A1110 Assignment 3
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Question: Consider the experiment of throwing a die.

- If a multiple of 3 comes up, throw the die again
- If any other number comes, toss a coin.

Find the conditional probability of the event 'the coin shows a tail', given that 'at least one die shows a 3'.
Solution:

CONTENTS

I. MARKOV CHAIN STATES

- Let us construct a Markov chain X_t with discrete time t.
- The states e_0, e_1 and e_2 describe the outcomes from the latest dice throw.
- The states e_3 and e_4 describe the outcomes of the latest coin toss.

II. STATES

Let $Y \in \{1,2,3,4,5,6\}$ denote the number obtained from a die throw.

i	**State** (e_i)
0	$Y = 3$
1	$Y = 6$
2	$\sum (Y = k)\,; k \in \{1,2,4,5\}$
3	Obtaining heads from coin toss
4	Obtaining tails from coin toss

TABLE 1: States in Markov Chain

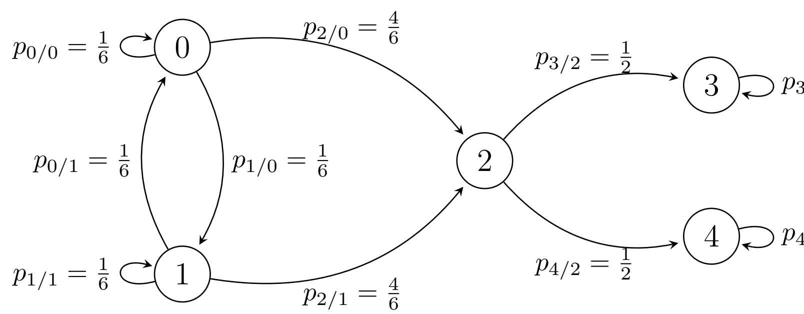

Fig. 1: Graph of Markov Chain

III. MARKOV GRAPH

IV. DESCRIPTION OF GRAPHS AND STATES

$p_{j/i}$ is the probability of moving from state e_i to e_j.

$$p_{j/i} = \Pr\left(\frac{X_{t+1} = j}{X_t = i}\right) \tag{1}$$

Absorbing States States e_3 and e_4 are absorbing states because $p_{3/3} = 1$ and $p_{4/4} = 1$. Once entered, they cannot be left.
Transient States States e_0, e_1 and e_2 are transient states, because they lead to other states which have no return path, For example, $p_{3/2} = \frac{1}{2}$ but $p_{2/3} = 0$. Their probability will reduce to 0 eventually.

V. TRANSITION PROBABILITY

State Probabilities in Next Step: State Probabilities in Next Step Let $P_i^{(t)}$ be the probability of state i at time t. Then the state vector is,

$$\mathbf{Q_t} = \begin{pmatrix} P_0^{(t)} \\ P_1^{(t)} \\ P_2^{(t)} \\ P_3^{(t)} \\ P_4^{(t)} \end{pmatrix} \tag{2}$$

The probabilities after one step in time are

$$P_0^{(t+1)} = \frac{1}{6} \times P_0^{(t)} + \frac{1}{6} \times P_1^{(t)} \tag{3}$$
$$P_1^{(t+1)} = \frac{1}{6} \times P_0^{(t)} + \frac{1}{6} \times P_1^{(t)} \tag{4}$$
$$P_2^{(t+1)} = \frac{4}{6} \times P_0^{(t)} + \frac{4}{6} \times P_1^{(t)} \tag{5}$$
$$P_3^{(t+1)} = \frac{1}{2} \times P_2^{(t)} + 1 \times P_3^{(t)} \tag{6}$$
$$P_4^{(t+1)} = \frac{1}{2} \times P_2^{(t)} + 1 \times P_4^{(t)} \tag{7}$$

VI. Transition Probability Matrix

The previous equations can be summarized as

$$\mathbf{Q_{t+1}} = \mathbf{P}\mathbf{Q_t} \tag{8}$$

Where $\mathbf{P}$ is the transition probability matrix. Its elements are values of $p_{i/j}$

$$\mathbf{P} = \begin{pmatrix} 1/6 & 1/6 & 0 & 0 & 0 \\ 1/6 & 1/6 & 0 & 0 & 0 \\ 4/6 & 4/6 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1 & 0 \\ 0 & 0 & 1/2 & 0 & 1 \end{pmatrix} \tag{9}$$

VII. Initial Condition and Limiting State

Initial Condition The given condition is that '3 occurs at least once'. Let the first occurrence of 3 be the initial state $\mathbf{Q_0}$.

$$\mathbf{Q_0} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{10}$$

Using equation (8), further states can be generated.

$$\mathbf{Q_1} = \mathbf{P}\mathbf{Q_0} = \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{4}{6} \\ 0 \\ 0 \end{pmatrix} \tag{11}$$

$$\mathbf{Q_2} = \mathbf{P}\mathbf{Q_1} = \mathbf{P}^2\mathbf{Q_0} = \begin{pmatrix} \frac{1}{18} \\ \frac{1}{18} \\ \frac{2}{9} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} \tag{12}$$

$$\vdots \tag{13}$$
$$\mathbf{Q_t} = \mathbf{P}^t\mathbf{Q_0} \tag{14}$$

Limiting Probabilities of States These can can be approximately calculated by taking large value of t,

$$\lim_{t\to\infty} \mathbf{Q_t} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0.5 \\ 0.5 \end{pmatrix} \tag{15}$$

Required Conditional Probability Probability of the coin showing tails, given that at least one die shows a 3,

$$\lim_{t\to\infty} P_4^{(t)} = 0.5 \tag{16}$$