NAME:

Math 331
May 10, 2018
W. Smith

Final Solutions

You must justify your answers by showing your work to receive credit. Be very careful with all that you write; please do not make me take off points for statements that you do not mean.

Part I. Do every problem in this part.

(8) **1.** **a.** State the *Mean Value Theorem*.

b. State the *Heine-Borel Theorem*, which is about compact sets of real numbers.

See the text or your notes for the statements.

(12) **2.** Give definitions of the following:

a. An *accumulation point* of a set of real numbers.

b. A function that is *differentiable* at a point.

c. A *Cauchy sequence* of real numbers.

d. A *closed* set of real numbers.

See the text or your notes for the statements.

(10) **3.** **a.** Identify the set of all accumulation points of the set $E = (0, 1] \cup \{3\}$. Explain your answer.

$E' = [0, 1]$, since for $x \in [0, 1]$ and $\delta > 0$, the set $E \cap (x - \delta, x + \delta)$ is infinite, and this fails for $x \notin [0, 1]$

b. Give an open cover of the set E from part **a.** that has no finite subcover.

One example is the collection $\{(1/n, \infty) : n \in \mathbb{N}\}$.

(10) **4.** Suppose that $f(x) \leq g(x)$ for all $x \in [a, b]$ and $f, g \in \mathcal{R}[a, b]$.

a. Explain why $U(P, f) \leq U(P, g)$ for every partition P of $[a, b]$. (Your explanation should be based on the definition of lower sum.)

Since $f(x) \leq g(x)$, $M_i(f) \leq M_i(g)$, and so
$U(P, f) = \sum M_i(f)(x_i - x_{i-1}) \leq \sum M_i(g)(x_i - x_{i-1}) = U(P, g)$.

b. Use part **a.** to prove that $\displaystyle\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$. Your proof should be based on the definitions.)

$$\int_a^b f(x)\,dx = \inf_P U(P,f) \leq \inf_P U(P,g) = \int_a^b g(x)\,dx$$

(27) **5.** Indicate by writing **T** or **F** whether each statement is true or false. Give no proofs.

a. If $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are both Cauchy sequences of real numbers, then $\{a_n+b_n\}_{n=1}^\infty$ is also a Cauchy sequence.

True. Cauchy sequences of real numbers are the same as convergent sequences, and sums of convergent sequences are convergent.

b. Every subset of a compact set is compact.

False. An example is $(0,1) \subset [0,1]$.

c. If $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$ are both uniformly continuous, then so is $f+g : E \to \mathbb{R}$.

True. This was HW #3.19

d. Let $\{a_n\}_{n=1}^\infty$ be a bounded sequence of real numbers that is also decreasing. Then $\{a_n\}_{n=1}^\infty$ converges.

True. See Theorem 1.16.

e. If $f : [-2,2] \to \mathbb{R}$ is continuous, then $g(x) = \sin(f(x)+3) \in \mathcal{R}[-2,2]$.

True. Since f is continuous, $g(x) = \sin(f(x)+3)$ is continuous, and hence $g \in \mathcal{R}[-2,2]$ by Theorem 5.4.

f. Every set of real numbers that is bounded and non-empty has at least one accumulation point.

False. A counterexample is any finite and non-empty set, such as the set $E = \{0\}$.

g. If f is a function that is defined on $(-1,1)$ and f is differentiable at $x=0$, then $\lim_{x\to 0} f(x) = f(0)$.

True. Since f is differentiable at $x=0$, f is continuous at $x=0$. Hence by definition $\lim_{x\to 0} f(x) = f(0)$.

h. If $\emptyset \neq A \subset B \subset [0,1]$, then $\inf A \leq \inf B$.

False. A counterexample is $A = \{1\}$, so $\inf A = 1$, and $B = [0,1]$, so $\inf B = 0$.

i. If $f : [0,1] \to \mathbb{R}$ is bounded, then $f \in \mathcal{R}[0,1]$ if and only if $\overline{\int_0^1} f(x)\,dx \leq \underline{\int_0^1} f(x)\,dx$.

