Dynamic Programming

Dynamic Programming – The IDEA

- A technique for solving problems with overlapping sub – problems.

- These sub – problems arise from a recurrence relating a solution to a given problem with solutions to its smaller sub - problems of the same type.

- Ex: Fibonacci Number Sequence

Dynamic Programming – The IDEA

- Dynamic Programming suggests:
  - Solve each of these sub – problems only once
  - Store the results in a table
  - Use these results to obtain solutions to the original problem.

Dynamic Programming – An EXAMPLE

FIBONACCI NUMBERS

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

Recurrence Relation:

$$F(n) = F(n-1) + F(n-2) \text{ for } n >= 2$$

$$F(0) = 0, F(1) = 1$$

Dynamic Programming – An EXAMPLE

- If the recurrence is directly used to compute the n^{th} Fibonacci number, we would have to recompute the same values again and again.
- Ex: $F(6) = F(5) + F(4)$

  $F(5) = F(4) + F(3)$

  $F(4) = F(3) + F(2)$ ……

Dynamic Programming – An EXAMPLE

- Instead, we can simply fill elements of a one – dimensional array with the n + 1 consecutive values of F(n) by starting with the initial elements and using the equation to produce all other elements.

1

Computing a Binomial Co - Efficient

What is Binomial Co – Efficient?

- ***Binomial Co - efficient***, denoted C(n, *k*) is the number of combinations (subsets) of *k* elements from an n-element set (0 <= *k* <= *n*).
- The name comes from the participation of these numbers in the binomial formula:

$$(a+b)^n = C(n, 0)a^n + \ldots + C(n, k)a^{n-k}b^k + \cdots + C(n, n)b^n.$$

Binomial Co – Efficient

- ***Properties:***
  - $C(n, k) = C(n-1, k-1) + C(n-1, k)$ for $n > k > 0$
  - $C(n, 0) = C(n, n) = 1$

The nature of the recurrence is such that it can be solved by Dynamic Programming.

Binomial Co – Efficient

To do this, we record the values of the binomial coefficients in a table of $n + 1$ rows and $k + 1$ columns, numbered from 0 to n and from 0 to k, respectively

	0	1	2	...	$k-1$	k
0	1					
1	1	1				
2	1	2	1			
$\vdots$						
k	1					1
$\vdots$						
$n-1$	1				$C(n-1, k-1)$	$C(n-1, k)$
n	1					$C(n, k)$

Binomial Co – Efficient – The ALGORITHM

ALGORITHM *Binomial*(n, k)

```
//Computes C(n, k) by the dynamic programming algorithm
//Input: A pair of nonnegative integers n ≥ k ≥ 0
//Output: The value of C(n, k)
for i ← 0 to n do
    for j ← 0 to min(i, k) do
        if j = 0 or j = i
            C[i, j] ← 1
        else C[i, j] ← C[i − 1, j − 1] + C[i − 1, j]
return C[n, k]
```

Binomial Co – Efficient – The EFFICIENCY

$$A(n,k) = \sum_{i=1}^{k}\sum_{j=1}^{i-1} 1 + \sum_{i=k+1}^{n}\sum_{j=1}^{k} 1 = \sum_{i=1}^{k}(i-1) + \sum_{i=k+1}^{n} k$$

$$= \frac{(k-1)k}{2} + k(n-k) \in \Theta(nk).$$

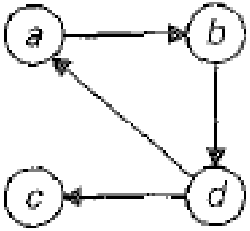
2

Warshall's Algorithm

What is Transitive Closure?

The transitive closure of a directed graph with n vertices can be defined as the n-by-n boolean matrix $T = \{t_{i,j}\}$, in which the element in the i^{th} row ($1 <= i <= n$) and the j^{th} column ($1 <= j <= n$) is 1 if there exists a nontrivial directed path (i.e., a directed path of a positive length) from the i^{th} vertex to the j^{th} vertex; otherwise, t_{ij} is 0.

What is Transitive Closure?



