Section 3.4 Exercise 41

Sol.

We claim that the interval $[0,1]$ contains a root of the equation $xe^x = 1$. Indeed, let $f(x) := xe^x - 1$. Then f is continuous, and $f(0) = -1 < 0$ while $f(1) = e - 1 > 0$. By the intermediate value theorem, there exists $x \in [0,1]$ such that $f(x) = 0$.

Section 3.4 Exercise 42

Sol.

We claim that the interval $[0,1]$ contains a root of the equation $x^3 - 6x^2 + 2.826 = 0$. Indeed, let $f(x) = x^3 - 6x^2 + 2.826$. Then f is continuous, and $f(0) = 2.826 > 0$ while $f(1) = 1 - 6 + 2.826 < 0$. By the intermediate value theorem, there exists $x \in [0,1]$ such that $f(x) = 0$.

Section 3.4 Exercise 44

Sol.

Let $g(x) = f(x) - x$. Then g is continuous on $[a,b]$. Also, we have $g(a) = f(a) - a \geq 0$ since $f(a) \in [a,b]$. On the other hand, we have $g(b) = f(b) - b \leq 0$, since $f(b) \in [a,b]$. By the intermediate value theorem, there exists $x \in [a,b]$ such that $g(x) = 0$, i.e. $f(x) = x$.

Section 4.1 Exercise 3

Sol.

We have

$$\frac{\sqrt{x} - \sqrt{x_0}}{x - x_0} = \frac{x - x_0}{(\sqrt{x} + \sqrt{x_0})(x - x_0)} = \frac{1}{\sqrt{x} + \sqrt{x_0}}.$$

By algebra of limits, the right-hand side has limit $\frac{1}{2\sqrt{x_0}}$ at x_0. It follows that $f(x) = \sqrt{x}$ is differentiable at x_0 and

$$f'(x_0) = \frac{1}{2\sqrt{x_0}}.$$

The function is not differentiable at 0, since

$$\frac{\sqrt{x}-\sqrt{0}}{x-0}=\frac{1}{\sqrt{x}}$$

does not have a limit at 0.

Section 4.1 Exercise 4

Sol.

Note that

$$\frac{x^2-x_0^2}{x-x_0}=x+x_0.$$

By algebra of limits, the right-hand side has limit $2x_0$ at x_0. It follows that $g(x)=x^2$ is differentiable at x_0 and $g'(x_0)=x_0^2$.

Section 4.1 Exercise 5

Sol.

If $h(x)=x^3\sin\frac{1}{x}$ for $x\neq 0$ and $h(0)=0$, then h is differentiable at every $x\neq 0$, and

$$h'(x)=3x^2\sin\frac{1}{x}-x\cos\frac{1}{x},$$

by algebra of limits and the chain rule. On the other hand, we have

$$\frac{h(x)-h(0)}{x-0}=x^2\sin\frac{1}{x}$$

which has limit equal to 0 at 0. It follows that h is differentiable at 0. Hence h is differentiable everywhere, with

$$h'(x)=3x^2\sin\frac{1}{x}-x\cos\frac{1}{x}$$

for every $x\neq 0$ and $h'(0)=0$. Note that h' is continuous at every $x\neq 0$, by algebra of continuous functions. Also, we have $\lim_{x\to 0}h'(x)=0=h'(0)$, so h' is also continuous at 0. It follows that h' is continuous everywhere. On the other hand, it is easy to see that h'' does not exist at 0.