

# ON SPLITTINGS OF DEFORMATIONS OF PAIRS OF COMPLEX STRUCTURES AND HOLOMORPHIC VECTOR BUNDLES

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**ABSTRACT.** We can show that the Kuranishi space of a pair  $(M, E)$  of a compact Kähler manifold  $M$  and its flat Hermitian vector bundle  $E$  is isomorphic to the direct product of the Kuranishi space of  $M$  and the Kuranishi space of  $E$ . We study non-Kähler case. We show that the Kuranishi space of a pair  $(M, E)$  of a complex parallelizable nilmanifold  $M$  and its trivial holomorphic vector bundle  $E$  is isomorphic to the direct product of the Kuranishi space of  $M$  and the Kuranishi space of  $E$ . We give examples of pairs  $(M, E)$  of nilmanifolds  $M$  with left-invariant abelian complex structures and their trivial holomorphic line bundles  $E$  such that the Kuranishi spaces of pairs  $(M, E)$  are not isomorphic to direct products of the Kuranishi spaces of  $M$  and the Kuranishi spaces of  $E$ .

## 1. INTRODUCTION

Let  $M$  be a compact complex manifold. By Kuranishi [12], we can describe the parameter space  $Kur_M$  of all sufficiently small deformations of complex structures on  $M$  as an analytic germ of an analytic space.  $Kur_M$  is called the Kuranishi space of  $M$ . By the analogue of Kuranishi's construction for deformations of holomorphic vector bundles over  $M$  with a fixed complex structure, we can describe the parameter space  $Kur_E$  of deformations of all sufficiently small holomorphic structures on a holomorphic vector bundle  $E$  ([6, 7, 11]).  $Kur_E$  is also called the Kuranishi space of  $E$ .

In [9], Huang studies deformations of a pair  $(M, E)$  of a complex manifold  $M$  and a holomorphic vector bundle  $E$  over  $M$ . Extending Kuranishi's construction to a pair  $(M, E)$ , we obtain the Kuranishi space  $Kur_{(M, E)}$  of a pair  $(M, E)$ .

The singularity of the Kuranishi space is considered as obstructions of deformations. We are interested in comparing the singularity of  $Kur_{(M, E)}$  with the singularity of  $Kur_M$  and  $Kur_E$ .

**Proposition 1.1.** *Let  $M$  be a compact Kähler manifold and  $E$  be a flat Hermitian vector bundle. Then we have*

$$Kur_{(M, E)} \cong Kur_M \times Kur_E$$

We study deformations of pairs of non-Kähler complex manifolds with trivial holomorphic vector bundles. A nilmanifold is a compact quotient  $\Gamma \backslash G$  of a simply connected nilpotent Lie group  $G$  by a discrete subgroup  $\Gamma$ . It is known that  $\Gamma \backslash G$  admits a Kähler structure if and only if  $G$  is abelian and  $\Gamma \backslash G$  is a torus ([1, 8]). Thus, if  $G$  is non-abelian and admits a left-invariant complex structure,  $\Gamma \backslash G$  is a non-Kähler complex manifold.

**Theorem 1.2.** *Let  $M = \Gamma \backslash G$  be a complex parallelizable nilmanifold and  $E$  be a trivial holomorphic vector bundle. Then we have*

$$Kur_{(M, E)} \cong Kur_M \times Kur_E$$

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Non-parallelizable case is more complicated, we give examples of complex nilmanifolds  $M = \Gamma \backslash G$  with the trivial line bundles  $E$  such that

$$Kur_{(M,E)} \not\cong Kur_M \times Kur_E.$$

Such examples are nilmanifolds with abelian complex structures. Thus, on nilmanifolds with abelian complex structures, we do not have a splitting

$$Kur_{(M,E)} \not\cong Kur_M \times Kur_E.$$

in general.

