General Comments:

- THE ORDER OF QUANTIFIERS MATTERS
  A lot of you, when supposing convergence of $\{a_n\}_{n=1}^{\infty}$ to A would state, "There exists $N \in \mathbb{N}$ such that if $n \geq N$ then $|a_n - A| < \epsilon$ for all $\epsilon > 0$." This, in general is not true.

  Consider the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$, which we have shown converges to 0 (page 34 of the text at the end of the paragraph following the definition of convergence).
  - "There exists $N \in \mathbb{N}$"
    Say $N = 100$.
  - "such that if $n \geq N$ then $\left|\frac{1}{n} - 0\right| < \epsilon$ for all $\epsilon > 0$."
    using $N = 100$ we see that if $n \geq N$, then $\frac{1}{n} \leq \frac{1}{N} \not< \frac{1}{101}$...so $N = 100$ does not work FOR ALL ϵ
  - We could keep on choosing bigger and bigger N, but no N will work FOR ALL ϵ. The order of quantifiers is wrong. The "magical" N we find depends on ϵ, so ϵ must be given first.
  - The general format for proving convergence of a sequence $\{a_n\}_{n=1}^{\infty}$ to A using the definition of convergence is:
    " Let $\epsilon > 0$ be given. Choose $N =$ (*something that you have done some scratch work to figure out what N works*). It follows that if $n \geq N$, then

    $$|a_n - A| \leq \textit{ (some more scratch work you have had to do to eventually show) } < \epsilon.$$

    Hence, by definition, $\{a_n\}_{n=1}^{\infty}$ converges to A as was to be shown."
  - This is a very "bland" format, but it is precise and once you get more comfortable with the process, you can perhaps color it up a little bit, if you so choose.
- NOTATION
  - $\{a_n\}_{n=1}^{\infty}$ denotes a sequence.
  - $\langle a_n \rangle$ does not denote a sequence. (It often denotes other things like a group generated by the single element a_n, for example.)
  - $[a_n], [a_n]_{n=1}^{\infty}$ do not denote sequences. I can't think off the top of my head what this notation might mean, but it probably already means something else.
  - $\{a_n\}$ does not denote a sequence. This denotes the set containing the single element a_n (which is even different from $\{a_n, a_n\}$, the set containing two copies of the single element a_n.
- KNOW YOUR DEFINITIONS
  - My undergrad analysis professor would say, "Recite to yourself in the mirror ten times a day:

    For every $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that if $n \geq N$ then $|a_n - L| < \epsilon$.

    Note the "change" from A to L. It still just denotes the real number that the sequence converges to.
  - That is the definition of a sequence $\{a_n\}_{n=1}^{\infty}$ converging to $L \in \mathbb{R}$. You also have the definition of a Cauchy sequence:

    For every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n, m \geq N,$ then $|a_n - a_m| < \epsilon$.
  - Know both definitions inside and out. Know the differences and similarities. Try to visualize or otherwise internalize what each definition is saying.

1. (Chapter 1, exercise 7)
Show that $\{a_n\}_{n=1}^{\infty}$ converges to A iff $\{a_n - A\}_{n=1}^{\infty}$ converges to 0.

Proof. First, we recall the definition of convergence of the indicated sequences:
The sequence $\{a_n\}_{n=1}^{\infty}$ converges to A iff for each $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\mid a_n - A \mid < \epsilon$. Similarly, the sequence $\{a_n - A\}_{n=1}^{\infty}$ converges to 0 iff for each $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\mid (a_n - A) - 0 \mid < \epsilon$. Since $\mid a_n - A \mid = \mid (a_n - A) - 0 \mid$, the desired result immediately follows. Note, given $\epsilon > 0$, the same $N \in \mathbb{N}$ will work for both sequences. □

2. (Chapter 1, exercise 10)
Prove that, if $\{a_n\}_{n=1}^{\infty}$ converges to A, then $\{\ |a_n|\ \}_{n=1}^{\infty}$ converges to $\ |A|\ $. Is the converse true? Justify your conclusion.

Proof. Let $\epsilon > 0$ be given. Since $\{a_n\}_{n=1}^{\infty}$ converges to A, there is an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\mid a_n - A \mid < \epsilon$. By part (iv) of Theorem 0.25, we have that $\Big|\ |a_n| - |A|\ \Big| \leq \mid a_n - A \mid$. Hence, if $n \geq N$,

$$\Big|\ |a_n| - |A|\ \Big| \leq \mid a_n - A \mid < \epsilon,$$

showing that $\{\ |a_n|\ \}_{n+=1}^{\infty}$ converges to $|A|$.

