Section 3.3 Exercise 19

Sol.

Let $f, g : D \to \mathbb{R}$ be uniformly continuous.

Let us prove that $f + g : D \to \mathbb{R}$ is also uniformly continuous. Let $\epsilon > 0$. Since $f : D \to \mathbb{R}$ is uniformly continuous, there exists $\delta_1 > 0$ such that for all $x, y \in D$ with $|x - y| < \delta_1$, we have $|f(x) - f(y)| < \epsilon/2$. Similarly, there exists $\delta_2 > 0$ such that for all $x, y \in D$ with $|x - y| < \delta_1$, we have $|g(x) - g(y)| < \epsilon/2$.
Let $\delta := \min(\delta_1, \delta_2) > 0$. Then for all $x, y \in D$ with $|x - y| < \delta$, we have

$$\begin{aligned} |(f+g)(x) - (f+g)(y)| &= |f(x) - f(y) + g(x) - g(y)| \\ &\leq |f(x) - f(y)| + |g(x) - g(y)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

as required.

It is not true in general that the product fg is uniformly continuous as well. Indeed, the function $f(x) = x$ is uniformly continuous on $\mathbb{R}$, but $f \cdot f$ is not.

Section 3.3 Exercise 20

Sol.

Let $f : A \to B$ and $g : B \to C$ be uniformly continuous. We show that $g \circ f : A \to C$ is also uniformly continuous.

Let $\epsilon > 0$. Since $g : B \to C$ is uniformly continuous, there exists $\delta_1 > 0$ such that for all $w, z \in B$ with $|w - z| < \delta_1$, we have $|g(w) - g(z)| < \epsilon$. Since $f : A \to B$ is uniformly continuous, there exists $\delta > 0$ such that for all $x, y \in A$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \delta_1$.
It follows that for all $x, y \in A$ with $|x - y| < \delta$, we have that $z = f(x)$ and $w = f(y)$ satisfy $|z - w| < \delta_1$, hence

$$|g(f(x)) - g(f(y))| = |g(z) - g(w)| < \epsilon,$$

as required.

Section 3.3 Exercise 21

Sol.

Let $f : [3.4, 5] \to \mathbb{R}$ defined by

$$f(x) = \frac{2}{x-3} \qquad (x \in [3.4, 5]).$$

Note that for all $x, y \in [3.4, 5]$, we have

$$|f(x) - f(y)| = \left| \frac{2}{x-3} - \frac{2}{y-3} \right| = 2\frac{|x-y|}{|x-3||y-3|} < 2\frac{|x-y|}{0.2^2},$$

since $|x-3| = x-3 \geq 3.4-3 = 0.4 > 0.2$, and similarly for $|y-3|$.
Now, let $\epsilon > 0$. Take $\delta := \frac{0.2^2}{2}\epsilon$. Then for all $x, y \in [3.4, 5]$ with $|x-y| < \delta$, we have

$$|f(x) - f(y)| < 2\frac{\delta}{0.2^2} = \epsilon,$$

as required.

Section 3.3 Exercise 22

Sol.

Let $f : (2, 7) \to \mathbb{R}$ defined by

$$f(x) = x^3 - x + 1 \qquad (x \in (2, 7)).$$

Note that for all $x, y \in (2, 7)$, we have

$$|f(x)-f(y)| = |x^3-y^3-x+y| = |(x-y)(x^2+xy+y^2)-(x-y)| = |x-y||x^2+xy+y^2-1|.$$

Using the triangle inequality and the fact that $x, y \in (2, 7)$, we get

$$|f(x) - f(y)| \leq |x-y|(7^2 + 7(7) + 7^2 + 1) = 148|x-y|.$$

It easily follows from this that f is uniformly continuous on $(2, 7)$, as in the previous exercise.

Section 3.3 Exercise 31

Sol.

Let $f, g : [a, b] \to \mathbb{R}$ be continuous, and let $T := \{x \in [a, b] : f(x) = g(x)\}$. We have to show that T is closed.

Let x_0 be an accumulation point of T. By Theorem 1.17, there is a sequence (x_n) of elements of T, each distinct from x_0, such that (x_n) converges to x_0. Then $f(x_n) = g(x_n)$ for all n, since each x_n belongs to T. Also, since f and g are both continuous at x_0, we have that $f(x_n) \to f(x_0)$ and $g(x_n) \to g(x_0)$. Hence $f(x_0) = g(x_0)$ and $x_0 \in T$.

This shows that every accumulation point of T belongs to T, so that T is closed, as required.

Section 3.3 Exercise 37

Sol.

Let $f : [a, b] \to \mathbb{R}$ be a function, and suppose that f has a limit at each $x \in [a, b]$. We have to show that f is bounded.

For each $x \in [a, b]$, we have that f has a limit at x, so that f is bounded in a neighborhood of x, by Theorem 2.3. Hence there exists a real number M_x and an interval I_x containing x such that $|f(y)| \leq M_x$ for all $y \in I_x$. Now, the collection of open intervals $\{I_x\}_{x\in[a,b]}$ is an open cover of $[a, b]$. Since $[a, b]$ is compact, there exists a finite subcover. In other words, there exist $x_1, \dots, x_n \in [a, b]$ such that

$$[a, b] \subset I_{x_1} \cup I_{x_2} \cup \cdots \cup I_{x_n}.$$

Now, let M be larger than all the numbers $M_{x_1}, M_{x_2}, \dots, M_{x_n}$. If $y \in [a, b]$, then y belongs to I_{x_i} for some i, hence $|f(y)| \leq M_i \leq M$. This shows that $|f(y)| \leq M$ for all $y \in [a, b]$, thus f is bounded, as required.

Section 3.3 Exercise 39

Sol.

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous with the property that for each $\epsilon > 0$, there exists $M > 0$ such that if $|x| \geq M$, then $|f(x)| < \epsilon$. We have to show that f is uniformly continuous.

Let $\epsilon > 0$. By the assumption of the problem, there exists $M > 0$ such that for all x with $|x| \geq M$, we have $|f(x)| < \epsilon/3$.
Now, the interval $[-M, M]$ is compact, so by a theorem from the last lecture notes, we have that f is uniformly continuous on $[-M, M]$. Hence there exists $\delta > 0$ such that for all $x, y \in [-M, M]$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon/3$.
We now claim that for all $x, y \in \mathbb{R}$ with $|x-y| < \delta$, we have $|f(x)-f(y)| < \epsilon$, which implies uniform continuity. Indeed, let $x, y \in \mathbb{R}$ with $|x - y| < \delta$. If both x, y belong to the interval $[-M, M]$, then we have

$$|f(x) - f(y)| < \frac{\epsilon}{3} < \epsilon.$$

On the other hand, if neither x or y belong to $[-M, M]$, then we have $|x| \geq M$ and $|y| \geq M$, so that

$$|f(x) - f(y)| \leq |f(x)| + |f(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon,$$

as required.
It remains to treat the case where one of x, y belongs to $[-M, M]$ and the other does not. Without loss of generality, suppose that $x < -M$ and $-M \leq y \leq M$. Note that $|-M| \geq M$ trivially, hence $|f(-M)| < \epsilon/3$. Also, since $|x-y| < \delta$, we must have $|-M-y| < \delta$, hence $|f(-M) - f(y)| < \epsilon/3$, since both $-M$ and y belong to $[-M, M]$. Combining, we get

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f(-M) + f(-M) - f(y)| \\ &\leq |f(x)| + |f(-M)| + |f(-M) - f(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \end{aligned}$$

as required.