

THE HEISENBERG ALGEBRA OF A VECTOR SPACE AND HOCHSCHILD HOMOLOGY

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ABSTRACT. We decategorify the Heisenberg 2-category of Gyenge-Koppensteiner-Logvinenko using Hochschild homology. We use this to generalise the Heisenberg algebra action of Grojnowski and Nakajima to all smooth and proper noncommutative varieties in the noncommutative geometry setting proposed by Kontsevich and Soibelman. For ordinary commutative varieties, we compute the resulting action on Chen-Ruan orbifold cohomology. As tools, we prove results about Heisenberg algebras of a graded vector space which might be of independent interest.

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1. INTRODUCTION

The Heisenberg algebra originated in quantum mechanics to describe the commutation relations between position and momentum operators. The ∞ -dimensional Heisenberg algebra $\underline{H}_{\mathbb{k}}$ has the generators $\{a(n)\}_{n \in \mathbb{Z} \setminus \{0\}}$ and the relation $[a(m), a(n)] = m\delta_{m,-n}$. It is important in many areas of mathematics and physics such as conformal field theory, string theory, and representation theory.

In algebraic geometry, much of its relevance is due to the following celebrated result obtained independently by Grojnowski and Nakajima in the 1990s:

Theorem (see [29], Theorem 3.1, [17], Theorem 7, and [30], Theorem 8.13). *Let X be a smooth projective surface over \mathbb{C} . Let $X^{[n]}$ be the Hilbert scheme of n points on X . Let χ be the pairing on $H^\bullet(X, \mathbb{Q})$ given by the cup product and then the direct image along $X \rightarrow pt$.*

For each $\alpha \in H^(X, \mathbb{Q})$ and $n > 0$, there are operators $A_\alpha(-n)$ and $A_\alpha(n)$ on $\bigoplus_{n=0}^\infty H^\bullet(X^{[n]}, \mathbb{Q})$ defined by certain correspondences on $X^{[N]} \times X^{[N-n]}$ and $X^{[N]} \times X^{[N+n]}$ for $N \geq 0$. These satisfy*

$$(1.1) \quad A_\alpha(m)A_\beta(n) = (-1)^{\deg(\alpha)\deg(\beta)}A_\beta(n)A_\alpha(m) + \delta_{m,-n}m\langle \alpha, \beta \rangle_\chi$$

and thus define an action of the Heisenberg algebra $\underline{H}_{H^\bullet(X, \mathbb{Q}), \chi}$ on the total cohomology $\bigoplus_{n=0}^\infty H^\bullet(X^{[n]}, \mathbb{Q})$. This action identifies $\bigoplus_{n=0}^\infty H^\bullet(X^{[n]}, \mathbb{Q})$ with the Fock space of $\underline{H}_{H^\bullet(X, \mathbb{Q}), \chi}$.

Here, the Heisenberg algebra $\underline{H}_{V, \chi}$ of a graded vector space V with a symmetric bilinear form χ is a generalisation introduced in [17, 30]. It has the generators $\{a_v(n)\}_{v \in V, n \in \mathbb{Z} \setminus \{0\}}$, the relations of linearity in v and the Heisenberg relation (1.1). The elements $a_v(n)$ are sometimes called the *creation* ($n > 0$) and *annihilation* ($n < 0$) operators. In the theorem above, these act by correspondences which add or remove, respectively, n points belonging to the prescribed cohomology class.

If $\dim X \geq 3$, $X^{[n]}$ is not well-behaved. Grojnowski conjectured in his paper [17, Footnote 3] that the result should hold for any smooth projective variety X if one replaces $X^{[n]}$ by the symmetric quotient orbifold X^n/S_n and uses equivariant K-theory. This was later proved in [35][41].

In this paper, we generalise this to all smooth and proper *noncommutative* varieties:

Theorem 1.1 (see Theorem 7.1). *Let \mathcal{V} be a smooth and proper DG category over an algebraically closed field \mathbb{k} of characteristic 0. Let χ be the Euler pairing on the Hochschild homology $\mathrm{HH}_\bullet(\mathcal{V})$.*

For each $\alpha \in \mathrm{HH}_\bullet(\mathcal{V})$ and $n > 0$, define operators $A_\alpha(-n)$ and $A_\alpha(n)$ on $\bigoplus_{n=0}^\infty \mathrm{HH}_\bullet(\mathcal{S}^n \mathcal{V})$ by

$$(1.2) \quad A_\alpha(-n): \mathrm{HH}_\bullet(\mathcal{S}^{N+n} \mathcal{V}) \xrightarrow{\mathrm{Res}_{\mathcal{S}_N \times \mathcal{S}_n}^{\mathcal{S}_N+n}} \mathrm{HH}_\bullet(\mathcal{S}^N \mathcal{V} \otimes \mathcal{S}^n \mathcal{V}) \cong \mathrm{HH}_\bullet(\mathcal{S}^N \mathcal{V}) \otimes \mathrm{HH}_\bullet(\mathcal{S}^n \mathcal{V}) \xrightarrow{\langle \psi_n(\alpha), - \rangle} \mathrm{HH}_\bullet(\mathcal{S}^N \mathcal{V}),$$

$$(1.3) \quad A_\alpha(n): \mathrm{HH}_\bullet(\mathcal{S}^N \mathcal{V}) \xrightarrow{(-) \otimes \psi_n(\alpha)} \mathrm{HH}_\bullet(\mathcal{S}^N \mathcal{V}) \otimes \mathrm{HH}_\bullet(\mathcal{S}^n \mathcal{V}) \cong \mathrm{HH}_\bullet(\mathcal{S}^N \mathcal{V} \otimes \mathcal{S}^n \mathcal{V}) \xrightarrow{\mathrm{Ind}_{\mathcal{S}_N \times \mathcal{S}_n}^{\mathcal{S}_N+n}} \mathrm{HH}_\bullet(\mathcal{S}^{N+n} \mathcal{V}),$$

where ψ_n are the maps (1.10) defined explicitly in §5.14. These operators satisfy

$$(1.4) \quad A_\alpha(m)A_\beta(n) - (-1)^{\deg(\alpha)\deg(\beta)}A_\beta(n)A_\alpha(m) = 0 \quad m, n > 0 \text{ or } m, n < 0,$$

$$(1.5) \quad A_\alpha(-m)A_\beta(n) - (-1)^{\deg(\alpha)\deg(\beta)}A_\beta(n)A_\alpha(-m) = \delta_{m,n}m\langle \alpha, \beta \rangle_\chi, \quad m, n > 0$$

and thus define an action of the Heisenberg algebra $H_{H_\bullet(\mathcal{V}), \chi}$ on $\bigoplus_{n=0}^\infty \mathrm{HH}_\bullet(\mathcal{S}^n \mathcal{V})$. This action identifies $\bigoplus_{n=0}^\infty \mathrm{HH}_\bullet(\mathcal{S}^n \mathcal{V})$ with the Fock space of $H_{H_\bullet(\mathcal{V}), \chi}$.

Comparing our operators $A_\alpha(\pm n)$ to those in [35][41], shows Theorem 1.1 to be a noncommutative analogue of their K -theoretic action. For commutative varieties, Baranovsky decomposition [6] gives an isomorphism from the Hochschild homology of their symmetric powers to the Chen-Ruan cohomology [13] [15] of the corresponding orbifold quotients, allowing us to prove:

Theorem 1.2 (see Theorem 4.3). *Let X be a smooth projective variety over \mathbb{C} and χ be the pairing*

$$(1.6) \quad \langle \alpha, \beta \rangle_\chi = \int_X K(\alpha) \wedge \beta \wedge \mathrm{td}_X$$

defined on $H^\bullet(X, \mathbb{C})$ in [34]. Here K sign twists each $H^{p,q}$ by $(-1)^q$ and td_X is the Todd class.

For each $\alpha \in \mathrm{HH}_\bullet(\mathcal{V})$ and $n > 0$, there are certain (see below) operators $A_\alpha(-n)$ and $A_\alpha(n)$ on the total orbifold cohomology $\bigoplus_{n=0}^\infty H_{\mathrm{orb}}^\bullet(X^n/S_n, \mathbb{C})$. These satisfy relations (1.2) and (1.3) and thus define an action of the Heisenberg algebra $H_{H^\bullet(X, \mathbb{C}), \chi}$ on $\bigoplus_{n=0}^\infty H_{\mathrm{orb}}^\bullet(X^n/S_n, \mathbb{C})$. This action identifies $\bigoplus_{n=0}^\infty H_{\mathrm{orb}}^\bullet(X^n/S_n, \mathbb{C})$ with the Fock space of $H_{H^\bullet(X, \mathbb{C}), \chi}$.

Noncommutative geometry comes in many flavors. Here we follow [25, 22, 20, 33, 14] where a noncommutative scheme is a small DG (or A_∞ -) category \mathcal{A} considered up to Morita equivalence. Such \mathcal{A} can be viewed as a DG enhanced triangulated category [8, 39, 38, 28]. The triangulated category enhanced by \mathcal{A} is $D_c(\mathcal{A})$, the compact derived category of \mathcal{A} -modules. If \mathcal{A} is a commutative algebra viewed as a DG category, then this is the compact derived category $D_c(\mathrm{Spec} \mathcal{A})$ of quasi-coherent sheaves on the scheme $\mathrm{Spec} \mathcal{A}$. On the other hand, the compact derived category of any quasi-compact quasi-separated scheme X can be enhanced by a (noncommutative) DG algebra [9].

The point of this approach is that we take any enhanced triangulated category \mathcal{A} and treat as if it were the compact derived category of a “noncommutative” scheme. A number of geometrical features can be read off at this abstract level [25]: smoothness, properness, polyvector fields, differential forms, Hodge and de Rham cohomologies, Hodge-to-de-Rham spectral sequence, etc. By this we mean that it is possible to define on the noncommutative level, in terms of \mathcal{A} , the notions which become the usual geometric notions listed above when \mathcal{A} is the derived category of a nice (commutative) scheme X . A beautiful summary is given in [20, 21]. One might be tempted to work with more sophisticated enhancements, but for the present paper this simple approach suffices.

When \mathcal{A} is the derived category of a smooth projective scheme X , by the global version [24, 37, 10] of the Hochschild-Kostant-Rosenberg (HKR) isomorphism [18] the Hochschild homology $\mathrm{HH}_\bullet(\mathcal{A})$ is isomorphic to the Hodge cohomology $H_{\mathrm{Hodge}}^{\bullet, \bullet}(X)$. In $\mathrm{char} = 0$, the Hodge-to-de-Rham spectral sequence degenerates and this is also isomorphic to the de Rham cohomology $H_{dR}^\bullet(X)$. When $\mathbb{k} = \mathbb{C}$, by Poincare lemma $H_{dR}^\bullet(X) \cong H^\bullet(X, \mathbb{C})$. Finally, the resulting isomorphism of $\mathrm{HH}_\bullet(\mathcal{A})$ and $H^\bullet(X, \mathbb{C})$ identifies the Euler pairing on the former with the pairing (1.6) on the latter [34].

It remains to do a similar translation for the symmetric powers $\mathcal{S}^n \mathcal{A}$ enhancing the derived categories $D([X^n/S_n])$. In [6], for any finite group G acting on a smooth quasi-projective variety Y Baranovsky constructed a decomposition identifying the Hochschild homology $\mathrm{HH}_\bullet([Y/G])$ of the smooth stack $[Y/G]$ and the Chen-Ruan orbifold cohomology $H_{\mathrm{orb}}^\bullet(Y/G, \mathbb{C})$. In our case, this gives

$$(1.7) \quad \mathrm{HH}_\bullet(\mathcal{S}^n \mathcal{A}) \cong H_{\mathrm{orb}}^\bullet(X^n/S_n, \mathbb{C}) \cong \bigoplus_{\underline{n} \vdash n} \mathrm{Sym}^{r_1(\underline{n})} H^\bullet(X, \mathbb{C}) \otimes \cdots \otimes \mathrm{Sym}^{r_n(\underline{n})} H^\bullet(X, \mathbb{C})$$

where \underline{n} is an unordered partition of n , $r_i(\underline{n})$ is its number of parts of size i , and $r(\underline{n}) = \sum r_i(\underline{n})$. In [3], Anno, Baranovsky, and the second author show that for any small DG category \mathcal{A}

$$(1.8) \quad \mathrm{HH}_\bullet(\mathcal{S}^n \mathcal{A}) \cong \bigoplus_{\underline{n} \vdash n} \mathrm{Sym}^{r_1(\underline{n})} \mathrm{HH}_\bullet(\mathcal{A}) \otimes \cdots \otimes \mathrm{Sym}^{r_n(\underline{n})} \mathrm{HH}_\bullet(\mathcal{A}).$$

by writing down two mutually inverse quasi-isomorphisms on the level of Hochschild complexes, see §5.14 for more detail. In the commutative case, applying the HKR isomorphism $\mathrm{HH}_\bullet(\mathcal{A}) \cong H^\bullet(X, \mathbb{C})$ to the noncommutative Baranovsky decomposition (1.8) recovers (1.7).

The induction and restriction functors $\mathrm{Ind}_{S_m \times S_n}^{S_{m+n}}$ and $\mathrm{Res}_{S_m \times S_n}^{S_{m+n}}$ give a Hopf algebra structure on $\bigoplus_{n \geq 0} \mathrm{HH}_\bullet(\mathcal{S}^n \mathcal{A})$. In [3], this structure is computed in terms of the decomposition (1.8). This gives a description of the operators $A_\alpha(\pm n)$ of Theorem 1.1 in terms of the decomposition (1.8). In the commutative case, this description defines the operators $A_\alpha(-n)$ and $A_\alpha(n)$ of Theorem 1.2 via the identification of the decompositions (1.8) and (1.7) provided by the HKR isomorphism.

Circumstances forced the authors to post this preprint to arXiv earlier than they would have wanted. In a future update, we will write down explicit formulas for the operators $A_\alpha(\pm n)$ of Theorem 1.2 in terms of the decomposition (1.7).

Thus Theorem 1.1 implies Theorem 1.2 via the HKR isomorphism and the noncommutative Baranovsky decomposition. To prove Theorem 1.1 we decategorify our Heisenberg algebra categorification of [42] using the Hochschild homology. In [42], for any smooth and proper DG category \mathcal{V} we constructed the *Heisenberg DG 2-category* $\mathbf{H}_\mathcal{V}$ of \mathcal{V} . In the language above, $\mathbf{H}_\mathcal{V}$ is a noncommutative scheme version of the Heisenberg algebra of \mathcal{V} . We also constructed its action $\Phi_\mathcal{V}$ on the 2-category of the symmetric powers $\mathcal{S}^n \mathcal{V}$, the *categorical Fock space* of \mathcal{V} . The idea is that instead of having to prove Theorem 1.2 for the Heisenberg algebra action on each given additive invariant of the stacks $[X^n/S_n]$ (K-theory, orbifold cohomology, Hochschild homology, etc) we would construct the (noncommutative) Heisenberg scheme of X and its action on $[X^n/S_n]$ themselves. This universal action could be decategorified with any additive invariant to produce an analogue of Theorem 1.2.

The decategorification process is far from automatic. In [42] we decategorified using the numerical Grothendieck group K_0^{num} . That meant constructing an injective algebra map

$$\pi: \underline{H}_{K_0^{\mathrm{num}}(\mathcal{V})} \hookrightarrow K_0^{\mathrm{num}}(\mathbf{H}_\mathcal{V}).$$

Ideally, one wants π to be an isomorphism, but injectivity is enough for any action of $\mathbf{H}_\mathcal{V}$ on any 2-category \mathcal{C} induce an action of $\underline{H}_{K_0^{\mathrm{num}}(\mathcal{V})}$ on $K_0^{\mathrm{num}}(\mathcal{C})$. Our universal action $\Phi_\mathcal{V}$ induced an action of $\underline{H}_{K_0^{\mathrm{num}}(\mathcal{V})}$ on $\bigoplus_{n \geq 0} K_0^{\mathrm{num}}(\mathcal{S}^n \mathcal{V})$. This induced an embedding of the Fock space of $\underline{H}_{K_0^{\mathrm{num}}(\mathcal{V})}$

$$\phi: F_{K_0^{\mathrm{num}}(\mathcal{V})} \hookrightarrow \bigoplus_{n \geq 0} K_0^{\mathrm{num}}(\mathcal{S}^n \mathcal{V}).$$

Here it turned out that K_0^{num} was not the best invariant to use: it fails the Künneth formula. This led to an example where the rank of $\bigoplus_{n \geq 0} K_0^{\mathrm{num}}(\mathcal{S}^n \mathcal{V})$ was strictly greater than that of $F_{K_0^{\mathrm{num}}(\mathcal{V})}$. So ϕ was not surjective, and for general reasons [42, Theorem 8.13] π couldn't be surjective either.

The referees of [42] pointed out that to claim our 2-categorical constructions to be a universal version of Theorem 1.2 we best show that they can be decategorified with other additive invariants. We agree and in this paper we prove:

Theorem 1.3 (Theorem 6.6 and Prop. 6.22). *Let \mathcal{V} be a smooth and proper DG category over \mathbb{k} . There exists an injective algebra homomorphism*

$$(1.9) \quad \pi: \underline{H}_{\mathrm{HH}_\bullet(\mathcal{V})} \hookrightarrow \mathrm{HH}_\bullet(\mathbf{H}_\mathcal{V}).$$

Roughly, this extends our previous decategorification from HH_0 to the whole Hochschild homology. The composition of π with $\mathrm{HH}_\bullet(\Phi_\mathcal{V})$ gives an action of $\underline{H}_{\mathrm{HH}_\bullet(\mathcal{V})}$ on $\bigoplus_{n=0}^\infty \mathrm{HH}_\bullet(\mathcal{S}^n \mathcal{V})$. In Prop. 6.22 we show that this action induces an injective morphism of $\underline{H}_{\mathrm{HH}_\bullet(\mathcal{V})}$ -modules

$$\phi: F_{\mathrm{HH}_\bullet(\mathcal{V})} \hookrightarrow \bigoplus_{n \geq 0} \mathrm{HH}_\bullet(\mathcal{S}^n \mathcal{V}).$$

The noncommutative Baranovsky decomposition (1.8) shows by dimension count that ϕ is an isomorphism. This completes the proof of Theorem 1.1, and encourages us to conjecture:

Conjecture 1.4. *The injective algebra homomorphism (1.9) is always an isomorphism.*

In proving Theorem 1.1, we were encouraged by recent results of Belmans, Fu, and Krug [7]. For \mathcal{V} a commutative smooth and proper variety, they computed $\dim \bigoplus_{n=0}^{\infty} \mathrm{HH}_{\bullet}(\mathcal{S}^n \mathcal{V})$ and showed that it matches the dimension of the Fock space of $\underline{H}_{\mathrm{HH}_{\bullet}(\mathcal{V})}$. They conjectured [7, Conj. 3.24] that the same holds in the noncommutative case and gave partial evidence. This matched our own expectations [42, Cor. 8.6] and led Anno, Baranovsky and the second author to prove it in [3]. Apparently, [7, Conj. 3.24] was also independently proved by Nordstrom via general considerations which do not yield explicit maps on the level of Hochschild complexes [31].

It remains to sum up our proof of Theorem 1.3. We define π by defining for each $\alpha \in \mathrm{HH}_{\bullet}(\mathcal{V})$ and $n \neq 0$ the class $A_{\alpha}(n) \in \mathrm{HH}_{\bullet}(\mathbf{H}_{\mathcal{V}})$. The decomposition (1.8) defines the linear map

$$(1.10) \quad \psi_n : \mathrm{HH}_{\bullet}(\mathcal{V}) \rightarrow \mathrm{HH}_{\bullet}(\mathcal{S}^n \mathcal{V})$$

as the inclusion of the summand indexed by the single part partition (n) of n . We apply the maps induced by the functors Ξ_P and Ξ_Q of [42, §6.1] to $\psi_n(\alpha)$ to obtain $A_{\alpha}(n)$ and $A_{\alpha}(-n)$. For these, we prove the commutation relation (1.2) and the Heisenberg relation (1.3). The commutation relation is easy because it holds tautologically for the classes $\psi_n(\alpha)$ which live in the graded commutative algebra $\bigoplus_{n \geq 0} \mathrm{HH}_{\bullet}(\mathcal{S}^n \mathcal{V})$. Proving the Heisenberg relation was, technically, the hardest step. Some 1-morphism identities we need only hold up to homotopy in $\mathbf{H}_{\mathcal{V}}$. This makes explicit computations with Hochschild chains difficult. To sidestep this, we construct two functors Ξ_{QP} and $\bigoplus_k \Xi_{QP}(\hat{k})$ from $\mathcal{S}^n \mathcal{V}^{\text{opp}} \otimes \mathcal{S}^m \mathcal{V}$ to $\mathbf{H}_{\mathcal{V}}$ and show that for any $\alpha, \beta \in \mathrm{HH}_{\bullet}(\mathcal{V})$ the images of $\psi_n(\alpha) \otimes \psi_m(\beta)$ under these two functors are the LHS and the RHS of the Heisenberg relation for $A_{\alpha}(-n)$ and $A_{\beta}(m)$. Homotopy equivalent functors induce the same map on Hochschild homology [23, Lemma 3.4], so we complete the argument by constructing in Theorem 6.21 a functorial homotopy equivalence

$$(1.11) \quad \bigoplus_k \Xi_{QP}(\hat{k}) \longrightarrow \Xi_{QP}.$$

To facilitate defining actions via generators and relations, we do some minor foundational work on Heisenberg algebras. In the literature, there is the *A-generators* definition [17, 30] of $H_{V,\chi}$ in terms of generators $\{a_v(n)\}_{v \in V, n \in \mathbb{Z} \setminus \{0\}}$ for a graded vector space V with a symmetric form χ . There is also the *PQ-generators* definition [12, 26, 42] in terms of generators $\{p_v^{(n)}, q_v^{(n)}\}_{v \in V, n \geq 0}$ for a lattice V with any form χ . The *A*-generators are linear in v , so for any basis e_1, \dots, e_n of V , $H_{V,\chi}$ is generated by $a_{e_i}(n)$ modulo only the Heisenberg relation. The *PQ*-generators are not linear in v and a similar basis reduction is a non-trivial result missing from the literature. In [26, Lemma 1.2] Krug proved a result which would imply the equivalence of *A*- and *PQ*-definitions for symmetric χ if this basis reduction was known for *PQ*-generators. In the present paper, we prove that:

Theorem 1.5 (see Defns. 3.20, 3.23, Prop. 3.22, and Theorems 3.24, 3.25, 3.26). *Let V be a graded vector space with a bilinear form χ . Let e_1, \dots, e_n be a basis of V . Then:*

- (1) *There is a definition of the Heisenberg algebra $\underline{H}_{V,\chi}^A$ with generators $\{a_v(n)\}_{v \in V, n \in \mathbb{Z} \setminus \{0\}}$ which reduces to [17, 30] when χ is symmetric and a definition of the Heisenberg algebra $\underline{H}_{V,\chi}^{PQ}$ with generators $\{p_v^{(n)}, q_v^{(n)}\}_{v \in V, n \geq 0}$ which reduces to [12, 26, 42] when V is a lattice.*
- (2) *$\underline{H}_{V,\chi}^A$ is generated by $a_{e_i}(n)$ modulo the relations (3.24) and (3.27).*
- (3) *$\underline{H}_{V,\chi}^{PQ}$ is generated by $p_{e_i}^{(n)}, q_{e_i}^{(n)}$ modulo the relations (3.31) and (3.34).*
- (4) *$\underline{H}_{V,\chi}^{PQ}$ is isomorphic to $\underline{H}_{V,\chi}^A$.*
- (5) *For non-degenerate χ , the algebra $\underline{H}_{V,\chi}^A$ does not depend on χ .*

The functorial homotopy equivalence (1.11) can be viewed as a functorial categorification of both the *A*-generator Heisenberg relation (3.27) and the *PQ*-generator Heisenberg relation (3.34). A non-functorial categorification of (3.34) appeared in [42, Theorem 6.3]. There was no hope of making it functorial directly as it related the symmetrised elements $P_a^{(n)}$ and $Q_b^{(m)}$ which are not functorial in $a, b \in \mathcal{V}$ for $n, m > 1$. Instead, we use its case $n = m = 1$ to iteratively construct the present, functorial categorification (1.11). Applying (1.11) to $\psi_n(\alpha) \otimes \psi_m(\beta)$ yields the *A*-generator Heisenberg relation (3.27) for $A_{\alpha}(\pm n)$ we prove in Theorem 6.9, while applying it to the product of the symmetrised powers $a^{(n)} \in \mathcal{S}^n \mathcal{V}^{\text{opp}}$ and $b^{(m)} \in \mathcal{S}^m \mathcal{V}$ recovers the categorified *PQ*-generator Heisenberg relation in [42, Theorem 6.3].

Acknowledgments. We would like to thank Rina Anno, Alexey Bondal, Ian Grojnowski, Bernhard Keller, and Hiraku Nakajima for useful discussions in the course of writing this paper. We would like to thank the anonymous referees of [42] for their many invaluable comments and for pushing us to investigate other additive invariants and, in particular, Hochschild homology. Without them, this paper would have never been written. We would also like to thank Alexey Bondal for pointing out a silly mistake with HH_0 in an earlier version of this paper. Á.Gy. was supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences. T.L. would like to thank the organisers of the conferences “Representations, Moduli and Duality” in Lausanne and “Corti60: A Tour through Algebraic Geometry” in Cortona for a stimulating environment in which some of the final technical breakthroughs in this project were made.

2. PRELIMINARIES

Throughout the paper \mathbb{k} denotes an algebraically closed field of characteristic 0.

2.1. Generalised binomial coefficients. Recall the definition of \mathbb{k} -valued binomial coefficients:

Definition 2.1. For any $z \in \mathbb{k}$ and any $k \in \mathbb{Z}_{\geq 0}$ define

$$(2.1) \quad \binom{z}{k} := \frac{z(z-1)\dots(z-k+1)}{k!}.$$

The expression in the numerator of (2.1) has k factors, and when $k = 0$ it is taken to be 1.

Remark 2.2. In the combinatorial case, i.e. when z is an integer $n \geq 0$, $\binom{n}{k}$ enumerates the number of ways to choose a k out of the total of n objects. In particular, in $\mathbb{C}[x, y]$ we have

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}.$$

Of particular interest to us is the following special case:

Definition 2.3. For any $z \in \mathbb{k}$ and any $k \in \mathbb{Z}_{\geq 0}$ define

$$s^k z := \binom{z+k-1}{k} = \frac{1}{k!}(z+k-1)(z+k-2)\cdots(z+1)z.$$

Remark 2.4. When z is an integer $n \geq 0$, we have $s^k n = \dim(S^k(\mathbb{C}^n))$ and for $n < 0$ we have $s^k n = (-1)^k \dim(\Lambda^k(\mathbb{C}^{-n}))$.

Remark 2.5. Below we give a list of some well-known combinatorial identities which we make use of in this paper. Since these only involve polynomials in finite number of variables z_1, \dots, z_m of finite degree, by the Combinatorial Nullstellensatz [2, Theorem 1.2] establishing them in the combinatorial case, i.e. for non-negative integer z_i , means establishing them for all $z_i \in \mathbb{k}$.

- “One set aside”: For any $z \in \mathbb{k}$ and $k \in \mathbb{Z}_{\geq 0}$ we have

$$(2.2) \quad \binom{z}{k} = \binom{z-1}{l} + \binom{z-1}{l-1}.$$

In the combinatorial case, we set one of the z objects aside and count first all the ways to choose k objects not including it and then all the ways including it.

- *Vandermonde’s identity*: For any $z_1, z_2 \in \mathbb{k}$ and $k \in \mathbb{Z}_{\geq 0}$ we have

$$(2.3) \quad \binom{z_1 + z_2}{k} = \sum_{i=0}^k \binom{z_1}{i} \binom{z_2}{k-i}.$$

In the combinatorial case, we divide z objects into two groups of z_1 and z_2 objects and then count all ways to choose k out of z objects by counting for each 2-partition of k into i and $k - i$ objects the ways to choose i out of z_1 objects and $k - i$ out of z_2 objects. Note that the “one set aside” identity is the instance of Vandermonde’s identity with $z_1 = 1$.

- *Generalised Vandermonde's identity:* For any $z_1, \dots, z_m \in \mathbb{k}$ and $k \in \mathbb{Z}_{\geq 0}$ we have

$$(2.4) \quad \binom{z_1 + \dots + z_m}{k} = \sum_{k_1 + \dots + k_m = k, k_i \geq 0} \binom{z_1}{k_1} \dots \binom{z_m}{k_m}.$$

In the combinatorial case, we divide z objects into m groups of z_1, \dots, z_m objects and then count all ways to choose k out of z objects by counting for each m -partition of k into $k_1 + \dots + k_m$ the ways to choose k_i out of z_i objects for each $1 \leq i \leq m$.

- *Negative binomial identity:* For any $z \in \mathbb{k}$ and $k \in \mathbb{Z}_{\geq 0}$ we have

$$(2.5) \quad \binom{-z}{k} = (-1)^k \binom{z+k-1}{k}.$$

In this paper we need the following generalisation of the binomial coefficients. In the combinatorial case, these are sometimes known as *multinomial* coefficients:

Definition 2.6. For any $z \in \mathbb{k}$ and any $k_1, \dots, k_m \in \mathbb{Z}_{\geq 0}$ define

$$(2.6) \quad \binom{z}{k_1, \dots, k_m} := \frac{z(z-1)\dots(z-(k_1+\dots+k_m)+1)}{k_1! \dots k_m!}.$$

For $m = 1$, we get the usual binomial coefficients. When $m \geq 1$, we have

$$(2.7) \quad \binom{z}{k_1, \dots, k_m} = \binom{z}{k_1} \binom{z-k_1}{k_2} \dots \binom{z-(k_1+\dots+k_{m-1})}{k_m}.$$

Remark 2.7. (1) Note that the permuting k_1, \dots, k_m leaves both sides of (2.7) invariant.

(2) In the combinatorial case, when $z = n$ for some $n \in \mathbb{Z}_{\geq 0}$, we have

$$\binom{n}{k_1, \dots, k_m} = \frac{n!}{(n-(k_1+\dots+k_m))! k_1! \dots k_m!}$$

which counts the ways to choose an ordered sequence of m unordered groups of k_1, \dots, k_m objects out of total of n objects. In particular, in $\mathbb{C}[x_1, \dots, x_m]$ we have

$$(x_1 + \dots + x_m)^n = \sum_{k_1 + \dots + k_m = n, k_i \geq 0} \binom{n}{k_1, \dots, k_m} x_1^{k_1} \dots x_m^{k_m}$$

whence “multinomial coefficients”.

(3) Note also that when $k_1 + \dots + k_m > n$ there is a zero in the numerator of (2.6). Hence

$$\binom{n}{k_1, \dots, k_m} = 0.$$

One can not choose several groups adding up to $> n$ objects out of the total of n objects.

Remark 2.8. The binomial coefficient identities listed in Remark 2.5 have generalisations for multinomial coefficients. In particular, we need:

- “*One set aside*”: For any $z \in \mathbb{k}$ and $k_1, \dots, k_m \in \mathbb{Z}_{\geq 0}$, we have

$$(2.8) \quad \binom{z}{k_1, \dots, k_m} = \binom{z-1}{k_1, \dots, k_m} + \sum_{i=1}^m \binom{z-1}{k_1, \dots, k_i-1, \dots, k_m}$$

where we use the convention that if $k_i = 0$ then $\binom{z-1}{k_1, \dots, k_i-1, \dots, k_m} = 0$. In the combinatorial case, we set one of z objects aside and then first count the ways to choose the m collections of k_1, \dots, k_m objects which do not include it, and then counting the ways where it is included in each of the m collections in turn.

- *Vandermonde's identity:* For any $z_1, z_2 \in \mathbb{k}$ and $k_1, \dots, k_m \in \mathbb{Z}_{\geq 0}$ we have

$$(2.9) \quad \binom{z_1 + z_2}{k_1, \dots, k_m} = \sum_{k_i = p_i + q_i, p_i, q_i \geq 0} \binom{z_1}{p_1, \dots, p_m} \binom{z_2}{q_1, \dots, q_m}.$$

In the combinatorial case, we divide z objects into two groups of z_1 and z_2 objects and then count the ways to choose k_1, \dots, k_m out of z objects by counting for each 2-partition of each k_i into $p_i + q_i$ all the ways to choose p_1, \dots, p_m out of z_1 objects and q_1, \dots, q_m out of z_2 objects.

- *Generalised Vandermonde's identity:* For any $z_1, \dots, z_n \in \mathbb{k}$ and $k_1, \dots, k_m \in \mathbb{Z}_{\geq 0}$ we have

$$(2.10) \quad \binom{z_1 + \dots + z_n}{k_1, \dots, k_m} = \sum_{k_i=k_{i1}+\dots+k_{in}, k_{ij}\geq 0} \binom{z_1}{k_{11}, \dots, k_{m1}} \dots \binom{z_n}{k_{1n}, \dots, k_{mn}}.$$

In the combinatorial case, we divide z objects into n groups of z_1, \dots, z_n objects and then count the ways to choose k_1, \dots, k_m out of z objects by counting for each n -partition of each k_i into $k_{i1} + \dots + k_{in}$ all the ways to choose k_{11}, \dots, k_{m1} out of the first group of z_1 objects, k_{12}, \dots, k_{m2} out of the second group of z_2 objects, etc.

2.2. Partitions.

Definition 2.9. Let $n \in \mathbb{Z}_{\geq 0}$. An (*unordered*) *partition* of n is an unordered collection $\underline{n} := \{n_1, \dots, n_m\}$ of strictly positive integers n_i with $\sum_{i=1}^m n_i = n$. We denote this by $\underline{n} \vdash n$.

The integers n_i are the *parts* of \underline{n} . We write $r(\underline{n})$ for the *length* of \underline{n} . By this we mean the total number of parts in \underline{n} , i.e. m . We write $r_k(\underline{n})$ for the number of parts of size k in \underline{n} , i.e. the number of i such that $n_i = k$. Finally, for any $n \in \mathbb{Z}_{\geq 0}$ we write $Part_n$ to be the set of all partitions of n and we set $Part := \coprod_{n \in \mathbb{Z}_{\geq 0}} Part_n$.

In this paper, we also need the following notion:

Definition 2.10. Let $n, m \in \mathbb{Z}_{\geq 0}$. An *ordered m -partition* (n_1, n_2, \dots, n_m) of n is an ordered m -tuple of non-negative integers n_i with $\sum_{i=1}^m n_i = n$. We denote this by $(n_1, \dots, n_m) \vdash n$.

Note that in these ordered partitions of fixed length we allow some parts to be of zero size.

We write $m\text{-}OrdPart_n$ for the set of all ordered m -partitions of n . This set can be viewed as a subset of \mathbb{Z}^m but note that there it is neither closed under addition nor multiplication. We further set $OrdPart_n := \coprod_{m \in \mathbb{Z}_{\geq 0}} m\text{-}OrdPart_n$ and $OrdPart := \coprod_{n \in \mathbb{Z}_{\geq 0}} OrdPart_n$.

For each $n, m \in \mathbb{Z}_{\geq 0}$ we have a natural forgetful map of sets

$$(2.11) \quad F: m\text{-}OrdPart_n \rightarrow Part_n$$

which takes an ordered m -partition, discards its parts of zero size, and forgets the ordering on the remaining parts. Its image in $Part_n$ are the length $\leq m$ partitions of n . Maps (2.11) combine into

$$(2.12) \quad F: OrdPart \rightarrow Part.$$

Let $\underline{n} \vdash n$. Then $(r_1(\underline{n}), \dots, r_n(\underline{n}))$ is an ordered n -partition of $r(\underline{n})$, the length of \underline{n} . We denote by $\underline{r}(\underline{n})$ the resulting unordered partition $F(r_1(\underline{n}), \dots, r_n(\underline{n}))$.

Proposition 2.11. Let $n, m \in \mathbb{Z}_{\geq 0}$ and let $\underline{n} \vdash n$ be a partition of n . Let $F^{-1}(\underline{n})$ be the pre-image of \underline{n} in $m\text{-}OrdPart_n$. Then

$$|F^{-1}(\underline{n})| = \binom{m}{r(\underline{n})}.$$

Note that, by definition, the generalised binomial coefficient $\binom{z}{k_1, \dots, k_m}$ only depends on the image of (k_1, \dots, k_m) in $Part$, and thus we can evaluate it on unordered partitions.

Proof. A pre-image of \underline{n} is obtained by distributing the parts of \underline{n} between m ordered positions and filling the rest with zeroes. As the parts of same size in \underline{n} are indistinguishable, the number of such distributions is the number of ways to choose $r_1(\underline{n}), \dots, r_n(\underline{n})$ positions out of m available. \square

Corollary 2.12. Let A be an abelian group and let $f: Part_n \rightarrow A$ be any maps of sets. Then

$$\sum_{(n_1, \dots, n_m) \vdash n} f(F(n_1, \dots, n_m)) = \sum_{\underline{n} \vdash n} \binom{m}{r(\underline{n})} f(\underline{n}).$$

2.3. The existing definitions of the Heisenberg algebra.

Definition 2.13. A *lattice* (M, χ) is a free abelian group M of finite rank with a bilinear form

$$\chi: M \times M \rightarrow \mathbb{Z}, \quad v, w \mapsto \langle v, w \rangle_\chi.$$

Apriori, we do not require the form χ to be symmetric or antisymmetric.

2.3.1. *A-generator definition.* This is the original definition in [30, §8.1] systematising the results of [29]. It takes the notion of the ∞ -dimensional Heisenberg Lie algebra [19, §9.13] and extends it to have the generators parametrised by elements of a vector space with a symmetric bilinear form.

We prefer to work with usual algebras, rather than Lie algebras. Similar to [12] and [26], by *the Heisenberg algebra* of a lattice or a vector space we mean the universal enveloping algebra of the corresponding Lie algebra where we identified the central charge with 1. This yields:

Definition 2.14. Let (V, χ) be a vector space with a symmetric bilinear form. The *Heisenberg algebra* $\underline{H}_{V,\chi}^A$ is the unital \mathbb{k} -algebra with generators $a_v(n)$ for $v \in V$ and integers $n \in \mathbb{Z} \setminus \{0\}$ modulo the following relations for all $v, w \in V$, $z \in \mathbb{k}$ and $n, m \in \mathbb{Z} \setminus \{0\}$:

$$(2.13) \quad a_{v+w}(n) = a_v(n) + a_w(n),$$

$$(2.14) \quad a_{zv}(n) = za_v(n),$$

$$(2.15) \quad [a_v(n), a_w(m)] = \delta_{m,-n} m \langle v, w \rangle_\chi,$$

where $[-, -]$ denotes the commutator.

The relations (2.13) and (2.14) are the relations of linearity in $v \in V$. The relation (2.15) is the *Heisenberg relation*. The ∞ -dimensional Heisenberg algebra (the case $V = \mathbb{k}$ with $\langle 1, 1 \rangle_\chi = 1$) is isomorphic to the algebra of differential operators of the polynomial ring $\mathbb{k}[x_1, x_2, \dots]$ via

$$(2.16) \quad a(n) \mapsto \begin{cases} nx_n, & n > 0, \\ \frac{\partial}{\partial x_n}, & n < 0, \end{cases}$$

The Heisenberg relation then corresponds to the identity

$$(2.17) \quad \frac{\partial}{\partial x_n} (nx_n f) = nx_n \frac{\partial}{\partial x_n} (f) + nf.$$

For a lattice, we use the same definition but without the scalar multiplication relation (2.14):

Definition 2.15. Let (M, χ) be a lattice with a symmetric bilinear form. The *Heisenberg algebra* $\underline{H}_{M,\chi}^A$ is the unital \mathbb{k} -algebra with generators $a_v(n)$ for $v \in M$ and $n \in \mathbb{Z} \setminus \{0\}$ modulo the following relations for all $v, w \in M$, $z \in \mathbb{k}$ and $n, m \in \mathbb{Z} \setminus \{0\}$:

$$(2.18) \quad a_{v+w}(n) = a_v(n) + a_w(n),$$

$$(2.19) \quad [a_v(n), a_w(m)] = \delta_{m,-n} m \langle v, w \rangle_\chi.$$

As the Heisenberg relation is bilinear in $v, w \in V$ we immediately have the *basis reduction* result:

Proposition 2.16. Let (V, χ) be a vector space (or a lattice) with a symmetric bilinear form and let e_1, \dots, e_l be a basis of V . The Heisenberg algebra $\underline{H}_{V,\chi}^A$ is isomorphic to the unital \mathbb{k} -algebra with generators $a_v(n)$ for all $v \in \{e_1, \dots, e_l\}$ and $n \in \mathbb{Z} \setminus \{0\}$ modulo the relations (2.13) and (2.15) for all $v, w \in \{e_1, \dots, e_l\}$ and $n, m \in \mathbb{Z}$.

