

A HIGHER RANK SHIFTED CONVOLUTION PROBLEM WITH APPLICATIONS TO L -FUNCTIONS

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ABSTRACT. While several instances of shifted convolution problems for $\mathrm{GL}(3) \times \mathrm{GL}(2)$ have been solved, the case where one factor is the classical divisor function and one factor is a $\mathrm{GL}(3)$ Fourier coefficient has remained open. We solve this case in the present paper. The proof involves two intertwined applications of different types of delta symbol methods. As an application we establish an asymptotic formula for central values of L -functions for a $\mathrm{GL}(3)$ automorphic form twisted by Dirichlet characters to moduli $q \leq Q$.

1. INTRODUCTION

1.1. Shifted convolution problems. A shifted convolution problem asks for an asymptotic formula for the product of two (usually) multiplicative arithmetic function whose arguments differ by an additive shift. It is therefore a measure of the correlation of the two functions. The most classical case for the divisor function $\tau = \mathbf{1} * \mathbf{1}$ is

$$\sum_{n \leq x} \tau(n)\tau(n+1),$$

which has been investigated from various points of view for a century. Definitive results exist also in the case when the divisor function is replaced with Fourier coefficients of $\mathrm{GL}(2)$ automorphic forms. In this case, the arithmetic function cannot be opened by a convolution formula, but a delta symbol can be used instead which has roughly the same strength. We recall that the divisor function can be seen as a Fourier coefficients of an Eisenstein series, and both types of arithmetic functions share a structurally similar Voronoi summation formula.

Shifted convolution problems with Fourier coefficients of higher rank automorphic forms, cuspidal or non-cuspidal, turn out to be extremely challenging and non-trivial results are unknown in most cases. Progress has been made in the case when one factor is associated with an automorphic form on $\mathrm{GL}(3)$ and the other is associated with an automorphic form on $\mathrm{GL}(2)$. The most factorable case

$$(1.1) \quad \sum_{n \leq x} \tau_3(n)\tau(n+1),$$

where $\tau_3 = \mathbf{1} * \mathbf{1} * \mathbf{1}$ denotes the ternary divisor function, was first treated by Hooley [Ho] who obtained the main term in the asymptotic formula. The first power saving error term was established by Deshouillers [De] based on the Kuznetsov formula, and the current record

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$O(x^{5/6+\theta/3+\varepsilon})$, $\theta = 7/64$ being an admissible constant towards the Ramanujan conjecture, for the error term of a smooth version of (1.1) is due to Topacogullari [To].

In the case when the divisor function τ in (1.1) is replaced with a $\text{GL}(2)$ Fourier coefficient, i.e.

$$(1.2) \quad \sum_{n \leq x} \tau_3(n) \lambda(n+1),$$

Pitt [Pi1] established a power saving bound for the corresponding shifted convolution problem, which is an important ingredient in his cuspidal version of the Titchmarsh divisor problem [Pi2]. The current record $O(x^{5/6+\theta/3+\varepsilon})$ for a smooth version is due to H. Tang [Ta], using ideas from [To]. On the other hand, when both factors in (1.1) are cuspidal, i.e.

$$(1.3) \quad \sum_{n \leq x} A(n, 1) \lambda(n+1)$$

for a $\text{GL}(3)$ Fourier coefficient $A(n, 1)$ and a $\text{GL}(2)$ Fourier coefficient $\lambda(n)$, Munshi obtained a power saving bound; the current record for a smooth version is $O(x^{21/22+\varepsilon})$ due to P. Xi [Xi].

One may argue that this is the hardest case, since none of the two arithmetic functions can be decomposed as a convolution of simpler functions, but this feature is only one aspect in the analysis. Munshi's proof of (1.3) uses Jutila's very flexible version of the circle method, which is only (directly) applicable if general exponential sums in at least one of the involved arithmetic functions have uniform square-root cancellation. This is not true for the divisor function and not known for $\text{GL}(3)$ Hecke eigenvalues. In particular, the last remaining case

$$(1.4) \quad \sum_{n \leq x} A(n, 1) \tau(n+1)$$

remained open and cannot be attacked by any of the methods used to treat (1.1), (1.2) or (1.3).

In this paper we solve this case, with a more general shift condition and complete uniformity in the bilinear shifting equation.

Theorem 1. *Let $h, \lambda_1, \lambda_2 \in \mathbb{Z} \setminus \{0\}$, $x \geq 1$. Let W, W_0 be smooth functions with compact support in $[1, 2]$. Let $A(n, 1)$ denote the Hecke eigenvalues of a cusp form F for the group $\text{SL}_3(\mathbb{Z})$. Then*

$$\sum_{\lambda_1 m - \lambda_2 n = h} A(n, 1) \tau(m) W_0\left(\frac{|\lambda_1|m}{x}\right) W\left(\frac{|\lambda_2|n}{x}\right) \ll_{F, W, W_0, \varepsilon} x^{41/42+\varepsilon}$$

for any $\varepsilon > 0$, uniformly in h, λ_1, λ_2 .

While the result is uniform in λ_1, λ_2 , we think of these coefficients as essentially fixed. If necessary, one can obtain additional small savings in λ_1, λ_2 (since the summation range becomes shorter), but we did not pursue this further.

With applications in mind, we also prove a slightly more flexible variation. For $A, B \geq 1$ and two functions v_1, v_2 (suppressed from the notation) let

$$\tau_{A,B}(n) := \sum_{ab=n} v_1\left(\frac{a}{A}\right) v_2\left(\frac{b}{B}\right).$$

Theorem 2. Let $h, \lambda_1, \lambda_2 \in \mathbb{Z} \setminus \{0\}$ and $x, A, B \geq 1$ such that $AB \asymp x/|\lambda_1|$. Let W, v_1, v_2 be smooth functions with compact support in $[1, 2]$. Let $A(n, 1)$ denote the Hecke eigenvalues of a cusp form F for the group $\mathrm{SL}_3(\mathbb{Z})$. Then

$$\sum_{\lambda_1 m - \lambda_2 n = h} A(n, 1) \tau_{A, B}(m) W\left(\frac{|\lambda_2|n}{x}\right) \ll_{F, W, v_1, v_2, \varepsilon} x^{41/42 + \varepsilon}$$

for any $\varepsilon > 0$, uniformly in $A, B, h, \lambda_1, \lambda_2$.

The proofs of Theorems 1 and 2 combine for the first time two different delta symbol methods – Jutila’s method and a modern version of the Kloosterman method – that are applied in an intertwined fashion. Jutila’s method gives the flexibility to choose moduli in a way that creates a bilinear structure, but it only approximates a delta-function in an L^2 -sense. On the other hand, exponential sums with divisor functions behave badly in an L^2 -sense, since they become very large on major arcs. Thus we invoke a second circle method to have a tool that is sensitive to the behaviour of these exponential sums. On the major arcs, the key observation is that an extra Kloosterman refinement is possible, i.e. a non-trivial (and in fact square-root saving) estimate over the fractions b/c for b modulo c . That this is possible is not obvious a priori, but depends on the interplay of the two circle methods. We import Munshi’s idea [Mu] to choose the moduli in Jutila’s method in a factorable way to create a bilinear structure. However, our arrangement of Poisson, Voronoi and Cauchy–Schwarz steps differs from all other previous treatments of $\mathrm{GL}(3) \times \mathrm{GL}(2)$ shifted convolution sums.

1.2. An application. That the problem (1.4) is not an artificial construction may be supported by the following application that establishes an asymptotic formula for a twisted moment of L -functions on $\mathrm{GL}(3)$.

Theorem 3. Let $Q \geq 1$ and let F be a cusp form for the group $\mathrm{SL}_3(\mathbb{Z})$. Let W be a smooth function with compact support in $[1, 2]$ and Mellin transform \widetilde{W} . Then

$$\sum_q W\left(\frac{q}{Q}\right) \sum_{\substack{\chi \pmod{q} \\ \chi \text{ primitive, even}}} L(1/2, F \times \chi) = \frac{\widetilde{W}(2)}{2\zeta(2)^2} Q^2 + O_{F, W, \varepsilon}(Q^{2-1/41+\varepsilon})$$

for any $\varepsilon > 0$.

A similar formula can be obtained when averaging over odd primitive characters. Theorem 3 features a moment containing roughly Q^2 terms for an L -function of conductor roughly Q^3 . Nevertheless, up until now, only a *lower* bound was available [Lu] which was a hard-earned result and is now over 20 years old. The connection of this moment to (1.4) comes from a divisor-switching trick. We take an unbalanced approximate functional equation, where the first term has length $Q^{2+\delta}$ and the root number term has length $Q^{1-\delta}$ for some very small $\delta > 0$. Then the root number term can be estimated trivially (and as long as we cannot average non-trivially hyper-Kloosterman sums over the modulus, we don’t have any better tools available). Applying orthogonality of characters, we are left with

$$\sum_{q \asymp Q} \sum_{\substack{n \asymp Q^{2+\delta} \\ n \equiv 1 \pmod{q}}} A(n, 1) \approx \sum_{q \asymp Q} \sum_{r \asymp Q^{1+\delta}} A(1 + rq, 1)$$

and the connection to (1.4) becomes clear.

As an aside we remark that also Luo's result [Lu] used crucially the idea of factorable moduli, and that Theorems 1, 2 and 3 become relatively straightforward if non-trivial averages of hyper-Kloosterman sums over the modulus were available.

2. PREPARATION

We will generally use the following standard conventions: the value of ε can change from line to line (any typically picks up divisor functions, logarithms etc. on the way), and we write $a \mid b^\infty$ to mean that all prime divisors of a divide b . Similarly, $(a, b^\infty) = \lim_{n \rightarrow \infty} (a, b^n)$.

2.1. Delta symbol methods. In this subsection we present two delta symbol methods. The first one is a very flexible method due to Jutila [Ju]. It gives, however, only an approximation to the constant function in an L^2 -sense.

Lemma 1. *Let $Q \geq 1$, $\omega : [1, Q] \rightarrow [0, \infty)$, $L = \sum_q \phi(q)\omega(q)$ such that $L \neq 0$. Let $\psi : [-1, 1] \rightarrow [0, 1]$ be a smooth function with $\int \psi = 1$ and $0 < \delta < 1/2$. For $\alpha \in \mathbb{R}$ define the 1-periodic function*

$$\chi(\alpha) = \frac{1}{\delta L} \sum_q \omega(q) \sum_{\substack{a \pmod{q} \\ (a, q) = 1}} \sum_{k \in \mathbb{Z}} \psi\left(\frac{1}{\delta}\left(\alpha - \frac{a}{q} + k\right)\right).$$

Then

$$\int_0^1 (1 - \chi(\alpha))^2 d\alpha \ll_\psi \frac{Q^2 \|w\|_\infty |\log \delta|^3}{L^2 \delta}.$$

Indeed, the ℓ -th Fourier coefficient of χ equals

$$(2.1) \quad \frac{1}{L} \sum_q \omega(q) r_q(\ell) \hat{\psi}(\delta \ell) \begin{cases} = 1, & \ell = 0, \\ \ll_\psi \frac{Q}{L} (1 + \delta|\ell|)^{-10} \tau(|\ell|) \|\omega\|_\infty, & \ell \neq 0, \end{cases}$$

where $\hat{\psi}$ denotes the Fourier transform and $r_q(\ell)$ the Ramanujan sum. The claim follows easily from Parseval.

We derive the following useful representation for $\alpha = \frac{b}{c} + z$ where $b, c \in \mathbb{Z}$, $c \neq 0$ and $z \in \mathbb{R}$. Opening the Ramanujan sum and applying Poisson summation we have

$$(2.2) \quad \begin{aligned} \chi\left(\frac{b}{c} + z\right) &= \frac{1}{L} \sum_q \omega(q) \sum_{d|q} d \mu\left(\frac{q}{d}\right) \sum_\ell \hat{\psi}(\delta d \ell) e\left(-\left(\frac{b}{c} + z\right) \ell\right) \\ &= \frac{1}{\delta L} \sum_q \omega(q) \sum_{d|q} \mu\left(\frac{q}{d}\right) \sum_{\ell \equiv bd \pmod{c}} \psi\left(\frac{1}{\delta}\left(\frac{\ell}{cd} + z\right)\right). \end{aligned}$$

The second delta symbol is a version of a Kloosterman refinement of the circle method in the style of Heath-Brown [HB, Section 3].

Lemma 2. *Let $C \geq 1$ and $n \in \mathbb{Z}$. Then*

$$\delta_{n=0} = \sum_{c \leq C} \sum_{\substack{b \pmod{c} \\ (b, c) = 1}} \int_{-\frac{1}{c(c+c')}}^{\frac{1}{c(c+c')}} e\left(\left(\frac{b}{c} + z\right)n\right) dz$$

where $\frac{b'}{c'}, \frac{b}{c}, \frac{b''}{c''}$ are consecutive Farey fractions of level C . We have

$$(2.3) \quad \int_{-\frac{1}{c(c+c'')}}^{\frac{1}{c(c+c'')}} e\left(\left(\frac{b}{c} + z\right)n\right) dz = \int_{-1/cC}^{1/cC} \frac{1}{c} \sum_{u \pmod{c}} \sum_{t \in I(c,z)} e\left(\frac{ut}{c}\right) e\left(\frac{ub}{c}\right) e\left(\left(\frac{b}{c} + z\right)n\right) dz$$

where

$$(2.4) \quad I(c, z) = \left(C - c, \max\left(\frac{1}{c|z|} - c, C\right) \right].$$

Proof. We decompose the interval $[0, 1]$ using Farey fractions of level C , so that

$$\int_0^1 e(\alpha n) d\alpha = \sum_{c \leqslant C} \sum_{\substack{b \pmod{c} \\ (b,c)=1}} \int_{\frac{b+b''}{c+c''}-\frac{b}{c}}^{\frac{b+b''}{c+c''}-\frac{b}{c}} e\left(\left(\frac{b}{c} + z\right)n\right) dz,$$

where $\frac{b'}{c'}, \frac{b}{c}, \frac{b''}{c''}$ are consecutive Farey fractions. Since $b''c - bc'' = bc' - b'c = 1$, the integral runs over the interval $[-(c(c+c'))^{-1}, (c(c+c''))^{-1}]$ and we obtain the first equality.

