Section 5.1 Exercise 2

Sol.

Let $f : [0,1] \to \mathbb{R}$ defined by $f(x) = x$. To prove that f is Riemann integrable on $[0,1]$, let $\epsilon > 0$. Take any partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[0,1]$ having the property that

$$x_i - x_{i-1} < \epsilon \qquad (i = 1, \ldots, n).$$

Now, clearly we have $M_i(f) = x_i$ and $m_i(f) = x_{i-1}$. Thus

$$\begin{aligned} U(P,f) - L(P,f) &= \sum_{i=1}^{n} (x_i - x_{i-1})(x_i - x_{i-1}) \\ &< \sum_{i=1}^{n} \epsilon (x_i - x_{i-1}) \\ &= \epsilon. \end{aligned}$$

It follows that f is Riemann integrable on $[0,1]$, by Theorem 5.2.
Now, to compute $\int_0^1 f(x)\,dx$, let $P = \{x_0, \ldots, x_n\}$ be an arbitrary partition of $[0,1]$, and note that for each i, we have

$$x_{i-1} \le \frac{x_{i-1} + x_i}{2} \le x_i.$$

Hence

$$L(P,f) = \sum_{i=1}^{n} x_{i-1}(x_i - x_{i-1}) \le \frac{1}{2} \sum_{i=1}^{n} (x_{i-1} + x_i)(x_i - x_{i-1}) = \frac{1}{2} \sum_{i=1}^{n} (x_i^2 - x_{i-1}^2) = \frac{1}{2}.$$

Similarly, we find $U(P,f) \ge 1/2$. Hence

$$L(P,f) \le \frac{1}{2} \le U(P,f)$$

for all partitions P of $[0,1]$. This implies that

$$\underline{\int_0^1} f(x)\,dx \le \frac{1}{2} \le \overline{\int_0^1} f(x)\,dx.$$

But since f is Riemann integrable, both the lower and upper integrals are equal to $\int_0^1 f(x)\,dx$, and the above gives $\int_0^1 f(x)\,dx = 1/2$.

Section 5.1 Exercise 4

Sol.

Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[0,1]$. For each i, since A is dense in $[0,1]$, the interval $[x_{i-1}, x_i]$ contains a point of A, and the value of f at that point is 0 by assumption, so $m_i(f) \le 0 \le M_i(f)$. We get

$$L(P,f) = \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}) \le 0$$

and

$$U(P,f) = \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}) \ge 0.$$

It follows that

$$\underline{\int_0^1} f(x)\,dx \le 0 \le \overline{\int_0^1} f(x)\,dx.$$

Since f is Riemann integrable, both the lower and upper integrals are equal to $\int_0^1 f(x)\,dx$, and the above gives $\int_0^1 f(x)\,dx = 0$.

Section 5.2 Exercise 8

Sol.

This is a very simple modification of Example 5.5. See the textbook for the details. We find

$$\int_1^3 x\,dx = 4.$$

Section 5.2 Exercise 9

Sol.

Suppose for a contradiction that $f(c) > 0$ for some $c \in [a,b]$. By the proof of Lemma 3.3 in the textbook, there exists an interval I containing c such that $f(x) > 0$ for all $x \in I \cap [a,b]$. Now, let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$ chosen so that one of the subintervals $[x_{i-1}, x_i]$ belongs to $I \cap [a,b]$. Then the corresponding $m_i(f)$ is positive, and we get that $L(P,f) > 0$. This gives

$$\underline{\int_a^b} f(x)\,dx \ge L(P,f) > 0.$$

Now, since f is continuous, it is Riemann integrable, hence

$$\int_a^b f(x)\,dx = \underline{\int_a^b} f(x)\,dx > 0$$

a contradiction.

Section 5.3 Exercise 14

Sol.

Let $f : [-b, b] \to \mathbb{R}$ be an odd integrable function. For $n \in \mathbb{N}$, let P_n be the partition consisting of the points

$$-b + k\frac{b}{n} \qquad (k = 0, 1, \ldots, 2n).$$

Then $\mu(P_n) = \frac{b}{n} \to 0$ as $n \to \infty$, so we know that the Riemann sums $S(P_n, f)$ converge to $\int_{-b}^{b} f(x)\, dx$, regardless of how we choose the marked points. Let us then choose the marked points to be symmetric on both sides of 0, so that $t_k = -t_{k+n}$ for $k = 0, \ldots, n$. Then we get

$$\begin{aligned} S(P_n, f) &= \frac{b}{n}\sum_{k=0}^{2n} f(t_k) \\ &= \frac{b}{n}\sum_{k=0}^{n} f(t_k) + \frac{b}{n}\sum_{k=n+1}^{2n} f(t_k) \\ &= \frac{b}{n}\sum_{k=0}^{n} f(-t_{k+n}) + \frac{b}{n}\sum_{k=n+1}^{2n} f(t_k) \\ &= -\frac{b}{n}\sum_{k=0}^{n} f(t_{k+n}) + \frac{b}{n}\sum_{k=n+1}^{2n} f(t_k) = 0. \end{aligned}$$

It follows that $\int_{-b}^{b} f(x)\, dx = 0$, as required. The case where f is even is similar. In this case, we get

$$S(P_n, f) = 2\frac{b}{n}\sum_{k=n+1}^{2n} f(t_k)$$

and this converges to

$$2\int_0^b f(x)\, dx$$

as required.

Section 5.4 Exercise 16

Sol.

For (a), we find the antiderivative $F(x) := \frac{x^3}{3} - \frac{x^2}{2}$, so

$$\int_0^3 (x^2 - x)\, dx = F(3) - F(0) = \frac{9}{2}.$$

For (b), we find the antiderivative $F(x) := x - \frac{x^4}{4} - \frac{x^3}{3}$, so

$$\int_{-2}^{4} (1 - x^3 - x^2)\, dx = F(4) - F(-2) = -78.$$

For (c), we find the antiderivative

$$F(x) := -\frac{1}{2} \cos x^2$$

so that

$$\int_{0}^{\pi/2} x \sin x^2\, dx = F(\pi/2) - F(0) = -\frac{1}{2} \cos \pi^2/4 + \frac{1}{2}.$$