Key Terms

- Partition
- Integrable
- Least Upper Bound Property
- supremum
- infrimum
- sequence
- convergence
- neighborhood
- cauchy sequence
- accumulation points
- Bolzano Weiestrass Theorem
- Sequential Limit Theorem
- sub-sequence
- monotone
- increasing
- decreasing
- limits
- continuity
- uniform continuity
- open
- closed
- compact
- Heine Borel Theorem
- Extreme Value Theorem
- Bolzano's Theormem
- connected

- Intermediate Value Theorem
- differentiable
- Chain Rule
- relative maximum
- Rolle's Theorem
- Mean Value Theorem
- Cauchy Mean Value Theorem
- L'Hospital's Rule
- partition
- integrable

Sample Problems

1. Let the sequence (a_n) converge to A and $(b_n - a_n)$ converge to 0. Using the ϵ and N argument show that (b_n) converges to A.
2. Using $\epsilon - N$ argument prove that the sequence $(\frac{n}{2n+1})_{n=1}^{\infty}$ converges and find its limit.
3. Define what it means for the sequence $\{a_n\}_{n=1}^{\infty}$ to converge to a real number A.
4. Suppose f: [a,b] $\to \mathbb{R}$ is a bounded function.
   - Define what it means for P to be a partition of [a,b]
   - Define a Lower Sum L(P,f)
   - Define a Upper Sum U(P,f)
   - Define the lower integral of f
   - Define the upper integral of f
   - Define what it means for a function to be integrable
5. State the following theorems
   - Mean Value Theorem
   - Extreme Value Theorem
   - Intermediate Value Theorem
   - Rolle's Theorem
6. Give an example of an open cover of the set [1,5) that has no finite subcover
7. Define f: $\mathbb{R} \to \mathbb{R}$ by f(x) = $x^2 - 5$. Use $\epsilon - \delta$ definition to prove that $\lim_{x\to 1} f(x) = -4$

8. State one of the Theorems that gives a necessary and sufficient condition for f to be Riemann integrable on the interval [a,b].

9. Suppose E $\subset \mathbb{R}$ is nonempty and that $E \cap [0,1] =$
   - Is it possible that the $\sup E = 0$
   - Is it possible that the $\sup E = 1$

10. Suppose f:[0,2] $\to \mathbb{R}$ is defined by f(x) = 1 - x^2
    - Explain how you can be sure that $f \in \mathbf{R}[0,2]$.
    - For P the partition of [0,2] given by P = $\{0, 0.5, 1, 2\}$ compute L(P,f)

11. Give the definitions of the following words
    - differentiable
    - uniformly continuous
    - continuous
    - closed open
    - compact

12. Prove that $g(x) = x^3 + x - 1$ has at least one root which lies in the open interval (0, 1).

13. Prove that if f: $[a,b] \to \mathbb{R}$ is continuous, then f is Riemann integrable on [a,b].

14. Prove that every Convergent Sequence is Cauchy

15. Answer these questions regarding to compact sets:
    - State the Heine Borel Theorem
    - $\{-1, 0, 1\}$: Is it compact?
    - $\{0\} \cup (1, 4]$ Is it compact?
    - $\{\frac{1}{n} : n \in \mathbb{N}\}$ Is it compact?

16. The following Statement is false. Explain why.
    There is a function $f \in \mathbf{R}(x)$ on [-1,1] and a partition P of [-1,1] such that L(P,f) = 1 and U(P,f) = 2 and $\int_{-1}^{1} f(x)dx = 3$

17. State True (T) or False (F) for the following:
    - If A is a non-empty and compact set of real numbers then A contains inf A and sup A.
    - If f:(2,10) $\to \mathbb{R}$ is uniformly continuous, then it is bounded
    - If A and B are compact sets of real numbers then so is $A \cup B$

- If A and B are open sets of real numbers then so is $A \cup B$
- Every monotone sequence of real numbers converges.

18. Define a function f: $\mathbb{R} \to \mathbb{R}$

$$\begin{cases} 0 & if x \in \mathbb{Q} \\ x^2 & if x \notin \mathbb{Q} \end{cases}$$

(a) Is f continuous at x = 0? Justify your answer. (Justification based on definition will receive the most points)
(b) Is f differentiable at x = 0? Justify your answer.

Problem 1: (20 pts)
(a) (5 pts) Complete the definition. Suppose $f : D \to \mathbb{R}$ and x_0 is ... Then f has a limit L at x_0 iff ...
(b) (5 pts) State the Bolzano-Weierstrass Theorem.
(c) (5 pts) Define what it means for the sequence $\{a_n\}_{n=1}^{\infty}$ to be Cauchy.
(d) (5 pts) State the Sequential Limit Theorem.

Problem 2: (14 pts) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 - 5$. Use $\epsilon - \delta$ definition to prove that $\lim_{x \to 1} f(x) = -4$.

