Topic 20: Maximum Flow -- Solutions

1. Why the Residual Graph? Suppose that f_1 and f_2 are flows in a network G (NOT in the residual graph). Assume that f_1 and f_2 individually satisfy the conservation property and capacity constraints, and we computed the augmented flow $f_1 \uparrow f_2$.

Note: the point of this problem is to see what goes wrong if we don't use the residual graph. So don't use the residual graph for the first two parts! Use G.

> ***Preamble:*** to understand the following, we need to understand that a flow is simply a mapping of edges (u, v) to real numbers, and that if we are working with G in both cases the augmentation of flow f_1 by f_2, written $f_1 \uparrow f_2$, adds up the numbers that each flow assigns to the edge. The general definition of $f_1 \uparrow f_2$ also subtracts the flow from an edge (v, u) going in the opposite direction, but this will not be the case in the present problem as both flows are in G and no anti-parallel edges are allowed.

(a) Does this augmented flow $f_1 \uparrow f_2$ necessarily satisfy the conservation property? Show why (proof) or why not (counter-example).

> Yes. Since f_1 satisfies conservation, the sum of the flows it assigns to edges going into any given vertex v must be equal to the sum of the flows it assigns to edges going out of v. Let's call these sums a and b, and write the fact that f_1 satisfies conservation as $\boldsymbol{a = b}$. Similarly, f_2 satisfies conservation, so the sum of flows it assigns to edges going into v is equal to the sum of flows assigned to edges going out of v. Let's call these sums c and d, and write the fact that f_2 satisfies conservation as $\boldsymbol{c = d}$. Then, since $f_1 \uparrow f_2$ simply adds up the flows on each given edge, the sum of flows assigned by $f_1 \uparrow f_2$ to edges going into v will be $a + c$ and the flows assigned by $f_1 \uparrow f_2$ to edges going out of v will be $b + d$. But adding the two equations derived from the conservation of the two component flows, we get $\boldsymbol{a + c = b + d}$, so $f_1 \uparrow f_2$ also satisfies the conservation property.

(b) Does $f_1 \uparrow f_2$ necessarily satisfy the capacity constraint? Show why (proof) or why not (counter-example).

> No. Suppose an edge has capacity c. f_1 can assign flow $a \le c$ to this edge, and f_2 can assign flow $b \le c$ to this edge, but the flow assigned by $f_1 \uparrow f_2$ will be a+b,

which is not necessarily ≤ c. For example, on an edge of capacity 5, a flow of 3 is legal and a flow of 4 is legal, but a flow of 3+4=7 is not legal.

(c) How does finding augmented flows in the residual graph G_f prevent the problem you identified in one of the above questions? Be specific.

G_f assigns capacities to edges that are no more than the residual (remaining) capacity in G. In terms of constants in part (b), if we have assigned flow *a* to an edge of capacity *c* in G, then the capacity of this edge in G_f will be *c*-*a*, so even if the flow f_2 assigns full capacity to this edge, (*c*-*a*) + *a* = *c*, which is within capacity in G.

2. Solving a Problem with Flow. Professor Kardashian's daughters, Khloe, Kim and Kourtney, have not been getting along lately. When walking to school (they go to the same school), each refuses to walk down any street block that another one of them has walked down, though strangely they have no problem crossing the same corner. (They have no problem with their brother, so you need not be concerned about him.) Professor Kardashian wants to figure out how to get his children to school. Fortunately both their house and the school are on corners. The professor has a map of his town. Show how to formulate the problem of determining whether all of his children can go to the same school as a maximum flow problem.

Note: If you answer with either "yes they can go to school" or "no they cannot go to school" you will get 0 points for this question, as you cannot solve the problem without the map. You are showing the Professor how to *model* the problem as a flow problem so that he can solve it using his map.

(a) How do you construct a graph for this problem? What do the vertices and edges represent?

Take his map and make a graph in which each corner (including home and school) are vertices, and the street blocks that connect corners are edges. The home is s and the school is t.

(b) What does a unit of flow represent? How do you assign capacity constraints to the edges?

A unit of "flow" is one child walking down a block. Since no child will walk down the same block as another, we assign each edge a capacity of 1.

(c) How do you derive the solution from this graph representation?

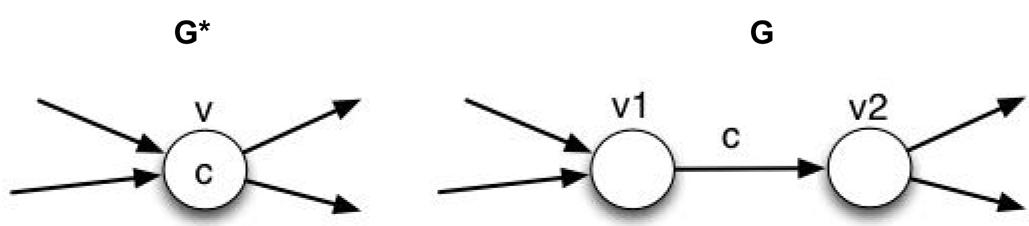
Run a flow algorithm to find the maximum flow. If the maximum flow is 3 or greater, the three daughters can walk to school; if it is 2, 1 or 0, the professor will have to drive them.

3. Challenge Problem: Extension to Vertex Capacities (read carefully). Suppose that, in addition to edge capacities, a flow network *G** has **vertex capacities**. We will extend the capacity function to work on vertices as well as edges: *c**(u,v) gives the usual edge capacity and *c**(*v*) gives the amount of flow that can pass through *v*. But we don't want to write a new algorithm.

We want to transform a flow network *G**=(*V**,*E**) with *c** that defines both edge and vertex capacities into an equivalent ordinary flow network *G*=(*V*,*E*) with *c* that is defined only on edges (without vertex capacities) such that a maximum flow in *G* has the same value as a maximum flow in *G**. Then we can run the algorithm we already have.

(a) Describe the appropriate transformation of problem representation. That is, **given *G**=(*V**,*E**) and *c**, how do you compute *G*=(*V*,*E*) and *c*?** Be as precise as possible.

Construct G as follows: For each vertex v $\in$ *G** with capacity c, make vertices v_1 and $v_2 \in G$ and a new edge (v_1, v_2) with capacity c (the same as v). Then rewire the graph so that all edges into v in *G** now go to v_1 in G, and all edges out of v in *G** now come out of v_2 in G (that is, replace (u, v) with (u, v_1) and (v, u) with (v_2, u)). Finally, make s_1 and t_2 be the new source and target vertices of G.



(b) What would you need to prove in order to prove that your solution is correct; that is, a solution computed on G will be a correct solution for *G?** (You don't need to actually do the proof: just identify everything you would have to prove.)

When flows computed on G are converted to flows in G*, we need to show that the latter respects

- edge capacities in G*
- vertex capacities in G*, and
- conservation constraints

Also, show

- flow equality: |f*| = |f|
- if flow f is maximal then f* is maximal.