**Theorem 1.3.** *Let  $M = \Gamma \backslash G$  be a complex nilmanifold equipped with an abelian complex structure and  $E$  be a trivial holomorphic vector bundle. Then if any small deformation of the complex structure is also abelian, then we have*

$$Kur_{(M,E)} \cong Kur_M \times Kur_E.$$

## 2. KURANISHI SPACES OF DIFFERENTIAL GRADED LIE ALGEBRAS

A differential graded Lie algebra (shortly DGLA)  $(L^*, d, [,])$  is a  $\mathbb{N}$ -graded vector space  $L^*$  equipped with a differential  $d$  and a graded Lie bracket  $[,]$  satisfying the Leibniz rule. An analytic DGLA is a normed DGLA  $(L^*, d, [,])$  whose cohomology  $H^i(L^*)$  is finite-dimensional for every  $i \in \mathbb{N}$ .

The Kuranishi space  $Kur_L$  of  $(L^*, d, [,])$  is an analytic germ of the analytic space  $K_L$  in  $H^1(L^*)$  at 0 defined by the following way. Take a Hodge decomposition  $L^i = \mathcal{H}^i \oplus dL^{i-1} \oplus A^i$  with  $H^i(L^*) \cong \mathcal{H}^i$ . Define the map  $\delta : L^i \rightarrow L^{i-1}$  by the extension of the inverse of  $d : A^{i-1} \rightarrow dL^{i-1}$  associated with the projection  $L^i \rightarrow dL^{i-1}$  and the inclusion  $A^{i-1} \subset L^i$ . Define the map  $F : L^1 \ni x \mapsto x + \frac{1}{2}\delta[x, x] \in L^1$ . Then, the inverse of  $F$  is defined on a small neighbourhood  $U$  of  $\mathcal{H}^1$  and define

$$K_L = \{x \in U \mid H([F^{-1}(x), F^{-1}(x)]) = 0\}$$

where  $H : L^1 \rightarrow \mathcal{H}^1$  is the projection.

$F^{-1}(x)$  can be written in as the Kuranishi series  $F^{-1}(x) = \sum x_i$  such that  $x_1 = x$  and

$$x_k = -\frac{1}{2} \sum_{i+j=k} \delta[x_i, x_j]$$

for  $k \geq 2$ . Considering each  $x_i$  as a homogenous polynomial of  $x$ ,

$$\sum H([x_i, x_j]) = 0$$

are defining equations of  $K_L$ .

For a DGLA homomorphism  $\phi : L_1^* \rightarrow L_2^*$  between analytic DGLAs, if  $\phi$  induces isomorphisms on 0-th and first cohomology and an injection on the second cohomology, then we have an isomorphism  $Kur_{L_1} \cong Kur_{L_2}$ . We say that a morphism  $\phi : L_1^* \rightarrow L_2^*$  is a quasi-isomorphism if  $\phi$  induces a cohomology isomorphism. If there is a quasi-isomorphism  $\phi : L_1^* \rightarrow L_2^*$ , then we have an isomorphism  $Kur_{L_1} \cong Kur_{L_2}$ .

## 3. KURANISHI SPACES OF JOINT DEFORMATIONS

Let  $M$  be a compact complex manifold and  $E$  a holomorphic vector bundle over  $M$  equipped with a Hermitian metric  $h$ . We know that the graded vector space  $A^{0,*}(M, T^{1,0}M)$  equipped with the Dolbeault differential  $\bar{\partial}_{T^{1,0}}$  and the Schouten–Nijenhuis bracket  $\llbracket \rrbracket_{SN}$  is a DGLA which governs a deformation theory of the complex structure on  $M$ . Kuranishi proves that the Kuranishi space  $Kur_X$  of the DGLA  $A^{0,*}(M, T^{1,0}M)$  is isomorphic to a parameter space of a complete deformation family of a complex manifold  $M$  (see [12] and [7]). We also know that graded vector

space  $A^{0,*}(M, \text{End}(E))$  equipped with the Dolbeault differential  $\bar{\partial}_{\text{End}(E)}$  and the bracket  $[\cdot]_{\text{End}(E)}$  induced by the wedge product on differential forms and the Lie bracket on  $\text{End}(E)$  is a DGLA which governs a deformation theory of the holomorphic structure on  $E$ . As similar to  $Kur_X$ , the Kuranishi space  $Kur_E$  of the DGLA  $A^{0,*}(M, \text{End}(E))$  is isomorphic to a parameter space of a complete deformation family of a complex manifold  $M$  (see [11] and [7]).

