1. (Chapter 4, exercise 6)
Suppose $f : (a, b) \to \mathbb{R}$ is differentiable at $x \in (a, b)$. Prove that

$$\lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h}$$

exists and equals $f'(x)$. Give an example of a function where this limit exists, but the function is not differentiable.

Student Solution:

Proof.

Lemma 0.1. *For a function $f : D \to \mathbb{R}$ differentiable at x_0,*

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \quad \Rightarrow \quad \lim_{t\to 0} \frac{f(x_0) - f(x_0 - t)}{t} = f'(x_0)$$

for $x_0 - t \in D$ and $t \neq 0$.

Proof. Let $x = x_0 - t$ for $t \in \mathbb{R}$ s.t. $x_0 \in D$ and $t \neq 0$. Then

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x_0 - t\to x_0} \frac{f(x_0 - t) - f(x_0)}{x_0 - t - x_0} = \lim_{t\to 0} \frac{f(x_0) - f(x_0 - t)}{t}.$$

□

Suppose $f : (a, b) \to \mathbb{R}$ is differentiable at $x \in (a, b)$.

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} &= \lim_{h\to 0} \frac{f(x+h) - f(x) + f(x) - f(x-h)}{2h} \\
&= \frac{1}{2} \lim_{h\to 0} \left(\frac{f(x+h) - f(x)}{h} \right) + \frac{1}{2} \lim_{h\to 0} \left(\frac{f(x) - f(x-h)}{h} \right) \\
&= \frac{1}{2} f'(x) + \frac{1}{2} f'(x) \\
&= f'(x),
\end{aligned}$$

as required. This limit exists based on my lemma and the "alternate definition of differentiable" in the text.

As an example would be $f(x) = |x|$ at $x = 0$. $\lim_{h\to 0} \frac{|x+h|+|x-h|}{2h} = \lim_{h\to 0} \frac{h-h}{2h} = \lim_{h\to 0} 0 = 0$. But $|x|$ is not differentiable at $x = 0$. □

2. (Chapter 4, exercise 9)
Suppose $f : (a, b) \to \mathbb{R}$ is continuous on (a, b) and differentiable at $x_0 \in (a, b)$. Define
$$g(x) = \frac{f(x) - f(x_0)}{x - x_0} \quad \text{for } x \in (a,b) \setminus \{x_0\}, \quad g(x_0) = f'(x_0).$$
Prove that g is continuous on (a, b).

Student Solution:

Proof. Since f is differentiable at x_0, then $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$. Thus $\lim_{x\to x_0} g(x) = g(x_0)$. Since f is continuous on (a, b), and x is continuous on (a, b), $f(x) - f(x_0)$ is continuous and $x - x_0$ is continuous. $\{$*by Thm 3.2 and the fact that* $x_0, f(x_0)$ *are constants*$\}$ Thus $\frac{f(x)-f(x_0)}{x-x_0}$ is continuous except at x_0. $\{$*again, by Thm 3.2*$\}$ Then, since $\lim_{x\to x_0} g(x) = g(x_0)$, g is continuous at x_0. $\{$*by Thm 3.1*$\}$ Thus g is continuous on $(a, b) \setminus \{x_0\} \cup \{x_0\}$, or on (a, b). ☐

3. (Chapter 4, exercise 19)
Show that $\cos(x) = x^3 + x^2 + 4x$ has exactly one root in $\left[0, \frac{\pi}{2}\right]$.

Student Solution 1:

Proof. Let $f(x) = x^3 + x^2 + 4x - \cos(x)$. all $\{$*terms of*$\}$ f are continuous, so f is continuous $\{$*by Thm 3.2*$\}$. $f(0) = 0 + 0 + 0 - 1 = -1 < 0$. $f(\pi/2) = (\pi/2)^3 + (\pi/2)^2 + 2\pi - 0 > 0$. By Balzano's Theorem, there is $z \in (0, \pi/2)$ s.t. $f(z) = 0$. All $\{$*terms*$\}$ of f are differentiable, so f is differentiable $\{$*by Thm 4.3*$\}$. Suppose there is $y, z \in (0, \pi/2)$ s.t. $f(y) = 0 = f(z)$. Thus, by Rolle's Theorem, there is $c \in (y, z) \subset (0, \pi/2)$ s.t $f'(c) = 0$. $f'(x) = 3x^2 + 2x + 4 + \sin(x)$, which is positive for every value in the interval $(0, \pi/2)$. This is a contradiction. THerefore, $f(x)$ has exactly one root in $[0, \pi/2]$. ☐

Student Solution 2:

Proof. Let $p(x) = x^3 + x^2 + 4x - \cos(x)$. Note that since $f(x) = \cos(x)$ and polynomial functions are differentiable, then their sums/differences are differentiable (hence continuous) by theorem 4.3. Note also that $[0, \pi/2]$ is connected by theorem 3.14. We have that $p(0) = -1 < 0 < \pi^3/8 + \pi^2/4 + 2\pi = p(\pi/2)$.

By the Intermediate Value Theorem, there exists $c \in (0, \pi/2)$ such that $p(c) = 0$. Thus c is a root, whence at least one root in $[0, \pi/2]$ exists. If another root exists, say $c' \in (0, \pi/2)$, then $p(c) = p(c') = 0$. Then by Rolle's Theorem, there exists $d \in (0, \pi/2)$ such that $f'(d) = 0$. But we have that $p'(d) = 3d^2 + 2d + 4 + \sin(d) \geq 4 > 0$. Hence it is not possible for $p'(x) = 0$ on $(0, \pi/2)$, when Rolle's Theorem is contradicted and no other root c' exists. Therefore there is exactly one root, namely c. ☐