Design and Analysis of Algorithms (UE17CS251)

Unit II - Brute Force

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Brute Force:
A straightforward approach, usually directly based on the problem statement and definitions of the concepts involved.

Examples:
1. Searching for a key of a given value in a list
2. Computing n!
3. Computing a^n ($a > 0$, n a nonnegative integer)
4. Hacking a password by matching all possible passwords.

- “Force” by the computer in terms of effort, but simple in strategy to implement.
- Trial and Error method of trying out in some order.
- Exhaustive effort rather than employing intellectual strategies.
- Often one of the easiest way to solve it.

```
Algorithm SequentialSearch(A[0..n-1], K)
    //Outputs the index of the first element of A that
    // matches K or -1 if there are no matching elements.
    i ← 0
    while (i < n) and (A[i] ≠ K) do
        i ← i + 1
    if (i < n) return i
    return -1
```

Input size: n.

Basic Operation: `(i < n) and (A[i] ≠ K)`

$\mathbf{C_{worst}(n) = n+1}$

$\mathbf{C_{best}(n) = 1}$

Let **‘p’** be the probability of the search key present in the array.

$C_{avg}(n) = p(n + 1) / 2 + (1 - p)(n + 1)$

`Algorithm SequentialSearch(A[0..n-1], K)`

Input Size: **`n`**

Basic Operation : **`(i < n) and (A[i] ≠ K)`**

$\mathbf{C_{worst}(n)}$ = Count of the basic operation at the max

= $\mathbf{n + 1} \in \Theta(n)$

$\mathbf{C_{best}(n)}$ = $\mathbf{1} \in \Theta(1)$

$\mathbf{C_{avg}(n)}$ = from $\mathbf{(n+1)/2}$ to $\mathbf{(n+1)}$ depending on the probability of search key being present in the input array.

$\mathbf{C_{avg}(n)} \in \Theta(n)$

Brute-Force String Matching:
In an **n**-characters **text**, search for the first occurrence of **m**-character **pattern**.

There are **n-m+1** substrings of length **m** in the text of length **n**.
Find the first such substring which matches with the pattern.

Find i, the index of the leftmost character of the first matching substring in the text such that

$$\begin{array}{ccccccccc} t_0 & \cdots & t_i & \cdots & t_{i+j} & \cdots & t_{i+m-1} & \cdots & t_{n-1} \\ & & \updownarrow & & \updownarrow & & \updownarrow & & \\ & & p_0 & \cdots & p_j & \cdots & p_{m-1} & \text{pattern } P & \end{array}$$

$$t_i = p_0, \ldots, t_{i+j} = p_j, \ldots, t_{i+m-1} = p_{m-1}$$

```
NaiveStringMatch(T[0..n-1], P[0..m-1])
     for i ← 0 to n-m
          j ← 0
          while (j < m and T[i+j] = P[j])
        j ← j + 1
     if(j = m) return i
   return -1
```

Analysis:

Input Size: **n, m**

Basic Operation : **(T[i+j] = P[j])**

...

Input Size: **`n, m`**

Basic Operation : **`(T[i+j] = P[j])`**

$C_{best}(n)$ = **0**, when n < m

= **1**, when n = m

= **min(m, n-m+1)** when n ≥ m

= **m**, when n ≥ 2m-1 (i.e., n-m+1 is at least m)

= **max(0, min(m, n-m+1))**, in general

$C_{best}(n)$ **∈ Θ(m)** with n ≥ 2m-1

$C_{worst}(n)$ = (**n - m + 1) * m ∈ Θ(nm)**

$C_{avg}(n)$ **∈ O(nm)**

A brute force way of sorting:

Find an arrangement of elements of an array, which is sorted.

Strategy: For every possible arrangements of an array, check if it's sorted, and return the arrangement which is sorted.

```
boolean SortByExhaustiveSearch( A[0..n-1] )
//Sorts array A by trying out all possible
arrangements of elements of the array.
//Input: An array A of orderable elements by ≤.
//Output: Sorted array A by ≤.
for each permutation p[0..n-1] of array A
     if (isSorted( p[0..n-1] ))
          return p[0..n-1]
```

Write an algorithm to check if the array is sorted.

```
boolean isSorted( A[0..n-1] )
//Checks if the array A is sorted.
//Input: An array A of orderable elements by ≤.
//Output: Return TRUE if array is sorted.
//         FALSE otherwise.
...
```

Write an algorithm to check if the array is sorted.

```
boolean isSorted( A[0..n-1] )
//Checks if the array A is sorted.
//Input: An array A of orderable elements by ≤.
//Output: Return TRUE if array is sorted.
//        FALSE otherwise.
for i ← 0 to n-2
     if(A[i] > A[i+1]) //not in order
          return FALSE
return TRUE
```

Sort by fixing the problems while checking for sortedness.

```
SortByCheckingSortedness( A[0..n-1] )
//Sorts by Checking sortedness.
//Input: An array A of orderable elements by ≤.
//Output: Sorted array A.
for i ← 0 to n-2
     if(A[i] > A[i+1])
          Swap A[i] with A[i+1]
```

Does it sort?

Sort by fixing the problems while checking for sortedness.

```
SortByCheckingSortedness2( A[0..n-1] )
//Sorts by Checking sortedness.
//Input: An array A of orderable elements by ≤.
//Output: Sorted array A.
while (TRUE)
   for i ← 0 to n-2
      if(A[i] > A[i+1])
            Swap A[i] with A[i+1]
   if(isSorted( A[0..n-1] ))
      return
```

Does it sort and that too in a finite amount of time?

