Key Terms

- convergence
- sequence
- neighborhood
- cauchy
- accumulation points
- Bolzanno Weierstrass theorem
- sub-seqeunce
- increase
- decrease
- monotone
- limits
- continuous
- uniformly continuous
- closed
- open
- compact
- Extreme Value Theorem connected
- Intermediate Value Theorem
- Derivative
- Chain Rule
- relative max (min)
- Rolle's theorem
- Cauchy Mean Value Theorem
- L'Hopital
- First Mean Value Theorem
- Second Mean Value Theorem

Chapter Problems

1. 2,7,9,10,11,16,24,25,26,26,28,32,24,35,38,40,47
2. 2,7,11,12,15,19,22,24
3. 2,3,5,6,14,15,19,22
4.
5.

Sample Problems

1. Assume that e^x is continuous on $\mathbb{R}$. Use this and any theorem to show that f(x) = x^2e^x is uniformly continuous on [0,1].
2. Find
$$\lim_{x\to 0} x^2 \sin(\frac{1}{x})$$
3. Use the L'Hospital's rule to find:
$$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{\sqrt{x}}$$
4. Use an $\epsilon - \mathbb{N}$ argument to find:
$$\lim_{x\to \infty} \frac{1}{n^2+1}$$
5. State the Bolzano's theorem in the context of letting $f : [a,b] \to \mathbb{R}$
6. Prove that $g(x) = x^3 + x - 1$ has at least one root which lies in the open interval (0, 1).
7. State the Mean Value Theorem
8. State the Least Upper Bound Property of Real numbers
9. Give the definitions of the following:
   - accumulation point of a set of real numbers
   - uniformly continuous functions
   - Cauchy seqeuence of real numbers
   - open set of real numbers
10. State True (T) or False (F) for the following:
    - If A is a non-empty and compact set of real numbers then A contains inf A and sup A.

- If f:(2,10) $\to \mathbb{R}$ is uniformly continuous, then it is bounded
- If A and B are compact sets of real numbers then so is $A \cup B$
- If A and B are open sets of real numbers then so is $A \cup B$
- Every monotone sequence of real numbers converges.

11. Use $\epsilon - \mathbb{N}$ argument to prove that the sequence $(\frac{n}{2n+1})$ converges and find its limit

12. (a) (2 pts) Define what it means for the sequence $\{a_n\}_{n=1}^{\infty}$ to converge to a real number A.

    (b) (2 pts) Define what it means for the sequence $\{a_n\}_{n=1}^{\infty}$ to be Cauchy.

    (c) (2 pts) Let S be a set of real numbers. Define what it means for A to be an accumulation point of S.

    (d) (2 pts) State the Least Upper Bound Property.

13. True or False (justify your answer).

    (a) (2 pts) $(A \setminus B) \cup (B \setminus A) \subset (A \cup B) \setminus (A \cap B)$.

    (b) (2 pts) There exists an infinite subset of $\mathbb{R}$ that has no accumulation points.

    (c) (2 pts) Every increasing sequence converges.

14. (a) (4 pts) Define the sequence $\{a_n\}_{n=1}^{\infty}$ by $a_1 = 15$ and $a_n = \sqrt{12 + a_{n-1}}$ for $n \geq 2$. Show that the sequence $\{a_n\}_{n=1}^{\infty}$ is decreasing and bounded.
    (b) (4 pts) Is this sequence convergent? Why or why not? If convergent, find its limit.

15. Suppose that (a_n) and (b_n) are sequences of real numbers, such that (a_n) converges to A and (b_n) converges to B. Prove that $(a_n + b_n)$ converges to A + B.

16. Assume that y is an upper bound for a nonempty bounded from above set A. Prove that $y = \sup A$ if and only if for each $\epsilon > 0$, there is $a \in A$ such that $y \geq a > y - \epsilon$.

17. (a) (2 pts) Give an example of a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers that is bounded but not convergetn.
    (b) (2 pts) Consider your example from (a). Does the set $\{a_n \mid n = 1, 2, ...\}$ have any accumulation points? Explain your answer.

18. (a) (6 pts) If $\{a_n\}_{n=1}^{\infty}$ converges to A and $\{b_n\}_{n=1}^{\infty}$ converges to B, prove that $\{a_n b_n\}_{n=1}^{\infty}$ converges to AB. (This is Theorem 1.9 from the textbook. To get credit you must prove it, not just quote the theorem.)
    (b) (2 pts) Can $\{a_n b_n\}_{n=1}^{\infty}$ converge without having that both $\{a_n\}_{n=1}^{\infty}$ converges and $\{b_n\}_{n=1}^{\infty}$ converges? Justify your answer.

19. Prove the compact set of real numbers is closed without the usage of the Heine Borel Theorem.

20. (a) Identify the set of all accumulation points of the set E = (0,1] $\cup\{3\}$ Explain your answer.
(b) Given an open cover of the set E that has no finite subcover

21. Suppose f,g: $D \to \mathbb{R}$ with x_0 as an accumulation point of D. Further suppose that f and g both have limits at x_0. Prove that f+g has a limit at x_0 and

$$\lim_{x\to x_0}(f+g)(x) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x)$$

22. Define a function f: $\mathbb{R} \to \mathbb{R}$

$$\begin{cases} 0 & if x \in \mathbb{Q} \\ x^2 & if x \notin \mathbb{Q} \end{cases}$$

(a) Is f continuous at x = 0? Justify your answer. (Justification based on definition will receive the most points)
(b) Is f differentiable at x = 0? Justify your answer.

23. Let f: $D \to \mathbb{R}$ be uniformly continuous and let (x_n) be a Cauchy seqeunce of points in D. Prove that $f(x_n)$ is a Cauchy sequence.