Let  $D$  be the Chern connection associated with  $h$  on the holomorphic vector bundle  $E$ . Denote by  $R \in A^{1,1}(M, \text{End}(E))$  the curvature of  $D$ . Consider the graded vector space

$$L^*(M, E) = A^{0,*}(M, T^{1,0}M) \oplus A^{0,*}(M, \text{End}(E)).$$

Define the operator  $\bar{\partial}_R : L^*(M, E) \rightarrow L^{*+1}(M, E)$  by

$$\bar{\partial}_R(\alpha, \beta) = (\bar{\partial}_{T^{1,0}M}\alpha, \bar{\partial}_{\text{End}(E)}\beta - \iota_\alpha R)$$

and the bilinear map  $[\cdot, \cdot]_D : L^*(M, E) \times L^*(M, E) \rightarrow L^*(M, E)$  by

$$[(\alpha_1, \beta_1), (\alpha_2, \beta_2)]_D = ([\alpha_1, \alpha_2]_{SN}, [\beta_1, \beta_2]_{\text{End}(E)} + \iota_{\alpha_1} D\beta_2 - (-1)^{pq} \iota_{\alpha_2} D\beta_1).$$

Then  $(L^*(M, E), \bar{\partial}_R, [\cdot, \cdot]_D)$  is a DGLA. In [9], Huang proves that the Kuranishi space  $Kur_{(M,E)}$  of the DGLA  $L^*(M, E)$  is isomorphic to a parameter space of a complete deformation family of a pair  $(M, E)$  of a complex manifold  $M$  and a holomorphic vector bundle  $E$ .

**Proposition 3.1.** *Let  $M$  be a compact Kähler manifold and  $E$  be a flat Hermitian vector bundle. Then we have*

$$Kur_{(M,E)} \cong Kur_M \times Kur_E$$

*Proof.* Since the Chern connection  $D$  is flat, we have  $\bar{\partial}_R = \bar{\partial}_{T^{1,0}M} \oplus \bar{\partial}_{\text{End}(E)}$ . Consider the subset  $A^{0,*}(M, T^{1,0}M) \oplus \ker \bar{\partial}_{\text{End}(E)}$  in the DGLA  $L^*(M, E)$ . Then, for  $(\alpha, \beta) \in A^{0,*}(M, T^{1,0}M) \oplus \ker \bar{\partial}_{\text{End}(E)}$ ,  $\iota_\alpha D\beta = 0$ . Hence, the subset  $A^{0,*}(M, T^{1,0}M) \oplus \ker \bar{\partial}_{\text{End}(E)}$  is a sub-DGLA in the DGLA  $L^*(M, E)$  such that  $A^{0,*}(M, T^{1,0}M) \oplus \ker \bar{\partial}_{\text{End}(E)}$  is a direct sum of the two DGLAs  $A^{0,*}(M, T^{1,0}M)$  and  $\ker \bar{\partial}_{\text{End}(E)}$ .

As [6], by the  $\partial\bar{\partial}$ -Lemma [5] on flat Hermitian vector bundle  $\text{End}(E)$ , the inclusion

$$\ker \bar{\partial}_{\text{End}(E)} \subset A^{0,*}(M, \text{End}(E))$$

is a quasi-isomorphism and hence  $Kur_E$  is isomorphic to the Kuranishi space of the DGLA  $\ker \bar{\partial}_{\text{End}(E)}$ . By this, the inclusion

$$A^{0,*}(M, T^{1,0}M) \oplus \ker \bar{\partial}_{\text{End}(E)} \subset L^*(M, E)$$

is also a quasi-isomorphism. Thus the Kuranishi space  $Kur_{(M,E)}$  is isomorphic to the Kuranishi space of  $A^{0,*}(M, T^{1,0}M) \oplus \ker \bar{\partial}_{\text{End}(E)}$ . Hence we have

$$Kur_{(M,E)} \cong Kur_M \times Kur_E.$$

□

**Remark 3.2.** *By more careful arguments, the similar statement for deformations of pairs of compact Kähler manifolds and polystable Higgs bundles with vanishing Chern classes is proved in [16].*