Sort by fixing the problems while checking for sortedness.

```
SortByCheckingSortedness3( A[0..n-1] )
//Sorts by Checking sortedness.
//Input: An array A of orderable elements by ≤.
//Output: Sorted array A.
for k ← 0 to n-2 //n-1 passes
   for i ← 0 to n-2 //n-1 consecutive pairs
      if(A[i] > A[i+1])
            Swap A[i] with A[i+1]
   if(isSorted( A[0..n-1] ))
      return
```

It should sort. Can it be improved?

$$A_0, \ldots, A_j \overset{?}{\leftrightarrow} A_{j+1}, \ldots, A_{n-i-1} \mid \underbrace{A_{n-i} \le \cdots \le A_{n-1}}_{\text{in their final positions}}$$

Algorithm BubbleSort(A[0..n-1])
//Sorts by Bubble Sort algorithm.
//Input: An array **A** of orderable elements by ≤.
//Output: Sorted array A.

```
for i ← 0 to n-2  //n-1 passes
   for j ← 0 to n-2-i  //last i elements are sorted
      if(A[j] > A[j+1])
            Swap A[j] with A[j+1]
return
```

Can it still be improved?

```
Algorithm BubbleSortImproved( A[0..n-1] )
//Sorts by an improved Bubble Sort algorithm.
//Input: An array A of orderable elements by ≤.
//Output: Sorted array A.
for i ← 0 to n-2  //n-1 passes
   anySwaps ← FALSE
   for j ← 0 to n-2-i  //last i elements are sorted
      if(A[j] > A[j+1])
            Swap A[j] with A[j+1]
            anySwaps ← TRUE
   if(anySwaps = FALSE)
      Break out of loop
```

```
Algorithm BubbleSort_Recursive(A[0..n-1])
//Sorts by an improved Bubble Sort algorithm.
//Input: An array A of orderable elements by ≤.
//Output: Sorted array A.
   anySwaps ← FALSE
   for i ← 0 to n-2
      if(A[i] > A[i+1])
            Swap A[i] with A[i+1]
            anySwaps ← TRUE
   if(anySwaps = TRUE)
      BubbleSort_Recursive(A[0..n-2])
```

5	1	12	-5	16	unsorted
5	1	12	-5	16	5 > 1, swap
1	5	12	-5	16	5 < 12, ok
1	5	12	-5	16	12 > -5, swap
1	5	-5	12	16	12 < 16, ok
1	5	-5	12	16	1 < 5, ok
1	5	-5	12	16	5 > -5, swap
1	-5	5	12	16	5 < 12, ok
1	-5	5	12	16	1 > -5, swap
-5	1	5	12	16	1 < 5, ok
-5	1	5	12	16	-5 < 1, ok
-5	1	5	12	16	sorted

Analysis of Bubble Sort:

```
Algorithm BubbleSort( A[0..n-1] )
//Sorts by Bubble Sort algorithm.
//Input: An array A of orderable elements by ≤.
//Output: Sorted array A.
for i ← 0 to n-2  //n-1 passes
    for j ← 0 to n-2-i  //last i elements are sorted
        if(A[j] > A[j+1])
            Swap A[j] with A[j+1]
return
```

`Algorithm BubbleSort( A[0..n-1] )`

Input Size: **`n`**

Basic Operation: (**`A[j] > a[j+1]`**)

$C_{worst}(n)$ = n * (n - 1) / 2 ∈ Θ(n^2)

$C_{best}(n)$ = n * (n - 1) / 2 ∈ Θ(n^2)

$$C(n) = \sum_{i=0}^{n-2} \sum_{j=0}^{n-2-i} 1 = \sum_{i=0}^{n-2} [(n-2-i) - 0 + 1]$$

$$= \sum_{i=0}^{n-2} (n-1-i) = \frac{(n-1)n}{2} \in \Theta(n^2).$$

Bubble Sort (improved version):

```
Algorithm BubbleSortImproved( A[0..n-1] )
```

//Sorts by an improved Bubble Sort algorithm.

//Input: An array **A** of orderable elements by ≤.

//Output: Sorted array A.

```
for i ← 0 to n-2  //n-1 passes
   anySwaps ← FALSE
   for j ← 0 to n-2-i  //last i elements are sorted
      if(A[j] > A[j+1])
            Swap A[j] with A[j+1]
            anySwaps ← TRUE
   if(anySwaps = FALSE)
      return
return
```

`Algorithm BubbleSortImproved( A[0..n-1] )`

Input Size: **`n`**

Basic Operation : (**`a[i] > a[i+1]`**)

