

ON SPLITTINGS OF DEFORMATIONS OF PAIRS OF COMPLEX STRUCTURES AND HOLOMORPHIC VECTOR BUNDLES

HISASHI KASUYA AND VALTO PURHO

ABSTRACT. We can show that the Kuranishi space of a pair (M, E) of a compact Kähler manifold M and its flat Hermitian vector bundle E is isomorphic to the direct product of the Kuranishi space of M and the Kuranishi space of E . We study non-Kähler case. We show that the Kuranishi space of a pair (M, E) of a complex parallelizable nilmanifold M and its trivial holomorphic vector bundle E is isomorphic to the direct product of the Kuranishi space of M and the Kuranishi space of E . We give examples of pairs (M, E) of nilmanifolds M with left-invariant abelian complex structures and their trivial holomorphic line bundles E such that the Kuranishi spaces of pairs (M, E) are not isomorphic to direct products of the Kuranishi spaces of M and the Kuranishi spaces of E .

1. INTRODUCTION

Let M be a compact complex manifold. By Kuranishi [12], we can describe the parameter space Kur_M of all sufficiently small deformations of complex structures on M as an analytic germ of an analytic space. Kur_M is called the Kuranishi space of M . By the analogue of Kuranishi's construction for deformations of holomorphic vector bundles over M with a fixed complex structure, we can describe the parameter space Kur_E of deformations of all sufficiently small holomorphic structures on a holomorphic vector bundle E ([6, 7, 11]). Kur_E is also called the Kuranishi space of E .

In [9], Huang studies deformations of a pair (M, E) of a complex manifold M and a holomorphic vector bundle E over M . Extending Kuranishi's construction to a pair (M, E) , we obtain the Kuranishi space $Kur_{(M,E)}$ of a pair (M, E) .

The singularity of the Kuranishi space is considered as obstructions of deformations. We are interested in comparing the singularity of $Kur_{(M,E)}$ with the singularity of Kur_M and Kur_E .

Proposition 1.1. *Let M be a compact Kähler manifold and E be a flat Hermitian vector bundle. Then we have*

$$Kur_{(M,E)} \cong Kur_M \times Kur_E$$

We study deformations of pairs of non-Kähler complex manifolds with trivial holomorphic vector bundles. A nilmanifold is a compact quotient $\Gamma \backslash G$ of a simply connected nilpotent Lie group G by a discrete subgroup Γ . It is known that $\Gamma \backslash G$ admits a Kähler structure if and only if G is abelian and $\Gamma \backslash G$ is a torus ([1, 8]). Thus, if G is non-abelian and admits a left-invariant complex structure, $\Gamma \backslash G$ is a non-Kähler complex manifold.

Theorem 1.2. *Let $M = \Gamma \backslash G$ be a complex parallelizable nilmanifold and E be a trivial holomorphic vector bundle. Then we have*

$$Kur_{(M,E)} \cong Kur_M \times Kur_E$$

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Non-parallelizable case is more complicated, we give examples of complex nilmanifods $M = \Gamma \backslash G$ with the trivial line bundles E such that

$$Kur_{(M,E)} \not\cong Kur_M \times Kur_E.$$

Such examples are nilmanifolds with abelian complex structures. Thus, on nilmanifolds with abelian complex structures, we do not have a splitting

$$Kur_{(M,E)} \not\cong Kur_M \times Kur_E.$$

in general.

Theorem 1.3. *Let $M = \Gamma \backslash G$ be a complex nillmanifold equipped with an abelian complex structure and E be a trivial holomorphic vector bundle. Then if any small deformation of the complex structure is also abelian, then we have*

$$Kur_{(M,E)} \cong Kur_M \times Kur_E.$$

2. KURANISHI SPACES OF DIFFERENTIAL GRADED LIE ALGEBRAS

A differential graded Lie algebra (shortly DGLA) $(L^*, d, [,])$ is a \mathbb{N} -graded vector space L^* equipped with a differential d and a graded Lie bracket $[,]$ satisfying the Leibniz rule. An analytic DGLA is a normed DGLA $(L^*, d, [,])$ whose cohomology $H^i(L^*)$ is finite-dimensional for every $i \in \mathbb{N}$.

