Section 4.1 Exercise 9

Sol.

Since f is differentiable at x_0, we have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

i.e.

$$\lim_{x \to x_0} g(x) = g(x_0).$$

It follows from Theorem 3.1 in the textbook that g is continuous at x_0. Also, for $x \neq x_0$, we have that

$$g(x) = \frac{f(x) - f(x_0)}{x - x_0}$$

is the quotient of two continuous functions with non-zero denominator, hence is continuous.

Section 4.2 Exercise 11

Sol.

Let $g(x) = 2x^2 - 3x + 6$ and $h(x) = \sqrt{x}$, so that $f(x) = h(g(x))$. Note that g is a polynomial hence is differentiable everywhere, and h is differentiable on $(0, +\infty)$, by Exercise 4.1.3. For $x \in (0, 1)$, we have $g(x) > 0$, thus $g((0, 1)) \subset (0, +\infty)$. It follows from the chain rule that f is differentiable on $(0, 1)$, and that

$$f'(x) = h'(g(x))g'(x) = \frac{4x - 3}{2\sqrt{2x^2 - 3x + 6}}.$$

Section 4.3 Exercise 16

Sol.

First note that for $x \in [0, 2]$, we have $2x - x^2 = x(2 - x) \geq 0$, so that $f(x) = \sqrt{2x - x^2}$ is well-defined. Also, we have that f is continuous on $[0, 2]$ and differentiable on $(0, 2)$, by the chain rule. Moreover, we have that $f(0) = \sqrt{2(0) - 0^2} = 0$ and $f(2) = \sqrt{2(2) - 2^2} = 0$, hence f satisfies the conditions of Rolle's theorem. Now, the chain rule gives

$$f'(c) = \frac{1 - c}{\sqrt{2c - c^2}}$$

and we see that $f'(c) = 0$ precisely when $c = 1$.

Section 4.3 Exercise 18

Sol.

Let $f(x) = x^3 - 3x + b$ for $x \in [-1, 1]$. Then f is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$. Suppose now that the equation $f(x) = 0$ has at least two solutions in $[-1, 1]$. Let $x_1 < x_2$ be points in $[-1, 1]$ such that $f(x_1) = f(x_2) = 0$. By Rolle's theorem, there exists x between x_1 and x_2 with $f'(x) = 0$. But for $x \in (0, 1)$, we have

$$f'(x) = 3x^2 - 3 < 0,$$

a contradiction. Thus the equation $f(x) = 0$ has at most one solution in $[-1, 1]$, as required.

Section 4.3 Exercise 20

Sol.

The first part follows from Rolle's theorem applied to the function $f(x) - 2$ on $[1, 2]$.

The second part follows from Rolle's theorem applied to the function $f(x) - 2x$ on $[0, 1]$.

The third part follows from the first part and the second part, together with Theorem 4.11 in the textbook.

Section 4.3 Exercise 21

Sol.

Let $h(x) = f(x) - g(x)$, for $x \in [0, 1]$. Then h is differentiable and $h(0) = f(0) - g(0) = 0$. Moreover, we have that $h'(x) = f'(x) - g'(x) > 0$, so that h is strictly increasing on $[0, 1]$, by Theorem 4.9. In particular, we get that $h(x) > h(0) = 0$ for all $x \in (0, 1]$, as required.

Section 4.3 Exercise 25

Sol.

Let $x, y \in (a, b)$ with $x < y$. By the mean-value theorem, there exists $c \in (x, y)$ such that

$$f(x) - f(y) = (x - y)f'(c).$$

Thus
$$|f(x) - f(y)| = |x - y||f'(c)| \leq M|x - y|.$$
Since this holds for every $x, y \in (a, b)$, we get that f is uniformly continuous on (a, b), by Weekly Exercise 1.4 of the March 23 notes.

Section 4.3 Exercise 28

Sol.

For $x \in [-1, 1]$, we have
$$f'(x) = 6x^2 + 6x - 36 = 6(x - 2)(x + 3) < 0,$$
so that f is strictly decreasing on $[-1, 1]$, by Theorem 4.9. In particular, it is injective.

Section 4.3 Exercise 30

Sol.

The function $f(x) = x^3$ is such an example.