$C_{worst}(n) = n * (n - 1) / 2 \in \Theta(n^2)$

$C_{best}(n) = (n - 1) \in \Theta(n)$

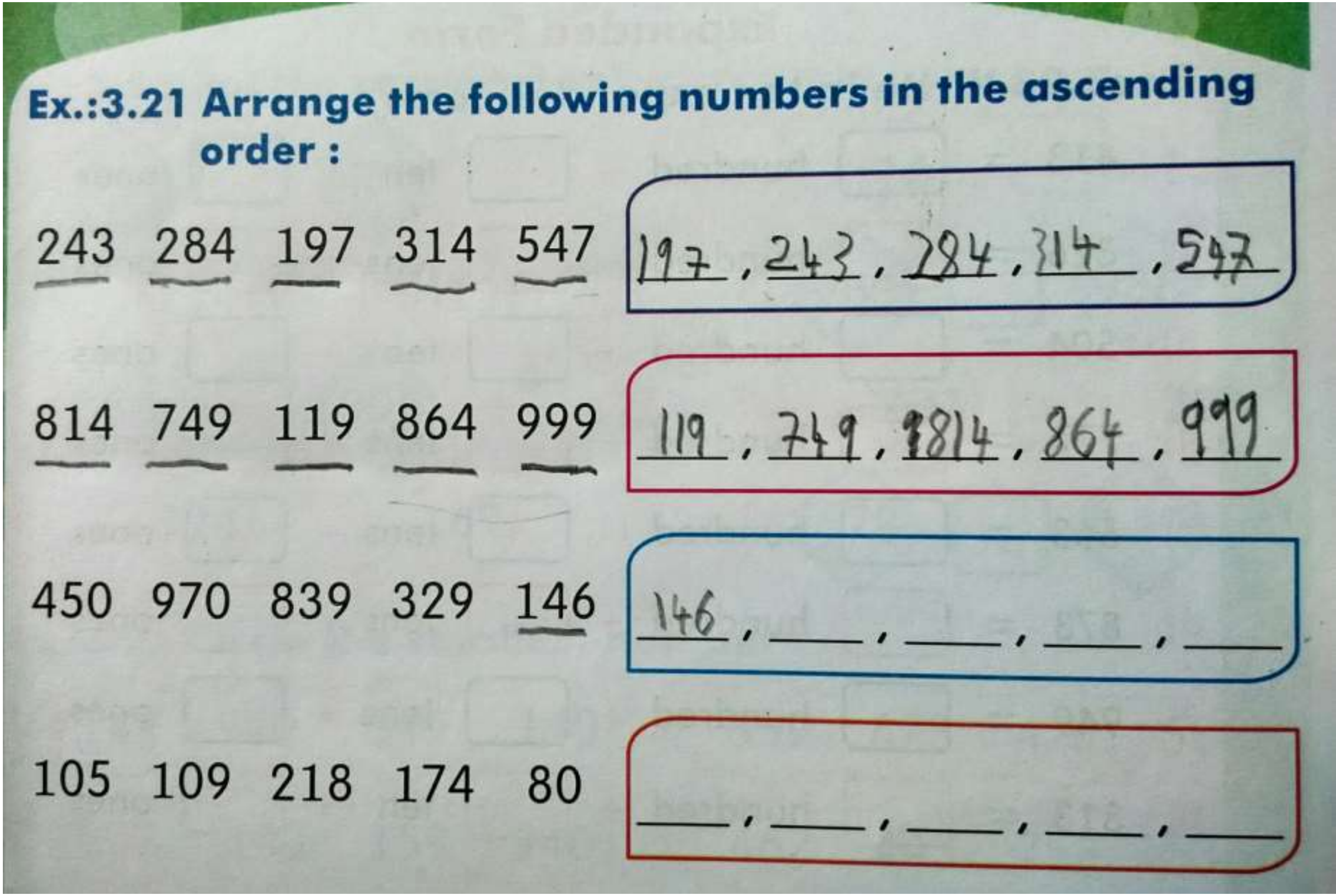
$C_{avg}(n) \in \Theta(?)$

Basic Operation : **`Swap`**

$C_{worst}(n) = n * (n - 1) / 2 \in \Theta(n^2)$

$C_{best}(n) = 0 \in \Theta(1)$

Yet another way of **sorting** by brute-force.

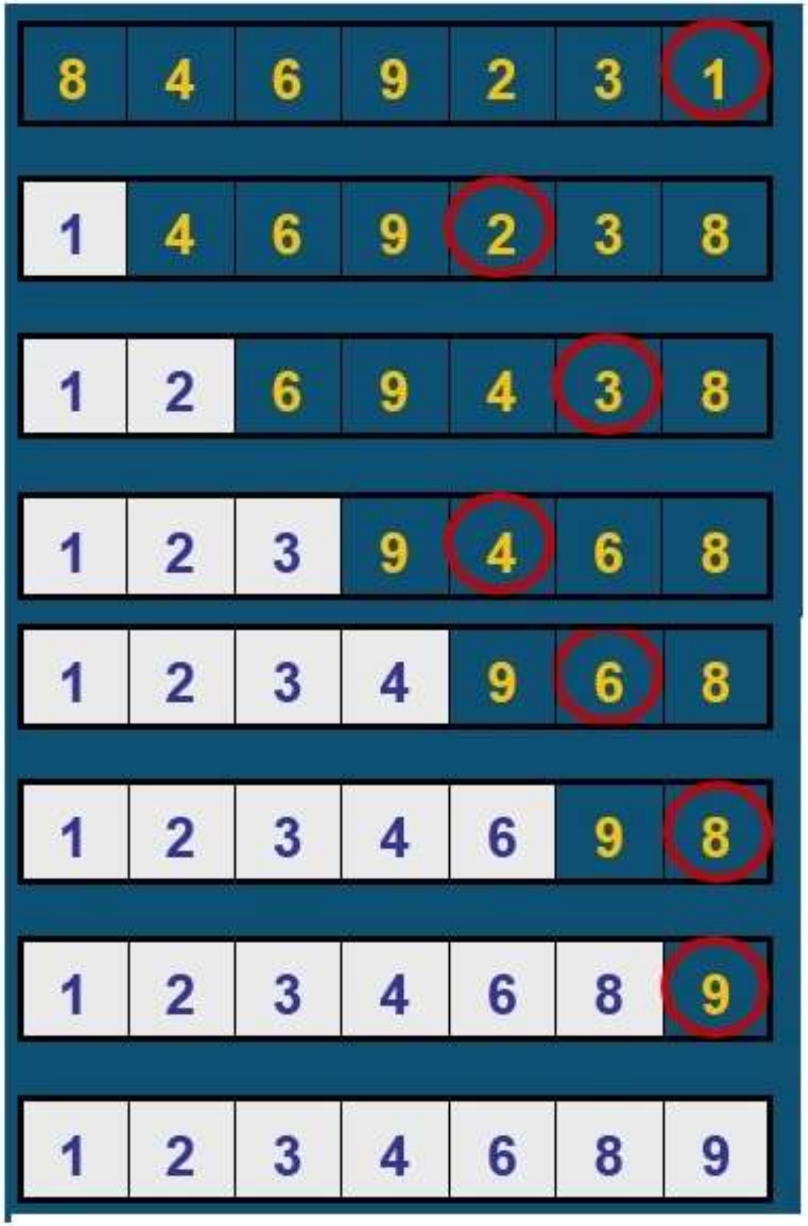


Ex.:3.21 Arrange the following numbers in the ascending order :

