Exercise 3.41

Let $f(x) = xe^x - 1$ and let the interval of length of 1 be [0,1]. Then at x = 0, f(0) = $0e^0 - 1 = -1$ and at x = - 1, $f(1) = 1e^1 - 1$. The value $e \approx 2.72$ then f(1) $\approx$ 1.72.
Then by the Bolzano's theorem since f(0) is a negative value and f(1) is a positive value it follows that there exists a value $c \in (0, 1)$ such that f(c) = 0.
Hence there is a root within the length of 1 for $xe^x = 1$

Exercise 3.42

Let $\phi(x) = x^3 - 6x^2 + 2.826$ and since we are dealing with an interval of length 1 let it be (0,1). Then for x = 0, $\phi(0) = 0^3 + 6(0)^2 + 2.826 = 2.826$ and let $\phi(1) = 1^3 - 6(1)^2 + 2.826 = -4.826$. Then by the Bolzano's Theorem since $\phi(0)$ and $\phi(1)$ have opposite signs then the exists c $\in (0, 1)$ such that $\phi(c) = 0$.
Therefore, there is a root within the interval of length of 1 such that $x^3 - 6x^2 + 2.826$.

Exercise 3.44

Let g(x) = f(x) - x. Then for x = a, then

$$g(a) = f(a) - a \geq a - a = 0$$

And for x = b then

$$g(b) = f(b) - b \leq b - b = 0$$

If g(a) > 0 and g(b) < 0 then by Bolaznno's theorem there exists a x $\in (a, b)$ such that g(x) = 0. Then if g(x) = 0 then there exists a fixed point such that f(x) = x.
Otherwise, if g(a) = 0 or g(b) = 0, then there exists a fixed point such that x = a for a $\in (a, b)$ or x = b for b $\in (a, b)$.

Exercise 4.3

Suppose f(x) = $\sqrt{x}$ for all $x > 0$. From the definition, let $x_0 = 0$ and then from the definition let us define T(x) = $\frac{f(x)-f(0)}{x-0}$. Then it follows that

$$\lim_{x \to x_0} T(x) = \lim_{x \to x_0} \frac{\sqrt{x} - 0}{x - 0} = \lim_{x \to 0} \frac{x^{0.5}}{x} = \lim_{x \to 0} x^{-0.5}.$$

If follows that the derivative of f(x) = $\sqrt{x}$ is $x^{-0.5}$. However, since the limit tends to ∞ then f is not differentible at 0.

Exercise 4.4

Let g(x) = x^2. Then from the definition of derivatives, let $x_0 \in \mathbb{D}$ and x_0 be an accumulation point such that $x_0 \neq 0$.
Suppose for each t $\in \mathbb{R}$ such that $x_0 + t \in \mathbb{D}$ and $t \neq 0$ then let us define Q(t) as

$$Q(t) = \frac{g(x_0 + t) - g(x_0)}{t}$$

Then by doing some algebraic work it follows that and taking the limit a t = 0 from the alternate definition:

$$\lim_{t\to 0} Q(t) = \lim_{t\to 0} \frac{(x_0+t)^2 - (x_0)^2}{t} = \lim_{t\to 0} 2x_0 + t = 2x_0$$

Thus the derivative of g(x) = x^2 is g'(x) = $2x$.

Exercise 4.5

Suppose we define h(x) = $x^3 \sin(\frac{1}{x})$ for $x \neq 0$ and h(0) = 0.
By applying the definition of derivatives given in Chapter 4, suppose we define T(x) = $\lim_{x\to 0} \frac{h(x)-h(0)}{x-0}$. Then it follows that

$$\lim_{x\to 0} \frac{x^3 \sin(\frac{1}{x}) - 0}{x - 0} = \lim_{x\to 0} x^2 \sin(\frac{1}{x})$$

By using the similar concept that was used in past homework problem 2.6, we know that there exists a derivative at $x_0 = 0$ and that the $\lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0$. Therefore, from theorem 3.1 it follows that if there is a limit then it is equivalent to being continuous at 0
However, using the product rule that was taught in Calculus course, it follows that the derivative of h(x) is

$$h'(x) = 2x \sin(\frac{1}{x}) - x \cos(\frac{1}{x})$$

Thus h' is continues everywhere but fails to have a derivative at 0.