**Section 2.2 Exercise 11**

**Sol.**

Let $f, g, h : D \to \mathbb{R}$ and let $x_0$ be an accumulation point of $D$. Suppose that $f(x) \le g(x) \le h(x)$ for all $x \in D$, and that both $f$ and $h$ have limits at $x_0$, with $\lim_{x\to x_0} f(x) = L = \lim_{x\to x_0} h(x)$. We have to prove that $g$ has a limit at $x_0$, and that

$$\lim_{x\to x_0} g(x) = L.$$

Let $(x_n)$ be a sequence converging to $x_0$, with $x_n \in D$ and $x_n \ne x_0$ for all $n$. Then $f(x_n) \to L$ and $h(x_n) \to L$. Moreover, we have

$$f(x_n) \le g(x_n) \le h(x_n)$$

for all $n$. By Exercise 9, Section 1.1, we get that $g(x_n) \to L$. Hence $\lim_{x\to x_0} g(x) = L$, as required.

**Section 2.2 Exercise 12**

**Sol.**

Let $f : D \to \mathbb{R}$ and let $x_0$ be an accumulation point of $D$. Suppose that $f$ has a limit at $x_0$, say $L$. We have to show that $\lim_{x\to x_0} |f(x)| = |L|$.

Let $(x_n)$ be a sequence converging to $x_0$, with $x_n \in D$ and $x_n \ne x_0$ for all $n$. Then $f(x_n) \to L$, so that $|f(x_n)| \to |L|$, by Exercise 10, Section 1.1. It follows that $\lim_{x\to x_0} |f(x)| = |L|$ as required.

**Section 2.2 Exercise 15**

**Sol.**

Let $f : D \to \mathbb{R}$ and let $x_0$ be an accumulation point of $D$. Suppose that for each $\epsilon > 0$, there is a neighborhood $Q$ of $x_0$ such that for all $x, y \in Q \cap D$, $x \ne x_0$, $y \ne x_0$, we have $|f(x) - f(y)| < \epsilon$. We have to show that $f$ has a limit at $x_0$.

Let $(x_n)$ be a sequence converging to $x_0$, with $x_n \in D$ and $x_n \ne x_0$ for all $n$. Let $\epsilon > 0$, and let $Q$ be a neighborhood of $x_0$ as above. Let $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset Q$, and let $N \in \mathbb{N}$ such that $|x_k - x_0| < \delta$ for all $k \ge N$. Then for $m, n \ge N$, we have that $x_m, x_n \in (x_0 - \delta, x_0 + \delta) \subset Q$, and $|f(x_m) - f(x_n)| < \epsilon$.

This shows that the sequence $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence, and thus must converge.

To summarize, we have proved that for every sequence $(x_n)$ converging to $x_0$ with $x_n \in D$ and $x_n \neq x_0$ for all $n$, the sequence $(f(x_n))$ converges. Note that this implies that for all such sequences $(x_n)$, the sequences $(f(x_n))$ have the same limit $L$, see the remark before the proof of Theorem 2.1 in the textbook. Finally, we get that

$$\lim_{x\to x_0} f(x) = L,$$

as required.

**Section 2.3 Exercise 19**

**Sol.**

Define $f : (0,1) \to \mathbb{R}$ by

$$f(x) := \frac{\sqrt{9-x}-3}{x}.$$

Multiplying and dividing by $\sqrt{9-x}+3$, we get

$$f(x) = \frac{9-x-9}{x(\sqrt{9-x}+3)} = \frac{-1}{\sqrt{9-x}+3}.$$

By algebra of limits, the denominator has limit 6 at 0, while the numerator has limit $-1$ at 0. It again follows from algebra of limits that $f(x)$ has limit $-1/6$ at 0.

**Section 2.3 Exercise 22**

**Sol.**

Consider $f : \mathbb{R}\setminus\{0\} \to \mathbb{R}$ defined by $f(x) = -1$ for $x < 0$ and $f(x) = 1$ for $x > 0$, and let $g := -f$. Then neither $f$ nor $g$ have limits at 0. On the other hand, the functions $f+g$, $f^2$ and $f/f$ all have limits at 0.