243 284 197 314 547	197 , 243 , 284 , 314 , 547
814 749 119 864 999	119 , 749 , 814 , 864 , 999
450 970 839 329 146	146 , ___ , ___ , ___ , ___
105 109 218 174 80	___ , ___ , ___ , ___ , ___

Selection Sort:

Example: 8 4 6 9 2 3 1



Selection Sort:

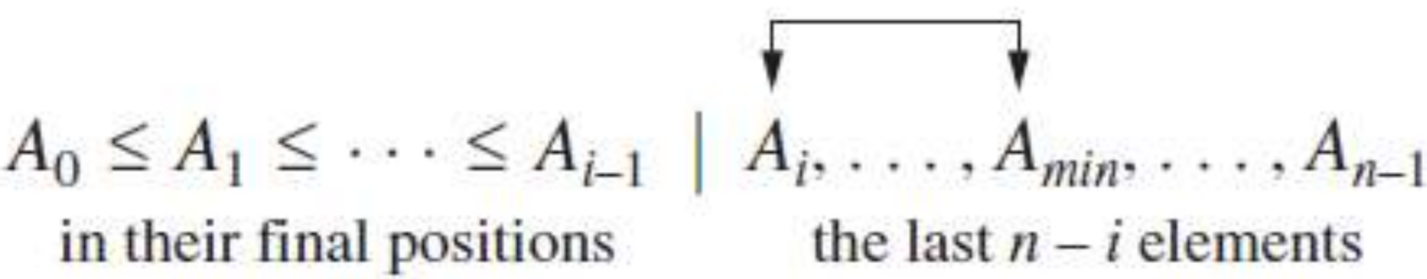
Find the smallest of the unsorted array and place it at the beginning of the unsorted array. Reduce the unsorted array by excluding the first one, which is already in its final position. Repeat sorting the unsorted array as long as there is only one element left in the unsorted array.

```
Algorithm SelectionSort_Recursive(A[0..n-1])
//Sorts a given array by Selection Sort.
//Input: An array A[0..n-1] of orderable elements.
//Output: Array A[0..n-1] sorted in ascending order.
   if(n ≤ 1) return
   min ← index of the smallest among A[0..n-1]
   Swap A[0] with A[min]
   SelectionSort_Recursive(A[1..n-1])
```

Selection Sort:

Find the smallest of the unsorted array and place it at the beginning of the unsorted array. Reduce the unsorted array by excluding the first one, which is already in its final position.

```
Algorithm SelectionSort(A[0..n-1])
//Sorts a given array by Selection Sort.
//Input: An array A[0..n-1] of orderable elements.
//Output: Array A[0..n-1] sorted in ascending order.
for i ← 0 to n-2
   min ← index of the smallest among A[i..n-1]
   Swap A[i] with A[min]
```



```
Algorithm SelectionSort(A[0..n-1])
//Sorts a given array by Selection Sort.
//Input: An array A[0..n-1] of orderable elements.
//Output: Array A[0..n-1] sorted in ascending order.
for i ← 0 to n-2
   min ← i
   for j ← i+1 to n-1
      if(A[j] < A[min]) min ← j
   Swap A[i] with A[min]
return A
```

$$A_0 \le A_1 \le \cdots \le A_{i-1} \;\Big|\; A_i, \ldots, A_{min}, \ldots, A_{n-1}$$

in their final positions　　　　the last $n - i$ elements

Analysis of Selection Sort:

Input Size: n (size of the list)

Basic Operation: Comparison (**`A[j] < A[min]`**)

$C_{best}(n) = C_{worst}(n) = C_{avg}(n)$

$$C(n) = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2} [(n-1) - (i+1) + 1] = \sum_{i=0}^{n-2} (n-1-i)$$

$$= \sum_{i=0}^{n-2} (n-1-i) = \frac{(n-1)n}{2} \in \Theta(n^2)$$

Polynomial Evaluation:

$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x^1 + a_0$ at $x = x_0$

```
Algorithm Polynomial(n, x0, a[0..n])
//Output: Value of the polynomial at x = x0.
p ← 0.0
for i ← n downto 0 do
      power ← 1
      for  j ← 1 to i do  //compute x^i
            power ← power * x0
      p ← p + a_i * power
return p
```

Input Size: **n**

Basic Operation : **power ← power * x_0**

C(n) = $n * (n + 1) / 2 \in \Theta(n^2)$

```
p ← 0.0
for i ← n downto 0 do
      power ← 1
      for  j ← 1 to i do  //compute x^i
            power ← power * x
      p ← p + a[i] * power
return p
-----------------------------------
p ← a[0]
power ← 1
for i ← 1 to n do
   power ← power * x
      p ← p + a[i] * power
return p
```

Polynomial Evaluation:

$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x^1 + a_0$ at $x = x_0$

```
Algorithm Polynomial2(n, x0, a[0..n])
//Output: Value of the polynomial at x = x0.
p ← a0
power ← 1
for i ← 1 to n do
      power ← power * x //compute x^i
      p ← p + ai * power
return p
```

Input Size: **n**

Basic Operation : **power ← power * x_0**

C(n) = **n ∈ Θ(n)**

Exhaustive Search:

A brute force solution to a problem involving search for an element with a special property, usually among combinatorial objects such as permutations, combinations, or subsets of a set.

