Assignment 3
12.13.1.15

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Question 15: Consider the experiment of throwing a die. If a multiple of 3 comes up, throw the die again and if any other number comes, toss a coin. Find the conditional probability of the event 'the coin shows a tail', given that 'at least one die shows a 3'.

Solution: Let us define some random variables.

Event	**Description**
$X_1 \in \{1, 2, 3, 4, 5, 6\}$	Number obtained from a die throw
$X_2 = 1$	Coin shows tails after experiment
$X_2 = 0$	Coin shows heads after experiment
$X_3 = 1$	At least one die shows a 3
$X_3 = 0$	No die shows a 3
$X_4 = 1$	Getting tails from a coin toss
$X_4 = 0$	Getting heads from a coin toss

TABLE I
RANDOM VARIABLES

When the die rolls a multiple of 3, recursion is generated. For $k \in \{1, 2, 3, 4, 5, 6\}$,

$$\Pr(X_{n+1} = k) = \sum_{i \in \{3,6\}} \Pr(X_1 = i) \times \Pr(X_n = k) \tag{1}$$

$$\Pr(X_{n+1} = k) = \frac{2}{6} \times \Pr(X_n = k) \tag{2}$$

$$\implies \Pr(X_n = k) = \left(\frac{1}{3}\right)^{n-1} \times \Pr(X_1 = k) \tag{3}$$

There is a recursion on the first occurrence of 3.

$$\Pr(Y = n) = \Pr(X_1 = 6) \times \Pr(Y = n - 1) \tag{4}$$

$$\implies \Pr(Y = n) = \left(\frac{1}{6}\right)^{n-1} \times \Pr(X_1 = 3) \tag{5}$$

For probability that 3 occurs at least once,

$$\sum_{n=1}^{\infty} \Pr(Y = n) = \sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^{n-1} \times \Pr(X_1 = 3) \tag{6}$$

$$\sum_{n=1}^{\infty} \Pr(Y = n) = \frac{\frac{1}{6}}{1 - \frac{1}{6}} \times \frac{1}{6} \tag{7}$$

$$\sum_{n=1}^{\infty} \Pr(Y = n) = \frac{1}{5} \tag{8}$$

Required conditional probability is,

$$\Pr\left((Z_1 = 1) \,|\, \sum_{n=1}^{\infty} \Pr(Y = n)\right) \tag{9}$$

$$= \frac{\sum_{n=1}^{\infty} \Pr((Y = n)(Z_1 = 1))}{\sum_{n=1}^{\infty} \Pr(Y = n)} \tag{10}$$

Consider first occurrence of 3 on n^{th} throw and m further throws.

$$\sum_{n=1}^{\infty} \Pr((Y = n)(Z_1 = 1)) \tag{11}$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \Pr(Y = n) \sum_{i \in \{1,2,4,5\}} \Pr(X_m = i) \times \Pr(Z_2 = 1) \tag{12}$$

$$= \sum_{n=1}^{\infty} \Pr(Y = n) \sum_{m=1}^{\infty} 4 \times \left(\frac{1}{3}\right)^{m-1} \left(\frac{1}{6}\right) \times \Pr(Z_2 = 1) \tag{13}$$

$$= \frac{1}{5} \times 4 \times \frac{3}{2} \times \frac{1}{6} \times \frac{1}{2} \tag{14}$$

$$\implies \sum_{n=1}^{\infty} \Pr(Y = n) \Pr(Z_1 = 1) = \frac{1}{10} \tag{15}$$

$$\therefore \Pr\left((Z_1 = 1) \,|\, \sum_{n=1}^{\infty} \Pr\left(Y = n\right)\right) = \frac{1}{10} \div \frac{1}{5} \tag{16}$$

$$= \frac{1}{2} \tag{17}$$