The Kuranishi space Kur_L of $(L^*, d, [,])$ is an analytic germ of the analytic space K_L in $H^1(L^*)$ at 0 defined by the following way. Take a Hodge decomposition $L^i = \mathcal{H}^i \oplus dL^{i-1} \oplus A^i$ with $H^i(L^*) \cong \mathcal{H}^i$. Define the map $\delta : L^i \rightarrow L^{i-1}$ by the extension of the inverse of $d : A^{i-1} \rightarrow dL^{i-1}$ associated with the projection $L^i \rightarrow dL^{i-1}$ and the inclusion $A^{i-1} \subset L^i$. Define the map $F : L^1 \ni x \mapsto x + \frac{1}{2}\delta[x, x] \in L^1$. Then, the inverse of F is defined on a small neighbourhood U of \mathcal{H}^1 and define

$$K_L = \{x \in U \mid H([F^{-1}(x), F^{-1}(x)]) = 0\}$$

where $H : L^1 \rightarrow \mathcal{H}^1$ is the projection.

$F^{-1}(x)$ can be written in as the Kuranishi series $F^{-1}(x) = \sum x_i$ such that $x_1 = x$ and

$$x_k = -\frac{1}{2} \sum_{i+j=k} \delta[x_i, x_j]$$

for $k \geq 2$. Considering each x_i as a homogenous polynomial of x ,

$$\sum H([x_i, x_j]) = 0$$

are defining equations of K_L .

For a DGLA homomorphism $\phi : L_1^* \rightarrow L_2^*$ between analytic DGLAs, if ϕ induces isomorphisms on 0-th and first cohomology and an injection on the second cohomology, then we have an isomorphism $Kur_{L_1} \cong Kur_{L_2}$. We say that a morphism $\phi : L_1^* \rightarrow L_2^*$ is a quasi-isomorphism if ϕ induces a cohomology isomorphism. If there is a quasi-isomorphism $\phi : L_1^* \rightarrow L_2^*$, then we have an isomorphism $Kur_{L_1} \cong Kur_{L_2}$.

3. KURANISHI SPACES OF JOINT DEFORMATIONS

Let M be a compact complex manifold and E a holomorphic vector bundle over M equipped with a Hermitian metric h . We know that the graded vector space $A^{0,*}(M, T^{1,0}M)$ equipped with the Dolbeault differential $\bar{\partial}_{T^{1,0}}$ and the Schouten–Nijenhuis bracket $[\cdot, \cdot]_{SN}$ is a DGLA which governs a deformation theory of the complex structure on M . Kuranishi proves that the Kuranishi space Kur_X of the DGLA $A^{0,*}(M, T^{1,0}M)$ is isomorphic to a parameter space of a complete deformation family of a complex manifold M (see [12] and [7]). We also know that graded vector

space $A^{0,*}(M, \text{End}(E))$ equipped with the Dolbeault differential $\bar{\partial}_{\text{End}(E)}$ and the bracket $[\cdot]_{\text{End}(E)}$ induced by the wedge product on differential forms and the Lie bracket on $\text{End}(E)$ is a DGLA which governs a deformation theory of the holomorphic structure on E . As similar to Kur_X , the Kuranishi space Kur_E of the DGLA $A^{0,*}(M, \text{End}(E))$ is isomorphic to a parameter space of a complete deformation family of a complex manifold M (see [11] and [7]).

Let D be the Chern connection associated with h on the holomorphic vector bundle E . Denote by $R \in A^{1,1}(M, \text{End}(E))$ the curvature of D . Consider the graded vector space

$$L^*(M, E) = A^{0,*}(M, T^{1,0}M) \oplus A^{0,*}(M, \text{End}(E)).$$

Define the operator $\bar{\partial}_R : L^*(M, E) \rightarrow L^{*+1}(M, E)$ by

$$\bar{\partial}_R(\alpha, \beta) = (\bar{\partial}_{T^{1,0}M}\alpha, \bar{\partial}_{\text{End}(E)}\beta - \iota_\alpha R)$$

and the bilinear map $[\cdot, \cdot]_D : L^*(M, E) \times L^*(M, E) \rightarrow L^*(M, E)$ by

$$[(\alpha_1, \beta_1), (\alpha_2, \beta_2)]_D = ([\alpha_1, \alpha_2]_{SN}, [\beta_1, \beta_2]_{\text{End}(E)} + \iota_{\alpha_1} D \beta_2 - (-1)^{pq} \iota_{\alpha_2} D \beta_1).$$

Then $(L^*(M, E), \bar{\partial}_R, [\cdot, \cdot]_D)$ is a DGLA. In [9], Huang proves that the Kuranishi space $\text{Kur}_{(M, E)}$ of the DGLA $L^*(M, E)$ is isomorphic to a parameter space of a complete deformation family of a pair (M, E) of a complex manifold M and a holomorphic vector bundle E .