(a)

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

(b)

$$T = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

(c)

a) Directed Graph
b) Adjacency Matrix
c) Transitive Closure

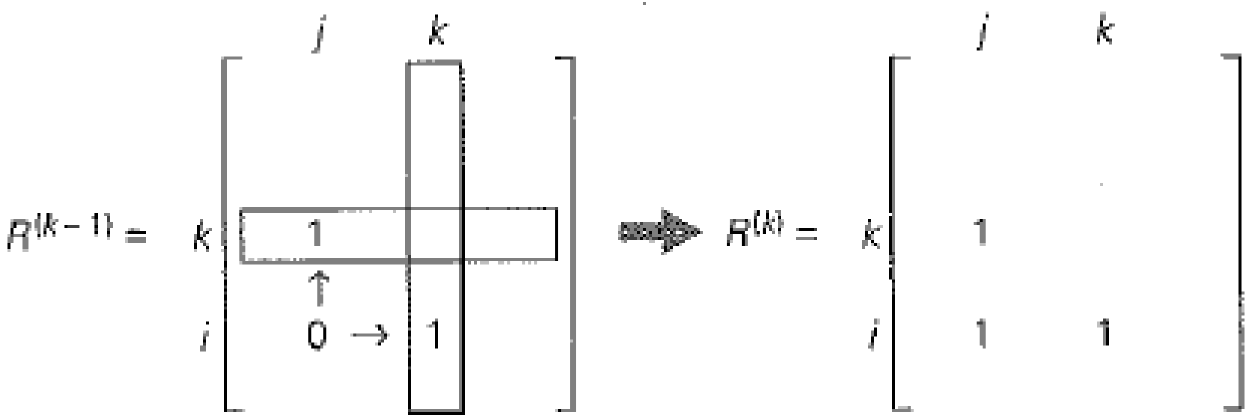
Warshall's Algorithm – The IDEA

- Warshall's Algorithm constructs the transitive closure of a digraph with n vertices through a series of n – by – n Boolean matrices.
- $R^{(0)}, R^{(1)}, \ldots, R^{(k-1)}, R^{(k)}, \ldots, R^{(n)}$
- The element $r_{ij}^{(k)}$ in the i^{th} row and j^{th} column of matrix $R^{(k)}$ ($k = 0, 1, \ldots, n$) is equal to 1 if and only if there exists a directed path (of a positive length) from the i^{th} vertex to the j^{th} vertex with each intermediate vertex, if any, numbered not higher than k.

Warshall's Algorithm – The IDEA

- Rules for generating elements of matrix $R^{(k)}$ from $R^{(k-1)}$:
    - If an element r_{ij} is 1 in $R^{(k)}$, it remains so in $R^{(k-1)}$.
    - If an element r_{ij} is 0 in $R^{(k-1)}$, it can be changed to 1 in $R^{(k)}$ if and only if the element in its row i and column k and the element in row k and column j are both 1's in $R^{(k-1)}$.

Warshall's Algorithm – The IDEA



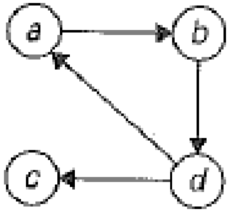
Warshall's Algorithm

ALGORITHM *Warshall*($A[1..n, 1..n]$)

//Implements Warshall's algorithm for computing the transitive closure
//Input: The adjacency matrix A of a digraph with n vertices
//Output: The transitive closure of the digraph

$R^{(0)} \leftarrow A$
for $k \leftarrow 1$ **to** n **do**
    for $i \leftarrow 1$ **to** n **do**
        for $j \leftarrow 1$ **to** n **do**
            $R^{(k)}[i, j] \leftarrow R^{(k-1)}[i, j]$ **or** $(R^{(k-1)}[i, k]$ **and** $R^{(k-1)}[k, j])$
return $R^{(n)}$

Warshall's Algorithm – An EXAMPLE



$$R^{(0)} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Ones reflect the existence of paths with no intermediate vertices ($R^{(0)}$ is just the adjacency matrix); boxed row and column are used for getting $R^{(1)}$.

$$R^{(1)} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & \mathbf{1} & 1 & 0 \end{bmatrix} \end{array}$$

Ones reflect the existence of paths with intermediate vertices numbered not higher than 1, i.e., just vertex *a* (note a new path from *d* to *b*); boxed row and column are used for getting $R^{(2)}$.

