****MAKE SURE YOU ARE DOING YOUR OWN WORK. THERE ARE A HANDFUL OF YOU WHO ARE TURNING IN IDENTICAL HOMEWORK SOLUTIONS, VIRTUALLY WORD FOR WORD, EVEN WITH THE SAME GRAMMATICAL ERRORS. COLLABORATING IS ENCOURAGED. COPYING IS VIOLATING ACADEMIC INTEGRITY.

1. (Chapter 4, exercise 20)
   Suppose $f : [0,2] \to \mathbb{R}$ is differentiable, $f(0) = 0, f(1) = 2$, and $f(2) = 2$. Prove that
   (a) there is c_1 such that $f'(c_1) = 0$,
   (b) there is c_2 such that $f'(c_2) = 2$, and
   (c) there is c_3 such that $f'(c_3) = \frac{3}{2}$.

Proof. First, since f is differentiable on $[0,2]$,by theorem 4.2 it is continuous on $[0,2]$, and hence continuous on $(0,2)$. For parts (a) and (b), we can then apply the Mean-Value Theorem as follows:

$$\text{There exists } c_1 \in (1,2) \text{ such that } f'(c_1) = \frac{f(2) - f(1)}{2-1} = \frac{2-2}{2-1} = 0, \text{ and}$$

$$\text{there exists } c_2 \in (0,1) \text{ such that } f'(c_2) = \frac{f(1) - f(0)}{1-0} = \frac{2-0}{1-0} = 2.$$

For part (c), we invoke theorem 4.11. Since f is differentiable on $[0,2]$, and from parts (a) and (b) there are $c_2 \in (0,1)$ and $c_1 \in (1,2)$ such that

$$f'(c_1) = 0 < \frac{3}{2} < 2 = f'(c_2),$$

by theorem 4.11 there is $c_3 \in (c_2, c_1) \subset (a,b)$ such that $f'(c_3) = \frac{3}{2}$ as desired. □

***COMMON MISTAKES ON THIS EXERCISE:
- not justifying use of MVT (get in the habit of establishing that the hypotheses of a theorem you want to use are satisfied before using the theorem)
- trying to apply IVT to f' in part (c), but you do not know f' is continuous (hence, the hypotheses of IVT are not necessarily satisfied, so you cannot use this theorem...notice the importance of the first comment here)

2. (Chapter 4, exercise 23)
   Use the Mean-Value Theorem to prove that

$$\sqrt{1+h} < 1 + \frac{1}{2}h \quad \text{for} \quad h > 0.$$

Proof. Let $f : [0,\infty) \to \mathbb{R}$ be given by $f(x) = \sqrt{1+x}$. Note, f is continuous on $[0,\infty)$ and differentiable on $(0,\infty)$, with $f'(x) = \frac{1}{2\sqrt{1+x}}$ by theorems 4.3 and 4.4. Observe, $\frac{1}{2\sqrt{1+x}} < \frac{1}{2}$ for all $x > 0$.

Given $h > 0$, since f is continuous and differentiable, the Mean-Value Theorem guarantees an $x_0 \in (0,h)$ such that $f'(x_0) = \frac{f(h)-f(0)}{h-0} = \frac{\sqrt{1+h}-1}{h}$. Hence, we have

$$\frac{\sqrt{1+h}-1}{h} = f'(x_0) = \frac{1}{2\sqrt{1+x_0}} < \frac{1}{2}$$

for all $h > 0$. This is equivalent to the desired inequality $\sqrt{1+h} < 1 + \frac{1}{2}h$ for all $h > 0$. □

***COMMON MISTAKES ON THIS EXERCISE:
- improper use of MVT; MVT does not ever tell you that $f'(h) = \frac{f(h)-f(x)}{h-x}$. That is, it never tells you the derivative at a point is equal to a slope involving that exact point.
- $\frac{1}{2\sqrt{1+h}}$ never equals $\frac{\sqrt{1+h}-1}{h}$...if you solve this equation, you only get the extraneous solution of $h = 0$
- an equality does not magically turn into an inequality when you change values of the inputs. You cannot first say $\frac{1}{2\sqrt{1+h}} = \frac{\sqrt{1+h}-1}{h}$ and then change it to $\frac{1}{2\sqrt{1+h}} > \frac{\sqrt{1+h}-1}{h}$ because $h > 0$.

3. (Chapter 4, exercise 25)
Suppose $f : (a,b) \to \mathbb{R}$ is differentiable and $|f'(x)| \leq M$ for all $x \in (a,b)$. Prove that f is uniformly continuous on (a,b). Give an example of a function $f : (0,1) \to \mathbb{R}$ that is differentiable and uniformly continuous on $(0,1)$ but such that f' is unbounded.

Proof. Since $f : (a,b) \to \mathbb{R}$ is differentiable, theorem 4.2 tells us that $f : (a,b) \to \mathbb{R}$ is also continuous. Hence, for any $x, y \in (a,b)$ with $x < y$, we can appeal to the Mean-Value Theorem to obtain $c \in (x,y)$ such that $f'(c) = \frac{f(x)-f(y)}{x-y}$. Note, by assumption, $\frac{|f(x)-f(y)|}{|x-y|} = |f'(c)| \leq M$. This yields $|f(x)-f(y)| \leq M|x-y|$ for all $x, y \in (a,b)$.

Let $\epsilon > 0$ be given. Set $\delta = \frac{\epsilon}{M}$. Then for $x, y \in (a,b)$ such that $|x-y| < \delta$, we have

$$|f(x) - f(y)| \leq M|x-y| < M\delta = M\frac{\epsilon}{M} = \epsilon,$$

showing f to be uniformly continuous by definition.

If we consider the function $f : (0,1) \to \mathbb{R}$ given by $f(x) = \sqrt{x}$, we see that the conclusion of the exercise is satisfied as f is uniformly continuous, but the hypothesis that $|f'(x)| \leq M$ for all $x \in (0,1)$ does not hold since $f'(x) = \frac{1}{2\sqrt{x}}$ is unbounded on $(0,1)$.

(*You can show f is uniformly continuous by definition or by use of theorem 3.8.) □

***COMMON MISTAKES ON THIS EXERCISE:

- improper use of MVT as mentioned above