Method:

- Generate a list of all potential solutions to the problem in a systematic manner.
- Evaluate potential solutions one by one, disqualifying infeasible ones and, for an optimization problem, keeping track of the best one found so far.
- When search ends, announce the solution(s) found.

Travelling Salesman Problem:

1. Given *n* cities and distances between each pair of cities, find the shortest tour that passes through all other cities and returns to the origin city.
2. In case of a weighted complete graph, it's about finding the shortest *Hamiltonian circuit*.

Eg: Driving time between some 10 cities of India (Cost Matrix).

000000	110189	050573	020948	109480	034435	028433	074836	091767	068406
109006	000000	079663	118195	079397	143304	083593	045792	037923	068146
051516	080265	000000	070149	121881	083636	044745	043763	042416	067450
021557	119539	069838	000000	095820	042397	037471	084186	111032	077756
110053	081231	121373	095977	000000	134475	085826	087690	100264	054016
034488	144238	082769	041728	134042	000000	062482	108885	123963	102455
028473	084770	045153	037117	085732	062772	000000	049417	078006	042987
075056	046162	044536	084245	086579	109354	049641	000000	031151	038399
092933	037994	042414	111566	099497	125053	078960	031010	000000	068113
068718	068844	068336	077907	055357	103016	043305	038648	068634	000000

Travelling Salesman Problem:

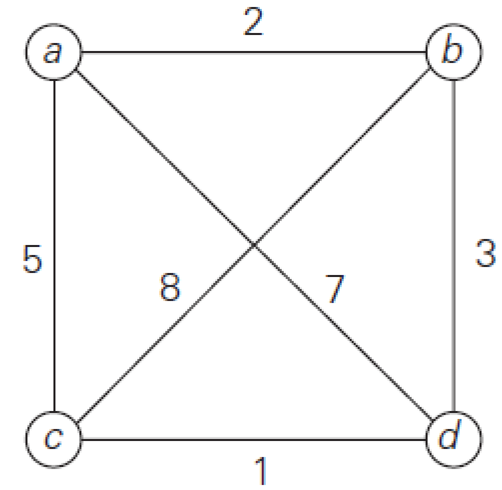
1. Bengaluru
2. New Delhi
3. Mumbai
4. Chennai
5. Kolkata
6. Kochi
7. Hyderabad
8. Bhopal
9. Udaipur
10. Raipur

Travelling Salesman Problem:

1. Bengaluru
2. Mumbai
3. Bhopal
4. Udaipur
5. New Delhi
6. Kolkata
7. Raipur
8. Hyderabad
9. Chennai
10. Kochi
11. Bengaluru

Shortest round trip takes **454201** sec.

Travelling Salesman Problem:



Tour	Length	
$a \to b \to c \to d \to a$	$l = 2 + 8 + 1 + 7 = 18$	
$a \to b \to d \to c \to a$	$l = 2 + 3 + 1 + 5 = 11$	optimal
$a \to c \to b \to d \to a$	$l = 5 + 8 + 3 + 7 = 23$	
$a \to c \to d \to b \to a$	$l = 5 + 1 + 3 + 2 = 11$	optimal
$a \to d \to b \to c \to a$	$l = 7 + 3 + 8 + 5 = 23$	
$a \to d \to c \to b \to a$	$l = 7 + 1 + 8 + 2 = 18$	

```
Algorithm Travelling Salesperson Problem
    mincost ← Infinity
    for each permutation of (n - 1) cities
        cost ← 0
        for each edge in the Hamiltonian circuit
            cost ← cost + cost of the edge
        if (cost < mincost)
            mincost ← cost
    return mincost
```

Input Size: **n**

Basic Operation : **addition of cost of an edge**

C(n) $= n * (n - 1)! = n! \in \Theta(n!)$

ALGORITHM **TravellingSalesmanProblem**

//Input: n x n adjacency matrix A. Assumed n > 1.

//Output: Min Cost Hamiltonian circuit.

//getPermutation(P[]) returns true with next permutation in lexicographic

// order, if it exists. Returns false otherwise.

```
mincost ← INFINITY
Permutation[1..n-1] ← [1, 2, 3, …, n-1] //1st permn.
do
  cost ← A[0, Permutation[1]] //1st edge of the circuit
  for i ← 1 to n-2
    cost ← cost + A[Permutation[i], Permutation[i+1]]
  cost ← cost + A[Permutation[n-1], 0] //last edge
  if (cost < mincost) mincost ← cost
while(getNextPermutation(Permutation[1..n-1]))
return mincost
```

Sorting by exhaustive search:

Find an arrangement of elements of an array, which is sorted.

Strategy: For every possible arrangements of an array, check if it’s sorted, and return the arrangement which is sorted.

```
boolean SortByBruteForce( a[0..n-1] )
//Sorts array a by trying out all possible arrangements of elements of the array.
//Input: An array a of orderable elements by ≤.
//Output: Sorted array a by ≤.
for each permutation p[0..n-1] of array a
     if (isSorted( p[0..n-1] ))
          return p[0..n-1]
```

The Assignment Problem:

Each one of *n* people are assigned one of *n* jobs, exactly one person per job. The cost of assigning person *i* to job *j* is C[i, j]. Find an assignment that minimizes the total cost.

$$
\begin{array}{c}
\\ \mathbf{P0} \\ \mathbf{P1} \\ \mathbf{P2} \\ \mathbf{P3}
\end{array}
\begin{array}{c}
\begin{array}{cccc}
\textbf{Job 0} & \textbf{Job 1} & \textbf{Job 2} & \textbf{Job 3}
\end{array} \\
\left[\begin{array}{cccc}
9 & 2 & 7 & 8 \\
6 & 4 & 3 & 7 \\
5 & 8 & 1 & 8 \\
7 & 6 & 9 & 4
\end{array}\right]
\end{array}
$$