Warshall's Algorithm – An EXAMPLE

$$R^{(2)} = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 0 & \mathbf{1} \\ b & 0 & 0 & 0 & 1 \\ c & 0 & 0 & 0 & 0 \\ d & 1 & 1 & 1 & \mathbf{1} \end{array}$$

Ones reflect the existence of paths with intermediate vertices numbered not higher than 2, i.e., *a* and *b* (note two new paths); boxed row and column are used for getting $R^{(3)}$.

$$R^{(3)} = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 0 & 1 \\ b & 0 & 0 & 0 & 1 \\ c & 0 & 0 & 0 & 0 \\ d & 1 & 1 & 1 & 1 \end{array}$$

Ones reflect the existence of paths with intermediate vertices numbered not higher than 3, i.e., *a*, *b*, and *c* (no new paths); boxed row and column are used for getting $R^{(4)}$.

$$R^{(4)} = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & \mathbf{1} & 1 & \mathbf{1} & 1 \\ b & \mathbf{1} & \mathbf{1} & \mathbf{1} & 1 \\ c & 0 & 0 & 0 & 0 \\ d & 1 & 1 & 1 & 1 \end{array}$$

Ones reflect the existence of paths with intermediate vertices numbered not higher than 4, i.e., *a*, *b*, *c*, and *d* (note five new paths).

Warshall's Algorithm

Efficiency: Θ (n^3)

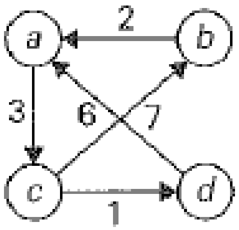
3

Floyd's Algorithm for All Pairs Shortest Path

All Pairs Shortest Path

- The *all-pairs shortest paths problem* asks to find the distances (the lengths of the shortest paths) from each vertex to all other vertices.
- It is convenient to record the lengths of shortest paths in an *n-by-n* matrix *D* called the *distance matrix*: the element d_{ij} in the i^{th} row and the j^{th} column of this matrix indicates the length of the shortest path from the i^{th} vertex to the j^{th} vertex ($1 <= i, j <= n$).

All pairs shortest path



(a)

$$W = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & \infty & 3 & \infty \\ 2 & 0 & \infty & \infty \\ \infty & 7 & 0 & 1 \\ 6 & \infty & \infty & 0 \end{bmatrix} \end{array}$$

(b)

$$D = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 10 & 3 & 4 \\ 2 & 0 & 5 & 6 \\ 7 & 7 & 0 & 1 \\ 6 & 16 & 9 & 0 \end{bmatrix} \end{array}$$

(c)

FIGURE 8.5 (a) Digraph. (b) Its weight matrix. (c) Its distance matrix.

Floyd's Algorithm – The IDEA

- Floyd's Algorithm computes the distance matrix of a weighted graph with n vertices through a series of n – by – n matrices:
- D(0), D(1), . . . , D(k-1), D(k), . . . , D(n)
- The element $d_{ij}^{(k)}$ in the i^{th} row and j^{th} column of matrix $D^{(k)}$ is equal to the length of the shortest path among all paths from the i^{th} vertex to the j^{th} vertex with each intermediate vertex, if any, numbered not higher than k.

Floyd's Algorithm – The IDEA

- $d_{ij}^{(k)}$ is equal to the length of the shortest path among all paths from i^{th} vertex v_i to j^{th} vertex v_j with their intermediate vertices not numbered higher than k.
- All such paths can be partitioned into two disjoint subsets: ones which do not use vk as the intermediate vertex and which do.

