Writing Problem

Sol.

Let S be a nonempty set of real numbers that is bounded from above, and let $x := \sup S$. We have to show that either x belongs to S or x is an accumulation point of S.

Suppose that x does not belong to S, and let I be an interval containing x. We want to show that I contains infinitely many elements of S.
Let $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subset I$. Note that $x - \epsilon$ is not an upper bound for S, so there exists $s_1 \in S$ such that $x - \epsilon < s_1$. Note that $s_1 \leq x$ since $x = \sup S$, and in fact we have $s_1 < x$ since $x \notin S$. In particular, we have that $s_1 \in I$. Moreover, we have that s_1 is not an upper bound for S, so there exists $s_2 \in S$ such that $s_1 < s_2$. Again, we have $s_2 < x$, and $s_2 \in I$. Continuing in this way, we obtain a sequence $s_1 < s_2 < s_3 < \cdots < x$ such that $s_n \in I \cap S$ for each n. This shows that I contains infinitely many elements of S. Finally, since I was an arbitrary interval containing x, we get that x is an accumulation point of S, as required.

The proof is the same if S is assumed to be bounded from below and $x = \inf S$.