1. (Chapter 5, exercise 7)
Suppose $g : [a, b] \to \mathbb{R}$ is continuous except at $x_0 \in (a, b)$ and bounded. Prove that $g \in R(x)$ on $[a, b]$. See Exercises 24 and 25 for generalizations of this result.

COMMENTS:
- A handful of students copied the proof for theorem 5.4 for this exercise (including writing the function as f instead of g). Theorem 5.4 is not fully applicable since g is not continuous on $[a, b]$.
- main ideas: use theorems 5.4 and 5.2 on the "good" parts of g; use theorem 5.2 and the fact that g is bounded on the "bad" parts to show g is Riemann integrable on $[a, b]$

Student Solution:

Proof. Let $g : [a, b] \to \mathbb{R}$ be continuous except at $x_0 \in (a, b)$ and bounded. Let $\epsilon > 0$,

$$M = \sup\{g(x) : x \in [a, b]\} \quad \text{and} \quad m = \inf\{g(x) : x \in [a, b]\}.$$

Let $L = M - m$. As g is continuous on $[a, x_0 - \epsilon/6L]$, by Theorem 5.4, g is Riemann integrable on $[a, x_0 - \epsilon/6L]$. Hence, by Theorem 5.2, there exists a partition of $[a, x_0 - \epsilon/6L]$, P_1 such that $U(P_1, g) - L(P_1, g) \le \epsilon/3$. Similarly, there exists a partition of $[x_0 + \epsilon/6L, b]$, P_2 such that $U(P_2, g) - L(P_2, g) \le \epsilon/3$. Let $P = P_1 \cup P_2$,

$$M_0 = \sup\{g(x) : x \in [x_0 - \epsilon/6L, x_0 + \epsilon/6L]\} \quad \text{and} \quad m_0 = \inf\{g(x) : x \in [x_0 - \epsilon/6L, x_0 + \epsilon/6L]\}.$$

$$\begin{aligned} U(P, g) &= U(P_1, g) + U(P_2, g) + M_0\big(x_0 + \epsilon/6L - x_0 + \epsilon/6L\big) \\ &= U(P_1, g) + U(P_2, g) + M_0(\epsilon/3L). \\ L(P, g) &= L(P_2, g) + L(P_1, g) + m_0\big(x_0 + \epsilon/6L - x_0 + \epsilon/6L\big) \\ &= L(P_1, g) + L(P_2, g) + m_0(\epsilon/3L). \end{aligned}$$

Thus,

$$\begin{aligned} U(P, g) - L(P, g) &= U(P_1, g) - L(P_1, g) + U(P_2, g) - L(P_2, g) + (M_0 - m_0)(\epsilon/3L) \\ &\le U(P_1, g) - L(P_1, g) + U(P_2, g) - L(P_2, g) + \epsilon/3 \\ &\le \epsilon/3 + \epsilon/3 + \epsilon/3 \\ &= \epsilon. \end{aligned}$$

Hence, by Theorem 5.2, g is Riemann integrable on $[a, b]$.

$\square$

2. (Chapter 5, exercise 9)
Assume $f : [a,b] \to \mathbb{R}$ is continuous and $f(x) \geq 0$ for all $x \in [a,b]$. Prove that if $\int_a^b f\ dx = 0$, then $f(x) = 0$ for all $x \in [a,b]$.

COMMENTS:
- One of the following solutions uses theorem 5.10. Since this is an exercise given after section 5.2, however, you can also do it without the use of theorem 5.10, as shown with the other solution.
- common mistakes:
  - f is Riemann integrable on $[a,b]$ iff $\underline{\int_a^b} f\ dx = \overline{\int_a^b} f\ dx$, and both of these are equal to $\int_a^b f\ dx$ if this is the case. This is by definition; it would not be something you deduce in your proof. The definitions of $\underline{\int_a^b} f\ dx$ and $\overline{\int_a^b} f\ dx$ were also misused. The first is the supremum of the lower sums, and the second is the infimum of the upper sums.
  - It is necessary to use the hypothesis that f is continuous. (Consider the function that is zero everywhere except at one point...)

Student Solution:

Proof. For the sake of contradiction, suppose $f(x_0) \neq 0$ from some $x_0 \in [a,b]$. Since f is continuous by the hypothesis, $\exists \delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \frac{f(x_0)}{2}$.

The above inequality is equivalent to the following:

$$-\frac{f(x_0)}{2} < f(x) - f(x_0) < \frac{f(x_0)}{2}.$$

Adding $f(x_0)$ then gives us

$$\frac{f(x_0)}{2} < f(x) < \frac{f(x_0)}{2}.$$

Note that $0 < \frac{f(x_0)}{2} < f(x)$ as $f(x_0) \neq 0$.

Since $f(x) \geq 0\ \forall x \in [a,b]$ and f is continuous, theorem 5.4 (gives us that) $f \in \mathcal{R}[a,b]$. We can then inspect $\int_a^b f(x)\ dx$. Invoking theorem 5.10 we can break $\int_a^b f(x)\ dx$ into three integrals and solve for each segment:

$$\int_a^b f(x)\ dx = \int_{[a,x_0-\delta]} f(x)\ dx + \int_{[x_0-\delta,x_0+\delta]} f(x)\ dx + \int_{[x_0+\delta,b]} f(x)\ dx.$$

While $\int_{[a,x_0-\delta]} f(x)\ dx \geq 0$ and $\int_{[x_0+\delta,b]} \geq 0$, $\int_{[x_0-\delta,x_0+\delta]} f(x)\ dx > 2\delta\left(\frac{f(x_0)}{2}\right) = \delta f(x_0) > 0$. Then $\int_a^b f(x)\ dx > 0$. But this is a contradiction as $\int_a^b f(x)\ dx = 0$ in the hypothesis. Thus, $f(x) = 0\ \forall x \in [a,b]$. □

Additional Solution:

Proof. Let $f : [a,b] \to \mathbb{R}$ be continuous with $f(x) \geq 0$ for all $x \in [a,b]$. Suppose $\int_a^b f\ dx = 0$. Suppose for the sake of contradiction that there exists $x_0 \in [a,b]$ such that $f(x_0) > 0$. Then, by continuity of f, there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \frac{f(x_0)}{2}$. Hence, on the interval $(x_0 - \delta, x_0 + \delta)$ we have $f(x) > \frac{f(x_0)}{2} > 0$.

Consider the partition of $[a,b]$ given by $P = \{a, x_0 - \delta, x_0 + \delta, b\}$. Note, $L(P,f) > 0$ as $f(x) > 0$ for $x \in (x_0 - \delta, x_0 + \delta)$. It follows that

$$0 < \sup\{L(P,f) : P \text{ a partition of } [a,b]\} = \underline{\int_a^b} f\ dx = \int_a^b f\ dx = 0,$$

which is a contradiction. Therefore, it must be true that $f(x) = 0$ for all $x \in [a,b]$. □