The converse of this statement, however, is not true. To see this, consider the sequence

$$\{a_n\}_{n=1}^{\infty} = \{(-1)^n\}_{n=1}^{\infty} = \{-1, 1, -1, 1, \ldots\}.$$

Observe $\{\ |a_n|\ \}_{n=1}^{\infty} = \{1\}_{n=1}^{\infty}$. Since this is a constant sequence of ones, it is clear that $\{\ |a_n|\ \}_{n=1}^{\infty}$ converges to 1. The sequence $\{a_n\}_{n=1}^{\infty}$, on the contrary, is divergent, as it oscillates between -1 and 1 indefinitely.

EXTRA:

Explicitly, suppose $\{(-1)^n\}_{n=1}^{\infty}$ converges to some real number L. Then it should follow that for any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that if $n \geq N$, we have $\mid (-1)^n - L \mid < \epsilon$. We first consider the case $L \neq 1, -1$. (These are actually the only reasonable guesses for L, but we are just being thorough here.) Then $\Big|\ 1 - |L|\ \Big| > 0$. Take $\epsilon = \frac{\left|\ 1-|L|\ \right|}{2}$. It follows from part (iv) of Theorem 0.25 that

$$\mid (-1)^n - L \mid \geq \Big|\ |(-1)^n| - |L|\ \Big| = \Big|\ 1 - |L|\ \Big| > \frac{\Big|\ 1 - |L|\ \Big|}{2},$$

showing $\{(-1)^n\}_{n=1}^{\infty}$ cannot converge to L.

Now if $L = 1$, take $\epsilon = \frac{1}{2}$ (or any positive number strictly less than 2). For $n = 2k - 1$, where $k \in \mathbb{N}$, we have

$$\Big|\ (-1)^{2k-1} - 1\ \Big| = |-1-1| = 2 > \frac{1}{2}.$$

Hence every odd term is more than ϵ away from the supposed limit L, showing the sequence cannot converge to 1.

Finally, if $L = -1$, again take $\epsilon = \frac{1}{2}$ (or any positive number strictly less than 2). For $n = 2k$, where $k \in \mathbb{N}$, we have

$$\Big|\ (-1)^{2k} - (-1)\ \Big| = |1+1| = 2 > \frac{1}{2}.$$

This shows that the sequence cannot converge to -1, as every even term is more than ϵ away from -1. We have thus exhausted all possible cases, and conclude that the sequence $\{(-1)^n\}_{n=1}^{\infty}$ is divergent.

This could also be done the following way:

Note, there is no interval of radius $\epsilon = \frac{1}{2}$ that contains both 1, and -1. Hence, the neighborhood $\left(A - \frac{1}{2}, A + \frac{1}{2}\right)$ cannot contain all but a finite number of terms of the sequence $\left\{(-1)^n\right\}_{n=1}^{\infty}$. (The neighborhood will miss infinitely many 1s, or infinitely many -1s, or both.) Therefore, by the lemma on page 35, the sequence $\left\{(-1)^n\right\}_{n=1}^{\infty}$ does not converge. $\square$

3. (Chapter 1, exercise 15)
Prove directly (do not use Theorem 1.8) that, if $\left\{a_n\right\}_{n=1}^{\infty}$ and $\left\{b_n\right\}_{n=1}^{\infty}$ are Cauchy, so is $\left\{a_n + b_n\right\}_{n=1}^{\infty}$.

Proof. Let $\epsilon > 0$ be given, and suppose $\left\{a_n\right\}_{n=1}^{\infty}$ and $\left\{b_n\right\}_{n=1}^{\infty}$ are Cauchy sequences. Then there exists $N_1 \in \mathbb{N}$ such that for all $n, m \geq N_1$ we have $|a_n - a_m| < \frac{\epsilon}{2}$, and there is an $N_2 \in \mathbb{N}$ such that for all $n, m \geq N_2$ we have $|b_n - b_m| < \frac{\epsilon}{2}$. Set $N = \max\{N_1, N_2\}$. If $n, m \geq N$, it follows that

$$\begin{aligned} |(a_n + b_n) - (a_m + b_m)| &= |(a_n - a_m) + (b_n - b_m)| && \text{by rearranging terms,} \\ &\leq |a_n - a_m| + |b_n - b_m| && \text{by the triangle inequality,} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} && \text{since } n, m \geq N, \\ &= \epsilon. \end{aligned}$$

Hence, $|(a_n + b_n) - (a_m + b_m)| < \epsilon$ and $\left\{a_n + b_n\right\}_{n=1}^{\infty}$ is Cauchy by definition. $\square$