Proposition 3.1. *Let M be a compact Kähler manifold and E be a flat Hermitian vector bundle. Then we have*

$$\text{Kur}_{(M, E)} \cong \text{Kur}_M \times \text{Kur}_E$$

Proof. Since the Chern connection D is flat, we have $\bar{\partial}_R = \bar{\partial}_{T^{1,0}M} \oplus \bar{\partial}_{\text{End}(E)}$. Consider the subset $A^{0,*}(M, T^{1,0}M) \oplus \ker \bar{\partial}_{\text{End}(E)}$ in the DGLA $L^*(M, E)$. Then, for $(\alpha, \beta) \in A^{0,*}(M, T^{1,0}M) \oplus \ker \bar{\partial}_{\text{End}(E)}$, $\iota_\alpha D \beta = 0$. Hence, the subset $A^{0,*}(M, T^{1,0}M) \oplus \ker \bar{\partial}_{\text{End}(E)}$ is a sub-DGLA in the DGLA $L^*(M, E)$ such that $A^{0,*}(M, T^{1,0}M) \oplus \ker \bar{\partial}_{\text{End}(E)}$ is a direct sum of the two DGLAs $A^{0,*}(M, T^{1,0}M)$ and $\ker \bar{\partial}_{\text{End}(E)}$.

As [6], by the $\partial\bar{\partial}$ -Lemma [5] on flat Hermitian vector bundle $\text{End}(E)$, the inclusion

$$\ker \bar{\partial}_{\text{End}(E)} \subset A^{0,*}(M, \text{End}(E))$$

is a quasi-isomorphism and hence Kur_E is isomorphic to the Kuranishi space of the DGLA $\ker \bar{\partial}_{\text{End}(E)}$. By this, the inclusion

$$A^{0,*}(M, T^{1,0}M) \oplus \ker \bar{\partial}_{\text{End}(E)} \subset L^*(M, E)$$

is also a quasi-isomorphism. Thus the Kuranishi space $\text{Kur}_{(M, E)}$ is isomorphic to the Kuranishi space of $A^{0,*}(M, T^{1,0}M) \oplus \ker \bar{\partial}_{\text{End}(E)}$. Hence we have

$$\text{Kur}_{(M, E)} \cong \text{Kur}_M \times \text{Kur}_E.$$

□

Remark 3.2. *By more careful arguments, the similar statement for deformations of pairs of compact Kähler manifolds and polystable Higgs bundles with vanishing Chern classes is proved in [16].*

Corollary 3.3. *Let M be a compact Kähler manifold and E be a flat Hermitian vector bundle. Assume that the canonical line bundle of M is holomorphically trivial. Then $\text{Kur}_{(M, E)}$ is cut out by polynomial equations of degree at most 2.*

Proof. By Proposition 3.1, we have $\text{Kur}_{(M, E)} \cong \text{Kur}_M \times \text{Kur}_E$. By Tian-Todorov theorem [21, 22] (see also [7]), Kur_M is smooth. As shown in [6], the quotient map

$$\ker \bar{\partial}_{\text{End}(E)} \rightarrow H_{\bar{\partial}_{\text{End}(E)}}^{0,*}(M, \text{End}(E))$$

is a quasi-isomorphism. This implies that Kur_E is isomorphic to

$$\left\{ \eta \in H_{\partial_{End(E)}}^{0,*}(M, \text{End}(E)) | [\eta, \eta] = 0 \right\}.$$

Hence, Kur_E is cut out by polynomial equations of degree at most 2. \square

4. NILMANIFOLDS

Let G be a n -dimensional simply connected Lie group with the Lie algebra \mathfrak{g} . A left-invariant complex structure on G can be identified with a complex structure on a real Lie algebra \mathfrak{g} i.e. a sub-algebra $\mathfrak{g}^{1,0}$ of the complexification $\mathfrak{g}_{\mathbb{C}}$ such that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$ where $\mathfrak{g}^{0,1} = \overline{\mathfrak{g}^{1,0}}$. Let $p^{1,0} : \mathfrak{g}_{\mathbb{C}} \rightarrow \mathfrak{g}^{1,0}$ be the projection. Consider $\mathfrak{g}^{1,0}$ as a $\mathfrak{g}^{0,1}$ -module via $p^{1,0}([X, Y])$ for $X \in \mathfrak{g}^{0,1}$, $Y \in \mathfrak{g}^{1,0}$. Then the cochain complex $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0}$ equipped with the bracket induced by the wedge product and the Lie bracket on $\mathfrak{g}^{1,0}$ is a DGLA.