**Corollary 3.3.** *Let  $M$  be a compact Kähler manifold and  $E$  be a flat Hermitian vector bundle. Assume that the canonical line bundle of  $M$  is holomorphically trivial. Then  $Kur_{(M,E)}$  is cut out by polynomial equations of degree at most 2.*

*Proof.* By Proposition 3.1, we have  $Kur_{(M,E)} \cong Kur_M \times Kur_E$ . By Tian-Todorov theorem [21, 22] (see also [7]),  $Kur_M$  is smooth. As shown in [6], the quotient map

$$\ker \bar{\partial}_{\text{End}(E)} \rightarrow H_{\bar{\partial}_{\text{End}(E)}}^{0,*}(M, \text{End}(E))$$

is a quasi-isomorphism. This implies that  $Kur_E$  is isomorphic to

$$\left\{ \eta \in H_{\partial_{\text{End}(E)}}^{0,*}(M, \text{End}(E)) \mid [\eta, \eta] = 0 \right\}.$$

Hence,  $Kur_E$  is cut out by polynomial equations of degree at most 2.  $\square$

#### 4. NILMANIFOLDS

Let  $G$  be a  $n$ -dimensional simply connected Lie group with the Lie algebra  $\mathfrak{g}$ . A left-invariant complex structure on  $G$  can be identified with a complex structure on a real Lie algebra  $\mathfrak{g}$  i.e. a sub-algebra  $\mathfrak{g}^{1,0}$  of the complexification  $\mathfrak{g}_{\mathbb{C}}$  such that  $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$  where  $\mathfrak{g}^{0,1} = \overline{\mathfrak{g}^{1,0}}$ . Let  $p^{1,0} : \mathfrak{g}_{\mathbb{C}} \rightarrow \mathfrak{g}^{1,0}$  be the projection. Consider  $\mathfrak{g}^{1,0}$  as a  $\mathfrak{g}^{0,1}$ -module via  $p^{1,0}([X, Y])$  for  $X \in \mathfrak{g}^{0,1}$ ,  $Y \in \mathfrak{g}^{1,0}$ . Then the cochain complex  $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0}$  equipped with the bracket induced by the wedge product and the Lie bracket on  $\mathfrak{g}^{1,0}$  is a DGLA.

Assume  $G$  has a lattice  $\Gamma$ . Consider the complex nilmanifold  $M = \Gamma \backslash G$  with the complex structure induced by a left-invariant complex structure on  $G$ . We have the inclusions

$$\begin{aligned} \bigwedge(\mathfrak{g}^{0,1})^* \otimes \bigwedge^p(\mathfrak{g}^{1,0})^* &\subset A^{p,*}(M) \\ \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} &\subset A^{0,*}(M, T^{1,0}M). \end{aligned}$$

We say that a left-invariant complex structure is of Calabi-Yau type if  $\bigwedge^n \mathfrak{g}^{1,0*}$  is a trivial  $\mathfrak{g}^{0,1}$ -module. Assume that a left-invariant complex structure on  $G$  is of Calabi-Yau type. If the inclusion  $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \bigwedge^p \mathfrak{g}^{1,0*} \subset A^{p,*}(M)$  induces a cohomology isomorphism for any integer  $p$ , then the inclusion  $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \subset A^{0,*}(M, T^{1,0}M)$  also induces a cohomology isomorphism (see [17]).

Assume that  $G$  is nilpotent. In this case any left-invariant complex structure on  $G$  is of Calabi-Yau type (see [2]). We call  $M = \Gamma \backslash G$  a complex nilmanifold. As an analogous of Nomizu's theorem ([15]) for the de Rham cohomology of nilmanifolds, the inclusion  $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \bigwedge^p \mathfrak{g}^{1,0*} \subset A^{p,*}(M)$  induces a cohomology isomorphism for any integer  $p$  if  $M$  has the structure of an iterated principal holomorphic torus bundle ([3, 18]). If  $(G, J)$  is a complex Lie group or  $\mathfrak{g}^{1,0}$  is abelian, then  $M$  has the structure of an iterated principal holomorphic torus bundle.

