MidTerm Exam #1

Monday, October 7, 2019

Name KEY

Solve all six problems. Show your work to receive full credit.

1. Let X denote the amount of time a book on two-hour reserve is checked out and suppose the cumulative distribution function is $F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{4} & 0 \le x < 2 \\ 1 & 2 \le x. \end{cases}$

   a. Calculate $P(X \le 1)$. *Give an exact answer (fraction or decimal).*

   $F(1) = \frac{1}{4}$

   b. Calculate $P(\frac{1}{2} \le X \le 1)$. *Give an exact answer (fraction or decimal).*

   $P(X \le 1) - P(0 \le X \le \frac{1}{2}) = \frac{1}{4} - F(\frac{1}{2}) = \frac{3}{16}$

   c. Calculate $P(X > \frac{3}{2})$. *Give an exact answer (fraction or decimal).*

   $1 - P(X \le \frac{3}{2}) = 1 - F(\frac{3}{2}) = 1 - \frac{9}{16} = \frac{7}{16}$

   d. Find the probability density function $f(x)$.

   $f(x) = \frac{x}{2}, \quad 0 \le x \le 2$

   e. Calculate the expected value. *Write your answer as a fraction (not a decimal).*

   $E(X) = \int_0^2 x \cdot \frac{x}{2}\, dx = \frac{1}{6}\left[x^3\right]_0^2 = \frac{4}{3}$

   f. Calculate the variance. *Write your answer as a fraction (not a decimal).*

   $E(X^2) = \int_0^2 x^2 \cdot \frac{x}{2}\, dx = \frac{1}{8}\left[x^4\right]_0^2 = 2$

   $V(X) = E(X^2) - E(X)^2 = 2 - \frac{16}{9} = \frac{2}{9}$

2. Time taken by a randomly selected applicant to fill out a certain form for a mortgage has a normal distribution with mean 10 minutes and standard deviation 2 minutes. Five individuals fill out a form on one day and six on another. Assuming the results for the two days are independent, what is the probability that the sample average amount of time taken on both days is at most 11 min? *Round your answer to four decimal places.*

$\mu = 10 \quad \sigma = 2$

Sample 1

$n = 5$

$\mu_1 = 10 \quad \sigma_1 = \frac{2}{\sqrt{5}}$

$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$

Sample 2

$n = 6$

$\mu_2 = 10 \quad \sigma_2 = \frac{2}{\sqrt{6}}$

P (sample 1 average ≤ 11) =

$P\left(Z \leq \frac{\sqrt{5}}{2}\right) \approx .8686$

P (Sample 2 average ≤ 11) =

$P\left(Z \leq \frac{\sqrt{6}}{2}\right) \approx .8888$

P (sample average on both days ≤ 11) $\approx (.8686)(.8888) \approx .7720$

3. Let $X_1, X_2, X_3, X_4, X_5, X_6$ denote the numbers of blue, brown, green, orange, red, and yellow *M&M* candies, respectively, in a sample of size 20. Then these X_i's have a multinomial distribution. Suppose that the color proportions are $p_1 = 0.24$, $p_2 = 0.13$, $p_3 = 0.16$, $p_4 = 0.2$, $p_5 = 0.13$, and $p_6 = 0.14$. What is the probability that the number of blue, green, or orange *M&M* candies is at least 10? *Round your answer to four decimal places.*

blue green orange

$Y = X_1 + X_3 + X_4$ is a binomial r.v. with success probability $p = .6$

$P(Y \geq 10) = 1 - P(Y < 10) = 1 - P(Y \leq 9) = 1 - B(9; 20, .6)$

$= .8720$

4. A random variable X has a standard beta distribution with parameters α and β if its probability density function is given by $f(x;\alpha,\beta)=\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$, $0\le x\le 1$, with mean $\mu=\dfrac{\alpha}{\alpha+\beta}$ and variance $\sigma^2=\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

Suppose that the proportion X of surface area covered by a certain species of plant in a randomly selected region has a standard beta distribution with $\alpha=5$ and $\beta=2$.

a. Compute μ and σ^2. *Round your answers to four decimal places.*

$$\mu=\frac{5}{7}\approx .7143 \qquad \sigma^2=\frac{10}{49.8}\approx .0255$$

b. Compute $P(.2\le X\le .4)$. *Round your answer to four decimal places.*

$$\int_{.2}^{.4}\frac{6!}{4!\,1!}x^4\cdot(1-x)^1\,dx=\int_{.2}^{.4}30(x^4-x^5)\,dx=30\left[\tfrac{1}{5}x^5-\tfrac{1}{6}x^6\right]_{.2}^{.4}=$$

$$\left[6x^5-5x^6\right]_{.2}^{.4}=\left[x^5\cdot(6-5x)\right]_{.2}^{.4}=.01024\cdot4-.00032\cdot5=.04096-.0016=.3936$$

c. What is the expected portion of the sampling region *not* covered by the plant species? *Write your answer as a fraction (not a decimal).* X: proportion/portion covered by plant

$$E(1-X)=1-E(X)=1-\int_0^1 30x(x^4-x^5)\,dx$$
$$=1-\int_0^1 30(x^5-x^6)\,dx$$
$$=1-30\left[\tfrac{1}{6}x^6-\tfrac{1}{7}x^7\right]_0^1$$
$$=1-30\left[\tfrac{1}{6}-\tfrac{1}{7}\right]$$
$$=1-30\cdot\tfrac{1}{42}$$
$$=1-\tfrac{5}{7}$$
$$=\tfrac{2}{7}$$

