Problem 24

Let $f : [a, b] \to \mathbb{R}$ be monotone. This follows that when $a < x < b$ then this implies that either $f(a) \leq f(x) \leq f(b)$ or $f(a) \geq f(x) \geq f(b)$ for all $x \in (a, b)$. This follows that either f(x) is increasing or decreasing.
Suppose that f is increasing. Let A(a) = $\inf\{f(y) : a < y\}$ and B(b) = $\sup\{f(y) : y < b\}$.
Suppose that $\lim_{x\to a} f(x) = A(a)$ and $\lim_{x\to b} f(x) = B(b)$.
Let there be an $\epsilon > 0$. Since A(a) + ϵ is not a lower bound for $\{f(y) : a < y\}$, then there is a real number n_1 such that $n_1 \in [a, b]$ and $a < n_1$ such tht $f(n_1) < A(a) + \epsilon$.
Let $\delta = n_1 - a$. Since by the definition of a limit, $0 < |x - a| < \delta, x_1 < n_1$, which follows that we have the following inequality: $A(a) - \epsilon < A(a) \leq f(a) < f(x) < f(n_1) < A(a)$.
Therefore, $\lim_{x\to a} f(x) \to A(a)$.
Furthermore, a similar proof can be constructed to see find that the $\lim_{x\to b} g(x)B(b)$.
Hence, there exists a limit at both a and b.