Writing Problem

Sol.

Let $f : [a, b] \to \mathbb{R}$ be monotone. We have to prove that f has a limit both at a and at b.

We treat the case where f is increasing and show that it has a limit at b. The other cases are similar.

Note that the set $\{f(y) : a \leq y < b\}$ is bounded from above by $f(b)$, so its supremum L exists. We prove that $\lim_{x \to x_0} f(x) = L$.
Let $\epsilon > 0$. Then $L - \epsilon$ is not an upper bound for the set $\{f(y) : a \leq y < b\}$, so there exists y with $a \leq y < b$ and $f(y) > L - \epsilon$. Let $\delta = b - y > 0$. If $x \in [a, b)$ and $0 < |x - b| < \delta$, then $x > b - \delta$ so that $x > y$. Hence

$$f(x) \geq f(y) > L - \epsilon.$$

But by definition of L, we also have $f(x) \leq L < L + \epsilon$. It follows that $|f(x) - L| < \epsilon$, as required.