Assume G has a lattice Γ . Consider the complex nilmanifold $M = \Gamma \backslash G$ with the complex structure induced by a left-invariant complex structure on G . We have the inclusions

$$\begin{aligned} \bigwedge(\mathfrak{g}^{0,1})^* \otimes \bigwedge^p(\mathfrak{g}^{1,0})^* &\subset A^{p,*}(M) \\ \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} &\subset A^{0,*}(M, T^{1,0}M). \end{aligned}$$

We say that a left-invariant complex structure is of Calabi-Yau type if $\bigwedge^n \mathfrak{g}^{1,0*}$ is a trivial $\mathfrak{g}^{0,1}$ -module. Assume that a left-invariant complex structure on G is of Calabi-Yau type. If the inclusion $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \bigwedge^p \mathfrak{g}^{1,0*} \subset A^{p,*}(M)$ induces a cohomology isomorphism for any integer p , then the inclusion $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \subset A^{0,*}(M, T^{1,0}M)$ also induces a cohomology isomorphism (see [17]).

Assume that G is nilpotent. In this case any left-invariant complex structure on G is of Calabi-Yau type (see [2]). We call $M = \Gamma \backslash G$ a complex nilmanifold. As an analogous of Nomizu's theorem ([15]) for the de Rham cohomology of nilmanifolds, the inclusion $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \bigwedge^p \mathfrak{g}^{1,0*} \subset A^{p,*}(M)$ induces a cohomology isomorphism for any integer p if M has the structure of an iterated principal holomorphic torus bundle ([3, 18]). If (G, J) is a complex Lie group or $\mathfrak{g}^{1,0}$ is abelian, then M has the structure of an iterated principal holomorphic torus bundle.

A complex nilmanifold $M = \Gamma \backslash G$ is complex parallelizable if and only if (G, J) is a complex Lie group. The complex structure on a complex nilmanifold $M = \Gamma \backslash G$ is called abelian if $\mathfrak{g}^{1,0}$ is abelian.

Theorem 4.1. *Let $M = \Gamma \backslash G$ be a complex parallelizable nillmanifold and E be a trivial holomorphic vector bundle. Then we have*

$$Kur_{(M,E)} \cong Kur_M \times Kur_E.$$

Proof. Take a trivialization $E \cong M \times \mathbb{C}^r$. The inclusions

$$\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \subset A^{0,*}(M, T^{1,0}M)$$

and

$$\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C}) \subset A^{0,*}(M, \text{End}(E))$$

are quasi-isomorphisms and hence Kur_M and Kur_E are isomorphic to the Kuranishi spaces of $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0}$ and $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$ respectively.

Consider the subset $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \oplus \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$ in the DGLA $L^*(M, E)$. We have $d = \bar{\partial}$ on $\bigwedge(\mathfrak{g}^{0,1})^*$. Thus, $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \oplus \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$ is a sub-DGLA in the DGLA $L^*(M, E)$.

such that $\Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \oplus \Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$ is a direct sum of the two DGLAs $\Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0}$ and $\Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$. Since the inclusion

$$\Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \oplus \Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C}) \subset L^*(M, E)$$

is a quasi-isomorphism, the Kuranishi space $Kur_{(M,E)}$ is isomorphic to the Kuranishi space of the direct sum $\Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \oplus \Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$. Hence we have

$$Kur_{(M,E)} \cong Kur_M \times Kur_E.$$

□

Corollary 4.2. *Let $M = \Gamma \backslash G$ be a complex parallelizable nilmanifold and E be a trivial holomorphic vector bundle. Assume that the Lie algebra \mathfrak{g} of G is ν -step and naturally graded. Then $Kur_{(M,E)}$ is cut out by polynomial equations of degree at most $\nu + 1$.*

Proof. By Theorem 4.1, we have $Kur_{(M,E)} \cong Kur_M \times Kur_E$. By [19], Kur_M is cut out by polynomial equations of degree at most ν . By [10], Kur_E is cut out by polynomial equations of degree at most $\nu + 1$. □

Theorem 4.3. *Let $M = \Gamma \backslash G$ be a complex nilmanifold equipped with an abelian complex structure and E be a trivial holomorphic vector bundle. Then if any small deformation of the complex structure is also abelian, then we have*

$$Kur_{(M,E)} \cong Kur_M \times Kur_E.$$

Proof. By the same argument of the proof of Theorem 4.1, Kur_X and Kur_E are isomorphic to the Kuranishi spaces of $\Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0}$ and $\Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$ respectively.