Proof. We give the proof for V being a vector space, the proof for the lattice is similar. The relations (2.13) and (2.14) are equivalent to the following basis reduction relation

$$(2.20) \quad a_v(n) = v_1 a_{e_1}(n) + \dots + v_l a_{e_l}(n) \quad \forall v \in V, n \in \mathbb{Z} \setminus \{0\},$$

where $v_i \in \mathbb{k}$ are the unique coefficients such that $v = v_1 e_1 + \dots + v_l e_l$.

We can thus replace relations (2.13) and (2.14) by the relation (2.20). Next, as the Heisenberg relation (2.15) is bilinear in $v, w \in V$, it is sufficient to only impose it for all $v, w \in \{e_1, \dots, e_l\}$. Now, for each $v \in V$ and $n \in \mathbb{Z} \setminus \{0\}$, the element $a_v(n)$ occurs in precisely one relation (2.20). Hence we can only take the generators $a_v(n)$ for $v \in \{e_1, \dots, e_l\}$ and the relations (2.20), and (2.15). Finally, the relation (2.20) is tautological for $v \in \{e_1, \dots, e_l\}$, so we can get rid of it entirely. \square

2.3.2. *PQ-generator definition.* Cautis and Licata used in [12] a different definition of the Heisenberg algebra with generators $p^{(n)}$ and $q^{(n)}$ for $n \geq 0$. For the ADE root lattices, they chose a basis of simple roots and only used these to parametrise the generators. Krug extended this definition in [26] to work with a basis of any vector space or a lattice with a bilinear form. In [42] we extended this to a basis independent definition for lattices by adding the additivity relation (2.22):

Definition 2.17. Let (M, χ) be a lattice. The *Heisenberg algebra* $\underline{H}_{M,\chi}^{PQ}$ is the unital \mathbb{k} -algebra with generators $\{p_a^{(n)}, q_a^{(n)}\}_{a \in M, n \geq 0}$ modulo the following relations for $a, b \in M$ and $n, m > 0$:

$$(2.21) \quad p_a^{(0)} = 1 = q_a^{(0)},$$

$$(2.22) \quad p_{a+b}^{(n)} = \sum_{k=0}^n p_a^{(k)} p_b^{(n-k)} \quad \text{and} \quad q_{a+b}^{(n)} = \sum_{k=0}^n q_a^{(k)} q_b^{(n-k)},$$

$$(2.23) \quad p_a^{(n)} p_b^{(m)} = p_b^{(m)} p_a^{(n)} \quad \text{and} \quad q_a^{(n)} q_b^{(m)} = q_b^{(m)} q_a^{(n)},$$

$$(2.24) \quad q_a^{(n)} p_b^{(m)} = \sum_{k=0}^{\min(m,n)} s^k \langle a, b \rangle_\chi p_b^{(m-k)} q_a^{(n-k)}.$$

Throughout the paper we use the convention that $p_a^{(n)} = q_b^{(n)} = 0$ for $n < 0$.

Neither the $p^{(n)}$ and $q^{(n)}$ generators nor the Heisenberg relation (2.24) are linear in M , so the basis reduction analogous to Theorem 2.16 is no longer immediate for $\underline{H}_{M,\chi}^{PQ}$. To our best knowledge, no such result appeared in the literature. By [26, Lemma 1.2], for symmetric χ such basis reduction result would be equivalent to the equivalence of A - and PQ -definitions of the Heisenberg algebra.

2.3.3. *Relation between A- and PQ-generators.* As explained in [12], the PQ -generators are obtained from the A -generators by exponentiation. In the Heisenberg algebra $\underline{H}_{V,\chi}^A$, set for any $v \in V$

$$A_v^+(t) = \sum_{n \geq 1} \frac{a_v(n)}{i} t^n \quad \text{and} \quad A_v^-(t) = \sum_{n \geq 1} \frac{a_v(-n)}{n} t^n,$$

and define $p_v^{(n)}$ and $q_v^{(n)}$ by

$$\sum_{n \geq 0} p_v^{(n)} t^n := \exp(A_v^+(t)), \quad \text{and} \quad \sum_{n \geq 0} q_v^{(n)} t^n := \exp(A_v^-(t)).$$

Explicitly, this yields:

$$(2.25) \quad \begin{aligned} p_v^{(0)} &:= 1, \\ p_v^{(1)} &:= a_v(1), \\ p_v^{(2)} &:= \frac{1}{2} a_v(2) + \frac{1}{2} a_v(1) a_v(1), \\ p_v^{(3)} &:= \frac{1}{3} a_v(3) + \frac{1}{2} a_v(1) a_v(2) + \frac{1}{6} a_v(1) a_v(1) a_v(1), \\ p_v^{(4)} &:= \frac{1}{4} a_v(4) + \frac{1}{6} a_v(1) a_v(3) + \frac{1}{8} a_v(2) a_v(2) + \frac{1}{12} a_v(1) a_v(1) a_v(2) + \frac{1}{24} a_v(1) a_v(1) a_v(1) a_v(1), \\ &\dots \\ p_v^{(n)} &:= \sum_{\underline{n} \vdash n} \frac{1}{r_1(\underline{n})! \dots r_n(\underline{n})!} \frac{1}{n_1 \dots n_{r(\underline{n})}} a_v(\underline{n}). \end{aligned}$$

From these, or by logarithmic power series expansion, it is clear that the subalgebra of $\underline{H}_{V,\chi}^A$ generated by $p_v^{(n)}$ and $q_v^{(n)}$ contains all $a_v(n)$ and hence is the whole of $\underline{H}_{V,\chi}^A$.

In [26, Lemma 1.2] Krug proved that for symmetric χ and any basis $\{e_1, \dots, e_l\}$ of V the elements $p_{e_i}^{(n)}, q_{e_i}^{(n)} \in \underline{H}_{V,\chi}^A$ satisfy the relations (2.21), (2.23), (2.24) and no others. This doesn't yet show that for lattices $\underline{H}_{M,\chi}^A \cong \underline{H}_{M,\chi}^{PQ}$, since in $\underline{H}_{M,\chi}^{PQ}$ we also have the additivity relation (2.22) which apriori might impose new relations when reducing to $p_{e_i}^{(n)}$ and $q_{e_i}^{(n)}$. However, it shows that for symmetric χ proving the PQ version of the basis reduction (which amounts to checking that (2.22) also reduces to the basis) would be equivalent to showing that $\underline{H}_{M,\chi}^A \cong \underline{H}_{M,\chi}^{PQ}$.

3. HEISENBERG ALGEBRA OF A GRADED VECTOR SPACE

In this section, for a graded vector space V with a bilinear form χ we give A - and PQ -generator definitions of the Heisenberg algebra $\underline{H}_{V,\chi}$. We prove these equivalent and prove the basis reduction for each. We also prove that for non-degenerate χ our definition is independent of the choice of χ .

3.1. Basis reduction for the lattice PQ Heisenberg algebras. We start by proving:

Theorem 3.1. *Let (M, χ) be a lattice and let e_1, \dots, e_l be a basis of M . Then the Heisenberg algebra \underline{H}_M^{PQ} is isomorphic to the unital \mathbb{k} -algebra with generators $p_a^{(n)}, q_a^{(n)}$ for all $a \in \{e_1, \dots, e_l\}$ and $n \geq 0$ modulo the relations (2.21), (2.23), and (2.24) for all $a, b \in \{e_1, \dots, e_l\}$ and $n, m \geq 0$.*

The formulas we get for lattices in this section explain our definitions for vector spaces in §3.2.

Lemma 3.2. *In presence of the relations (2.21) and (2.23), the relation (2.22) is equivalent to*

$$(3.1) \quad p_{a_1+a_2+\dots+a_k}^{(n)} = \sum_{(n_1, \dots, n_k) \vdash n} p_{a_1}^{(n_1)} p_{a_2}^{(n_2)} \dots p_{a_k}^{(n_k)} \quad \forall n \geq 0, k \geq 1, a_1, \dots, a_k \in M,$$

and an analogous relation for q 's. The sum is over all ordered k -partitions (n_1, \dots, n_k) , see §2.2.

Proof. The relation (2.22) is the case $k = 2$ of the relation (3.1), so it suffices to show that the former implies the latter. We show this by induction on k . The base is the case $k = 1$ which is tautologically true. Suppose the relation (3.1) holds for $l \leq k - 1$. Then

$$\begin{aligned} p_{a_1+a_2+\dots+a_k}^{(n)} &= \sum_{j=0}^n p_{a_1+a_2+\dots+a_{k-1}}^{(n-j)} p_{a_k}^{(j)} = \\ &= \sum_{j=0}^n \left(\sum_{n_1+\dots+n_{k-1}=n-j} p_{a_1}^{(n_1)} p_{a_2}^{(n_2)} \dots p_{a_{k-1}}^{(n_{k-1})} \right) p_{a_k}^{(j)} = \sum_{(n_1, \dots, n_k) \vdash n} p_{a_1}^{(n_1)} p_{a_2}^{(n_2)} \dots p_{a_k}^{(n_k)}, \end{aligned}$$

where the first equality is by (2.22), the second is by the induction assumption, and third is by noting that summing over all ordered k -partitions of n is the same as summing over the size j of the last part of the partition, and then summing over all ordered $(k-1)$ -partitions of $n-j$. \square

Setting all a_i in (3.1) to be the same yields a formula for $p_{ka}^{(n)}$ for $k \geq 0$. We want it to work for any $k \in \mathbb{Z}$ and, ultimately, any $k \in \mathbb{k}$. This requires the generalised binomial coefficients, see §2.1.

Definition 3.3. Let $n \geq 0$ and $\underline{n} = \{n_1, \dots, n_m\}$ be any unordered partition. We write

$$p_a^{(\underline{n})} := p_a^{(n_1)} \dots p_a^{(n_m)} \quad \text{and} \quad q_a^{(\underline{n})} := q_a^{(n_1)} \dots q_a^{(n_m)} \quad \forall a \in M.$$

Lemma 3.4. *In presence of the relations (2.21) and (2.23), the relation (2.22) implies the relation*

$$(3.2) \quad p_{ka}^{(\underline{n})} = \sum_{\underline{n} \vdash n} \binom{k}{r(n)} p_a^{(\underline{n})} \quad \forall k \in \mathbb{Z}, a \in M,$$

and an analogous relation for q 's. Here the sum is taken over all unordered partitions \underline{n} of n .

Proof. For $k \geq 1$ this follows from Lemma 3.2 by setting $a_i = a$ in (3.1) and applying Cor. 2.12 to the map $p_{(a)}^{(-)}: \text{Part} \rightarrow \underline{H}_M$. For general k , we proceed by induction on n . By (2.22), we have

$$(3.3) \quad p_{ka}^{(n)} = p_{(k-1)a+a}^{(n)} = \sum_{i=0}^n p_{(k-1)a}^{(n-i)} p_a^{(i)}.$$

We claim that using (3.2) for each term $p_{(k-1)a}^{(n-i)}$ on the RHS to replace it by the corresponding sum turns (3.3) into the relation (3.2) for $p_{ka}^{(n)}$. All the summands on the RHS except for $p_{(k-1)a}^{(n)}$ only involve terms $p_{(k-1)a}^{(m)}$ with $m < n$. Hence, in presence of the relation (3.2) for $m < n$ and $k \in \mathbb{Z}$, the relation (3.2) for $p_{(k-1)a}^{(n)}$ and for $p_{ka}^{(n)}$ are equivalent. We can thus do both upwards and downwards induction on $k \in \mathbb{Z}$ starting for each n with $k = 1$ where the relation (3.2) holds tautologically.

For the claim, consider each summand $p_{(k-1)a}^{(n-i)} p_a^{(i)}$ on the RHS with $i \geq 1$. Replacing $p_{(k-1)a}^{(n-i)}$ according to the relation (3.2), we obtain a sum in which for any partition $\underline{n} \vdash n$ the term $p_a^{(\underline{n})}$

occurs with the coefficient $\binom{k-1}{r(\underline{n} \setminus \{i\})}$ if \underline{n} contains a part of size i and 0 otherwise. Note that $\underline{r}(\underline{n} \setminus \{i\})$ is just $\underline{r}(\underline{n})$ with $r_i(\underline{n})$ decreased by 1. On the other hand, $p_{(k-1)a}^{(n)}$ contributes the term $p_a^{(n)}$ with coefficient $\binom{k-1}{r(\underline{n})}$. The claim now follows from the multinomial identity (2.8). \square

Corollary 3.5. *In presence of the relations (2.21) and (2.23), the relation (2.22) in the definition of $\underline{H}_{M,\chi}^{PQ}$ implies the following relation for any decomposition $a = \sum_{i=1}^m k_i a_i$ with $k_i \in \mathbb{Z}$ and $a_i \in M$*

$$(3.4) \quad p_{\sum k_i a_i}^{(n)} = \sum_{(n_1, \dots, n_m) \vdash n} \sum_{\underline{n}_1 \vdash n_1, \dots, \underline{n}_m \vdash n_m} \binom{k_1}{r(\underline{n}_1)} \cdots \binom{k_m}{r(\underline{n}_m)} p_{a_1}^{(n_1)} \cdots p_{a_m}^{(n_m)}$$

and a similar relation for q 's.

Proof. Follows from Lemmas 3.2 and 3.4. \square

Proposition 3.6. *Let (M, χ) be a lattice and e_1, \dots, e_m be a basis of M . In presence of the relations (2.21) and (2.23), the relation (2.22) in the definition of $\underline{H}_{M,\chi}^{PQ}$ is equivalent to having for any $a \in M$ the relation (3.4) with respect to its basis decomposition $a = \sum_{i=1}^m k_i e_i$.*

Proof. By Cor. 3.5 relations (2.22) imply relations (3.4) with respect to all decompositions. In particular, the basis ones. For the converse, let $a, b \in M$ and let $a = \sum k_j e_j$ and $b = \sum l_j e_j$ be their basis decompositions. We need to prove (2.22) for a and b . Use (3.4) for the basis decompositions of a and b to replace all terms on the RHS of (2.22) by the corresponding sums. The RHS becomes

$$\sum_{n=n_1+\dots+n_m} \sum_{n_i=n_{i1}+n_{i2}} \sum_{\underline{n}_{ij} \vdash n_{ij}} \binom{k_1}{r(\underline{n}_{11})} \binom{l_1}{r(\underline{n}_{12})} \cdots \binom{k_m}{r(\underline{n}_{m1})} \binom{l_m}{r(\underline{n}_{m2})} p_{e_1}^{(n_{11} \cup n_{12})} \cdots p_{e_m}^{(n_{m1} \cup n_{m2})}.$$

We can rewrite this as

$$\sum_{n=n_1+\dots+n_m} \sum_{\underline{n}_i \vdash n_i} \left(\sum_{n_1=\underline{n}_{11} \cup \underline{n}_{12}} \binom{k_1}{r(\underline{n}_{11})} \binom{l_1}{r(\underline{n}_{12})} \right) \cdots \left(\sum_{n_m=\underline{n}_{m1} \cup \underline{n}_{m2}} \binom{k_m}{r(\underline{n}_{m1})} \binom{l_m}{r(\underline{n}_{m2})} \right) p_{e_1}^{(n_1)} \cdots p_{e_m}^{(n_m)}.$$

By multinomial Vandermonde's identity (2.9) this is the sum in (3.4) for the basis decomposition of $a + b$. Thus (2.22) holds for a and b if (3.4) hold for the basis decompositions of a , b and $a + b$. \square

It remains to show that in presence of all the other relations in the definition of $\underline{H}_{M,\chi}^{PQ}$, the Heisenberg relation (2.24) for all $a, b \in M$ reduces to having it only for all a, b in a basis of M . We proceed in two steps: additivity and scalar multiplication.

Lemma 3.7. *In presence of the relations (2.21), (2.22) and (2.23) in the definition of the Heisenberg algebra of a lattice (M, χ) , having the Heisenberg relation (2.24) for pairs $a_1, b \in M$ and $a_2, b \in M$ implies having it for the pair $a_1 + a_2, b \in M$. Similarly, having (2.24) for pairs $a, b_1 \in M$ and $a, b_2 \in M$ implies having it for the pair $a, b_1 + b_2 \in M$.*

Proof. We only prove the first assertion. The second one is proved similarly.

Let $x_1 := \langle a_1, b \rangle_\chi$ and $x_2 := \langle a_2, b \rangle$. Let $x := \langle a_1 + a_2, b \rangle_\chi = x_1 + x_2$. We have:

$$\begin{aligned} q_{a_1+a_2}^{(n)} p_b^{(m)} &= \sum_{j=0}^n q_{a_1}^{(n-j)} q_{a_2}^{(j)} p_b^{(m)} = \sum_{j=0}^n \sum_{i_2=0}^{\min(j,m)} \binom{x_2 + i_2 - 1}{i_2} q_{a_1}^{(n-j)} p_b^{(m-i_2)} q_{a_2}^{(j-i_2)} = \\ &= \sum_{j=0}^n \sum_{i_2=0}^{\min(j,m)} \sum_{i_1=0}^{\min(n-j, m-i_2)} \binom{x_2 + i_2 - 1}{i_2} \binom{x_1 + i_1 - 1}{i_1} p_b^{(m-i_1-i_2)} q_{a_1}^{(n-j-i_1)} q_{a_2}^{(j-i_2)}. \end{aligned}$$

where the first equality is due to (2.22) and the latter two are due to (2.24) for a_1, b and a_2, b .

We now reindex to sum over $i := i_1 + i_2$, $j' := j - i_2$ and i_1 . This turns the sum above into:

$$(3.5) \quad \sum_{i=0}^{\min(n,m)} \sum_{j'=0}^{n-i} \sum_{i_1=0}^i \binom{x_2 + (i - i_1) - 1}{i - i_1} \binom{x_1 + i_1 - 1}{i_1} p_b^{(m-i)} q_{a_1}^{(n-i-j')} q_{a_2}^{(j')},$$

By the negative binomial identity (2.5) and Vandermonde's identity (2.3) we have

$$\sum_{i_1=0}^i \binom{x_2 + (i - i_1) - 1}{i - i_1} \binom{x_1 + i_1 - 1}{i_1} = \sum_{i_1=0}^i (-1)^i \binom{-x_2}{i - i_1} \binom{-x_1}{i_1} = (-1)^i \binom{-x}{i} = \binom{x + i - 1}{i}.$$

It follows that (3.5) is further equal to

$$(3.6) \quad \sum_{i=0}^{\min(n,m)} \sum_{j'=0}^{n-i} \binom{x+i-1}{i} p_b^{(m-i)} q_{a_1}^{(n-i-j')} q_{a_2}^{(j')} = \sum_{i=0}^{\min(n,m)} \binom{x+i-1}{i} p_b^{(m-i)} q_{a_1+a_2}^{(n-i)}$$

where the final equality is due to (2.22). This shows (2.24) for the pair $a_1 + a_2, b \in M$, as desired. \square

Lemma 3.8. *In presence of the relations (2.21), (2.22) and (2.23) in the definition of the Heisenberg algebra of a lattice (M, χ) , having the Heisenberg relation (2.24) for pair $a, b \in M$ implies having it for the pairs $ka, b \in M$ and $a, kb \in M$ for any $k \in \mathbb{Z}$.*

Proof. We only prove the first assertion. Let $x := \langle a, b \rangle_\chi$. By Lemma 3.4 the relations (2.21), (2.22) and (2.23) imply the relation (3.2). Hence for the LHS of the relation (2.24) for $ka, b \in M$ we have

$$(3.7) \quad q_{ka}^{(n)} p_b^{(m)} = \sum_{\underline{n} \vdash n} \binom{k}{r(\underline{n})} q_a^{(\underline{n})} p_b^{(m)} = \sum_{\underline{n} \vdash n} \sum_{i=0}^{\min(n,m)} \sum_{\substack{i \leq n, \\ \underline{i} \vdash i}} \binom{k}{r(\underline{n})} \left(\prod_{j=1}^{|n|} \binom{x+i_j-1}{i_j} \right) p_b^{(m-i)} q_a^{(\underline{n}-\underline{i})}$$

where the third sum is over all choices $i_j \leq n_j$ for each n_j in \underline{n} yielding a partition $\underline{i} \vdash i$ and where $\underline{n} - \underline{i}$ is the complementary partition of $n - i$ formed by $n_j - i_j$. The first equality is due to the relation (3.2) and the second due to the relation (2.24) for $a, b \in M$.

On the other hand, for the RHS of the Heisenberg relation (2.24) for $ka, b \in M$ we have

$$(3.8) \quad \sum_{i=0}^{\min(m,n)} \binom{kx+i-1}{i} p_b^{(m-i)} q_{ka}^{(n-i)} = \sum_{i=0}^{\min(m,n)} \sum_{\substack{\underline{n-i} \vdash n-i \\ \underline{i} \vdash i}} \binom{k}{r(\underline{n-i})} \left(\binom{kx+i-1}{i} p_b^{(m-i)} q_a^{(\underline{n-i})} \right).$$

We need to show that (3.7) equals (3.8). This is equivalent to the equality of coefficients of each term $p_b^{(m-i)} q_a^{(\underline{n}-\underline{i})}$ in both of these expressions. Since these coefficients are polynomial expressions of finite degree in k , it suffices to establish the equality of (3.7) and (3.8) for an infinite number of $k \in \mathbb{k}$. Thus, we can assume k to be a non-negative integer.

When $k \in \mathbb{Z}_{\geq 0}$, we can apply Corollary (2.12) to the following expression in (3.7)

$$\sum_{\substack{i \leq n, \\ \underline{i} \vdash i}} \left(\prod_{j=1}^{|n|} \binom{x+i_j-1}{i_j} \right) p_b^{(m-i)} q_a^{(\underline{n}-\underline{i})}$$

viewed as a map $Part_n \rightarrow \underline{H}_M$. Thus

$$\begin{aligned} (3.7) &= \sum_{i=0}^{\min(n,m)} \sum_{n_1+\dots+n_k=n} \sum_{0 \leq i_j \leq n_j} \left(\prod_{j=1}^k \binom{x+i_j-1}{i_j} \right) p_b^{(m-i)} q_a^{(n_1-i_1)} \dots q_a^{(n_k-i_k)} = \\ &= \sum_{i=0}^{\min(n,m)} \sum_{k_1+\dots+k_k=n-i} \sum_{i_1+\dots+i_k=i} \left(\prod_{j=1}^k \binom{x+i_j-1}{i_j} \right) p_b^{(m-i)} q_a^{(k_1)} \dots q_a^{(k_k)} = \\ &= \sum_{i=0}^{\min(n,m)} \sum_{\substack{\underline{n-i} \vdash n-i \\ \underline{i} \vdash i}} \binom{k}{r(\underline{n-i})} \left(\sum_{i_1+\dots+i_k=i} \left(\prod_{j=1}^k \binom{x+i_j-1}{i_j} \right) \right) p_b^{(m-i)} q_a^{(\underline{n-i})}. \end{aligned}$$

The first and third equalities are due to Corollary (2.12) and the second equality is reindexing.

By the negative binomial identity (2.5) and the generalised Vandermonde's identity (2.4)

$$\sum_{i_1+\dots+i_k=i} \left(\prod_{j=1}^k \binom{x+i_j-1}{i_j} \right) = (-1)^i \sum_{i_1+\dots+i_k=i} \left(\prod_{j=1}^k \binom{-x}{i_j} \right) = (-1)^i \binom{-kx}{i} = \binom{kx+i-1}{i}.$$

We conclude that

$$\begin{aligned} & \sum_{i=0}^{\min(n,m)} \sum_{\underline{n-i} \vdash n-i} \binom{k}{r(\underline{n-i})} \left(\sum_{i_1+\dots+i_k=i} \left(\prod_{j=1}^k \binom{x+i_j-1}{i_j} \right) \right) p_b^{(m-i)} q_a^{(n-i)} = \\ &= \sum_{i=0}^{\min(n,m)} \sum_{\underline{n-i} \vdash n-i} \binom{k}{r(\underline{n-i})} \binom{kx+i-1}{i} p_b^{(m-i)} q_a^{(n-i)} = (3.8). \end{aligned}$$

□

Proof of Theorem 3.1. By definition, $\underline{H}_{M,\chi}$ is the unital \mathbb{k} -algebra with generators $p_a^{(n)}, q_a^{(n)}$ for $a \in M$ and $n \geq 0$ modulo the relations (2.21), (2.22), (2.23), (2.24). By Prop. 3.6, (2.22) can be replaced by (3.4) for each $a \in M$. This expresses each $p_a^{(n)}$ and $q_a^{(n)}$ in terms of $p_{e_i}^{(\bullet)}$ and $q_{e_i}^{(\bullet)}$. By Lemmas 3.7 and 3.8 we only need the Heisenberg relation (2.24) for $a, b \in \{e_1, \dots, e_l\}$. With the relation (3.4) we only need (2.23) for $a, b \in \{e_1, \dots, e_l\}$.

Thus $\underline{H}_{M,\chi}$ is isomorphic to the unital \mathbb{k} -algebra with generators $p_a^{(n)}, q_a^{(n)}$ for $a \in M$ and $n \geq 0$ modulo the relations (2.23), (2.24) for $a, b \in \{e_1, \dots, e_l\}$ and (2.21), (3.4) for all $a \in M$. For any $a \notin \{e_1, \dots, e_l\}$ generators $p_a^{(n)}$ and $q_a^{(n)}$ occur in just one of the relations (2.21), (3.4) which express them in terms of 1, $p_{e_i}^{(\bullet)}$ and $q_{e_i}^{(\bullet)}$. For $a \in \{e_1, \dots, e_l\}$, the relations (3.4) are tautological. We conclude that $\underline{H}_{M,\chi}^{PQ}$ is isomorphic to the unital \mathbb{k} -algebra with generators $p_a^{(n)}, q_a^{(n)}$ for all $a \in \{e_1, \dots, e_l\}$ and relations (2.21), (2.23) and (2.24) as desired. □

3.2. Vector space definition. Let V be a vector space and χ be a bilinear form on V .

3.2.1. A -generator definition. The existing A -generator definition of $\underline{H}_{V,\chi}^A$ (Defn. 2.14) does not make sense for nonsymmetric forms. This can be fixed replacing the Heisenberg relation (2.19) by two relations: one saying that generators $a_v(n)$ and $a_w(m)$ commute for $m, n > 0$ or $m, n < 0$ and the other saying how to commute $a_v(n)$ with $n < 0$ past $a_w(m)$ with $m > 0$.

Definition 3.9. Let (V, χ) be a vector space with a bilinear form. The *Heisenberg algebra* $\underline{H}_{V,\chi}^A$ is the unital \mathbb{k} -algebra with generators $a_v(n)$ for $v \in V$ and $n \in \mathbb{Z} \setminus \{0\}$ modulo the relations:

$$(3.9) \quad a_v(n)a_w(m) = a_w(m)a_v(n) \quad \forall v, w \in V \text{ and either } m, n \in \mathbb{Z}_{>0} \text{ or } m, n \in \mathbb{Z}_{<0}$$

$$(3.10) \quad a_{v+w}(n) = a_v(n) + a_w(n) \quad \forall v, w \in V \text{ and } n \in \mathbb{Z} \setminus \{0\},$$

$$(3.11) \quad a_{zv}(n) = za_v(n) \quad \forall v \in V, z \in \mathbb{k}, n \in \mathbb{Z} \setminus \{0\},$$

$$(3.12) \quad a_v(-n)a_w(m) = a_w(m)a_v(-n) + \delta_{n,m}m\langle v, w \rangle_\chi \quad \forall v, w \in V \text{ and } n, m \in \mathbb{Z}_{>0}.$$

As the new relations (3.9) and (3.12) are still bilinear in V , the basis reduction is still immediate:

Proposition 3.10. Let (V, χ) be a vector space with a bilinear form and e_1, \dots, e_l be a basis. The Heisenberg algebra \underline{H}_V^A is isomorphic to the unital \mathbb{k} -algebra with generators $a_v(n)$ for $v \in \{e_1, \dots, e_l\}$ and $n \in \mathbb{Z} \setminus \{0\}$ modulo the relations (3.9), (3.12) for $v, w \in \{e_1, \dots, e_l\}$ and $n, m \in \mathbb{Z} \setminus \{0\}$.

Proof. Same as the proof of Proposition 2.16. □

3.2.2. PQ -generator definition. To extend the PQ -generator definition from lattices (Defn. 2.17) to vector spaces we need a new scalar multiplication relation compatible with the existing ones. That is, it would add no new relations when reducing to a basis of V .

For lattices, by Lemma 3.4 the additivity relation (2.22) implies the following relation in $\underline{H}_{M,\chi}^{PQ}$

$$(3.13) \quad p_{ka}^{(n)} = \sum_{\underline{n} \vdash n} \binom{k}{r(\underline{n})} p_a^{(n)}, \quad \forall a \in M, k \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0}.$$

It is natural to expect that for vector spaces this must hold for each $z \in \mathbb{k}$, not merely each $k \in \mathbb{Z}$. It turns out that this is all we need to do – impose (3.13) as a separate relation for all $z \in \mathbb{k}$:

Definition 3.11. Let (V, χ) be a vector space with a bilinear form. The *Heisenberg algebra* $\underline{H}_{V,\chi}^{PQ}$ is the unital \mathbb{k} -algebra with generators $p_v^{(n)}$, $q_v^{(n)}$ for $v \in V$ and $n \in \mathbb{Z}_{\geq 0}$ modulo the relations:

$$(3.14) \quad p_v^{(0)} = 1 = q_v^{(0)} \quad \forall v \in V,$$

$$(3.15) \quad p_v^{(n)} p_w^{(m)} = p_w^{(m)} p_v^{(n)} \quad \text{and} \quad q_v^{(n)} q_w^{(m)} = q_w^{(m)} q_v^{(n)} \quad \forall v, w \in V \text{ and } m, n \in \mathbb{Z}_{>0},$$

$$(3.16) \quad p_{v+w}^{(n)} = \sum_{k=0}^n p_v^{(k)} p_w^{(n-k)} \quad \text{and} \quad q_{v+w}^{(n)} = \sum_{k=0}^n q_v^{(k)} q_w^{(n-k)} \quad \forall v, w \in V \text{ and } n \in \mathbb{Z}_{>0},$$

$$(3.17) \quad p_{zv}^{(n)} = \sum_{\underline{n} \vdash n} \binom{z}{r(\underline{n})} p_v^{(\underline{n})} \quad \text{and} \quad q_{zv}^{(n)} = \sum_{\underline{n} \vdash n} \binom{z}{r(\underline{n})} q_v^{(\underline{n})} \quad \forall v \in V, z \in \mathbb{k}, n \in \mathbb{Z}_{>0},$$

$$(3.18) \quad q_v^{(n)} p_w^{(m)} = \sum_{k=0}^{\min(m,n)} s^k \langle a, b \rangle_\chi p_w^{(m-k)} q_v^{(n-k)} \quad \forall v, w \in V \text{ and } m, n \in \mathbb{Z}_{>0}.$$

We proceed as in Section 3.1 to establish the basis reduction for this new definition:

Lemma 3.12. Let (V, χ) be a vector space with a bilinear form and $\{e_1, \dots, e_l\}$ be a basis of V . In presence of the relations (3.14), (3.15), the relations (3.16), (3.17) are equivalent to the following basis decomposition relation for each $v \in V$. Let $v = \sum_{i=1}^l v_i e_i \in V$ for some $v_i \in \mathbb{k}$, then:

$$(3.19) \quad p_v^{(n)} = \sum_{(n_1, \dots, n_l) \vdash n} \sum_{\underline{n}_1 \vdash n_1, \dots, \underline{n}_l \vdash n_l} \binom{v_1}{r(\underline{n}_1)} \cdots \binom{v_l}{r(\underline{n}_l)} p_{e_1}^{(\underline{n}_1)} \cdots p_{e_l}^{(\underline{n}_l)}$$

and a similar relation for q 's.

Proof. Clearly, the relations (3.16) and (3.17) imply (3.19). For the converse, (3.19) implies the additivity relation (3.16) by the same argument as in the proof of Prop. 3.6. It remains to show that (3.19) implies (3.17). Let $z \in \mathbb{k}$, $v \in V$ and let $v = \sum_{i=1}^l v_i e_i$ with $v_i \in \mathbb{k}$. We need to show that (3.17) holds for z and v . Use (3.19) for v on all the terms in the RHS of (3.17). Use (3.19) for zv on the single term in the LHS of (3.17). This turns the LHS and RHS of (3.17) into sums of monomials of form $p_{e_1}^{(\bullet)} \cdots p_{e_l}^{(\bullet)}$ with coefficients which are polynomials in z of finite degree. To show the equality of these coefficients, it suffices to show them to be equal on all $z \in \mathbb{Z}_{\geq 0}$.

When $z = k$ for some $k \in \mathbb{Z}_{\geq 0}$, we have for the RHS of the (3.17):

$$\begin{aligned} \sum_{\underline{n} \vdash n} \binom{k}{r(\underline{n})} p_v^{(\underline{n})} &= \sum_{(n_1, \dots, n_k) \vdash n} p_v^{(n_1)} \cdots p_v^{(n_k)} = \\ &= \sum_{(n_1, \dots, n_k) \vdash n} \prod_{i=1}^k \left(\sum_{(n_{i1}, \dots, n_{il}) \vdash n_i} \sum_{\underline{n}_{ij} \vdash n_{ij}} \binom{v_1}{r(\underline{n}_{i1})} \cdots \binom{v_l}{r(\underline{n}_{il})} p_{e_1}^{(n_{i1})} \cdots p_{e_l}^{(n_{il})} \right) = \\ &= \sum_{\substack{(n_{ij}) \vdash n \\ 1 \leq i \leq k, 1 \leq j \leq l}} \sum_{\underline{n}_{ij} \vdash n_{ij}} \prod_{i=1}^k \prod_{j=1}^l \binom{v_j}{r(\underline{n}_{ij})} p_{e_j}^{(n_{ij})} = \\ &= \sum_{\substack{(n_{ij}) \vdash n \\ 1 \leq i \leq k, 1 \leq j \leq l}} \sum_{\underline{n}_{ij} \vdash n_{ij}} \prod_{j=1}^l \left(\left(\prod_{i=1}^k \binom{v_j}{r(\underline{n}_{ij})} \right) p_{e_j}^{(n_{1j} \cup \dots \cup n_{kj})} \right) = \\ &= \sum_{(n_1, \dots, n_l) \vdash n} \sum_{\underline{n}_j \vdash n_j} \prod_{j=1}^l \left(\sum_{\substack{\underline{n}_{1j} \cup \dots \cup \underline{n}_{kj} = \underline{n}_j \\ i=1}} \prod_{i=1}^k \binom{v_j}{r(\underline{n}_{ij})} \right) p_{e_j}^{(n_j)} = \\ &= \sum_{(n_1, \dots, n_l) \vdash n} \sum_{\underline{n}_j \vdash n_j} \prod_{j=1}^l \left(\binom{kv_j}{r(\underline{n}_j)} p_{e_j}^{(n_j)} \right) = \\ &= p_{kv}^{(n)}, \end{aligned}$$

as desired. Here the first equality is by Cor. (2.12), the second is by (3.19) for v , the third, fourth, and fifth by regrouping and reindexing, the sixth is by the generalised Vandermonde identity for multinomials (2.10), and the last is by (3.19) for kv . \square

The next two results shows that in presence of all the other relations in the definition of $\underline{H}_{V,\chi}^{PQ}$, the Heisenberg relation (3.18) reduces to a basis:

Lemma 3.13. *In presence of the relations (3.14), (3.15), (3.16), and (3.17), the Heisenberg relation (2.24) for $v_1, w \in V$ and $v_2, w \in V$ implies (2.24) for $v_1 + v_2, w \in V$. Similarly, (2.24) for $v, w_1 \in V$ and $v, w_2 \in V$ implies (2.24) for $v, w_1 + w_2 \in V$.*

Proof. Identical to the proof of Lemma 3.7. \square

Lemma 3.14. *In presence of the relations (3.14), (3.15), (3.16), and (3.17), the Heisenberg relation (3.18) for $v, w \in V$ implies (3.18) for $zv, w \in V$ and $v, zw \in V$ for any $z \in \mathbb{k}$.*

Proof. Identical to the proof of Lemma 3.8. \square

We can now prove the basis reduction for the PQ -generators of the Heisenberg algebra:

Theorem 3.15. *Let (V, χ) be a vector space with a bilinear form and e_1, \dots, e_l be a basis of V . The Heisenberg algebra $\underline{H}_{V,\chi}^{PQ}$ is isomorphic to the unital \mathbb{k} -algebra with generators $p_{e_i}^{(n)}, q_{e_i}^{(n)}$ for $n \geq 0$ modulo the relations (3.14), (3.15), and (3.18) for all $v, w \in \{e_1, \dots, e_l\}$ and $n, m \geq 0$.*

Proof. By definition, $\underline{H}_{V,\chi}^{PQ}$ is the unital \mathbb{k} -algebra with generators $p_v^{(n)}, q_v^{(n)}$ for all $v \in V$ and $n \geq 0$ modulo the relations (3.14), (3.15), (3.16), (3.17), and (3.18). By Lemma 3.12, we can replace (3.16) and (3.17) by (3.19) for all $v \in V$. This expresses each $p_v^{(n)}$ and $q_v^{(n)}$ in terms of $p_{e_i}^{(\bullet)}$ and $q_{e_i}^{(\bullet)}$. By Lemmas 3.13 and 3.14 we only need the Heisenberg relation (3.18) for $v, w \in \{e_1, \dots, e_l\}$. With the relation (3.19), we only need (3.15) for $v, w \in \{e_1, \dots, e_l\}$.

Thus $\underline{H}_{V,\chi}^{PQ}$ is isomorphic to the unital \mathbb{k} -algebra with generators $p_v^{(n)}, q_v^{(n)}$ for $v \in V$ and $n \geq 0$ modulo the relations (3.15), (3.18) for $v, w \in \{e_1, \dots, e_l\}$ and (3.14), (3.19) for $v \in V$. For any $v \notin \{e_1, \dots, e_l\}$ generators $p_v^{(n)}$ and $q_v^{(n)}$ occur in precisely one of the relations (3.14) or (3.19) which expresses them in terms of 1, $p_{e_i}^{(\bullet)}$ and $q_{e_i}^{(\bullet)}$. For $v \in \{e_1, \dots, e_l\}$, the relation (3.19) is tautological.