True. $\underline{\int_0^1} f(x)\,dx \leq \overline{\int_0^1} f(x)\,dx$ always, so $\overline{\int_0^1} f(x)\,dx \leq \underline{\int_0^1} f(x)\,dx$ iff $\overline{\int_0^1} f(x)\,dx = \underline{\int_0^1} f(x)\,dx$ iff $f \in \mathcal{R}[0,1]$.

Part II. Do just 5 of the 7 problems in this part. Each problem is worth 15 points.

(12) **6.** Suppose $f, g : D \to \mathbb{R}$ with x_0 an accumulation point of D. Further suppose that f and g have limits at x_0. Prove that $f + g$ has a limit at x_0, and

$$\lim_{x \to x_0} (f + g)(x) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x).$$

(This is part of Theorem 2.4 in the text. Don't just refer to that theorem – prove it!)

See the text or your notes for the proof.

(12) **7.** Prove that a compact set of real numbers is bounded. (This is part of the easy direction of the proof of the Heine-Borel Theorem. Do not just quote that theorem; prove this part of it from the definitions.)

See the text or your notes for the proof.

(12) **8.** [Homework problem #5.4] A set $A \subset [0, 1]$ is dense in $[0, 1]$ iff every open interval that intersects $[0, 1]$ contains a point of A. Suppose $f \in \mathcal{R}[0, 1]$ and $f(x) = 0$ for all $x \in A$. Show that $\int_0^1 f(x)\, dx = 0$.

See your solution to the HW problem.

(12) **9.** Define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q}, \\ x^2, & \text{if } x \notin \mathbb{Q}. \end{cases}$

a. Is f continuous at $x = 0$? Justify your answer. (Justifications based on the definition will receive the most points.)

Yes, f is continuous at $x = 0$. Since $0 \le f(x) \le x^2$ and $\lim_{x \to 0} 0 = 0 = \lim_{x \to 0} x^2$, we get that $\lim_{x \to 0} f(x) = 0 = f(0)$. Hence f is continuous at $x = 0$ by definition.

b. Is f differentiable at $x = 0$? Justify your answer.

Yes, f is differentiable at $x = 0$. $\lim_{h \to 0} \dfrac{f(h) - f(0)}{h} = 0 = f'(0)$, from the Squeeze Theorem since
$-|h| \le \dfrac{f(h) - f(0)}{h} \le |h|$, and $\lim_{h \to 0} -|h| = 0 = \lim_{h \to 0} |h|$.

(12) **10.** Let $f : D \to \mathbb{R}$ be uniformly continuous, and let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence of points in D. Prove that $\{f(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence. (This is a lemma proved in class. Don't just refer to that lemma – give the proof, based on the definitions.)

See your class notes, or look at the proof of Theorem 3.5 in the text.

(12) **11.** **a.** (4 points) State a necessary and sufficient condition for $f : [a, b] \to \mathbb{R}$ to be Riemann integrable. (See part **b.** before choosing which condition.)

Probably the easiest is Theorem 5.2.

b. (8 points) Use the condition in part **a.** to prove that if $f : [a, b] \to \mathbb{R}$ is continuous, then $f \in \mathcal{R}[a, b]$. [This is Theorem 5.4 in the text.]

See your class notes or the proof in the text.

(12) **12.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and suppose that $f'(x)$ exists for all $x \neq 0$ and that $\lim_{x \to 0} f'(x) = A$. Prove that f is differentiable at 0, and if possible find $f'(0)$.
(Hint: Start with the definition of $f'(0)$ as a limit, and then try to use the Mean Value Theorem.)

This was a problem on the posted sample final, with solution also posted. Here is the solution again:

Assume $h > 0$. By the Mean Value Theorem applied with the interval $[0, h]$, there is a point $c_h \in (0, h)$ such that $f(h) - f(0) = f'(c_h) \cdot (h - 0)$. Hence $\dfrac{f(h) - f(0)}{h} = \dfrac{f'(c_h) \cdot (h - 0)}{h} = f'(c_h) \to A$ as $h \to 0$ with $h > 0$, since $h \to 0$ implies $c_h \to 0$ and $\lim_{x \to 0} f'(x) = A$. The same argument works for $h < 0$, but using the interval $[h, 0]$ in that case. Hence $\lim_{h \to 0} \dfrac{f(h) - f(0)}{h} = A = f'(0)$.