NAME:

Math 331
March 22, 2018
W. Smith

QUIZ 2

(4) **1.** Give the definitions of:

a. A function *uniformly continuous* on its domain $D \subset \mathbb{R}$

A function $f : D \to \mathbb{R}$ is uniformly continuous if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x, y \in D$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

b. A *compact* set of real numbers (Give the *definition*, don't state the Heine-Borel Theorem.)

A set $E \subset \mathbb{R}$ is compact if every open cover of E has a finite subcover.

(3) **2.** State the *Heine-Borel Theorem*, which has to do with when a set of real numbers is compact.

A set $E \subset \mathbb{R}$ is compact if and only if it is closed and bounded.

(4) **3.** Let $E = \left\{ \dfrac{1}{n} : n \in \mathbb{N} \right\}$ and define $f : E \to \mathbb{R}$ by $f(1/n) = (-1)^n$.
Is f continuous? **You must justify your answer!** Little credit will be given for a correct answer without proper justification.

Each point $\frac{1}{n}$ of E is an isolated point, since for each n, $\left(\dfrac{1}{n} - \dfrac{1}{10n^2}, \dfrac{1}{n} + \dfrac{1}{10n^2} \right) \bigcap E = \left\{ \dfrac{1}{n} \right\}$. We've seen that every function is continuous at any isolated points in its domain, so f is continuous.

(6) **4.** Are the following sets of real numbers compact? For each, just say "yes" or "no"; no explanation is required.

a. $\{-1, 0, 1\}$

Yes, it is compact. We've seen that every finite set is compact.

b. $\{0\} \cup (1, 4]$

No, it is not compact. It is not closed since 1 is an accumulation point of the set that is not in the set. By the Heine-Borel Theorem, compact sets are always closed, so this set is not compact.

c. $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

No, it is not compact. It is not closed since 0 is an accumulation point of the set that is not in the set. By the Heine-Borel Theorem, compact sets are always closed, so this set is not compact.

(4) **5.** Give an example of an open cover of the set $[0, \infty)$ that has no finite subcover.

One example is $\{(-1, n)\}_{n=1}^{\infty}$. Each set is open, and $\displaystyle\bigcup_{n=1}^{\infty}(-1, n) \supset [0, \infty)$, so it is an open cover. There is no finite subcover, since any finite union of these sets is bounded, while $[0, \infty)$ is unbounded.

(4) **6.** Define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} x\sin(1/x), & x \neq 0; \\ 0, & x = 0. \end{cases}$

a. Explain why f is continuous at each point of $\mathbb{R} \setminus \{0\}$. (You can cite theorems and you may use that $\sin(x)$ is continuous on all of $\mathbb{R}$ to show continuity at these points.)

We've seen that every quotient of polynomials is continuous at all points where the denominator is not zero. Hence $1/x$ is continuous at each point of $\mathbb{R} \setminus \{0\}$. We've also seen that $\sin x$ is continuous on all of $\mathbb{R}$, and that the composition of continuous functions is continuous. Hence $\sin(1/x)$ is continuous at each point of $\mathbb{R} \setminus \{0\}$. Finally, we know that the product of continuous functions is continuous, so $x\sin(1/x)$ is continuous at each point of $\mathbb{R} \setminus \{0\}$.

b. Show that f is continuous at $x = 0$. (A different argument is required here; one way is to argue from the definition of continuity.)

Let $\varepsilon > 0$ and set $\delta = \varepsilon$, so that $\delta > 0$ also. Suppose that $|x - 0| < \delta$. Then

$$|f(x) - f(0)| = |x\sin(1/x) - 0| \leq |x| < \delta = \varepsilon.$$

Hence f is continuous at $x = 0$, by the definition of continuity.