5. The joint probability density function modeling the distribution of weights of almonds, cashews, and peanuts in a 1-pound can of mixed nuts, based on the weight of almonds, *X*, and the weight of cashews, *Y*, is $f(x,y) = 24xy$, for $0 \le x \le 1$, $0 \le y \le 1$, $0 \le x+y \le 1$ and $f(x,y) = 0$ otherwise. The marginal probability density functions are $f_X(x) = 12x(1-x)^2$, $0 \le x \le 1$, and $f_Y(y) = 12y(1-y)^2$, $0 \le y \le 1$.

a. Calculate $f_X(\frac{3}{4})$, $f_Y(\frac{3}{4})$, and $f(\frac{3}{4},\frac{3}{4})$ to show that *X* and *Y* are not indepedent.

$$f_X(\tfrac{3}{4}) = 9/16 \qquad f_Y(\tfrac{3}{4}) = 9/16 \qquad f(\tfrac{3}{4},\tfrac{3}{4}) = 24\cdot\frac{9}{16}$$

$$\underbrace{\frac{9}{16}\cdot\frac{9}{16}}_{81/256} \neq \underbrace{24\cdot\frac{9}{16}}_{27/2}$$

b. Calculate μ_X and μ_Y. *Use symmetry and write your answers as fractions (not decimals).*

$$\mu_X = \int_0^1 12x^2(1-x)^2\,dx = 12\int_0^1 (x^2 - 2x^3 + x^4)\,dx = 12\left[\tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5}\right] = 4 - 6 + \frac{12}{5} = \frac{2}{5} = \mu_Y$$

c. Calculate σ_X and σ_Y. *Use symmetry and write your answers as fractions (not decimals).*

$$E(X^2) = 12\int_0^1 (x^3 - 2x^4 + x^5)\,dx = 12\left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6}\right) = \frac{1}{5}$$

$$\sigma_X^2 = E(X^2) - E(X)^2 = \frac{1}{5} - \frac{4}{25} = \frac{1}{25} \Rightarrow \sigma_X = \sigma_Y = \frac{1}{5}$$

d. Calculate $E(XY)$. *Write your answer as a fraction (not a decimal).*

$$\iint_{0 \le x+y \le 1} xy(24xy)\,dA = \int_0^1\int_0^{1-x} 24x^2y^2\,dx\,dy = 24\int_0^1 y^2\cdot\frac{1}{3}(1-y)^3\,dy = 8\int_0^1 (y^2 - 3y^3 + 3y^4 - y^5)\,dy$$

$$= 8\left[\frac{1}{3} - \frac{3}{4} + \frac{3}{5} - \frac{1}{6}\right]$$

$$= \frac{8}{3} - 6 + \frac{24}{5} - \frac{4}{3}$$

$$= \frac{2}{15}$$

e. Calculate the covariance $\text{Cov}(X,Y)$ between the random variables *X* and *Y* and the correlation coefficient ρ of *X* and *Y*. *Write your answers as fractions (not decimals).*

$$\text{Cov}(X,Y) = E(XY) - \mu_X\mu_Y = \frac{2}{15} - \frac{4}{25} = -\frac{2}{75}$$

$$\rho = \frac{\text{Cov}(X,Y)}{\sigma_X\sigma_Y} = \frac{-\frac{2}{75}}{\frac{1}{25}} = -\frac{2}{3}$$

6. Let $X_1, \ldots, X_n$ represent a random sample from a Rayleigh distribution with probability density function $f(x;\theta) = \frac{x}{\theta} e^{-x^2/(2\theta)}$ for $x > 0$, and expected value given by $E(X^2) = 2\theta$.

a. Show that $\hat{\theta} = \frac{\sum_{i=1}^{n} X_i^2}{2n}$ is an unbiased estimator for θ.

$$E(\hat{\theta}) = E\left(\frac{\sum X_i^2}{2n}\right) = \frac{\sum E(X_i^2)}{2n} = \frac{\sum_{i=1}^{n} 2\theta}{2n} = \frac{n\cdot(2\theta)}{2n} = \theta$$ ✓

b. Show that the maximum likelihood estimator of θ is $\hat{\theta} = \frac{\sum_{i=1}^{n} X_i^2}{2n}$.

$$f(x_1,\ldots,x_n;\hat{\theta}_1,\ldots,\hat{\theta}_n) \geq f(x_1,\ldots,x_n;\theta_1,\ldots,\theta_n)$$

maximize as function of $\theta_1, \ldots, \theta_n$

Our distribution: $f(x_1,\ldots,x_n;\theta) = f(x_1;\theta)\cdots f(x_n;\theta)$ rs $\Rightarrow$ ind. X_i's

maximize as function of θ

$$f(x_1,\ldots,x_n;\theta) = \frac{x_1}{\theta} e^{-x_1^2/(2\theta)} \cdots \frac{x_n}{\theta} e^{-x_n^2/(2\theta)} = \frac{x_1\cdots x_n}{\theta^n} \cdot e^{\frac{-1}{2\theta}\sum x_i^2}$$

$$\ln f = (\sum \ln x_i) + \frac{-1}{2\theta}\sum x_i^2 - n\ln\theta$$

$$\frac{d(\ln f)}{d\theta} = -\frac{n}{\theta} + \frac{\sum x_i^2}{2\theta^2} = 0 \iff \theta = \frac{\sum x_i^2}{2n} \Rightarrow$$

$\hat{\theta} = \left(\sum_{i=1}^{n} x_i^2\right)/2n$ is a maximum likelihood estimator (mle)