Strategy:

1. Generate all legitimate assignments,
2. Compute cost of each assignment, and
3. Select the cheapest one.

Cost Matrix

$$
\begin{array}{c|cccc}
 & \text{Job 0} & \text{Job 1} & \text{Job 2} & \text{Job 3} \\
\hline
P0 & 9 & 2 & 7 & 8 \\
P1 & 6 & 4 & 3 & 7 \\
P2 & 5 & 8 & 1 & 8 \\
P3 & 7 & 6 & 9 & 4
\end{array}
$$

Assignment (col.#s)	**Total Cost**	
1, 2, 3, 4	**9+4+1+4 =**	**18**
1, 2, 4, 3	**9+4+8+9 =**	**30**
1, 3, 2, 4	**9+3+8+4 =**	**24**
1, 3, 4, 2	**9+3+8+6 =**	**26**

```
Algorithm AssignmentProblem(C[0..n-1, 0..n-1])
//Input: n x n cost matrix C.
//Output: Cost of the cheapest assignment.
mincost ← Infinity
for each permutation of n jobs
    cost ← 0, person ← 0
    for each job in the assignment
        cost ← cost + C[person, job]
        person ← person + 1
    if (cost < mincost) mincost = cost
return mincost
```

Input Size: `n`, **Basic Operation :** `cost + C[person, job]`

C(n) = n * n! ∈ Θ(n*n!)

```
Algorithm AssignmentProblem(C[0..n-1, 0..n-1])
//Input: n x n cost matrix C.
//Output: Cost of the cheapest assignment.
mincost ← Infinity
Permn[0..n-1] ← [0,1,2,…,n-1] //1st permn.
do
   cost ← 0
   for i ← 0 to n-1
      cost ← cost + C[i, Permn[i]]
   if (cost < mincost) mincost = cost
while(getNextPermn(Permn[0..n-1]))
return mincost
```

Input Size: `n`, **Basic Operation :** `cost + C[i,Permn[i]]`

C(n) = n * n! ∈ Θ(n*n!)

$$C(n) = n * n! \in \Theta(n*n!)$$

Knapsack Problem:

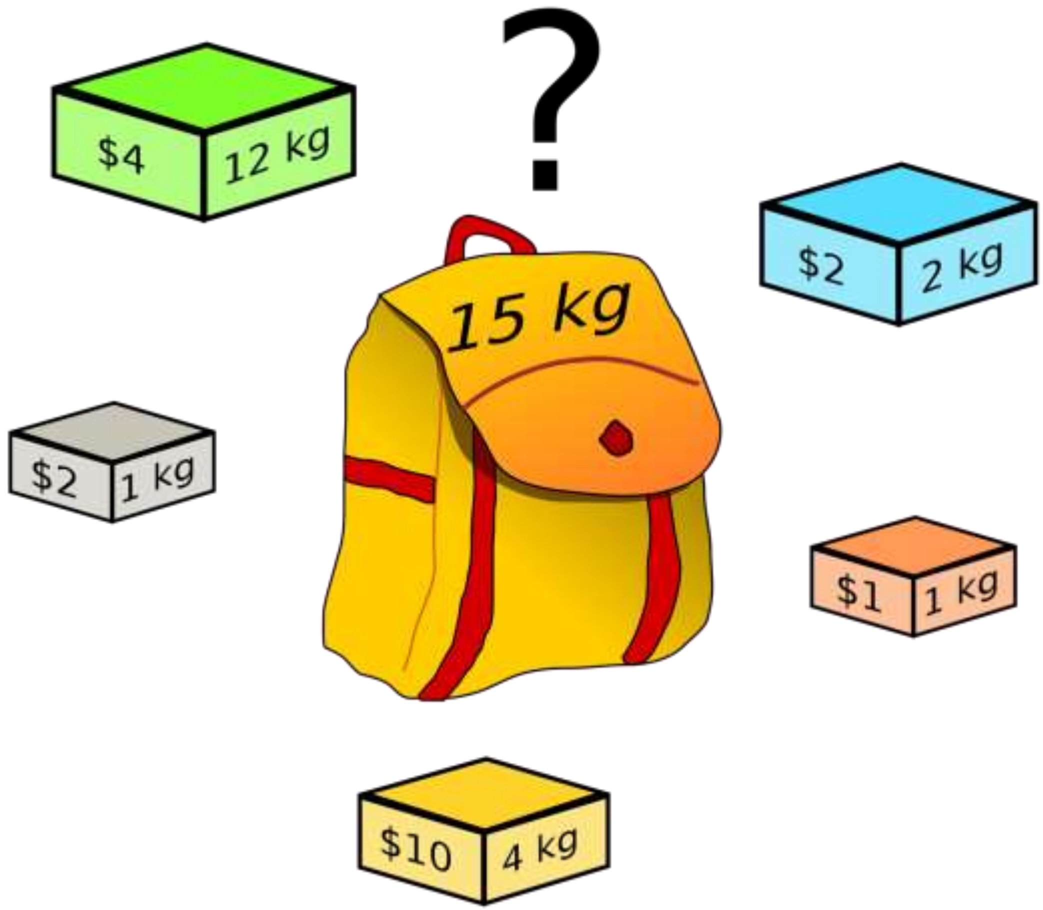
Given n items:

weights: w_1 w_2 … w_n

values: v_1 v_2 … v_n

a knapsack of capacity W

Find most valuable subset of the items that fit into the knapsack.



