xi := \frac{g^* \omega_i}{\pi^* dz},$$

which gives a holomorphic map $\xi : Y \rightarrow \mathbb{P}^1 \setminus \{\infty\}$, since $g^* \omega_i$ is a holomorphic section of $\pi^* \Omega_{\mathbb{C}_{>\delta}}^1$. Hence $N_\xi(r, [\infty]) = 0$. By (4.6) one has

$$N_\xi(r, [0]) = o(T_g(r, L)) + O(\log r)\|.$$

If $g(y) \in Z_i$, then $g^* \omega_i(y) = 0$ as a section of $\pi^* \Omega_{\mathbb{C}_{>\delta}}^1$. Hence $\xi(y) = 0$, therefore

$$\overline{N}_g(r, Z_i) \leq N_\xi(r, [0]).$$

These two estimates yield

$$\overline{N}_g(r, Z_i) = o(T_g(r, L)) + O(\log r)\|.$$

Case 2. $g^* \omega_i = 0$. By the morphism $q^* \Omega_{\bar{X}}(\log D) \rightarrow \Omega_{\bar{S}}(\log E)$, we regard ω_i as a section of $H^0(\bar{S}, \Omega_{\bar{S}}(\log E))$. Then we have $g^* \omega_i = 0$ as sections of Ω_Y . Let $u : Z \rightarrow S$ be a smooth modification of an irreducible component of $Z_i \cap S$. Since $u(Z) \subset Z_i$, the definition of Z_i yields $u^* \omega_i = 0$ in $H^0(Z, \Omega_Z)$. Therefore by Lemma 11.2, we have $\overline{N}_g(r, u(Z)) = o(T_g(r)) + O(\log r)\|$. This shows $\overline{N}_g(r, Z_i) = o(T_g(r, L)) + O(\log r)\|$. Hence we have proved (11.9).

Now we identify \mathbb{D}^* with $\mathbb{C}_{>1}$ by taking a transformation $z \mapsto \frac{1}{z}$. By Theorem A and (11.9), we have an extension $\bar{g} : \bar{Y} \rightarrow \bar{S}$. Thus we have an extension $\bar{f} : \mathbb{D} \rightarrow \bar{X}$. Indeed, given a sequence (a_n) in $\mathbb{C}_{>\delta}$ satisfying $a_n \rightarrow \infty$, we may take a sequence (b_n) in Y such that $p(b_n) = a_n$ and $b_n \rightarrow \beta \in \bar{Y}$. Then $\bar{p}(\beta) = \infty$ and $f(a_n) = \pi \circ g(b_n) \rightarrow \pi \circ \bar{g}(\beta)$. Thus $(f(a_n))$ converges for every sequence (a_n) satisfying $a_n \rightarrow \infty$. Thus f has continuous extension $\bar{f} : \mathbb{C}_{>\delta} \cup \{\infty\} \rightarrow \bar{X}$, which is holomorphic by Riemann's removable singularity theorem. \square

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