Problem 3: (15 pts) Suppose x is an accumulation point of the set $\{a_n \mid n \in \mathbb{N}\}$. Prove that there is a subsequence of $\{a_n\}_{n=1}^{\infty}$ that converges to x.

Problem 4: (15 pts) Suppose that $f : \mathbb{R} \to \mathbb{R}$, $f(x) < 0$ if $x < 0$, $f(x) > 0$ if $x > 0$ and $\lim_{x \to 0} f(x)$ exists. Call that limit L. Prove that $L = 0$. (*Hint:* Use the Sequential Limit Theorem.)

Problem 5: (12 pts) Suppose $E \subset R$ is non-empty and that $E \cup [0, 1] = \varnothing$, where $[0, 1]$ denotes a closed interval.
(a) (6 pts) Is it possible that sup $E = 1$? If yes, give an example of such a set. If no, explain why not.
(b) (6 pts) Is it possible that sup $E = 0$? If yes, give an example of such a set. If no, explain why not.

Problem 6: (24 pts) Indicate by writing **T** or **F** whether each statement is true or false. **Give no proofs.**
(1) The function $f : (0, 1) \to \mathbb{R}$, defined by $f(x) = x \cos\left(\frac{1}{x}\right)$ does not have a limit at 0.
(2) The function $f : (0, 1) \to \mathbb{R}$, defined by $f(x) = \cos\left(\frac{1}{x}\right)$ does not have a limit at 0.
(3) If x_0 is an accumulation point of a set $S \subset \mathbb{R}$, then $x_0 \in S$.
(4) Every subsequence of a Cauchy sequence is Cauchy.
(5) If $\varnothing \neq A \subset B \subset \mathbb{R}$ then inf $A \leq$ inf B?
(6) Let A be the limit of the sequence $\{a_n\}_{n=1}^{\infty}$. Then every neighborhood of A contains all but finitely many members of the sequence $\{a_n\}_{n=1}^{\infty}$.
(7) Let A be an accumulation point of the set $\{a_n \mid n \in \mathbb{N}\}$. Then every neighborhood of A contains all but finitely many elements of the set $\{a_n \mid n \in \mathbb{N}\}$.
(8) If x and y are real numbers with $x \neq y$, then there is a neighborhood P of x and a neighborhood Q of y such that $P \cap Q = \varnothing$.

Problem 1: (15 pts) Give the following definitions

(a) (5 pts) Assume $f : D \to \mathbb{R}$, $x_0 \in D$ and x_0 is an accumulation point of D. Define what it means for f to be differentiable at x_0.
(b) (5 pts) Assume $f : D \to \mathbb{R}$, and $E \subset D$. Define what it means for f to be uniformly continuous on E.
(c) (5 pts) Assume $f : D \to \mathbb{R}$, and $x_0 \in D$. Define what it means for f to be continuous at x_0.

Problem 2: (10 pts) Give an example of an open cover of the set $[1, 5)$ that has no finite subcover.

Problem 3: (15 pts) State the following theorems:

(a) (5pts) The Mean Value Theorem
(b) (5pts) The Extreme Value Theorem
(c) (5pts) The Intermediate Value Theorem

Problem 4: (12 pts) Prove that the equation $x^3 + 3x + 1 = 0$ has exactly one root in the interval $[-2, 2]$.

Problem 5: (10 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that the set $A = \{x \in \mathbb{R} \mid f(x) = 0\}$ is a closed subset of $\mathbb{R}$.

Problem 6: (10 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0 & x = 0 \end{cases}$.
Show that $f'(x)$ exists for all $x \in \mathbb{R}$, but the function $f' : \mathbb{R} \to \mathbb{R}$ is not continuous at 0.

Problem 7: (24 pts) Indicate by writing **T** or **F** whether each statement is true or false. **Give no proofs.**
(1) Every uniformly continuous function is differentiable. (F)
(2) If $f : J \to \mathbb{R}$ is defined by $f(n) = n^2$, then f is uniformly continuous. (T)
(3) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $E \subset \mathbb{R}$ is open, then $f(E)$ is open. (F)
(4) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(0) = 0$, then f is not one-to-one. (F)
(5) If $f : (2, 3) \to \mathbb{R}$ is uniformly continuous, then $\lim_{x\to 2} f(x)$ exists. (T)
(6) A union of any collection of closed sets of real numbers is a closed set. (F)
(7) Let $f : [a, b] \to \mathbb{R}$ be continuous. Then the image of f is a closed interval. (T)
(8) If a set is not open, it is closed. (F)

Addendum from Quiz 2. Prove that the curves $f(x) = 2x^3$ and $g(x) = 3x^2 - 2$ intersect on the interval $[-1, 1]$. Justify your answer.