Consider the subset $\Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \oplus \Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$ in the DGLA $L^*(M, E)$. Then $\Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \oplus \Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$ is a sub-DGLA in the DGLA $L^*(M, E)$ such that the inclusion

$$\Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \oplus \Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C}) \subset L^*(M, E)$$

is a quasi-isomorphism. We have $d = \bar{\partial}$ on $\Lambda(\mathfrak{g}^{0,1})^*$. Thus $\Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \oplus \Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$ is not a direct-sum as a DGLA. We study Kuranishi spaces of DGLAs $\Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0}$, $\Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$ and $\Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \oplus \Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$ precisely.

Consider a Kuranishi series $\sum x_i$ of the DGLA $\Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \oplus \Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$. Write $x_i = \varphi_i + \psi_i$ with $\varphi_i \in \Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0}$ and $\psi_i \in \Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$. Then $\sum \varphi_i$ is a Kuranishi series of the DGLA $\Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0}$. [4, Theorem 4, Proposition 1] says that for an abelian complex structure, a Kuranishi series $\sum \varphi_i$ of $\Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0}$ defines an abelian deformation if and only if $\sum \varphi_i = \varphi_1$ and

$$\iota_{\varphi_1} \alpha = 0$$

for any $\alpha \in (\mathfrak{g}^{0,1})^*$. Thus, by the assumption, we have $\sum \varphi_i = \varphi_1$ and $[\varphi_1, (\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0}]_d = 0$. Thus $\sum \psi_i$ is a Kuranishi series of $\Lambda(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$ and defining equations of $Kur_{(M,E)}$ are only the equations of Kur_E . Hence we have we have

$$Kur_{(M,E)} \cong Kur_X \times Kur_E.$$

□

If the Lie algebra \mathfrak{g} of G is isomorphic to one of the following Lie algebras:

$$\mathfrak{n}_3 = (0, 0, 0, 0, 0, 12 + 34), \mathfrak{n}_8 = (0, 0, 0, 0, 0, 12), \mathfrak{n}_9 = (0, 0, 0, 0, 12, 14 + 25),$$

then any left-invariant complex structure on G is abelian ([23, Theorem 8]). Thus in such case, we have

$$Kur_{(M,E)} \cong Kur_M \times Kur_E.$$

by Theorem 4.3.

5. NON-SPLITTING EXAMPLES

We consider real 6-dimensional nilmanifolds $M = \Gamma \backslash G$ with abelian complex structures. Kuranishi spaces Kur_M of them are computed in [13].

Example 5.1. Consider the direct product $G = H_3(\mathbb{R}) \times H_3(\mathbb{R})$ of two copies of the 3-dimensional real Heisenberg group $H_3(\mathbb{R})$. Then, we have $\mathfrak{g} = \langle X_1, X_2, Y_1, Y_2, Z_1, Z_2 \rangle$ such that $[X_1, Y_1] = Z_1, [X_2, Y_2] = Z_2$. Define

$$\mathfrak{g}^{1,0} = \langle W_1 = \frac{1}{2}(X_1 - \sqrt{-1}Y_1), W_2 = \frac{1}{2}(X_2 - \sqrt{-1}Y_2), W_3 = \frac{1}{2}(Z_1 - \sqrt{-1}Z_2) \rangle.$$

Then $\mathfrak{g}^{1,0}$ is abelian. We consider the DGLAs $L_1 = \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0}$, $L_2 = \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_1(\mathbb{C})$ and $L_3 = \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \oplus \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_1(\mathbb{C})$. We see that $Kur_{L_3} \not\cong Kur_{L_1} \times Kur_{L_2}$. We have

$$[\overline{W_1}, W_1] = -\frac{1}{2}\sqrt{-1}(W_3 + \overline{W_3}) \quad \text{and} \quad [\overline{W_2}, W_2] = \frac{1}{2}(W_3 - \overline{W_3}).$$

Thus, $H^1(L_1) = \langle \overline{w_1} \otimes W_1, \overline{w_2} \otimes W_2, \overline{w_3} \otimes W_3, \overline{w_1} \otimes W_2 + \sqrt{-1}\overline{w_2} \otimes W_1 \rangle$. A Kuranishi series of L_1 is given by

$$(t_1\overline{w_1} \otimes W_1 + t_2\overline{w_2} \otimes W_2 + t_3\overline{w_3} \otimes W_3 + t_4(\overline{w_2} \otimes W_1 + \sqrt{-1}\overline{w_1} \otimes W_2) - t_3t_4(\overline{w_1} \otimes W_2 - \sqrt{-1}\overline{w_2} \otimes W_1)).$$