We conclude that $\underline{H}_{V,\chi}^{PQ}$ is isomorphic to the unital \mathbb{k} -algebra with generators $p_{e_i}^{(n)}, q_{e_i}^{(n)}$ for $n \geq 0$ and relations (3.14), (3.15), (3.18), as desired. \square

We also get the compatibility of our PQ -definitions for vector spaces and for lattices:

Theorem 3.16. *Let (M, χ) be a lattice. We have an isomorphism of \mathbb{k} -algebras*

$$(3.20) \quad \underline{H}_{M,\chi}^{PQ} \cong \underline{H}_{M \otimes \mathbb{Z}\mathbb{k}, \chi}^{PQ}.$$

Proof. Let $\{e_1, \dots, e_l\}$ be a basis of the lattice M . Then it is also a basis of the vector space $M \otimes \mathbb{Z}\mathbb{k}$. By Theorem 3.1 and Theorem 3.15 both sides of (3.20) are isomorphic to the unital \mathbb{k} -algebra with generators $p_a^{(n)}, q_a^{(n)}$ for $n \geq 0$ and $a \in \{e_1, \dots, e_m\}$ and relations (3.14), (3.15) and (3.18). \square

3.2.3. *Equivalence of A - and PQ -generator definitions.* The basis reduction results for the A - and PQ -generator definitions of the Heisenberg algebra of a vector space let us prove their equivalence:

Theorem 3.17. *Let (V, χ) be a vector space with a bilinear form. There is an isomorphism*

$$(3.21) \quad \phi: \underline{H}_{V,\chi}^{PQ} \xrightarrow{\sim} \underline{H}_{V,\chi}^A.$$

Proof. Choose a basis $\{e_1, \dots, e_l\}$ of V . By the basis reduction for PQ -generators (Theorem 3.15), $\underline{H}_{V,\chi}^{PQ}$ is generated by $p_{e_i}^{(n)}$ and $q_{e_i}^{(n)}$ modulo the relations (3.14), (3.15), and (3.18). Define the map

$$\underline{H}_{V,\chi}^{PQ} \xrightarrow{\sim} \underline{H}_{V,\chi}^A$$

on $p_{e_i}^{(n)}$ and $q_{e_i}^{(n)}$ via the exponentiation formula described in §2.3.3. In \underline{H}_V^A , set for any $v \in V$

$$A_v^+(t) = \sum_{n \geq 1} \frac{a_v(n)}{n} t^n \quad \text{and} \quad A_v^-(t) = \sum_{n \geq 1} \frac{a_v(-n)}{n} t^n,$$

and define $\phi(p_{e_i}^{(n)})$ and $\phi(q_{e_i}^{(n)})$ by

$$\sum_{n \geq 0} \phi(p_{e_i}^{(n)}) t^n := \exp(A_{e_i}^+(t)), \quad \text{and} \quad \sum_{n \geq 0} \phi(q_{e_i}^{(n)}) t^n := \exp(A_{e_i}^-(t)).$$

We need to show that $\phi(p_{e_i}^{(n)})$ and $\phi(q_{e_i}^{(n)})$ satisfy the relations (3.14), (3.15), and (3.18). As the constant term of the exponential series is 1, we have $\phi(p_{e_i}^{(0)}) = 1 = \phi(q_{e_i}^{(0)})$ and so (3.14) holds. As all coefficients of $A_{e_i}^+(t)$ (resp. $A_{e_i}^-(t)$) commute with each other, so do all the coefficients of $\exp(A_{e_i}^+(t))$ (resp. $\exp(A_{e_i}^-(t))$). Thus the commutation relation (3.15) also holds.

To show that the Heisenberg relation (3.18) holds, we follow the method of [12] and [26] in our more general situation. By the Heisenberg relation (3.12) we have in $\underline{H}_{V,\chi}^A$,

$$[A_{e_i}^-(t), A_{e_j}^+(u)] = \sum_{k \geq 1} \langle e_i, e_j \rangle \frac{t^k u^k}{k} = -\langle e_i, e_j \rangle \log(1 - tu).$$

In particular, $[A_{e_i}^-(t), A_{e_j}^+(u)]$ commutes with $A_{e_i}^-(t)$ and $A_{e_j}^+(u)$, so

$$(3.22) \quad \exp(A_{e_j}^-(t)) \exp(A_{e_k}^+(u)) = \exp([A_{e_j}^-(t), A_{e_k}^+(u)]) \exp(A_{e_k}^+(u)) \exp(A_{e_j}^-(t)).$$

Since

$$\exp([A_{e_j}^-(t), A_{e_k}^+(u)]) = (1 - tu)^{-\langle e_i, e_j \rangle} = \sum_{k \geq 0} \binom{-\langle e_i, e_j \rangle}{k} t^k u^k = \sum_{k \geq 0} s^k \langle e_i, e_j \rangle t^k u^k,$$

the relation (3.18) follows by comparing the coefficients of $t^n u^m$ on both sides of (3.22).

Thus $\phi: \underline{H}_{V,\chi}^{PQ} \xrightarrow{\sim} \underline{H}_{V,\chi}^A$ is well-defined. Inspecting the formula (2.25) for $\phi(p_{e_i}^{(n)})$, we see that $\{a_{e_i}(n)\}_{n \geq 0}$ lie in the image of ϕ . A similar argument with $\phi(q_{e_i}^{(n)})$ shows that $\{a_{e_i}(n)\}_{n \leq 0}$ also lie in the image of ϕ . By the basis reduction for A -generators, we conclude that ϕ is surjective.

For injectivity, define a \mathbb{Z} -grading on the generators of \underline{H}_V^{PQ} by $\deg(p_{e_i}^{(n)}) = \deg(q_{e_i}^{(n)}) = n$ and on the generators of \underline{H}_V^A by $\deg(a_{e_i}(n)) = |n|$. The relations on generators do not respect these gradings, but do respect the induced filtrations. We thus get filtrations on \underline{H}_V^{PQ} and \underline{H}_V^A : $(\underline{H}_V^{PQ})_n$ comprises all linear combinations of products of $p_{e_i}^{(x)}$ and $q_{e_j}^{(y)}$ with total degree $\leq n$ and $(\underline{H}_V^A)_n$ all linear combinations of products of $a_{e_i}(x)$ with total degree $\leq n$.

The map ϕ respects these filtrations and for each $n \geq 0$ restricts to a surjective map

$$\phi: (\underline{H}_V^{PQ})_n \twoheadrightarrow (\underline{H}_V^A)_n.$$

We conclude the argument by observing that $(\underline{H}_V^{PQ})_n$ and $(\underline{H}_V^A)_n$ are finite-dimensional vector spaces of the same dimension: their bases are given by all the monomials of form

$$p_{e_{i_1}}^{(x_1)} \cdots p_{e_{i_N}}^{(x_N)} q_{e_{j_1}}^{(y_1)} \cdots q_{e_{j_M}}^{(y_M)}$$

with $i_{\bullet}, j_{\bullet} \in \{1, \dots, l\}$, $x_{\bullet}, y_{\bullet} \in \mathbb{Z}_{\geq 0}$ and $\sum x_{\bullet} + \sum y_{\bullet} = n$ and all the monomials of form

$$a_{e_{i_1}}(x_1) \cdots a_{e_{i_N}}(x_N) a_{e_{j_1}}(-y_1) \cdots a_{e_{j_M}}(-y_M)$$

with $i_{\bullet}, j_{\bullet} \in \{1, \dots, l\}$, $x_{\bullet}, y_{\bullet} \in \mathbb{Z}_{\geq 0}$ and $\sum x_{\bullet} + \sum y_{\bullet} = n$. \square

For any vector space V with bilinear form χ , we use Theorem 3.17 to consider $\underline{H}_{V,\chi}^A$ and $\underline{H}_{V,\chi}^{PQ}$ to be the same algebra $\underline{H}_{V,\chi}$ which has two different sets of generators related by the formula (2.25).

3.2.4. Independence of χ . For non-degenerate χ , we show that $\underline{H}_{V,\chi}$ doesn't depend on χ .

Proposition 3.18. *Let (V, χ) be a vector space with a bilinear form and let $S, T \in GL(V)$. Define the form $S\chi T$ by $\langle v, w \rangle_{S\chi T} := \langle Sv, Tw \rangle_{\chi}$. Then*

$$\underline{H}_{V,\chi} \cong \underline{H}_{V,S\chi T}.$$

Proof. Define a map $\underline{H}_{V,\chi} \rightarrow \underline{H}_{V,S\chi T}$ on the generators by

$$(3.23) \quad q_v^{(n)} \mapsto q_{Sv}^{(n)} \quad \text{and} \quad p_v^{(n)} \mapsto p_{Tv}^{(n)}.$$

This respects the relations (3.14)-(3.18), and hence gives a well-defined map $\underline{H}_{V,\chi} \rightarrow \underline{H}_{V,S\chi T}$. As S and T are invertible, this map is a bijection on the sets of generators and hence an isomorphism. \square

Theorem 3.19. Let (V, χ) be a vector space with a bilinear form. If χ is non-degenerate, then the Heisenberg algebra $\underline{H}_{V,\chi}$ is independent of χ .

Proof. Choose a basis $e_1, \dots, e_l \in V$. Let X be the matrix of χ with respect to this basis and let ι be the diagonal form $\langle e_i, e_j \rangle_\iota = \delta_{i,j}$. We have

$$\langle v, w \rangle_\chi = \langle v, Xw \rangle_\iota.$$

If χ is non-degenerate, X is invertible and by Prop. 3.18 the algebras $\underline{H}_{V,\chi}$ and $\underline{H}_{V,\iota}$ are isomorphic. \square

3.3. Graded vector space definition. In this section, we extend the results of §3.2 to graded vector spaces. This is straightforward for the A -generators, but non-trivial for the PQ -generators.

Throughout this section, a *graded vector space* means a vector space V with \mathbb{Z} - or \mathbb{Z}_2 -grading. Our definitions only use the parity of the degree, so the resulting Heisenberg algebra only depends on the \mathbb{Z}_2 -grading. However, for a \mathbb{Z} -graded vector space this algebra will have a natural \mathbb{Z} -grading.

By a *bilinear form* χ on a graded vector space V we mean a map $V \otimes_{\mathbb{k}} V \rightarrow \mathbb{k}$ of *graded* vector spaces. Note that this means that for homogeneous elements $\langle v, w \rangle_\chi \neq 0$ if and only $\deg(v) + \deg(w) = 0$. As Example 3.21 in Section 3.3.1 shows, if any odd degree v pairs non-trivially with any even degree w , then the Heisenberg relation implies that $a_v(n) = 0$ for all $n \neq 0$.

3.3.1. A -generator definition. As the parametrisation of A -generators by the elements of V is linear, it makes sense to set $\deg(a_v(n)) = \deg(v)$ for all $n \neq 0$. This agrees with the definition of the Heisenberg super Lie algebra in [30, §8.1]. With this in mind, the A -generator definition is the same in the non-graded case, but with the commutation up the standard degree-based sign twist:

Definition 3.20. Let (V, χ) be a graded vector space V with a bilinear form χ . The *Heisenberg algebra* $\underline{H}_{V,\chi}^A$ is the unital graded \mathbb{k} -algebra with generators $a_v(n)$ of degree $\deg(v)$ for all homogeneous $v \in V$ and $n \in \mathbb{Z} \setminus \{0\}$ modulo the relations:

$$(3.24) \quad a_v(n)a_w(m) = (-1)^{\deg(w)\deg(v)}a_w(m)a_v(n) \quad v, w \in V \text{ and } m, n \in \mathbb{Z}_{>0} \text{ or } m, n \in \mathbb{Z}_{<0}$$

$$(3.25) \quad a_{v+w}(n) = a_v(n) + a_w(n) \quad v, w \in V \text{ and } n \in \mathbb{Z} \setminus \{0\},$$

$$(3.26) \quad a_{zv}(n) = za_v(n) \quad v \in V, z \in \mathbb{k}, n \in \mathbb{Z} \setminus \{0\},$$

$$(3.27) \quad a_v(-n)a_w(m) = (-1)^{\deg(w)\deg(v)}a_w(m)a_v(-n) + \delta_{n,m}m\langle v, w \rangle_\chi \quad v, w \in V \text{ and } n, m \in \mathbb{Z}_{>0}.$$

Denote by V^{odd} the sum of all odd degree graded parts of V and by V^{even} the sum of all even degree ones. We use similar notation for any graded vector space and, in particular, for \underline{H}_V .

By Defn. 3.9, the even part $(\underline{H}_V)^{even}$ is the Heisenberg algebra $\underline{H}_{V^{even}}$ in the sense of §3.2. The odd part $(\underline{H}_V)^{odd}$ is a Clifford algebra [30, §8.1]. Since χ is a graded pairing, $(\underline{H}_V)^{even}$ and $(\underline{H}_V)^{odd}$ commute. The following example shows why χ has to be at least \mathbb{Z}_2 -graded:

Example 3.21. Let $v \in (\underline{H}_V)^{even}$, $w \in (\underline{H}_V)^{odd}$ and $n \in \mathbb{Z} \setminus \{0\}$. We have

$$\begin{aligned} 0 &= a_v(-n)a_w(n)a_w(n) = (a_w(n)a_v(-n) + n\langle v, w \rangle)a_w(n) = a_w(n)a_v(-n)a_w(n) + n\langle v, w \rangle a_w(n) = \\ &= a_w(n)(a_w(n)a_v(-n) + n\langle v, w \rangle) + n\langle v, w \rangle a_w(n) = a_w(n)a_w(n)a_v(-n) + 2n\langle v, w \rangle a_w(n) = \\ &= 2n\langle v, w \rangle a_w(n). \end{aligned}$$

A similar computation with $a_w(-n)a_w(-n)a_v(n)$ shows that $2n\langle w, v \rangle a_w(n) = 0$. If χ is graded, $\langle w, v \rangle = \langle v, w \rangle = 0$ for any even v and odd w . Were χ not to be graded, we must have $a_w(n) = 0$ for any odd w which pairs non-trivially, on either side, with an even v . Thus for non-graded χ the Heisenberg algebra $\underline{H}_{V,\chi}^A$ coincides with $\underline{H}_{V^{even} \oplus K,\chi}^A$ where $K \subset V^{odd}$ consists of all elements which pair trivially on both sides with V^{even} . Note that on $V^{even} \oplus K$ the pairing χ is \mathbb{Z}_2 -graded.

Since the relations in Definition 3.9 are still linear, the basis reduction is still immediate:

Proposition 3.22. Let (V, χ) be a graded vector space with bilinear form and $\{e_1, \dots, e_l\}$ be a homogeneous basis. $\underline{H}_{V,\chi}^A$ is isomorphic to the unital \mathbb{k} -algebra with generators $a_{e_i}(n)$ for $n \in \mathbb{Z} \setminus \{0\}$ and the relations (3.24) and (3.27) for $v, w \in \{e_1, \dots, e_l\}$ and $n, m \in \mathbb{Z} \setminus \{0\}$.

Proof. Same as the proof of Prop. 2.16. \square

3.3.2. *PQ-generator definition.* In the graded vector space case, we obtain our PQ -generators from the A -generators by the same exponentiation formulas detailed in Section 2.3.3:

$$\begin{aligned} A_v^+(t) &:= \sum_{n \geq 1} \frac{a_v(n)}{i} t^n & \text{and} & \quad A_v^-(t) := \sum_{n \geq 1} \frac{a_v(-n)}{n} t^n, \\ \sum_{n \geq 0} p_v^{(n)} t^n &:= \exp(A_v^+(t)) & \text{and} & \quad \sum_{n \geq 0} q_v^{(n)} t^n := \exp(A_v^-(t)). \end{aligned}$$

The formulas for $p_v^{(n)}$ and $q_v^{(n)}$ only involve exponentiating $a_v(n)$ for $n > 0$ and $n < 0$, respectively. For $v \in V^{\text{even}}$ these commute, so the explicit formula is the same as in the non-graded case:

$$(3.28) \quad p_v^{(n)} = \sum_{\underline{n} \vdash n} \frac{1}{r_1(\underline{n})! \dots r_n(\underline{n})!} \frac{1}{n_1 \dots n_{r(\underline{n})}} a_v(\underline{n}), \quad q_v^{(n)} = \sum_{\underline{n} \vdash n} \frac{1}{r_1(\underline{n})! \dots r_n(\underline{n})!} \frac{1}{n_1 \dots n_{r(\underline{n})}} a_v(-\underline{n}),$$

where as before by $a_v(\pm \underline{n})$ we mean $a_v(\pm n_1) \dots a_v(\pm n_{r(\underline{n})})$.

On the other hand, for $v \in V^{\text{odd}}$ the generators $a_v(n)$ for $n > 0$ and for $n < 0$ anticommute. Hence $A_v^+(t)$ and $A_v^-(t)$ square to zero. Thus $\sum p_v^{(n)} t^n = 1 + A_v^+(t)$ and $\sum q_v^{(n)} t^n = 1 + A_v^-(t)$, so

$$(3.29) \quad p_v^{(n)} = a_v(n) \quad \text{and} \quad q_v^{(n)} = a_v(-n).$$

Thus for homogeneous $v, w \in V$ and $n, m \geq 0$, the relations between $p_v^{(n)}, q_v^{(n)}$ and $p_w^{(m)}, q_w^{(m)}$ should be the non-graded PQ relations (3.15), (3.18) when $v, w \in V^{\text{even}}$ and the graded A relations (3.24), (3.27) when $v, w \in V^{\text{odd}}$. When $v \in V^{\text{even}}$ and $w \in V^{\text{odd}}$ and vice versa, we have $\langle v, w \rangle = 0$ and both these sets of relations reduce to $p_v^{(n)}, q_v^{(n)}$ commuting with $p_w^{(m)}, q_w^{(m)}$ for all $n, m > 0$.

It is clear from (3.28) that for $v \in V^{\text{even}}$ the elements $p_v^{(n)}, q_v^{(n)}$ are not homogeneous for $n > 1$. Thus apriori \underline{H}_V^{PQ} only has the structure of a filtered algebra. However, the canonical isomorphism $\underline{H}_V^A \cong \underline{H}_V^{PQ}$ of Theorem 3.25 induces a grading on \underline{H}_V^{PQ} which agrees with the filtration.

Definition 3.23. Let (V, χ) be a graded vector space with a bilinear form. The *Heisenberg algebra* $\underline{H}_{V,\chi}^{PQ}$ is the unital filtered \mathbb{k} -algebra with generators $p_v^{(n)}, q_v^{(n)}$ of degree $\leq n \deg(v)$ for homogeneous $v \in V$ and $n \in \mathbb{Z}_{\geq 0}$ modulo the relations:

$$(3.30) \quad p_v^{(0)} = 1 = q_v^{(0)} \quad v \in V,$$

$$(3.31) \quad \begin{aligned} p_v^{(n)} p_w^{(m)} &= (-1)^{\deg v \deg w} p_w^{(m)} p_v^{(n)} & v, w \in V \text{ and } m, n \in \mathbb{Z}_{>0}, \\ q_v^{(n)} q_w^{(m)} &= (-1)^{\deg v \deg w} q_w^{(m)} q_v^{(n)} & v, w \in V \text{ and } m, n \in \mathbb{Z}_{>0}, \end{aligned}$$

$$(3.32) \quad \begin{aligned} p_{v+w}^{(n)} &= \begin{cases} p_v^{(n)} + p_w^{(n)} & v, w \in V^{\text{odd}} \\ \sum_{k=0}^n p_v^{(k)} p_w^{(n-k)} & v, w \in V^{\text{even}} \end{cases} & n \in \mathbb{Z}_{>0}, \\ q_{v+w}^{(n)} &= \begin{cases} q_v^{(n)} + q_w^{(n)} & v, w \in V^{\text{odd}} \\ \sum_{k=0}^n q_v^{(k)} q_w^{(n-k)} & v, w \in V^{\text{even}} \end{cases} & n \in \mathbb{Z}_{>0}, \end{aligned}$$

$$(3.33) \quad p_{zv}^{(n)} = \begin{cases} z p_v^{(n)} & v \in V^{\text{odd}} \\ \sum_{\underline{n} \vdash n} \binom{z}{r(\underline{n})} p_v^{(n)} & v \in V^{\text{even}} \end{cases} \quad z \in \mathbb{k}, n \in \mathbb{Z}_{>0},$$

$$q_{zv}^{(n)} = \begin{cases} z q_v^{(n)} & v \in V^{\text{odd}} \\ \sum_{\underline{n} \vdash n} \binom{z}{r(\underline{n})} q_v^{(n)} & v \in V^{\text{even}} \end{cases} \quad z \in \mathbb{k}, n \in \mathbb{Z}_{>0},$$

$$(3.34) \quad q_v^{(n)} p_w^{(m)} = \begin{cases} -p_w^{(m)} q_v^{(n)} + \delta_{n,m} m \langle v, w \rangle_\chi & v, w \in V^{\text{odd}}, \\ \sum_{k=0}^{\min(m,n)} s^k \langle a, b \rangle_\chi p_w^{(m-k)} q_v^{(n-k)} & \text{otherwise,} \end{cases} \quad m, n \in \mathbb{Z}_{>0},$$

Thus the graded PQ -relations are a mix of the non-graded PQ relations and graded A relations. We prove the basis reduction by using the linearity of the latter and the techniques we already developed for the former in the non-graded case:

Theorem 3.24. *Let (V, χ) be a graded vector space with a bilinear form and e_1, \dots, e_l be a homogenous basis. \underline{H}_V^{PQ} is isomorphic to the unital \mathbb{k} -algebra with generators $p_{e_i}^{(n)}, q_{e_i}^{(n)}$ for $n \geq 0$ and the relations (3.30), (3.31), and (3.34) for $v, w \in \{e_1, \dots, e_l\}$ and $n, m \geq 0$.*

Proof. We proceed as in the proof of Theorem 3.24. By definition, \underline{H}_V is the unital \mathbb{k} -algebra with generators $p_v^{(n)}, q_v^{(n)}$ for all homogeneous $v \in V$ and $n \geq 0$ modulo the relations (3.30), (3.31), (3.32), (3.33), and (3.34) for all homogeneous $v, w \in V$.

First, for $v, w \in V^{\text{even}}$ by Lemma 3.12 we can replace (3.32), (3.33) by basis decomposition relation (3.19) for homogeneous $v \in V^{\text{even}}$. For $v, w \in V^{\text{odd}}$, we can replace (3.32), (3.33) by the linear basis decomposition relation $p_{\sum z_i e_i}^{(n)} = \sum z_i p_{e_i}^{(n)}$ and $q_{\sum z_i e_i}^{(n)} = \sum z_i q_{e_i}^{(n)}$ for homogeneous $v = \sum z_i e_i$ in V^{odd} . In both cases, these express each $p_v^{(n)}$ and $q_v^{(n)}$ in terms of $p_{e_i}^{(\bullet)}$ and $q_{e_i}^{(\bullet)}$.

Next, we only need Heisenberg relation (3.34) for $v, w \in \{e_1, \dots, e_l\}$. For $v, w \in V^{\text{even}}$ this is by Lemmas 3.13 and 3.14. For $v, w \in V^{\text{odd}}$, the relation (3.34) is linear in v and w , so this is immediate from the linear basis decomposition relation. Finally, for $v \in V^{\text{odd}}$ and $w \in V^{\text{even}}$, or vice versa, we have $\langle v, w \rangle = 0$ so (3.34) just says that $q_v^{(n)}$ and $p_w^{(m)}$ commute. This follows from the basis decomposition relation, since we can express $q_v^{(n)}$ and $p_w^{(m)}$ in terms of $q_{e_i}^{(\bullet)}$ with $e_i \in V^{\text{odd}}$ and in terms of $p_{e_i}^{(\bullet)}$ with $e_i \in V^{\text{even}}$, respectively. By a similar argument, we also only need the commutation relations (3.31) for $v, w \in \{e_1, \dots, e_l\}$.

Thus \underline{H}_V^{PQ} is isomorphic to the unital \mathbb{k} -algebra with generators $p_v^{(n)}, q_v^{(n)}$ for $v \in V$ and $n \geq 0$ and the relations (3.31), (3.34) for $v, w \in \{e_1, \dots, e_l\}$ and (3.30) and the basis decomposition for $v \in V$. For any $v \notin \{e_1, \dots, e_l\}$ the generators $p_v^{(n)}$ and $q_v^{(n)}$ occur in precisely one of (3.30) or the basis decomposition, expressing them in terms of 1, $p_{e_i}^{(\bullet)}$ and $q_{e_i}^{(\bullet)}$. For $v \in \{e_1, \dots, e_l\}$, the basis decomposition relation is tautological. We conclude that \underline{H}_V^{PQ} is isomorphic to the unital \mathbb{k} -algebra with generators $p_{e_i}^{(n)}, q_{e_i}^{(n)}$ for all $n \geq 0$ and relations (3.30), (3.31), and (3.34), as desired. \square

3.3.3. Equivalence of A - and PQ - generator definitions.

Theorem 3.25. *Let (V, χ) be a graded vector space with a bilinear form. There is an isomorphism*

$$(3.35) \quad \phi: \underline{H}_{V,\chi}^{PQ} \xrightarrow{\sim} \underline{H}_{V,\chi}^A.$$

Proof. Choose a homogeneous basis $\{e_1, \dots, e_l\}$ of V . By Theorem 3.24, $\underline{H}_{V,\chi}^{PQ}$ is generated by $p_{e_i}^{(n)}$ and $q_{e_i}^{(n)}$ modulo the relations (3.30), (3.31), (3.34). Define the map ϕ on $p_{e_i}^{(n)}$ and $q_{e_i}^{(n)}$ as follows. In \underline{H}_V^A , set for any $v \in V$

$$A_v^+(t) := \sum_{n \geq 1} \frac{a_v(n)}{n} t^n \quad \text{and} \quad A_v^-(t) := \sum_{n \geq 1} \frac{a_v(-n)}{n} t^n.$$

For any $e_i \in V$ define $\phi(p_{e_i}^{(n)})$ and $\phi(q_{e_i}^{(n)})$ by

$$\sum_{n \geq 0} \phi(p_{e_i}^{(n)}) t^n := \exp(A_{e_i}^+(t)), \quad \text{and} \quad \sum_{n \geq 0} \phi(q_{e_i}^{(n)}) t^n := \exp(A_{e_i}^-(t)).$$

Note that for $e_i \in V^{\text{odd}}$ this implies $\phi(p_{e_i}^{(n)}) = a_{e_i}(n)$ $\phi(q_{e_i}^{(n)}) = a_{e_i}(-n)$ for all $n > 0$.

To show that ϕ is well-defined, we need to show that the images $\phi(p_{e_i}^{(n)})$ and $\phi(q_{e_i}^{(n)})$ satisfy the relations (3.30), (3.31), (3.34). As the constant term of the exponential series is 1, (3.30) holds. For any e_i and e_j , as all coefficients of $A_{e_i}^+(t)$ skew-commute with all coefficients of $A_{e_j}^+(t)$, so do all coefficients of $\exp(A_{e_i}^+(t))$ and of $\exp(A_{e_j}^+(t))$. The same holds for $\exp(A_{e_i}^-(t))$ and $\exp(A_{e_j}^-(t))$. Thus (3.31) holds for $\phi(p_{e_i}^{(n)})$ and $\phi(q_{e_i}^{(n)})$.

When e_i and e_j not both in V^{odd} , the same proof as in Theorem 3.25 shows that (3.34) holds for $\phi(q_{e_i}^{(n)})$ and $\phi(p_{e_j}^{(m)})$ because we still have

$$[A_{e_i}^-(t), A_{e_j}^+(u)] = \sum_{k \geq 1} \langle e_i, e_j \rangle \frac{t^k u^k}{k} = -\langle e_i, e_j \rangle \log(1 - tu).$$

When $e_i, e_j \in V^{odd}$, (3.34) is the A-Heisenberg relation. Since $\phi(q_{e_i}^{(n)}) = a_{e_i}(-n)$ and $\phi(p_{e_j}^{(m)}) = a_{e_j}(-m)$, this holds trivially.

Thus $\phi: \underline{H}_V^{PQ} \xrightarrow{\sim} \underline{H}_V^A$ is well-defined. It is surjective because we can invert the exponentiation formulas and injective by the same dimension count argument as in Theorem 3.25. \square

3.3.4. Independence of χ .

Theorem 3.26. *Let (V, χ) be a graded vector space with a bilinear form. For non-degenerate χ , $\underline{H}_{V,\chi}$ is independent of χ .*

Proof. Same as for Theorem 3.19. \square

4. GENERALISED GROJNOWSKI-NAKAJIMA ACTION

Geometrical relevance of the Heisenberg algebras came to prominence with the following famous result by Grojnowski and Nakajima:

Theorem 4.1 (see [29], Theorem 3.1, [17], Theorem 7, and [30], Theorem 8.13). *Let X be a smooth projective surface over \mathbb{C} . Let $X^{[n]}$ be the Hilbert scheme of n points on X . Let χ be the pairing on $H^\bullet(X, \mathbb{Q})$ given by taking the cup product and then the direct image along $X \rightarrow pt$.*

The Heisenberg algebra $H_{H^\bullet(X, \mathbb{Q}), \chi}$ acts on the total cohomology $\bigoplus_{n=0}^\infty H^\bullet(X^{[n]}, \mathbb{Q})$ of the Hilbert schemes of points on X . This action identifies $\bigoplus_{n=0}^\infty H^\bullet(X^{[n]}, \mathbb{Q})$ with the Fock space of $H_{H^\bullet(X, \mathbb{Q}), \chi}$.

The Hilbert schemes $X^{[n]}$ are smooth and this result tells us that their cohomology is determined by the cohomology of X in a straightforward way: the Fock space of the Heisenberg algebra of a graded vector space V is the sum of its graded symmetric powers $\bigoplus_n S^n V$.

When $\dim(X) \geq 3$, $X^{[n]}$ is badly singular and this is no longer true. Grojnowski conjectured [17, Footnote 3] that Theorem 4.1 should hold for a smooth projective variety X of any dimension if one replaced $X^{[n]}$ by the symmetric quotient orbifold $S^n X = X^n / S_n$, where S_n is the permutation group which acts on X^n by permuting the factors, and replaced rational cohomology $H^\bullet(-, \mathbb{Q})$ by equivariant K -theory. This conjecture was later proved by Segal [35] and Wang [41].

Baranovsky decomposition [6] allows to translate our results from Hochschild homology to the orbifold cohomology introduced by Chen and Ruan [13]. They constructed a rationally graded ring with a rather intricately defined product and grading structures, cf. [1, §4] [15, §2]. We need it as a target of the Heisenberg algebra action, so the product structure is irrelevant and we are only interested in its natural \mathbb{Z}_2 -grading [15, Defn. 1.8]. Thus the following simple definition suffices:

Definition 4.2 (see [1], Remark 4.18, [15], Defn. 1.1). *Let Y be a smooth complex variety and G a finite group acting on Y . The *orbifold cohomology* of Y/G is the vector space*

$$H_{orb}^\bullet(Y/G, \mathbb{C}) := \left(\bigoplus_{g \in G} H^\bullet(Y_g, \mathbb{C}) \right)_G,$$

with its natural $\mathbb{Z}/2$ -grading. Here $Y_g := \{y \in Y \mid g.y = y\}$ is the fixed point locus of $g \in G$ and $(-)_G$ denotes taking coinvariants under the action of G on the cohomology induced by each $h \in G$ acting as $X_g \xrightarrow{h.(-)} X_{hgh^{-1}}$.

The following generalises Grojnowski-Nakajima action to all smooth projective varieties:

Theorem 4.3. *Let X be a smooth projective variety over \mathbb{C} and χ be the pairing*

$$(4.1) \quad \langle \alpha, \beta \rangle_\chi = \int_X K(\alpha) \wedge \beta \wedge \text{td}_X$$

defined on $H^\bullet(X, \mathbb{C})$ in [34]. Here K sign twists each $H^{p,q}$ by $(-1)^q$ and td_X is the Todd class.

For each $\alpha \in \mathrm{HH}_\bullet(\mathcal{V})$ and $n > 0$, there are certain operators $A_\alpha(-n)$ and $A_\alpha(n)$ on the total orbifold cohomology $\bigoplus_{n=0}^\infty H_{\mathrm{orb}}^\bullet(X^n/S_n, \mathbb{C})$. These satisfy relations (1.2) and (1.3) and thus define an action of the Heisenberg algebra $\mathrm{HH}_\bullet(\mathcal{V}, \chi)$ on $\bigoplus_{n=0}^\infty H_{\mathrm{orb}}^\bullet(X^n/S_n, \mathbb{C})$. This action identifies $\bigoplus_{n=0}^\infty H_{\mathrm{orb}}^\bullet(X^n/S_n, \mathbb{C})$ with the Fock space of $H_{H^\bullet(X, \mathbb{C}), \chi}$.

Proof. We prove this theorem by deducing it from its noncommutative analogue, Theorem 7.1.

Let \mathcal{V} be the standard DG enhancement of the derived category $D(X)$. By [42, Lemma 4.46], the symmetric power $\mathcal{S}^n \mathcal{V}$ (see §5.2) is the DG enhancement of the derived category $D([X^n/S_n])$ of the symmetric quotient stack $[X^n/S_n]$. By the HKR theorem (see §5.11), $H^\bullet(X, \mathbb{C})$ is isomorphic as a \mathbb{Z}_2 -graded vector space to the Hochschild homology $\mathrm{HH}_\bullet(\mathcal{V})$. Moreover, since X is a smooth projective variety, \mathcal{V} is a smooth and proper DG category. Thus the Euler pairing exists on $\mathrm{HH}_\bullet(\mathcal{V})$ and is non-degenerate (see §5.12). By [34, Prop. 3], the HKR isomorphism identifies the Euler pairing on $\mathrm{HH}_\bullet(\mathcal{V})$ with the pairing (4.1) on $H^\bullet(X, \mathbb{C})$. By [6, Theorem 1.1] the orbifold cohomology $H_{\mathrm{orb}}^\bullet(X^n/S_n, \mathbb{C})$ is isomorphic as a \mathbb{Z}_2 -graded vector space to the Hochschild homology $\mathrm{HH}_\bullet(\mathcal{S}^n \mathcal{V})$.

In view of these identifications, the assertion of the theorem follows from Theorem 7.1. \square

5. PRELIMINARIES ON DG CATEGORIES AND HOCHSCHILD HOMOLOGY

5.1. DG categories. A general introduction can be found in [4, Section 2][40] and the technicalities relevant to this paper in [42, §4]. We use freely the concepts and notation introduced therein.

For the convenience of the reader, we summarise some notation below. Let \mathcal{A} be a small DG category over \mathbb{k} . We write $\mathcal{M}\text{-}\mathcal{A}$, $\mathcal{A}\text{-Mod}$, and $\mathcal{A}\text{-Mod-}\mathcal{A}$ to denote the DG categories of right and left \mathcal{A} -modules and of \mathcal{A} - \mathcal{A} -bimodules. Similarly, we write $\mathcal{M}\text{-}\mathbb{k}$ for the DG category of DG \mathbb{k} -modules. We view DG algebras as DG categories with a single object and ordinary associative algebras as DG algebras concentrated in degree 0.

We view a small DG category as a Morita enhanced triangulated category. Its underlying triangulated category $D_c(\mathcal{A})$, the derived category of compact DG \mathcal{A} -modules. When \mathcal{A} is a commutative associative algebra A we can equivalently view it as an affine scheme $X = \mathrm{Spec} A$. The triangulated category $D_c(A)$ is then equivalent to $D_c(X)$, the compact derived category of perfect complexes of quasi-coherent sheaves on X . Following [25], we also view any small DG category \mathcal{A} as a noncommutative scheme whose derived category of perfect complexes is $D_c(\mathcal{A})$. We say that such a noncommutative scheme is commutative when $D_c(\mathcal{A}) \cong D_c(X)$ for some scheme X over \mathbb{k} .

This notion is most meaningful when \mathcal{A} possesses the following two properties [25, §8.1-8.2]: we say that \mathcal{A} is *smooth* if \mathcal{A} is a perfect object in the category of \mathcal{A} - \mathcal{A} -bimodules and that \mathcal{A} is *proper* when \mathcal{A} is a perfect object in the category of \mathbb{k} -modules, i.e. when the total cohomology of any Hom-complex in \mathcal{A} is finite dimensional. If \mathcal{A} is a commutative scheme X , then \mathcal{A} is smooth (resp. proper) if and only if X is smooth (resp. proper). Thus a smooth and proper noncommutative scheme for us is a smooth and proper DG category. These are our main objects of interest for which we construct an analogue of the Grojnowski-Nakajima Heisenberg algebra action described in §4.

5.2. Strong group actions and DG categories. We need to work with symmetric powers of DG categories. We realise this technically as follows. Let \mathcal{A} be a small DG category. A *strong* action of a finite group G on \mathcal{A} is an embedding of G into the group of DG automorphisms of \mathcal{A} .

Definition 5.1 ([42], Defn. 4.45). The *semi-direct product* $\mathcal{A} \rtimes G$ is the following DG category:

- $\mathrm{Ob} \mathcal{A} \rtimes G = \mathrm{Ob} \mathcal{A}$,
- For any $a, b \in \mathrm{Ob}(\mathcal{A} \rtimes G)$ their morphism complex is

$$\mathrm{Hom}_{\mathcal{A} \rtimes G}^i(a, b) := \{(\alpha, g) \mid \alpha \in \mathrm{Hom}_{\mathcal{A}}^i(g.a, b), g \in G\}$$

with $\deg_{\mathcal{A} \rtimes G}(\alpha, g) = \deg_{\mathcal{A}} \alpha$ and $d_{\mathcal{A} \rtimes G}(\alpha, g) = (d_{\mathcal{A}} \alpha, g)$,

- The composition in $\mathcal{A} \rtimes G$ is given by

$$(\alpha_1, g_1) \circ (\alpha_2, g_2) = (\alpha_1 \circ g_1 \cdot \alpha_2, g_1 g_2).$$

- For any $a \in \mathrm{Ob}(\mathcal{A} \rtimes G)$ the identity morphism of a is $(\mathrm{id}_a, 1_G)$.

The point of this definition is that modules over $\mathcal{A} \rtimes G$ are G -equivariant modules over \mathcal{A} :

Lemma 5.2 ([42], Lemma 4.49). *There are mutually inverse isomorphisms of categories*

$$\begin{aligned} \text{Mod-}(\mathcal{A} \rtimes G) &\leftrightarrows \text{Mod}^G\text{-}\mathcal{A}, \\ \mathcal{H}\text{perf}(\mathcal{A} \rtimes G) &\leftrightarrows \mathcal{H}\text{perf}^G\text{-}\mathcal{A}. \end{aligned}$$

Each autoequivalence $g: \mathcal{A} \xrightarrow{\sim} \mathcal{A}$ with which G acts on \mathcal{A} extends to an autoequivalence $\mathcal{A} \rtimes G$:

Definition 5.3. Let \mathcal{A} be a small DG category and G be a group acting strongly on \mathcal{A} . For any $g \in G$ define the autoequivalence

$$g: \mathcal{A} \rtimes G \xrightarrow{\sim} \mathcal{A} \rtimes G$$

to have the same action on objects as $g: \mathcal{A} \xrightarrow{\sim} \mathcal{A}$ and to act on morphisms by $g(\alpha, f) = (g(\alpha), gfg^{-1})$.

In $\mathcal{A} \rtimes G$ the autoequivalence g is isomorphic to the identity functor. The natural isomorphism $\text{id}_{\mathcal{A} \rtimes G} \rightarrow g$ is given for $a \in \mathcal{A}$ by the isomorphism $g: a \rightarrow g.a$. Further technical aspects of semi-direct products $\mathcal{A} \rtimes G$ can be found in [42, §4.8]. We also need one more technical result:

Definition 5.4. Let G be a group and $H \leq G$ be a subgroup. Write Q for the set of left cosets of H in G . The group G acts on Q by left multiplication. For any $g \in G$ and $q \in Q$ write $g.q \in Q$ to denote this action. For any $g \in G$ write $\text{Fix}(g)$ for the fixed point set of the action of g on Q .

For every $q \in Q$ choose a representative $r_q \in G$ of the corresponding coset. Then for every $g \in G$ and $q \in Q$ write $h_{g,q}$ for the unique element of H such that $gr_q = r_{g.q}h_{g,q}$.

Lemma 5.5. *Let \mathcal{A} be a small DG category, G be a group acting strongly on \mathcal{A} , and $H \leq G$ be a subgroup. Let Q , $\text{Fix}(g)$, r_q , and $h_{g,q}$ be as in Definition 5.4. Let*

$$\text{Res}_H^G: \text{Mod}(\mathcal{A} \rtimes G) \rightarrow \text{Mod}(\mathcal{A} \rtimes H)$$

to be the restriction of scalars functor. Then

$$(5.1) \quad \text{Res}_H^G(a) \cong \bigoplus_{q \in Q} r_q^{-1}.a \quad \forall a \in \mathcal{A}.$$

$$(5.2) \quad \text{Res}_H^G(\alpha): \bigoplus_{q \in G/H} r_q^{-1}.a \rightarrow \bigoplus_{q \in G/H} r_q^{-1}.b \quad \forall \alpha \in \text{Hom}_{\mathcal{A}}(a, b),$$

is the sum of morphisms $r_q^{-1}(\beta): r_q^{-1}.a \rightarrow r_q^{-1}.b$ over all $q \in Q$.

$$(5.3) \quad \text{Res}_H^G(g): \bigoplus_{q \in Q} r_q^{-1}.a \rightarrow \bigoplus_{q \in Q} r_q^{-1}.(g.a) \quad \forall g \in G$$

is the sum of morphisms $h_{g,q}: r_q^{-1}.a \rightarrow r_{g.q}^{-1}.(g.a)$ over all $q \in Q$.

