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Math 331
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**Final Solutions**

You must justify your answers by showing your work to receive credit. Be very careful with all that you write; please do not make me take off points for statements that you do not mean.

**Part I.** Do every problem in this part.

(8) **1.** **a.** State the *Mean Value Theorem*.

**b.** State the *Least Upper Bound Property* of real numbers.

See the text or your notes for the statements.

(12) **2.** Give definitions of the following:

**a.** An *accumulation point* of a set of real numbers.

**b.** A *uniformly continuous* function.

**c.** A *Cauchy sequence* of real numbers.

**d.** An *open* set of real numbers.

See the text or your notes for the statements.

(10) **3.** Suppose that $f(x) \le g(x)$ for all $x \in [a, b]$ and $f, g \in \mathcal{R}[a, b]$.

**a.** Explain why $L(P, f) \le L(P, g)$ for every partition $P$ of $[a, b]$. (Your explanation should be based on the definition of lower sum.)

Since $f(x) \le g(x)$, $m_i(f) \le m_i(g)$, and so
$L(P, f) = \sum m_i(f)(x_i - x_{i-1}) \le \sum m_i(g)(x_i - x_{i-1}) = L(P, g)$.

**b.** Use part **a.** to prove that $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$. Your proof should be based on the definitions.)

$$\int_a^b f(x)\,dx = \sup_P L(P, f) \le \sup_P L(P, g) = \int_a^b g(x)\,dx$$

(10) **4.** Suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are sequences of real numbers, such that $\{a_n\}_{n=1}^{\infty}$ converges to $A$ and $\{b_n\}_{n=1}^{\infty}$ converges to $B$. Prove that $\{a_n + b_n\}_{n=1}^{\infty}$ converges to $A + B$. (Your proof should be based on the definition involving $\varepsilon$ of a convergent sequence.)

This is Theorem 1.8 of the text, proved there and in class using a "standard $\varepsilon/2$" argument.

(40) **5.** Indicate by writing **T** or **F** whether each statement is true or false. Give no proofs.

**a.** If $A$ is a non-empty and compact set of real numbers, then $A$ contains $\inf A$ and $\sup A$.

True. This is Exercise #3.35 of the text, which you prepared for discussion in class.

**b.** If $f : (2, 10) \to \mathbb{R}$ is uniformly continuous, then $f$ is bounded.

True. Theorem 3.9 of the text states that a uniformly continuous function on a bounded domain is bounded.

**c.** There is a function $f$ that is differentiable on (0,1) with $f'(x) = \dfrac{7x^2 - \sin x}{\sqrt{e^x + 3}}$.

True, since $f'$ is continuous the Fundamental Theorem of Calculus tells us that $\displaystyle\int_0^x f'(t)\,dt$ will be the solution.

**d.** If $A$ and $B$ are compact sets of real numbers, then $A \cup B$ is also a compact set.

True, as discussed in class.

**e.** If $A$ and $B$ are open sets of real numbers, then $A \cup B$ is also an open set.

True, as discussed in class.

**f.** Every monotone sequence of real numbers converges.

False. The sequence $a_n = n$ is a counterexample. (However, *bounded* monotone sequences do converge.)

**g.** If $f \in \mathcal{R}[0, 1]$ and $K$ is a compact subset of $[0, 1]$, then $f(K)$ is compact.

False. A counterexample is $f(x) = x$, for $0 \le x < 1$, and $f(1) = 0$.

**h.** If $f$ is a continuous function on $[0, 1]$ and $g(x) = 3(f(x))^2 + x^5 - 7$, then $g \in \mathcal{R}[0, 1]$.

True. $g$ is continuous, so $g \in \mathcal{R}[0, 1]$ by Theorem 5.4.

**i.** Every set of real numbers that is bounded and non-empty has at least one accumulation point.

False. Any non-empty and finite set is a counterexample, since to have an accumulation point a set must be infinite.

**j.** If $f : \mathbb{R} \to \mathbb{R}$ and $f([0, 1])$ is an open set, then $f$ is not continuous.

True. If $f$ were continuous, then $f([0, 1])$ would be compact and non-empty. But the only set of real numbers that is both compact and open is the empty set.