We have $Kur_{L_1} \cong \mathbb{C}^4$. We have $H^1(L_2) \cong H^1(\mathfrak{g}^{0,1}) \cong \langle \overline{w_1}, \overline{w_2}, \overline{w_3} \rangle$ and $Kur_{L_2} \cong \mathbb{C}^3$. On the other hand, for a Kuranishi series

$$(t_1\overline{w_1} \otimes W_1 + t_2\overline{w_2} \otimes W_2 + t_3\overline{w_3} \otimes W_3 + t_4(\overline{w_2} \otimes W_1 + \sqrt{-1}\overline{w_1} \otimes W_2) + t_3t_4(\overline{w_2} \otimes W_1 - \sqrt{-1}\overline{w_1} \otimes W_2), s_1\overline{w_1} + s_2\overline{w_2} + s_3\overline{w_3})$$

of L_3 , we have the defining equation $t_4s_3 = 0$. Thus $Kur_{L_3} \not\cong Kur_{L_1} \times Kur_{L_2}$.

Example 5.2. Consider $G = H_3(\mathbb{C})$ the 3-dimensional complex Heisenberg group. We take a real basis $X_1, X_2, X_3, X_4, Z_1, Z_2$ such that $[X_1, X_3] = -\frac{1}{2}Z_1$, $[X_1, X_4] = [X_2, X_3] = -\frac{1}{2}Z_2$, $[X_1, X_3] = \frac{1}{2}Z_1$. Define

$$\mathfrak{g}^{1,0} = \langle W_1 = X_1 - \sqrt{-1}X_2, W_2 = X_3 + \sqrt{-1}X_4, W_3 = Z_1 + \sqrt{-1}Z_2 \rangle.$$

Then $\mathfrak{g}^{1,0}$ is abelian. We consider the DGLAs $L_1 = \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0}$, $L_2 = \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_1(\mathbb{C})$ and $L_3 = \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \oplus \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_1(\mathbb{C})$. We see that $Kur_{L_3} \not\cong Kur_{L_1} \times Kur_{L_2}$. We have

$$[\overline{W_1}, W_1] = -\sqrt{-1}(W_3 + \overline{W_3}), \quad [\overline{W_2}, W_2] = -(W_3 - \overline{W_3}) \quad \text{and} \quad [\overline{W_1}, W_2] = -W_3.$$

Thus, $H^1(L_1) = \langle \overline{w_1} \otimes W_1, \overline{w_2} \otimes W_1, \overline{w_3} \otimes W_1, \overline{w_1} \otimes W_2, \overline{w_3} \otimes W_2, \overline{w_3} \otimes W_3 \rangle$. A Kuranishi series of L_1 is given by

$$t_1\overline{w_1} \otimes W_1 + t_2\overline{w_2} \otimes W_1 + t_3\overline{w_3} \otimes W_1 + t_4\overline{w_3} \otimes W_2 + t_5\overline{w_3} \otimes W_2 + t_6\overline{w_3} \otimes W_3 + t_1t_6\overline{w_2} \otimes W_2.$$

We have the defining equation $t_3 = 0$ and hence $Kur_{L_1} \cong \mathbb{C}^5$. We have $H^1(L_2) \cong H^1(\mathfrak{g}^{0,1}) \cong \langle \overline{w_1}, \overline{w_2}, \overline{w_3} \rangle$ and $Kur_{L_2} \cong \mathbb{C}^3$. On the other hand, for a Kuranishi series

$$(t_1\overline{w_1} \otimes W_1 + t_2\overline{w_2} \otimes W_1 + t_3\overline{w_3} \otimes W_1 + t_4\overline{w_3} \otimes W_2 + t_5\overline{w_3} \otimes W_2 + t_6\overline{w_3} \otimes W_3 + t_1t_6\overline{w_2} \otimes W_2, s_1\overline{w_1} + s_2\overline{w_2} + s_3\overline{w_3})$$

of L_3 , we have the defining equation $t_1s_3 = 0$. Thus $Kur_{L_3} \not\cong Kur_{L_1} \times Kur_{L_2}$.

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GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY

Email address: `kasuya@math.nagoya-u.ac.jp`

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, OSAKA UNIVERSITY, OSAKA, JAPAN

Email address: `valto.purho@pm.me`