Proof. Let $a \in \mathcal{A}$. We have $\text{Res}_H^G(a) = {}_a(\mathcal{A} \rtimes G) \in \text{Mod-}(\mathcal{A} \rtimes H)$. By [42, Eq.(4.48)], this decomposes as ${}_a(\mathcal{A} \rtimes G) \cong \bigoplus_{g \in G} {}_a\mathcal{A}_g$. Right action of $h \in H$ sends each ${}_a\mathcal{A}_g$ to ${}_a\mathcal{A}_{gh}$ by precomposing with h , so we can regroup this direct sum into a direct sum of $(\mathcal{A} \rtimes H)$ -modules as

$$\bigoplus_{g \in G} {}_a\mathcal{A}_g \cong \bigoplus_{q \in Q} \left(\bigoplus_{h \in H} {}_a\mathcal{A}_{r_q h} \right).$$

Since $\text{Hom}_{\mathcal{A}}(r_q h.(-), a) \cong \text{Hom}_{\mathcal{A}}(h.(-), r_q^{-1}.a)$, we further have

$$\bigoplus_{q \in Q} \left(\bigoplus_{h \in H} {}_a\mathcal{A}_{r_q h} \right) \cong \bigoplus_{q \in Q} \left(\bigoplus_{h \in H} r_q^{-1}.{}_a\mathcal{A}_h \right) \cong \bigoplus_{q \in Q} r_q^{-1}.(\mathcal{A} \rtimes H),$$

where the second isomorphism is by [42, Eq.(4.48)] again.

This establishes the first assertion. The remaining assertions are established by chasing

$$\text{id}_{\bigoplus r_q^{-1}.a} = \sum \text{id}_{r_q^{-1}.a} \in \bigoplus_{q \in Q} r_q^{-1}.(\mathcal{A} \rtimes H)$$

through the isomorphisms above and the maps ${}_a(\mathcal{A} \rtimes G) \rightarrow {}_b(\mathcal{A} \rtimes G)$ and ${}_a(\mathcal{A} \rtimes G) \rightarrow {}_{g.a}(\mathcal{A} \rtimes G)$ given by postcompositions with α and g , respectively. \square

Let \mathcal{A} be a small DG category. For any $n \geq 1$ the permutation group S_n strongly acts on the DG category $\mathcal{A}^{\otimes n}$ by permuting factors of objects and of morphisms: for any $a_1, \dots, a_n \in \mathcal{A}$

$$(5.4) \quad \sigma(a_1 \otimes \cdots \otimes a_n) = a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)}.$$

It is necessary to invert σ to ensure that we get a left action of S_n and thus an embedding of S_n into the group of automorphisms of \mathcal{A} . Similarly, given $\alpha_i \in \text{Hom}_{\mathcal{A}}(a_i, a'_i)$ for $1 \leq i \leq n$ we have

$$(5.5) \quad \sigma(\alpha_1 \otimes \cdots \otimes \alpha_n) = \alpha_{\sigma^{-1}(1)} \otimes \cdots \otimes \alpha_{\sigma^{-1}(n)}.$$

Definition 5.6. Let \mathcal{A} be a small DG category. For any $n > 0$ define the n -th symmetric power $\mathcal{S}^n \mathcal{A}$ of the enhanced triangulated category \mathcal{A} to be the semi-direct product $\mathcal{A} \rtimes S_n$.

By Lemma 5.2, the underlying triangulated category $D_c(\mathcal{S}^n \mathcal{A}) = H^0 \mathcal{H}\text{perf}(\mathcal{S}^n \mathcal{A})$ of $\mathcal{S}^n \mathcal{A}$ coincides with other definitions of n -th symmetrical powers for DG categories [16, Section 2.2.7].

5.3. Hochschild homology.

Definition 5.7. Let \mathcal{A} be a small DG category. Its *Hochschild homology* is

$$(5.6) \quad \text{HH}_\bullet(\mathcal{A}) := H^\bullet(\mathcal{A} \overset{\mathbf{L}}{\otimes}_{\mathcal{A}-\mathcal{A}} \mathcal{A}),$$

where $\mathcal{A} \overset{\mathbf{L}}{\otimes}_{\mathcal{A}-\mathcal{A}} \mathcal{A} \in D(\mathbb{k})$ and $\overset{\mathbf{L}}{\otimes}_{\mathcal{A}-\mathcal{A}}$ is the derived functor of the DG functor

$$(5.7) \quad \otimes_{\mathcal{A}-\mathcal{A}}: \mathcal{A}\text{-Mod-}\mathcal{A} \otimes_k \mathcal{A}\text{-Mod-}\mathcal{A} \rightarrow \mathcal{M}\text{od-}\mathbb{k}$$

where we tensor the left \mathcal{A} -action with the right \mathcal{A} -action and vice versa. More precisely, $\mathcal{A}\text{-}\mathcal{A}$ -bimodules are, equivalently, right $\mathcal{A}^{\text{opp}} \otimes_k \mathcal{A}$ -modules or left $\mathcal{A} \otimes_k \mathcal{A}^{\text{opp}}$ -modules. We can view them as both left and right $\mathcal{A}^{\text{opp}} \otimes_k \mathcal{A}$ -modules via the canonical isomorphism

$$\begin{aligned} \mathcal{A} \otimes_k \mathcal{A}^{\text{opp}} &\xrightarrow{\sim} \mathcal{A}^{\text{opp}} \otimes_k \mathcal{A}, \\ a \otimes b &\mapsto b \otimes a \end{aligned}$$

and (5.7) is the functor of tensoring over this module structure. Explicitly, it sends any pair $E, F \in \mathcal{A}\text{-Mod-}\mathcal{A}$ to the complex of \mathbb{k} -modules

$$(5.8) \quad E \otimes_{\mathcal{A}-\mathcal{A}} F := E \otimes_k F / \{e \otimes a.f.b - b.e.a \otimes f \mid \forall e \in E, f \in F, \text{ and } a, b \in \mathcal{A}\}.$$

We compute $\overset{\mathbf{L}}{\otimes}_{\mathcal{A}-\mathcal{A}}$ by taking an h-projective resolution in either variable. Using the bar-resolution $\bar{\mathcal{A}}$ of the diagonal bimodule \mathcal{A} , see e.g. [5, Section 2.11], we see that $\text{HH}_\bullet(\mathcal{A})$ are isomorphic to the cohomologies of the convolution of the *Hochschild complex* $\text{HC}_\bullet(\mathcal{A})$ over $\mathcal{M}\text{od-}\mathbb{k}$:

$$(5.9) \quad \cdots \rightarrow \bigoplus_{a, b, c \in \mathcal{A}} \text{Hom}_{\mathcal{A}}^\bullet(c, a) \otimes_k \text{Hom}_{\mathcal{A}}^\bullet(b, c) \otimes_k \text{Hom}_{\mathcal{A}}^\bullet(a, b) \rightarrow \bigoplus_{a, b \in \mathcal{A}} \text{Hom}_{\mathcal{A}}^\bullet(b, a) \otimes_k \text{Hom}_{\mathcal{A}}^\bullet(a, b) \rightarrow \bigoplus_{a \in \mathcal{A}} \text{Hom}_{\mathcal{A}}^\bullet(a, a),$$

with the differentials defined by

$$\begin{aligned} \alpha_0 \otimes \alpha_1 \otimes \cdots \otimes \alpha_n &\mapsto \sum_{i=0}^{n-1} (-1)^i \alpha_0 \otimes \cdots \otimes \alpha_i \alpha_{i+1} \otimes \cdots \otimes \alpha_n + \\ &\quad + (-1)^{n+|\alpha_n|(|\alpha_0|+\cdots+|\alpha_{n-1}|)} \alpha_n \alpha_0 \otimes \alpha_1 \otimes \cdots \otimes \alpha_{n-1} \end{aligned}$$

For example, $\alpha_0 \otimes \alpha_1 \otimes \alpha_2 \mapsto \alpha_0 \alpha_1 \otimes \alpha_2 - \alpha_0 \otimes \alpha_1 \alpha_2 + (-1)^{|\alpha_3|(|\alpha_1||\alpha_2|)} \alpha_2 \alpha_0 \otimes \alpha_1$.

Example 5.8. Let A be an associative algebra. Then each term of (5.9) is also concentrated in degree 0. Thus (5.9) is a complex of \mathbb{k} -modules. In particular, since (5.9) is concentrated in non-positive degrees, so is $\text{HH}_\bullet(A)$. We also have $\text{HH}_0(A) = A/[A, A]$. This is a vector space quotient and $[A, A]$ is the subspace of commutators, and not the ideal generated by them.

This recovers the original definition of the Hochschild complex [11, §IX.4] [18] with a minor difference. Originally the i -th Hochschild homology $\text{HH}_i(A)$ was defined as $\text{Tor}_{\mathcal{A}-\mathcal{A}}^i(A, A)$ and (5.9) was a chain complex with $A^{\otimes i}$ in degree i . In DG setup this is unnatural, so we follow Shklyarov [36] and others in our present conventions. Thus, $\text{HH}_i(A)$ in the original definition for associative algebras is $\text{HH}_{-i}(A)$ in our conventions, matching the fact that $\text{Tor}_{\mathcal{A}-\mathcal{A}}^i(A, A)$ is $H^{-i}(A \overset{\mathbf{L}}{\otimes}_{\mathcal{A}-\mathcal{A}} A)$.

The key properties of Hochschild homology are:

- *Self-opposite:* $\text{HH}_\bullet(\mathcal{A}) \cong \text{HH}_\bullet(\mathcal{A}^{\text{opp}})$.
- *Functoriality:* a DG functor $F: \mathcal{A} \rightarrow \mathcal{B}$ induces a map $F^{\text{HH}}: \text{HH}_\bullet(\mathcal{A}) \rightarrow \text{HH}_\bullet(\mathcal{B})$.

- *Homotopy invariance:* If $F: \mathcal{A} \rightarrow \mathcal{B}$ is a quasi-equivalence, then F^{HH} is an isomorphism and if functors $F, G: \mathcal{A} \rightarrow \mathcal{B}$ are homotopy equivalent, then $F^{\text{HH}} = G^{\text{HH}}$ [23, Lemma 3.4].
- *Künneth formula:*

$$(5.10) \quad \text{HH}_\bullet(A \otimes_{\mathbb{k}} B) \cong \text{HH}_\bullet(A) \otimes_{\mathbb{k}} \text{HH}_\bullet(B).$$

- *Morita invariance:* the Yoneda embedding $\mathcal{A} \hookrightarrow \text{Perf}(\mathcal{A})$ into the DG category of perfect \mathcal{A} -modules induces an isomorphism $\text{HH}_\bullet(\mathcal{A}) \cong \text{HH}_\bullet(\text{Perf}(\mathcal{A}))$.

5.4. Opposite category. Let \mathcal{A} be a DG category. The isomorphism $\text{HH}_\bullet(\mathcal{A}) \cong \text{HH}_\bullet(\mathcal{A}^{\text{opp}})$ is induced by the isomorphism of Hochschild complexes

$$(5.11) \quad \text{HC}_\bullet(\mathcal{A}) \xrightarrow{\sim} \text{HC}_\bullet(\mathcal{A}^{\text{opp}})$$

defined for any

$$\alpha_0 \otimes \alpha_1 \otimes \cdots \otimes \alpha_n \in \text{Hom}_{\mathcal{A}}(a_1, a_0) \otimes \text{Hom}_{\mathcal{A}}(a_2, a_1) \otimes \cdots \otimes \text{Hom}_{\mathcal{A}}(a_0, a_n) \in \text{HC}_n(\mathcal{A})$$

by $\alpha_0 \otimes \alpha_1 \otimes \cdots \otimes \alpha_n \mapsto (-1)^{\sum_{i < j} \deg(\alpha_i) \deg(\alpha_j)} \alpha_n \otimes \alpha_{n-1} \otimes \cdots \otimes \alpha_0$. Note that the image lies in $\text{Hom}_{\mathcal{A}^{\text{opp}}}(a_n, a_0) \otimes \text{Hom}_{\mathcal{A}^{\text{opp}}}(a_{n-1}, a_n) \otimes \cdots \otimes \text{Hom}_{\mathcal{A}^{\text{opp}}}(a_0, a_1) = \text{HC}_n(\mathcal{A}^{\text{opp}})$.

5.5. Functoriality. Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a DG functor between two DG categories. The map $F^{\text{HH}}: \text{HH}_\bullet(\mathcal{A}) \rightarrow \text{HH}_\bullet(\mathcal{B})$ is induced by the closed degree zero map of Hochschild complexes

$$(5.12) \quad F: \text{HC}_\bullet(\mathcal{A}) \rightarrow \text{HC}_\bullet(\mathcal{B})$$

defined for any $\alpha_0 \otimes \alpha_1 \otimes \cdots \otimes \alpha_n \in \text{HC}_n(\mathcal{A})$ by $\alpha_0 \otimes \alpha_1 \otimes \cdots \otimes \alpha_n \mapsto F(\alpha_0) \otimes F(\alpha_1) \otimes \cdots \otimes F(\alpha_n)$.

5.6. Künneth isomorphism. Let \mathcal{A} and \mathcal{B} be two DG categories. The Künneth formula isomorphism (5.10) is induced by the closed degree zero map of Hochschild complexes

$$(5.13) \quad K: \text{HC}_\bullet(\mathcal{A}) \otimes_{\mathbb{k}} \text{HC}_\bullet(\mathcal{B}) \xrightarrow{\sim} \text{HC}_\bullet(\mathcal{A} \otimes_{\mathbb{k}} \mathcal{B})$$

defined using the shuffle product as follows. For any Hochschild chains

$$\alpha = \alpha_0 \otimes \cdots \otimes \alpha_n \in \text{HC}_n(\mathcal{A}) \quad \text{and} \quad \beta = \beta_0 \otimes \cdots \otimes \beta_m \in \text{HC}_m(\mathcal{B})$$

we have

$$(5.14) \quad K(\alpha \otimes \beta) = \sum_{\sigma \in S_{n,m}} (-1)^\sigma (-1)^{\deg_\sigma(\alpha, \beta)} (\alpha_0 \otimes \beta_0) \otimes \cdots \otimes (\alpha_i \otimes \text{id}) \otimes \dots (\text{id} \otimes \beta_j) \otimes \dots$$

where in the summand indexed by the shuffle $\sigma \in S_{n,m}$ the factors $(\alpha_i \otimes \text{id})$ and $(\text{id} \otimes \beta_j)$ occur in the positions $\sigma(i)$ and $\sigma(n+j)$, respectively. The second sign is computed by setting $d_\sigma(\alpha, \beta)$ to be the sum of $\deg(\alpha_i) \deg(\beta_j)$ for all $0 \leq i \leq n$ and $0 \leq j \leq m$ such that the factor containing α_i occurs to the right of the factor containing β_j .

The Künneth map (5.13) is a homotopy equivalence of twisted complexes over $\text{Mod-}\mathbb{k}$ [36, §2.4]. In particular, its induced map on cohomologies is an isomorphism, establishing (5.10).

5.7. Hochschild homology and direct sums. In this paper, we need to work with Hochschild chains of morphisms whose objects decompose as direct sums:

Definition 5.9. Let \mathcal{A} be a small DG category. For every $a \in \mathcal{A}$ fix a direct sum decomposition

$$a \cong s(a)_1 \oplus \cdots \oplus s(a)_{m(a)},$$

which may be trivial, i.e. $m(a) = 1$ and $s(a)_1 = a$. For each $1 \leq i \leq m(a)$, let $\iota_i: s(a)_i \rightarrow a$ and $\pi_i: a \rightarrow s(a)_i$ be the inclusion and projection morphisms. Let $p_i: a \rightarrow a$ be the idempotent $\pi_i \circ \iota_i$.

Every Hom-complex of \mathcal{A} decomposes as:

$$\text{Hom}_{\mathcal{A}}(a, b) = \bigoplus_{i,j} p_i \text{Hom}_{\mathcal{A}}(a, b) p_j.$$

The elements of each $p_i \text{Hom}_{\mathcal{A}}(a, b) p_j$ are said to be *single component* morphisms as they each go from a single summand of a to a single summand of b . Correspondingly, each term of the Hochschild complex $\text{HC}_\bullet(\mathcal{A})$ decomposes further into a direct sum of tensor products of $p_i \text{Hom}_{\mathcal{A}}(a, b) p_j$'s:

$$(5.15) \quad \text{HC}_n(\mathcal{A}) = \bigoplus_{\substack{a_0, a_1, \dots, a_n \in \mathcal{A} \\ 1 \leq s_j, t_j \leq m(a_j)}} p_{t_0} \text{Hom}_{\mathcal{A}}^\bullet(a_1, a_0) p_{s_1} \otimes p_{t_1} \text{Hom}_{\mathcal{A}}^\bullet(a_2, a_1) p_{s_2} \otimes \cdots \otimes p_{t_n} \text{Hom}_{\mathcal{A}}^\bullet(a_0, a_n) p_{s_0}.$$

This a *single component chain* decomposition of the terms of $\mathrm{HC}_\bullet(\mathcal{A})$ and the elements of the summands in (5.15) are *single component chains* consisting of single component morphisms.

For each summand in (5.15) we say that it is *continuous* if $s_0 = t_0, \dots, s_n = t_n$ and *discontinuous* otherwise. Its elements are, correspondingly, *continuous* and *discontinuous* single component chains. The Hochschild differential preserves continuity or discontinuity of a chain and therefore we obtain a decomposition of the Hochschild twisted complex $\mathrm{HC}_\bullet(\mathcal{A})$ into a direct sum of twisted complexes

$$(5.16) \quad \mathrm{HC}_\bullet(\mathcal{A}) = \mathrm{HC}_\bullet(\mathcal{A})_{\text{cts}} \oplus \mathrm{HC}_\bullet(\mathcal{A})_{\text{dis}},$$

its continuous and discontinuous components.

Finally, define a closed, degree zero twisted complex map

$$(5.17) \quad \underline{\mathrm{Rdc}}: \mathrm{HC}_\bullet(\mathcal{A}) \rightarrow \mathrm{HC}_\bullet(\mathcal{A})$$

by setting its action on each summand in (5.15) to send each $\alpha_0 \otimes \cdots \otimes \alpha_n$ with $\alpha_i = p_{t_i} \alpha_i p_{s_{i+1}}$ to

$$\pi_{t_0} \alpha_0 \iota_{t_1} \otimes \cdots \otimes \pi_{t_i} \alpha_i \iota_{t_{i+1}} \otimes \cdots \otimes \pi_{t_n} \alpha_n \iota_{t_0}.$$

In other words, the map $\underline{\mathrm{Rdc}}$ kills the discontinuous chains and discards the redundant summands in the continuous chains. A continuous single component Hochschild chain is a chain of morphisms between objects going from a single direct summand to a single direct summand, and with each morphism starting at the direct summand where the previous morphism ended. The map $\underline{\mathrm{Rdc}}$ discards all the summands not involved and reduces a chain to the chain of morphisms between the single direct summands which are involved.

Theorem 5.10. *Let \mathcal{A} be a small DG category. For every $a \in \mathcal{A}$ fix its decomposition into direct summands as in Definition 5.9.*

The map $\underline{\mathrm{Rdc}}: \mathrm{HC}_\bullet(\mathcal{A}) \rightarrow \mathrm{HC}_\bullet(\mathcal{A})$ is homotopic to the identity map of twisted complexes. In particular, for any chain $\alpha \in \mathrm{HC}_\bullet(\mathcal{A})$ which defines a class $[\alpha] \in \mathrm{HH}_\bullet(\mathcal{A})$ we have $[\alpha] = [\underline{\mathrm{Rdc}}(\alpha)]$.

Proof. The desired homotopy between $\mathrm{id}_{\mathrm{HC}_\bullet}$ and $\underline{\mathrm{Rdc}}$ is the degree -1 twisted complex map

$$h : \mathrm{HC}_\bullet(\mathcal{A}) \rightarrow \mathrm{HC}_\bullet(\mathcal{A})$$

whose action on each summand in (5.15) sends each $\alpha_0 \otimes \cdots \otimes \alpha_n$ with $\alpha_i = p_{t_i} \alpha_i p_{s_{i+1}}$ to

$$\sum_{i=0}^n (-1)^i \pi_{t_0} \alpha_0 \otimes \alpha_1 \otimes \cdots \otimes \alpha_i \otimes \iota_{t_{i+1}} \otimes \pi_{t_{i+1}} \alpha_{i+1} \iota_{t_{i+2}} \otimes \cdots \otimes \cdots \otimes \pi_{t_n} \alpha_n \iota_{s_0}.$$

□

5.8. Hochschild homology, DG bicategories and enhanced triangulated categories.

Functoriality of Hochschild homology implies that we have a 1-functor:

$$(5.18) \quad \mathrm{HH}: \mathbf{dgCat}^1 \rightarrow \mathrm{gr-Vect}_k,$$

sending any small DG category \mathcal{A} to $\mathrm{HH}_\bullet(\mathcal{A})$ and any DG functor $\mathcal{A} \rightarrow \mathcal{B}$ to the induced map $\mathrm{HH}_\bullet(\mathcal{A}) \rightarrow \mathrm{HH}_\bullet(\mathcal{B})$. As quasi-equivalences induce isomorphisms on HH_\bullet we further have a 1-functor

$$(5.19) \quad \mathrm{HH}: \mathbf{Ho}(\mathbf{dgCat}^1) \rightarrow \mathrm{gr-Vect}_k.$$

This and the Künneth formula allow us to define:

Definition 5.11. Let \mathbf{M} be any $\mathbf{Ho}(\mathbf{dgCat})$ -enriched bicategory. Define $\mathrm{HH}_\bullet(\mathbf{M})$ to be a graded k -linear 1-category with the same objects as \mathbf{M} and morphism spaces

$$\mathrm{Hom}_{\mathrm{HH}_\bullet(\mathbf{M})}(N, N') = \mathrm{HH}_\bullet(\mathrm{Hom}_{\mathbf{M}}(N, N')).$$

The composition in $\mathrm{HH}_\bullet(\mathbf{M})$ is induced by the composition in \mathbf{M} as follows

$$\begin{aligned} & \mathrm{HH}_\bullet(\mathrm{Hom}_{\mathbf{M}}(N', N'')) \otimes \mathrm{HH}_\bullet(\mathrm{Hom}_{\mathbf{M}}(N, N')) \xrightarrow{\sim} \\ & \xrightarrow{\sim} \mathrm{HH}_\bullet(\mathrm{Hom}_{\mathbf{M}}(N', N'') \otimes \mathrm{Hom}_{\mathbf{M}}(N, N')) \xrightarrow{\text{1-composition in } \mathbf{M}} \\ & \xrightarrow{\text{1-composition in } \mathbf{M}} \mathrm{HH}_\bullet(\mathrm{Hom}_{\mathbf{M}}(N, N'')). \end{aligned}$$

Identity morphisms are given by the classes of identity 1-morphisms in \mathbf{M} .

There is a natural 1-functor

$$\mathbf{dgCat}^1 \rightarrow \mathrm{HH}_\bullet(\mathbf{dgCat}^{dg})$$

which is identity on objects and sends each DG functor $\mathcal{A} \rightarrow \mathcal{B}$ to its Euler character in $HH_0(\mathcal{DGFun}(\mathcal{A}, \mathcal{B}))$. The 1-functor (5.18) lifts to a 1-functor

$$(5.20) \quad \mathrm{HH}: \mathrm{HH}_\bullet(\mathbf{dgCat}^{dg}) \rightarrow \mathrm{gr}\text{-}\mathrm{Vect}_\mathbb{k},$$

which sends any small DG category \mathcal{A} to $\mathrm{HH}_\bullet(\mathcal{A})$ and whose action on the morphism spaces

$$\mathrm{HH}_\bullet(\mathcal{DGFun}(\mathcal{A}, \mathcal{B})) \rightarrow \mathrm{Hom}_\mathbb{k}(\mathrm{HH}_\bullet(\mathcal{A}), \mathrm{HH}_\bullet(\mathcal{B}))$$

is adjoint to the composite map

$$\mathrm{HH}_\bullet(\mathcal{DGFun}(\mathcal{A}, \mathcal{B})) \otimes_\mathbb{k} \mathrm{HH}_\bullet(\mathcal{A}) \xrightarrow{K} \mathrm{HH}_\bullet(\mathcal{DGFun}(\mathcal{A}, \mathcal{B}) \otimes_\mathbb{k} \mathcal{A}) \xrightarrow{\mathrm{HH}_\bullet(\mathrm{eval})} \mathrm{HH}_\bullet(\mathcal{B}).$$

To construct a similar lift of (5.19) it is best to restrict to the full subcategory $\mathbf{Ho}(\mathbf{dgCat}^1)$ consisting of all DG categories of the form $\mathcal{H}perf(\mathcal{A})$. This subcategory is equivalent to the 1-categorical truncation $\mathbf{Mor}(\mathbf{dgCat}^1)$ of the strict 2-category \mathbf{EnhCat}_{kc} of Morita enhanced triangulated categories, and \mathbf{EnhCat}_{kc} lifts to the DG bicategory $\mathbf{EnhCat}_{kc}^{dg}$ of Morita enhanced triangulated categories, cf. [42, §4.4]. The objects of $\mathbf{EnhCat}_{kc}^{dg}$ are small DG categories viewed as enhanced triangulated categories and its 1-morphisms categories are

$$\mathrm{Hom}_{\mathbf{EnhCat}_{kc}^{dg}}(\mathcal{A}, \mathcal{B}) := (\mathcal{A}\text{-}\overline{\mathcal{M}\text{od}}\text{-}\mathcal{B})_{\mathcal{B}\text{-}\mathcal{P}erf},$$

where $\mathcal{A}\text{-}\overline{\mathcal{M}\text{od}}\text{-}\mathcal{B}$ is the bar-category of $\mathcal{A}\text{-}\mathcal{B}$ -bimodules, cf. [5, §3.2]. We similarly have a 1-functor

$$\mathbf{Mor}(\mathbf{dgCat}^1) \rightarrow \mathrm{HH}_\bullet(\mathbf{EnhCat}_{kc}^{dg})$$

which is identity on objects and sends any Morita quasifunctor $F: \mathcal{A} \rightarrow \mathcal{B}$ to the Euler class of any bimodule $M \in \mathcal{A}\text{-}\overline{\mathcal{M}\text{od}}\text{-}\mathcal{B}$ representing it. This lifts to a 1-functor

$$(5.21) \quad \mathrm{HH}: \mathrm{HH}_\bullet(\mathbf{EnhCat}_{kc}^{dg}) \rightarrow \mathrm{gr}\text{-}\mathrm{Vect}_\mathbb{k}$$

which sends any small DG category \mathcal{A} to

$$\mathrm{HH}_\bullet(\mathcal{A}) \cong \mathrm{HH}_\bullet(\mathcal{H}perf(\mathcal{A})) \cong \mathrm{HH}_\bullet(\mathcal{P}erf(\mathcal{A}))$$

and whose action on the morphism spaces

$$\mathrm{HH}_\bullet((\mathcal{A}\text{-}\overline{\mathcal{M}\text{od}}\text{-}\mathcal{B})_{\mathcal{B}\text{-}\mathcal{P}erf}) \rightarrow \mathrm{Hom}_\mathbb{k}(\mathrm{HH}_\bullet(\mathcal{A}), \mathrm{HH}_\bullet(\mathcal{B}))$$

is adjoint to the composite map

$$\mathrm{HH}_\bullet(\mathcal{P}erf(\mathcal{A})) \otimes_\mathbb{k} \mathrm{HH}_\bullet(\mathcal{A}\text{-}\overline{\mathcal{M}\text{od}}\text{-}\mathcal{B})_{\mathcal{B}\text{-}\mathcal{P}erf} \xrightarrow{K} \mathrm{HH}_\bullet(\mathcal{P}erf(\mathcal{A}) \otimes_\mathbb{k} (\mathcal{A}\text{-}\overline{\mathcal{M}\text{od}}\text{-}\mathcal{B})_{\mathcal{B}\text{-}\mathcal{P}erf}) \xrightarrow{\mathrm{HH}_\bullet(\otimes)} \mathrm{HH}_\bullet(\mathcal{P}erf(\mathcal{B})).$$

5.9. Hochschild homology with coefficients in a bimodule.

We need the following:

Definition 5.12. Let \mathcal{A} be a small DG-category and let $M \in \mathcal{A}\text{-Mod-}\mathcal{A}$. The *Hochschild homology of \mathcal{A} with coefficients in M* is

$$(5.22) \quad \mathrm{HH}_\bullet(\mathcal{A}; M) := H^\bullet(M \overset{\mathbf{L}}{\otimes}_{\mathcal{A}\text{-}\mathcal{A}} \mathcal{A}).$$

See [27, §1.1.3] [6, §3, Step 4] [32, §3.2] for other variations of this notion.

As before, using the bar-resolution $\bar{\mathcal{A}}$ of \mathcal{A} we see that $\mathrm{HH}_\bullet(\mathcal{A}; M)$ are isomorphic to the cohomologies of the convolution of the *Hochschild complex* $\mathrm{HC}_\bullet(\mathcal{A}; M)$ with coefficients in M :

$$(5.23) \quad \cdots \rightarrow \bigoplus_{a,b,c \in \mathcal{A}} {}_a M_c \otimes_k \mathrm{Hom}_\mathcal{A}^\bullet(b, c) \otimes_k \mathrm{Hom}_\mathcal{A}^\bullet(a, b) \rightarrow \bigoplus_{a,b \in \mathcal{A}} {}_a M_b \otimes_k \mathrm{Hom}_\mathcal{A}^\bullet(a, b) \rightarrow \bigoplus_{a \in \mathcal{A}} {}_a M_a$$

with the differentials defined by

$$\begin{aligned} m_0 \otimes \alpha_1 \otimes \cdots \otimes \alpha_n &\mapsto m_0 \cdot \alpha_1 \otimes \cdots \otimes \alpha_{n-1} + \\ &+ \sum_{i=1}^{n-1} (-1)^i m_0 \otimes \cdots \otimes \alpha_i \alpha_{i+1} \otimes \cdots \otimes \alpha_n + \\ &+ (-1)^{n+|\alpha_n|(|m_0|+\cdots+|\alpha_{n-1}|)} \alpha_n \cdot m_0 \otimes \alpha_1 \otimes \cdots \otimes \alpha_{n-1} \end{aligned}$$

Let $F: \mathcal{A} \rightarrow \mathcal{A}$ be a DG functor. Define the bimodule ${}_F\mathcal{A} \in \mathcal{A}\text{-Mod-}\mathcal{A}$ by setting

$${}_a({}_F\mathcal{A})_b := \mathrm{Hom}_\mathcal{A}(b, Fa),$$

and letting \mathcal{A} act naturally on the right and via F on the left:

$$\alpha \cdot (\beta) \cdot \gamma = F(\alpha)\beta\gamma \quad \forall \beta \in \text{Hom}_{\mathcal{A}}(b, Fa), \alpha \in \text{Hom}_{\mathcal{A}}(a, c), \gamma \in \text{Hom}_{\mathcal{A}}(d, b).$$

Definition 5.13. Let \mathcal{A} be a small DG category and $F: \mathcal{A} \rightarrow \mathcal{A}$ be a DG functor. The F -twisted Hochschild complex $\text{HC}_\bullet(\mathcal{A}; F)$ and the F -twisted Hochschild homology $\text{HH}_\bullet(\mathcal{A}; F)$ are the Hochschild complex $\text{HH}_\bullet(\mathcal{A}; {}_F\mathcal{A})$ and the Hochschild homology $\text{HH}_\bullet(\mathcal{A}; {}_F\mathcal{A})$.

Definition 5.14. Let \mathcal{A}, \mathcal{B} be small DG categories and $F: \mathcal{A} \rightarrow \mathcal{A}$, $G: \mathcal{B} \rightarrow \mathcal{B}$ be DG functors. Let $H: \mathcal{A} \rightarrow \mathcal{B}$ be a DG functor and $\eta: HF \rightarrow GH$ be a natural transformation. Define the map

$$(5.24) \quad (H, \eta): \text{HH}_\bullet(\mathcal{A}; F) \rightarrow \text{HH}_\bullet(\mathcal{B}; G)$$

by setting

$$\alpha_0 \otimes \alpha_1 \otimes \cdots \otimes \alpha_n \mapsto (\eta \circ H\alpha_0) \otimes H\alpha_1 \otimes \cdots \otimes H\alpha_n.$$

We similarly denote by (H, η) the induced map on Hochschild homologies. If $HF = GH$ and $\eta = \text{id}$, we write H for the map (H, η) . It is simply termwise application of H to the chains of $\text{HH}_\bullet(\mathcal{A}; F)$.

5.10. Hochschild homology and strong group actions. We need the following observation:

Lemma 5.15. Let \mathcal{A} be a DG category with a strong action of a group G . For any $g \in G$ the map

$$g: \text{HH}_\bullet(\mathcal{A} \rtimes G) \rightarrow \text{HH}_\bullet(\mathcal{A} \rtimes G)$$

induced by the autoequivalence g of $\mathcal{A} \rtimes G$ of Definition 5.3 is the identity map.

Proof. This follows from [23, Lemma 3.4] since the autoequivalence g is isomorphic to the identity functor, cf. §5.2. \square

The notion of a functor twisted Hochschild homology detailed in §5.9 is useful in the context of strong categorical group actions. Let \mathcal{A} be a DG category and G a finite group acting strongly on \mathcal{A} . For any $g, h \in G$ we have $hg = (hgh^{-1})h$, and so by Definition 5.14 we get an isomorphism

$$(5.25) \quad h: \text{HC}_\bullet(\mathcal{A}; g) \rightarrow \text{HC}_\bullet(\mathcal{A}; hgh^{-1}).$$

Definition 5.16. Let \mathcal{A} be a DG category with a strong action of a group G . For any $g \in G$ define

$$(5.26) \quad \xi_g: \text{HC}_\bullet(\mathcal{A}; g) \rightarrow \text{HC}_\bullet(\mathcal{A} \rtimes G),$$

by setting

$$\alpha_0 \otimes \cdots \otimes \alpha_m \mapsto (g^{-1}(\alpha_0), g^{-1}) \otimes (\alpha_1, \text{id}) \otimes \cdots \otimes (\alpha_{m-1}, \text{id}) \otimes (\alpha_m, \text{id}).$$

The maps ξ_\bullet are compatible with the shuffle product K detailed in Section 5.6:

Lemma 5.17. Let \mathcal{A} and \mathcal{B} be DG categories and G and H be groups acting strongly on them. Then $G \times H$ acts strongly on $\mathcal{A} \times \mathcal{B}$, and for any $g \in G$ and $h \in H$ the following square commutes:

$$(5.27) \quad \begin{array}{ccc} \text{HC}^\bullet(\mathcal{A}; g) \otimes \text{HC}^\bullet(\mathcal{B}; h) & \xrightarrow{K} & \text{HC}^\bullet(\mathcal{A} \otimes \mathcal{B}; g \times h) \\ \downarrow \xi_g \otimes \xi_h & & \downarrow \xi_{g \times h} \\ \text{HC}^\bullet(\mathcal{A} \rtimes G) \otimes \text{HC}^\bullet(\mathcal{B} \rtimes H) & \xrightarrow{K} & \text{HC}^\bullet((\mathcal{A} \otimes \mathcal{B}) \rtimes (G \times H)). \end{array}$$

Proof. A straightforward verification. The key point is that ξ_g and ξ_h insert g and h into the first element of each basic chain, while K tensors the first elements of the two basic chains and makes this the first element of each summand of their shuffle product. \square

The following unassuming technical fact lies at the heart of the proof of Theorem 6.9:

Lemma 5.18. Let \mathcal{A} be a DG category with a strong action of a group G . Let $H \leq G$ be a subgroup and let Q , $\text{Fix}(g)$, r_q , and $h_{g,q}$ be as in Defn. 5.4. For every $g \in G$ the following diagram commutes:

$$(5.28) \quad \begin{array}{ccc} \text{HH}_\bullet(\mathcal{A}; g) & \xrightarrow{\sum_{q \in \text{Fix}(g)} r_q^{-1}} & \bigoplus_{q \in \text{Fix}(g)} \text{HH}_\bullet(\mathcal{A}; h_{g,q}) \\ \downarrow \xi_g & & \downarrow \sum_{q \in \text{Fix}(g)} \xi_{h_{g,q}} \\ \text{HH}_\bullet(\mathcal{A} \rtimes G) & \xrightarrow{\text{Res}_H^G} & \text{HH}_\bullet(\mathcal{A} \rtimes H). \end{array}$$

where ξ_g and $\xi_{h_{g,q}}$ are morphisms (5.26).

Proof. Let $\underline{\alpha} \in \mathrm{HH}_\bullet(\mathcal{A}; g)$ and lift it to a chain $\underline{\alpha} \in \mathrm{HC}_\bullet(\mathcal{A}; g)$. Write $\underline{\alpha}$ as a sum $\sum_i \underline{\alpha}_i$ of

$$\underline{\alpha}_i := \alpha_{i0} \otimes \cdots \otimes \alpha_{in_i}, \quad \text{where } \alpha_{ij} \text{ is a morphism } \begin{cases} a_{i1} \rightarrow g.a_{i0} & j = 0, \\ a_{i(j+1)} \rightarrow a_{ij} & j = 1, \dots, n_i - 1, \\ a_{i0} \rightarrow a_{in_i} & j = n_i, \end{cases}$$

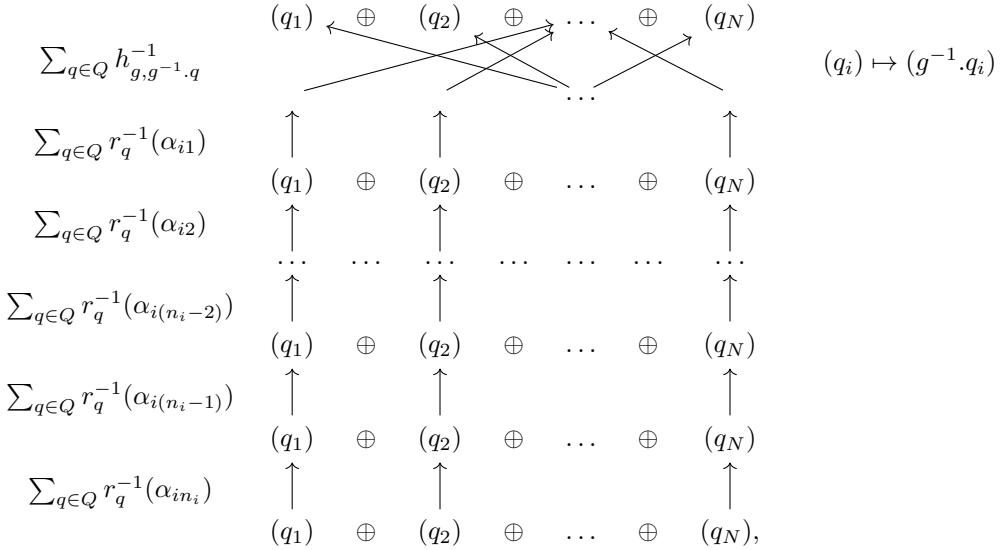
for some objects $a_{ij} \in \mathcal{A}$. By Definition 5.26, we have

$$(5.29) \quad \xi_g(\underline{\alpha}_i) = ((\mathrm{id}, g^{-1}) \circ (\alpha_{i0}, \mathrm{id})) \otimes (\alpha_{i1}, \mathrm{id}) \otimes \cdots \otimes (\alpha_{in_i}, \mathrm{id}).$$

We now apply Res_H^G to the chain on the RHS of (5.29). By Lemma 5.5, we have

$$\mathrm{Res}_H^G(a_{ij}) = \bigoplus_{q \in Q} r_q^{-1}.a_{ij}, \quad \text{and} \quad \mathrm{Res}_H^G(\alpha_{ij}, \mathrm{id}) = \sum_{q \in Q} r_q^{-1}(\alpha_{ij}).$$

Finally, $\mathrm{Res}_H^G(\mathrm{id}, g^{-1})$ sends each $r_q^{-1}.(g.a_{i0})$, the summand indexed by $q \in Q$ in $\mathrm{Res}_H^G(g.a_{i0})$, to $r_{g^{-1}.q}^{-1}(a_{i0})$, the summand indexed by $g^{-1}.q$ in $\mathrm{Res}_H^G(a_{i0})$, via the morphism $h_{g,g^{-1}.q}^{-1}$. By definition $gr_{g^{-1}.q} = r_q h_{g,g^{-1}.q}$, so $h_{g,g^{-1}.q}^{-1} = r_{g^{-1}.q}^{-1} g^{-1} r_q$. Writing $Q = \{q_1, \dots, q_N\}$, we can depict the way the morphisms of the chain $\mathrm{Res}_H^G(\xi_g(\underline{\alpha}_i))$ act on the direct summands of its objects as follows



Each (q_i) represents the q_i -indexed summand of the corresponding direct sum and each arrow represents a non-zero component of the morphism between the corresponding direct sums.