From the conditions $c' \equiv -c'' \equiv \bar{b} \pmod{c}$ and $C - c < c', c'' \leqslant C$, we see that there is a unique pair (c', c'') which determines $\bar{b} \pmod{c}$. Since $c + c', c + c'' > C$, we can write

$$\int_{-\frac{1}{c(c+c'')}}^{\frac{1}{c(c+c'')}} e\left(\left(\frac{b}{c} + z\right)n\right) dz = \int_{-\frac{1}{cC}}^{\frac{1}{cC}} \mathbf{1}_{\bar{b} \equiv t \pmod{c} \text{ for some } t \in I(c,z)} e\left(\left(\frac{b}{c} + z\right)n\right) dz$$

where $I(c, z)$ is as in (2.4). Detecting the congruence $\bar{b} \equiv t \pmod{c}$ with additive characters gives the second equality. \square

With the same notation we conclude for a smooth, one-periodic function f (by decomposing into its Fourier series) that

$$(2.5) \quad \int_0^1 f(z) dz = \sum_{c \leqslant C} \int_{-1/cC}^{1/cC} \frac{1}{c} \sum_{u \pmod{c}} \sum_{t \in I(c,z)} e\left(\frac{ut}{c}\right) \sum_{\substack{b \pmod{c} \\ (b,c)=1}} e\left(\frac{ub}{c}\right) f\left(\frac{b}{c} + z\right) dz.$$

2.2. Voronoi summation. The following two Voronoi summation formulae are well-known.

Lemma 3. *Let $c \in \mathbb{N}$, $b \in \mathbb{Z}$, $(b,c) = 1$, w a smooth function with compact support in $(0, \infty)$. Then*

$$\sum_n \tau(n) e\left(\frac{b}{c}n\right) w(n) = \frac{1}{c} \int_0^\infty w(\xi) \left(\log \frac{\xi}{c^2} + 2\gamma \right) d\xi + \frac{1}{c} \sum_{\pm} \sum_n \tau(n) e\left(\pm \frac{\bar{b}}{c}n\right) \int_0^\infty w(\xi) J^\pm\left(\frac{\sqrt{n}\xi}{c}\right) d\xi$$

where $J^-(\xi) = \sum_{\pm} e(\pm 2\xi) v_{\pm}(\xi)$, $J^+(\xi) = v_0(\xi)$ with

$$\xi^j v_{\pm}^{(j)}(\xi) \ll_j \frac{1 + |\log \xi|}{1 + \xi^{1/2}}, \quad v_0(\xi) \ll_A \frac{1 + |\log \xi|}{1 + \xi^A}$$

for any $j, A \geqslant 0$.

For future reference, we analyze the integral transform in the case in the following special case.

Lemma 4. Let $X \geq 1$, $|Z| \leq 1$ and W be a fixed smooth function with support in $[1, 2]$. Let $n, c \in \mathbb{N}$. For

$$w(\xi) = w_{X,Z}(\xi) = W\left(\frac{\xi}{X}\right)e(\xi Z)$$

we have

$$\begin{aligned} \int_0^\infty w(\xi) J^-\left(\frac{\sqrt{n\xi}}{c}\right) d\xi &\ll_{A,\varepsilon} \frac{X(Xnc)^\varepsilon}{1+X|Z|} \left(1 + \frac{nX}{c^2(1+X|Z|)^2}\right)^{-A}, \\ \int_0^\infty w(\xi) J^+\left(\frac{\sqrt{n\xi}}{c}\right) d\xi &\ll_{A,\varepsilon} X(Xnc)^\varepsilon \left(1 + \frac{nX}{c^2}\right)^{-A} \end{aligned}$$

for all $\varepsilon, A > 0$.

Proof. Put $P = Xnc$. The second bound follows by trivial estimation. As long as $|Z| \leq P^\varepsilon/x$, the first bound follows by a simple integration by parts argument.

Let us now consider the first bound when $|Z| \geq P^\varepsilon/X$. In this case we are looking at

$$\int_0^\infty W\left(\frac{\xi}{X}\right) v_\pm\left(\frac{\sqrt{n\xi}}{c}\right) e\left(\xi Z \pm \frac{2\sqrt{n\xi}}{c}\right) d\xi.$$

There is at most one stationary point at $\xi = n(cZ)^{-2}$. If $X \not\asymp n(cZ)^{-2}$ we apply integration by parts in the form of [BKY, Lemma 8.1] with

$$U = Q = (\beta - \alpha) = X, \quad R = |Z| + \frac{n^{1/2}}{X^{1/2}c}, \quad X = \frac{1 + |\log \frac{nX}{c^2}|}{1 + (nXc^{-2})^{1/4}}, \quad Y = \frac{\sqrt{nX}}{c}$$

to bound the integral by

$$\begin{aligned} &\ll_A x \frac{1 + |\log \frac{nX}{c^2}|}{1 + (nXc^{-2})^{1/4}} \left[\left(\left(X|Z| + \frac{\sqrt{nX}}{c} \right) \left(\frac{\sqrt{nX}}{c} \right)^{-1/2} \right)^{-A} + \left(X|Z| + \frac{\sqrt{nX}}{c} \right)^{-A} \right] \\ &\ll_A X \frac{(Xnc)^\varepsilon}{1 + (nXc^{-2})^{1/4}} \left(X|Z| + \frac{\sqrt{nX}}{c} \right)^{-A/2} \end{aligned}$$

which is stronger than claimed (after changing the constant A).

Assume now $x \asymp n(cz)^{-2}$ in which case the target bound is $P^\varepsilon|Z|^{-1}$. Put

$$V = V_1 = Q = X, \quad Y = \frac{\sqrt{nX}}{c}, \quad X = \frac{1 + |\log \frac{nX}{c^2}|}{1 + (nXc^{-2})^{1/4}}.$$

Since $Y \asymp X|Z| \gg P^\varepsilon$, we can apply [BKY, Proposition 8.2] to obtain the bound

$$\ll \frac{X}{\sqrt{YQ^{-2}}} \ll P^\varepsilon|Z|^{-1}.$$

This completes the proof. \square

Lemma 5. Let $c \in \mathbb{N}$, $b \in \mathbb{Z}$, $(b, c) = 1$, w a smooth function with compact support in $(0, \infty)$. Then

$$\sum_n A(n, 1) e\left(\frac{b}{c}n\right) w(n) = \frac{1}{c^2} \sum_{\pm} \sum_{n_2} \sum_{n_1|c} n_1 A(n_1, n_2) S\left(\bar{b}, \pm n_2, \frac{c}{n_1}\right) \int_0^\infty w(y) V^\pm\left(\frac{n_1^2 n_2 y}{c^3}\right) dy$$

where

$$V^\pm(\xi) = R(\xi) \frac{e(\pm 3\xi^{1/3})}{\xi^{1/3}} + \frac{S(\xi)}{\xi^{1/2}}$$

with

$$\xi^k \frac{d^k}{d\xi^k} R(\xi) \ll_k \mathbf{1}_{\xi \gg 1}, \quad \xi^k \frac{d^k}{d\xi^k} S(\xi) \ll_k \mathbf{1}_{\xi \ll 1}.$$

Indeed, the function V^\pm is the inverse Mellin transform of

$$G^\pm(s) = \frac{\pi^{3/2-3s}}{2} \left(\prod_{j=1}^3 \frac{\Gamma(\frac{s+\alpha_j}{2})}{\Gamma(\frac{1-s-\alpha_j}{2})} \pm \frac{1}{i} \prod_{j=1}^3 \frac{\Gamma(\frac{s+1+\alpha_j}{2})}{\Gamma(\frac{2-s-\alpha_j}{2})} \right),$$

where $\{\alpha_1, \alpha_2, \alpha_3\}$ is the Langlands parameter of the underlying cusp form F . The bound for $\xi \ll 1$ follows from shifting the contour to the left using that $\max_j |\Re \alpha_j| < 1/2$, while the bound for $\xi \gg 1$ follows from [Bl, Lemma 6].

As an analogue of Lemma 3 we state the following structurally similar formula for a convolution, which follows easily from two applications of Poisson summation.

Lemma 6. *Let w be a smooth function with compact support in $(0, \infty)^2$, $c \in \mathbb{N}$ and $b \in \mathbb{Z}$ with $(b, c) = 1$. Then*

$$\sum_{r,q} w(r, q) e\left(\frac{rqb}{c}\right) = \frac{1}{c} \sum_{r, q \in \mathbb{Z}} e\left(-\frac{rq\bar{b}}{c}\right) \int_{\mathbb{R}^2} w(x, y) e\left(\frac{xr + yq}{c}\right) dx dy.$$

2.3. Bounds for Hecke eigenvalues. We will frequently use the following bounds which follow from the Hecke relation $A(n_1, n_2) = \sum_{d|(n_1, n_2)} \mu(d) A(n_1/d, 1) A(1, n_2/d)$, Rankin-Selberg theory and trivial bounds towards the Ramanujan conjecture: we have

$$(2.6) \quad \sum_{n \asymp X} |A(n, m)| \ll_\varepsilon Xm^{1/2}(Xm)^\varepsilon, \quad \sum_{n \asymp X} \sum_{m \asymp Y} |A(n, m)| \ll_\varepsilon (XY)^{1+\varepsilon}$$

for $X, Y \geq 1$, $\varepsilon > 0$.

2.4. Character sums. Both the delta symbol methods and the Voronoi summation formulae create various character sums for which it is important to have best possible bounds – at least in generic cases – uniformly in several auxiliary parameters.

For $c \in \mathbb{N}$, $n_1 \mid c$, $h, d, n_2 \in \mathbb{Z}$ we define the character sum

$$(2.7) \quad \Sigma_{h,d,n_1,n_2}(c) := \sum_{\substack{b \pmod{c} \\ (b,c)=1}} e\left(\frac{bh + \bar{b}d}{c}\right) S\left(\bar{b}, n_2, \frac{c}{n_1}\right).$$

Lemma 7. *Let $n_1 \mid c, h, d, n_2 \in \mathbb{Z}$. We decompose uniquely $c = c_1 c_2$ with c_1 squarefree, c_2 squarefull and $(c_1, c_2) = 1$. Then*

$$\Sigma_{h,d,n_1,n_2}(c) \ll_\varepsilon c^{1+\varepsilon} c_2^{1/2} \frac{\sqrt{(n_1, c_1, d, h)}}{\sqrt{n_1}} \leq c^{1+\varepsilon} c_2^{1/2}$$

for any $\varepsilon > 0$.

Remark 1. With more work it should be possible to remove the factor $c_2^{1/2}$, at least in typical cases. The present bound suffices for our purposes.

Proof. We first consider the case when $c = p$ is prime. If $p \mid n_1$, then the sum becomes a Kloosterman sum and

$$\Sigma_{h,d,n_1,n_2}(p) = \sum_{\substack{b \pmod{p} \\ (b,p)=1}} e\left(\frac{bh + \bar{b}d}{p}\right) \ll \sqrt{p} \sqrt{(p, h, d)}.$$

If $p \nmid n_1$ and $p \mid n_2$, then

$$\Sigma_{h,d,n_1,n_2}(p) = \sum_{\substack{b,x \pmod{c} \\ (bx,c)=1}} e\left(\frac{hb + \bar{b}d + \bar{b}xn_1}{c}\right) \ll \sqrt{p} \sqrt{(p, h, d)}.$$

If $p \nmid n_1 n_2$, $p \mid h$, then

$$\Sigma_{h,d,n_1,n_2}(p) = \sum_{\substack{b,x \pmod{p} \\ (bx,p)=1}} e\left(\frac{\bar{b}d + \bar{b}n_1x + n_2n_1\bar{x}}{p}\right) \ll \sum_{\substack{x \pmod{p} \\ (x,p)=1}} (d + n_1x, p) \ll p.$$

If $p \nmid n_1 n_2 h$, then

$$\Sigma_{h,d,n_1,n_2}(p) = \sum_{\substack{b,x \pmod{p} \\ (bx,p)=1}} e\left(\frac{hb + \bar{b}(d + n_1x) + n_2n_1\bar{x}}{p}\right) = \sum_{\substack{b,x \pmod{p} \\ (bx,p)=1}} e\left(\frac{b + \bar{b}h(d + xn_1^2 n_2) + \bar{x}}{p}\right) \ll p$$

by the bounds of Adolphson–Sperber [AS] with the Newton polygon $\{(1, 0), (-1, 0), (-1, 1), (0, -1)\}$ if $p \nmid d$, and by Deligne’s bound for hyper-Kloosterman sums if $p \mid d$. The desired bound (without the factor $c_2^{1/2}$) follows now from the Chinese remainder theorem if c is squarefree.

On the other hand, we always have

$$\Sigma_{h,d,n_1,n_2}(c) \ll \sum_{\substack{b \pmod{c} \\ (b,c)=1}} \left| S\left(\bar{b}, n_2, \frac{c}{n_1}\right) \right| \ll c \sqrt{\frac{c}{n_1}}$$

which again by the Chinese remainder theorem concludes the proof in all cases. \square

We will also need to estimate the character sum

$$(2.8) \quad \mathcal{T}(h, d_1, d_2, n_1, n_2, p_1, p_2, t) := \sum_{x \pmod{[p_1, p_2]t}} e\left(\frac{xn_2}{[p_1, p_2]t}\right) \Sigma_{h,d_1,n_1,x}(p_1t) \overline{\Sigma_{h,d_2,n_1,x}(p_2t)}$$

for two primes p_1, p_2 and $n_1 \mid t$. The precedent of the following lemma is [Mu, Lemma 10 & 11] which however requires some corrections as noted in [Xi]. Our version needs more refined bounds.

Lemma 8. *Let p_1, p_2 be two primes, $n_1 \mid t = t_1 t_2$ with t_1 squarefree, t_2 squarefull, $(t_1, t_2) = 1$. Assume $(hn_1, p_1 p_2) = 1$ and let $\varepsilon > 0$.*

If $p_1 \neq p_2$, then \mathcal{T} vanishes unless $(n_2, p_1 p_2) = 1$ in which case we have

$$\mathcal{T}(h, d_1, d_2, n_1, n_2, p_1, p_2, t) \ll_\varepsilon p_1^{3/2} p_2^{3/2} t^{5/2+\varepsilon} (hn_2, t_1)^{1/2} t_2^{1/2}.$$

If $p_1 = p_2$, then we have

$$(2.9) \quad \mathcal{T}(h, d_1, d_2, n_1, n_2, p_1, p_1, t) \ll_\varepsilon p_1^3 t^{5/2+\varepsilon} (hn_2 p_1, t_1)^{1/2} t_2^{1/2}.$$

If $p_1 = p_2$ and $n_2 = 0$, then we have an improved bound

$$\mathcal{T}(h, d_1, d_2, n_1, n_2, p_1, p_1, t) \ll_{\varepsilon} p_1^2 t^{3+\varepsilon} (d_1 - d_2, p_1).$$

Proof. Consider the case $p_1 \neq p_2$ first. We write $t = g_1 g_2 \tau_1 \tau_2$ where $g_1 \mid p_1^\infty$, $g_2 \mid p_2^\infty$, $(\tau_1 \tau_2, p_1 p_2) = 1$, τ_1 squarefree and τ_2 is squarefull with $(\tau_1, \tau_2) = 1$. Then \mathcal{T} factors into a product of exponential sums modulo $p_1 g_1, p_2 g_2, \tau_1, \tau_2$. If $g \mid p_1 p_2 t$, we will generally use the notation $g' = p_1 p_2 t / g$ for the co-divisor.

The sum modulo $p_1 g_1$ is given by

$$\sum_{x \pmod{p_1 g_1}} e\left(\frac{x n_2 \overline{(p_1 g_1)'}}{p_1 g_1}\right) \sum_{\substack{b \pmod{p_1 g_1} \\ (b, p_1)=1}} e\left(\frac{(bh + \bar{b}d_1)\overline{(p_1 g_1)'p_2}}{p_1 g_1}\right) S(\bar{b}, x n_1^2 \overline{(p_1 g_1)'}^2 p_2^2, p_1 g_1).$$

If $p_1 \mid n_2$, the sum over x vanishes since $(n_1, p_1) = 1$. Otherwise, we open the Kloosterman sum, sum over x and obtain

$$p_1 g_1 \sum_{\substack{b \pmod{p_1 g_1} \\ (b, p_1)=1}} e\left(\frac{(bh + \bar{b}d_1)\overline{(p_1 g_1)'p_2} - \overline{b n_2} n_1^2 p_2^2 \overline{(p_1 g_1)'}}{p_1 g_1}\right) \ll (p_1 g_1)^{3/2}$$

since $(h, p_1) = 1$ and $g_1 \mid p_1^\infty$.