**Part II.** Do any 5 of the 7 problems in this part. Each problem is worth 15 points.

(15) **6.** [Homework problem #1.36] Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers. Prove that $\{a_n\}_{n=1}^{\infty}$ has a convergent subsequence. (*Hint*: You may want to use the Bolzano-

Weierstrass Theorem.)

See the posted homework solutions.

(15) **7.** [Homework problem #5.9] Assume $f : [a, b] \to \mathbb{R}$ is continuous and $f(x) \geq 0$ for all $x \in [a, b]$. Prove that if $\int_a^b f(x)\,dx = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

See the posted homework solutions.

(15) **8.** Prove that a compact set of real numbers is closed. (This is part of the easy direction of the proof of the Heine-Borel Theorem. Do not just quote that theorem; prove this part of it from the definitions.)

See you class notes, or the relevant part of the proof of Theorem 3.7 in the text.

(15) **9.** Let $E = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ and define $f : E \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{-1}{n}, & \text{if } x = \frac{1}{n} \text{ where } n \text{ is odd;} \\ \frac{+1}{n}, & \text{if } x = \frac{1}{n} \text{ where } n \text{ is even;} \\ 0, & \text{if } x = 0. \end{cases}$$

**a.** At what points of $E$ is $f$ continuous? Justify your answer!

$f$ is continuous at all points of $E$. For the points $\frac{1}{n} \in E$, this is because they are isolated points of $E$. For the point 0, it is because $\lim_{n\to\infty} f(1/n) = 0 = f(0)$.

**b.** At what points of $E$ is $f$ differentiable? Justify your answer!

$f$ is not differentiable at any points. Since the points $1/n$ are not accumulation points of $E$, the notion of derivative is not even defined at these points. 0 is an accumulation point, but $\lim\limits_{n\to\infty} \dfrac{f(1/n) - f(0)}{1/n - 0} = \lim\limits_{n\to\infty} (-1)^n$ does not exist, so $f$ is not differentiable at 0.

(15) **10.** [Midterm 1 problem **6.**] Suppose that $f : \mathbb{R} \to \mathbb{R}$, $f(x) < 0$ if $x < 0$, $f(x) > 0$ if $x > 0$ and $\lim\limits_{x\to 0} f(x)$ exists. Prove that $\lim\limits_{x\to 0} f(x) = 0$.

See the posted Midterm 1 solution.

(15) **11.** **a.** State a necessary and sufficient condition for $f : [a, b] \to \mathbb{R}$ to be Riemann integrable. (See part **b.** before choosing which condition.)

Theorem 5.2 probably is easiest.

**b.** Use the condition in part **a.** to prove that if $f : [a, b] \to \mathbb{R}$ is monotone, then $f \in \mathcal{R}[a, b]$.

This is Theorem 5.3 in the text. See the proof there or in your notes.

(15) **12.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and suppose that $f'(x)$ exists for all $x \neq 0$ and that $\lim_{x\to 0} f'(x) = A$. Prove that $f$ is differentiable at 0, and if possible find $f'(0)$.
(Hint: Start with the definition of $f'(0)$ as a limit, and then try to use the Mean Value Theorem.)

This was a problem on the posted sample final, with solution also posted. Here is the solution again:

Assume $h > 0$. By the Mean Value Theorem applied with the interval $[0, h]$, there is a point $c_h \in (0, h)$ such that $f(h) - f(0) = f'(c_h) \cdot (h - 0)$. Hence $\dfrac{f(h) - f(0)}{h} = \dfrac{f'(c_h) \cdot (h - 0)}{h} = f'(c_h) \to A$ as $h \to 0$ with $h > 0$, since $h \to 0$ implies $c_h \to 0$ and $\lim_{x\to 0} f'(x) = A$. The same argument works for $h < 0$, but using the interval $[h, 0]$ in that case. Hence $\lim_{h\to 0} \dfrac{f(h) - f(0)}{h} = A = f'(0)$.