SOLUTIONS — MATH 331 HMWK 7

10 points per problem — Wayne Smith

6 pts for correct mathematical work, 4 pts for writing — Due 3/8/18

GENERAL COMMENTS:

- Everything is getting better.
- Keep proof reading your solutions aloud so you can hear your grammar mistakes (and just where words do not make sense together). This will also help you break up run-on sentences.
- Acknowledge who you have worked with, and what sources you have consulted for assistance.
- I have anonymously included some student solutions for this homework set.

1. (Chapter 3, exercise 19)
   Let $f, g : D \to \mathbb{R}$ be uniformly continuous. Prove that $f + g : D \to \mathbb{R}$ is uniformly continuous. What can be said about fg? Justify.

Proof. Let $\epsilon > 0$ be given. Since f is uniformly continuous, we know there exists $\delta_1 > 0$ such that if $|x - y| < \delta_1$, where $x, y \in D$, then $|f(x) - f(y)| < \frac{\epsilon}{2}$. Similarly, since g is uniformly continuous, there exists $\delta_2 > 0$ such that if $|x - y| < \delta_2$, where $x, y \in D$, then $|g(x) - g(y)| < \frac{\epsilon}{2}$. Set $\delta = \min\{\delta_1, \delta_2\}$. Thus, if $|x - y| < \delta$, with $x, y \in D$, we have

$$\begin{aligned}
|(f+g)(x) - (f+g)(y)| &= |f(x) + g(x) - \big(f(y) + g(y)\big)| \\
&= |f(x) - f(y) + g(x) - g(y)| \\
&\leq |f(x) - f(y)| + |g(x) - g(y)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon,
\end{aligned}$$

showing that $(f + g) : D \to \mathbb{R}$ is uniformly continuous.

Given that $f, g : D \to \mathbb{R}$ are uniformly continuous, it is not guaranteed, however, that fg must be uniformly continuous. We simply consider the example $f(x) = g(x) = x$, where both are defined on all of $\mathbb{R}$. Then both f and g are uniformly continuous. However, $fg(x) = x^2$ is not uniformly continuous on $\mathbb{R}$. Note, by Theorem 3.8, if D is compact, then it would follow that fg is uniformly continuous on D. $\square$

2. (Chapter 3, exercise 29)
If $D \subset \mathbb{R}$ is bounded, prove that $\overline{D}$ is bounded.

Proof.
Student solution 1:
Suppose $D \subset \mathbb{R}$ is bounded. It follows that $\exists a, b \in \mathbb{R}$ s.t. $D \subset [a, b]$. Let $\overline{D} = D \cup D'$. I will prove $D' \subset [a, b]$.
Suppose $\exists x_0 \in D'$ s.t. $x_0 \in \mathbb{R} \setminus [a, b]$. This is a contradiction because if $x_0 \in \mathbb{R} \setminus [a, b]$, $\exists$ a neighborhood of x_0, call it P, s.t. $x_0 \in P \subset \mathbb{R} \setminus [a, b]$. It follows that $P \cap D = \emptyset$, but this is impossible because $x_0 \in D'$. Thus, $\forall x_0 \in D', x_0 \in [a, b]$, so D' is bounded and $D' \cup D = \overline{D}$ is bounded.

Student solution 2:
We show that if $x_0 \in D'$, then $\inf(D) \leq x_0 \leq \sup(D)$. Suppose that $x_0 > \sup(D)$. Then for all $\delta > 0$, $(x_0 - \delta, x_0 + \delta)$ contains infinitely many $x \in D$. Choose $\delta = x_0 - \sup(D) > 0$. Then $(x_0 - \delta, x_0 + \delta) = (\sup(D), 2x_0 - \sup(D))$ and so $(\sup(D), 2x_0 - \sup(D)) \cap D = \emptyset$, since there exist no $x \in D$ such that $x > \sup(D)$, per the definition of supremum. This contradicts the assumption that $x_0 \in D'$, so $x \leq \sup(D)$, as a result.
Similarly, suppose that $x_0 \in D'$ is less than $\inf(D)$. Then for all $\delta > 0$, $|(x_0 - \delta, x_0 + \delta) \cap D| = \infty$. Let $\delta = \inf(D) - x_0 > 0$. So $(x_0 - \delta, x_0 + \delta) = (2x_0 - \inf(D), \inf(D))$ and $(x_0 - \delta, x_0 + \delta) \cap D = \emptyset$ for this choice of $\delta > 0$. This contradicts the fact that $x_0 \in D'$. So $\inf(D) \leq x_0$.
In turn, if $x_0 \in D'$, then $\inf(D) \leq x_0 \leq \sup(D)$, so D' is bounded. In the trivial case, where D has no accumulation points, the proof is done, since $D' = \emptyset, D$ is bounded and $\overline{D} = D \cup D' = D$.

Maureen's write up:
By definition, $\overline{D} = D \cup D'$. There are two cases.
Case I: D is finite.
In this case, D does not have any accumulation points. Hence, $\overline{D} = D$, which is bounded.
Case II: D is infinite.
Suppose D is infinite and nonempty. (If D is empty, we are back in case I.) Then by the least upper bound property of $\mathbb{R}$ and Theorem 0.20, D has both a supremum, denoted M, and an infimum, denoted N. Thus, $N \leq d \leq M$ for all $d \in D$, and $D \subset [N, M]$. By exercise 1.22, $M, N \in D$ or $M, N \in D'$. We claim $N \leq d \leq M$ for all $d \in \overline{D}$, that is $\overline{D} \subset [N, M]$.
Let a be an accumulation point of D. Then we know every neighborhood of a contains at least one point of D other than a. Suppose for the sake of contradiction that $a < N$. (The case for $a > M$ is similar and will be omitted.) Then there exists an $\epsilon > 0$ such that $a < a + \epsilon < N$, and hence, $(a - \epsilon, a + \epsilon) \cap [N, M] \subset (a - \epsilon, a + \epsilon) \cap D = \emptyset$. Thus, we have arrived at a contradiction and it must be true that $\overline{D} \subset [N, M]$. This shows that $\overline{D}$ is bounded, as was to be shown. □

3. (Chapter 3, exercise 34)
Find an open cover of $(1, 2)$ with no finite subcover.

Proof. Let $O_n = \left(1 + \frac{1}{n}, 2\right)$. Then the family $\{O_n\}_{n \in \mathbb{N}}$ of open sets forms an open cover of the interval $(1, 2)$ as $(1, 2) \subseteq \cup_{n=2}^{\infty} O_n$. Let $\{O_{n_i}\}_{1 \leq i \leq k}$ be any finite subcover. It follows that $\cup_{i=1}^{k} O_{n_i} = O_N$, where $N = \max\{n_1, n_2, \ldots, n_k\}$. However, $O_N = \left(1 + \frac{1}{N}, 2\right)$, does not contain $\left(1, 1 + \frac{1}{N}\right)$, and hence does not cover $(1, 2)$. □