The sum modulo $p_2 g_2$ can be estimated in the same way.

Recall that $n_1 \mid t$ and $(n_1, p_1 p_2) = 1$ and decompose $n_1 = n_{11} n_{12}$ with $n_{11} \mid \tau_1$ and $n_{12} \mid \tau_2$. Then the exponential sum modulo τ_1 becomes

$$(2.10) \quad \begin{aligned} & \sum_{x \pmod{\tau_1}} e\left(\frac{x n_2 \overline{\tau_1'}}{\tau_1}\right) \sum_{\substack{b_1 \pmod{\tau_1} \\ (b_1, \tau_1)=1}} e\left(\frac{(b_1 h + \bar{b}_1 d_1) \overline{\tau_1'} p_2}{\tau_1}\right) S\left(\overline{b_1}, x \overline{\tau_1'}^2 n_{12}^2 p_2^2, \frac{\tau_1}{n_{11}}\right) \\ & \times \sum_{\substack{b_2 \pmod{\tau_1} \\ (b_2, \tau_1)=1}} e\left(\frac{-(b_2 h + \bar{b}_2 d_2) \overline{\tau_1'} p_1}{\tau_1}\right) S\left(\overline{b_2}, x \overline{\tau_1'}^2 n_{12}^2 p_1^2, \frac{\tau_1}{n_{11}}\right). \end{aligned}$$

Recall that τ_1 is squarefree. We apply the Chinese remainder theorem and consider the sum in (2.10) modulo a prime $p \mid \tau_1$.

If $p \mid n_{11}$, then the exponential sum in (2.10) becomes

$$\sum_{x \pmod{p}} e\left(\frac{x n_2 \overline{p'}}{p}\right) \sum_{\substack{b_1 \pmod{p} \\ (b_1, p)=1}} e\left(\frac{(b_1 h + \bar{b}_1 d_1) \overline{p'} p_2}{p}\right) \sum_{\substack{b_2 \pmod{p} \\ (b_2, p)=1}} e\left(\frac{-(b_2 h + \bar{b}_2 d_2) \overline{p'} p_1}{p}\right) \ll p^3 \mathbf{1}_{p \mid n_2}$$

which satisfies the crude bound $p^{5/2}(p, n_2)^{1/2}$.

If $p \nmid n_{11}$, then $n_1 = n_{12}$ and the exponential sum in (2.10) becomes

$$\begin{aligned} & \sum_{x \pmod{p}} e\left(\frac{x n_2 \overline{p'}}{p}\right) \sum_{\substack{b_1 \pmod{p} \\ (b_1, p)=1}} e\left(\frac{(b_1 h + \bar{b}_1 d_1) \overline{p'} p_2}{p}\right) S(\overline{b_1}, x \overline{p'}^2 n_1^2 p_2^2, p) \\ & \times \sum_{\substack{b_2 \pmod{p} \\ (b_2, p)=1}} e\left(\frac{-(b_2 h + \bar{b}_2 d_2) \overline{p'} p_1}{p}\right) S(\overline{b_2}, x \overline{p'}^2 n_1^2 p_1^2, p). \end{aligned}$$

Opening the Kloosterman sums and summing over x , we obtain

$$(2.11) \quad \sum_{\substack{y_1, y_2 \pmod{p} \\ (y_1 y_2, p) = 1 \\ p' n_2 + (y_1 p_2^2 + y_2 p_1^2) n_1^2 \equiv 0 \pmod{p}}} p \sum_{\substack{b_1 \pmod{p} \\ (b_1, p) = 1}} e\left(\frac{(b_1 h + \bar{b}_1 d_1) \bar{p}' p_2 + \bar{b}_1 y_1}{p}\right) \\ \times \sum_{\substack{b_2 \pmod{p} \\ (b_2, p) = 1}} e\left(\frac{-(b_2 h + \bar{b}_2 d_2) \bar{p}' p_1 + \bar{b}_2 y_2}{p}\right).$$

If $p \mid h$, then (2.11) becomes

$$\sum_{\substack{y_1, y_2 \pmod{p} \\ (y_1 y_2, p) = 1 \\ p' n_2 + (y_1 p_2^2 + y_2 p_1^2) n_1^2 \equiv 0 \pmod{p}}} p \sum_{\substack{b_1 \pmod{p} \\ (b_1, p) = 1}} e\left(\frac{\bar{b}_1 d_1 \bar{p}' p_2 + \bar{b}_1 y_1}{p}\right) \sum_{\substack{b_2 \pmod{p} \\ (b_2, p) = 1}} e\left(\frac{-\bar{b}_2 d_2 \bar{p}' p_1 + \bar{b}_2 y_2}{p}\right) \\ \ll p \sum_{\substack{y_1, y_2 \pmod{p} \\ (y_1 y_2, p) = 1}} (y_1 d_1 p_2 + p', p) (y_2 d_2 p_1 + p', p) \ll p^3 = p^{5/2} (p, h)^{1/2}.$$

If $p \nmid h$, we change variables and write $y_1 = x$, $b_1 = y p' \bar{h} p_2$, $b_2 = -z p' \bar{h} p_1$. Then (2.11) becomes

$$(2.12) \quad p \sum_{\substack{x, y, z \pmod{p} \\ (xyz, p) = 1 \\ (n_1^2 p_1^2 x + p' n_2, p) = 1}} e\left(y + \frac{b(x)}{y} + z + \frac{c(x)}{z}\right)$$

where

$$b(x) = \frac{h p_2}{p'^2} \left(\frac{p'}{x} + d_1 p_2 \right), \quad c(x) = \frac{h p_1^2}{p'^2} \left(\frac{p' p_1 n_1^2}{n_1^2 p_2^2 x + p' n_2} + d_2 \right).$$

We have

$$\frac{c(x)}{b(x)} = \frac{d_2 n_2 p_1^2 p' x + n_1^2 p_1^3 p' x + d_2 n_1^2 p_1^2 p_2^2 x^2}{n_2 p_2 (p')^2 + d_1 n_2 p_2^2 p' x + n_1^2 p_2^3 p' x + d_1 n_1^2 p_2^4 x^2}$$

which is the constant one rational function if and only if

$$p \mid p_1^3 - p_2^3, p \mid d_2 p_1^2 - d_1 p_2^2, p \mid n_2$$

by our current assumption $p \nmid h n_1$. In particular, by Bombieri–Sperber [BS, Theorem 4] we conclude that (2.12) is $\ll p^{5/2}$ if $p \nmid n_2$. Otherwise, we use Weil’s bound for the y, z -sums and bound the x -sum trivially getting the estimate $\ll p^3$ for (2.12). We summarize the preceding discussion as

$$\sum_{x \pmod{p}} (\dots) \ll p^{5/2} (p, h n_2)^{1/2}$$

in all cases for $p \mid \tau_1$.

Finally, for the exponential sum modulo τ_2 we have

$$(2.13) \quad \sum_{x \pmod{\tau_2}} e\left(\frac{x n_2 \bar{\tau}_2'}{\tau_2}\right) \sum_{\substack{b_1 \pmod{\tau_2} \\ (b_1, \tau_2)=1}} e\left(\frac{(b_1 h + \bar{b}_1 d_1) \bar{\tau}_2' p_2}{\tau_2}\right) S\left(\bar{b}_1 n_{11}, x \bar{\tau}_2'^2 n_{11} p_2^2, \frac{\tau_2}{n_{12}}\right) \\ \times \sum_{\substack{b_2 \pmod{\tau_2} \\ (b_2, \tau_2)=1}} e\left(\frac{-(b_2 h + \bar{b}_2 d_2) \bar{\tau}_2' p_1}{\tau_2}\right) S\left(\bar{b}_2 n_{11}, x \bar{\tau}_2'^2 n_{11} p_1^2, \frac{\tau_2}{n_{12}}\right).$$

Expanding the Kloosterman sum and then summing over x gives

$$\frac{\phi(\tau_2/n_{12})^2}{\tau_2^2} \sum_{\substack{y_1, y_2 \pmod{\tau_2} \\ \tau_2' n_2 + (y_1 p_2^2 + y_2 p_1^2) n_1 \equiv 0 \pmod{\tau_2} \\ (b_1, \tau_2)=1}} \sum_{b_1 \pmod{\tau_2}} e\left(\frac{(b_1 h + \bar{b}_1 d_1) \bar{\tau}_2' p_2 + \bar{b}_1 y_1 n_1}{\tau_2}\right) \\ \times \sum_{\substack{b_2 \pmod{\tau_2} \\ (b_2, \tau_2)=1}} e\left(\frac{-(b_2 h + \bar{b}_2 d_2) \bar{\tau}_2' p_1 + \bar{b}_2 y_2 n_1}{\tau_2}\right),$$

which by Weil's bound can be bounded by

$$\frac{\tau_2^{2+\varepsilon}}{n_{12}^2} \sum_{\substack{y_1, y_2 \pmod{\tau_2} \\ \tau_2' n_2 + (y_1 p_2^2 + y_2 p_1^2) n_1 \equiv 0 \pmod{\tau_2}}} \sqrt{(h, y_1 d_1 p_2 + \tau_2' n_1, \tau_2)(h, y_2 d_2 p_1 + \tau_2' n_1, \tau_2)} \\ \ll \frac{\tau_2^{2+\varepsilon}}{n_{12}^2} \left(\sum_{\substack{y_1, y_2 \pmod{\tau_2} \\ \tau_2' n_2 + (y_1 p_2^2 + y_2 p_1^2) n_1 \equiv 0 \pmod{\tau_2}}}(y_1 d_1 p_2 + \tau_2' n_1, \tau_2) \right)^{1/2} \\ \times \left(\sum_{\substack{y_1, y_2 \pmod{\tau_2} \\ \tau_2' n_2 + (y_1 p_2^2 + y_2 p_1^2) n_1 \equiv 0 \pmod{\tau_2}}}(y_2 d_2 p_1 + \tau_2' n_1, \tau_2) \right)^{1/2}.$$

By symmetry it suffices to analyze one of the parentheses, say the first. The congruence determines y_2 modulo $\tau_2/(\tau_2, n_1)$. For a given value $\tau = (y_1 d_1 p_2 + \tau_2' n_1, \tau_2) \mid \tau_2$ there are at most $\tau_2(d_1, n_1, \tau)/\tau$ choices for y_1 , so that we can bound the previous display by

$$\ll \frac{\tau_2^{2+\varepsilon}}{n_{12}^2} \tau_2^{1+\varepsilon} (n_1, \tau_2)^2 \ll \tau_2^{3+\varepsilon},$$

noting that $(\tau_2, n_1) = (\tau_2, n_{12})$.

Combining all previous estimates, we have a final bound

$$(p_1 g_1)^{3/2} (p_2 g_2)^{3/2} \tau_1^{5/2+\varepsilon} (h n_2, \tau_1)^{1/2} \tau_2^{3+\varepsilon} \ll p_1^{3/2} p_2^{3/2} t^{5/2+\varepsilon} (h n_2, t_1)^{1/2} t_2^{1/2}$$

for \mathcal{T} in the case $p_1 \neq p_2$.

Next we consider $p_1 = p_2$. We write $t = g_1 \tau_1 \tau_2$ where $g_1 \mid p_1^\infty$ and $(\tau_1 \tau_2, p_1) = 1$ and τ_1, τ_2 have the same meaning as before. Note that we have $(n_1, p_1 g_1) = 1$. Modulo $p_1 g_1$, the

exponential sum becomes

$$\begin{aligned} & \sum_{x \pmod{p_1g_1}} e\left(\frac{x n_2 \bar{\tau}_1 \bar{\tau}_2}{p_1 g_1}\right) \sum_{\substack{b_1 \pmod{p_1 g_1} \\ (b_1, p_1) = 1}} e\left(\frac{(b_1 h + \bar{b}_1 d_1) \bar{\tau}_1 \bar{\tau}_2}{p_1 g_1}\right) S(\bar{b}_1, x \bar{\tau}_1 \bar{\tau}_2^2 n_1^2, p_1 g_1) \\ & \quad \times \sum_{\substack{b_2 \pmod{p_1 g_1} \\ (b_2, p_1) = 1}} e\left(\frac{-(b_2 h + \bar{b}_2 d_2) \bar{\tau}_1 \bar{\tau}_2}{p_1 g_1}\right) S(\bar{b}_2, x \bar{\tau}_1 \bar{\tau}_2^2 n_1^2, p_1 g_1), \end{aligned}$$

which after summing over x becomes in the same way as before

$$\begin{aligned} (2.14) \quad & \sum_{\substack{y_1, y_2 \pmod{p_1 g_1} \\ (y_1 y_2, p_1) = 1}} p_1 g_1 \sum_{\substack{b_1 \pmod{p_1 g_1} \\ (b_1, p_1) = 1}} e\left(\frac{(b_1 h + \bar{b}_1 d_1) \bar{\tau}_1 \bar{\tau}_2 + \bar{b}_1 y_1}{p_1 g_1}\right) \\ & \quad \times \sum_{\substack{b_2 \pmod{p_1 g_1} \\ (b_2, p_1) = 1}} e\left(\frac{-(b_2 h + \bar{b}_2 d_2) \bar{\tau}_1 \bar{\tau}_2 + \bar{b}_2 y_2}{p_1 g_1}\right). \end{aligned}$$

Applying Weil's bound for the sums over b_1, b_2 and using that $(hn_1, p_1) = 1$, we see that the above can be bounded by

$$p_1 g_1 \sum_{\substack{y_1, y_2 \pmod{p_1 g_1} \\ (y_1 y_2, p_1) = 1}} p_1 p_2 \ll (p_1 g_1)^3.$$

Modulo τ_1 , the exponential sum is of the form (2.10) with $\tau'_1 = p_1 g_1 \tau_2$, $p_1 = p_2 = 1$ and the same proof gives the bound

$$\tau_1^{5/2+\varepsilon} (hn_2, \tau_1)^{1/2}.$$

Modulo τ_2 , the exponential sum is of the form (2.13) with $\tau'_2 = p_1 g_1 \tau_1$, $p_1 = p_2 = 1$ and so we obtain the bound $\tau_2^{3+\varepsilon}$. Combining all these estimates, we have a final bound

$$(p_1 g_1)^3 \tau_1^{5/2+\varepsilon} (hn_2, \tau_1)^{1/2} \tau_2^{3+\varepsilon}$$

which gives (2.9) for \mathcal{T} in the case $p_1 = p_2$ since $g_1^{1/2} (hn_2, \tau_1)^{1/2} \leq (hn_2 p_1, t_1)^{1/2}$. (The bound (2.9) can be improved in the p_1 -aspect in many cases, but the above suffices for our purposes as long as $n_2 \neq 0$.)

