Section 1.1 Exercise 2

Sol.

Let $x, y \in \mathbb{R}$ with $x \neq y$, and let $\delta := |x-y|/2$. Let $P := (x-\delta, x+\delta)$ and $Q := (y-\delta, y+\delta)$. Then P and Q are neighborhoods of x and y respectively. We claim that $P \cap Q = \emptyset$. Indeed, suppose for a contradiction that there exists $z \in P \cap Q$. Then $|z-x| < \delta$ and $|z-y| < \delta$, and we get

$$|x-y| = |x-z+z-y| \leq |x-z| + |z-y| < \delta + \delta = |x-y|,$$

by the triangle inequality, a contradiction. Thus $P \cap Q = \emptyset$ as required.

Section 1.1 Exercise 7

Sol.

Suppose that (a_n) converges to A. Then for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - A| < \epsilon$. Writing $|a_n - A| = |(a_n - A) - 0|$ makes it clear that the sequence $(a_n - A)$ then converges to 0, as required. The converse is proved similarly.

Section 1.1 Exercise 9

Sol.

Let $\epsilon > 0$. Since (a_n) converges to A, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have $|a_n - A| < \epsilon$. Similarly, since (b_n) converges to A, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, we have $|b_n - A| < \epsilon$. Thus, for $n \geq N := \max(N_1, N_2)$, we have

$$|a_n - A| < \epsilon \qquad \text{and} \qquad |b_n - A| < \epsilon,$$

so that

$$A - \epsilon < a_n \leq c_n \leq b_n < A + \epsilon$$

and thus $|c_n - A| < \epsilon$. This shows that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|c_n - A| < \epsilon$, as required.

Section 1.1 Exercise 10

Sol.

Suppose that (a_n) converges to A. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - A| < \epsilon$. In particular, by the reverse triangle inequality, we get

$$||a_n| - |A|| \leq |a_n - A| < \epsilon,$$

and this holds for all $n \geq N$. It follows that $(|a_n|)$ converges to $|A|$, as required.

The converse is false, as can be seen by taking $a_n = (-1)^n$ for all n. Then $(|a_n|)$ converges to 1, but the sequence (a_n) diverges.

Section 1.1 Exercise 11

Sol.

Let $\epsilon > 0$. Let N be as in the statement. Then for all $n \geq N$, we have

$$|a_n - \alpha| = |\alpha - \alpha| = 0 < \epsilon.$$

This shows that (a_n) converges to α, as required.