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Math 331
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QUIZ 3

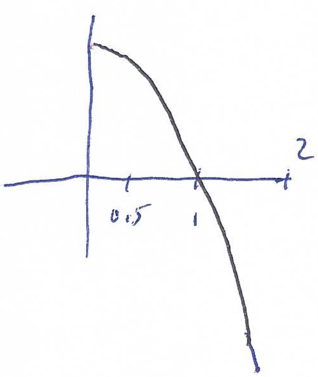
(10) **1.** Suppose $f : [0, 2] \to \mathbb{R}$ is defined by $f(x) = 1 - x^2$.

a. Explain how you can be sure that $f \in \mathcal{R}[0, 2]$. You can answer by referencing a theorem, but you should then state the theorem.

We know every continuous function on $[a,b]$ is in $\mathcal{R}[a,b]$.
f is a polynomial, and so continuous. Hence $f \in \mathcal{R}[0,2]$.

or We know every monotone function on $[a,b]$ is in $\mathcal{R}[a,b]$.
f is decreasing on $[0,2]$, (its graph is a downward opening parabola with vertex at $(0,1)$)
so $f \in \mathcal{R}[0,2]$.

b. For P the partition of $[0, 2]$ given by $P = \{0, 0.5, 1, 2\}$, compute $L(P, f)$. Your answer should be an ordinary number, but you do not have to simplify (combine terms).



$m_1 = \inf\{f(x) : 0 \le x \le 0.5\} = f(0.5) = 1 - .25 = .75$

$m_2 = \inf\{f(x) : 0.5 \le x \le 1\} \overset{\text{see graph}}{=} f(1) = 0$

$m_3 = \inf\{f(x) : 1 \le x \le 2\} = f(2) = 1 - 2^2 = -3$

$L(P,f) = m_1(0.5 - 0) + m_2(1 - 0.5) + m_3(2 - 1)$
$= (.75)\cdot(0.5) + 0\cdot(0.5) + (-3)(1)$
$= \frac{3}{8} + 0 - 3 = -\frac{21}{8}$

(5) **2.** The following statement is *false*. Explain why.

There is a function $f \in \mathcal{R}[-1, 1]$ and a partition P of $[-1, 1]$ such that $L(P, f) = 3$, $U(P, f) = 6$, and $\int_{-1}^{1} f(x)\,dx = 2$.

It is always true that $L(P,f) \le \int_a^b f\,dx \le U(P,f)$
The first inequality is broken if $L(P,f) = 3$ and $\int_{-1}^{1} f\,dx = 2$.

(10) **3.** **a.** Give an example of a bounded function that is *not* Riemann integrable on $[0, 1]$.

The example presented in class was

$$f(x) = \begin{cases} 1 & , \quad x \in \mathbb{Q} \\ 0 & , \quad x \in [0,1] \setminus \mathbb{Q} \end{cases}$$

b. Explain how it can be demonstrated that your answer to **a.** is not in $\mathcal{R}[0, 1]$. (For the most credit, your explanation should be based on the definitions, for example definitions of upper and lower Riemann integrals, etc.)

For the given function in a), and any partition P of $[0,1]$,

$m_i = \inf \{ f(x) : x_{i-1} \leq x \leq x_i \} = 0$, since every interval contains some irrational numbers.

So $\quad L(P,f) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}) = \sum_{i=1}^{n} 0 \cdot (x_i - x_{i-1}) = 0,$

and $\quad \underline{\int_0^1} f\,dx = \sup_P L(P,f) = 0.$

Similarly, $\quad M_i = \sup \{ f(x) : x_{i-1} \leq x \leq x_i \} = 1$, since every interval contains some rational number.

So $\quad U(P,f) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) = \sum_{i=1}^{n} (x_i - x_{i-1}) = 1 - 0 = 1$

and $\quad \overline{\int_0^1} f\,dx = \inf_P U(P,f) = 1$, since $U(P,f) = 1$ always.

Hence $\quad \underline{\int_0^1} f\,dx \neq \overline{\int_0^1} f\,dx$, which means $f \notin \mathcal{R}[0,1]$.