MATH 331 (Introduction to Real Analysis), Spring 2020
Midterm Exam 1 Solution
Monday, February 10, 2020

Name: ____________________

Student ID Number: ____________________

I understand it is against the rules to cheat or engage in other academic misconduct during this test.

(SIGN HERE)

Question 1	20	
Question 2	20	
Question 3	20	
Question 4	20	
Question 5	20	
Total	100	

- There are 5 questions. Make sure your exam contains all these questions.
- You must show your work on all problems. The correct answer with no supporting work may result in no credit. **Put a box around your FINAL ANSWER for each problem and cross out any work that you don't want to be graded.**
- If you need more room, use the backs of the pages and indicate that you have done so.
- Any student found engaging in academic misconduct will receive a score of 0 on this exam.
- You have 50 minutes to complete the exam.

Problem 1 (20 pts).

Let $A \subset \mathbb{R}$.

a. Give the definition of $\sup A$ and $\inf A$.

Sol. $\sup A$ is the least upper bound of A, if it exists. $\inf A$ is the greatest lower bound of A, if it exists.

b. Suppose that $A \cap [0, 1] = \emptyset$. Is it possible that $\sup A = 0$?

If yes, give an example of such a set. If no, explain why.

Sol. Yes, consider $A = (-1, 0)$ for example.

c. Suppose that $A \cap [0, 1] = \emptyset$. Is it possible that $\sup A = 1$?

If yes, give an example of such a set. If no, explain why.

Sol. No. Suppose $\sup A = 1$. By Homework 1 Exercise 44, for any $\epsilon > 0$, there exists $a \in A$ such that

$$1 - \epsilon < a \leq 1.$$

Taking ϵ small enough then gives an element $a \in A$ with $0 \leq a \leq 1$, contradicting the fact that $A \cap [0, 1] = \emptyset$.

Problem 2 (20 pts).

Let $A \subset \mathbb{R}$.

a. Give the definition of accumulation point of A.

Sol. A real number a is an accumulation point of A if every interval containing a contains infinitely many elements of A.

b. State the Bolzano-Weierstrass theorem.

Sol. Every bounded infinite set of real numbers has at least one accumulation point.

c. Give an example of a set A that has exactly two accumulation points.

Sol. We can take

$$A := \left\{ \frac{1}{n} \right\}_{n \in \mathbb{N}} \cup \left\{ 1 - \frac{1}{n} \right\}_{n \in \mathbb{N}}$$

for example. The set A has two accumulation points, 0 and 1.

Problem 3 (20 pts).

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.

a. Define what it means for $(a_n)_{n\in\mathbb{N}}$ to converge to a real number a.

Sol. The sequence $(a_n)_{n\in\mathbb{N}}$ converges to a if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$|a_n - a| < \epsilon.$$

b. Suppose that $(a_n)_{n\in\mathbb{N}}$ converges to a, and define a new sequence by

$$b_n := \frac{a_n + a_{n+1}}{2} \qquad (n \in \mathbb{N}).$$

Prove that $(b_n)_{n\in\mathbb{N}}$ also converges to a.

Sol. Let $\epsilon > 0$. Since $a_n \to a$, there exists $N \in \mathbb{N}$ such that for all $k \geq N$, we have

$$|a_k - a| < \epsilon.$$

Now, let $n \geq N$. Then also $n+1 \geq N$, and we have

$$|b_n - a| = \left|\frac{a_n - a + a_{n+1} - a}{2}\right| \leq \frac{|a_n - a|}{2} + \frac{|a_{n+1} - a|}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that $b_n \to a$, as required.

Problem 4 (20 pts).

Using arithmetic on sequences, find the limits of the following sequences:

a. $a_n = \frac{n^3 - 7n + 4}{-2n^3 + 2n^2 + 1}$.

Sol. Dividing both the numerator and the denominator by n^3, we see that

$$a_n \to \frac{1 - 7(0) + 4(0)}{-2 + 2(0) + 1(0)} = -\frac{1}{2}.$$

b. $a_n = \frac{(-1)^n \, n \sin(n^3)}{n^2}$.

Sol. We have that a_n is the product of the bounded sequence $(-1)^n \sin(n^3)$ and the sequence $1/n$, which converges to 0. By a result seen in class, we have that $a_n \to 0$.

c. $a_n = \sqrt{n+1} - \sqrt{n}$.

Noting that

$$a_n = \frac{1}{\sqrt{n+1} + \sqrt{n}},$$

it easily follows that $a_n \to 0$. See Example 1.5 in the textbook.

d. $a_n = \frac{\cos(\sin n)}{\sqrt{n}}$.

Sol. We have that a_n is the product of the bounded sequence $\cos(\sin n)$ and the sequence $1/\sqrt{n}$. By a result seen in class, we have that $a_n \to 0$.

Problem 5 (20 pts).

True or False? Give a **short** justification.

a. Every nonempty set of real numbers that is bounded from above has a greatest lower bound.

Sol. FALSE. For example, take $A = \{-1, -2, -3, \dots\}$.

b. If $(a_n)_{n\in\mathbb{N}}$ is a sequence, then $a_n \to a$ if and only if $|a_n| \to |a|$.

Sol. FALSE. The direct implication is true, but the converse is false, as seen by taking $a_n = (-1)^n$.

c. Every Cauchy sequence is convergent.

Sol. TRUE. We proved this in class.

d. Every infinite set of real numbers has a least one accumulation point.

Sol. FALSE. For example, $\mathbb{N}$ is an infinite set that does not have an accumulation point. This is true if we require that the set is bounded though, as stated by the Bolzano-Weierstrass theorem.