A complex nilmanifold  $M = \Gamma \backslash G$  is complex parallelizable if and only if  $(G, J)$  is a complex Lie group. The complex structure on a complex nilmanifold  $M = \Gamma \backslash G$  is called abelian if  $\mathfrak{g}^{1,0}$  is abelian.

**Theorem 4.1.** *Let  $M = \Gamma \backslash G$  be a complex parallelizable nilmanifold and  $E$  be a trivial holomorphic vector bundle. Then we have*

$$Kur_{(M,E)} \cong Kur_M \times Kur_E.$$

*Proof.* Take a trivialization  $E \cong M \times \mathbb{C}^r$ . The inclusions

$$\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \subset A^{0,*}(M, T^{1,0}M)$$

and

$$\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C}) \subset A^{0,*}(M, \text{End}(E))$$

are quasi-isomorphisms and hence  $Kur_M$  and  $Kur_E$  are isomorphic to the Kuranishi spaces of  $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0}$  and  $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$  respectively.

Consider the subset  $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \oplus \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$  in the DGLA  $L^*(M, E)$ . We have  $d = \bar{\partial}$  on  $\bigwedge(\mathfrak{g}^{0,1})^*$ . Thus,  $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \oplus \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$  is a sub-DGLA in the DGLA  $L^*(M, E)$

such that  $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \oplus \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$  is a direct sum of the two DGLAs  $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0}$  and  $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$ . Since the inclusion

$$\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \oplus \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C}) \subset L^*(M, E)$$

is a quasi-isomorphism, the Kuranishi space  $Kur_{(M,E)}$  is isomorphic to the Kuranishi space of the direct sum  $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \oplus \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$ . Hence we have

$$Kur_{(M,E)} \cong Kur_M \times Kur_E.$$

□

**Corollary 4.2.** *Let  $M = \Gamma \backslash G$  be a complex parallelizable nilmanifold and  $E$  be a trivial holomorphic vector bundle. Assume that the Lie algebra  $\mathfrak{g}$  of  $G$  is  $\nu$ -step and naturally graded. Then  $Kur_{(M,E)}$  is cut out by polynomial equations of degree at most  $\nu + 1$ .*

*Proof.* By Theorem 4.1, we have  $Kur_{(M,E)} \cong Kur_M \times Kur_E$ . By [19],  $Kur_M$  is cut out by polynomial equations of degree at most  $\nu$ . By [10],  $Kur_E$  is cut out by polynomial equations of degree at most  $\nu + 1$ . □

**Theorem 4.3.** *Let  $M = \Gamma \backslash G$  be a complex nilmanifold equipped with an abelian complex structure and  $E$  be a trivial holomorphic vector bundle. Then if any small deformation of the complex structure is also abelian, then we have*

$$Kur_{(M,E)} \cong Kur_M \times Kur_E.$$

*Proof.* By the same argument of the proof of Theorem 4.1,  $Kur_X$  and  $Kur_E$  are isomorphic to the Kuranishi spaces of  $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0}$  and  $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$  respectively.

Consider the subset  $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \oplus \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$  in the DGLA  $L^*(M, E)$ . Then  $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \oplus \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$  is a sub-DGLA in the DGLA  $L^*(M, E)$  such that the inclusion

$$\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \oplus \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C}) \subset L^*(M, E)$$

is a quasi-isomorphism. We have  $d = \bar{d}$  on  $\bigwedge(\mathfrak{g}^{0,1})^*$ . Thus  $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \oplus \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$  is not a direct-sum as a DGLA. We study Kuranishi spaces of DGLAs  $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0}$ ,  $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$  and  $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \oplus \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$  precisely.