Knapsack Problem:

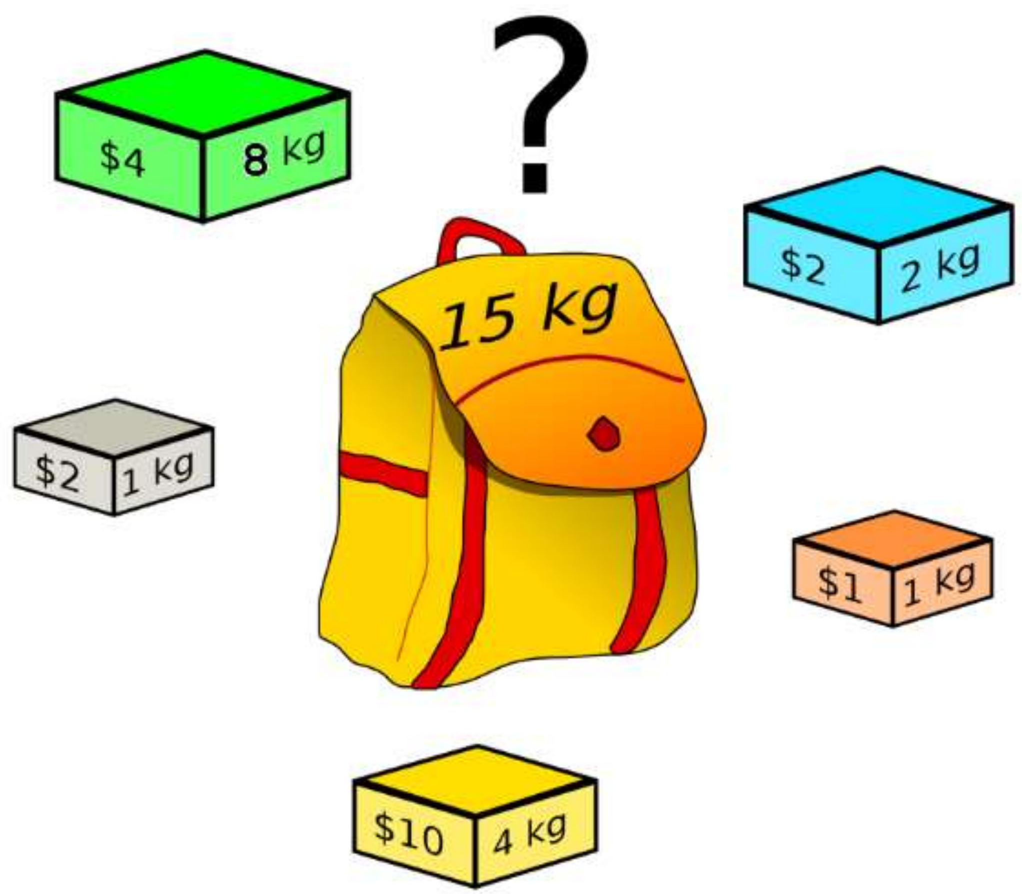
Given n items:

weights: $w_1 \quad w_2 \ \ldots \ w_n$

values: $v_1 \qquad v_2 \ \ldots$

a knapsack of capacity W

Find most valuable subset of the items that fit into the knapsack.



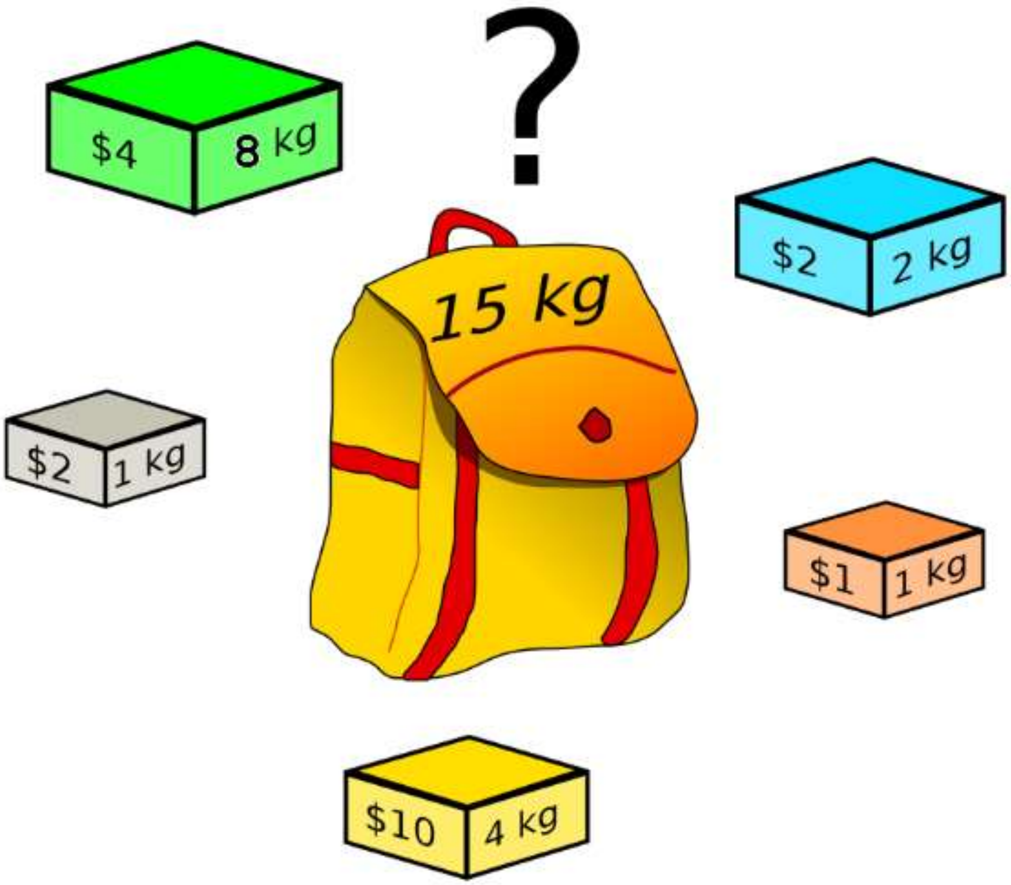
Ans: {B,O,Y,W} 8kg, $15

What if the green object weighs 8 kg instead of 12 kg?