Now we improve (2.9) if in addition $n_2 = 0$. We revisit the sum modulo $p_1 g_1$ and note that (2.14) becomes

$$(2.15) \quad \sum_{\substack{y_1 \pmod{p_1 g_1} \\ (y_1, p_1) = 1}} p_1 g_1 \sum_{\substack{b_1 \pmod{p_1 g_1} \\ (b_1, p_1) = 1}} e\left(\frac{(b_1 h + \bar{b}_1 d_1) \bar{\tau}_1 \bar{\tau}_2 + \bar{b}_1 y_1}{p_1 g_1}\right) \sum_{\substack{b_2 \pmod{p_1 g_1} \\ (b_2, p_1) = 1}} e\left(\frac{-(b_2 h + \bar{b}_2 d_2) \bar{\tau}_1 \bar{\tau}_2 - \bar{b}_2 y_1}{p_1 g_1}\right).$$

Then the sum over y_1 gives $p_1g_1\mathbf{1}_{b_1 \equiv b_2 \pmod{p_1g_1}}$ so that (2.15) equals

$$(p_1g_1)^2 \sum_{\substack{b_1 \pmod{p_1g_1} \\ (b_1, p_1)=1}} e\left(\frac{\overline{b_1\tau_1\tau_2}(d_1 - d_2)}{p_1g_1}\right) \ll (p_1g_1)^2(d_1 - d_2, p_1g_1).$$

Modulo $\tau_1\tau_2$, we use the same bound as before getting a final bound

$$(p_1g_1)^2(d_1 - d_2, p_1g_1)(\tau_1\tau_2)^{3+\varepsilon} \ll p_1^2(d_1 - d_2, p_1)(g_1\tau_1\tau_2)^{3+\varepsilon}.$$

This completes the proof. \square

Since both Lemma 7 and 8 feature the squarefull part of the modulus we record for future reference the following bound. Let $f \in \mathbb{N}$ and write $f = f_1f_2$ with f_1 squarefree, f_2 squarefull, $(f_1, f_2) = 1$ and use the same notation for $d = d_1d_2$. Then

$$\begin{aligned} \sum_{\substack{c \leq C \\ c \text{ squarefull}}} c^{1/2}(c, f) &= \sum_{d|f} d \sum_{\substack{d|c \leq C \\ c \text{ squarefull}}} c^{1/2} = \sum_{d|f} d \sum_{\delta|d^\infty} \sum_{\substack{c \leq C/(d_1^2d_2\delta) \\ c \text{ squarefull}}} (\delta d_1^2d_2c)^{1/2} \\ (2.16) \quad &\ll C \sum_{d|f} d_2^{1/2} \sum_{\delta|d^\infty} \delta^{-1/2} \ll Cf_2^{1/2}f^\varepsilon. \end{aligned}$$

3. PROOF OF THEOREM 1

We start by observing that without loss of generality we can and do assume for the proof of Theorems 1 and 2 that λ_1, λ_2, h are pairwise coprime. Indeed, if not, then they all must have common divisor $d > 1$, otherwise the equation $\lambda_1m - \lambda_2n = h$ has no solution. We can divide the entire equation by d , which in effect amounts to replacing x with x/d in the weight functions W and W_0 . Hence the proof in the case λ_1, λ_2, h pairwise coprime implies a fortiori the case of a non-trivial common divisor.

For the rest of the argument all implied constants may depend on a small ε and a large A , where applicable, without displaying this in the \ll -notation.

We recall that the main object of interest is

$$\sum_{\lambda_1m - \lambda_2n = h} A(n, 1)W\left(\frac{|\lambda_2|n}{x}\right)\tau(m)W_0\left(\frac{|\lambda_1|m}{x}\right).$$

In the notation of Lemma 1 this equals $S_1 + S_2$ where

$$\begin{aligned} (3.1) \quad S_1 &= \sum_{n,m} A(n, 1)W\left(\frac{|\lambda_2|n}{x}\right)\tau(m)W_0\left(\frac{|\lambda_1|m}{x}\right) \int_0^1 \chi(\alpha)e((\lambda_1m - \lambda_2n - h)\alpha)d\alpha, \\ S_2 &= \sum_{n,m} A(n, 1)W\left(\frac{|\lambda_2|n}{x}\right)\tau(m)W_0\left(\frac{|\lambda_1|m}{x}\right) \int_0^1 (1 - \chi(\alpha))e((\lambda_1m - \lambda_2n - h)\alpha)d\alpha. \end{aligned}$$

Recall that the function χ depends on a choice of data

$$Q, \quad \delta, \quad \omega$$

(which determine L), and a function ψ which we fix once and for all. To simplify the notation, we make the general assumptions

$$\|\omega\|_\infty \ll x^\varepsilon, \quad \log Q, |\log \delta| \asymp \log x, \quad L = Q^{2+o(1)}.$$

3.1. Estimation of S_2 . In this subsection we estimate S_2 . The final bound will be (3.16) below. We choose a parameter C with

$$\log C \asymp \log x$$

and invoke Lemma 2 (cf. also (2.5)) along with (2.1) to rewrite S_2 as

$$(3.2) \quad \begin{aligned} & \sum_{n,m} A(n,1) W\left(\frac{|\lambda_2|n}{x}\right) \tau(m) W_0\left(\frac{|\lambda_1|m}{x}\right) \\ & \times \sum_{c \leqslant C} \sum_{\substack{b \pmod{c} \\ (b,c)=1}} \int_{-\frac{1}{c(c+c')}}^{\frac{1}{c(c+c'')}} \left(1 - \chi\left(\frac{b}{c} + z\right)\right) e\left((\lambda_1 m - \lambda_2 n - h)\left(\frac{b}{c} + z\right)\right) dz. \end{aligned}$$

Finally we choose another parameter

$$C_0 < C, \quad \log C_0 \asymp \log x$$

and split the previous sum into ‘‘major arcs’’ with $c \leqslant C_0$ and ‘‘minor arcs’’ $c \geqslant C_0$, which we call $S_{2,0}$ and $S_{2,1}$ respectively.

We estimate the minor arc contribution as follows:

$$\begin{aligned} |S_{2,1}| & \leqslant \left(\sum_{C_0 < c \leqslant C} \sum_{\substack{b \pmod{c} \\ (b,c)=1}} \int_{-\frac{1}{c(c+c')}}^{\frac{1}{c(c+c'')}} \left|1 - \chi\left(\frac{b}{c} + z\right)\right|^2 dz \right)^{1/2} \\ & \times \left(\sum_{C_0 < c \leqslant C} \sum_{\substack{b \pmod{c} \\ (b,c)=1}} \int_{-\frac{1}{c(c+c')}}^{\frac{1}{c(c+c'')}} \left| \sum_n A(n,1) W\left(\frac{|\lambda_2|n}{x}\right) e\left(-\lambda_2 n\left(\frac{b}{c} + z\right)\right) \right|^2 dz \right)^{1/2} \\ & \times \max_{C_0 < c \leqslant C} \max_{\substack{b \pmod{c} \\ (b,c)=1}} \max_{-\frac{1}{c(c+c')} < z < \frac{1}{c(c+c'')}} \left| \sum_m \tau(m) W_0\left(\frac{|\lambda_1|m}{x}\right) e\left(\lambda_1 m\left(\frac{b}{c} + z\right)\right) \right|. \end{aligned}$$

Since the intervals do not overlap and $c + c', c + c'' > C$, we can replace the b, c -sum and z -integral in the first two lines by an integral over $[0, 1]$, take max over z in the last line in a larger range $|z| \leqslant (cC)^{-1}$ and obtain

$$\begin{aligned} S_{2,1} & \ll \left(\int_0^1 |1 - \chi(z)|^2 dz \right)^{1/2} \left(\sum_n \left| A(n,1) W\left(\frac{|\lambda_2|n}{x}\right) \right|^2 \right)^{1/2} \\ & \max_{C_0 < c \leqslant C} \max_{\substack{b \pmod{c} \\ (b,c)=1}} \max_{|z| \leqslant (cC)^{-1}} \left| \sum_m \tau(m) W_0\left(\frac{|\lambda_1|m}{x}\right) e\left(\lambda_1 m\left(\frac{b}{c} + z\right)\right) \right|. \end{aligned}$$

Let us write

$$\frac{\lambda_1}{c} = \frac{\lambda'_1}{\tilde{c}}, \quad \frac{\lambda_2}{c} = \frac{\lambda'_2}{\tilde{c}}$$

where the right hand sides are in lowest terms. We apply Lemma 1 for the first factor, the Rankin-Selberg bound for the second factor and Lemmata 3 and 4 with \tilde{c} in place of c and

$X = x/|\lambda_1|$, $Z = z|\lambda_1|$ for the last factor, which gives the bound

$$(3.3) \quad \begin{aligned} & \frac{x^{1+\varepsilon}}{|\lambda_1|\tilde{c}} \left(1 + \sum_m \tau(m) \left[\left(1 + \frac{mx}{|\lambda_1|\tilde{c}^2} \right)^{-A} + \frac{1}{1+x|z|} \left(1 + \frac{mx}{|\lambda_1|\tilde{c}^2(1+x|z|)^2} \right)^{-A} \right] \right) \\ & \ll \frac{x^{1+\varepsilon}}{|\lambda_1|\tilde{c}} \left(1 + \frac{\tilde{c}^2(1+x|z|)|\lambda_1|}{x} \right) = x^\varepsilon \left(\frac{x}{|\lambda_1|\tilde{c}} + \tilde{c}(1+x|z|) \right) \end{aligned}$$

for the m -sum. In this way we obtain

$$(3.4) \quad \begin{aligned} S_{2,1} & \ll x^\varepsilon \frac{Q}{L\delta^{1/2}} \left(\frac{x}{|\lambda_2|} \right)^{1/2} \max_{C_0 < c \leqslant C} \max_{|z| \leqslant (cC)^{-1}} \left(\frac{x}{|\lambda_1|\tilde{c}} + \tilde{c}(1+x|z|) \right) \\ & \ll x^\varepsilon \frac{1}{Q\delta^{1/2}} \left(\frac{x}{|\lambda_2|} \right)^{1/2} \left(\frac{x}{C_0} + C \right). \end{aligned}$$

We now return to (3.2) and estimate the major arc contribution $S_{2,0}$ where $c \leqslant C_0$. For each $c \leqslant C_0$, we use (2.3) so that

$$\begin{aligned} S_{2,0} & = \sum_{c \leqslant C_0} \int_{-1/cC}^{1/cC} \frac{1}{c} \sum_{u \pmod{c}} \sum_{t \in I(c,z)} e\left(\frac{ut}{c}\right) \sum_{\substack{b \pmod{c} \\ (b,c)=1}} e\left(\frac{u\bar{b}-hb}{c}\right) e(-hz) \\ & \quad \times \sum_n A(n,1) W\left(\frac{|\lambda_2|n}{x}\right) e(-\lambda_2 nz) e\left(\frac{-\lambda_2 nb}{c}\right) \\ & \quad \times \sum_m \tau(m) e(\lambda_1 mz) e\left(\frac{\lambda_1 mb}{c}\right) W_0\left(\frac{|\lambda_1|m}{x}\right) \left(1 - \chi\left(\frac{b}{c} + z\right)\right) dz. \end{aligned}$$

We start by dualizing the m -sum by Lemma 3. We also apply Lemma 4 and see that the dual sum is, up to a negligible error, restricted to

$$m \ll x^\varepsilon \frac{\tilde{c}^2(1+x|z|)^2}{x/|\lambda_1|} \ll x^\varepsilon |\lambda_1| \left(\frac{C_0^2}{x} + \frac{x}{C^2} \right).$$

Let us assume

$$(3.5) \quad |\lambda_1|^{1/2} C_0 \leqslant x^{1/2-\eta}, \quad C \geqslant (|\lambda_1|x)^{1/2+\eta}$$

for some fixed $\eta > 0$. Then the dual sum is negligible. Moreover, a simple integration by parts argument shows that also the main term is negligible unless

$$(3.6) \quad |z| \ll x^{\varepsilon-1}$$

for any $\varepsilon > 0$. Finally we note that the main term is independent of b and the t -sum can be bounded by $\frac{c}{1+|u|}$ for $|u| \leqslant c/2$. Thus we obtain

$$\begin{aligned} S_{2,0} & \ll \sum_{c \leqslant C_0} \int_{|z| \leqslant x^{-1+\varepsilon}} \frac{x \log x}{|\lambda_1|\tilde{c}} \sum_{|u| \leqslant c/2} \frac{1}{1+|u|} \\ & \quad \times \left| \sum_{\substack{b \pmod{c} \\ (b,c)=1}} e\left(\frac{u\bar{b}-hb}{c}\right) \sum_n A(n,1) W\left(\frac{|\lambda_2|n}{x}\right) e(-\lambda_2 nz) e\left(\frac{-\lambda_2 nb}{c}\right) \left(1 - \chi\left(\frac{b}{c} + z\right)\right) \right| dz. \end{aligned}$$

We now dualize the n -sum using Lemma 5 getting

$$\begin{aligned} S_{2,0} &\ll \sum_{c \leq C_0} \int_{|z| \leq x^{-1+\varepsilon}} \frac{x^2 \log x}{|\lambda_1 \lambda_2| \tilde{c} \check{c}^2} \sum_{|u| \leq c/2} \frac{1}{1+|u|} \left| \sum_{\substack{b \pmod{c} \\ (b,c)=1}} e\left(\frac{u\bar{b}-hb}{c}\right) \sum_{\pm} \sum_{n_2} \sum_{n_1 \mid \check{c}} n_1 A(n_1, n_2) \right. \\ &\quad \times S\left(-\bar{\lambda}_2' \bar{b}, \pm n_2, \frac{\check{c}}{n_1}\right) \int_0^\infty W(y) e(-\operatorname{sgn}(\lambda_2) y z x) V^\pm\left(\frac{n_1^2 n_2 x y}{\check{c}^3 |\lambda_2|}\right) dy \left. \left(1 - \chi\left(\frac{b}{c} + z\right)\right) \right| dz. \end{aligned}$$

A simple integration by parts argument using the bounds in Lemma 5 (recall (3.6)) shows that

$$(3.7) \quad \int_0^\infty W(y) e(-\operatorname{sgn}(\lambda_2) y z x) V^\pm\left(\frac{n_1^2 n_2 x y}{\check{c}^3 |\lambda_2|}\right) dy \ll x^\varepsilon \left(1 + \frac{n_1^2 n_2 x}{|\lambda_2| \check{c}^3}\right)^{-A} \left(\frac{n_1^2 n_2 x}{|\lambda_2| \check{c}^3}\right)^{-1/2}$$

for any $\varepsilon, A > 0$. We conclude

$$\begin{aligned} S_{2,0} &\ll \sum_{c \leq C_0} \int_{|z| \leq x^{-1+\varepsilon}} \frac{x^{3/2+\varepsilon}}{|\lambda_1| |\lambda_2|^{1/2} \tilde{c} \check{c}^{1/2}} \sum_{\pm} \sum_{n_2} \frac{1}{n_2^{1/2}} \sum_{n_1 \mid \check{c}} |A(n_1, n_2)| \left(1 + \frac{n_1^2 n_2 x}{|\lambda_2| \check{c}^3}\right)^{-A} \\ &\quad \sum_{|u| \leq c/2} \frac{1}{1+|u|} \left| \sum_{\substack{b \pmod{c} \\ (b,c)=1}} e\left(\frac{u\bar{b}-hb}{c}\right) S\left(-\bar{b}, \pm \bar{\lambda}_2' n_2, \frac{\check{c}}{n_1}\right) \left(1 - \chi\left(\frac{b}{c} + z\right)\right) \right| dz. \end{aligned}$$

We split this term into two parts according to the term $(1 - \chi(\frac{b}{c} + z))$ and call them $S_{2,0,0}$ and $S_{2,0,1}$. For the contribution of the first summand we insert the bound for Lemma 7 with $n_1(\lambda_2, c)$ in place of n_1 (and with the same notation $c = c_1 c_2$ where c_2 is the squarefull part of c) and obtain (recall (2.6) and (2.16))

$$\begin{aligned} (3.8) \quad S_{2,0,0} &\ll \sum_{c \leq C_0} \frac{x^{1/2+\varepsilon} c c_2^{1/2}}{|\lambda_1| |\lambda_2|^{1/2} \tilde{c} \check{c}^{1/2}} \sum_{n_2} \frac{1}{n_2^{1/2}} \sum_{n_1 \mid \check{c}} |A(n_1, n_2)| \left(1 + \frac{n_1^2 n_2 x}{|\lambda_2| \check{c}^3}\right)^{-A} \\ &\ll \sum_{c \leq C_0} \frac{x^{1/2+\varepsilon} c c_2^{1/2}}{|\lambda_1| |\lambda_2|^{1/2} \tilde{c} \check{c}^{1/2}} \sum_{n_1 \mid \check{c}} n_1^{1/2} \frac{|\lambda_2|^{1/2} \check{c}^{3/2}}{x^{1/2} n_1} \ll x^\varepsilon \sum_{c \leq C_0} \frac{c c_2^{1/2}(c, \lambda_1)}{|\lambda_1|} \ll x^\varepsilon \frac{C_0^2}{|\lambda_1|^{1/2}}. \end{aligned}$$

(Here we could tighten the estimate slightly if λ_1 is assumed to be squarefree or close to squarefree.)