Consider a Kuranishi series  $\sum x_i$  of the DGLA  $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \oplus \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$ . Write  $x_i = \varphi_i + \psi_i$  with  $\varphi_i \in \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0}$  and  $\psi_i \in \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$ . Then  $\sum \varphi_i$  is a Kuranishi series of the DGLA  $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0}$ . [4, Theorem 4, Proposition 1] says that for an abelian complex structure, a Kuranishi series  $\sum \varphi_i$  of  $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0}$  defines an abelian deformation if and only if  $\sum \varphi_i = \varphi_1$  and

$$\iota_{\varphi_1} \alpha = 0$$

for any  $\alpha \in (\mathfrak{g}^{0,1})^*$ . Thus, by the assumption, we have  $\sum \varphi_i = \varphi_1$  and  $[\varphi_1, (\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0}]_d = 0$ . Thus  $\sum \psi_i$  is a Kuranishi series of  $\bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_r(\mathbb{C})$  and defining equations of  $Kur_{(M,E)}$  are only the equations of  $Kur_E$ . Hence we have we have

$$Kur_{(M,E)} \cong Kur_X \times Kur_E.$$

□

If the Lie algebra  $\mathfrak{g}$  of  $G$  is isomorphic to one of the following Lie algebras:

$$\mathfrak{n}_3 = (0, 0, 0, 0, 0, 12 + 34), \mathfrak{n}_8 = (0, 0, 0, 0, 0, 12), \mathfrak{n}_9 = (0, 0, 0, 0, 12, 14 + 25),$$

then any left-invariant complex structure on  $G$  is abelian ([23, Theorem 8]). Thus in such case, we have

$$Kur_{(M,E)} \cong Kur_M \times Kur_E.$$

by Theorem 4.3.

## 5. NON-SPLITTING EXAMPLES

We consider real 6-dimensional nilmanifolds  $M = \Gamma \backslash G$  with abelian complex structures. Kuranishi spaces  $Kur_M$  of them are computed in [13].

**Example 5.1.** Consider the direct product  $G = H_3(\mathbb{R}) \times H_3(\mathbb{R})$  of two copies of the 3-dimensional real Heisenberg group  $H_3(\mathbb{R})$ . Then, we have  $\mathfrak{g} = \langle X_1, X_2, Y_1, Y_2, Z_1, Z_2 \rangle$  such that  $[X_1, Y_1] = Z_1, [X_2, Y_2] = Z_2$ . Define

$$\mathfrak{g}^{1,0} = \langle W_1 = \frac{1}{2}(X_1 - \sqrt{-1}Y_1), W_2 = \frac{1}{2}(X_2 - \sqrt{-1}Y_2), W_3 = \frac{1}{2}(Z_1 - \sqrt{-1}Z_2) \rangle.$$

Then  $\mathfrak{g}^{1,0}$  is abelian. We consider the DGLAs  $L_1 = \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0}$ ,  $L_2 = \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_1(\mathbb{C})$  and  $L_3 = \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \oplus \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_1(\mathbb{C})$ . We see that  $Kur_{L_3} \not\cong Kur_{L_1} \times Kur_{L_2}$ . We have

$$[\overline{W_1}, W_1] = -\frac{1}{2}\sqrt{-1}(W_3 + \overline{W_3}) \quad \text{and} \quad [\overline{W_2}, W_2] = \frac{1}{2}(W_3 - \overline{W_3}).$$

Thus,  $H^1(L_1) = \langle \overline{w_1} \otimes W_1, \overline{w_2} \otimes W_2, \overline{w_3} \otimes W_3, \overline{w_1} \otimes W_2 + \sqrt{-1}\overline{w_2} \otimes W_1 \rangle$ . A Kuranishi series of  $L_1$  is given by

$$t_1\overline{w_1} \otimes W_1 + t_2\overline{w_2} \otimes W_2 + t_3\overline{w_3} \otimes W_3 + t_4(\overline{w_2} \otimes W_1 + \sqrt{-1}\overline{w_1} \otimes W_2) - t_3t_4(\overline{w_1} \otimes W_2 - \sqrt{-1}\overline{w_2} \otimes W_1).$$

