Writing Problem

Sol.

For (a), we claim that the sequence (a_n), where $a_n = 5 + \frac{1}{n}$, converges to 5. Indeed, let $\epsilon > 0$, and let $N \in \mathbb{N}$ be an integer greater than $1/\epsilon$. Then for $n \geq N$, we have

$$|a_n - 5| = \frac{1}{n} \leq \frac{1}{N} < \epsilon,$$

as required.

For (b), we claim that the sequence (a_n), where $a_n = \frac{2-2n}{n}$, converges to -2.
Indeed, let $\epsilon > 0$. Note that

$$|a_n - (-2)| = \left|\frac{2-2n}{n} + 2\right| = \left|\frac{2}{n} - 2 + 2\right| = \frac{2}{n}.$$

Therefore, if we take $N \in \mathbb{N}$ such that $N > 2/\epsilon$, then we have, for $n \geq N$,

$$|a_n - (-2)| = \frac{2}{n} \leq \frac{2}{N} < \epsilon,$$

as required.

For (c), we claim that the sequence (a_n), where $a_n = 2^{-n}$, converges to 0. Indeed, let $\epsilon > 0$. Let $N \in \mathbb{N}$ be an integer such that $2^N > 1/\epsilon$. For instance, we can take $N > \log(1/\epsilon)/\log 2$. Then for $n \geq N$, we have

$$|a_n - 0| = \frac{1}{2^n} \leq \frac{1}{2^N} < \epsilon,$$

as required.

For (d), we claim that the sequence (a_n), where $a_n = 3n/(2n+1)$, converges to $3/2$.
Indeed, let $\epsilon > 0$. First note that

$$\begin{aligned}\left|a_n - \frac{3}{2}\right| &= \left|\frac{3n}{2n+1} - \frac{3}{2}\right| \\ &= \left|\frac{6n - 3(2n+1)}{2(2n+1)}\right| \\ &= \left|\frac{-3}{4n+2}\right| \\ &= \frac{3}{4n+2} \leq \frac{4}{4n} = \frac{1}{n}.\end{aligned}$$

Let $N \geq 2$ be an integer such that $N > 1/\epsilon$. Then for $n \geq N$, we have

$$\left| a_n - \frac{3}{2} \right| \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon,$$

as required.