To deal with $S_{2,0,1}$, we insert (2.2) getting

$$\begin{aligned} (3.9) \quad S_{2,0,1} &\ll \sum_{c \leq C_0} \int_{|z| \leq x^{-1+\varepsilon}} \frac{x^{3/2+\varepsilon}}{|\lambda_1| |\lambda_2|^{1/2} \tilde{c} \check{c}^{1/2}} \sum_{\pm} \sum_{n_2} \sum_{n_1 \mid \check{c}} \frac{|A(n_1, n_2)|}{n_2^{1/2}} \left(1 + \frac{n_1^2 n_2 x}{|\lambda_2| \check{c}^3}\right)^{-A} \sum_{|u| \leq c/2} \frac{1}{1+|u|} \\ &\quad \times \left| \sum_{\substack{b \pmod{c} \\ (b,c)=1}} e\left(\frac{u\bar{b}-hb}{c}\right) S\left(-\bar{b}, \pm \bar{\lambda}_2' n_2, \frac{\check{c}}{n_1}\right) \frac{1}{\delta L} \sum_q \omega(q) \sum_{d|q} \mu\left(\frac{q}{d}\right) \sum_{\ell \equiv bd \pmod{c}} \psi\left(\frac{1}{\delta} \left(\frac{\ell}{cd} + z\right)\right) \right| dz. \end{aligned}$$

We first treat the contribution $\ell = 0$. In this case we must have $c \mid d$, and Lemma 7 (recall the notation $c = c_1 c_2$ where c_2 is the squarefull part of c) implies the bound (cf. (3.8))

$$(3.10) \quad \ll x^\varepsilon \sum_{c \leq C_0} \frac{c c_2^{1/2}}{|\lambda_1|} (c, \lambda_1) \frac{1}{\delta L} \sum_{c|q} \omega(q) \tau(q) \ll x^\varepsilon \sum_{c \leq C_0} \frac{c_2^{1/2}(c, \lambda_1)}{|\lambda_1|} \frac{Q}{\delta L} \ll x^\varepsilon \frac{C_0}{|\lambda_1|^{1/2} \delta Q}.$$

From now on we assume $\ell \neq 0$. Since $(b, c) = 1$, we must have $(\ell, c) = (d, c) = g_1$, say. The second line in (3.9) equals

$$(3.11) \quad \begin{aligned} & \frac{1}{\delta L} \sum_{\substack{b \pmod{c} \\ (b,c)=1}} e\left(\frac{u\bar{b} - hb}{c}\right) S\left(-\bar{b}, \pm \overline{\lambda_2} n_2, \frac{\check{c}}{n_1}\right) \\ & \times \sum_{g_1 g_2 = c} \sum_{\substack{(\ell, g_2) = 1 \\ \ell \neq 0}} \sum_r \mu(r) \sum_{\substack{(d, g_2) = 1 \\ d \equiv \bar{b}\ell \pmod{g_2}}} \omega(dg_1 r) \psi\left(\frac{1}{\delta} \left(\frac{\ell}{cd} + z\right)\right). \end{aligned}$$

It is at this point that we choose the function ω . To this end we write $Q = Q_1 Q_2$ with two parameters $1 \leq Q_1, Q_2 \leq Q$ and write

$$(3.12) \quad \omega(q) = \sum_{\substack{\frac{1}{2} Q_1 \leq p \leq Q_1 \\ p \text{ prime} \\ p \nmid h\lambda_1\lambda_2}} \sum_{\substack{t \in \mathbb{N} \\ pt = q}} \rho\left(\frac{t}{Q_2}\right)$$

where $\rho : [1/2, 1] \rightarrow [0, 1]$ is a fixed smooth nonzero function. Analyzing the condition $pt = dg_1 r$ yields three terms corresponding to $p \mid r$; $p \nmid r, p \mid g_1$; and $p \nmid rg_1, p \mid d$ (hence $(p, c) = 1$). Changing variables, this gives

$$(3.13) \quad \begin{aligned} & \sum_{g_1 g_2 = c} \sum_{\substack{(\ell, g_2) = 1 \\ \ell \neq 0}} \sum_r \mu(r) \sum_{\substack{(d, g_2) = 1 \\ d \equiv \bar{b}\ell \pmod{g_2}}} \omega(dg_1 r) \psi\left(\frac{1}{\delta} \left(\frac{\ell}{cd} + z\right)\right) \\ & = - \sum_{\substack{\frac{1}{2} Q_1 \leq p \leq Q_1 \\ p \text{ prime} \\ p \nmid h\lambda_1\lambda_2}} \sum_{g_1 g_2 = c} \sum_{\substack{\ell \neq 0 \\ (\ell, g_2) = 1}} \sum_{(p, r) = 1} \mu(r) \sum_{\substack{(d, g_2) = 1 \\ d \equiv \bar{b}\ell \pmod{g_2}}} \rho\left(\frac{dg_1 r}{Q_2}\right) \psi\left(\frac{1}{\delta} \left(\frac{\ell}{cd} + z\right)\right) \\ & + \sum_{\substack{\frac{1}{2} Q_1 \leq p \leq Q_1 \\ p \text{ prime} \\ p \nmid h\lambda_1\lambda_2}} \sum_{pg_1 g_2 = c} \sum_{\substack{\ell \neq 0 \\ (\ell, g_2) = 1}} \sum_{(p, r) = 1} \mu(r) \sum_{\substack{(d, g_2) = 1 \\ d \equiv \bar{b}\ell \pmod{g_2}}} \rho\left(\frac{dg_1 r}{Q_2}\right) \psi\left(\frac{1}{\delta} \left(\frac{\ell}{cd} + z\right)\right) \\ & + \sum_{\substack{\frac{1}{2} Q_1 \leq p \leq Q_1 \\ p \text{ prime} \\ p \nmid ch\lambda_1\lambda_2}} \sum_{g_1 g_2 = c} \sum_{\substack{\ell \neq 0 \\ (\ell, g_2) = 1}} \sum_{(p, r) = 1} \mu(r) \sum_{\substack{(d, g_2) = 1 \\ d \equiv \bar{b}\ell \pmod{g_2}}} \rho\left(\frac{dg_1 r}{Q_2}\right) \psi\left(\frac{1}{\delta} \left(\frac{\ell}{cdp} + z\right)\right). \end{aligned}$$

We now prepare for the next important step, Poisson summation in d . This can only be done efficiently if we have no arithmetic condition in the sum over t in (3.12), in particular we cannot restrict to t prime as in [Mu].

We apply a smooth partition of unity and localize $d \asymp D$ for some parameter $1 \leq D \leq Q_2/g_1 r$ with a smooth weight function $v(d/D)$. We make the general assumption

$$(3.14) \quad \delta \gg x^{-1+\varepsilon}$$

so that $z/\delta \ll 1$. We remember the size condition $\ell \ll cD\varpi\delta$ with $\varpi \in \{1, p\}$, depending on the summand. For $\varpi \in \{1, p\}$ we have

$$\sum_{\substack{(d,g_2)=1 \\ d \equiv \bar{\varpi}b\ell \pmod{g_2}}} v\left(\frac{d}{D}\right) \rho\left(\frac{dg_1r}{Q_2}\right) \psi\left(\frac{1}{\delta}\left(\frac{\ell}{cd\varpi} + z\right)\right)$$

After Poisson summation, this becomes

$$\frac{1}{g_2} e\left(\frac{\bar{\varpi}b\ell d}{g_2}\right) \int_{\mathbb{R}} v\left(\frac{\xi}{D}\right) \rho\left(\frac{\xi g_1 r}{Q_2}\right) \psi\left(\frac{1}{\delta}\left(\frac{\ell}{c\xi\varpi} + z\right)\right) e\left(\frac{-\xi d}{g_2}\right) d\xi.$$

Integration by parts shows that the integral is $\ll_A D(1 + dD/g_2)^{-A}$ for every $A > 0$. The character sum over b becomes

$$\begin{aligned} & \sum_{\substack{b \pmod{c} \\ (b,c)=1}} e\left(\frac{u\bar{b} - hb}{c}\right) S\left(-\bar{b}, \pm \bar{\lambda}_2' n_2, \frac{\check{c}}{n_1}\right) e\left(\frac{\bar{\varpi}b\ell d}{g_2}\right) \\ &= \sum_{\substack{b \pmod{c} \\ (b,c)=1}} e\left(\frac{-hb + \bar{b}\bar{\varpi}(\ell dg_1 + \varpi u)}{c}\right) S\left(-\bar{b}, \pm \bar{\lambda}_2' n_2, \frac{c}{n_1(c, \lambda_2)}\right) \ll c^{1+\varepsilon} c_2^{1/2} \end{aligned}$$

by Lemma 7 (with the usual notation that c_2 denotes the squarefull part of c), so that we can conclude that the contribution from third summand in (3.13) (which is the hardest) to (3.11) is bounded by

$$\begin{aligned} & \ll \frac{1}{\delta L} Q_1 \sum_{g_1 g_2 = c} \sum_{r \leq Q_2} \max_{D \ll \frac{Q_2}{g_1 r}} (cDQ_1\delta) \frac{1}{g_2} c^{1+\varepsilon} c_2^{1/2} D\left(1 + \frac{g_2}{D}\right) \\ & \ll \frac{x^\varepsilon}{Q^2} Q_1 \sum_{g_1 g_2 = c} \sum_{r \leq Q_2} \left(\frac{Q_2}{g_1 r} Q_1\right) c^2 c_2^{1/2} \left(\frac{Q_2}{g_1 g_2 r} + 1\right) \\ & \ll \frac{x^\varepsilon}{Q} Q_1 c^2 c_2^{1/2} \left(\frac{Q_2}{c} + 1\right) \ll x^\varepsilon c_2^{1/2} \left(c + \frac{c^2}{Q_2}\right). \end{aligned}$$

The other two summands in (3.13) are dominated by this quantity. Substituting back into (3.9), together with (2.6), (2.16) and (3.10), we obtain the total bound

$$(3.15) \quad S_{2,0,1} \ll x^\varepsilon \left(\sum_{c \leq C_0} \frac{c_2^{1/2}(c, \lambda_1)}{|\lambda_1|} \left(c + \frac{c^2}{Q_2}\right) + \frac{C_0}{|\lambda_1|^{1/2} \delta Q} \right) \ll \frac{x^\varepsilon}{|\lambda_1|^{1/2}} \left(C_0^2 + \frac{C_0^3}{Q_2} + \frac{C_0}{\delta Q}\right).$$

Combining this with (3.4) and (3.8) together with $\delta \gg x^{-1+\varepsilon}$, we arrive at the final bound

$$(3.16) \quad S_2 \ll x^{1+\varepsilon} \left(\frac{x}{|\lambda_2|^{1/2} Q C_0} + \frac{C}{|\lambda_2|^{1/2} Q} + \frac{C_0^2}{|\lambda_1|^{1/2} x} + \frac{C_0^3}{|\lambda_1|^{1/2} x Q_2} + \frac{C_0}{|\lambda_1|^{1/2} Q} \right).$$

3.2. Estimation of S_1 . We now estimate S_1 , defined in (3.1). The final bound is (3.27) below. The first steps follow Munshi [Mu, Section 4] with a different choice of \mathcal{Q} , but at some point we need to diverge from his analysis.

Using the definition given in Lemma 1, we see that

$$\begin{aligned} S_1 &= \frac{1}{\delta L} \sum_{\substack{q \in \mathbb{N} \\ q \leq Q}} \omega(q) \sum_{\substack{a \pmod{q} \\ (a,q)=1}} \sum_{m,n} A(n,1) W\left(\frac{|\lambda_2|n}{x}\right) \tau(m) W_0\left(\frac{|\lambda_1|m}{x}\right) \\ &\quad \times \int_{\mathbb{R}} e\left(\left(\frac{a}{q} + z\right)(\lambda_1 m - \lambda_2 n - h)\right) \psi\left(\frac{z}{\delta}\right) dz. \end{aligned}$$

As before we write

$$\frac{\lambda_1}{q} = \frac{\lambda'_1}{\tilde{q}}, \quad \frac{\lambda_2}{q} = \frac{\lambda'_2}{\check{q}}$$

in lowest terms and recall that in the decomposition $q = pt$ of (3.12) we have $p \nmid h\lambda_1\lambda_2$.

We now apply Lemma 3 and Lemma 5 so that

$$\begin{aligned} S_1 &= \frac{1}{\delta L} \sum_{q \leq Q} \sum_{\substack{a \pmod{q} \\ (a,q)=1}} \int \omega(q) \frac{x^2}{|\lambda_1\lambda_2|\tilde{q}\check{q}^2} \left(\int_0^\infty W_0(\xi) e(\operatorname{sgn}(\lambda_1)z\xi x) \left(\log \frac{\xi x}{\check{q}^2} + 2\gamma \right) d\xi \right. \\ &\quad \left. + \sum_{\pm} \sum_m \tau(m) e\left(\pm \frac{\lambda'_1 am}{\tilde{q}}\right) \int_0^\infty W_0(\xi) e(\operatorname{sgn}(\lambda_1)z\xi x) J^\pm\left(\frac{\sqrt{m\xi x}}{|\lambda_1|^{1/2}\tilde{q}}\right) d\xi \right) \\ &\quad \times \sum_{\pm} \sum_{n_2} \sum_{n_1 \mid \check{q}} n_1 A(n_1, n_2) S\left(-\overline{\lambda'_2 a}, \pm n_2, \frac{\check{q}}{n_1}\right) \\ &\quad \times \int_0^\infty W(y) e(-\operatorname{sgn}(\lambda_2)zyx) V^\pm\left(\frac{n_1^2 n_2 y x}{|\lambda_2| \check{q}^3}\right) dy e\left(\frac{-ah}{q}\right) e(-zh) \psi\left(\frac{z}{\delta}\right) dz. \end{aligned}$$

We now make the final choice

$$(3.17) \quad \delta = x^{-1+\varepsilon}$$

so that the exponentials $e(\operatorname{sgn}(\lambda_1)z\xi x)$ and $e(-\operatorname{sgn}(\lambda_2)zyx)$ are almost flat, but (3.14) is satisfied. In particular, by Lemma 5 (cf. (3.7)), the y -integrals is bounded by

$$\ll x^\varepsilon \frac{\check{q}^{3/2} |\lambda_2|^{1/2}}{x^{1/2} n_1 n_2^{1/2}} \left(1 + \frac{n_1^2 n_2 x}{\check{q}^3 |\lambda_2|}\right)^{-A}$$

for any $\varepsilon, A > 0$ (where ε in (3.17) and hence in (3.14), which is the same ε as in (3.6), has to be chosen accordingly in terms of the present ε and A).

