Writing Problem

Sol.

Let $f : D \to \mathbb{R}$ and let x_0 be an accumulation point of D. Assume that L_1 and L_2 are limits of f at x_0. We are asked to prove that $L_1 = L_2$ directly from the definition of limit.

Suppose for a contradiction that $L_1 \neq L_2$, and let $\epsilon := |L_1 - L_2|/2 > 0$. Then there exists $\delta_1 > 0$ such that for all $x \in D$ with $0 < |x - x_0| < \delta_1$, we have

$$|f(x) - L_1| < \epsilon.$$

Similarly, there exists $\delta_2 > 0$ such that for all $x \in D$ with $0 < |x - x_0| < \delta_2$, we have

$$|f(x) - L_2| < \epsilon.$$

Let $\delta := \min(\delta_1, \delta_2) > 0$, and take $x \in D$ with $0 < |x - x_0| < \delta$. Then we have

$$|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |f(x) - L_1| + |f(x) - L_2| < 2\epsilon.$$

This gives $|L_1 - L_2| < |L_1 - L_2|$, a contradiction.

Hence L_1 must be equal to L_2, as required.