Problem 1: (8 pts) Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function.
(a) (2 pts) Define what it means for P to be a partition of $[a, b]$. What is a marked partition?
(b) (2 pts) Define a lower sum $L(P, f)$.
(c) (2 pts) Define a Riemann sum $S(P, f)$.
(d) (2 pts) Define the lower integral of f.

Problem 2: (4 pts) State one of the Theorems that gives a necessary and sufficient condition for f to be Riemann integrable on the interval $[a, b]$. (One of Theorems 5.2, 5.5, 5.6, or 5.7.)

Problem 3: (4 pts) The following statement is false. Explain why.
There is a function $f \in R(x)$ on $[-1, 1]$ and a partition P of $[-1, 1]$ such that $L(P, f) = 1$, $U(P, f) = 2$, and $\int_{-1}^{1} f(x)dx = 3$.

Problem 4: (10 pts) Prove that if $f : [a, b] \to \mathbb{R}$ is continuous, then f is Riemann integrable on $[a, b]$. (This is Theorem 5.4 from the book, prove it).

Problem 5: (5 pts) A set $A \subset [0, 1]$ is dense in $[0, 1]$ iff every open interval that intersects $[0, 1]$ contains a point of A. Suppose $f : [0, 1] \to \mathbb{R}$ is integrable and $f(x) = 0$ for all $x \in A$ with A dense in $[0, 1]$. Show that $\int_0^1 f(x)dx = 0$.

Problem 6: (6 pts) Either give an example, or explain why there is no such example of ...
(a) (3 pts) ... a continuous function $f : [0, 1] \to \mathbb{R}$ that is not Riemann integrable.
(b) (3 pts) ... a Riemann integrable function $g : [0, 1] \to \mathbb{R}$ that is not continuous.

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Math 331
May 12, 2015
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Final Solutions

You must justify your answers by showing your work to receive credit. Be very careful with all that you write; please do not make me take off points for statements that you do not mean.

Part I. Do every problem in this part.

(8) **1.** **a.** State the *Mean Value Theorem*.

b. State the *Least Upper Bound Property* of real numbers.

See the text or your notes for the statements.

(12) **2.** Give definitions of the following:

a. An *accumulation point* of a set of real numbers.

b. A *uniformly continuous* function.

c. A *Cauchy sequence* of real numbers.

d. An *open* set of real numbers.

See the text or your notes for the statements.

(10) **3.** Suppose that $f(x) \le g(x)$ for all $x \in [a, b]$ and $f, g \in \mathcal{R}[a, b]$.

a. Explain why $L(P, f) \le L(P, g)$ for every partition P of $[a, b]$. (Your explanation should be based on the definition of lower sum.)

Since $f(x) \le g(x)$, $m_i(f) \le m_i(g)$, and so
$L(P, f) = \sum m_i(f)(x_i - x_{i-1}) \le \sum m_i(g)(x_i - x_{i-1}) = L(P, g)$.

b. Use part **a.** to prove that $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$. Your proof should be based on the definitions.)

$$\int_a^b f(x)\,dx = \sup_P L(P, f) \le \sup_P L(P, g) = \int_a^b g(x)\,dx$$

(10) **4.** Suppose that $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are sequences of real numbers, such that $\{a_n\}_{n=1}^\infty$ converges to A and $\{b_n\}_{n=1}^\infty$ converges to B. Prove that $\{a_n + b_n\}_{n=1}^\infty$ converges to $A + B$. (Your proof should be based on the definition involving ε of a convergent sequence.)

This is Theorem 1.8 of the text, proved there and in class using a "standard $\varepsilon/2$" argument.

(40) **5.** Indicate by writing **T** or **F** whether each statement is true or false. Give no proofs.

a. If A is a non-empty and compact set of real numbers, then A contains $\inf A$ and $\sup A$.

True. This is Exercise #3.35 of the text, which you prepared for discussion in class.

b. If $f : (2, 10) \to \mathbb{R}$ is uniformly continuous, then f is bounded.

True. Theorem 3.9 of the text states that a uniformly continuous function on a bounded domain is bounded.

c. There is a function f that is differentiable on (0,1) with $f'(x) = \dfrac{7x^2 - \sin x}{\sqrt{e^x + 3}}$.

True, since f' is continuous the Fundamental Theorem of Calculus tells us that $\displaystyle\int_0^x f'(t)\,dt$ will be the solution.

d. If A and B are compact sets of real numbers, then $A \cup B$ is also a compact set.

True, as discussed in class.

e. If A and B are open sets of real numbers, then $A \cup B$ is also an open set.

True, as discussed in class.

f. Every monotone sequence of real numbers converges.

False. The sequence $a_n = n$ is a counterexample. (However, *bounded* monotone sequences do converge.)

g. If $f \in \mathcal{R}[0, 1]$ and K is a compact subset of $[0, 1]$, then $f(K)$ is compact.