We put the variables n_1, n_2 in dyadic ranges $H \leq n_1 \leq 2H$, $N \leq n_2 \leq 2N$ and denote by $S_1(N, H)$ the corresponding contribution to S_1 . In particular, we may at the cost of a negligible error assume that $H^2 N \ll Q^3 |\lambda_2| x^{\varepsilon-1}$. Using in addition the bounds in Lemma 4 we see that

$$\begin{aligned} S_1(N, H) &\ll \frac{x^{3/2+\varepsilon}}{|\lambda_1| |\lambda_2|^{1/2} \delta L Q^{3/2}} \int_{z \ll \delta} \left| \sum_{\substack{q=pt \asymp Q \\ p \asymp Q_1, t \asymp Q_2 \\ p \nmid h\lambda_1\lambda_2}} (t, \lambda_1) (t, \lambda_2)^{1/2} \sum_{\substack{n_1 \asymp H \\ n_1 \mid \check{q}}} \sum_{n_2 \asymp N} \frac{|A(n_1, n_2)|}{\sqrt{n_2}} \right. \\ &\quad \left. \times \sum_{m \in \mathbb{Z}} \Sigma_{h, m} \overline{\lambda'_1(t, \lambda_1)}, n_1(t, \lambda_2), \pm n_2 \overline{\lambda'_2}(q) \Omega_{z, n_1, n_2}(m, q) \right| dz \end{aligned}$$

for any $\varepsilon > 0$ where

$$(3.18) \quad \Omega_{z,n_1,n_2}(m,q) \ll \tau(m) \left(1 + \frac{|m|x}{|\lambda_1|\tilde{q}^2}\right)^{-A} \left(1 + \frac{n_1^2 n_2 x}{|\lambda_2|\tilde{q}^3}\right)^{-A}$$

for any $A > 0$.

The typical case is that n_2 has the maximal length $N \approx q^3/x$, while $H \approx 1$, but we will first provide a trivial bound that is useful if H is big. To this end, we estimate all sums trivially and decompose $q = q_1 q_2$ uniquely with q_1 squarefree, q_2 squarefull and $(q_1, q_2) = 1$. Using Lemma 7 and (3.18), we see that

$$\begin{aligned} S_1(N, H) &\ll \frac{x^{3/2+\varepsilon}}{|\lambda_1||\lambda_2|^{1/2}Q^{7/2}} \sum_{\substack{q=pt \asymp Q \\ p \asymp Q_1, t \asymp Q_2}} (t, \lambda_1)(t, \lambda_2)^{1/2} \sum_{\substack{n_1 \asymp H \\ n_1 \mid \tilde{q}}} \sum_{n_2 \asymp N} \frac{|A(n_1, n_2)|}{\sqrt{n_2}} \frac{q\sqrt{q_2}}{\sqrt{n_1(t, \lambda_2)}} \\ &\quad \times \sum_{m \in \mathbb{Z}} \left(1 + \frac{|m|x}{|\lambda_1|\tilde{q}^2} + \frac{H^2 Nx}{|\lambda_2|\tilde{q}^3}\right)^{-A} \sqrt{(q_1, n_1(t, \lambda_2), m(t, \lambda_1))} \\ &\ll \frac{x^{3/2+\varepsilon}}{|\lambda_1||\lambda_2|^{1/2}Q^2} \sum_{n_1 \asymp H} \sum_{n_2 \asymp N} \frac{|A(n_1, n_2)|}{\sqrt{n_1 n_2}} \sum_{\substack{q \asymp Q \\ n_1 \mid \tilde{q}}} \frac{(q, \lambda_1)}{\sqrt{q_1}} \left(\sqrt{(q, \lambda_2)(q_1, n_1)} + \frac{|\lambda_1|\tilde{q}^2\sqrt{(q_1, \lambda_1)}}{x}\right) \left(1 + \frac{H^2 Nx}{|\lambda_2|\tilde{q}^3}\right)^{-A}. \end{aligned}$$

Here the first term in the penultimate parenthesis corresponds to $m = 0$. Let $g = (\lambda_2, q)$. For any $n \in \mathbb{N}$ we have

$$\begin{aligned} \sum_{\substack{q \asymp Q \\ n \mid \tilde{q}}} \frac{(q, \lambda_1)\sqrt{(q, \lambda_2)(q_1, n)}}{q_1^{1/2}} &\leq |\lambda_1|g^{1/2} \sum_{\substack{\tilde{q} \asymp Q/g \\ n \mid \tilde{q}}} \frac{\sqrt{(\tilde{q}_1, n)}}{\tilde{q}_1^{1/2}} \leq |\lambda_1|g^{1/2} \sum_{r \asymp Q/gn} \frac{(r, n^\infty)^{1/2}}{r_1^{1/2}} \\ &\leq |\lambda_1|g^{1/2} \sum_{\substack{\nu \mid n^\infty \\ \nu \leq Q}} \nu^{1/2} \sum_{r \asymp Q/gn\nu} \frac{1}{r_1^{1/2}} \ll |\lambda_1|g^{1/2} \sum_{\substack{\nu \mid n^\infty \\ \nu \leq Q}} \nu^{1/2} \left(\frac{Q}{gn\nu}\right)^{1/2} \ll Q^{1/2+\varepsilon} \frac{|\lambda_1|}{n^{1/2}} \end{aligned}$$

where \tilde{q}_1 , q_1 , r_1 denotes the respective squarefree part of \tilde{q}, q, r . Moreover,

$$\sum_{\substack{q \asymp Q \\ n_1 \mid \tilde{q}}} \frac{(q, \lambda_1)|\lambda_1|\tilde{q}^2\sqrt{(q_1, \lambda_1)}}{q_1^{1/2}x} \ll |\lambda_1| \sum_{\substack{q \asymp Q \\ n_1 \mid \tilde{q}}} \frac{q^2}{xq_1^{1/2}} \ll \frac{|\lambda_1|Q^{5/2}}{x\sqrt{n_1}},$$

and so

$$\begin{aligned} (3.19) \quad S_1(N, H) &\ll \frac{x^{3/2+\varepsilon}}{|\lambda_1||\lambda_2|^{1/2}Q^2} \sum_{n_1 \asymp H} \sum_{n_2 \asymp N} \frac{|A(n_1, n_2)|}{\sqrt{n_1 n_2}} \left(\frac{|\lambda_1|Q^{1/2}}{\sqrt{n_1}} + \frac{|\lambda_1|Q^{5/2}}{xn_1^{1/2}}\right) \left(1 + \frac{H^2 Nx}{|\lambda_2|Q^3}\right)^{-A} \\ &\ll \frac{x^{3/2+\varepsilon}}{|\lambda_1||\lambda_2|^{1/2}Q^2} \sqrt{HN} \left(\frac{|\lambda_1|Q^{1/2}}{H^{1/2}} + \frac{|\lambda_1|Q^{5/2}}{xH^{1/2}}\right) \left(1 + \frac{H^2 Nx}{|\lambda_2|Q^3}\right)^{-A} \\ &\ll x^\varepsilon \left(\frac{x}{H} + \frac{Q^2}{H}\right) \end{aligned}$$

for any $\varepsilon > 0$.

We keep this in mind for future reference and continue with a more sophisticated argument. From now on let us assume

$$(3.20) \quad Q_1 \geq 10H$$

so that $n_1 \mid \check{q}$ with $n_1 \asymp H$ and $p \asymp Q_1, p \nmid \lambda_2$ implies $n_1 \mid \check{t} := t/(t, \lambda_2)$. It then follows that

$$\begin{aligned} S_1(N, H) &\ll \frac{x^{3/2+\varepsilon}}{|\lambda_1||\lambda_2|^{1/2}\delta LQ^{3/2}} \int_{z \ll \delta} \sum_{t \asymp Q_2} (t, \lambda_1)(t, \lambda_2)^{1/2} \sum_{\substack{n_1 \asymp H \\ n_1 \mid \check{t}}} \sum_{n_2 \asymp N} \frac{|A(n_1, n_2)|}{\sqrt{n_2}} \\ &\quad \times \left| \sum_{m \in \mathbb{Z}} \sum_{\substack{p \asymp Q_1 \\ p \text{ prime} \\ p \nmid h\lambda_1\lambda_2}} \Sigma_{h, m\overline{\lambda'_1}(t, \lambda_1), n_1(t, \lambda_2), \pm n_2\overline{\lambda'_2}}(pt) \Omega_{z, n_1, n_2}(m, pt) \right| dz. \end{aligned}$$

While most sums are estimated trivially, it is important to keep the m, p -sums both inside the absolute values (unlike the treatments in [Mu, Xi] for instance). We apply the Cauchy–Schwarz inequality to bound $S_1(N, H)$ by

$$\begin{aligned} &\frac{x^{3/2+\varepsilon}}{|\lambda_1||\lambda_2|^{1/2}\delta LQ^{3/2}} \int_{z \ll \delta} \left(\sum_{n_1 \asymp H} \sum_{n_2 \asymp N} \frac{|A(n_1, n_2)|^2}{n_2} \sum_{\substack{t \asymp Q_2 \\ n_1 \mid \check{t}}} (t, \lambda_2) \right)^{1/2} \left(\sum_{n_1 \asymp H} \sum_{\substack{t \asymp Q_2 \\ n_1 \mid \check{t}}} (t, \lambda_1)^2 \right. \\ &\quad \times \left. \sum_{n_2 \asymp N} v\left(\frac{n_2}{N}\right) \left| \sum_m \sum_{\substack{p \asymp Q_1 \\ p \text{ prime} \\ p \nmid h\lambda_1\lambda_2}} \Sigma_{h, m\overline{\lambda'_1}(t, \lambda_1), n_1(t, \lambda_2), \pm n_2\overline{\lambda'_2}}(pt) \Omega_{z, n_1, n_2}(m, pt) \right|^2 \right)^{1/2} dz \end{aligned}$$

where v is some non-negative smooth function with support on $[1/3, 3]$ and $v(x) = 1$ on $[1/2, 2]$. Recalling that λ_1, λ_2 are coprime, we can recast the above sum

$$\begin{aligned} (3.21) \quad &\frac{x^{3/2+\varepsilon}}{|\lambda_1||\lambda_2|^{1/2}\delta LQ^{3/2}} \int_{z \ll \delta} \left(\sum_{n_1 \asymp H} \sum_{n_2 \asymp N} \frac{|A(n_1, n_2)|^2}{n_2} \sum_{g_2 \mid \lambda_2} g_2 \sum_{\substack{\check{t} \asymp Q_2/g_2 \\ n_1 \mid \check{t}}} 1 \right)^{1/2} \left(\sum_{n_1 \asymp H} \sum_{\substack{t \asymp Q_2 \\ n_1 \mid \check{t}}} (t, \lambda_1)^2 \right. \\ &\quad \times \left. \sum_{n_2 \asymp N} v\left(\frac{n_2}{N}\right) \left| \sum_m \sum_{\substack{p \asymp Q_1 \\ p \text{ prime} \\ p \nmid h\lambda_1\lambda_2}} \Sigma_{h, m\overline{\lambda'_1}(t, \lambda_1), n_1(t, \lambda_2), \pm n_2\overline{\lambda'_2}}(pt) \Omega_{z, n_1, n_2}(m, pt) \right|^2 \right)^{1/2} dz \\ &\ll \frac{x^{3/2+\varepsilon}}{|\lambda_1||\lambda_2|^{1/2}\delta Q_1^{7/2}Q_2^3} \int_{z \ll \delta} \left(\sum_{g_2 \mid \lambda_2} \sum_{\check{t} \asymp Q_2/g_2} \sum_{g_1 \mid (\check{t}, \lambda_1)} g_1^2 \sum_{n_1 \mid \check{t}} \sum_{n_2 \asymp N} v\left(\frac{n_2}{N}\right) \right. \\ &\quad \times \left. \left| \sum_m \sum_{\substack{p \asymp Q_1 \\ p \text{ prime} \\ p \nmid h\lambda_1\lambda_2}} \Sigma_{h, m\overline{\lambda'_1}g_1, n_1g_2, \pm n_2\overline{\lambda'_2}}(pg_2\check{t}) \Omega_{z, n_1, n_2}(m, pg_2\check{t}) \right|^2 \right)^{1/2} dz. \end{aligned}$$

Expanding the square and changing the order of summation, the n_2 -sum becomes

$$\begin{aligned} (3.22) \quad &\sum_{m_1, m_2} \sum_{\substack{p_1, p_2 \asymp Q_1 \\ p_1, p_2 \text{ prime} \\ (p_1p_2, h\lambda_1\lambda_2) = 1}} \sum_{n_2} v\left(\frac{n_2}{N}\right) \Omega_{z, n_1, n_2}(m_1, p_1g_2\check{t}) \overline{\Omega_{z, n_1, n_2}(m_2, p_2g_2\check{t})} \\ &\quad \times \Sigma_{h, m_1\overline{\lambda'_1}g_1, n_1g_2, \pm n_2\overline{\lambda'_2}}(p_1g_2\check{t}) \overline{\Sigma_{h, m_2\overline{\lambda'_1}g_1, n_1g_2, \pm n_2\overline{\lambda'_2}}(p_2g_2\check{t})}. \end{aligned}$$

From (3.18), we see that the above is negligible unless

$$(3.23) \quad m_1, m_2 \ll x^\varepsilon \frac{|\lambda_1|Q^2}{g_1^2 x} \text{ and } N \ll x^\varepsilon \frac{|\lambda_2|Q^3}{g_2^3 Hx}.$$