Floyd's Algorithm – The IDEA

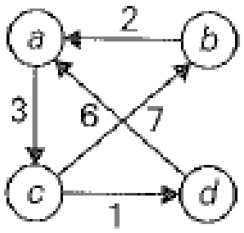
- Taking into account both these subsets lead to the following recurrence:

$$d_{ij}^{(k)} = \min\{d_{ij}^{(k-1)},\ d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\} \quad \text{for } k \geq 1, \qquad d_{ij}^{(0)} = w_{ij}.$$

Floyd's Algorithm

```
ALGORITHM  Floyd(W[1..n, 1..n])
    //Implements Floyd's algorithm for the all-pairs shortest-paths problem
    //Input: The weight matrix W of a graph with no negative-length cycle
    //Output: The distance matrix of the shortest paths' lengths
    D ← W //is not necessary if W can be overwritten
    for k ← 1 to n do
        for i ← 1 to n do
            for j ← 1 to n do
                D[i, j] ← min{D[i, j], D[i, k] + D[k, j]}
    return D
```

Floyd's Algorithm – An EXAMPLE



$$D^{(0)} = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & \infty & 3 & \infty \\ b & 2 & 0 & \infty & \infty \\ c & \infty & 7 & 0 & 1 \\ d & 6 & \infty & \infty & 0 \end{array}$$

Lengths of the shortest paths with no intermediate vertices ($D^{(0)}$ is simply the weight matrix).

$$D^{(1)} = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & \infty & 3 & \infty \\ b & 2 & 0 & \mathbf{5} & \infty \\ c & \infty & 7 & 0 & 1 \\ d & 6 & \infty & \mathbf{9} & 0 \end{array}$$

Lengths of the shortest paths with intermediate vertices numbered not higher than 1, i.e. just *a* (note two new shortest paths from *b* to *c* and from *d* to *c*).

Floyd's Algorithm – An EXAMPLE

$$D^{(2)} = \begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & \infty & 3 & \infty \\ 2 & 0 & 5 & \infty \\ \mathbf{9} & 7 & 0 & 1 \\ 6 & \infty & 9 & 0 \end{bmatrix} \end{array}$$

Lengths of the shortest paths with intermediate vertices numbered not higher than 2, i.e. a and *b* (note a new shortest path from *c* to *a*).

$$D^{(3)} = \begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & \mathbf{10} & 3 & \mathbf{4} \\ 2 & 0 & 5 & \mathbf{6} \\ 9 & 7 & 0 & 1 \\ 6 & \mathbf{16} & 9 & 0 \end{bmatrix} \end{array}$$

Lengths of the shortest paths with intermediate vertices numbered not higher than 3, i.e. a, *b*, and *c* (note four new shortest paths from *a* to *b*, from a to *d*, from *b* to *d*, and from *d* to *b*).

$$D^{(4)} = \begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 10 & 3 & 4 \\ 2 & 0 & 5 & 6 \\ \mathbf{7} & 7 & 0 & 1 \\ 6 & 16 & 9 & 0 \end{bmatrix} \end{array}$$

Lengths of the shortest paths with intermediate vertices numbered not higher than 4, i.e. a, *b*, *c*, and *d* (note a new shortest path from *c* to *a*).

Floyd's Algorithm

Efficiency: Θ (n^3)

4

Knapsack Problem and Memory Functions

Knapsack Problem

- Given '*n*' items of known weights $w_1, w_2, w_3, \ldots, w_n$, and values $v_1, v_2, v_3, \ldots, v_n$ and a knapsack of capacity W, find the most valuable subset of items that fit into the knapsack.
- We have already solved this problem using Exhaustive Search.
- Now, we shall see how to solve the same problem using Dynamic Programming.

Knapsack Problem

- Let us consider an instance defined by the first I items $1 <= i <= n$, with weights $w_1, w_2, w_3, . . . , w_i$, and values $v_1, v_2, v_3, . . . , v_i$ and a knapsack of capacity j, $1 <= j <= W$.
- Let $V[i, j]$ be the value of an optimal solution to this instance.
- We can divide all the subsets of the first i items that fit into knapsack of capacity j into two categories: those that do not include the i^{th} item and those that do.

Knapsack Problem

- Among the subsets that do not include the i^{th} item, the value of an optimal subset is, by definition, V[i - 1, j].
- Among the subsets that do include the i^{th} item (hence, j - w_i >= 0), an optimal subset is made up of this item and an optimal subset of the first i - 1 items that fit into the knapsack of capacity j- w_i. The value of such an optimal subset is v_i + V[i - 1, j - w_i].

