NAME:

Math 331
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Final

You must justify your answers by showing your work to receive credit. Be very careful with all that you write; please do not make me take off points for statements that you do not mean.

Part I. Do every problem in this part.

(8) **1.** **a.** State the *Mean Value Theorem.*

b. State the *Least Upper Bound Property* of real numbers.

(12) **2.** Give definitions of the following:

a. An *accumulation point* of a set of real numbers.

b. A *uniformly continuous* function.

c. A *Cauchy sequence* of real numbers.

d. An *open* set of real numbers.

(10) **3.** Suppose that $f(x) \leq g(x)$ for all $x \in [a, b]$ and $f, g \in \mathcal{R}[a, b]$.

a. Explain why $L(P, f) \leq L(P, g)$ for every partition P of $[a, b]$. (Your explanation should be based on the definition of lower sum.)

b. Use part **a.** to prove that $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$. Your proof should be based on the definitions.)

(10) **4.** Suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are sequences of real numbers, such that $\{a_n\}_{n=1}^{\infty}$ converges to A and $\{b_n\}_{n=1}^{\infty}$ converges to B. Prove that $\{a_n + b_n\}_{n=1}^{\infty}$ converges to $A + B$. (Your proof should be based on the definition involving ε of a convergent sequence.)

(40) **5.** Indicate by writing **T** or **F** whether each statement is true or false. Give no proofs.

a. If A is a non-empty and compact set of real numbers, then A contains $\inf A$ and $\sup A$.

b. If $f : (2, 10) \to \mathbb{R}$ is uniformly continuous, then f is bounded.

c. There is a function f that is differentiable on (0,1) with $f'(x) = \dfrac{7x^2 - \sin x}{\sqrt{e^x + 3}}$.

d. If A and B are compact sets of real numbers, then $A \cup B$ is also a compact set.

e. If A and B are open sets of real numbers, then $A \cup B$ is also an open set.

f. Every monotone sequence of real numbers converges.

g. If $f \in \mathcal{R}[0, 1]$ and K is a compact subset of $[0, 1]$, then $f(K)$ is compact.

h. If f is a continuous function on $[0, 1]$ and $g(x) = 3(f(x))^2 + x^5 - 7$, then $g \in \mathcal{R}[0, 1]$.

i. Every set of real numbers that is bounded and non-empty has at least one accumulation point.

j. If $f : \mathbb{R} \to \mathbb{R}$ and $f([0, 1])$ is an open set, then f is not continuous.

Part II. Do any 5 of the 7 problems in this part. Each problem is worth 15 points.

(15) **6.** [Homework problem #1.36] Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers. Prove that $\{a_n\}_{n=1}^{\infty}$ has a convergent subsequence. (*Hint*: You may want to use the Bolzano-Weierstrass Theorem.)

(15) **7.** [Homework problem #5.9] Assume $f : [a, b] \to \mathbb{R}$ is continuous and $f(x) \geq 0$ for all $x \in [a, b]$. Prove that if $\int_a^b f(x)\, dx = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

(15) **8.** Prove that a compact set of real numbers is closed. (This is part of the easy direction of the proof of the Heine-Borel Theorem. Do not just quote that theorem; prove this part of it from the definitions.)

(15) **9.** Let $E = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ and define $f : E \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{-1}{n}, & \text{if } x = \frac{1}{n} \text{ where } n \text{ is odd;} \\ \frac{+1}{n}, & \text{if } x = \frac{1}{n} \text{ where } n \text{ is even;} \\ 0, & \text{if } x = 0. \end{cases}$$

a. At what points of E is f continuous? Justify your answer!

b. At what points of E is f differentiable? Justify your answer!

(15) **10.** [Midterm 1 problem **6.**] Suppose that $f : \mathbb{R} \to \mathbb{R}$, $f(x) < 0$ if $x < 0$, $f(x) > 0$ if $x > 0$ and $\lim_{x\to 0} f(x)$ exists. Prove that $\lim_{x\to 0} f(x) = 0$.

(15) **11.** **a.** State a necessary and sufficient condition for $f : [a, b] \to \mathbb{R}$ to be Riemann integrable. (See part **b.** before choosing which condition.)

b. Use the condition in part **a.** to prove that if $f : [a, b] \to \mathbb{R}$ is monotone, then $f \in \mathcal{R}[a, b]$.

(15) **12.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and suppose that $f'(x)$ exists for all $x \neq 0$ and that $\lim_{x \to 0} f'(x) = A$. Prove that f is differentiable at 0, and if possible find $f'(0)$.
(Hint: Start with the definition of $f'(0)$ as a limit, and then try to use the Mean Value Theorem.)