Decompose, as described in §5.7, the chain $\mathrm{Res}_H^G(\xi_g(\underline{\alpha}_i))$ into a sum of single component chains of Defn. 5.9. By above, the continuous chains correspond bijectively to $\mathrm{Fix}(g) \subseteq Q$, the elements fixed by g . For each $q \in \mathrm{Fix}(g)$, the corresponding chain goes through all q -indexed summands:

$$(5.30) \quad ((h_{g,q}^{-1}, \mathrm{id}) \circ (r_q^{-1}(\alpha_{i1}), \mathrm{id})) \otimes (r_q^{-1}(\alpha_{i2}), \mathrm{id}) \otimes \cdots \otimes (r_q^{-1}(\alpha_{in_i}), \mathrm{id}).$$

This equals the image of $r_q^{-1}(\underline{\alpha}_i)$ under the map $\xi_{h_{g,q}}$.

Repeating this for each $\mathrm{Res}_H^G(\xi_g(\underline{\alpha}_i))$, we see that the continuous part of $\mathrm{Res}_H^G(\xi_g(\underline{\alpha}))$ is

$$\mathrm{Res}_H^G(\xi_g(\underline{\alpha}))_{\mathrm{cts}} = \sum_i \sum_{q \in \mathrm{Fix}g} \xi_{h_{g,q}} r_q^{-1}(\underline{\alpha}_i) = \sum_{q \in \mathrm{Fix}g} \xi_{h_{g,q}} r_q^{-1}(\underline{\alpha}).$$

This is the image of $\underline{\alpha}$ under $(\sum_{q \in \mathrm{Fix}(g)} \xi_{h_{g,q}}) \circ (\sum_{q \in \mathrm{Fix}(g)} r_q^{-1})$ going around the upper right half of (5.28) on the level of Hochschild complexes. By Theorem 5.10, the classes of $\mathrm{Res}_H^G(\xi_g(\underline{\alpha}))$ and $\mathrm{Res}_H^G(\xi_g(\underline{\alpha}))_{\mathrm{cts}}$ are equal in the Hochschild homology. It follows that the square (5.28) commutes. \square

5.11. Hochschild-Kostant-Rosenberg isomorphism. The famous HKR (Hochschild-Kostant-Rosenberg) theorem interprets the Hochschild homology as a non-commutative analogue of the Hodge cohomology of a smooth algebraic variety. Its original, local version [18, Theorem 5.2] states that for a smooth and commutative associative algebra A we have

$$\mathrm{HH}_{-n}(A) \cong \Omega_A^n,$$

where Ω_A^i is the A -module of \mathbb{k} -linear Kähler differential i -forms. Viewing A as a smooth affine variety $X = \text{Spec } A$, this means that

$$\text{HH}_{-n}(A) \cong H_X^0(\Omega_X^n),$$

where Ω_X^i is the sheaf of differential i -forms on X . The global version of the HKR theorem [24, Theorem 4.6.2] [37, Cor. 2.6] [10, Theorem 4.9] states that for any DG algebra or category A which Morita enhances a smooth, quasi-projective variety X we have

$$\text{HH}_{-n}(A) \cong \bigoplus_{i-j=n} H_X^i(\Omega_X^j).$$

By Morita enhancement we mean A such that $D_c(A) \cong D_c(X)$.

Thus, the total Hochschild homology of A is isomorphic as \mathbb{Z}_2 -graded vector space to the total algebraic Hodge cohomology of X . Since the Hodge-to-de-Rham spectral sequence degenerates in characteristic 0, this is further isomorphic to the total algebraic de Rham cohomology of X :

$$\text{HH}_\bullet(A) \cong H_{\text{Hedge}}^{\bullet, \bullet}(X) \cong H_{dR}^\bullet(X).$$

When $\mathbb{k} = \mathbb{C}$ by Poincaré Lemma this is further isomorphic to the topological cohomology $H^\bullet(X, \mathbb{C})$.

5.12. Euler pairing on Hochschild homology. If A is proper, then its Hochschild homology carries a graded bilinear form

$$\chi : \text{HH}_\bullet(A) \otimes_{\mathbb{k}} \text{HH}_\bullet(A) \rightarrow \mathbb{k},$$

called the *Euler pairing*, see [36]. We briefly summarise its construction.

For a smooth and proper DG category \mathcal{A} , the standard Hom-pairing on \mathcal{A}

$$\begin{aligned} \mathcal{A} \otimes_{\mathbb{k}} \mathcal{A}^{\text{opp}} &\rightarrow \text{Mod-}\mathbb{k} \\ (a, b) &\mapsto \text{Hom}_{\mathcal{A}}(a, b) \end{aligned}$$

restricts to $\mathcal{A} \otimes_{\mathbb{k}} \mathcal{A}^{\text{opp}} \rightarrow \mathcal{H}\text{perf}(k)$ and therefore induces a map

$$\text{HH}_\bullet(\mathcal{A} \otimes_{\mathbb{k}} \mathcal{A}^{\text{opp}}) \rightarrow \text{HH}_\bullet(\mathcal{H}\text{perf}(k))$$

which the canonical isomorphisms

$$\begin{aligned} \text{HH}_\bullet(\mathcal{A}) \otimes_{\mathbb{k}} \text{HH}_\bullet(\mathcal{A}) &\cong \text{HH}_\bullet(\mathcal{A}) \otimes_{\mathbb{k}} \text{HH}_\bullet(\mathcal{A}^{\text{opp}}) \cong \text{HH}_\bullet(\mathcal{A} \otimes_{\mathbb{k}} \mathcal{A}^{\text{opp}}) \\ \text{HH}_\bullet(\mathcal{H}\text{perf}(k)) &\cong \text{HH}_\bullet(k) \cong \mathbb{k} \end{aligned}$$

turn into a \mathbb{k} -bilinear pairing on $\text{HH}_\bullet(\mathcal{A})$ known as the *Euler pairing*:

$$(5.31) \quad \chi^H : \text{HH}_\bullet(\mathcal{A}) \otimes_{\mathbb{k}} \text{HH}_\bullet(\mathcal{A}) \rightarrow \mathbb{k}$$

For a smooth and proper \mathcal{A} it is shown in [36, Theorem 4] that this pairing is non-degenerate.

5.13. Euler character.

Definition 5.19. Let \mathcal{A} be a small DG category. For any $a \in \mathcal{A}$ the *Euler character* $\text{eu}(a) \in \text{HH}_0(\mathcal{A})$ is the class of $\text{id}_a \in \text{Hom}_{\mathcal{A}}^\bullet(a, a)$.

HH_\bullet is stable under Morita equivalences [23]. Hence $\text{HH}_\bullet(\mathcal{A}) \cong \text{HH}_\bullet(\mathcal{H}\text{perf}(\mathcal{A}))$, so any element of $\mathcal{H}\text{perf}(\mathcal{A})$ has a class in $\text{HH}_0(\mathcal{A})$. An explicit formula for this class was given in [36, Theorem 1] and is as follows. Any $F \in \mathcal{H}\text{perf}(\mathcal{A})$ is homotopy equivalent to a homotopy direct summand of an element of $\mathcal{P}\text{re-Tr}(\mathcal{A})$. It thus suffices, for any twisted complex $(a_i, \alpha_{ij}) \in \mathcal{P}\text{re-Tr}(\mathcal{A})$ and any homotopy idempotent $\pi : (a_i, \alpha_{ij}) \rightarrow (a_i, \alpha_{ij})$ to give a formula for the class of π in $\text{HH}_0(\mathcal{A})$:

$$(5.32) \quad \text{eu}(\pi) = \sum_{l=0}^{\infty} (-1)^l \text{strace}(\pi \otimes (\alpha_{\bullet\bullet})^{\otimes l}),$$

where $\text{strace} : \text{HH}(\mathcal{P}\text{re-Tr } \mathcal{A}) \rightarrow \text{HH}(\mathcal{A})$ is the supertrace map

$$\beta_{\bullet\bullet}^1 \otimes \beta_{\bullet\bullet}^2 \otimes \cdots \otimes \beta_{\bullet\bullet}^n \mapsto \sum_{j \in \mathbb{Z}} (-1)^j \beta_{ji_1}^1 \otimes \beta_{i_1 i_2}^2 \otimes \cdots \otimes \beta_{i_n j}^n,$$

where the sign is as described in [36, §3.2].

Assignment of the Euler character gives a functorial map

$$\text{eu} : \mathcal{H}\text{perf}(\mathcal{A}) \rightarrow \text{HH}_0(\mathcal{A})$$

which is a non-commutative analogue of the Chern character map in the following sense. For any $\alpha: E \rightarrow F$ in $\mathcal{H}perf(\mathcal{A})$, $\text{Cone}(\alpha)$ is the convolution of the twisted complex

$$E \xrightarrow[\deg. 0]{\alpha} F.$$

Applying the formula above to this twisted complex we see that

$$\text{eu}(\text{Cone}(\alpha)) = \text{eu}(F) - \text{eu}(E).$$

It follows that the Euler character map induces \mathbb{k} -module morphism

$$(5.33) \quad \text{eu}: K_0(\mathcal{A}) \rightarrow \text{HH}_0(\mathcal{A})$$

where $K_0(\mathcal{A}) := K_0(H^0(\mathcal{H}perf(\mathcal{A})))$. For a scheme X , set \mathcal{A} to be the DG category of perfect complexes of injective sheaves on X , then $K_0(X) \cong K_0(\mathcal{A})$. On a smooth variety X , the Hochschild-Kostant-Rosenberg theorem identifies $\text{HH}_0(X)$ with its Hodge cohomology $\bigoplus_i H^{i,i}(X)$. The map (5.33) is then identified with the Chern character map

$$K_0(X) \rightarrow \bigoplus_i H^{i,i}(X).$$

The Euler pairing (5.31) on $\text{HH}_0(\mathcal{A})$ was defined via the Hom-pairing on \mathcal{A} , so its composition with the Euler character map (5.33) gives the usual Euler pairing on $K_0(\mathcal{A})$. As the pairing on $\text{HH}_0(\mathcal{A})$ is non-degenerate, the map (5.33) kills the kernel of the Euler pairing on $K_0(\mathcal{A})$. It therefore factors through the projection to the numerical Grothendieck group:

$$(5.34) \quad \text{eu}: K_0^{\text{num}}(\mathcal{A}) \rightarrow \text{HH}_0(\mathcal{A}).$$

5.14. Noncommutative Baranovsky decomposition. In [6] Baranovsky gave a decomposition of the Hochschild homology of the orbifold stack $[X/G]$ where a finite group G acts on a smooth, quasi-projective variety X with generically trivial stabilisers. In [3], Anno, Baranovsky, and the second author give a non-commutative version of this result for symmetric quotient stacks:

$$(5.35) \quad \text{HH}_\bullet(\mathcal{S}^n \mathcal{A}) \cong \bigoplus_{\underline{n} \vdash n} \text{HH}_\bullet\left(\mathcal{A}^{r(\underline{n})}\right)_{S_{r_1(\underline{n})} \times \cdots \times S_{r_n(\underline{n})}},$$

where \underline{n} is an unordered partition of n , $r_i(\underline{n})$ is its number of parts of size i , and $r(\underline{n})$ is the total number of parts. The subscript denotes taking the coinvariants of the action induced by the action of $S_{r_1(\underline{n})} \times \cdots \times S_{r_n(\underline{n})}$ on $\mathcal{A}^{r(\underline{n})} \cong \mathcal{A}^{r_1(\underline{n})} \otimes \cdots \otimes \mathcal{A}^{r_n(\underline{n})}$. By the Künneth formula, it follows that

$$(5.36) \quad \text{HH}_\bullet(\mathcal{S}^n \mathcal{A}) \cong \bigoplus_{\underline{n} \vdash n} \text{Sym}^{r_1(\underline{n})} \text{HH}_\bullet(\mathcal{A}) \otimes \cdots \otimes \text{Sym}^{r_n(\underline{n})} \text{HH}_\bullet(\mathcal{A}).$$

The decomposition (5.36) has two steps. The first is a pair of quasi-inverse quasi-isomorphisms

$$(5.37) \quad \text{HC}_\bullet(\mathcal{S}^n \mathcal{A}) \xrightleftharpoons[]{} (\bigoplus_{\sigma \in S_n} \text{HC}_\bullet(\mathcal{A}^n; \sigma))_{S_n}$$

where the subscript denotes the coinvariants under the action of S_n by the maps

$$\tau: \text{HC}_\bullet(\mathcal{A}^n; \sigma) \xrightarrow{(5.25)} \text{HC}_\bullet(\mathcal{A}^n; \tau\sigma\tau^{-1}) \quad \tau \in S_n.$$

The rightward quasi-isomorphism (5.37) sends

$$(\alpha_0, \sigma_0) \otimes \cdots \otimes (\alpha_m, \sigma_m) \mapsto \alpha_0 \otimes \sigma_0(\alpha_1) \otimes \sigma_0\sigma_1(\alpha_2) \otimes \cdots \otimes \sigma_0 \dots \sigma_{m-1}(\alpha_m) \in \text{HC}_m(\mathcal{A}^n; (\sigma_0 \dots \sigma_m)^{-1}).$$

It is a straightforward generalisation of [6, Proposition 3.5] and it works identically for the strong action of any finite group G on any DG category \mathcal{A} . In the language of G -equivariant objects in $\mathcal{H}perf \mathcal{A}$, it appeared also in [32, Theorem 4.3]. The leftward quasi-isomorphism (5.37) is the roof

$$\left(\bigoplus_{\sigma \in S_n} \text{HC}_\bullet(\mathcal{A}^n; \sigma) \right)_{S_n} \xrightarrow{\sum_{\sigma \in S_n} \xi_\sigma} (\text{HC}_\bullet(\mathcal{S}^n \mathcal{A}))_{S_n} \xleftarrow{q} (\text{HC}_\bullet(\mathcal{S}^n \mathcal{A})),$$

where the maps ξ_σ are as in Definition 5.26, the group S_n acts on $\mathcal{S}^n \mathcal{A}$ as in Definition 5.3, and q is the quotient map. Note that q is a quasi-isomorphism by Lemma 5.15.

For each conjugacy class $\underline{n} \vdash n$ of S_n , choose a representative $\sigma_{\underline{n}}$. Taking the coinvariants of the action of S_n on $\bigoplus_{\sigma \in S_n} \mathrm{HC}_\bullet(\mathcal{A}^n; \sigma)$ reduces it to $\bigoplus_{\underline{n} \vdash n} \mathrm{HC}_\bullet(\mathcal{A}^n; \sigma_{\underline{n}})_{C(\sigma_{\underline{n}})}$, where $C(\sigma_{\underline{n}})$ is the centraliser of \underline{n} . The second step of the (5.36) is a pair of quasi-inverse quasi-isomorphisms

$$(5.38) \quad \mathrm{HC}_\bullet(\mathcal{A}^n; \sigma_{\underline{n}})_{C(\sigma_{\underline{n}})} \xrightleftharpoons[g]{f} \mathrm{HC}_\bullet(\mathcal{A}^{r(\underline{n})})_{S_{r_1(\underline{n})} \times \dots \times S_{r_n(\underline{n})}}.$$

These quasi-isomorphisms are the key result of [3]. Below, we give the explicit formulas for them in the specific case of $\underline{n} = (n)$ which we need in this paper. General formulas in [3] are similar, but involve more cumbersome notation. We introduce the notation we need. Let $t = (1 \dots n) \in S_n$ and assume without loss of generality that $\sigma_{(n)} = t$. The long cycle t acts on \mathcal{V}^n by sending

$$a_1 \otimes a_2 \otimes \dots \otimes a_n \rightarrow a_n \otimes a_1 \otimes \dots \otimes a_{n-1}$$

on objects and correspondingly on the morphisms. The centraliser $C(t)$ is the cyclic subgroup generated by t , so for the partition (n) we can rewrite (5.38) as

$$(5.39) \quad \mathrm{HC}_\bullet(\mathcal{A}^n; t) \xrightleftharpoons[g]{f} \mathrm{HC}_\bullet(\mathcal{A})$$

Let $m \geq 1$ and let

$$\underline{\alpha}_0 \otimes \dots \otimes \underline{\alpha}_{m-1} \in \mathrm{HC}_{m-1}(\mathcal{A}^n; t).$$

By definition of $\mathrm{HC}_{m-1}(\mathcal{A}^n; t)$, for $0 \leq i \leq m-1$ the morphism $\underline{\alpha}_i$ can be postcomposed with $\underline{\alpha}_{i+1}$ in \mathcal{A}^n and $t(\underline{\alpha}_m)$ can be postcomposed with $\underline{\alpha}_0$.

Every chain in $\mathrm{HC}_{m-1}(\mathcal{A}^n; t)$ can be decomposed as a sum of basic chains where

$$\underline{\alpha}_i = \alpha_{(i+1)1} \otimes \dots \otimes \alpha_{(i+1)n} \quad \alpha_{(i+1)j} \in \mathcal{A}.$$

The index shift is because we want to use the matrix notation: the chain above can be visualised as

$$(5.40) \quad \begin{aligned} & (\alpha_{11} \quad \otimes \quad \alpha_{12} \quad \otimes \quad \dots \quad \otimes \quad \alpha_{1n}) \\ & \quad \otimes \\ & (\alpha_{21} \quad \otimes \quad \alpha_{22} \quad \otimes \quad \dots \quad \otimes \quad \alpha_{2n}) \\ & \quad \otimes \\ & \quad \dots \\ & \quad \otimes \\ & (\alpha_{m1} \quad \otimes \quad \alpha_{m2} \quad \otimes \quad \dots \quad \otimes \quad \alpha_{mn}). \end{aligned}$$

and we denote it simply by $m \times n$ matrix

$$(5.41) \quad \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & & & \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mn} \end{pmatrix}$$

For any $1 \leq i \leq m$ and $1 \leq j \leq n$ the morphism α_{ij} can be postcomposed with $\alpha_{(i+1)j}$. The morphism α_{mj} can be postcomposed with $\alpha_{1(j+1)}$ for $1 \leq j \leq n-1$ and with α_{11} for $j = n$.

Note that for $n = 1$ we simply have $\mathrm{HC}_\bullet(\mathcal{A}^n; t) = \mathrm{HC}_\bullet(\mathcal{A})$, so the chain $\alpha_1 \otimes \dots \otimes \alpha_m \in \mathrm{HC}_{m-1}(\mathcal{A})$ is represented in the above notation by the column vector

$$(5.42) \quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \dots \\ \alpha_m \end{pmatrix}$$

We thus switch from denoting chains by $\alpha_0 \otimes \dots \otimes \alpha_{m-1}$ to denoting them by $\alpha_1 \otimes \dots \otimes \alpha_m$.

Definition 5.20. The quasi-isomorphism f in (5.39) is the map

$$(5.43) \quad \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & & & \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mn} \end{pmatrix} \mapsto \frac{1}{n} \sum_{i=1}^n \begin{pmatrix} \alpha_{1(i+1)} \dots \alpha_{mi-1} \alpha_{1i} \\ \alpha_{2i} \\ \dots \\ \alpha_{mi} \end{pmatrix}$$

Definition 5.21. The quasi-isomorphism g in (5.39) is the quotient of the map

$$(5.44) \quad g: \mathrm{HC}_\bullet(\mathcal{A}) \rightarrow \mathrm{HC}_\bullet(\mathcal{A}^n; t),$$

given by

$$(5.45) \quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \dots \\ \alpha_m \end{pmatrix} \mapsto \sum_{c \in \{1, \dots, n\}^n \text{ with } c_1=1} (-1)^{\sigma_c} \begin{pmatrix} \beta_{11} & \beta_{12} & \dots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \dots & \beta_{2n} \\ \dots & & & \\ \beta_{m1} & \beta_{m2} & \dots & \beta_{mn} \end{pmatrix}$$

Here:

- The summation is taken over all $c = (c_1, \dots, c_n)$ with $c_i \in \{1, \dots, n\}$ and $c_1 = 1$. We think of each c_i as a choice of a column position in the i -th row of the matrix (β_{ij}) and we always choose the first position in the first row.
- Define $\sigma_c \in S_m$ to send each i to the number of c_k with $c_k < c_i$ or with $c_k = c_i$ and $k \leq i$.
- Define β_{ij} to be $\alpha_{\sigma_c(i)}$ if $j = c_i$ and the identity map otherwise.

In other words, we take the matrix (β_{ij}) , start at β_{11} , and follow the composable order of its entries (down each column and then to the top of the column to its right), placing $\alpha_1, \dots, \alpha_n$, in that order, in the chosen column position in each row. The remaining entries are filled with the identity maps. We obtain a matrix each whose row contains exactly one α_i . The permutation σ_c corresponds to the new ordering on α_i 's given by counting from the top to the bottom row.

Example 5.22. For $n = 1$ and $m = 3$ the quasi-isomorphism g is the identity map

$$(5.46) \quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \mapsto \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}.$$

For $n = 2$ and $m = 3$ the quasi-isomorphism g is the map

$$(5.47) \quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \mapsto \begin{pmatrix} \alpha_1 & \text{id} \\ \alpha_2 & \text{id} \\ \alpha_3 & \text{id} \end{pmatrix} + \begin{pmatrix} \alpha_1 & \text{id} \\ \text{id} & \alpha_2 \\ \text{id} & \alpha_3 \end{pmatrix} + \begin{pmatrix} \alpha_1 & \text{id} \\ \alpha_2 & \text{id} \\ \text{id} & \alpha_3 \end{pmatrix} - \begin{pmatrix} \alpha_1 & \text{id} \\ \text{id} & \alpha_3 \\ \alpha_2 & \text{id} \end{pmatrix}.$$

For $n = 3$ and $m = 3$ the quasi-isomorphism g is the map

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \mapsto \begin{pmatrix} \alpha_1 & \text{id} & \text{id} \\ \alpha_2 & \text{id} & \text{id} \\ \alpha_3 & \text{id} & \text{id} \end{pmatrix} + \begin{pmatrix} \alpha_1 & \text{id} & \text{id} \\ \alpha_2 & \text{id} & \text{id} \\ \text{id} & \alpha_3 & \text{id} \end{pmatrix} + \begin{pmatrix} \alpha_1 & \text{id} & \text{id} \\ \alpha_2 & \text{id} & \text{id} \\ \text{id} & \text{id} & \alpha_3 \end{pmatrix} - \begin{pmatrix} \alpha_1 & \text{id} & \text{id} \\ \text{id} & \alpha_3 & \text{id} \\ \alpha_2 & \text{id} & \text{id} \end{pmatrix} + \begin{pmatrix} \alpha_1 & \text{id} & \text{id} \\ \text{id} & \alpha_2 & \text{id} \\ \text{id} & \alpha_3 & \text{id} \end{pmatrix} + \begin{pmatrix} \alpha_1 & \text{id} & \text{id} \\ \text{id} & \alpha_2 & \text{id} \\ \text{id} & \text{id} & \alpha_3 \end{pmatrix} - \begin{pmatrix} \alpha_1 & \text{id} & \text{id} \\ \text{id} & \text{id} & \alpha_3 \\ \alpha_2 & \text{id} & \text{id} \end{pmatrix} - \begin{pmatrix} \alpha_1 & \text{id} & \text{id} \\ \text{id} & \text{id} & \alpha_3 \\ \text{id} & \alpha_2 & \text{id} \end{pmatrix} + \begin{pmatrix} \alpha_1 & \text{id} & \text{id} \\ \text{id} & \text{id} & \alpha_2 \\ \text{id} & \text{id} & \alpha_3 \end{pmatrix}$$

Map g in (5.44) is compatible with the shuffle product K detailed in Section 5.6:

Lemma 5.23. Let \mathcal{A} and \mathcal{B} be DG categories and let $n \geq 1$. The following square commutes:

$$(5.48) \quad \begin{array}{ccc} \mathrm{HC}_\bullet(\mathcal{A}) \otimes \mathrm{HC}_\bullet(\mathcal{B}) & \xrightarrow{K} & \mathrm{HC}_\bullet(\mathcal{A} \otimes \mathcal{B}) \\ \downarrow g \otimes g & & \downarrow g \\ \mathrm{HC}_\bullet(\mathcal{A}^n; t_n) \otimes \mathrm{HC}_\bullet(\mathcal{B}^n; t_n) & \xrightarrow{K} & \mathrm{HC}_\bullet(\mathcal{A}^n \otimes \mathcal{B}^n; t_n \times t_n) \cong \mathrm{HC}_\bullet((\mathcal{A} \otimes \mathcal{B})^n; t_n), \end{array}$$

where $t_n \in S_n$ is the long cycle $(1 \dots n)$.

Proof. Let $\alpha \in \mathrm{HC}_\bullet(\mathcal{A})$ and $\beta \in \mathrm{HC}_\bullet(\mathcal{B})$ be two basic chains of lengths p and q . Going around the lower left half of (5.48) sends $\alpha \otimes \beta$ to a sum with the following summands. We choose one position in each non-first row of a $p \times n$ matrix and of a $q \times n$ matrix, place α_i and β_i into these chosen positions as described in the definition of g , and set the remaining entries to be id. Finally, we choose a shuffle $\sigma \in S_{p-1, q-1}$ to shuffle the two matrices together into a $(p+q-1) \times n$ matrix.

The upper right half of (5.48) yields a sum with the following summands. Choose a shuffle $\sigma \in S_{p-1, q-1}$ to shuffle α and β together. Choose one position in each non-first row of a $(p+q-1) \times n$ matrix, place the elements of the shuffled chain into them, and set the remaining entries to id.

These two sets of choices are naturally bijective, and the summands produced by corresponding choices are equal. We conclude that (5.48) commutes on $\alpha \otimes \beta$. \square

6. HOCHSCHILD HOMOLOGY DECATEGORIFICATION OF THE HEISENBERG 2-CATEGORY

In this section, we decategorify the Heisenberg categorification of [42] via Hochschild homology.

6.1. Decategorification map for the Heisenberg 2-category \mathbf{H}_V . Let \mathcal{V} be a smooth and proper DG category. In [42], we constructed for any such \mathcal{V} :

- (1) the $\mathbf{Ho}(\mathbf{dgCat})$ -enriched bicategory \mathbf{H}_V , called the *Heisenberg 2-category* of \mathcal{V} [42, §5],
- (2) the $\mathbf{Ho}(\mathbf{dgCat})$ -enriched bicategory \mathbf{F}_V , called the *categorical Fock space* of \mathcal{V} [42, §7.2],
- (3) the homotopy strong 2-functor $\Phi_V: \mathbf{H}_V \rightarrow \mathbf{F}_V$ which gives categorical action of \mathbf{H}_V on \mathbf{F}_V [42, §7.3–7.5, Theorem 7.38].

As the first step in these constructions, we replace \mathcal{V} by $\mathcal{H}\text{perf}(\mathcal{V})$, the category of its h -projective perfect modules. This does not change its underlying triangulated category $D_c(\mathcal{V})$. Hence we can assume that \mathcal{V} has a homotopy Serre functor and is homotopy Karoubi complete.

In [42, §8] we decategorified these constructions using the numerical Grothendieck group $K_0^{\text{num}}(-)$. Here we decategorify them using the Hochschild homology $\text{HH}_\bullet(-)$. This offers significant simplifications compared to working with the numerical Grothendieck groups. There, lack of Hom-finiteness of $K_0(\mathbf{H}_V, \mathbb{k})$ prevented us from passing to $K_0^{\text{num}}(\mathbf{H}_V, \mathbb{k})$ by factoring out the kernel of the Euler pairing [42, §6.4]. Instead we had to artificially reproduce that by taking the further factorisation of $K_0(\mathbf{H}_V, \mathbb{k})$ by the ad-hoc ideal described in [42, Defn. 6.26].

Here, we decategorify by first applying $\text{HH}_\bullet(-)$ to a DG bicategory to get a graded \mathbb{k} -linear 1-category as per Definition 5.11. We then flatten this 1-category into an algebra as follows:

Definition 6.1. Let \mathcal{C} be a \mathbb{k} -linear 1-category. Define the unital \mathbb{k} -algebra $\mathcal{A}\text{lg}(\mathcal{C})$ to be

$$\mathcal{A}\text{lg}(\mathcal{C}) := \left\{ (f_{ab}) \in \prod_{a,b \in \mathcal{C}} \text{Hom}_{\mathcal{C}}(a, b) \mid \forall a \in \mathcal{C} \text{ we have } f_{ab} \neq 0 \text{ for only finite number of } b \in \mathcal{C} \right\}$$

with the addition given by the \mathbb{k} -linearity of \mathcal{C} and the multiplication by the composition in \mathcal{C} . Note that this algebra is unital with the identity element $1 = (\text{id}_{aa})_{a \in \mathcal{C}}$.

The analogue of [42, Cor. 8.8] is now automatic. The 2-functor $\Phi_V: \mathbf{H}_V \rightarrow \mathbf{F}_V$ induces a 1-functor

$$\text{HH}_\bullet(\mathbf{H}_V) \rightarrow \text{HH}_\bullet(\mathbf{F}_V).$$

By definition, \mathbf{F}_V is the 1-full subcategory of $\mathbf{EnhCat}_{kc}^{dg}$ supported at $\mathcal{S}^N \mathcal{V}$ for $N \geq 0$. We can therefore compose the above with the 1-functor (5.21) to get a 1-functor

$$\text{HH}_\bullet(\mathbf{H}_V) \rightarrow \text{gr-Vect}_{\mathbb{k}}$$

whose image lies in the full subcategory of $\text{gr-Vect}_{\mathbb{k}}$ supported at $\text{HH}_\bullet(\mathcal{S}^N \mathcal{V})$ for $N \geq 0$. Hence, applying the flattening $\mathcal{A}\text{lg}$ of Definition 6.1 on this 1-functor yields an algebra homomorphism

$$\mathcal{A}\text{lg}(\text{HH}_\bullet(\mathbf{H}_V)) \rightarrow \text{End} \left(\bigoplus_N \text{HH}_\bullet(\mathcal{S}^N \mathcal{V}) \right).$$

For brevity we write HH_{alg} for the composition of $\mathcal{A}\text{lg}$ and HH_\bullet , so the above can be rewritten as

$$(6.1) \quad \text{HH}_{alg}(\Phi_V): \text{HH}_{alg}(\mathbf{H}_V) \rightarrow \text{End} \left(\bigoplus_N \text{HH}_\bullet(\mathcal{S}^N \mathcal{V}) \right).$$

The main goal of this section is to construct an injective decategorification map

$$(6.2) \quad \underline{H}_{\text{HH}_\bullet(\mathcal{V})} \rightarrow \text{HH}_{alg}(\mathbf{H}_V),$$

where $\underline{H}_{\text{HH}_\bullet(\mathcal{V})}$ is the Heisenberg algebra of $\text{HH}_\bullet(\mathcal{V})$ with the Euler pairing described in §5.12.

6.2. Reduction to $A\text{HH}_{\mathbf{H}_V}$ and idempotent modification. Applying Defn. 6.1 to a 1-category \mathcal{C} whose set of objects is \mathbb{Z} we obtain a \mathbb{k} -algebra $\mathcal{A}\text{lg}(\mathcal{C})$ which has a natural \mathbb{Z} -grading:

$$\mathcal{A}\text{lg}(\mathcal{C}) = \bigoplus_{n \in \mathbb{Z}} \left(\prod_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{C}}(i, i+n) \right).$$

We get a functor $\mathcal{A}\text{lg}$ from the category whose objects are the 1-categories with the object set \mathbb{Z} and whose morphisms are the functors which are identity on objects to the category of \mathbb{Z} -graded \mathbb{k} -algebras. This admits a left adjoint:

Definition 6.2. Let A be a \mathbb{Z} -graded algebra. Define $\mathcal{C}\text{at}(A)$ to be the 1-category with:

- the set of objects \mathbb{Z} ,
- $\text{Hom}_{\mathcal{C}\text{at}(A)}(i, j) = (A)_{j-i}$,
- the composition given by multiplication in A ,
- the identity morphisms given by $1 \in A_0$.

The adjunction unit

$$(6.3) \quad A \rightarrow \mathcal{A}\text{lg}(\mathcal{C}\text{at}(A))$$

sends any $a \in A_n$ for $n \in \mathbb{Z}$ to $(\alpha_{ij}) \in \prod_{i,j \in \mathbb{Z}} A_{i-j}$ with $\alpha_{ij} = a$ if $j - i = n$ and $\alpha_{ij} = 0$ otherwise. The adjunction counit

$$(6.4) \quad \mathcal{C}\text{at}(\mathcal{A}\text{lg}(\mathcal{C})) \rightarrow \mathcal{C}$$

is the functor which on objects is the identity map and on morphisms is the map

$$\prod_{k \in \mathbb{Z}} \text{Hom}_C(k, k + j - i) \rightarrow \text{Hom}_{\mathcal{C}}(i, j)$$

which is the projection to the $k = i$ factor.

Definition 6.3. Let \mathcal{V} be any smooth and proper DG category. Define the \mathbb{Z} -graded \mathbb{k} -algebra

$$\text{AHH}_{\mathbf{H}_{\mathcal{V}}} := \bigoplus_{n \in \mathbb{Z}} \text{HH}_{\bullet}(\text{Hom}_{\mathbf{H}_{\mathcal{V}}}(0, n))$$

whose multiplication

$$\text{HH}_{\bullet}(\text{Hom}_{\mathbf{H}_{\mathcal{V}}}(0, n)) \otimes_{\mathbb{k}} \text{HH}_{\bullet}(\text{Hom}_{\mathbf{H}_{\mathcal{V}}}(0, m)) \rightarrow \text{HH}_{\bullet}(\text{Hom}_{\mathbf{H}_{\mathcal{V}}}(0, n + m))$$

is the corresponding composition map in $\text{HH}_{\bullet}(\mathbf{H}_{\mathcal{V}})$

$$\text{HH}_{\bullet}(\text{Hom}_{\mathbf{H}_{\mathcal{V}}}(0, n)) \otimes_{\mathbb{k}} \text{HH}_{\bullet}(\text{Hom}_{\mathbf{H}_{\mathcal{V}}}(n, n + m)) \rightarrow \text{HH}_{\bullet}(\text{Hom}_{\mathbf{H}_{\mathcal{V}}}(0, n + m))$$

and whose identity element is the Euler class of the identity 1-morphism $1_0 \in \text{Hom}_{\mathbf{H}_{\mathcal{V}}}(0, 0)$.

Lemma 6.4. $\text{HH}_{\bullet}(\mathcal{V})$ is isomorphic to $\mathcal{C}\text{at}(\text{AHH}_{\mathbf{H}_{\mathcal{V}}})$.

Proof. We observe that, by construction, we have

$$\text{Hom}_{\mathbf{H}_{\mathcal{V}}}(i, j) = \text{Hom}_{\mathbf{H}_{\mathcal{V}}}(0, j - i) \quad \forall i, j \in \mathbb{Z}$$

for 1-morphisms categories of $\mathbf{H}_{\mathcal{V}}$. □

Define a \mathbb{Z} -grading on $\underline{\text{HH}}_{\bullet}(\mathcal{V})$ by setting $\deg(a_{\alpha}(n)) = n$ for all $n \in \mathbb{Z} \setminus \{0\}$ and $\alpha \in \text{HH}_{\bullet}(\mathcal{V})$. By adjunction of $\mathcal{C}\text{at}$ and $\mathcal{A}\text{lg}$, constructing (6.2) is equivalent to constructing a 1-functor

$$(6.5) \quad \mathcal{C}\text{at}(\underline{\text{HH}}_{\bullet}(\mathcal{V})) \rightarrow \text{HH}_{\bullet}(\mathbf{H}_{\mathcal{V}}).$$

Thus to construct (6.5), and hence (6.2), it suffices to construct an algebra homomorphism

$$(6.6) \quad \pi: \underline{\text{HH}}_{\bullet}(\mathcal{V}) \rightarrow \text{AHH}_{\mathbf{H}_{\mathcal{V}}}.$$

which we also call the decategorification map. In the rest of this section, we proceed to construct π . Once constructed, the homomorphism (6.2) is obtained from π as the composition

$$(6.7) \quad \underline{\text{HH}}_{\bullet}(\mathcal{V}) \xrightarrow{\text{adj. unit}} \mathcal{A}\text{lg}(\mathcal{C}\text{at}(\underline{\text{HH}}_{\bullet}(\mathcal{V}))) \xrightarrow{\mathcal{A}\text{lg}(\mathcal{C}\text{at}(\pi))} \mathcal{A}\text{lg}(\mathcal{C}\text{at}(\text{AHH}_{\mathbf{H}_{\mathcal{V}}})) \cong \text{HH}_{alg}(\mathbf{H}_{\mathcal{V}}).$$

Thus (6.2) is injective if and only if π is injective. In §6.6 we show that π is injective, and the main conjecture of this paper is that π is also surjective.

On the other hand, (6.2) sends $a_{\alpha}(n) \in \underline{\text{HH}}_{\bullet}(\mathcal{V})$ to the element of $\prod_{i \in \mathbb{Z}} \text{HH}_{\bullet}(\text{Hom}_{\mathbf{H}_{\mathcal{V}}}(i, i + n))$ consisting of $\pi(a_{\alpha}(n))$ in every factor. This is clearly never surjective — in $\text{HH}_{alg}(\mathbf{H}_{\mathcal{V}})$ we have $1 = \sum_{n \in \mathbb{Z}} 1_n$ where 1_n are the orthogonal idempotents given by the Euler classes of the identity 1-morphisms $1_n \in \text{Hom}_{\mathbf{H}_{\mathcal{V}}}(0, n)$. These idempotents never lie in the image of (6.2).

This led us in [42] to consider the idempotent-modified non-unital version $H_{K_0^{\text{num}}(\mathcal{V})}$ of the Heisenberg algebra $\underline{\text{H}}_{K_0^{\text{num}}(\mathcal{V})}$. In present terms, the idempotent modified $H_{\text{HH}_{\bullet}(\mathcal{V})}$ is the subalgebra

$$H_{\text{HH}_{\bullet}(\mathcal{V})} := \bigoplus_{i,j \in \mathbb{Z}} (\underline{\text{HH}}_{\bullet}(\mathcal{V}))_{j-i} \subset \mathcal{A}\text{lg}(\mathcal{C}\text{at}(\underline{\text{HH}}_{\bullet}(\mathcal{V}))).$$

It is a flattening of the 1-category $\text{Cat}(\underline{H}_{\text{HH}_\bullet}(\mathcal{V}))$ different from Alg : we take a direct sum of the Hom-spaces instead of the direct product with a finiteness condition. The resulting algebra is not unital as the identity element would have to be an infinite sum of the identity elements of $\text{Hom}_{\text{Cat}(\underline{H}_{\text{HH}_\bullet}(\mathcal{V}))}(n, n)$ for all $n \in \mathbb{Z}$ and this lies outside the direct sum flattening $H_{\text{HH}_\bullet}(\mathcal{V})$. More generally, $\underline{H}_{\text{HH}_\bullet}(\mathcal{V})$ doesn't naturally contain the original algebra $H_{\text{HH}_\bullet}(\mathcal{V})$.

Enlarging the direct sum flattening to the Alg -flattening solves this. The adjunction unit embeds $\underline{H}_{\text{HH}_\bullet}(\mathcal{V})$ as a subalgebra into $\text{Alg}(\text{Cat}(\underline{H}_{\text{HH}_\bullet}(\mathcal{V})))$. By above, $\text{Alg}(\text{Cat}(\underline{H}_{\text{HH}_\bullet}(\mathcal{V})))$ contains $H_{\text{HH}_\bullet}(\mathcal{V})$. Constructing the decategorification map π as in (6.6), and thus the Alg -flattening homomorphism

$$(6.8) \quad \text{Alg}(\text{Cat}(\underline{H}_{\text{HH}_\bullet}(\mathcal{V}))) \xrightarrow{\text{Alg}(\text{Cat}(\pi))} \text{HH}_{\text{alg}}(\mathbf{H}_\mathcal{V}),$$

we construct homomorphisms from both $\underline{H}_{\text{HH}_\bullet}(\mathcal{V})$ and its idempotent-modified version $H_{\text{HH}_\bullet}(\mathcal{V})$ into $\text{HH}_{\text{alg}}(\mathbf{H}_\mathcal{V})$. Neither would ever be surjective, however (6.8) is injective if and only π is.