Knapsack Problem – Recurrence and Initial Condition

$$V[i,\ j] = \begin{cases} \max\{V[i-1,\ j],\ v_i + V[i-1,\ j-w_i]\} & \text{if } j - w_i \geq 0 \\ V[i-1,\ j] & \text{if } j - w_i < 0. \end{cases}$$

$$V[0,\ j] = 0 \text{ for } j \geq 0 \text{ and } V[i,\ 0] = 0 \text{ for } i \geq 0.$$

Our goal is to find V[n, W]

Table For Solving Knapsack Problem

		0	$j-w_i$	j	W
	0	0	0	0	0
	$i-1$	0	$V[i-1, j-w_i]$	$V[i-1, j]$	
w_i, v_i	i	0		$V[i, j]$	
	n	0			goal

Table For Solving Knapsack Problem

- For i, j > 0, to compute the entry in the ith row and the jth column, V[i, j], we compute the maximum of the entry in the previous row and the same column and the sum of v_i and the entry in the previous row and w_i columns to the left.
- The table can be filled either row by row or column by column.

Knapsack Problem - Example

item	weight	value
1	2	$12
2	1	$10
3	3	$20
4	2	$15

capacity $W = 5$

		capacity j					
	i	0	1	2	3	4	5
	0	0	0	0	0	0	0
$w_1 = 2, v_1 = 12$	1	0	0	12	12	12	12
$w_2 = 1, v_2 = 10$	2	0	10	12	22	22	22
$w_3 = 3, v_3 = 20$	3	0	10	12	22	30	32
$w_4 = 2, v_4 = 15$	4	0	10	15	25	30	37

Knapsack Problem - Efficiency

- The time efficiency and space efficiency are both Θ (nW).
- The time needed to find the composition of an optimal solution is O(n + W).

Memory Functions

- Dynamic Programming deals with problems which satisfy a recurrence relation with overlapping sub problems.
- The top - down approach to solve such a recurrence leads to an algorithm that solves common sub problems more than once.
- The dynamic programming approach (bottom – up) solves these sub problems only once.
- But the disadvantage is that sometimes it solves sub problems which are not necessary for solving the problem for the given instance.
- The goal is to build a method which solves only those sub problems which are necessary.

Memory Functions

- This method solves a problem in the top – down manner but maintains a table similar to the one maintained by the bottom – up approach.
- Initially, all the table's entries are initialized with a special "null" symbol to indicate that they have not yet been calculated. Thereafter, whenever a new value needs to be calculated, the method checks the corresponding entry in the table first:
  - if this entry is not "null," it is simply retrieved from the table;
  - otherwise, it is computed by the recursive call whose result is then recorded in the table.

Memory Functions – The ALGORITHM

ALGORITHM *MFKnapsack*(i, j)

```
//Implements the memory function method for the knapsack problem
//Input: A nonnegative integer i indicating the number of the first
//        items being considered and a nonnegative integer j indicating
//        the knapsack's capacity
//Output: The value of an optimal feasible subset of the first i items
//Note: Uses as global variables input arrays Weights[1..n], Values[1..n],
//and table V[0..n, 0..W] whose entries are initialized with −1's except for
//row 0 and column 0 initialized with 0's
if V[i, j] < 0
    if j < Weights[i]
        value ← MFKnapsack(i − 1, j)
    else
        value ← max(MFKnapsack(i − 1, j),
                     Values[i] + MFKnapsack(i − 1, j − Weights[i]))
    V[i, j] ← value
return V[i, j]
```

Memory Functions - EXAMPLE

item	weight	value
1	2	\$12
2	1	\$10
3	3	\$20
4	2	\$15

capacity $W = 5$

		capacity j					
	i	0	1	2	3	4	5
	0	0	0	0	0	0	0
$w_1 = 2, v_1 = 12$	1	0	0	12	12	12	12
$w_2 = 1, v_2 = 10$	2	0	–	12	22	–	22
$w_3 = 3, v_3 = 20$	3	0	–	–	22	–	32
$w_4 = 2, v_4 = 15$	4	0	–	–	–	–	37

Knapsack Problem with Memory Functions – Efficiency

- Same as the efficiency of Bottom – Up algorithm except for a constant factor gain.