Applying Poisson summation modulo $[p_1, p_2]g_2\check{t}$, the inner sum becomes

$$(3.24) \quad \begin{aligned} & \frac{N}{[p_1, p_2]g_2\check{t}} \sum_{n_2} \mathcal{T}(h, m_1\overline{\lambda'_1}g_1, m_2\overline{\lambda'_1}g_1, n_1g_2, \pm n_2\overline{\lambda'_2}, p_1, p_2, g_2\check{t}) \\ & \times \int_{\mathbb{R}} v(x)\Omega_{z, n_1, Nx}(m_1, p_1g_2\check{t})\overline{\Omega_{z, n_1, Nx}(m_2, p_2g_2\check{t})}e\left(-\frac{xn_2}{[p_1, p_2]g_2\check{t}}\right)dx \end{aligned}$$

using the notation (2.8). We write

$$t = g_2\check{t} = t_1 t_2$$

with t_1 squarefree, t_2 squarefull and $(t_1, t_2) = 1$. Note that we can choose representatives of $\overline{\lambda'_1}(\bmod \tilde{q}), \overline{\lambda'_2}(\bmod \tilde{q})$ such that $(\overline{\lambda'_1}\overline{\lambda'_2}, t) = 1$. Then Lemma 8 tells us that \mathcal{T} is bounded by

$$\begin{cases} p_1^2(m_1 - m_2, p_1)t^{3+\varepsilon}, & p_1 = p_2, n_2 = 0, \\ p_1^3 t^{5/2+\varepsilon}(hn_2 p_1, t_1)^{1/2} t_2^{1/2}, & p_1 = p_2, n_2 \neq 0, \\ p_1^{3/2} p_2^{3/2} t^{5/2+\varepsilon}(hn_2, t_1)^{1/2} t_2^{1/2}, & p_1 \neq p_2, n_2 \neq 0, \\ 0, & p_1 \neq p_2, n_2 = 0. \end{cases}$$

Integration by parts shows that the contribution from $|n_2| \geq N_2 := x^\varepsilon \frac{[p_1, p_2]g_2\check{t}}{N}$ is negligible, so that (3.24) is bounded by

$$(3.25) \quad \begin{aligned} & \left(\frac{N}{p_1 t} p_1^2(m_1 - m_2, p_1)t^{3+\varepsilon} + \frac{N}{p_1 t} \sum_{1 \leq |n_2| \leq N_2} p_1^3 t^{5/2+\varepsilon}(hn_2 p_1, t_1)^{1/2} t_2^{1/2} \right) \mathbf{1}_{p_1=p_2} \\ & + \frac{N}{p_1 p_2 t} \sum_{1 \leq |n_2| \leq N_2} p_1^{3/2} p_2^{3/2} t^{5/2+\varepsilon}(hn_2, t_1)^{1/2} t_2^{1/2} \mathbf{1}_{p_1 \neq p_2} \\ & \ll x^\varepsilon \left(N p_1(m_1 - m_2, p_1)t^2 \mathbf{1}_{p_1=p_2} + p_1^3 t^{5/2}(hp_1, t_1)^{1/2} t_2^{1/2} \mathbf{1}_{p_1=p_2} + p_1^{3/2} p_2^{3/2} t^{5/2}(h, t_1)^{1/2} t_2^{1/2} \right). \end{aligned}$$

We assume

$$(3.26) \quad Q \geq (x|\lambda_1|)^{1/2}$$

and sum this first over m_1, m_2 , keeping in mind (3.23) and the fact that $|\lambda_1|Q^2/g_1^2 x \geq 1$, and then over p_1, p_2 . In this way we bound (3.22) by

$$\begin{aligned} & \sum_{p_1 \asymp Q_1} \frac{x^\varepsilon |\lambda_1| Q^2}{g_1^2 x} N p_1^2 t^2 + \frac{x^\varepsilon |\lambda_1|^2 Q^4}{g_1^4 x^2} \left(\sum_{p_1 \asymp Q_1} \left(N p_1 t^2 + p_1^3 t^{5/2}(hp_1, t_1)^{1/2} t_2^{1/2} \right) \right. \\ & \quad \left. + \sum_{p_1, p_2 \asymp Q} p_1^{3/2} p_2^{3/2} t^{5/2}(h, t_1)^{1/2} t_2^{1/2} \right), \end{aligned}$$

where the first term is the contribution of $m_1 = m_2$ in the first term of (3.25).

Substituting this back into (3.21), we see that $S_1(N, H)$ is bounded by

$$\begin{aligned} &\ll \frac{x^{3/2+\varepsilon}}{|\lambda_1||\lambda_2|^{1/2}Q_1^{7/2}Q_2^3} \left(\sum_{g_2|\lambda_2} \sum_{\check{t} \asymp Q_2/g_2} \sum_{g_1|(\check{t}, \lambda_1)} g_1^2 \left(1 + \frac{H^2 g_2^3 N x}{|\lambda_2| Q^3} \right)^{-A} \frac{|\lambda_1|^2 Q^4}{g_1^4 x^2} \left[\sum_{p_1 \asymp Q_1} \frac{g_1^2 x}{|\lambda_1| Q^2} (N p_1^2 t^2) \right. \right. \\ &\quad \left. \left. + \sum_{p_1 \asymp Q_1} (N p_1 t^2 + p_1^3 t^{5/2} (h p_1, t_1)^{1/2} t_2^{1/2}) + \sum_{p_1, p_2 \asymp Q_1} p_1^{3/2} p_2^{3/2} t^{5/2} (h, t_1)^{1/2} t_2^{1/2} \right] \right)^{1/2} \\ &\ll \frac{x^{1/2+\varepsilon}}{|\lambda_2|^{1/2} Q_1^{3/2} Q_2} \left(\sum_{g_2|\lambda_2} \sum_{\check{t} \asymp Q_2/g_2} \sum_{g_1|(\check{t}, \lambda_1)} \left[\left(\frac{x Q_1}{|\lambda_1| Q^2} + 1 \right) \frac{|\lambda_2| Q^3}{g_2^3 H^2 x} Q_1^2 t^2 + Q_1^5 t^{5/2} (t_1, h)^{1/2} t_2^{1/2} \right] \right)^{1/2} \\ &\ll \frac{x^{1/2+\varepsilon}}{|\lambda_2|^{1/2} Q_1^{3/2} Q_2} \left(\sum_{t \asymp Q_2} \left[\left(\frac{x Q_1}{|\lambda_1| Q^2} + 1 \right) \frac{|\lambda_2| Q^3}{x} Q_1^2 t^2 + Q_1^5 t^{5/2} (t_1, h)^{1/2} t_2^{1/2} \right] \right)^{1/2} \\ &\ll \frac{x^{1/2+\varepsilon}}{|\lambda_2|^{1/2} Q_1^{3/2} Q_2} \left(\frac{|\lambda_2|}{|\lambda_1|} Q^2 + \frac{|\lambda_2| Q^3}{x} Q_1^2 Q_2^3 + Q_1^5 Q_2^{7/2} \right)^{1/2} \\ &\ll x^\varepsilon \left(\frac{x^{1/2} Q}{|\lambda_1|^{1/2} Q_1^{1/2}} + \frac{Q^2}{Q_1} + \frac{x^{1/2} Q}{|\lambda_2|^{1/2} Q_2^{1/4}} \right) \end{aligned}$$

under the assumptions (3.20) and (3.26).

On the other hand, if $H \gg Q_1$, we apply (3.19) to see that

$$S_1(N, H) \ll x^\varepsilon \left(\frac{x}{Q_1} + \frac{Q^2}{Q_1} \right).$$

This is dominated by the previous bound under (3.26), and so we obtain the final bound

$$(3.27) \quad S_1 \ll x^\varepsilon \left(\frac{x^{1/2} Q}{|\lambda_1|^{1/2} Q_1^{1/2}} + \frac{Q^2}{Q_1} + \frac{x^{1/2} Q}{|\lambda_2|^{1/2} Q_2^{1/4}} \right),$$

provided that (3.26) holds.

3.3. The endgame. By the Cauchy–Schwarz inequality we have (recall that $(\lambda_1, \lambda_2) = 1$)

$$\begin{aligned} &\sum_{\lambda_1 m - \lambda_2 n = h} A(n, 1) \tau(m) W_0 \left(\frac{|\lambda_1| m}{x} \right) W \left(\frac{|\lambda_2| n}{x} \right) \ll x^\varepsilon \sum_{\substack{\lambda_2 n \equiv h \pmod{|\lambda_1|} \\ n \ll x/|\lambda_2|}} |A(n, 1)| \\ &\ll x^\varepsilon \left(\frac{x}{|\lambda_2|} \right)^{1/2} \left(1 + \frac{x}{|\lambda_2| |\lambda_1|} \right)^{1/2} \ll x^\varepsilon \left(\frac{x}{|\lambda_2| |\lambda_1|^{1/2}} + \frac{x^{1/2}}{|\lambda_2|^{1/2}} \right). \end{aligned}$$

Hence we can assume $|\lambda_1| \ll x^{1/21}$ (otherwise we use the previous trivial bound), in which case we choose C, C_0, Q_1, Q_2 as

$$(3.28) \quad C_0 = x^{19/42-\eta}, \quad C = x^{23/42+\eta}, \quad Q_1 = x^{4/21}, \quad Q_2 = q^{8/21}$$

for some fixed, but arbitrarily small $\eta > 0$, so $Q = x^{12/21}$. For the present situation, we could choose $\eta = 0$, but these values are designed to work also for the proof of Theorem 2.

With this choice, we see that (3.5) and (3.26) hold and from (3.16) and (3.27) we find that

$$S_1, S_2 \ll x^{41/42+\eta+\varepsilon}.$$

Since η can be arbitrarily small, this completes the proof of Theorem 1.

4. PROOF OF THEOREM 2

We indicate the modifications of the previous proof necessary for the proof of Theorem 2. The only difference is that the classical Voronoi summation formula (Lemma 3) is replaced with Lemma 6. This has the same structure except that $rq = 0$ can come from the three sources $r = q = 0$, $r \neq q = 0$ and $q \neq r = 0$. In our application the “half-diagonal” terms $rq = 0$ but $(r, q) \neq (0, 0)$ will not play a major role, since we may assume that A, B in $\tau_{A,B}(m)$ are roughly of equal size, otherwise we apply Voronoi summation as follows. We have

$$\begin{aligned} \mathcal{S}_{A,B}(x) &:= \sum_{\lambda_1 m - \lambda_2 n = h} A(n, 1) \tau_{A,B}(m) W\left(\frac{|\lambda_2|n}{x}\right) \\ &= \sum_a v_1\left(\frac{a}{A}\right) \sum_{\lambda_2 n \equiv -h \pmod{\lambda_1 a}} A(n, 1) W\left(\frac{|\lambda_2|n}{x}\right) v_2\left(\frac{\lambda_2 n + h}{\lambda_1 a B}\right). \end{aligned}$$

Since λ_1, λ_2, h are pairwise coprime, the congruence is void unless $(\lambda_2, a) = 1$. Hence we obtain

$$\begin{aligned} &\sum_{(a, \lambda_2)=1} v_1\left(\frac{a}{A}\right) \sum_{n \equiv -\overline{\lambda_2}h \pmod{\lambda_1 a}} A(n, 1) W\left(\frac{|\lambda_2|n}{x}\right) v_2\left(\frac{\lambda_2 n + h}{\lambda_1 a B}\right) \\ &= \sum_{\substack{\lambda_1 | a \\ (a, \lambda_2)=1}} v_1\left(\frac{a}{|\lambda_1| A}\right) \frac{1}{a} \sum_{\alpha | a} \sum_{\substack{b \pmod{\alpha} \\ (b, \alpha)=1}} \sum_n A(n, 1) e\left(\frac{(n + \overline{\lambda_2}h)b}{\alpha}\right) W\left(\frac{|\lambda_2|n}{x}\right) v_2\left(\frac{\lambda_2 n + h}{a B}\right) \end{aligned}$$

A standard application of the Voronoi summation formula (Lemma 5, cf. also (3.7)) as before bounds this by

$$\sum_{\substack{(a, \lambda_2)=1 \\ \lambda_1 | a \asymp |\lambda_1| A}} \frac{1}{a} \sum_{\alpha | a} \frac{1}{\alpha^2} \sum_{\pm} \sum_{n_2} \sum_{n_1 | \alpha} |\Sigma_{\overline{\lambda_2}h, 0, n_1, \pm n_2}(\alpha)| n_1 |A(n_1, n_2)| \frac{x}{|\lambda_2|} \left(\frac{n_1^2 n_2 x}{|\lambda_2| \alpha^3}\right)^{-1/2} \left(1 + \frac{n_1^2 n_2 x}{|\lambda_2| \alpha^3}\right)^{-K}$$

using the notation (2.7). By Lemma 7 (with the usual notation $\alpha = \alpha_1 \alpha_2$ where α_2 is the squarefull part of α) we obtain for any $K > 1$ the bound

$$\begin{aligned} &x^{1/2+\varepsilon} \sum_{\lambda_1 | a \asymp |\lambda_1| A} \frac{1}{a} \sum_{\alpha | a} \left(\frac{\alpha \alpha_2}{|\lambda_2|}\right)^{1/2} \sum_{n_2} \sum_{n_1 | \alpha} \frac{|A(n_1, n_2)|}{n_2^{1/2}} \left(1 + \frac{n_1^2 n_2 x}{|\lambda_2| \alpha^3}\right)^{-K} \\ &\ll x^\varepsilon \sum_{\lambda_1 | a \asymp |\lambda_1| A} \frac{1}{a} \sum_{\alpha | a} \alpha^2 \alpha_2^{1/2} \ll x^\varepsilon |\lambda_1|^{3/2} A^2. \end{aligned}$$

Exchanging the roles of A and B , we obtain the bound

$$(4.1) \quad \mathcal{S}_{A,B}(x) \ll x^\varepsilon |\lambda_1|^{3/2} \min(A^2, B^2).$$

After this preliminary bound, we follow the proof of Theorem 1 to write $\mathcal{S}_{A,B}(x) = S_1 + S_2$ where

$$S_1 = \sum_{n,m} A(n, 1) W\left(\frac{|\lambda_2|n}{x}\right) \tau_{A,B}(m) \int_0^1 \chi(\alpha) e((\lambda_1 m - \lambda_2 n - h)\alpha) d\alpha,$$

$$S_2 = \sum_{n,m} A(n, 1) W\left(\frac{|\lambda_2|n}{x}\right) \tau_{A,B}(m) \int_0^1 (1 - \chi(\alpha)) e((\lambda_1 m - \lambda_2 n - h)\alpha) d\alpha.$$

Note that Lemma 6 together with $AB \asymp x/|\lambda_1|$ gives

$$\begin{aligned} & \sum_m \tau_{A,B}(m) e\left(\lambda_1 m \left(\frac{b}{c} + z\right)\right) \\ &= \frac{1}{\tilde{c}} \sum_{a,b} e\left(\frac{-abb\lambda'_1}{\tilde{c}}\right) \int_{\mathbb{R}^2} v_1\left(\frac{u}{A}\right) v_2\left(\frac{w}{B}\right) e(uw\lambda_1 z) e\left(\frac{ua+wb}{\tilde{c}}\right) du dw \\ &\ll \frac{AB}{\tilde{c}} \sum_{a,b} \left(1 + \frac{Aa}{\tilde{c}} + \frac{Bb}{\tilde{c}}\right)^{-K} + \frac{1}{1+x|z|} \left(1 + \frac{Aa}{\tilde{c}(1+x|z|)} + \frac{Bb}{\tilde{c}(1+x|z|)}\right)^{-K} \\ &\ll \frac{AB}{\tilde{c}} + A + B + \tilde{c}(1+x|z|) \end{aligned}$$

for any $K > 0$ as an analogue of (3.3). Thus we have

$$\max_{C_0 \leq c \leq C} \max_{\substack{b \pmod{c} \\ (b,c)=1}} \max_{|z| \ll (cC)^{-1}} \left| \sum_m \tau_{A,B}(m) e\left(\lambda_1 m \left(\frac{b}{c} + z\right)\right) \right| \ll x^\varepsilon \left(\frac{x}{C_0} + A + B + C \right),$$

so that the estimate corresponding to (3.4) becomes

$$S_{2,1} \ll x^\varepsilon \frac{1}{Q^{1/2}} \left(\frac{x}{|\lambda_2|} \right)^{1/2} \left(\frac{x}{C_0} \left(1 + \frac{(A+B)C_0}{x}\right) + C \right).$$

For $c \leq C_0$ and $z \leq (cC)^{-1}$ we see that the contribution from $ab \neq 0$ is negligible unless

$$a \ll x^\varepsilon \frac{\tilde{c}(1+x|z|)}{A} \ll x^\varepsilon \left(\frac{C_0}{A} + \frac{x}{AC} \right), \quad b \ll x^\varepsilon \frac{\tilde{c}(1+x|z|)}{B} \ll x^\varepsilon \left(\frac{C_0}{B} + \frac{x}{BC} \right).$$

Thus, if we choose

$$(4.2) \quad C_0 \leq x^{-\eta} \min(A, B), \quad C \geq \max(x/A, x/B)x^\eta = |\lambda_1| \max(A, B)x^\eta$$

(since $AB = x/|\lambda_1|$) for some very small $\eta > 0$, then the contribution from $ab \neq 0$ is negligible. Moreover, the contribution from $a = 0$ or $b = 0$ is $\ll A + B \ll AB/C_0$, thus dominated by the contribution from $a = b = 0$. As in (3.6) we choose $z \ll x^{-1+\varepsilon}$. Therefore, under the assumptions in (4.2), we see the same bounds in (3.8) and (3.15) hold, and we conclude

$$(4.3) \quad S_2 \ll x^{1+\varepsilon} \left(\frac{x}{|\lambda_2|^{1/2} Q C_0} + \frac{\max(A, B)}{|\lambda_2|^{1/2} Q} + \frac{C}{|\lambda_2|^{1/2} Q} + \frac{C_0^2}{|\lambda_1|^{1/2} x Q_2} + \frac{C_0^3}{|\lambda_1|^{1/2} x Q_2} + \frac{C_0}{|\lambda_1|^{1/2} Q} \right)$$

as an analogue of (3.16).