We have  $Kur_{L_1} \cong \mathbb{C}^4$ . We have  $H^1(L_2) \cong H^1(\mathfrak{g}^{0,1}) \cong \langle \overline{w_1}, \overline{w_2}, \overline{w_3} \rangle$  and  $Kur_{L_2} \cong \mathbb{C}^3$ . On the other hand, for a Kuranishi series

$$(t_1\overline{w_1} \otimes W_1 + t_2\overline{w_2} \otimes W_2 + t_3\overline{w_3} \otimes W_3 + t_4(\overline{w_2} \otimes W_1 + \sqrt{-1}\overline{w_1} \otimes W_2) + t_3t_4(\overline{w_2} \otimes W_1 - \sqrt{-1}\overline{w_1} \otimes W_2), \\ s_1\overline{w_1} + s_2\overline{w_2} + s_3\overline{w_3})$$

of  $L_3$ , we have the defining equation  $t_4s_3 = 0$ . Thus  $Kur_{L_3} \not\cong Kur_{L_1} \times Kur_{L_2}$ .

**Example 5.2.** Consider  $G = H_3(\mathbb{C})$  the 3-dimensional complex Heisenberg group. We take a real basis  $X_1, X_2, X_3, X_4, Z_1, Z_2$  such that  $[X_1, X_3] = -\frac{1}{2}Z_1$ ,  $[X_1, X_4] = [X_2, X_3] = -\frac{1}{2}Z_2$ ,  $[X_1, X_3] = \frac{1}{2}Z_1$ . Define

$$\mathfrak{g}^{1,0} = \langle W_1 = X_1 - \sqrt{-1}X_2, W_2 = X_3 + \sqrt{-1}X_4, W_3 = Z_1 + \sqrt{-1}Z_2 \rangle.$$

Then  $\mathfrak{g}^{1,0}$  is abelian. We consider the DGLAs  $L_1 = \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0}$ ,  $L_2 = \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_1(\mathbb{C})$  and  $L_3 = \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{g}^{1,0} \oplus \bigwedge(\mathfrak{g}^{0,1})^* \otimes \mathfrak{gl}_1(\mathbb{C})$ . We see that  $Kur_{L_3} \not\cong Kur_{L_1} \times Kur_{L_2}$ . We have

$$[\overline{W_1}, W_1] = -\sqrt{-1}(W_3 + \overline{W_3}), \quad [\overline{W_2}, W_2] = -(W_3 - \overline{W_3}) \quad \text{and} \quad [\overline{W_1}, W_2] = -W_3.$$

Thus,  $H^1(L_1) = \langle \overline{w_1} \otimes W_1, \overline{w_2} \otimes W_1, \overline{w_3} \otimes W_1, \overline{w_1} \otimes W_2, \overline{w_3} \otimes W_2, \overline{w_3} \otimes W_3 \rangle$ . A Kuranishi series of  $L_1$  is given by

$$t_1\overline{w_1} \otimes W_1 + t_2\overline{w_2} \otimes W_1 + t_3\overline{w_3} \otimes W_1 + t_4\overline{w_3} \otimes W_2 + t_5\overline{w_3} \otimes W_2 + t_6\overline{w_3} \otimes W_3 + t_1t_6\overline{w_2} \otimes W_2.$$

We have the defining equation  $t_3 = 0$  and hence  $Kur_{L_1} \cong \mathbb{C}^5$ . We have  $H^1(L_2) \cong H^1(\mathfrak{g}^{0,1}) \cong \langle \overline{w_1}, \overline{w_2}, \overline{w_3} \rangle$  and  $Kur_{L_2} \cong \mathbb{C}^3$ . On the other hand, for a Kuranishi series

$$(t_1\overline{w_1} \otimes W_1 + t_2\overline{w_2} \otimes W_1 + t_3\overline{w_3} \otimes W_1 + t_4\overline{w_3} \otimes W_2 + t_5\overline{w_3} \otimes W_2 + t_6\overline{w_3} \otimes W_3 + t_1t_6\overline{w_2} \otimes W_2, \\ s_1\overline{w_1} + s_2\overline{w_2} + s_3\overline{w_3})$$

of  $L_3$ , we have the defining equation  $t_1s_3 = 0$ . Thus  $Kur_{L_3} \not\cong Kur_{L_1} \times Kur_{L_2}$ .

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