6.3. Construction of the map π .

$$\pi: \underline{H}_{\text{HH}_\bullet}(\mathcal{V}) \rightarrow \text{AHH}_{\mathbf{H}_\mathcal{V}}.$$

must agree with the numerical Grothendieck group decategorification map π [42, §6.1-6.4]. When it is necessary to differentiate between the two, we write π_{HH} for the former and π_K for the latter. The two maps must agree as follows: for any $a \in K_0^{\text{num}}(\mathcal{V})$ and $n \geq 1$ the Euler classes of the images of $p_a^{(n)}$ and $q_a^{(n)}$ under π_K must coincide with the images of $p_{\text{eu}(a)}^{(n)}$ and $q_{\text{eu}(a)}^{(n)}$ under π_{HH} :

$$(6.9) \quad \begin{array}{ccc} K_0^{\text{num}}(\mathcal{V}) & \xrightarrow{\begin{matrix} p_\bullet^{(n)} \\ q_\bullet^{(n)} \end{matrix}} & \underline{H}_{K_0^{\text{num}}(\mathcal{V})} \xrightarrow{\pi_K} \bigoplus_{n \in \mathbb{Z}} K_0^{\text{num}}(\text{Hom}_{\mathbf{H}_\mathcal{V}}(0, n)) \\ \downarrow \text{eu} & & \downarrow \text{eu} \\ \text{HH}_\bullet(\mathcal{V}) & \xrightarrow{\begin{matrix} p_\bullet^{(n)} \\ q_\bullet^{(n)} \end{matrix}} & \underline{H}_{\text{HH}_\bullet}(\mathcal{V}) \xrightarrow{\pi_{\text{HH}}} \bigoplus_{n \in \mathbb{Z}} \text{HH}_\bullet(\text{Hom}_{\mathbf{H}_\mathcal{V}}(0, n)). \end{array}$$

Recall the construction of the map π_K from [42, §6.1-6.4]. For each $n \geq 1$ and each $a \in \mathcal{V}$, we assign to a a class $\psi_n(a) \in K_0^{\text{num}}(\mathcal{H}\text{perf}(\mathcal{S}^n \mathcal{V}))$ which is the class of the twisted complex which homotopy splits the symmetrising idempotent

$$e := \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \in \text{Hom}_{\mathcal{S}^n \mathcal{V}}(a^{\otimes n}, a^{\otimes n}).$$

We then use the homomorphisms defined by the 2-functors $\Xi^P, \Xi^Q: \mathbf{H}\text{perf}(\mathbf{Sym}_\mathcal{V}) \rightarrow \mathbf{H}_\mathcal{V}$ constructed in [42, §6.1-6.4] to obtain the classes $P_a^{(n)}$ and $Q_a^{(n)}$:

$$(6.10) \quad \begin{array}{ccc} a \in K_0^{\text{num}}(\mathcal{V}) & \xrightarrow{\psi_n} & K_0^{\text{num}}(\mathcal{S}^n \mathcal{V}) \xrightarrow{\Xi^P} K_0^{\text{num}}(\text{Hom}_{\mathbf{H}_\mathcal{V}}(0, n)) \ni P_a^{(n)} \\ \downarrow \sim & & \downarrow \\ K_0^{\text{num}}(\mathcal{V}^{\text{opp}}) & \xrightarrow{\psi_n} & K_0^{\text{num}}(\mathcal{S}^n \mathcal{V}^{\text{opp}}) \xrightarrow{\Xi^Q} K_0^{\text{num}}(\text{Hom}_{\mathbf{H}_\mathcal{V}}(0, -n)) \ni Q_a^{(n)}. \end{array}$$

We choose objects $\{a_1, \dots, a_n\} \subset \mathcal{V}$ whose classes give a basis of $K_0^{\text{num}}(\mathcal{V})$. We can do this as we replaced \mathcal{V} by $\mathcal{H}\text{perf } \mathcal{V}$ at the outset. By Theorem 3.1, $\underline{H}_{K_0^{\text{num}}(\mathcal{V})}$ is generated by $p_{a_i}^{(n)}$ and $q_{a_i}^{(n)}$ subject to relations (2.23), (2.24). In [42, Theorem 6.3] we verify that these hold for $P_a^{(n)}$ and $Q_a^{(n)}$ and thus (6.10) extends to an algebra homomorphism $\pi_K: \underline{H}_{K_0^{\text{num}}(\mathcal{V})} \rightarrow \bigoplus_{n \in \mathbb{Z}} K_0^{\text{num}}(\text{Hom}_{\mathbf{H}_\mathcal{V}}(0, n))$.

There is no hope of extending ψ_n in (6.10) from a set-theoretic assignment defined on the classes of objects of \mathcal{V} to an additive map: in the Heisenberg algebra $\underline{H}_{K_0^{\text{num}}(\mathcal{V})}$ the parametrisation of the generators $p_v^{(n)}$ and $q_v^{(n)}$ is not additive in $v \in K_0^{\text{num}}(\mathcal{V})$. The parametrisation is additive for $a_\bullet(n)$ and $a_\bullet(-n)$ generators, but for them the definition of ψ_n would be very complicated.

This is another aspect where working with the Hochschild homology offers a simplification: working with $a_\bullet(n)$ and $a_\bullet(-n)$ generators is easy enough. To construct the decategorification map π , we construct below in §6.4 for any $n \geq 1$ \mathbb{k} -linear maps

$$(6.11) \quad \psi_n: \text{HH}_\bullet(\mathcal{V}) \rightarrow \text{HH}_\bullet(\mathcal{S}^n \mathcal{V}) \quad \text{and} \quad \psi_n: \text{HH}_\bullet(\mathcal{V}^{\text{opp}}) \rightarrow \text{HH}_\bullet(\mathcal{S}^n \mathcal{V}^{\text{opp}}),$$

We then define the following assignment:

Definition 6.5. For any $\alpha \in \mathrm{HH}_\bullet(\mathcal{V})$ and $n \geq 1$, we use the compositions

$$(6.12) \quad \begin{array}{ccc} \mathrm{HH}_\bullet(\mathcal{V}) & \xrightarrow{\psi_n} & \mathrm{HH}_\bullet(\mathcal{S}^n\mathcal{V}) \xrightarrow{\Xi^P} \mathrm{HH}_\bullet(\mathrm{Hom}_{\mathbf{H}_\mathcal{V}}(0, n)). \\ \sim \downarrow^{(5.11)} & & \\ \mathrm{HH}_\bullet(\mathcal{V}^\mathrm{opp}) & \xrightarrow{\psi_n} & \mathrm{HH}_\bullet(\mathcal{S}^n\mathcal{V}^\mathrm{opp}) \xrightarrow{\Xi^Q} \mathrm{HH}_\bullet(\mathrm{Hom}_{\mathbf{H}_\mathcal{V}}(0, -n)) \end{array}$$

and the isomorphism (5.11) to define

$$\begin{aligned} \mathsf{A}_\alpha(n) &:= \Xi^P(\psi_n(\alpha)) \in \mathrm{HH}_\bullet(\mathrm{Hom}_{\mathbf{H}_\mathcal{V}}(0, n)), \\ \mathsf{A}_\alpha(-n) &:= \Xi^Q(\psi_n(\alpha)) \in \mathrm{HH}_\bullet(\mathrm{Hom}_{\mathbf{H}_\mathcal{V}}(0, -n)), \end{aligned}$$

We then have:

Theorem 6.6. *There exists the unique algebra homomorphism*

$$\pi_{\mathrm{HH}}: \underline{H}_{\mathrm{HH}_\bullet(\mathcal{V})} \rightarrow \mathrm{AH}\mathrm{H}_{\mathbf{H}_\mathcal{V}}$$

which sends $a_\alpha(n)$ and $a_\alpha(-n)$ to $\mathsf{A}_\alpha(n)$ and $\mathsf{A}_\alpha(-n)$ for any $\alpha \in \mathrm{HH}_\bullet(\mathcal{V})$ and $n \geq 1$.

As before, we write simply π for π_{HH} where no confusion is possible.

Proof. By Defn. 3.20, $\underline{H}_{\mathrm{HH}_\bullet(\mathcal{V})}$ is the unital \mathbb{k} -algebra generated by the elements $a_\alpha(n)$ and $a_\alpha(-n)$ subject to the linearity relations (3.25), (3.26), the commutation relation (3.24), and the Heisenberg relation (3.27). For the elements, $\mathsf{A}_\alpha(n)$ and $\mathsf{A}_\alpha(-n)$ the linearity relations hold by the linearity of (6.12) and in §6.5 we prove that the commutation and the Heisenberg relations hold as well. \square

6.4. Linear maps ψ_n .

The \mathbb{k} -linear maps

$$\begin{aligned} \psi_n: \mathrm{HH}_\bullet(\mathcal{V}) &\rightarrow \mathrm{HH}_\bullet(\mathcal{S}^n\mathcal{V}) \\ \psi_n: \mathrm{HH}_\bullet(\mathcal{V}^\mathrm{opp}) &\rightarrow \mathrm{HH}_\bullet(\mathcal{S}^n\mathcal{V}^\mathrm{opp}) \end{aligned}$$

must produce the map π_{HH} which makes (6.9) commute. The maps Ξ^P and Ξ^Q in (6.10) and (6.12) commute with taking the Euler class, so for any $a \in \mathcal{V}$, the class of the symmetrising idempotent e_n of $a^{\otimes n}$ in $\mathrm{HH}_0(\mathcal{S}^n\mathcal{V})$ must be related in the algebra $\bigoplus_{m \geq 0} \mathrm{HH}_\bullet(\mathcal{S}^m\mathcal{V})$ to $\psi_m(\mathrm{id}_a)$ for $m \leq n$ via the same combinatorial formulas which express $p_a^{(n)}$ in terms of $a_a(m)$:

$$(6.13) \quad [e_n] = \frac{1}{n!} \sum_{\underline{n} \vdash n} |\underline{n}| \psi_{n_1}(\mathrm{id}_a) \dots \psi_{n_r(\underline{n})}(\mathrm{id}_a),$$

where $|\underline{n}|$ is the size of the conjugacy class \underline{n} .

In $\mathrm{HH}_0(\mathcal{S}^n\mathcal{V})$, the class $[e_n]$ is the sum of the classes $\frac{1}{n!}[\sigma]$ for each $\sigma \in S_n$. Since HH_0 is commutative, the classes of conjugate $\sigma \in S_n$ are equal and thus

$$(6.14) \quad [e_n] = \frac{1}{n!} \sum_{\underline{n} \vdash n} |\underline{n}| [\sigma_{\underline{n}}],$$

where we choose a representative $\sigma_{\underline{n}}$ of every conjugacy class $\underline{n} \in n$. The first step (5.37) of the Baranovsky decomposition implies that $[\sigma_{\underline{n}}]$ lies in the summand of the decomposition indexed by \underline{n} . The algebra structure on $\bigoplus_{m \geq 0} \mathrm{HH}_\bullet(\mathcal{S}^m\mathcal{V})$ is induced by the functors $\mathcal{S}^{m_1}\mathcal{V} \otimes \mathcal{S}^{m_2}\mathcal{V} \rightarrow \mathcal{S}^{m_1+m_2}\mathcal{V}$ which are in turn induced by the inclusion $S_{m_1} \times S_{m_2} \hookrightarrow S_{m_1+m_2}$. For the Baranovsky decomposition, this means that the product of summands indexed by $\underline{m}_1 \vdash m_1$ and $\underline{m}_2 \vdash m_2$ lies in the summand indexed by $\underline{m}_1 \cup \underline{m}_2 \vdash m_1 + m_2$. Comparing (6.13) with (6.14) and keeping the algebra structure in mind, we conclude that we must have $\psi_n([\mathrm{id}_a]) = [t]$, where $t := (1 \dots n)$ is the long cycle representing the conjugacy class (n) , the partition with 1 part of size n .

In particular, $\psi_a(\mathrm{id}_v)$ lies in the summand of the Baranovsky decomposition of $\mathrm{HH}_\bullet(\mathcal{S}^n\mathcal{V})$ indexed by (n) . This summand is just $\mathrm{HH}_\bullet(\mathcal{V})$. This motivates the following definition:

Definition 6.7. Define the \mathbb{k} -linear maps

$$\begin{aligned} \psi_n: \mathrm{HH}_\bullet(\mathcal{V}) &\rightarrow \mathrm{HH}_\bullet(\mathcal{S}^n\mathcal{V}) \\ \psi_n: \mathrm{HH}_\bullet(\mathcal{V}^\mathrm{opp}) &\rightarrow \mathrm{HH}_\bullet(\mathcal{S}^n\mathcal{V}^\mathrm{opp}) \end{aligned}$$

to be the inclusions of the summand indexed by (n) in the Baranovsky decomposition (5.36).

Explicitly, ψ_n is given for \mathcal{V} by the composition of the following maps:

$$(6.15) \quad \mathrm{HC}_\bullet(\mathcal{V}) \xrightarrow{(5.44)} \mathrm{HC}_\bullet(\mathcal{V}^n; t) \xrightarrow{\xi_t} \mathrm{HC}_\bullet(\mathcal{S}^n \mathcal{V}),$$

and similarly for \mathcal{V}^{opp} . In particular, on HC_0 it is the map $\mathrm{HC}_0(\mathcal{V}) \rightarrow \mathrm{HC}_0(\mathcal{S}^n \mathcal{V})$ given by

$$\alpha \mapsto (\mathrm{id} \otimes \otimes \cdots \otimes \mathrm{id} \otimes \alpha, t^{-1}) \quad \forall a \in \mathcal{V}, \alpha \in \mathrm{Hom}_{\mathcal{V}}(a, a).$$

Thus we have, as desired

$$\psi_n([\mathrm{id}_a]) = [t^{-1}] = [t].$$

Note that for any $\sigma \in S_n$, we have $[\sigma] = [\sigma^{-1}]$ in $\mathrm{HH}_0(\mathcal{S}^n \mathcal{V})$. This is because any HH_0 is a commutative algebra, and if we decompose σ as a product of transpositions, then σ^{-1} is the product of the same transpositions in the reverse order.

6.5. Commutation and Heisenberg relations. We now prove the commutation relations (3.24) and the Heisenberg relation (3.27) for the elements $\mathsf{A}_\alpha(n)$ and $\mathsf{A}_\alpha(-n)$ defined in Definition 6.12.

The commutation relations (3.24) are easy to establish because they only involve the images of the elements $\psi_n(\alpha)$ under one of the functors Ξ^P and Ξ^Q . Since these functors are monoidal, it is enough to verify the corresponding relations for the elements $\psi_n(\alpha)$ in $\bigoplus_{n \geq 0} \mathrm{HH}_\bullet(\mathcal{S}^n \mathcal{V})$:

Proposition 6.8. *For any $n, m \geq 1$ and $\alpha, \beta \in \mathrm{HH}_\bullet(\mathcal{V})$ the following relations hold in $\mathrm{AHH}_{\mathbf{H}_\mathcal{V}}$:*

$$(6.16) \quad \mathsf{A}_\alpha(n) \mathsf{A}_\beta(m) = (-1)^{\deg(\alpha) \deg(\beta)} \mathsf{A}_\beta(m) \mathsf{A}_\alpha(n),$$

$$(6.17) \quad \mathsf{A}_\alpha(-n) \mathsf{A}_\beta(-m) = (-1)^{\deg(\alpha) \deg(\beta)} \mathsf{A}_\beta(-m) \mathsf{A}_\alpha(-n).$$

Proof. By the definition of $\mathsf{A}_\alpha(\pm n)$ and $\mathsf{A}_\beta(\pm m)$ we need to establish that:

$$\Xi^P(\psi_n(\alpha)) \Xi^P(\psi_m(\beta)) = (-1)^{\deg(\alpha) \deg(\beta)} \Xi^P(\psi_m(\beta)) \Xi^P(\psi_n(\alpha)),$$

$$\Xi^Q(\psi_n(\alpha)) \Xi^Q(\psi_m(\beta)) = (-1)^{\deg(\alpha) \deg(\beta)} \Xi^Q(\psi_m(\beta)) \Xi^Q(\psi_n(\alpha)),$$

where again we use implicitly the isomorphism $\mathrm{HC}_\bullet(\mathcal{V}) \cong \mathrm{HC}_\bullet(\mathcal{V}^{\text{opp}})$ established in §5.4

The maps Ξ^P and Ξ^Q in (6.12) are induced by the DG 2-functors

$$\Xi^P: \mathbf{Hperf}(\mathbf{Sym}_{\mathcal{V}}) \rightarrow \mathbf{H}_\mathcal{V} \quad \text{and} \quad \Xi^Q: \mathbf{Hperf}(\mathbf{Sym}_{\mathcal{V}^{\text{opp}}}) \rightarrow \mathbf{H}_\mathcal{V}$$

constructed in [42, §6.1]. They thus package up into algebra homomorphisms

$$\Xi^P: \bigoplus_{n \geq 0} \mathrm{HH}_\bullet(\mathcal{S}^n \mathcal{V}) \longrightarrow \mathrm{AHH}_{\mathbf{H}_\mathcal{V}} \quad \text{and} \quad \Xi^P: \bigoplus_{n \geq 0} \mathrm{HH}_\bullet(\mathcal{S}^n \mathcal{V}^{\text{opp}}) \longrightarrow \mathrm{AHH}_{\mathbf{H}_\mathcal{V}}.$$

It suffices therefore to establish in $\bigoplus_{n \geq 0} \mathrm{HH}_\bullet(\mathcal{S}^n \mathcal{V})$ and $\bigoplus_{n \geq 0} \mathrm{HH}_\bullet(\mathcal{S}^n \mathcal{V}^{\text{opp}})$ the relationship

$$(6.18) \quad \psi_n(\alpha) \psi_m(\beta) = (-1)^{\deg(\alpha) \deg(\beta)} \psi_m(\beta) \psi_n(\alpha),$$

This is trivial, as ψ_\bullet are degree preserving and the algebra $\bigoplus_{n \geq 0} \mathrm{HH}_\bullet(\mathcal{S}^n \mathcal{V})$ is super-commutative for any small DG category \mathcal{V} . For super-commutativity, recall that the algebra structure

$$\mathrm{HH}_\bullet(\mathcal{S}^n \mathcal{V}) \otimes \mathrm{HH}_\bullet(\mathcal{S}^m \mathcal{V}) \rightarrow \mathrm{HH}_\bullet(\mathcal{S}^{n+m} \mathcal{V})$$

is induced by the functor $\mathcal{S}^n \mathcal{V} \otimes \mathcal{S}^m \mathcal{V} \rightarrow \mathcal{S}^{n+m} \mathcal{V}$ induced by the subgroup inclusion $S_n \times S_m \subset S_{n+m}$.

Let $\tau \in S_{n+m}$ be the permutation

$$\tau(i) := \begin{cases} m+i, & 1 \leq i \leq n, \\ i-n, & n+1 \leq i \leq n+m, \end{cases}$$

which swaps the first n and the last m elements. For any

$$a_1 \otimes \cdots \otimes a_n \otimes b_1 \otimes \cdots \otimes b_m \in \mathcal{S}^{n+m} \mathcal{V}$$

we have in $\mathcal{S}^{n+m} \mathcal{V}$ an isomorphism

$$a_1 \otimes \cdots \otimes a_n \otimes b_1 \otimes \cdots \otimes b_m \xrightarrow{(\mathrm{id}, \tau)} b_1 \otimes \cdots \otimes b_m \otimes a_1 \otimes \cdots \otimes a_n.$$

These give an endofunctor $T: \mathcal{S}^{n+m} \mathcal{V} \rightarrow \mathcal{S}^{n+m} \mathcal{V}$ naturally isomorphic to $\mathrm{id}_{\mathcal{S}^{n+m} \mathcal{V}}$. The induced map

$$T: \mathrm{HH}_\bullet(\mathcal{S}^{n+m} \mathcal{V}) \rightarrow \mathrm{HH}_\bullet(\mathcal{S}^{n+m} \mathcal{V})$$

for any $\eta \in \mathrm{HH}_\bullet(\mathcal{S}^n \mathcal{V})$ and $\zeta \in \mathrm{HH}_\bullet(\mathcal{S}^m \mathcal{V})$ sends $\eta \zeta \mapsto (-1)^{\deg(\eta)} (-1)^{\deg(\zeta)} \zeta \eta$. But as the functor T is isomorphic to $\mathrm{id}_{\mathcal{S}^{n+m} \mathcal{V}}$ the induced map must be the identity map. \square

The Heisenberg relation requires more work to establish:

Theorem 6.9. *For any $n, m \geq 1$ and $\alpha, \beta \in \text{HH}_\bullet(\mathcal{V})$ the following relation hold in $\text{AHH}_{\mathbf{H}_\mathcal{V}}$:*

$$(6.19) \quad A_\alpha(-n)A_\beta(m) = (-1)^{\deg(\alpha)\deg(\beta)}A_\beta(m)A_\alpha(-n) + \delta_{n,m}m \langle \alpha, \beta \rangle.$$

Since this relation involves the images of the elements $\psi_n(\alpha)$ and $\psi_m(\beta)$ under both the functors Ξ^P and Ξ^Q , it can not come from some relation already existing in $\text{HH}_\bullet(\mathcal{S}^{n+m}\mathcal{V})$. Ideally, we would have wanted to prove it by constructing two homotopy equivalent functors

$$\mathcal{V}^{\text{opp}} \otimes \mathcal{V} \rightarrow \text{Hom}_{\mathbf{H}_\mathcal{V}}(0, m - n)$$

whose induced maps on Hochschild homology send $\alpha \otimes \beta \in \text{HH}_\bullet(\mathcal{V}^{\text{opp}}) \otimes \text{HH}_\bullet(\mathcal{V})$ to the LHS and the RHS of (6.19). The Heisenberg relation would then follow since homotopy equivalent functors induce the same map on Hochschild homology [23, Lemma 3.4]. But this is impossible: our construction of π , and hence of classes $A_\alpha(-n)$ and $A_\beta(m)$, is not functorial. It involves the \mathbb{k} -linear maps $\psi_n : \text{HH}_\bullet(\mathcal{V}) \rightarrow \text{HH}_\bullet(\mathcal{S}^n\mathcal{V})$ which do not come from any functors $\mathcal{V} \rightarrow \mathcal{S}^n\mathcal{V}$ when $n > 1$.

However, once we have the classes $\psi_n(\alpha)$ the rest of π is functorial. We get $A_\alpha(-n)$ and $A_\alpha(n)$ by applying to $\psi_n(\alpha)$ the maps induced on the Hochschild homology by the functors Ξ_Q and Ξ_P , respectively. To prove the Heisenberg relation we construct two homotopy equivalent functors

$$(6.20) \quad \mathcal{S}^n\mathcal{V}^{\text{opp}} \otimes \mathcal{S}^m\mathcal{V} \rightarrow \text{Hom}_{\mathbf{H}_\mathcal{V}}(0, m - n)$$

whose induced maps send $\psi_n(\alpha) \otimes \psi_m(\beta)$ for any $\alpha, \beta \in \text{HC}_\bullet(\mathcal{V})$ to the LHS and the RHS of (6.19). Here and below we implicitly use the isomorphism $\text{HC}_\bullet(\mathcal{V}) \cong \text{HC}_\bullet(\mathcal{V}^{\text{opp}})$ defined in Section 5.4.

The first functor, Ξ_{QP} , is straightforward: it is a 1-composition of 2-functors Ξ_Q and Ξ_P . To be precise, note that our 2-category $\mathbf{H}_\mathcal{V}$ has object set \mathbb{Z} , and by construction

$$\text{Hom}_{\mathbf{H}_\mathcal{V}}(n, m) = \text{Hom}_{\mathbf{H}_\mathcal{V}}(0, m - n) \quad \forall n, m \in \mathbb{Z}.$$

It has a natural structure of monoidal 2-category given on the object set \mathbb{Z} by addition $n \otimes n' = n + n'$ and on 1-morphism categories by the 1-composition functor

$$\begin{aligned} & \text{Hom}_{\mathbf{H}_\mathcal{V}}(n, m) \otimes \text{Hom}_{\mathbf{H}_\mathcal{V}}(n', m') \\ & \quad \parallel \\ & \text{Hom}_{\mathbf{H}_\mathcal{V}}(n + m', m + m') \otimes \text{Hom}_{\mathbf{H}_\mathcal{V}}(n' + n, m' + n) \\ & \quad \downarrow \text{1-comp} \\ & \text{Hom}_{\mathbf{H}_\mathcal{V}}(n + n', m + m'). \end{aligned}$$

We therefore have a 2-functor

$$(6.21) \quad \Xi_{QP} : \text{Sym}_{\mathcal{V}^{\text{opp}}} \otimes \text{Sym}_\mathcal{V} \xrightarrow{\Xi_Q \otimes \Xi_P} \mathbf{H}_\mathcal{V} \otimes \mathbf{H}_\mathcal{V} \rightarrow \mathbf{H}_\mathcal{V}.$$

Definition 6.10. For any $n, m \geq 0$ define the DG functor

$$(6.22) \quad \Xi_{QP} : \mathcal{S}^n\mathcal{V}^{\text{opp}} \otimes \mathcal{S}^m\mathcal{V} \rightarrow \text{Hom}_{\mathbf{H}_\mathcal{V}}(0, m - n)$$

to be the action of the 2-functor (6.21) on the 1-morphism category $\text{Hom}_{\mathcal{S}^n\mathcal{V}^{\text{opp}} \otimes \mathcal{S}^m\mathcal{V}}(0 \otimes 0, n \otimes m)$.

Example 6.11. Let $(a_n \otimes \cdots \otimes a_1) \otimes (b_1 \otimes \cdots \otimes b_m) \in \mathcal{S}^n\mathcal{V}^{\text{opp}} \otimes \mathcal{S}^m\mathcal{V}$. The functor Ξ_{QP} takes it to the 1-composition of $\Xi_Q(a_n \otimes \cdots \otimes a_1)$ and $\Xi_P(b_1 \otimes \cdots \otimes b_m)$. That is

$$\Xi_{QP}((a_n \otimes \cdots \otimes a_1) \otimes (b_1 \otimes \cdots \otimes b_m)) = Q_{a_n} \dots Q_{a_1} P_{b_1} \dots P_{b_m} \in \text{Hom}_{\mathbf{H}_\mathcal{V}}(0, m - n).$$

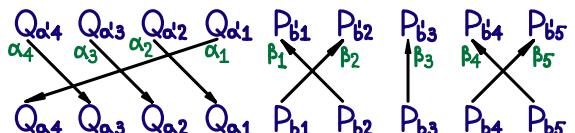
Similarly, given any morphisms

$$\begin{aligned} \alpha : a_n \otimes \cdots \otimes a_1 &\rightarrow a'_n \otimes \cdots \otimes a'_1 \\ \beta : b_1 \otimes \cdots \otimes b_m &\rightarrow b'_1 \otimes \cdots \otimes b'_m \end{aligned}$$

in $\mathcal{S}^n\mathcal{V}^{\text{opp}}$ and $\mathcal{S}^m\mathcal{V}$ we have

$$\Xi_{QP}(\alpha \otimes \beta) = \Xi_Q(\alpha) \circ_1 \Xi_P(\beta).$$

Let $\sigma = (1234) \in S_4$, $\tau = (12)(45) \in S_5$. Let $a_i, a'_i, b_j, b'_j \in \mathcal{V}$, $\alpha_i \in \text{Hom}_\mathcal{V}(a'_i, a_{\sigma^{-1}(i)})$ and $\beta_i \in \text{Hom}_\mathcal{V}(b_{\tau^{-1}(i)}, b'_i)$. Then $\Xi_{QP}((\alpha_4 \otimes \alpha_3 \otimes \alpha_2 \otimes \alpha_1, \sigma) \otimes (\beta_1 \otimes \beta_2 \otimes \beta_3 \otimes \beta_4 \otimes \beta_5, \tau))$ is



Before defining the second functor (6.20) we give some intuition for its construction. It corresponds to the following iterative procedure:

Definition 6.12 (Commutation-annihilation procedure). Let $(a_n \otimes \cdots \otimes a_1) \otimes (b_1 \otimes \cdots \otimes b_m) \in \mathcal{S}^n \mathcal{V}^{\text{opp}} \otimes \mathcal{S}^m \mathcal{V}$ and take the 1-morphism the 2-functor $\Xi_{\mathbf{Q}\mathbf{P}}$ sends it to:

$$(6.23) \quad Q_{a_n} \dots Q_{a_1} P_{b_1} \dots P_{b_m} \in \text{Hom}_{\mathbf{H}_{\mathcal{V}}}(0, m-n).$$

We now apply to it the following iterative procedure. Locate the rightmost Q which has P to its right. At this first step, it is $Q_{a_1} P_{b_1}$. Take this pair, and apply the homotopy equivalence

$$(6.24) \quad P_b Q_a \oplus (\text{Hom}(a, b) \otimes_{\mathbb{k}} 1) \xrightarrow{[\times, \psi_2]} Q_a P_b,$$

defined in [42, §5.4]. Here, it breaks (6.23) into two summands. In one Q_{a_1} is commuted past P_{b_1} , while in the other Q_{a_1} and P_{b_1} annihilate each other and the rest is tensored by $\text{Hom}_{\mathcal{V}}(a_1, b_1)$:

$$(6.25) \quad \begin{array}{c} Q_{a_n} \dots Q_{a_1} P_{b_1} \dots P_{b_m} \\ \uparrow \\ Q_{a_n} \dots Q_{a_2} P_{b_1} Q_{a_1} P_{b_2} \dots P_{b_m} \oplus Q_{a_n} \dots Q_{a_2} P_{b_2} \dots P_{b_m} \otimes \text{Hom}_{\mathcal{V}}(a_1, b_1) \end{array}$$

We now take each of the newly obtained summands and apply the same procedure to them. We locate the rightmost QP subword, if one exists, and apply to it the homotopy equivalence (6.24). This replaces the old summand by two new ones, where the QP pair in question is commuted and is annihilated, respectively. We repeat this until none of the summands contain a QP subword.

The result is a homotopy equivalence into (6.23) from a direct sum whose summands have form

$$(6.26) \quad P_{b_1} \dots \widehat{P_{b_{j_1}}} \dots \widehat{P_{b_{j_k}}} \dots P_{b_m} Q_{a_n} \dots \widehat{Q_{a_{i_1}}} \dots \widehat{Q_{a_{i_k}}} \dots Q_{a_1} \otimes \text{Hom}_{\mathcal{V}}(a_{i_1}, b_{j_{\sigma(1)}}) \otimes \cdots \otimes \text{Hom}_{\mathcal{V}}(a_{i_k}, b_{j_{\sigma(k)}}).$$

where a hat indicates that we skip this element and where $\sigma \in S_k$. We obtain each summand by starting at (6.23), taking each Q (from right to left) and starting to move it past all the P s. At each P we choose whether to commute the Q past it or to annihilate the Q against it and start on the next Q . In (6.26) the elements $Q_{a_{i_1}}, \dots, Q_{a_{i_k}}$ were annihilated against the elements $P_{b_{j_1}}, \dots, P_{b_{j_k}}$ in the order specified by σ . The remaining Q s were successfully commuted past all the P s.

Example 6.13. Let $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathcal{V}$. Write (a_i, b_j) for $\text{Hom}_{\mathcal{V}}(a_i, b_j)$. For

$$Q_{a_3} Q_{a_2} Q_{a_1} P_{b_1} P_{b_2} P_{b_3} \in \text{Hom}_{\mathbf{H}_{\mathcal{V}}}(0, 0)$$

the first few steps of the commutation-annihilation procedure described above are

$$\begin{aligned} & Q_{a_3} Q_{a_2} Q_{a_1} P_{b_1} P_{b_2} P_{b_3} \\ & \uparrow \\ & (Q_{a_3} Q_{a_2} P_{b_1} Q_{a_1} P_{b_2} P_{b_3}) \oplus (Q_{a_3} Q_{a_2} P_{b_2} P_{b_3} \otimes (a_1, b_1)) \\ & \uparrow \\ & (Q_{a_3} Q_{a_2} P_{b_1} P_{b_2} Q_{a_1} P_{b_3}) \oplus (Q_{a_3} Q_{a_2} P_{b_1} P_{b_3} \otimes (a_1, b_2)) \oplus (Q_{a_3} P_{b_2} Q_{a_2} P_{b_3} \otimes (a_1, b_1)) \oplus (Q_{a_3} P_{b_3} \otimes (a_1, b_1) \otimes (a_2, b_2)) \\ & \uparrow \\ & (Q_{a_3} Q_{a_2} P_{b_1} P_{b_2} P_{b_3} Q_{a_1}) \oplus (Q_{a_3} Q_{a_2} P_{b_1} P_{b_2} \otimes (a_1, b_3)) \oplus (Q_{a_3} P_{b_1} Q_{a_2} P_{b_3} \otimes (a_1, b_2)) \oplus (Q_{a_3} P_{b_2} \otimes (a_1, b_2) \otimes (a_2, b_1)) \oplus \\ & \oplus (Q_{a_3} P_{b_2} P_{b_3} Q_{a_2} \otimes (a_1, b_1)) \oplus (Q_{a_3} P_{b_2} \otimes (a_1, b_1) \otimes (a_2, b_3)) \oplus (P_{b_3} Q_{a_3} \otimes (a_1, b_1) \otimes (a_2, b_2)) \oplus ((a_1, b_1) \otimes (a_2, b_2) \otimes (a_3, b_3)) \\ & \uparrow \\ & (Q_{a_3} P_{b_1} Q_{a_2} P_{b_2} P_{b_3} Q_{a_1}) \oplus (Q_{a_3} P_{b_2} P_{b_3} Q_{a_1} \otimes (a_2, b_1)) \oplus (Q_{a_3} P_{b_1} Q_{a_2} P_{b_2} \otimes (a_1, b_3)) \oplus (Q_{a_3} P_{b_2} \otimes (a_1, b_3) \otimes (a_2, b_1)) \oplus \\ & \oplus (Q_{a_3} P_{b_1} P_{b_3} Q_{a_2} \otimes (a_1, b_2)) \oplus (Q_{a_3} P_{b_1} \otimes (a_1, b_2) \otimes (a_2, b_3)) \oplus (P_{b_3} Q_{a_3} \otimes (a_1, b_2) \otimes (a_2, b_1)) \oplus ((a_1, b_2) \otimes (a_2, b_1) \otimes (a_3, b_3)) \oplus \\ & \oplus (P_{b_2} Q_{a_3} P_{b_3} Q_{a_2} \otimes (a_1, b_1)) \oplus (P_{b_2} Q_{a_3} \otimes (a_1, b_1) \otimes (a_3, b_2)) \oplus (P_{b_2} Q_{a_3} \otimes (a_1, b_1) \otimes (a_2, b_3)) \oplus ((a_1, b_1) \otimes (a_2, b_3) \otimes (a_3, b_2)) \oplus \\ & \oplus (P_{b_3} Q_{a_3} \otimes (a_1, b_1) \otimes (a_2, b_2)) \oplus ((a_1, b_1) \otimes (a_2, b_2) \otimes (a_3, b_3)) \end{aligned}$$

Its result is the direct sum whose summands, grouped by the number of annihilations, are:

- (1) $P_{b_1} P_{b_2} P_{b_3} Q_{a_3} Q_{a_2} Q_{a_1}$,
- (2) $(P_{b_2} P_{b_3} Q_{a_3} Q_{a_2} \otimes (a_1, b_1)) \oplus (P_{b_1} P_{b_3} Q_{a_3} Q_{a_2} \otimes (a_1, b_2)) \oplus (P_{b_1} P_{b_2} Q_{a_3} Q_{a_2} \otimes (a_1, b_3)) \oplus$
 $\oplus (P_{b_2} P_{b_3} Q_{a_3} Q_{a_1} \otimes (a_2, b_1)) \oplus (P_{b_1} P_{b_3} Q_{a_3} Q_{a_1} \otimes (a_2, b_2)) \oplus (P_{b_1} P_{b_2} Q_{a_3} Q_{a_1} \otimes (a_2, b_3)) \oplus$
 $\oplus (P_{b_2} P_{b_3} Q_{a_2} Q_{a_1} \otimes (a_3, b_1)) \oplus (P_{b_1} P_{b_3} Q_{a_2} Q_{a_1} \otimes (a_3, b_2)) \oplus (P_{b_1} P_{b_2} Q_{a_2} Q_{a_1} \otimes (a_3, b_3)) \oplus$
- (3) $(P_{b_3} Q_{a_3} \otimes (a_1, b_1) \otimes (a_2, b_2)) \oplus (P_{b_2} Q_{a_3} \otimes (a_1, b_1) \otimes (a_2, b_3)) \oplus (P_{b_3} Q_{a_2} \otimes (a_1, b_1) \otimes (a_3, b_2)) \oplus$
 $\oplus (P_{b_2} Q_{a_2} \otimes (a_1, b_1) \otimes (a_3, b_3)) \oplus (P_{b_3} Q_{a_3} \otimes (a_1, b_2) \otimes (a_2, b_1)) \oplus (P_{b_1} Q_{a_3} \otimes (a_1, b_2) \otimes (a_2, b_3)) \oplus$
 $\oplus (P_{b_3} Q_{a_2} \otimes (a_1, b_2) \otimes (a_3, b_1)) \oplus (P_{b_1} Q_{a_2} \otimes (a_1, b_2) \otimes (a_3, b_3)) \oplus (P_{b_2} Q_{a_3} \otimes (a_1, b_3) \otimes (a_2, b_1)) \oplus$
 $\oplus (P_{b_1} Q_{a_3} \otimes (a_1, b_3) \otimes (a_2, b_2)) \oplus (P_{b_2} Q_{a_2} \otimes (a_1, b_3) \otimes (a_3, b_1)) \oplus (P_{b_1} Q_{a_2} \otimes (a_1, b_3) \otimes (a_3, b_2)) \oplus$

$$(4) \quad \begin{aligned} & \oplus (\mathsf{P}_{b_3} \mathsf{Q}_{a_1} \otimes (a_2, b_1) \otimes (a_3, b_2)) \oplus (\mathsf{P}_{b_2} \mathsf{Q}_{a_1} \otimes (a_2, b_1) \otimes (a_3, b_3)) \oplus (\mathsf{P}_{b_3} \mathsf{Q}_{a_1} \otimes (a_2, b_2) \otimes (a_3, b_1)) \oplus \\ & \oplus (\mathsf{P}_{b_1} \mathsf{Q}_{a_1} \otimes (a_2, b_2) \otimes (a_3, b_3)) \oplus (\mathsf{P}_{b_2} \mathsf{Q}_{a_1} \otimes (a_2, b_3) \otimes (a_3, b_1)) \oplus (\mathsf{P}_{b_1} \mathsf{Q}_{a_1} \otimes (a_2, b_3) \otimes (a_3, b_2)) \\ & \oplus ((a_1, b_1) \otimes (a_2, b_2) \otimes (a_3, b_3)) \oplus ((a_1, b_1) \otimes (a_2, b_3) \otimes (a_3, b_2)) \oplus ((a_1, b_2) \otimes (a_2, b_1) \otimes (a_3, b_3)) \oplus \\ & \oplus ((a_1, b_2) \otimes (a_2, b_3) \otimes (a_3, b_2)) \oplus ((a_1, b_3) \otimes (a_2, b_1) \otimes (a_3, b_2)) \oplus ((a_1, b_3) \otimes (a_2, b_2) \otimes (a_3, b_1)). \end{aligned}$$

We now make this procedure functorial. We begin with the annihilation:

Definition 6.14. Let $n, m > 0$ and let $\min(n, m) \geq k \geq 0$. Define the DG functor

$$(6.27) \quad (\hat{k}): \mathcal{S}^n \mathcal{V}^{\text{opp}} \otimes \mathcal{S}^m \mathcal{V} \longrightarrow \mathcal{H}\text{perf}(\mathcal{S}^{n-k} \mathcal{V}^{\text{opp}} \otimes \mathcal{S}^{m-k} \mathcal{V})$$

as the composition

$$\begin{aligned} \mathcal{S}^n \mathcal{V}^{\text{opp}} \otimes \mathcal{S}^m \mathcal{V} & \xrightarrow{\text{Res}_{S_{n-k} \times S_k \times S_{m-k}}^{S_n \times S_m}} \mathcal{H}\text{perf}(\mathcal{S}^{n-k} \mathcal{V}^{\text{opp}} \otimes (((\mathcal{V}^{\text{opp}})^{\otimes k} \otimes \mathcal{V}^{\otimes k}) \rtimes S_k) \otimes \mathcal{S}^{m-k} \mathcal{V}) \cong \\ & \cong \mathcal{H}\text{perf}(\mathcal{S}^{n-k} \mathcal{V}^{\text{opp}} \otimes \mathcal{S}^k (\mathcal{V}^{\text{opp}} \otimes \mathcal{V}) \otimes \mathcal{S}^{m-k} \mathcal{V}) \xrightarrow{\text{Hom}_{\mathcal{V}}(-, -)} \\ & \rightarrow \mathcal{H}\text{perf}(\mathcal{S}^{n-k} \mathcal{V}^{\text{opp}} \otimes \mathcal{S}^k (\mathcal{H}\text{perf} \mathbb{k}) \otimes \mathcal{S}^{m-k} \mathcal{V}) \xrightarrow{(-) \otimes \cdots \otimes (-)} \\ & \rightarrow \mathcal{H}\text{perf}(\mathcal{S}^{n-k} \mathcal{V}^{\text{opp}} \otimes \mathcal{H}\text{perf} \mathbb{k} \otimes \mathcal{S}^{m-k} \mathcal{V}) \cong \\ & \cong \mathcal{H}\text{perf}(\mathcal{S}^{n-k} \mathcal{V}^{\text{opp}} \otimes \mathcal{S}^{m-k} \mathcal{V}). \end{aligned}$$

The first composant is the restriction of scalars functors induced by the group embedding

$$S_{n-k} \times S_k \times S_{m-k} \hookrightarrow S_{n-k} \times S_k \times S_k \times S_{m-k} \hookrightarrow S_n \times S_m$$

where S_k embeds into $S_k \times S_k$ as $\sigma \mapsto (\iota \sigma \iota, \sigma)$ with ι the order reversing involution $1 \dots n \mapsto n \dots 1$.

