A1110 Assignment 3
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Question: Consider the experiment of throwing a die.

- If a multiple of 3 comes up, throw the die again
- If any other number comes, toss a coin.

Find the conditional probability of the event 'the coin shows a tail', given that 'at least one die shows a 3'.
Solution:

CONTENTS

I. MARKOV CHAIN STATES

- Let us construct a Markov chain X_n with discrete time n.
- The states e_0 and e_1 describe the outcomes from the latest dice throw.
- The states e_2 and e_3 describe the outcomes of the latest coin toss.

II. STATES

Let $Y \in \{1,2,3,4,5,6\}$ denote the number obtained from a die throw.

i	**State** (e_i)
0	$Y = 3$ OR $Y = 6$
1	$\sum (Y = k)\,; k \in \{1,2,4,5\}$
2	Obtaining heads from coin toss
3	Obtaining tails from coin toss

TABLE 1: States in Markov Chain

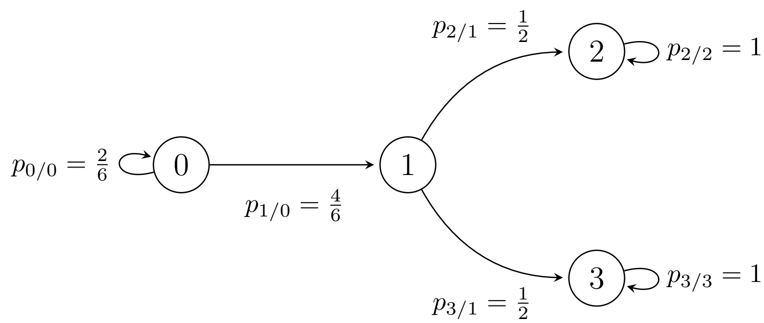

Fig. 1: Graph of Markov Chain

III. MARKOV GRAPH

IV. DESCRIPTION OF GRAPHS AND STATES

$p_{j/i}$ is the probability of moving from state e_i to e_j.

$$p_{j/i} = \Pr\left(\frac{X_{n+1} = j}{X_n = i}\right) \tag{1}$$

Absorbing States States e_2 and e_3 are absorbing states because $p_{2/2} = 1$ and $p_{3/3} = 1$. Once entered, they cannot be left.
Transient States States e_0 and e_1 are transient states, because they lead to other states which have no return path, For example, $p_{2/1} = \frac{1}{2}$ but $p_{1/2} = 0$. Their probability will reduce to 0 eventually.

V. TRANSITION PROBABILITY

State Probabilities in Next Step: State Probabilities in Next Step Let $P_i^{(n)}$ be the probability of state i at time n. Then the state vector is,

$$\mathbf{Q_n} = \begin{pmatrix} P_0^{(n)} \\ P_1^{(n)} \\ P_2^{(n)} \\ P_3^{(n)} \end{pmatrix} \tag{2}$$

The probabilities after one step in time are

$$P_0^{(n+1)} = \frac{2}{6} \times P_0^{(n)} \tag{3}$$

$$P_1^{(n+1)} = \frac{4}{6} \times P_0^{(n)} \tag{4}$$

$$P_2^{(n+1)} = \frac{1}{2} \times P_1^{(n)} + 1 \times P_2^{(n)} \tag{5}$$

$$P_3^{(n+1)} = \frac{1}{2} \times P_1^{(n)} + 1 \times P_3^{(n)} \tag{6}$$

VI. Transition Probability Matrix

The previous equations can be summarized as

$$\mathbf{Q_{n+1}} = \mathbf{P}\mathbf{Q_n} \tag{7}$$

Where $\mathbf{P}$ is the transition probability matrix. Its elements are values of $p_{i/j}$

$$\mathbf{P} = \begin{pmatrix} 2/6 & 0 & 0 & 0 \\ 4/6 & 0 & 0 & 0 \\ 0 & 1/2 & 1 & 0 \\ 0 & 1/2 & 0 & 1 \end{pmatrix} \tag{8}$$

VII. Initial Condition and Limiting State

Initial Condition The given condition is that '3 occurs at least once'. Let the first occurrence of 3 be the initial state $\mathbf{Q_0}$.

$$\mathbf{Q_0} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{9}$$

Using equation (7), further states can be generated.

$$\mathbf{Q_1} = \mathbf{P}\mathbf{Q_0} = \begin{pmatrix} \frac{2}{6} \\ \frac{4}{6} \\ 0 \\ 0 \end{pmatrix} \tag{10}$$

$$\mathbf{Q_2} = \mathbf{P}\mathbf{Q_1} = \mathbf{P}^2\mathbf{Q_0} = \begin{pmatrix} \frac{4}{9} \\ \frac{8}{9} \\ \frac{5}{24} \\ \frac{5}{12} \end{pmatrix} \tag{11}$$

$$\vdots \tag{12}$$

$$\mathbf{Q_n} = \mathbf{P}^n\mathbf{Q_0} \tag{13}$$

VIII. Eigen Decomposition

Now to find the eigen values, let λ be the eigen value for the transition matrix P

$$\implies | \, P - \lambda I_4 \, | = 0 \tag{14}$$

$$\begin{pmatrix} 2/6 - \lambda & 0 & 0 & 0 \\ 4/6 & -\lambda & 0 & 0 \\ 0 & 1/2 & 1-\lambda & 0 \\ 0 & 1/2 & 0 & 1-\lambda \end{pmatrix} = 0 \tag{15}$$

$$\implies (\frac{2}{6} - \lambda)(-\lambda)(1-\lambda^2) = 0 \tag{16}$$

$$\implies \lambda = \frac{2}{6}, 0, 1, 1 \tag{17}$$

$\therefore$ eigen values

Now, finding the eigen vectors, say X

$$\mathbf{X} = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \tag{18}$$

$$\implies (P - \lambda I)X = 0 \tag{19}$$

1) $\lambda = \frac{2}{6}$

$$\mathbf{X} = \begin{pmatrix} \frac{-2}{3} \\ \frac{-4}{3} \\ 1 \\ 1 \end{pmatrix} \tag{20}$$

2) $\lambda = 0$

$$\mathbf{X} = \begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix} \tag{21}$$

3) $\lambda = 1$

$$\mathbf{X} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{22}$$

$$\tag{23}$$

Now, forming a eigenvector matrix

$$\mathbf{S} = \begin{pmatrix} -2/3 & 0 & 0 & 0 \\ -4/3 & -2 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \tag{24}$$

Now, we can write

$$P = SDS^{-1} \tag{25}$$

Where D is eigenvalue matrix,

$$\mathbf{D} = \begin{pmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{26}$$

To, find

$$\lim_{n\to\infty} Q_n \tag{27}$$

We know,

$$Q_n = P^n Q_0 \tag{28}$$

and

$$P^n = (SDS^{-1})(SDS^{-1})\dots(SDS^{-1}) \tag{29}$$

$$\implies P^n = SD^nS^{-1} \tag{30}$$

$$\implies \lim_{n\to\infty} P^n = \lim_{n\to\infty} SD^nS^{-1} \tag{31}$$

Now,

$$\lim_{\mathbf{n}\to\infty} \mathbf{D^n} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{32}$$

$$\implies \lim_{\mathbf{n}\to\infty} \mathbf{SD^nS^{-1}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{P^n} \tag{33}$$

$$\implies \mathbf{Q_n} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{Q_0} \tag{34}$$

$$\implies \mathbf{Q_n} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{35}$$

Required Conditional Probability Probability of the coin showing tails, given that at least one die shows a 3,

$$\lim_{n\to\infty} P_3^{(n)} = 0 \tag{36}$$