Section 0.1 Exercise 1

Sol.

(a) $\{1, 2, 3, 4, 5\}$.
(b) $\{-5, -4, -3, -2, -1, 0, 1, 2\}$.
(c) $\{1, 2, 3, 4, 5\}$.
(d) $\{2, 3, 4\}$.

Section 0.1 Exercise 2

Sol.

(a) $(1/2, 1)$.
(b) $[-1, 7]$.

Section 0.1 Exercise 4

Sol.

We prove that

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C).$$

Let $x \in A \setminus (B \cup C)$. Then $x \in A$ but $x \notin (B \cup C)$, so that $x \notin B$ and $x \notin C$. It follows that $x \in A \setminus B$ and $x \in A \setminus C$, i.e., $x \in (A \setminus B) \cap (A \setminus C)$. This shows that

$$A \setminus (B \cup C) \subset (A \setminus B) \cap (A \setminus C).$$

For the reverse inclusion, suppose that $x \in (A \setminus B) \cap (A \setminus C)$, so that $x \in A$ but $x \notin B$ and $x \notin C$. Then $x \notin B \cup C$, and we get that $x \in A \setminus (B \cup C)$. This shows that

$$(A \setminus B) \cap (A \setminus C) \subset A \setminus (B \cup C)$$

and the two sets have to be equal, namely

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$$

as required.

Section 0.1 Exercise 6

Sol.

Suppose that $A \subset B$. We prove that $(C \setminus B) \subset (C \setminus A)$.

Let $x \in (C \setminus B)$. Then $x \in C$, but $x \notin B$. In particular, we have that $x \notin A$, since otherwise x would be in B, because $A \subset B$. This shows that $x \in (C \setminus A)$, as required.

It is not true in general that $A \subset B$ implies $(C \setminus A) \subset (C \setminus B)$, as can be seen taking $C = \{1, 2, 3\}$, $A = \{1\}$ and $B = \{1, 2\}$. Then $C \setminus B = \{3\}$ but $C \setminus A = \{2, 3\}$.

Section 0.1 Exercise 10

Sol.

(a) $\{0\}$.
(b) $\mathbb{R}$.
(c) $[0, 1]$.
(d) $(-1, 3)$.

Section 0.1 Exercise 12

Sol.

(a) $\mathbb{R} \setminus \{0\}$.
(b) $\mathbb{R} \setminus [1, 2]$.

Section 0.2 Exercise 13

Sol.

Let $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = 2n - 1$.
The image of f is the set of all positive odd integers, $B := \{1, 3, 5, \dots\}$.
The function f is injective; indeed, if $f(n) = f(m)$, then $2n - 1 = 2m - 1$ which gives $m = n$.
The function f is not surjective; for example 2 is not in the image of f.
The domain of f^{-1} is B and for $n \in B$, we have $f^{-1}(n) = (n + 1)/2$.

Section 0.2 Exercise 17

Sol.

Consider $A = \{1\}$, $B = \{1, 2\}$, $C = \{1, 2\}$, and define $f : A \to B$ by $f(1) = 1$, and $g : B \to C$ by $g(1) = 1$ and $g(2) = 1$. Then g is not injective, but $g \circ f : A \to C$ is trivially injective.

Section 0.4 Exercise 37

Sol.

Suppose that there exists a bijection $f : A \to \mathcal{P}(A)$. As in the hint, define $C := \{x \in A : x \notin f(x)\}$. Then C is a subset of A, so $C \in \mathcal{P}(A)$. Since f is surjective, we must have that $C = f(a)$ for some $a \in A$. There are two cases: $a \in C$ or $a \notin C$.

Suppose that $a \in C$. Then by definition of C, we must have $a \notin f(a)$. But $f(a) = C$, so that $a \notin C$, a contradiction.
Suppose that $a \notin C$. But again by definition of C, this implies that $a \in f(a)$, i.e. $a \in C$, again a contradiction.
Since both cases lead to a contradiction, we deduce that there is no bijection $f : A \to \mathcal{P}(A)$.

Section 0.5 Exercise 40

Sol.

Suppose that $x, y \geq 0$. Then

$$0 \leq (\sqrt{x} - \sqrt{y})^2 = x + y - 2\sqrt{x}\sqrt{y}.$$

Rearranging, this gives

$$\sqrt{xy} = \sqrt{x}\sqrt{y} \leq \frac{x+y}{2}$$

as required.

Section 0.5 Exercise 41

Sol.

Suppose that $0 < a < b$. Multiplying by a on both sides of the inequality $0 < a$ and using Axiom A11, we get $0 < a^2$. Note that we used the fact that $0x = 0$ for all $x \in \mathbb{R}$, as proved in class.
On the other hand, multiplying by a on both sides of the inequality $a < b$ gives $a^2 < ab$, again using Axiom A11. But $ab < b^2$, as can be seen multiplying the inequality $a < b$ by b. This gives $a^2 < b^2$, as required.

The other inequality is proved similarly.