The second composant is induced by the factor permuting isomorphism

$$(\mathcal{V}^{\text{opp}})^{\otimes k} \otimes \mathcal{V}^{\otimes k} \cong (\mathcal{V}^{\text{opp}} \otimes \mathcal{V})^{\otimes k}$$

which sends $a_k \otimes \cdots \otimes a_1 \otimes b_1 \otimes \cdots \otimes b_k$ to $a_1 \otimes b_1 \otimes \cdots \otimes a_k \otimes b_k$.

The third composant is induced by the functor

$$\text{Hom}_{\mathcal{V}}(-, -): \mathcal{V}^{\text{opp}} \otimes \mathcal{V} \rightarrow \mathcal{H}\text{perf} \mathbb{k}$$

which sends $a \otimes b \mapsto \text{Hom}_{\mathcal{V}}(a, b)$.

The fourth composant is induced by the functor

$$(-) \otimes \cdots \otimes (-): \mathcal{S}^k \mathcal{H}\text{perf} \mathbb{k} \rightarrow \mathcal{H}\text{perf} \mathbb{k}$$

which tensors k complexes of \mathbb{k} -modules together.

The last composant is the isomorphism given by the restriction of scalars along the Yoneda embedding $\mathbb{k} \rightarrow \mathcal{H}\text{perf} \mathbb{k}$.

Lemma 6.15. Let $n, m > 0$ and $\min(n, m) \geq k \geq 0$. Let $a_1, \dots, a_n \in \mathcal{V}$ and $b_1, \dots, b_m \in \mathcal{V}$. Then

$$(6.28) \quad \begin{aligned} & (\hat{k})((a_n \otimes \cdots \otimes a_1) \otimes (b_1 \otimes \cdots \otimes b_m)) \cong \\ & \bigoplus_{\substack{1 \leq i_1 < \cdots < i_k \leq n, \\ 1 \leq j_1 < \cdots < j_k \leq m, \\ \sigma \in S_k}} (a_n \otimes \cdots \widehat{a_{i_k}} \cdots \widehat{a_{i_1}} \cdots \otimes a_1) \otimes (b_1 \otimes \cdots \widehat{b_{j_k}} \cdots \widehat{b_{j_1}} \cdots \otimes b_m) \otimes \\ & \quad \otimes \text{Hom}_{\mathcal{V}}(a_{i_1}, b_{j_{\sigma(1)}}) \otimes \cdots \otimes \text{Hom}_{\mathcal{V}}(a_{i_k}, b_{j_{\sigma(k)}}). \end{aligned}$$

Furthermore, let $\alpha_i \in \text{Hom}_{\mathcal{V}}(a'_i, a_i)$ for $1 \leq i \leq n$ and $\beta_j \in \text{Hom}_{\mathcal{V}}(b_j, b'_j)$ for $1 \leq j \leq m$. Then in terms of the direct sum decomposition (6.28)

$$(\hat{k})((\alpha_n \otimes \cdots \otimes \alpha_1) \otimes (\beta_1 \otimes \cdots \otimes \beta_m))$$

maps the summand indexed by $1 \leq i_1 < \cdots < i_k \leq n$, $1 \leq j_1 < \cdots < j_k \leq m$, and $\sigma \in S_k$ to the summand with the same index via the map

$$(\alpha_m \otimes \cdots \widehat{\alpha_{i_k}} \cdots \widehat{\alpha_{i_1}} \cdots \otimes \alpha_1) \otimes (\beta_1 \otimes \cdots \widehat{\beta_{j_{\sigma(1)}}} \cdots \widehat{\beta_{j_{\sigma(k)}}} \cdots \otimes \beta_m) \otimes (\beta_{j_{\sigma(1)}} \circ \alpha_1) \otimes \cdots \otimes (\beta_{j_{\sigma(k)}} \circ \alpha_k).$$

Similarly, let $\eta \in S_n$ and $\zeta \in S_m$. Then

$$(\hat{k})(\eta \otimes \zeta)$$

maps the summand indexed by $1 \leq i_1 < \dots < i_k \leq n$, $1 \leq j_1 < \dots < j_k \leq m$, and $\sigma \in S_k$ to the one indexed by $1 \leq \eta(i_{\tau^{-1}(1)}) < \dots < \eta(i_{\tau^{-1}(k)}) \leq n$, $1 \leq \zeta(j_{v^{-1}(1)}) < \dots < \zeta(j_{v^{-1}(k)}) \leq m$, and $v\sigma\tau^{-1} \in S_k$ via the isomorphism

$$\phi \otimes \tau \otimes \chi$$

where

- $\tau \in S_k$ reorders $\eta(i_1), \dots, \eta(i_k)$ in the increasing order,
- $v \in S_k$ reorders $\zeta(j_1), \dots, \zeta(j_k)$ in the increasing order,
- $\phi \in S_{n-k}$ reorders $\eta(n) \dots \widehat{\eta(i_k)} \dots \widehat{\eta(i_1)} \dots \eta(1)$ in the decreasing order,
- $\chi \in S_{m-k}$ reorders $\zeta(1) \dots \widehat{\zeta(j_1)} \dots \widehat{\zeta(j_k)} \dots \zeta(m)$ in the increasing order.

Proof. For any $p > q > 0$ the left cosets of $S_{p-q} \otimes S_q \hookrightarrow S_p$ are enumerated by the partitions of $\{1, \dots, p\}$ into q and $p-q$ elements. This can be viewed as choices of $1 \leq i_1 < \dots < i_q \leq p$ and as a representative of each coset we can take either of the permutations

$$\begin{aligned} 1 \dots p &\mapsto 1 \dots \widehat{i_1} \dots \widehat{i_q} \dots p \ i_1 \dots i_q, \\ 1 \dots p &\mapsto p \dots \widehat{i_q} \dots \widehat{i_1} \dots 1 \ i_q \dots i_1. \end{aligned}$$

For any $p > 0$ the left cosets of $S_p \hookrightarrow S_p \times S_p$ with the embedding as in Definition 6.14 are enumerated by permutations $\sigma \in S_p$. As a representative of each coset we can take $\text{id} \times \sigma$ or $\sigma \times \text{id}$.

Hence the left cosets of $S_{n-k} \times S_k \times S_{m-k} \hookrightarrow S_n \times S_m$ are given by choices of $1 \leq i_1 < \dots < i_k \leq n$, $1 \leq j_1 < \dots < j_k \leq m$, $\sigma \in S_k$. The coset corresponding to $(\underline{i}, \underline{j}, \sigma)$ has the representative

$$(6.29) \quad \rho_{\underline{i}, \underline{j}, \sigma} := \text{the product of } \begin{cases} 1 \dots n \mapsto n \dots \widehat{i_k} \dots \widehat{i_1} \dots 1 \ i_k \dots i_1 & \in S_n, \\ 1 \dots m \mapsto j_{\sigma(1)} \dots j_{\sigma(k)} \ 1 \dots \widehat{j_1} \dots \widehat{j_k} \dots m & \in S_m. \end{cases}$$

The group $S_n \times S_m$ acts on the set of left cosets of $S_{n-k} \times S_k \times S_{m-k}$ by left multiplication. On the coset representatives $\rho_{\underline{i}, \underline{j}, \sigma}$, for any $\eta \times \zeta \in S_n \times S_m$ we have

$$(\eta \times \zeta) \rho_{(\underline{i}, \underline{j}, \sigma)} = \rho_{(\eta(\underline{i}), \zeta(\underline{j}), v\sigma\tau^{-1})} (\phi \times \tau \times \chi) \quad (\phi \times \tau \times \chi) \in S_{n-k} \times S_k \times S_{m-k}$$

with τ , v , ϕ , and χ as above. By Lemma 5.5 the first composable $\text{Res}_{S_{n-k} \times S_k \times S_{m-k}}^{S_n \times S_m}$ of (\hat{k}) sends

$$(a_n \otimes \dots \otimes a_1) \otimes (b_1 \otimes \dots \otimes b_m)$$

to the direct sum of representable modules

$$\bigoplus_{\substack{1 \leq i_1 < \dots < i_k \leq n, \\ 1 \leq j_1 < \dots < j_k \leq m, \\ \sigma \in S_k}} (a_n \otimes \dots \widehat{a_{i_k}} \dots \widehat{a_{i_1}} \dots \otimes a_1) \otimes a_{i_k} \otimes \dots \otimes a_{i_1} \otimes b_{j_{\sigma(1)}} \otimes \dots \otimes b_{j_{\sigma(k)}} \otimes (b_1 \times \dots \widehat{b_{j_1}} \dots \widehat{b_{j_k}} \dots \otimes b_m).$$

The second composable sends each summand to

$$(a_n \otimes \dots \widehat{a_{i_k}} \dots \widehat{a_{i_1}} \dots \otimes a_1) \otimes (a_{i_1} \otimes b_{j_{\sigma(1)}}) \otimes \dots \otimes (a_{i_k} \otimes b_{j_{\sigma(k)}}) \otimes (b_1 \times \dots \widehat{b_{j_1}} \dots \widehat{b_{j_k}} \dots \otimes b_m).$$

The third sends it further to

$$(a_n \otimes \dots \widehat{a_{i_k}} \dots \widehat{a_{i_1}} \dots \otimes a_1) \otimes \text{Hom}_{\mathcal{V}}(a_{i_1}, b_{j_{\sigma(1)}}) \otimes \dots \otimes \text{Hom}_{\mathcal{V}}(a_{i_k}, b_{j_{\sigma(k)}}) \otimes (b_1 \otimes \dots \widehat{b_{j_1}} \dots \widehat{b_{j_k}} \dots \otimes b_m).$$

The fourth views the factor

$$\text{Hom}_{\mathcal{V}}(a_{i_1}, b_{j_{\sigma(1)}}) \otimes \dots \otimes \text{Hom}_{\mathcal{V}}(a_{i_k}, b_{j_{\sigma(k)}})$$

as a representable $(\mathcal{H}\text{perf } \mathbb{k})$ -module instead of a representable $\mathcal{S}^k(\mathcal{H}\text{perf } \mathbb{k})$ -module. Finally the fifth views it as a perfect \mathbb{k} -module instead of a representable $(\mathcal{H}\text{perf } \mathbb{k})$ -module.

The remaining assertions follow similarly by Lemma 5.5. \square

We now define a 2-functor corresponding to each group of summands with the same number of annihilations in the result of the commutation-annihilation procedure of Definition 6.12.

Definition 6.16. For any $k \geq 0$ define a 2-functor

$$(6.30) \quad \Xi_{\mathbf{PQ}}(\hat{k}): \mathbf{Sym}_{\mathcal{V}^{\text{opp}}} \otimes \mathbf{Sym}_{\mathcal{V}} \rightarrow \mathbf{H}_{\mathcal{V}} \otimes \mathbf{H}_{\mathcal{V}} \rightarrow \mathbf{H}_{\mathcal{V}}$$

on objects to be the map $(n, m) \mapsto m - n$ for all $n, m \in \mathbb{Z}$ and on 1-morphisms to be the functor

$$\text{Hom}_{\mathbf{Sym}_{\mathcal{V}^{\text{opp}}} \otimes \mathbf{Sym}_{\mathcal{V}}}((r, s), (r + n, s + m)) \rightarrow \text{Hom}_{\mathbf{H}_{\mathcal{V}}}(s - r, (s - r) + (m - n)) \quad \forall r, s, n, m \in \mathbb{Z}$$

defined for $k > \min(n, m)$ to be zero and for $k \leq \min(n, m)$ to be the composition

$$\begin{aligned} \mathcal{S}^n \mathcal{V}^{\text{opp}} \otimes \mathcal{S}^m \mathcal{V} &\xrightarrow{(\hat{k})} \mathcal{H}\text{perf}(\mathcal{S}^{n-k} \mathcal{V}^{\text{opp}} \otimes \mathcal{S}^{m-k} \mathcal{V}) \cong \mathcal{H}\text{perf}(\mathcal{S}^{n-k} \mathcal{V}^{\text{opp}}) \otimes \mathcal{H}\text{perf}(\mathcal{S}^{m-k} \mathcal{V}) \cong \\ &\cong \mathcal{H}\text{perf}(\mathcal{S}^{m-k} \mathcal{V}) \otimes \mathcal{H}\text{perf}(\mathcal{S}^{n-k} \mathcal{V}^{\text{opp}}) \xrightarrow{\Xi_P \circ \Xi_Q} \text{Hom}_{\mathbf{H}_V}(s-r, (s-r)+(m-n)). \end{aligned}$$

We finally define the second functor (6.20) which we show to be homotopy equivalent to Ξ_{QP} :

Definition 6.17. Define the functor

$$(6.31) \quad \bigoplus \Xi_{PQ}(\hat{k}): \mathcal{S}^n \mathcal{V}^{\text{opp}} \otimes \mathcal{S}^m \mathcal{V} \rightarrow \text{Hom}_{\mathbf{H}_V}(0, m-n)$$

to be the action of the 2-functor $\bigoplus_{k=0}^{\infty} \Xi_{PQ}(\hat{k})$ on the 1-morphism category $\text{Hom}_{\mathbf{Sym}_{V^{\text{opp}}} \otimes \mathbf{Sym}_V}((0, 0), (n, m))$.

Corollary 6.18. For any object

$$(a_n \otimes \cdots \otimes a_1) \otimes (b_1 \otimes \cdots \otimes b_m) \in \mathcal{S}^n \mathcal{V}^{\text{opp}} \otimes \mathcal{S}^m \mathcal{V}$$

the 1-morphism

$$\bigoplus \Xi_{PQ}(\hat{k})((a_n \otimes \cdots \otimes a_1) \otimes (b_1 \otimes \cdots \otimes b_m))$$

is isomorphic to the result of the commutation-annihilation procedure of Definition 6.12 applied to

$$\Xi_{QP}((a_n \otimes \cdots \otimes a_1) \otimes (b_1 \otimes \cdots \otimes b_n)).$$

Proof. This follows from Lemma 6.15 and the description of the summands of the result of the commutation-annihilation procedure given in (6.26). \square

We need to show that the homotopy equivalence given by the commutation-annihilation procedure

$$\bigoplus \Xi_{PQ}(\hat{k})((a_n \otimes \cdots \otimes a_1) \otimes (b_1 \otimes \cdots \otimes b_n)) \xrightarrow{\sim} \Xi_{QP}((a_n \otimes \cdots \otimes a_1) \otimes (b_1 \otimes \cdots \otimes b_n))$$

define a natural transformation $\bigoplus \Xi_{PQ}(\hat{k}) \rightarrow \Xi_{QP}$. First, we give a direct definition:

Definition 6.19. Let

$$\begin{aligned} \underline{a} &= (a_n \otimes \cdots \otimes a_1) \in \mathcal{S}^n \mathcal{V}^{\text{opp}}, \\ \underline{b} &= (b_1 \otimes \cdots \otimes b_m) \in \otimes \mathcal{S}^m \mathcal{V}. \end{aligned}$$

Define a 2-morphism in \mathbf{H}_V

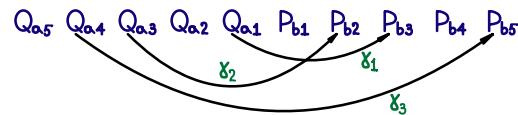
$$(6.32) \quad \phi: \bigoplus \Xi_{PQ}(\hat{k})(\underline{a} \otimes \underline{b}) \longrightarrow \Xi_{PQ}(\underline{a} \otimes \underline{b})$$

by setting for $0 \leq k \leq \min(n, m)$, $1 \leq i_1 < \cdots < i_k \leq n$, $1 \leq j_1 < \cdots < j_k \leq m$, and $\sigma \in S_k$ the component of ϕ on the corresponding summand of $\Xi_{PQ}(\hat{k})(\underline{a} \otimes \underline{b})$ to be the adjoint of the morphism

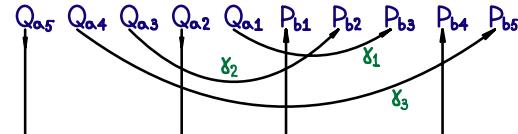
$$(6.33) \quad \begin{aligned} &\text{Hom}_V(a_{i_1}, b_{j_{\sigma(1)}}) \otimes \cdots \otimes \text{Hom}_V(a_{i_k}, b_{j_{\sigma(k)}}) \\ &\downarrow \\ &\text{Hom}_{\mathbf{H}_V} \left(P_{b_1} \dots \widehat{P_{b_{j_1}}} \dots P_{b_m} Q_{a_n} \dots \widehat{Q_{a_{i_1}}} \dots Q_{a_1}, \quad Q_{a_n} \dots Q_{a_1} P_{b_1} \dots P_{b_m} \right) \end{aligned}$$

which sends each $\gamma_1 \otimes \cdots \otimes \gamma_k$ to the 2-morphism in \mathbf{H}_V defined by the following planar diagram:

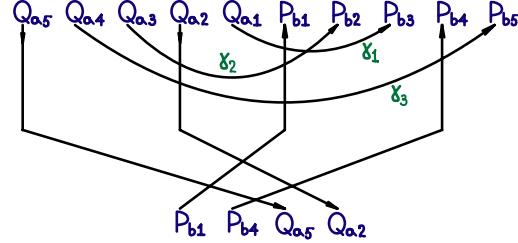
- (1) First, draw the annihilation strands. Start at the top of the diagram, and for each $1 \leq l \leq k$ draw a circular strand counterclockwise from Q_{a_l} to $P_{j_{\sigma(l)}}$ and decorate it by γ_l . Draw these so that each annihilation strand dips below the previous one, i.e. the height of the l -th strand arc is less than that of the $(l+1)$ -st for all $1 \leq l < k$.



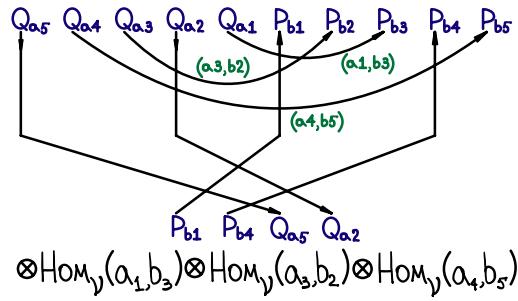
- (2) Next, begin drawing the commutation strands. From each of the remaining Qs and Ps draw a vertical line down to some horizontal level line below all the annihilation strands.



- (3) Finish by starting at that level and joining the commutation strand of each Q_{a_i} or P_{b_j} at the top of the diagram to the same Q_{a_i} or P_{b_j} at the bottom of the diagram in a straight line.



Denote the component of ϕ on each summand of $\Xi_{PQ}(\hat{k})(\underline{a} \otimes \underline{b})$ by the same diagram as its adjoint (6.33) only with each annihilation strand decorated by $(a_{i_l}, b_{j_{\sigma(l)}})$ instead of γ_l :



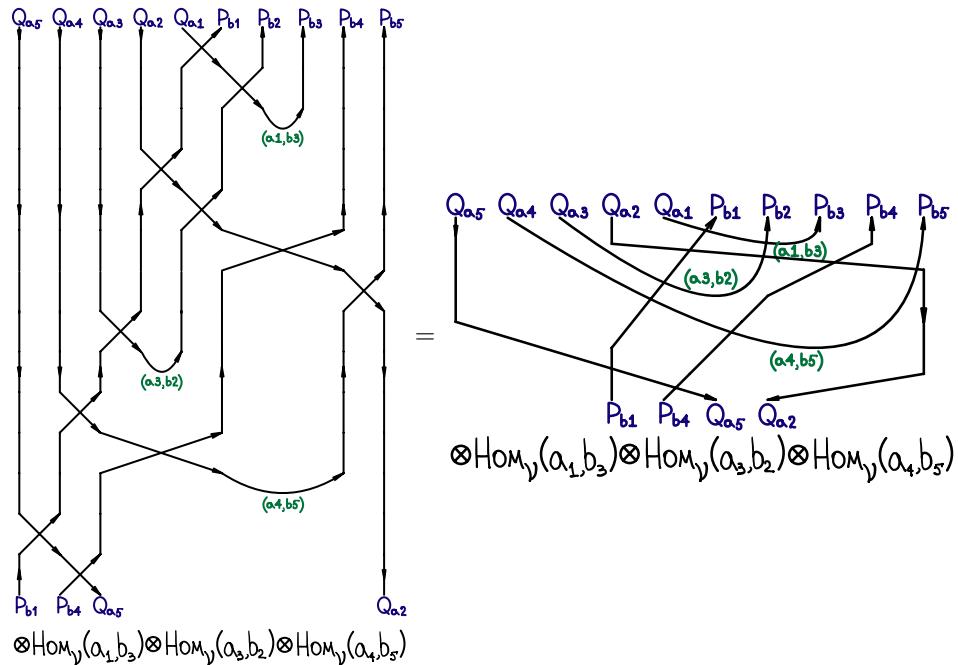
Proposition 6.20. For any $\underline{a} = (a_n \otimes \cdots \otimes a_1) \in S^n \mathcal{V}^{\text{OPP}}$ and $\underline{b} = (b_1 \otimes \cdots \otimes b_m) \in \otimes \mathcal{S}^m \mathcal{V}$, the 2-morphism $\phi: \bigoplus \Xi_{PQ}(\hat{k})(\underline{a} \otimes \underline{b}) \rightarrow \Xi_{QP}(\underline{a} \otimes \underline{b})$ of Defn. 6.19 equals in \mathbf{H}_V the one constructed by the commutation-annihilation procedure of Defn. 6.12. In particular, it is a homotopy equivalence.

Proof. This follows from the pitchfork and triple move relations in \mathbf{H}_V [42, Lemma 5.5].

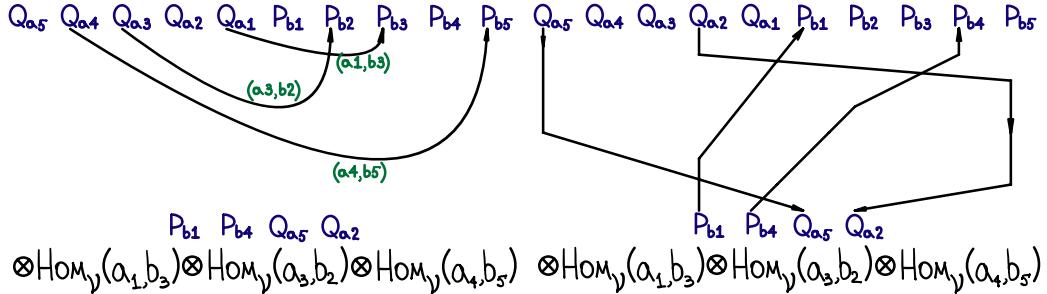
We can prove it separately for each summand of $\bigoplus \Xi_{PQ}(\hat{k})(\underline{a} \otimes \underline{b})$. For a given summand, the commutation-annihilation procedure construct the following 2-morphism into $\Xi_{QP}(\underline{a} \otimes \underline{b})$. We begin at the top of the diagram at Q_{a_1} and perform rightward crossings \nearrow until:

- If the Q_{a_1} -strand is a commutation strand – until it moves to the right of all P -strands,
- If the Q_{a_1} -strand is an annihilation strand – until it reaches the P_{b_j} -strand it is paired with. The two strands are then terminated by a cup marked by (a_1, b_j) .

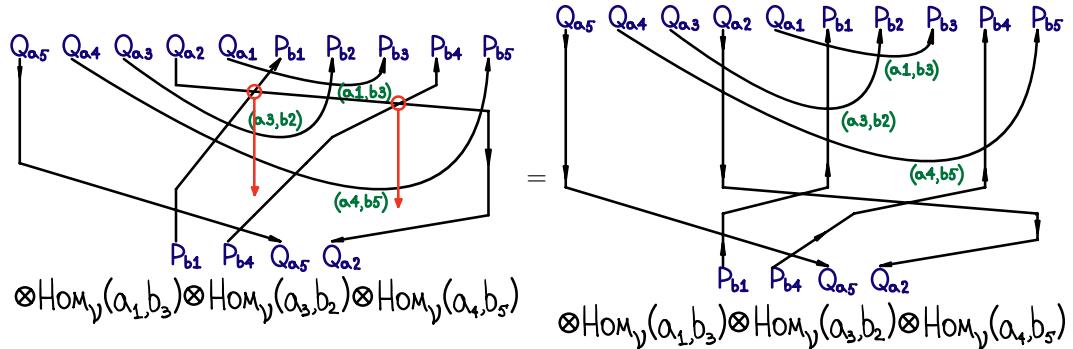
We then repeat the same for Q_{a_2} , etc.



If we take this planar diagram and consider all the annihilation strands separately and all the commutation strands separately, then the two configurations match, possibly up to some pitchfork relations, the configurations of the annihilation and of the commutation strands in the planar diagram defining the corresponding component of the 2-morphism ϕ of Definition 6.19.



Now taking all the commutation strand crossings in the commutation-annihilation planar diagram and using triple moves to commute them downwards past all the annihilation strands produces the planar diagram in the definition of ϕ .



We conclude that the two planar diagrams define the same 2-morphism in \mathbf{H}_V . \square

The following can be viewed as a functorial categorification of the PQ Heisenberg relation (3.34):

Theorem 6.21. *The 2-morphisms ϕ of Definition 6.19 define a DG natural transformation*

$$(6.34) \quad \phi: \bigoplus \Xi_{PQ}(\hat{k}) \longrightarrow \Xi_{QP}$$

of DG functors $\mathcal{S}^n\mathcal{V}^{OPP} \otimes \mathcal{S}^m\mathcal{V} \rightarrow \text{Hom}_{\mathbf{H}_V}(0, m-n)$. By Prps. 6.20, ϕ is a homotopy equivalence.

A non-functorial categorification of the Heisenberg relation appeared in [42, Theorem 6.3]. However, it relates the symmetrised elements $P_a^{(n)}$ and $Q_b^{(m)}$ and as these are not functorial in $a, b \in V$ for $n, m > 1$, there was little hope of making it functorial directly. Instead, we used its basic case $n = m = 1$ to iteratively construct the present, functorial categorification (6.34). It is clear that applying (6.34) to the product of the symmetrised powers $a^{(n)} \in \mathcal{S}^n\mathcal{V}^{OPP}$ and $b^{(m)} \in \mathcal{S}^m\mathcal{V}$, defined via the twisted complexes analogous to those defining $P_a^{(n)}$ and $Q_b^{(m)}$ in [42, Definition 6.2], recovers the non-functorial categorification of [42, Theorem 6.3].

Proof of Theorem 6.21. We need to show that for all

$$\begin{aligned} \underline{a} &= (a_n \otimes \cdots \otimes a_1) \text{ and } \underline{a}' = (a'_n \otimes \cdots \otimes a'_1) \in \mathcal{S}^n\mathcal{V}^{OPP}, \\ \underline{b} &= (b_1 \otimes \cdots \otimes b_m) \text{ and } \underline{b}' = (b'_1 \otimes \cdots \otimes b'_m) \in \mathcal{S}^m\mathcal{V} \end{aligned}$$

and each morphism

$$(6.35) \quad \underline{\alpha} \otimes \underline{\beta}: \underline{a} \otimes \underline{b} \rightarrow \underline{a}' \otimes \underline{b}' \quad \text{in } \mathcal{S}^n\mathcal{V}^{OPP} \otimes \mathcal{S}^m\mathcal{V}$$

the corresponding square commutes in $\text{Hom}_{\mathbf{H}_V}(0, m-n)$:

$$(6.36) \quad \begin{array}{ccc} \Xi_{QP}(\underline{a} \otimes \underline{b}) & \xrightarrow{\Xi_{QP}(\underline{\alpha} \otimes \underline{\beta})} & \Xi_{QP}(\underline{a}' \otimes \underline{b}') \\ \phi_{\underline{a} \otimes \underline{b}} \uparrow & & \phi_{\underline{a}' \otimes \underline{b}'} \uparrow \\ \bigoplus \Xi_{PQ}(\hat{k})(\underline{a} \otimes \underline{b}) & \xrightarrow{\bigoplus \Xi_{PQ}(\hat{k})(\underline{\alpha} \otimes \underline{\beta})} & \bigoplus \Xi_{PQ}(\hat{k})(\underline{a}' \otimes \underline{b}'). \end{array}$$

The action of Ξ_{QP} on the morphism spaces of $\mathcal{S}^n\mathcal{V}^{opp} \otimes \mathcal{S}^m\mathcal{V}$ is clear. The functor $\Xi_{PQ}(\hat{k})$ is defined in terms of the functor (\hat{k}) whose definition is more involved. However, the action of (\hat{k}) on the morphism spaces of $\mathcal{S}^n\mathcal{V}^{opp} \otimes \mathcal{S}^m\mathcal{V}$ is described explicitly by Lemma 6.15. We thus prove the functoriality of ϕ by explicitly verifying for each generating morphism of $\mathcal{S}^n\mathcal{V}^{opp} \otimes \mathcal{S}^m\mathcal{V}$ that (6.36) commutes in $\text{Hom}_{H_V}(0, m - n)$. This reduces, once more, to applying the pitchfork and triple move relations [42, Lemma 5.5] and the dot sliding relations [42, Lemma 5.4].

Indeed, it suffices to prove that (6.36) commutes for each direct summand of $\bigoplus \Xi_{PQ}(\hat{k})(\underline{a} \otimes \underline{b})$. Fix $0 \leq k \leq \min(n, m)$. By Lemma 6.15, each $\Xi_{PQ}(\hat{k})(\underline{a} \otimes \underline{b})$ is itself a direct sum indexed by choices of $1 \leq i_1 < \dots < i_k \leq n$, $1 \leq j_1 < \dots < j_k \leq m$ and $\sigma \in S_k$. Fix any such $(\underline{i}, \underline{j}, \sigma)$.

The morphisms (6.35) are generated by composition from the following four basic types:

- (1) $\underline{\alpha} \otimes \text{id}$ with $\underline{\alpha} = \text{id}^{\otimes(n-i)} \otimes \alpha_i \otimes \text{id}^{\otimes(i-1)}$ for some $1 \leq i \leq n$ and $\alpha_i \in \text{Hom}_{\mathcal{V}}(a'_i, a_i)$,
- (2) $\text{id} \otimes \underline{\beta}$ with $\underline{\beta} = \text{id}^{\otimes(j-1)} \otimes \beta_j \otimes \text{id}^{\otimes(m-j)}$ for some $1 \leq j \leq m$ and $\beta \in \text{Hom}_{\mathcal{V}}(b_i, b'_i)$,
- (3) $\eta \otimes \text{id}$ with $\eta = (i(i+1)) \in S_n$ for some $1 \leq i \leq n-1$,
- (4) $\text{id} \otimes \zeta$ with $\zeta = (j(j+1)) \in S_m$ for some $1 \leq j \leq m-1$.

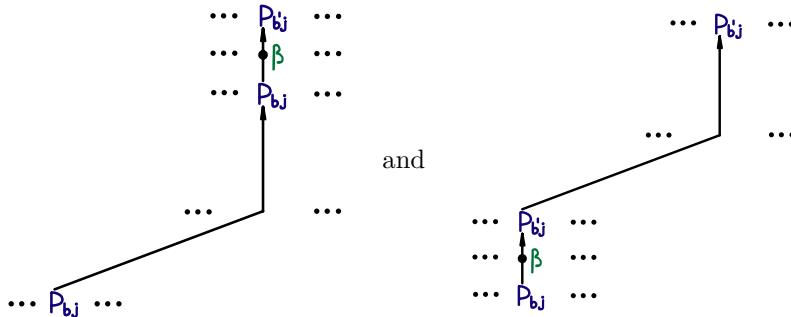
We only give the proofs for the types (2) and (4), the proofs for the other two types are similar.

Morphisms of type (2):

Consider $\phi_{\underline{a} \otimes \underline{b}}$ restricted to the $(\underline{i}, \underline{j}, \sigma)$ direct summand of $\Xi_{PQ}(\hat{k})(\underline{a} \otimes \underline{b})$. There are two cases:

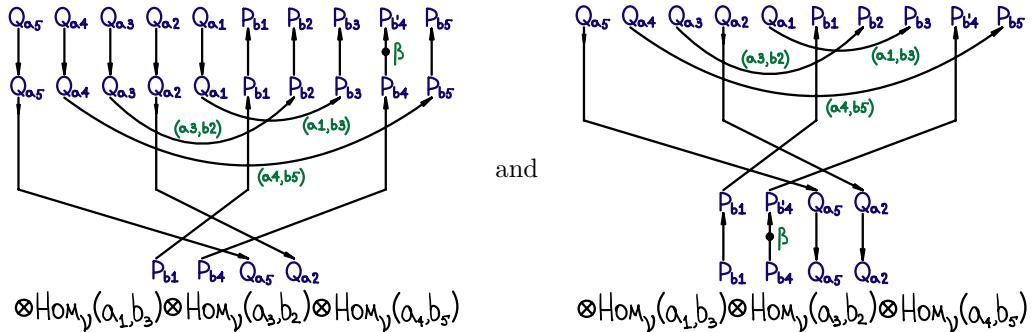
- (1) *The strand entering $P_{b,j}$ is a commutation strand ($j \notin \{j_1, \dots, j_k\}$):*

Then on the $(\underline{i}, \underline{j}, \sigma)$ direct summand $\Xi_{QP}(\text{id} \otimes \beta) \circ \phi_{\underline{a} \otimes \underline{b}}$ and $\phi_{\underline{a}' \otimes \underline{b}'} \circ \bigoplus \Xi_{PQ}(\hat{k})(\text{id} \otimes \beta)$, the two compositions around the upper-left and the lower-right halves of the square (6.36), are defined by two planar diagrams which only differ in the following parts:



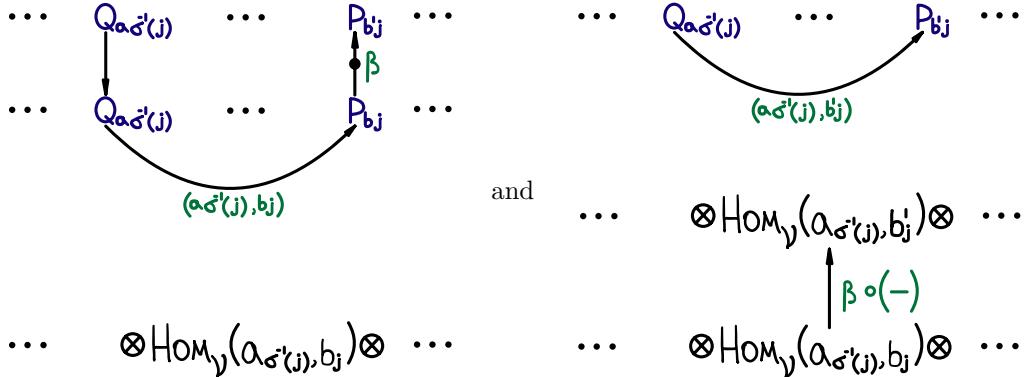
Sliding the dot down the strand turns the left diagram into the right one. The identical remainder of the two diagrams, not depicted above, crosses the depicted parts. As it slides, the dot travels through these crossings. By the dot sliding relations [42, Lemma 5.4], this doesn't change the 2-morphism defined by the diagram. We conclude that the corresponding 2-morphisms are equal and thus (6.36) commutes on this direct summand.

Here and below, we only draw the relevant parts in which the diagrams differ nontrivially. For example, here the rest is identical since $\Xi_{QP}(\text{id} \otimes \beta)$ and $\Xi_{PQ}(\hat{k})(\text{id} \otimes \beta)$ both consist of a row of parallel vertical strands one of which is adorned by β , while $\phi_{\underline{a} \otimes \underline{b}}$ and $\phi_{\underline{a}' \otimes \underline{b}'}$ are restricted to the same indexed summand and hence are defined by the identical diagrams. Thus, away from the β -adorned strand, we are post-composing the diagram of $\phi_{\underline{a} \otimes \underline{b}}$ and pre-composing the same diagram of $\phi_{\underline{a}' \otimes \underline{b}'}$ with a row of unadorned parallel vertical strands:



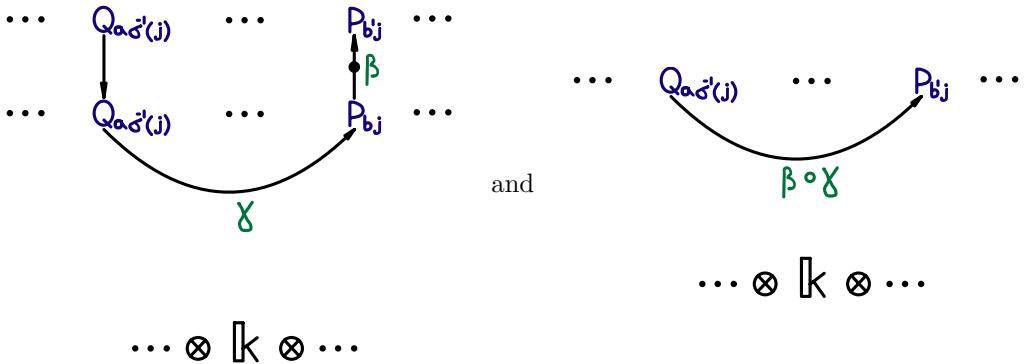
(2) *The strand entering P_{b_j} is an annihilation strand ($j \in \underline{j}$):*

Then on the $(\underline{\iota}, \underline{j}, \sigma)$ direct summand the two compositions around (6.36) are defined by two planar diagrams which only differ in:



Here and below, by abuse of notation, by $\sigma^{-1}(j)$ we denote the pre-image of j under the permutation of $\{j_1, \dots, j_k\}$ defined by $\sigma \in S_k$. In other words, if $j = j_l$, we mean $j_{\sigma^{-1}(l)}$.

