Solutions | Math 331 Hmwk 13

10 points per problem | Wayne Smith

6 pts for correct mathematical work, 4 pts for writing | Due 5/1/18

1. (Chapter 5, exercise 18)
Suppose f and g are differentiable on $[a, b]$ and f' and g' are integrable on $[a, b]$. Prove that $f'g$ and $g'f$ are integrable on $[a, b]$ and that
$$\int_a^b f'g\,dx = f(b)g(b) - f(a)g(a) - \int_a^b g'f\,dx.$$
Of course, this is the *integration-by-parts formula.*

Proof. Suppose f and g are differentiable on $[a, b]$. Then by Theorem 4.2, we have that f and g are continuous on $[a, b]$. Furthermore, from Theorem 5.4 it follows that $f, g \in \mathcal{R}(x)$ on $[a, b]$. Hence, we have that f, g, f', and g' are all Riemann integrable on $[a, b]$. By Theorem 5.12 we now also have that $f'g$ and $g'f$ are Riemann integrable on $[a, b]$.

Recall, from Theorem 4.3, since f and g are differentiable on $[a, b]$, we have that fg is differentiable on $[a, b]$ with
$$(fg)' = f'g + g'f,$$
which, by Theorem 5.9, is Riemann integrable on $[a, b]$. Integrating both sides and applying the Fundamental Theorem of Integral Calculus to the left hand side, we obtain
$$(fg)(b) - (fg)(a) = \int_a^b (fg)'\,dx = \int_a^b (f'g + g'f)\,dx.$$
Applying Theorem 5.9 to the right hand side of this and rearranging terms, we arrive at the desired result of
$$\int_a^b f'g\,dx = f(b)g(b) - f(a)g(a) - \int_a^b g'f\,dx.$$
□

2. (Chapter 5, exercise 27)
Suppose f and g are integrable on $[a, b]$. Define $h(x) = \max\{f(x), g(x)\}$. Prove that h is integrable on $[a, b]$.

Proof. Note, for $w, z \in \mathbb{R}$ we have $\max\{w, z\} = \frac{1}{2}(w + z) + \frac{1}{2}|w - z|$. (Convince yourself of this using the figure below for assistance.) Now, suppose f and g are integrable on $[a, b]$ and define
$$h(x) = \max\{f(x), g(x)\} = \frac{1}{2}\big(f(x) + g(x)\big) + \frac{1}{2}\big|f(x) - g(x)\big|.$$

By Theorem 5.9, linear combinations of integrable functions are integrable, and by Theorem 5.12, the absolute value of and integrable function is an integrable function. Hence, $h \in \mathcal{R}(x)$ on $[a, b]$, as was to be shown.

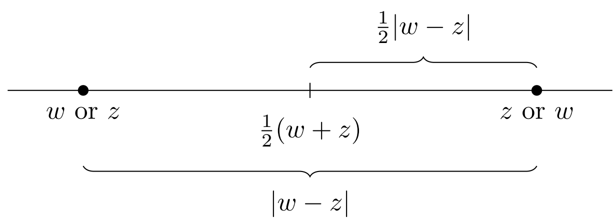

Figure 1. "Proof" that $\max\{w, z\} = \frac{1}{2}(w + z) + \frac{1}{2}|w - z|$.

□