Section 1.2 Exercise 16

Sol.

Suppose that the sequences (a_n) and (b_n) are Cauchy. In particular, the sequences (a_n) and (b_n) are bounded, so there exist $M_1, M_2 \in \mathbb{R}$ such that $|a_n| \leq M_1$ and $|b_n| \leq M_2$ for all n.
Now, let $\epsilon > 0$. Let $N_1 \in \mathbb{N}$ such that for all $m, n \geq N_1$, we have

$$|a_n - a_m| < \frac{\epsilon}{2M_2}.$$

Similarly, let $N_2 \in \mathbb{N}$ such that for all $m, n \geq N_2$, we have

$$|b_n - b_m| < \frac{\epsilon}{2M_1}.$$

Let $N := \max(N_1, N_2)$. Then for all $m, n \geq N$, we have

$$\begin{aligned} |a_n b_n - a_m b_m| &= |a_n(b_n - b_m) + b_m(a_n - a_m)| \\ &\leq |a_n||b_n - b_m| + |b_m||a_n - a_m| \\ &\leq M_1|b_n - b_m| + M_2|a_n - a_m| \\ &\leq M_1\frac{\epsilon}{2M_1} + M_2\frac{\epsilon}{2M_2} = \epsilon. \end{aligned}$$

It follows that $(a_n b_n)$ is Cauchy, as required.

Section 1.2 Exercise 24

Sol.

Suppose that (a_n) converges to a and $\{a_n\}_{n\in\mathbb{N}}$ is infinite. We want to show that a is an accumulation point of the set $\{a_n\}_{n\in\mathbb{N}}$.

Let I be an interval containing a. We can choose $\epsilon > 0$ sufficiently small so that $(a - \epsilon, a + \epsilon) \subset I$. Now, since (a_n) converges to a, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$|a_n - a| < \epsilon.$$

The above inequality implies that $a_n \in (a-\epsilon, a+\epsilon)$, for all $n \geq N$. Now, the set $\{a_n\}_{n\geq N}$ must be infinite, and the above shows that each of the infinitely many elements of $\{a_n\}_{n\geq N}$ belongs to the interval $(a - \epsilon, a + \epsilon)$ and hence belongs to I. In particular, the interval I contains infinitely many elements of $\{a_n\}_{n\in\mathbb{N}}$. Since I was an arbitrary interval containing a, we get that a is an accumulation point of $\{a_n\}_{n\in\mathbb{N}}$, as required.

Section 1.3 Exercise 25

Sol.

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences such that $(a_n)_{n\in\mathbb{N}}$ and $(a_n+b_n)_{n\in\mathbb{N}}$ converge. It is easy to see that the sequence $(-a_n)_{n\in\mathbb{N}}$ also converges. By a theorem proved in class, it follows that the sequence $(b_n)_{n\in\mathbb{N}}$ converges, since

$$b_n = (a_n + b_n) + (-a_n).$$

Section 1.3 Exercise 26

Sol.

It suffices to consider the sequences $(a_n) = (1, -1, 1, -1, \dots)$ and $(b_n) = (-1, 1, -1, 1, \dots)$. Then (a_n) and (b_n) diverge, but $(a_n + b_n)$ is the sequence $(0, 0, 0, 0, \dots)$, which obviously converges.

Section 1.3 Exercise 28

Sol.

Suppose that (a_n) converges to a and that $a_n \geq 0$ for all n.

We first consider the case $a = 0$. In this case, let $\epsilon > 0$, and let $N \in \mathbb{N}$ such that $a_n < \epsilon^2$ for all $n \geq N$. We then have $\sqrt{a_n} < \epsilon$ for $n \geq N$. This shows that $(\sqrt{a_n})$ converges to $0 = \sqrt{a}$, as required.

Now, for the case $a > 0$, let $\epsilon > 0$, and let $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$|a_n - a| < \sqrt{a}\epsilon.$$

It follows that for $N \geq N$, we have

$$|\sqrt{a_n} - \sqrt{a}| = \frac{|a_n - a|}{\sqrt{a_n} + \sqrt{a}} \leq \frac{|a_n - a|}{\sqrt{a}} < \epsilon.$$

This shows that $(\sqrt{a_n})$ converges to $\sqrt{a}$, as required.

Section 1.3 Exercise 32

Sol.

Combining the various results seen in class, we get the following limits:

(a) The limit is 1.

(b) The limit is 0 (use theorem 1.13).

(c) The limit is 0 (use theorem 1.13).

(d) The limit is 0.

(e) The limit is $-1/4$ (see Example 1.5).

(f) The limit is 0 (use theorem 1.13).