By definition, these 2-morphisms are the left adjoints of the maps which send each $\gamma \in \text{Hom}_V(a_{\sigma^{-1}(j)}, b_j)$ to the 2-morphisms defined by the diagrams only in



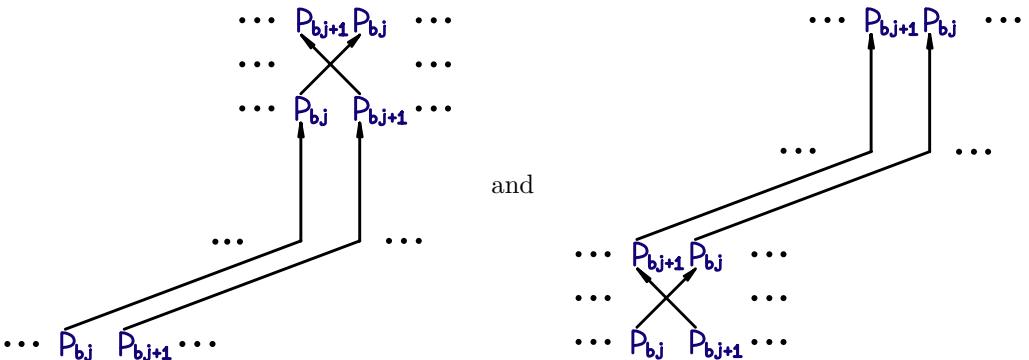
Slide down the β -dot down the string and merge it with the γ -dot turns the left diagram into the right diagram. By the dot sliding relations, the corresponding 2-morphisms are equal. Hence their left adjoints are also equal, and (6.36) commutes on this direct summand.

Morphisms of type (4):

Consider $\phi_{\underline{a} \otimes \underline{b}}$ restricted to the $(\underline{\iota}, \underline{j}, \sigma)$ direct summand of $\Xi_{PQ}(\hat{k})$ ($\underline{a} \otimes \underline{b}$). There are five cases:

(1) *The strand entering P_{b_j} is a commutation strand ($j \notin \underline{j}$),
the strand entering $P_{b_{j+1}}$ is a commutation strand ($j+1 \notin \underline{j}$):*

Then on the $(\underline{\iota}, \underline{j}, \sigma)$ direct summand the two compositions around (6.36) are defined by two planar diagrams which only differ in:

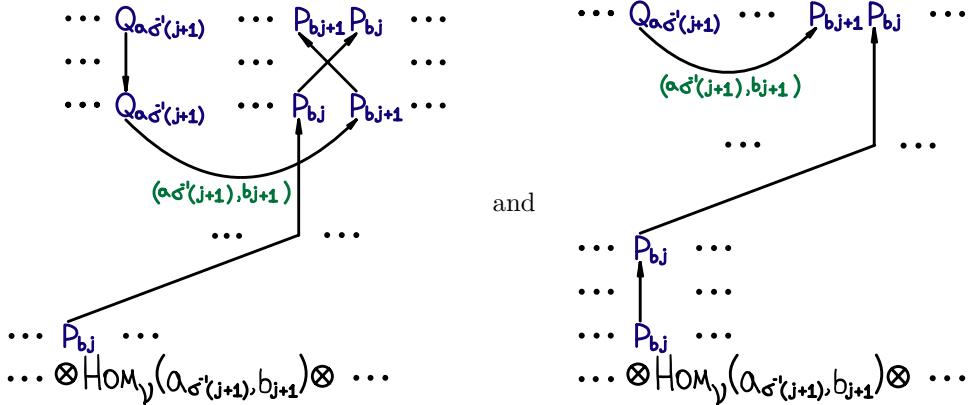


Sliding the upward crossing down the parallel strands turns the left diagram into the right one. On its way down, the crossing travels through the identical remainder of the

two diagrams. By the triple move relations [42, Lemma 5.5], this doesn't change the corresponding 2-morphism. Thus (6.36) commutes on this direct summand.

- (2) *The strand entering P_{b_j} is a commutation strand ($j \notin \underline{j}$), the strand entering $P_{b_{j+1}}$ is an annihilation ($j+1 \in \underline{j}$):*

Then on the $(\underline{i}, j, \sigma)$ direct summand the two compositions around (6.36) are defined by two planar diagrams which only differ in:

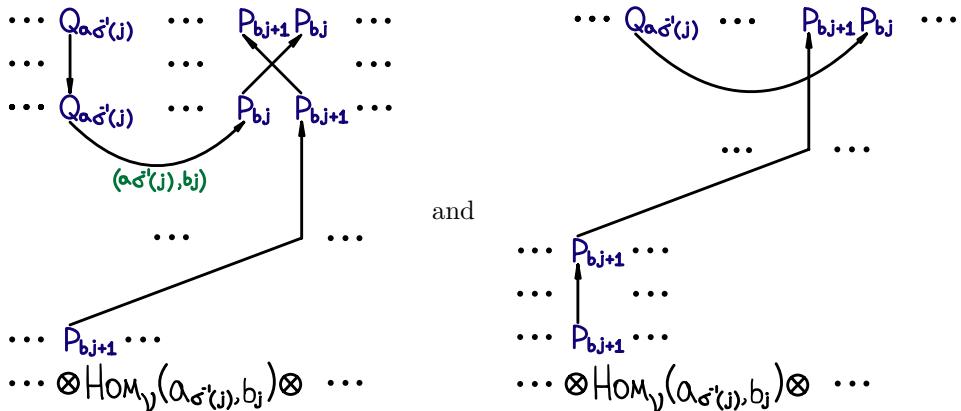


Here for the first time $\text{id} \otimes \zeta$ acts non-trivially on our chosen direct summand. We have $\zeta = (j(j+1))$, so by Lemma 6.15 the morphism $\bigoplus \Xi_{PQ}(\hat{k})(\text{id} \otimes \zeta)$ maps the $(\underline{i}, j, \sigma)$ summand of $\bigoplus \Xi_{PQ}(\underline{a} \otimes \underline{b})$ to the $(\underline{i}, \zeta(\underline{j}), \zeta(\sigma))$ summand of $\bigoplus \Xi_{PQ}(\underline{a} \otimes \zeta(\underline{b}))$. Recall that \underline{j} is a choice of $1 \leq j_1 < \dots < j_k \leq m$ and $\zeta(\underline{j})$ is $\zeta(j_1), \dots, \zeta(j_k)$ reordered in the increasing order. As $j \notin \underline{j}$ and $j+1 \in \underline{j}$, $\zeta(j_1), \dots, \zeta(j_k)$ is j_1, \dots, j_k with $j+1$ replaced by j . No reordering needed, so $\zeta(\sigma) = \sigma$. Similarly, $\zeta(1), \dots, \widehat{\zeta(j_1)}, \dots, \widehat{\zeta(j_k)}, \dots, \zeta(m)$ are $1, \dots, \widehat{j_1}, \dots, \widehat{j_k}, \dots, m$ with $j+1$ replaced by j , with no reordering needed. Thus, by Lemma 6.15, $\bigoplus \Xi_{PQ}(\hat{k})(\text{id} \otimes \zeta)$ maps one summand to the other by the identity map.

On the left diagram, slide the lower crossing up until it reaches the other crossing. By the triple move relations this does not change the corresponding 2-morphism. Once one crossing reaches the other, replace their composition by two parallel upward strands to get the right diagram. By the symmetric group relations for upward strands [42, Lemma 5.5], this doesn't change the 2-morphism either. Thus (6.36) commutes on this direct summand.

- (3) *The strand entering P_{b_j} is an annihilation strand ($j \in \underline{j}$), the strand entering $P_{b_{j+1}}$ is an commutation strand ($j+1 \notin \underline{j}$):*

Then on the $(\underline{i}, j, \sigma)$ direct summand the two compositions around (6.36) are defined by two planar diagrams which only differ in:



Similarly, ζ sends $(\underline{i}, j, \sigma)$ to $(\underline{i}, \zeta(j), \sigma)$ where $\zeta(\underline{j})$ is \underline{j} with j replaced by $j+1$. Again, $\bigoplus \Xi_{PQ}(\hat{k})(\text{id} \otimes \zeta)$ acts by the identity map.

Take the left diagram and slide the upward crossing down the parallel strands to obtain the right diagram. By the triple move relations, this doesn't change the corresponding 2-morphism. Thus (6.36) commutes on this direct summand.

- (4) The strand entering P_{b_j} is an annihilation strand ($j \in \underline{j}$),
the strand entering $P_{b_{j+1}}$ is an annihilation strand ($j+1 \in \underline{j}$),
and these two strands cross ($\sigma^{-1}(j) > \sigma^{-1}(j+1)$):

Then on the $(\underline{i}, \underline{j}, \sigma)$ direct summand the two compositions around (6.36) are defined by two planar diagrams which only differ in:

$$\cdots \otimes \text{Hom}_V(a_{\sigma'(j)}, b_j) \otimes \text{Hom}_V(a_{\sigma'(j+1)}, b_{j+1}) \otimes \cdots$$

$$\cdots \otimes \text{Hom}_V(a_{\sigma'(j)}, b_j) \otimes \text{Hom}_V(a_{\sigma'(j+1)}, b_{j+1}) \otimes \cdots$$

Here applying ζ to \underline{j} swaps j and $j+1$, and we need to reorder by applying ζ again. Thus $\zeta(\underline{j}) = \underline{j}$, but σ becomes $\zeta\sigma$. Moreover, applying ζ to $1, \dots, \hat{j_1}, \dots, \hat{j_k}, \dots, m$ changes nothing. Thus $\bigoplus \Xi_{PQ}(\hat{k})(\text{id} \otimes \zeta)$ maps the $(\underline{i}, \underline{j}, \sigma)$ summand of $\bigoplus \Xi_{PQ}(\underline{a} \otimes \underline{b})$ to the $(\underline{i}, \underline{j}, \zeta\sigma)$ summand of $\bigoplus \Xi_{PQ}(\underline{a} \otimes \zeta(\underline{b}))$ by the identity map.

Take the left diagram, slide the lower crossing up until it reaches the other crossing, and then replace the composition of the two crossings by two upward parallel strands to obtain the right diagram. By the triple move and the symmetric group relations, this doesn't change the corresponding 2-morphism. Thus (6.36) commutes on this direct summand.

- (5) The strand entering P_{b_j} is an annihilation strand ($j \in \underline{j}$),
the strand entering $P_{b_{j+1}}$ is an annihilation strand ($j+1 \in \underline{j}$),
and these two strands do not cross ($\sigma^{-1}(j) < \sigma^{-1}(j+1)$):

$$\cdots \otimes \text{Hom}_V(a_{\sigma'(j)}, b_j) \otimes \text{Hom}_V(a_{\sigma'(j+1)}, b_{j+1}) \otimes \cdots$$

$$\cdots \otimes \text{Hom}_V(a_{\sigma'(j)}, b_j) \otimes \text{Hom}_V(a_{\sigma'(j+1)}, b_{j+1}) \otimes \cdots$$

Again, ζ sends $(\underline{i}, \underline{j}, \sigma)$ to $(\underline{i}, \underline{j}, \zeta\sigma)$ and $\bigoplus \Xi_{PQ}(\hat{k})(\text{id} \otimes \zeta)$ acts by the identity map.

Take the left diagram and slide the upward crossing down the parallel strands to obtain the right diagram. By the triple move relations, this doesn't change the corresponding 2-morphism. Thus (6.36) commutes on this direct summand.

□

Its functoriality makes the Heisenberg relation categorification (6.34) applicable to Hochschild homology. We now complete the proof of the desired Heisenberg relation (6.19) by applying (6.34) to the product of the Hochschild homology classes $\psi_n(\alpha)$ and $\psi_m(\beta)$:

Proof of Theorem 6.9. Let $\alpha, \beta \in \text{HC}_\bullet(\mathcal{V})$. Implicitly using the isomorphism $\text{HC}_\bullet(\mathcal{V}) \cong \text{HC}_\bullet(\mathcal{V}^{\text{opp}})$ defined in Section 5.4, we also denote by α the corresponding element in $\text{HC}_\bullet(\mathcal{V}^{\text{opp}})$.

By [23, Lemma 3.4], it follows from Theorem 6.21 that in $\text{HH Hom}_{\mathbf{H}_V}(0, m-n)$ we have

$$\Xi_{PQ}(K(\psi_n(\alpha) \otimes \psi_m(\beta))) = \bigoplus_{k=0}^{k=\min(n,m)} \Xi_{PQ}(\hat{k})(K(\psi_n(\alpha) \otimes \psi_m(\beta))),$$

where K is the shuffle product map

$$\mathrm{HC}_\bullet(\mathcal{S}^n \mathcal{V}^{\mathrm{opp}}) \otimes \mathrm{HC}_\bullet(\mathcal{S}^m \mathcal{V}) \xrightarrow{(5.14)} \mathrm{HC}_\bullet(\mathcal{S}^n \mathcal{V}^{\mathrm{opp}} \otimes \mathcal{S}^m \mathcal{V}).$$

It follows from the definition (6.21) of the functor Ξ_{PQ} that

$$\Xi_{\mathrm{PQ}}(K(\psi_n(\alpha) \otimes \psi_m(\beta))) = \Xi_{\mathrm{Q}}(\psi_n(\alpha)) \Xi_{\mathrm{P}}(\psi_m(\beta)) = \mathsf{A}_\alpha(-n) \mathsf{A}_\beta(m).$$

Since $\hat{0}$ is the identity functor, we have $\Xi_{\mathrm{PQ}}(\hat{0}) = \Xi_{\mathrm{PQ}}$ and similarly

$$\Xi_{\mathrm{PQ}}(K(\psi_n(\alpha) \otimes \psi_m(\beta))) = \Xi_{\mathrm{P}}(\psi_m(\beta)) \Xi_{\mathrm{Q}}(\psi_n(\alpha)) = \mathsf{A}_\beta(m) \mathsf{A}_\alpha(-n).$$

It now suffices to establish the following claim: in the Hochschild homology

$$(6.37) \quad \forall k \geq 1 \quad \Xi_{\mathrm{PQ}}(\hat{k})(K(\psi_n(\alpha) \otimes \psi_m(\beta))) = \begin{cases} n \langle \alpha, \beta \rangle & \text{if } k = n = m, \\ 0 & \text{otherwise.} \end{cases}$$

By definition, $K(\psi_n(\alpha) \otimes \psi_m(\beta))$ is the image of $\alpha \otimes \beta$ under the composition

$$\begin{aligned} \mathrm{HC}_\bullet(\mathcal{V}^{\mathrm{opp}}) \otimes \mathrm{HC}_\bullet(\mathcal{V}) &\xrightarrow{g_n \otimes g_m} \mathrm{HC}_\bullet((\mathcal{V}^{\mathrm{opp}})^{\otimes n}; t_n) \otimes \mathrm{HC}_\bullet(\mathcal{V}^{\otimes m}; t_m) \xrightarrow{\xi_{t_n} \otimes \xi_{t_m}} \\ &\rightarrow \mathrm{HC}_\bullet(\mathcal{S}^n \mathcal{V}^{\mathrm{opp}}) \otimes \mathrm{HC}_\bullet(\mathcal{S}^m \mathcal{V}) \xrightarrow{K} \mathrm{HC}_\bullet(\mathcal{S}^n \mathcal{V}^{\mathrm{opp}} \otimes \mathcal{S}^m \mathcal{V}), \end{aligned}$$

where $t_n = (1 \dots n) \in S_n$ and $t_m = (1 \dots m) \in S_m$ are the long cycles. By Lemma 5.23, this equals

$$\begin{aligned} \mathrm{HC}_\bullet(\mathcal{V}^{\mathrm{opp}}) \otimes \mathrm{HC}_\bullet(\mathcal{V}) &\xrightarrow{g_n \otimes g_m} \mathrm{HC}_\bullet((\mathcal{V}^{\mathrm{opp}})^{\otimes n}; t_n) \otimes \mathrm{HC}_\bullet(\mathcal{V}^{\otimes m}; t_m) \xrightarrow{K} \\ &\rightarrow \mathrm{HC}_\bullet((\mathcal{V}^{\mathrm{opp}})^{\otimes n} \otimes \mathcal{V}^{\otimes m}; t_n \times t_m) \xrightarrow{\xi_{t_n \times t_m}} \mathrm{HC}_\bullet(\mathcal{S}^n \mathcal{V}^{\mathrm{opp}} \otimes \mathcal{S}^m \mathcal{V}), \end{aligned}$$

where we implicitly identify $\mathcal{S}^n \mathcal{V}^{\mathrm{opp}} \otimes \mathcal{S}^m \mathcal{V}$ with $(\mathcal{V}^{\mathrm{opp}})^{\otimes n} \otimes \mathcal{V}^{\otimes m} \rtimes (S_n \times S_m)$.

By definition, $\Xi_{\mathrm{PQ}}(\hat{k}) = (\Xi_{\mathrm{P}} \circ_1 \Xi_{\mathrm{Q}}) \circ (\hat{k})$. The annihilation functor (\hat{k}) was itself defined in Definition 6.14 as a composition of functors which begins with the functor

$$\mathcal{S}^n \mathcal{V}^{\mathrm{opp}} \otimes \mathcal{S}^m \mathcal{V} \xrightarrow{\mathrm{Res}_{S_{n-k} \times S_k \times S_{m-k}}^{S_n \times S_m}} \mathcal{H}\mathrm{perf}(\mathcal{S}^{n-k} \mathcal{V}^{\mathrm{opp}} \otimes ((\mathcal{V}^{\mathrm{opp}})^{\otimes k} \otimes \mathcal{V}^{\otimes k}) \rtimes S_k) \otimes \mathcal{S}^{m-k} \mathcal{V},$$

where $S_{n-k} \times S_k \times S_{m-k}$ embeds into $S^n \times S^m$ as described in Definition 6.14.

We conclude that $\Xi_{\mathrm{PQ}}(\hat{k})(K(\psi_n(\alpha) \otimes \psi_m(\beta)))$ is the image of $\alpha \otimes \beta \in \mathrm{HC}_\bullet(\mathcal{V}^{\mathrm{opp}}) \otimes \mathrm{HC}_\bullet(\mathcal{V})$ under a composition of maps which includes

$$(6.38) \quad \begin{array}{c} \mathrm{HC}_\bullet((\mathcal{V}^{\mathrm{opp}})^{\otimes n} \otimes \mathcal{V}^{\otimes m}; t_n \times t_m) \\ \downarrow \xi_{t_n \times t_m} \\ \mathrm{HC}_\bullet(\mathcal{S}^n \mathcal{V}^{\mathrm{opp}} \otimes \mathcal{S}^m \mathcal{V}) \\ \downarrow \mathrm{Res}_{S_{n-k} \times S_k \times S_{m-k}}^{S_n \times S_m} \\ \mathrm{HC}_\bullet(\mathcal{S}^{n-k} \mathcal{V}^{\mathrm{opp}} \otimes ((\mathcal{V}^{\mathrm{opp}})^{\otimes k} \otimes \mathcal{V}^{\otimes k}) \rtimes S_k) \otimes \mathcal{S}^{m-k} \mathcal{V}. \end{array}$$

By Lemma 5.18, on HH_\bullet the map (6.38) is a sum indexed by the elements of $\mathrm{Fix}_Q(t_n \times t_m)$, the fixed set of the action of $t_n \times t_m$ on the set Q of the left cosets of $S_{n-k} \times S_k \times S_{m-k}$ in $S_n \times S_m$. However, $\mathrm{Fix}_Q(t_n \times t_m) = \emptyset$ unless $k = 0$ or $k = n = m$. This can be seen from the explicit description of the action of $S_n \times S_m$ on Q in the proof of Lemma 6.15. Alternatively, since

$$S_{n-k} \times S_k \times S_{m-k} \leq S_{n-k} \times S_k \times S_k \times S_{m-k} \leq S_n \times S_m,$$

it suffices to consider the action of $t_n \times t_m$ on the cosets of $S_{n-k} \times S_k \times S_k \times S_{m-k}$. We have

$$\mathrm{Fix}_{S_n \times S_m / S_{n-k} \times S_k \times S_{m-k}}(t_n \times t_m) = \mathrm{Fix}_{S_n / S_{n-k} \times S_k}(t_n) \times \mathrm{Fix}_{S_m / S_k \times S_{m-k}}(t_m),$$

and unless $k = 0$ or $k = n = m$ one of the two sets in the cartesian product on the RHS is empty.

Thus the chain $\Xi_{\mathrm{PQ}}(\hat{k})(K(\psi_n(\alpha) \otimes \psi_m(\beta)))$ vanishes in HH_\bullet unless $k = 0$ or $k = n = m$. This shows most of (6.37). It remains to show that when $n = m$ we have

$$(6.39) \quad \Xi_{\mathrm{PQ}}(\hat{n})(K(\psi_n(\alpha) \otimes \psi_n(\beta))) = \langle \alpha, \beta \rangle [\mathbb{1}] \in \mathrm{HH}_\bullet(\mathrm{Hom}_{\mathbf{H}_V}(0, 0)),$$

where $\mathbb{1} \in \mathrm{Hom}_{\mathbf{H}_V}(0, 0)$ is the identity 1-morphism. By Definition 6.17, the functor $\Xi_{\mathrm{PQ}}(\hat{n})$ is

$$\mathcal{S}^n \mathcal{V}^{\mathrm{opp}} \otimes \mathcal{S}^n \mathcal{V} \xrightarrow{(\hat{n})} \mathcal{H}\mathrm{perf}(\mathbb{k}) \xrightarrow{\sim} \mathcal{H}\mathrm{perf}(\mathbb{k}) \otimes \mathcal{H}\mathrm{perf}(\mathbb{k}) \xrightarrow{\Xi_{\mathrm{P}} \circ_1 \Xi_{\mathrm{Q}}} \mathrm{Hom}_{\mathbf{H}_V}(0, 0).$$

Each of Ξ_P and Ξ_Q send $\mathbb{k} \in \mathcal{H}perf(\mathbb{k})$ to $1 \in \text{Hom}_{\mathbf{H}\mathcal{V}}(0, 0)$. Thus the latter two terms in the composition send $\mathbb{k} \in \mathcal{H}perf(\mathbb{k})$ to $1 \circ_1 1 = 1 \in \text{Hom}_{\mathbf{H}\mathcal{V}}(0, 0)$. Therefore, on the level of Hochschild homology they become the linear map $\mathbb{k} \rightarrow \text{HH}_\bullet(\text{Hom}_{\mathbf{H}\mathcal{V}}(0, 0))$ which sends 1 to the $[1]$.

It remains to show that in $\text{HH}_\bullet(\mathcal{H}perf \mathbb{k}) = \mathbb{k}$ we have

$$(\hat{n})(K(\psi_n(\alpha) \otimes \psi_n(\beta))) = n \langle \alpha, \beta \rangle.$$

By Definition 6.14, the map induced by (\hat{n}) on the Hochschild homology is

$$\begin{aligned} \text{HH}_\bullet(\mathcal{S}^n \mathcal{V}^{\text{opp}} \otimes \mathcal{S}^n \mathcal{V}) &\xrightarrow{\text{Res}_{S_n}^{S_n \times S_n}} \text{HH}_\bullet(\mathcal{S}^n(\mathcal{V}^{\text{opp}} \otimes \mathcal{V})) \xrightarrow{\text{Hom}_{\mathcal{V}}(-, -)} \\ &\longrightarrow \text{HH}(\mathcal{S}^n(\mathcal{H}perf \mathbb{k})) \xrightarrow{(-) \otimes \dots \otimes (-)} \text{HH}_\bullet(\mathcal{H}perf \mathbb{k}). \end{aligned}$$

Consider the diagram

$$(6.40) \quad \begin{array}{ccc} \text{HH}_\bullet(\mathcal{V}^{\text{opp}}) \otimes \text{HH}_\bullet(\mathcal{V}) & \xrightarrow{\psi_n \otimes \psi_n} & \text{HH}_\bullet(\mathcal{S}^n \mathcal{V}^{\text{opp}}) \otimes \text{HH}_\bullet(\mathcal{S}^n \mathcal{V}) \\ \downarrow K & & \downarrow K \\ \text{HH}_\bullet(\mathcal{V}^{\text{opp}} \otimes \mathcal{V}) & \xrightarrow{\psi_n} & \text{HH}_\bullet(\mathcal{S}^n(\mathcal{V}^{\text{opp}} \otimes \mathcal{V})) \\ \downarrow \text{Hom}_{\mathcal{V}}(-, -) & & \downarrow \mathcal{S}^n \text{Hom}_{\mathcal{V}}(-, -) \\ \text{HH}_\bullet(\mathcal{H}perf \mathbb{k}) & \xrightarrow{\psi_n} & \text{HH}_\bullet(\mathcal{S}^n(\mathcal{H}perf \mathbb{k})) \\ \searrow & & \searrow \\ & & \text{HH}_\bullet(\mathcal{H}perf \mathbb{k}). \end{array}$$

If we start with $\alpha \otimes \beta \in \text{HH}_\bullet(\mathcal{V}^{\text{opp}}) \otimes \text{HH}_\bullet(\mathcal{V})$, then going around the upper right perimeter of the diagram produces $(\hat{n})(K(\psi_n(\alpha) \otimes \psi_n(\beta)))$, while going around the lower left perimeter produces the Euler pairing $\langle \alpha, \beta \rangle$, as per its definition in §5.12.

The triangle at the bottom of (6.40) commutes because $\text{HH}_\bullet(\mathcal{H}perf) \cong \mathbb{k}$, so it suffices to check that it commutes on $1 \in \mathbb{k}$, i.e. on the class $[\text{id}_{\mathbb{k}}]$, which is trivial. The middle square in (6.40) commutes because our construction of the map $\psi_n: \text{HC}_\bullet(\mathcal{A}) \rightarrow \text{HC}_\bullet(\mathcal{S}^n \mathcal{A})$ is functorial in \mathcal{A} .

It remains to show that the top square in (6.40) commutes up to the factor of n . For this, consider its refinement to the diagram

$$\begin{array}{ccccc} \text{HH}_\bullet(\mathcal{V}^{\text{opp}}) \otimes \text{HH}_\bullet(\mathcal{V}) & \xrightarrow{g_n \otimes g_n} & \text{HH}_\bullet((\mathcal{V}^{\text{opp}})^{\otimes n}; t) \otimes \text{HH}_\bullet(\mathcal{V}^{\otimes n}; t) & \xrightarrow{\xi_t \otimes \xi_t} & \text{HH}_\bullet(\mathcal{S}^n \mathcal{V}^{\text{opp}}) \otimes \text{HH}_\bullet(\mathcal{S}^n \mathcal{V}) \\ \downarrow K & & \downarrow K & & \downarrow K \\ & & \text{HH}_\bullet((\mathcal{V}^{\text{opp}})^{\otimes n} \otimes \mathcal{V}^{\otimes n}; t \times t) & \xrightarrow{\xi_{t \times t}} & \text{HH}_\bullet(\mathcal{S}^n \mathcal{V}^{\text{opp}} \otimes \mathcal{S}^n \mathcal{V}) \\ \downarrow \sim & & \downarrow \sim & & \downarrow \text{Res}_{S_n}^{S_n \times S_n} \\ \text{HH}_\bullet(\mathcal{V}^{\text{opp}} \otimes \mathcal{V}) & \xrightarrow{g_n} & \text{HH}_\bullet((\mathcal{V}^{\text{opp}} \otimes \mathcal{V})^{\otimes n}; t) & \xrightarrow{\xi_t} & \text{HH}_\bullet(\mathcal{S}^n(\mathcal{V}^{\text{opp}} \otimes \mathcal{V})), \end{array}$$

where t is the long cycle $(1 \dots n) \in S_n$.

The left square and the top right square commute by Lemmas 5.23 and 5.17, respectively. It remains to show that the bottom right square commutes up to the factor of n . For this, we apply Lemma 5.18 to the composition $\text{Res}_{S_n}^{S_n \times S_n} \circ \xi_{t \times t}$.

The set Q of the left cosets of S_n in $S_n \times S_n$ can be identified with S_n . The coset corresponding to each $\sigma \in Q$ has a representative $r_\sigma = \text{id} \times \sigma \in S_n \times S_n$. Moreover, for every $\sigma \in Q$ we have

$$(t \times t)(\text{id} \times \sigma) = (\text{id} \times t\sigma t^{-1})(t \times t).$$

Thus the action of $t \times t$ on Q is given by $(t \times t).\sigma = t\sigma t^{-1}$ and we can set $h_{t \times t, \sigma} = t \times t$ in the terminology of Definition 5.4. The fixed set $\text{Fix}(t \times t)$ of this action is the subset $\{\text{id}, t, t^2, \dots, t^{n-1}\} \subset Q$. The representatives of these fixed cosets are the elements

$$\text{id}, \text{id} \times t, \text{id} \times t^2, \dots, \text{id} \times t^{n-1} \in S_n \times S_n.$$

These give isomorphisms (5.25) on twisted Hochschild complexes

$$(\text{id} \times t^i): \text{HC}_\bullet((\mathcal{V}^{\text{opp}})^{\otimes n} \otimes \mathcal{V}^{\otimes n}; t \times t) \longrightarrow \text{HC}_\bullet((\mathcal{V}^{\text{opp}})^{\otimes n} \otimes \mathcal{V}^{\otimes n}; t \times t)$$

whose induced maps on the Hochschild homology are the identity maps.

Therefore, applying Lemma 5.18 we get that on the Hochschild homology

$$\text{Res}_{S_n}^{S_n \times S_n} \circ \xi_{t \times t} = \sum_{i=0}^{n-1} \xi_t \circ (1 \times t^i)^{-1} = \sum_{i=0}^{n-1} \xi_t = n\xi_t,$$

as desired. \square

6.6. Injectivity of π . Having proved for $\mathbf{A}_\bullet(n) \in \underline{H}_{\text{HH}_\bullet(\mathcal{V})}$ the commutation and the Heisenberg relations, we obtain by Theorem 6.6 the desired Hochschild homology categorification map. It is injective for the same tautological reason as its numerical Grothendieck group counterpart in [42]:

Proposition 6.22. *The decategorification map*

$$\pi: \underline{H}_{\text{HH}_\bullet(\mathcal{V})} \rightarrow \text{AHH}_{\mathbf{H}_\mathcal{V}}$$

constructed in §6.3-6.5 is injective.

Proof. The 2-functor $\Phi_{\mathcal{V}}$ of [42, Theorem 7.30] sends each object $n \in \mathbb{Z}$ of $\mathbf{H}_\mathcal{V}$ to the object $\mathcal{S}^n \mathcal{V}$ of $\mathbf{F}_\mathcal{V}$. Since for $n < 0$ these are zero categories, $\Phi_{\mathcal{V}}$ sends all $\text{Hom}_{\mathbf{H}_\mathcal{V}}(n, n-m)$ to zero for $m > n$. In particular, it kills $\text{Hom}_{\mathbf{H}_\mathcal{V}}(0, -n)$ for $n > 0$. Let I_- be the left ideal in $\underline{H}_{\text{HH}_\bullet(\mathcal{V})}$ generated by $a_\alpha(-n)$ for $n > 0$ and any $\alpha \in \text{HH}_\bullet(\mathcal{V})$. By construction, π maps each $a_\alpha(-n)$ to $\mathbf{A}_\alpha(-n) \in \text{HH}_\bullet(\text{Hom}_{\mathbf{H}_\mathcal{V}}(0, -n))$. It follows that the image of I_- under the \mathbb{k} -algebra homomorphism

$$(6.41) \quad \underline{H}_{\text{HH}_\bullet(\mathcal{V})} \xrightarrow{(6.2)} \text{HH}_{\text{alg}}(\mathbf{H}_\mathcal{V}) \xrightarrow{\text{HH}_{\text{alg}}(\Phi_{\mathcal{V}})} \text{End} \left(\bigoplus_{n \geq 0} \text{HH}_\bullet(\mathcal{S}^n \mathcal{V}) \right)$$

kills $1 \in \mathbb{k} \cong \text{HH}_\bullet(\mathcal{S}^0 \mathcal{V})$. Since the quotient $\underline{H}_{\text{HH}_\bullet(\mathcal{V})}/I_-$ is the Fock space representation $F_{\text{HH}_\bullet(\mathcal{V})}$, the action of $\underline{H}_{\text{HH}_\bullet(\mathcal{V})}$ on $1 \in \mathbb{k} \cong \text{HH}_\bullet(\mathcal{S}^0 \mathcal{V})$ induces a map of $\underline{H}_{\text{HH}_\bullet(\mathcal{V})}$ representations

$$(6.42) \quad \phi: F_{\text{HH}_\bullet(\mathcal{V})} \rightarrow \bigoplus_{n \geq 0} \text{HH}_\bullet(\mathcal{S}^n \mathcal{V}).$$

The map ϕ is non-zero since (6.41) is a unital algebra homomorphism, so $1 \in \underline{H}_{\text{HH}_\bullet(\mathcal{V})}$ acts as the identity map. By irreducibility of the Fock space representation, ϕ is injective. By faithfulness of the Fock space representation, the morphism (6.41) is injective, and hence so is the morphism (6.2). Since (6.2) is the composition (6.7), it follows that π is also injective. \square

7. NONCOMMUTATIVE GENERALISED GROJNOWSKI-NAKAJIMA ACTION

In this section, we use the Hochschild homology decategorification constructed in §6 to prove:

Theorem 7.1. *Let \mathcal{V} be a smooth and proper DG category over an algebraically closed field \mathbb{k} of characteristic 0. Let χ be the Euler pairing on the Hochschild homology $\text{HH}_\bullet(\mathcal{V})$.*

For each $\alpha \in \text{HH}_\bullet(\mathcal{V})$ and $n > 0$, define operators $A_\alpha(-n)$ and $A_\alpha(n)$ on $\bigoplus_{n=0}^\infty \text{HH}_\bullet(\mathcal{S}^n \mathcal{V})$ by

$$(7.1) \quad A_\alpha(-n): \text{HH}_\bullet(\mathcal{S}^{N+n} \mathcal{V}) \xrightarrow{\text{Res}_{\mathcal{S}_N \times \mathcal{S}_n}^{\mathcal{S}_{N+n}}} \text{HH}_\bullet(\mathcal{S}^N \mathcal{V} \otimes \mathcal{S}^n \mathcal{V}) \cong \text{HH}_\bullet(\mathcal{S}^N \mathcal{V}) \otimes \text{HH}_\bullet(\mathcal{S}^n \mathcal{V}) \xrightarrow{\langle \psi_n(\alpha), - \rangle} \text{HH}_\bullet(\mathcal{S}^N \mathcal{V}),$$

$$(7.2) \quad A_\alpha(n): \text{HH}_\bullet(\mathcal{S}^N \mathcal{V}) \xrightarrow{(-) \otimes \psi_n(\alpha)} \text{HH}_\bullet(\mathcal{S}^N \mathcal{V}) \otimes \text{HH}_\bullet(\mathcal{S}^n \mathcal{V}) \cong \text{HH}_\bullet(\mathcal{S}^N \mathcal{V} \otimes \mathcal{S}^n \mathcal{V}) \xrightarrow{\text{Ind}_{\mathcal{S}_N \times \mathcal{S}_n}^{\mathcal{S}_{N+n}}} \text{HH}_\bullet(\mathcal{S}^{N+n} \mathcal{V}).$$

These operators satisfy

$$(7.3) \quad A_\alpha(m)A_\beta(n) - (-1)^{\deg(\alpha)\deg(\beta)}A_\beta(n)A_\alpha(m) = 0 \quad m, n > 0 \text{ or } m, n < 0,$$

$$(7.4) \quad A_\alpha(-m)A_\beta(n) - (-1)^{\deg(\alpha)\deg(\beta)}A_\beta(n)A_\alpha(-m) = \delta_{m,n}m\langle\alpha, \beta\rangle_\chi, \quad m, n > 0$$

and thus define an action of the Heisenberg algebra $H_{HH_\bullet(\mathcal{V}),\chi}$ on $\bigoplus_{n=0}^\infty HH_\bullet(\mathcal{S}^n\mathcal{V})$. This action identifies $\bigoplus_{n=0}^\infty HH_\bullet(\mathcal{S}^n\mathcal{V})$ with the Fock space of $H_{HH_\bullet(\mathcal{V}),\chi}$.

Proof. In [42], we constructed for any smooth and proper \mathcal{V} its Heisenberg 2-category $\mathbf{H}_\mathcal{V}$, its 2-categorical Fock space $\mathbf{F}_\mathcal{V}$, and the 2-categorical action $\Phi_\mathcal{V}$ of $\mathbf{H}_\mathcal{V}$ on $\mathbf{F}_\mathcal{V}$. In §6.1, we defined the functor HH_{alg} of taking the Hochschild homology of a 2-category and flattening it into an algebra. We also demonstrated that applying it to $\Phi_\mathcal{V}$ yields an algebra homomorphism

$$HH_{alg}(\mathbf{H}_\mathcal{V}) \xrightarrow{(6.1)} \text{End}\left(\bigoplus_{n \geq 0} HH_\bullet(\mathcal{S}^n\mathcal{V})\right).$$

In §6.2 we constructed from the decategorification map π (see §6.3–6.5) an algebra homomorphism

$$(7.5) \quad \underline{H}_{HH_\bullet(\mathcal{V}),\chi} \xrightarrow{(6.7)} HH_{alg}(\mathbf{H}_\mathcal{V}).$$

The composition of (6.7) and (6.1) gives the action of $\underline{H}_{HH_\bullet(\mathcal{V}),\chi}$ on $\bigoplus_{N=0}^\infty HH_\bullet(\mathcal{S}^N\mathcal{V})$.

We need to show that this identifies $\bigoplus_{N=0}^\infty HH_\bullet(\mathcal{S}^N\mathcal{V})$ with the Fock space $F_{HH_\bullet(\mathcal{V})}$ of $\underline{H}_{HH_\bullet(\mathcal{V}),\chi}$. In the proof of Proposition 6.22, we constructed an injective map $\phi: F_{HH_\bullet(\mathcal{V})} \rightarrow \bigoplus_{n \geq 0} HH_\bullet(\mathcal{S}^n\mathcal{V})$ of $\underline{H}_{HH_\bullet(\mathcal{V}),\chi}$ -representations. By the decomposition (5.36), the dimensions of $F_{HH_\bullet(\mathcal{V})}$ and $\bigoplus_{n \geq 0} HH_\bullet(\mathcal{S}^n\mathcal{V})$ are equal. Since ϕ is injective, it must therefore also be an isomorphism.

It remains to show that for any $\alpha \in HH_\bullet(\mathcal{V})$ and $n > 0$ the images of the generators $a_\alpha(-n)$ and $a_\alpha(n)$ of $\underline{H}_{HH_\bullet(\mathcal{V}),\chi}$ under the composition of (6.1) and (6.7) are the operators $A_\alpha(-n)$ and $A_\alpha(n)$ defined in (7.1) and (7.2). By construction, these images are $\Phi_\mathcal{V}(\Xi_Q(\psi_n(\alpha)))$ and $\Phi_\mathcal{V}(\Xi_P(\psi_n(\alpha)))$. By [42, Lemma 8.4], the composition $\Phi_\mathcal{V} \circ \Xi_P$ is homotopy equivalent to the functor

$$(7.6) \quad \begin{array}{c} \mathcal{H}perf(\mathcal{S}^n\mathcal{V}) \\ \downarrow \text{unit} \\ \mathcal{DGFun}(\mathcal{H}perf(\mathcal{S}^N\mathcal{V}), \mathcal{H}perf(\mathcal{S}^n\mathcal{V}) \otimes \mathcal{H}perf(\mathcal{S}^N\mathcal{V})) \\ \downarrow \otimes_{\mathbb{k}} \circ (-) \\ \mathcal{DGFun}(\mathcal{H}perf(\mathcal{S}^N\mathcal{V}), \mathcal{H}perf(\mathcal{S}^n\mathcal{V} \otimes \mathcal{S}^N\mathcal{V})) \\ \downarrow \text{Ind}_{n,N}^{n+N} \circ (-) \\ \mathcal{DGFun}(\mathcal{H}perf(\mathcal{S}^N\mathcal{V}), \mathcal{H}perf(\mathcal{S}^{N+n}\mathcal{V})). \end{array}$$

By its definition in §5.8, it follows that the corresponding map

$$(7.7) \quad HH_\bullet(\mathcal{S}^n\mathcal{V}) \rightarrow \text{Hom}_{\mathbb{k}}(HH_\bullet(\mathcal{S}^N\mathcal{V}), HH_\bullet(\mathcal{S}^{N+n}\mathcal{V}))$$

sends $\psi_n(\alpha)$ to the operator $A_\alpha(n)$ defined in (7.2). By construction of $\Phi_\mathcal{V}$ in [42, §7], for any $E \in \mathcal{H}perf(\mathcal{S}^n\mathcal{V})$ the composition $\Phi_\mathcal{V} \circ \Xi_Q(E)$ is the right adjoint of $\Phi_\mathcal{V} \circ \Xi_P(E)$. It follows that $\Phi_\mathcal{V} \circ \Xi_Q$ sends $\psi_n(\alpha)$ to the operator $A_\alpha(-n)$ defined in (7.1). \square

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