For S_1 , we see that for $q \leq Q, z \leq x^{-1+\varepsilon}$, the main terms in the dual summation when $ab = m = 0$ are of size

$$\frac{AB}{\tilde{q}} \left(1 + \frac{\tilde{q}}{A} + \frac{\tilde{q}}{B}\right) \ll \frac{AB}{\tilde{q}} \left(1 + \frac{Q}{A} + \frac{Q}{B}\right).$$

so that the trivial bound becomes (cf. (3.19) where the first term comes from $m = 0$)

$$S_1(N, H) \ll x^\varepsilon \left(\frac{x}{H} \left(1 + \frac{Q}{A} + \frac{Q}{B}\right) + \frac{Q^2}{H} \right).$$

If we choose

$$(4.4) \quad Q \gg |\lambda_1| \max(A, B),$$

then the arguments after the Cauchy steps remain the same since we have $\tilde{q}^2/(x/|\lambda_1|) = \tilde{q}^2/AB \geq 1$ terms for the dual variables a, b .

Thus the analogue of (3.27) becomes

$$(4.5) \quad S_1 \ll x^\varepsilon \left(\frac{x^{1/2}Q}{|\lambda_1|^{1/2}Q_1^{1/2}} + \frac{Q^2}{Q_1} + \frac{x^{1/2}Q}{|\lambda_2|^{1/2}Q_2^{1/4}} \right) + x^\varepsilon \left(\frac{xQ}{Q_1 \min(A, B)} + \frac{Q^2}{Q_1} \right).$$

With (4.1), (4.3) and (4.5) we can conclude the proof in a similar way. Again we can assume without loss of generality $|\lambda_1| \leq x^{1/21}$. Moreover, by (4.1) we can assume $\min(A, B) \geq x^{19/42}$ and hence $|\lambda_1| \max(A, B) \leq x^{23/42}$. Now we make the same choice as in (3.28), which satisfies (4.2) and (4.4), getting again

$$S_1, S_2 \ll x^{41/42+\eta+\varepsilon}.$$

This completes the proof.

Remark 2. We have assumed that the weight functions W, W_0, v_1, v_2 in Theorems 1 and 2 are fixed. During the proofs they are subject to finitely many integrations by parts, and hence it is clear that the implicit constants depend on some suitable Sobolev norms of these weight functions.

5. PROOF OF THEOREM 3

Let F be a cusp form for the group $\mathrm{SL}(3, \mathbb{Z})$ and χ an even primitive Dirichlet character modulo q . We start with a standard approximate functional equation [IK, Theorem 5.3, Proposition 5.4]

$$L(1/2, F \times \chi) = \sum_n \frac{A(n, 1)\chi(n)}{\sqrt{n}} V\left(\frac{n}{q^{3/2}X}\right) + \frac{\tau(\chi)^3}{q^{3/2}} \sum_n \frac{\overline{A(n, 1)\chi(n)}}{\sqrt{n}} V\left(\frac{nX}{q^{3/2}}\right)$$

where

$$\tau(\chi) = \sum_{h \pmod q} \chi(h)e(h/q)$$

is the standard Gauss sum, $X > 0$ is a parameter at our disposal and V is a smooth function depending on F satisfying

$$y^j V^{(j)}(y) \ll_{j,A} (1+y)^{-A}$$

for any $j, A \geq 0$ and

$$V(y) = 1 + O(y^{1/21}), \quad y \rightarrow 0$$

using the constant $5/14$ [KS, Proposition 1] towards the Ramanujan conjecture for the archimedean Langlands parameters of F . (The exponent $\frac{1}{21} = \frac{1}{3}(\frac{1}{2} - \frac{5}{14})$ suffices for our purpose, but could be improved by modifying [IK, Proposition 5.4].) We will optimize X later, for now we assume

$$Q^{1/2} \leq X \leq Q^{2/3}.$$

If f is any function on characters, then by Möbius inversion we have

$$\sum_{\substack{\chi \pmod q \\ \chi \text{ primitive, even}}} f(\chi) = \frac{1}{2} \sum_{\substack{\chi \pmod q \\ \chi \text{ primitive}}} (1 + \chi(-1))f(\chi) = \frac{1}{2} \sum_{d|q} \mu\left(\frac{q}{d}\right) \sum_{\chi \pmod d} (1 + \chi(-1))f(\chi^*)$$

where χ^* denotes the character modulo q induced from χ . Hence

$$\mathcal{L}(Q) := \sum_q W\left(\frac{q}{Q}\right) \sum_{\substack{\chi \pmod{q} \\ \chi \text{ primitive, even}}} L(1/2, F \times \chi) = T_1 + T_2$$

where

$$T_1 = \frac{1}{2} \sum_{\pm} \sum_q W\left(\frac{q}{Q}\right) \sum_{d|q} \mu\left(\frac{q}{d}\right) \phi(d) \sum_{\substack{n \equiv \pm 1 \pmod{d} \\ (n,q)=1}} \frac{A(n,1)}{\sqrt{n}} V\left(\frac{n}{q^{3/2}X}\right)$$

and

$$T_2 = \sum_q W\left(\frac{q}{Q}\right) \sum_{(n,q)=1} \overline{\frac{A(n,1)}{\sqrt{n}}} V\left(\frac{nX}{q^{3/2}}\right) \mathcal{K}(n; d, q)$$

with

$$\mathcal{K}(n; d, q) = \frac{1}{2} \sum_{d|q} \mu\left(\frac{q}{d}\right) \sum_{\chi \pmod{d}} (1 + \chi(-1)) \frac{\tau(\chi^*)^3}{q^{3/2}} \bar{\chi}(n).$$

By [IK, Lemma 3.1] we have

$$\begin{aligned} \mathcal{K}(n; d, q) &= \frac{1}{2q^{3/2}} \sum_{d|q} \mu\left(\frac{q}{d}\right) \sum_{\chi \pmod{d}} (1 + \chi(-1)) \left(\mu\left(\frac{q}{d}\right) \chi\left(\frac{q}{d}\right) \tau(\chi) \right)^3 \bar{\chi}(n) \\ &= \frac{1}{2q^{3/2}} \sum_{\substack{d|q \\ (d, \frac{q}{d})=1}} \mu^2\left(\frac{q}{d}\right) \sum_{\pm} \sum_{\chi \pmod{d}} \sum_{h_1, h_2, h_3 \pmod{d}} \chi\left(\pm h_1 h_2 h_3 \bar{n} \frac{q^3}{d^3}\right) e\left(\frac{h_1 + h_2 + h_3}{d}\right) \\ &= \frac{1}{2q^{3/2}} \sum_{\substack{d|q \\ (d, \frac{q}{d})=1}} \mu^2\left(\frac{q}{d}\right) \phi(d) \sum_{\pm} \sum_{\substack{h_1, h_2 \pmod{d} \\ (h_1 h_2, d)=1}} e\left(\frac{h_1 + h_2 \pm \sqrt{h_1 h_2 n q^3 / d^3}}{d}\right). \end{aligned}$$

By Deligne's bound for hyper-Kloosterman sums, we obtain

$$\mathcal{K}(n; d, q) \ll q^{1/2+\varepsilon},$$

and so by trivial estimates

$$(5.1) \quad T_2 \ll \frac{Q^{9/4+\varepsilon}}{X^{1/2}}.$$

For T_1 , we single out the contribution $\pm = +$, $n = 1$ which equals

$$\begin{aligned} (5.2) \quad &\frac{1}{2} \sum_q W\left(\frac{q}{Q}\right) \sum_{d|q} \mu\left(\frac{q}{d}\right) \phi(d) \left(1 + O\left(\frac{1}{(Q^{3/2}X)^{1/21}}\right)\right) \\ &= \frac{1}{2} \int_{(3)} \sum_{d,q} \frac{\mu(q)\phi(d)}{(dq)^s} \widetilde{W}(s) Q^s \frac{ds}{2\pi i} + O\left(\frac{Q^2}{(Q^{3/2}X)^{1/21}}\right) \\ &= \frac{1}{2} \int_{(3)} \frac{\zeta(s-1)}{\zeta(s)^2} \widetilde{W}(s) Q^s \frac{ds}{2\pi i} + O\left(\frac{Q^2}{(Q^{3/2}X)^{1/21}}\right) \\ &= \frac{\widetilde{W}(2)}{2\zeta(2)^2} Q^2 + O\left(Q^{3/2} + \frac{Q^2}{(Q^{3/2}X)^{1/21}}\right) \end{aligned}$$

by a standard contour shift argument.

We now make a number of technical adjustments in preparation for applying Theorem 2.

We open the Euler ϕ -function by writing $\phi(d) = \sum_{d_1 d_2 = d} \mu(d_2) d_1$, and we write $q = d_1 d_2 d'$. Next we apply a smooth partitions of unity to the d_1 -sum and the n -sum by attaching weight functions

$$v\left(\frac{n}{N}\right) \left(\frac{n}{N}\right)^{1/2} v\left(\frac{d_1}{D}\right) \left(\frac{d_1}{D}\right)^{-1}$$

where

$$(5.3) \quad N \ll Q^{3/2+\varepsilon} X$$

up to a negligible error. We open the existing weight function $V(n/q^{3/2}X)$ by Mellin inversion as follows

$$V\left(\frac{n}{q^{3/2}X}\right) = \int_{(0)} \left(\frac{N}{Q^{3/2}}\right)^{-s} \tilde{V}(s) \left(\frac{nQ^{3/2}}{Nq^{3/2}X}\right)^{-s} \frac{ds}{2\pi i}.$$

Since $\tilde{V}(s) \ll (1+|s|)^{-A}$, we can truncate the integral at $|\Im s| \leq Q^\varepsilon$ at the cost of a negligible error, pull it outside and in this way separate the variables n, q at essentially no cost. By slight abuse of notation, we replace $v(x)$ with $v(x)x^{-s}$ (without changing the notation). By Remark 2 this only results in additional Q^ε -powers.

Thus we are left with bounding

$$T_1^\pm := \frac{D}{N^{1/2}} \sum_{d_2, d'} \left| \sum_{d_1} W\left(\frac{d_1 d_2 d'}{Q}\right) v\left(\frac{d_1}{D}\right) \sum_{\substack{n \equiv \pm 1 \pmod{d_1 d_2} \\ (n, d')=1}}^* A(n, 1) v\left(\frac{n}{N}\right) \right|$$

(up to an outside integration of length Q^ε), where \sum^* indicates that in $+$ -term the summand $n = 1$ has been removed. At this point we record a trivial bound: we write $n = \pm 1 + d_1 d_2 r$ and note the removal of the $n = 1$ term implies $n \gg d_1 d_2$, so $r \asymp N/d_1 d_2$. Thus we obtain

$$\begin{aligned} T_1^\pm &\ll \frac{D}{N^{1/2}} \sum_{d_2, d'} \sum_{d_1} W\left(\frac{d_1 d_2 d'}{Q}\right) v\left(\frac{d_1}{D}\right) \sum_{r \asymp N/d_1 d_2} |A(1 + d_1 d_2 r, 1)| \\ &\ll \frac{D}{N^{1/2}} N^{1+\varepsilon} \frac{Q}{D} \ll Q^{1+\varepsilon} N^{1/2}. \end{aligned}$$

This is more than sufficient for $N \leq 10$, say, and for larger N , which we assume from now on, the removal of the $n = 1$ term is invisible; hence we remove the asterisk from the sum.

Finally we remove the coprimality condition $(n, d') = 1$. Using the Hecke relations [BL, (2.2)], we write the n -sum as

$$\begin{aligned} &\sum_{f|d'} \mu(f) \sum_{fn \equiv \pm 1 \pmod{d_1 d_2}} A(fn, 1) v\left(\frac{fn}{N}\right) \\ &= \sum_{f_1|f_2|f|d'} \mu(f) \mu(f_1) \mu(f_2) A\left(\frac{f}{f_2}, \frac{f_2}{f_1}\right) \sum_{f_1 f_2 f n \equiv \pm 1 \pmod{d_1 d_2}} A(n, 1) v\left(\frac{f_1 f_2 f n}{N}\right) \end{aligned}$$

We conclude (for $N \geq 10$) that

$$T_1^\pm \ll \frac{D}{N^{1/2}} \sum_{d_2 f_1 g_1 g_2 g_3 \ll Q/D} |A(g_2, g_1)| \left| \sum_{d_1} v\left(\frac{d_1}{D}\right) W\left(\frac{d_1 d_2 f_1 g_1 g_2 g_3}{Q}\right) \sum_{B n \equiv \pm 1 \pmod{d_1 d_2}} A(n, 1) v\left(\frac{B n}{N}\right) \right|$$

with

$$B = f_1^3 g_1^2 g_2.$$

We write the congruence as an equation

$$Bn - d_2m = \pm 1, \quad m = rd_1,$$

and we can insert a redundant smooth weight function localizing the new variable $r \asymp N/Dd_2$. Hence we are in a position to apply Theorem 2, getting

$$(5.4) \quad T_1^\pm \ll Q^\varepsilon \frac{D}{N^{1/2}} \frac{Q}{D} N^{41/42} = Q^{1+\varepsilon} N^{10/21} \ll Q^{12/7+\varepsilon} X^{10/21}$$

by (5.3).

We choose $X = Q^{45/82}$ and combine (5.1), (5.2) and (5.4) to finish the proof of Theorem 3.

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