1. (Chapter 0, exercise 40)
If $x \geq 0$ and $y \geq 0$, prove that $\sqrt{xy} \leq \dfrac{x+y}{2}$. (*Hint:* Use the fact that $\left(\sqrt{x}-\sqrt{y}\right)^2 \geq 0$.)

Proof. Let $x \geq 0$ and $y \geq 0$. Then, by Theorem 0.23, $\sqrt{x} \geq 0$ and $\sqrt{y} \geq 0$ and we may consider $\left(\sqrt{x}-\sqrt{y}\right)^2$. Given $0 \leq \left(\sqrt{x}-\sqrt{y}\right)^2 = x - 2\sqrt{xy} + y$, it follows from property 8 of $\mathbb{R}$ that $2\sqrt{xy} \leq x + y$. Furthermore, by property 11 of $\mathbb{R}$, we have $\sqrt{xy} \leq \frac{x+y}{2}$, as was to be shown. □

2. (Chapter 0, exercise 41)
If $0 < a < b$, prove that $0 < a^2 < b^2$ and $0 < \sqrt{a} < \sqrt{b}$.

Proof. Suppose $0 < a < b$. Then by property 11 for the real numbers, we have the following three inequalities:

$$0(a) < a(a), \qquad a(a) < b(a), \qquad a(b) < b(b).$$

Property 2 gives us that $ba = ab$. Hence, putting the three above inequalities together we obtain

$$0 < a^2 < ab < b^2,$$

which implies $0 < a^2 < b^2$ as desired.

For the second part of the exercise, Theorem 0.23 tells us $0 < \sqrt{a}$ and $0 < \sqrt{b}$. It remains to show that $\sqrt{a} < \sqrt{b}$. Suppose for the sake of contradiction that $\sqrt{b} \leq \sqrt{a}$. By what we have just shown, it follows that $(\sqrt{b})^2 \leq (\sqrt{a})^2$, that is, $b \leq a$. This, however, contradicts the assumption that $0 < a < b$. Hence, it must be true that $0 < \sqrt{a} < \sqrt{b}$. □

3. (Chapter 1, exercise 3)
Suppose x is a real number and $\epsilon > 0$. Prove that $(x-\epsilon, x+\epsilon)$ is a neighborhood of each of its members; in other words, if $y \in (x-\epsilon, x+\epsilon)$, then there is $\delta > 0$ such that $(y-\delta, y+\delta) \subset (x-\epsilon, x+\epsilon)$.

Proof. Let $x \in \mathbb{R}, \epsilon > 0$, and consider $(x-\epsilon, x+\epsilon)$. For any $y \in (x-\epsilon, x+\epsilon)$ it follows that $x-\epsilon < y < x+\epsilon$. Hence,

$$0 < x+\epsilon-y \qquad \text{and} \qquad 0 < y-x+\epsilon.$$

Set $\delta = \min\left\{\dfrac{x+\epsilon-y}{2}, \dfrac{y-x+\epsilon}{2}\right\}$. We claim that $(y-\delta, y+\delta) \subset (x-\epsilon, x+\epsilon)$. It is sufficient to show

$$\text{(i) } x-\epsilon < y-\delta \qquad \text{and} \qquad \text{(ii) } y+\delta < x+\epsilon.$$

Observe, (i) is equivalent to $\delta < y-x+\epsilon$. By definition of δ,

$$\delta \le \frac{y-x+\epsilon}{2} < y-x+\epsilon,$$

and (i) holds. Similarly, (ii) is equivalent to $\delta < x+\epsilon-y$, and by definition of δ,

$$\delta \le \frac{x+\epsilon-y}{2} < x+\epsilon-y,$$

showing that (ii) holds as well and the claim is proved. □

Picture idea for this exercise:

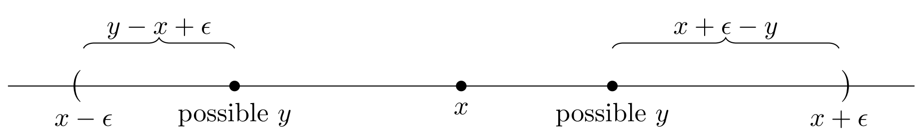

FIGURE 1

*Take minimum of the two distances and cut in half to REALLY ensure that the symmetric interval about y is contained in $(x-\epsilon, x+\epsilon)$.