False. A counterexample is $f(x) = x$, for $0 \le x < 1$, and $f(1) = 0$.

h. If f is a continuous function on $[0, 1]$ and $g(x) = 3(f(x))^2 + x^5 - 7$, then $g \in \mathcal{R}[0, 1]$.

True. g is continuous, so $g \in \mathcal{R}[0, 1]$ by Theorem 5.4.

i. Every set of real numbers that is bounded and non-empty has at least one accumulation point.

False. Any non-empty and finite set is a counterexample, since to have an accumulation point a set must be infinite.

j. If $f : \mathbb{R} \to \mathbb{R}$ and $f([0, 1])$ is an open set, then f is not continuous.

True. If f were continuous, then $f([0, 1])$ would be compact and non-empty. But the only set of real numbers that is both compact and open is the empty set.

Part II. Do any 5 of the 7 problems in this part. Each problem is worth 15 points.

(15) **6.** [Homework problem #1.36] Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers. Prove that $\{a_n\}_{n=1}^{\infty}$ has a convergent subsequence. (*Hint*: You may want to use the Bolzano-

Weierstrass Theorem.)

See the posted homework solutions.

(15) **7.** [Homework problem #5.9] Assume $f : [a,b] \to \mathbb{R}$ is continuous and $f(x) \geq 0$ for all $x \in [a,b]$. Prove that if $\int_a^b f(x)\,dx = 0$, then $f(x) = 0$ for all $x \in [a,b]$.

See the posted homework solutions.

(15) **8.** Prove that a compact set of real numbers is closed. (This is part of the easy direction of the proof of the Heine-Borel Theorem. Do not just quote that theorem; prove this part of it from the definitions.)

See you class notes, or the relevant part of the proof of Theorem 3.7 in the text.

(15) **9.** Let $E = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ and define $f : E \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{-1}{n}, & \text{if } x = \frac{1}{n} \text{ where } n \text{ is odd;} \\ \frac{+1}{n}, & \text{if } x = \frac{1}{n} \text{ where } n \text{ is even;} \\ 0, & \text{if } x = 0. \end{cases}$$

a. At what points of E is f continuous? Justify your answer!

f is continuous at all points of E. For the points $\frac{1}{n} \in E$, this is because they are isolated points of E. For the point 0, it is because $\lim_{n\to\infty} f(1/n) = 0 = f(0)$.

b. At what points of E is f differentiable? Justify your answer!

f is not differentiable at any points. Since the points $1/n$ are not accumulation points of E, the notion of derivative is not even defined at these points. 0 is an accumulation point, but $\displaystyle\lim_{n\to\infty} \frac{f(1/n) - f(0)}{1/n - 0} = \lim_{n\to\infty} (-1)^n$ does not exist, so f is not differentiable at 0.

(15) **10.** [Midterm 1 problem **6.**] Suppose that $f : \mathbb{R} \to \mathbb{R}$, $f(x) < 0$ if $x < 0$, $f(x) > 0$ if $x > 0$ and $\lim\limits_{x\to 0} f(x)$ exists. Prove that $\lim\limits_{x\to 0} f(x) = 0$.

See the posted Midterm 1 solution.

(15) **11.** **a.** State a necessary and sufficient condition for $f : [a,b] \to \mathbb{R}$ to be Riemann integrable. (See part **b.** before choosing which condition.)

Theorem 5.2 probably is easiest.

b. Use the condition in part **a.** to prove that if $f : [a,b] \to \mathbb{R}$ is monotone, then $f \in \mathcal{R}[a,b]$.

This is Theorem 5.3 in the text. See the proof there or in your notes.

(15) **12.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and suppose that $f'(x)$ exists for all $x \neq 0$ and that $\lim_{x\to 0} f'(x) = A$. Prove that f is differentiable at 0, and if possible find $f'(0)$.
(Hint: Start with the definition of $f'(0)$ as a limit, and then try to use the Mean Value Theorem.)

This was a problem on the posted sample final, with solution also posted. Here is the solution again:

Assume $h > 0$. By the Mean Value Theorem applied with the interval $[0, h]$, there is a point $c_h \in (0, h)$ such that $f(h) - f(0) = f'(c_h) \cdot (h - 0)$. Hence $\dfrac{f(h) - f(0)}{h} = \dfrac{f'(c_h) \cdot (h - 0)}{h} = f'(c_h) \to A$ as $h \to 0$ with $h > 0$, since $h \to 0$ implies $c_h \to 0$ and $\lim_{x\to 0} f'(x) = A$. The same argument works for $h < 0$, but using the interval $[h, 0]$ in that case. Hence $\lim_{h\to 0} \dfrac{f(h) - f(0)}{h} = A = f'(0)$.