MATH 331 (Introduction to Real Analysis), Spring 2020
Midterm Exam 2
Friday, March 6, 2020

Name: ______________________________

Student ID Number: ______________________________

I understand it is against the rules to cheat or engage in other academic misconduct during this test.

(SIGN HERE)

Question 1	20	
Question 2	20	
Question 3	20	
Question 4	20	
Question 5	20	
Total	100	

- There are 5 questions. Make sure your exam contains all these questions.
- You must show your work on all problems. The correct answer with no supporting work may result in no credit. **Put a box around your FINAL ANSWER for each problem and cross out any work that you don't want to be graded.**
- If you need more room, use the backs of the pages and indicate that you have done so.
- Any student found engaging in academic misconduct will receive a score of 0 on this exam.
- You have 50 minutes to complete the exam.

Problem 1 (20 pts).

a. Prove that if a sequence $(a_n)_{n\in\mathbb{N}}$ converges to a, then every subsequence of $(a_n)_{n\in\mathbb{N}}$ also converges to a.

b. Let $0 < b < 1$ and let $a_n = b^n$ for all $n \in \mathbb{N}$. Use **a.** to show that the sequence $(a_n)_{n\in\mathbb{N}}$ converges to 0.

Problem 2 (20 pts).

a. Give the definition of limit of a function at a point.

b. Let $f : \mathbb{R} \to \mathbb{R}$. Suppose that $f(x) > 0$ for all $x > 0$ and $f(x) < 0$ for all $x < 0$. Prove that if f has a limit L at 0, then L must be equal to 0.

Problem 3 (20pts).

a. Give the definition of a continuous function at a point.

b. Let $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Use the definition from **a.** to prove that f is continuous at 0.

c. Explain why f is also continuous at every $x \neq 0$.

(You can use without proof the fact that $\sin x$ is continuous on all of $\mathbb{R}$)

Problem 4 (20pts).

a. Give the definition of a compact set $E \subset \mathbb{R}$.

b. State the Heine–Borel theorem.

c. Let E_1 and E_2 be compact sets. Prove that $E_1 \cup E_2$ is compact.

Problem 5 (20pts).

True or False? Write a short justification.

a. There exists a function $f : (0, 1) \to \mathbb{R}$ such that f is continuous at $x_0 \in (0, 1)$ if and only if $x_0 \notin \mathbb{Q}$.

b. If U and V are two open sets, then $U \cup V$ is also open.

c. Every sequence has a convergent subsequence.

d. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two functions. If both f and fg are continuous at every point, then g is continuous at every $x_0 \in \mathbb{R}$ such that $f(x_0) \neq 0$.