Knapsack Problem:

What if the green object weighs 8 kg instead of 12 kg?

Ans:
{G,B,Y,W}
15kg, $18



Example:

Knapsack capacity W=16

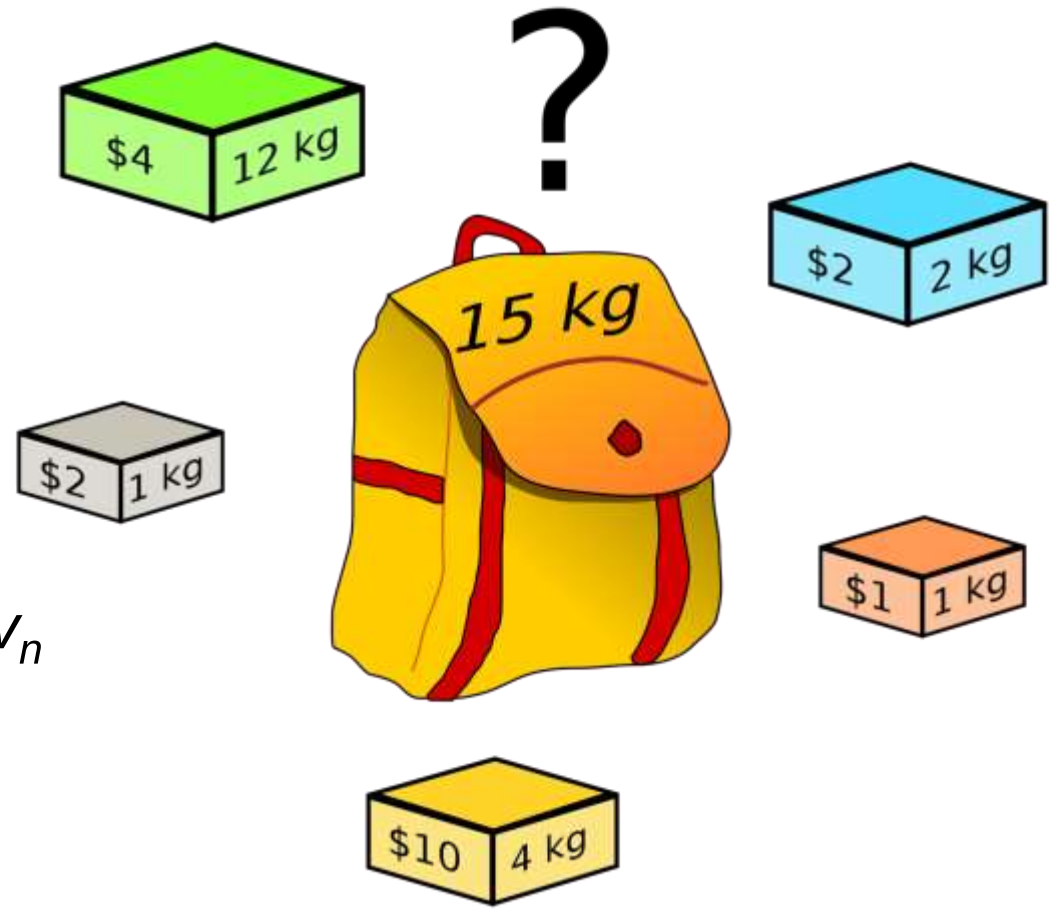
item	weight	value
1	2	$20
2	5	$30
3	10	$50
4	5	$10

{2,3} with value **$80** is optimal.

C(n) $\in$ $\mathbf{\Omega(2^n)}$

Subset	**Total weight**	**Total value**
{}	**0**	**$0**
{1}	**2**	**$20**
{2}	**5**	**$30**
{3}	**10**	**$50**
{4}	**5**	**$10**
{1,2}	**7**	**$50**
{1,3}	**12**	**$70**
{1,4}	**7**	**$30**
{2,3}	**15**	**$80**
{2,4}	**10**	**$40**
{3,4}	**15**	**$60**
{1,2,3}	**17**	**-**

Write a brute-force algorithm to find an optimal solution for the 0/1 Knapsack Problem.



Given n items:

weights: $w_1 \quad w_2 \; \ldots \; w_n$

values: $v_1 \quad v_2 \; \ldots \; v_n$

a knapsack of capacity W

Find most valuable subset of the items that fit into the knapsack.

Multiplication of Large Integers:

Consider the problem of multiplying two (large) n-digit integers represented by arrays of their digits such as:

A = 12345678901357986429   B = 87654321284820912836

Brute-Force Strategy:

$a_1 \ a_2 \ \ldots \ a_n * b_1 \ b_2 \ \ldots \ b_n$

$d_{10} d_{11} d_{12} \ \ldots \ d_{1n}$

$d_{20} \ d_{21} d_{22} \ \ldots \ d_{2n}$

… … … … … … …

$d_{n0} d_{n1} d_{n2} \ \ldots \ d_{nn}$

```
   2135 * 4014
   8540
  2135+
 0000++
8540+++
-------
8569890
```

Write a brute-force algorithm to multiply two arbitrarily large (of n digits) integers.

```
            12345678 * 32165487
            86419746
           98765424+
          49382712++
         61728390+++
        74074068++++
       12345678+++++
      24691356++++++
     37037034+++++++
_______________________________
     397104745215186
```

Basic Operation:
single-digit multiplication

C(n) = n^2 one-digit multiplications

C(n) $\in$ $\boldsymbol{\Theta(n^2)}$

May the **Force** be with you!

</ Brute Force >