SOLUTIONS | MATH 331 HMWK 5
10 points per problem | Wayne Smith
6 pts for correct mathematical work, 4 pts for writing | Due 2/15/18

General Comments:

- PLEASE PROOF READ YOUR SOLUTIONS OUT LOUD TO YOURSELF TO MAKE SURE THE WORDS MAKE SENSE. There are still a lot of run-on sentences, misuses of "such that," "it follows that," "suppose," "without loss of generality," etc. I also saw a few, "For $\forall \epsilon > 0$...." This reads, "For for all epsilon greater than zero...."
- Everyone did pretty well on the first one. Almost everyone did an $\epsilon - \delta$ proof. I have also included a proof using Theorem 2.1 so you can see how the machinery of this theorem works. (It really a is very helpful and powerful tool, so worth getting used to.)
- The second problem was done well, for the most part. You were asked to prove that the limit you found was indeed the limit. Some people did $\epsilon - \delta$ (and successfully so). Many of you who used Theorem 2.4 got a point knocked off because Theorem 2.4 does not address how to deal with square roots. If you think you cited something done in class that would address that, though, come to me and I will give you that point back. I did not have my notes on me from class when I was grading.
- THIS IS WHERE JUST ABOUT EVERYONE LOST POINTS
  - Limits do not exist "at an interval." Limits of functions exist at a point. Some of you wrote, "Suppose f has a limit at $[a, b]$." This is also problematic because the exercise asks you to show a limit exists, so you cannot start off supposing it already does.
  - Some of you successfully did this exercise by breaking up into increasing and decreasing cases (each of which needed to show that the limit at a and b both exist − so there are kind of four different cases total), and using the definitions of supremum, infemum, and limit of a function at a point.
  - Professor Smith, I believe, stated in class that using Theorem 2.1 on this exercise would lead to an easier approach. (Oftentimes important/helpful things are just stated in class and not written on the board.)
  - It looks like most of you tried reformulating the proof of Lemma 2.7, or just even copied the proof. However,
    1. This lemma only deals with f increasing (not monotone in general, although a similar argument should work to show the result for a decreasing function).
    2. This lemma, assuming f is increasing, is utilizing what are called the limit inferior ($\liminf$, the supremum of the infemums, $L(x) = \sup\{f(y) : y < x\}$) and limit superior ($\limsup$, the infemum of the supremems, $U(x) = \inf\{f(y) : x < y\}$). So when a lot of you tried adjusting this proof for the exercise, you were confusing inf, sup, lower, and upper bounds.
    3. Remarks after the proof of this lemma explicitly state that this result DOES NOT show limits exist at the endpoints of the interval, and actually refers the reader to this exercise.

1. (Chapter 2, exercise 11)
Suppose f, g, and $h : D \to \mathbb{R}$ where x_0 is an accumulation point of D, $f(x) \le g(x) \le h(x)$ for all $x \in D$, and f and h have limits at x_0 with $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x)$. Prove that g has a limit at x_0 and
$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = \lim_{x\to x_0} h(x).$$

Proof. Denote the limit of f and h at x_0 as $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = L$. By Theorem 2.1 (and its proof), we know for each sequence $\{x_n\}_{n=1}^{\infty}$ converging to x_0 with $x_n \in D$ and $x_n \ne x_0$ for all n, the sequences $\{f(x_n)\}_{n=1}^{\infty}$ and $\{h(x_n)\}_{n=1}^{\infty}$ converge to L. Since $f(x_n) \le g(x_n) \le h(x_n)$ for all n, by exercise 9 in chapter 1 (or the squeeze/sandwich theorem), we know $\{g(x_n)\}_{n=1}^{\infty}$ converges to L. Thus, for every sequence $\{x_n\}_{n=1}^{\infty}$ converging to x_0 (with $x_n \in D$ and $x_n \ne x_0$ for all n), we have that $\{g(x_n)\}_{n=1}^{\infty}$ converges to L. Again by Theorem 2.1 (this time in the other direction), we conclude that $\lim_{x\to x_0} g(x) = L$.

$\epsilon - \delta$ version:

Let $\epsilon > 0$ be given. Since f and h have the same limit at x_0, say L, we know there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that
$$\text{if } 0 < |x - x_0| < \delta_1 \text{ and } x \in D, \text{ then } |f(x) - L| < \epsilon$$
and
$$\text{if } 0 < |x - x_0| < \delta_2 \text{ and } x \in D, \text{ then } |h(x) - L| < \epsilon.$$
Now, let $\delta = \min\{\delta_1, \delta_2\}$. Since $f(x) \le g(x) \le h(x)$ for all $x \in D$, it follows that if $0 < |x - x_0| < \delta$ and $x \in D$, we have
$$L - \epsilon < f(x) \le g(x) \le h(x) < L + \epsilon.$$
Hence, $|g(x) - L| < \epsilon$ and $\lim_{x\to x_0} g(x) = L$ as desired. □

2. (Chapter 2, exercise 18)

   Define $g : (0,1) \to \mathbb{R}$ by $g(x) = \dfrac{\sqrt{1+x}-1}{x}$. Prove that g has a limit at 0 and find it.

   Proof. First, note
$$g(x) = \frac{\sqrt{1+x}-1}{x}\left(\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}\right) = \frac{1}{\sqrt{1+x}+1}.$$
   Let $\{x_n\}_{n=1}^{\infty}$ be a sequence that converges to 0 with $x_n \in (0,1)$ for all n. Then
$$\{g(x_n)\}_{n=1}^{\infty} = \left\{\frac{1}{\sqrt{1+x_n}+1}\right\}_{n=1}^{\infty}.$$
   Thus we have
$$\begin{aligned}
\lim_{n\to\infty} g(x_n) &= \lim_{n\to\infty} \frac{1}{\sqrt{1+x_n}+1} \\
&= \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty}(\sqrt{1+x_n}+1)} && \text{by Theorem 1.11} \\
&= \frac{1}{\lim_{n\to\infty}\sqrt{1+x_n}+\lim_{n\to\infty} 1} && \text{by Theorem 1.8} \\
&= \frac{1}{\sqrt{\lim_{n\to\infty}(1+x_n)}+1} && \text{by Chapter 1, exercise 28} \\
&= \frac{1}{\sqrt{1+0}+1} \\
&= \frac{1}{2}.
\end{aligned}$$
   By Theorem 2.1, it follows that $\lim_{x\to x_0} g(x) = \frac{1}{2}$. $\square$

3. (Chapter 2, exercise 24)
Let $f : [a, b] \to \mathbb{R}$ be monotone. Prove that f has a limit both at a and at b.

(Comment by W. Smith: I mentioned in class that Theorem 2.1 could be used to do this exercise, so I wrote this solution to show what I had in mind. A direct proof, without using Theorem 2.1, is just as easy. You may want to formulate that proof as well, for practice.)

Proof. Let $f : [a, b] \to \mathbb{R}$ be monotone. Then $a < x < b$ implies $f(a) \leq f(x) \leq f(b)$ or $f(a) \geq f(x) \geq f(b)$ for all $x \in (a, b)$. Note, this implies f is bounded as $|f(x)| \leq M = \max\{|f(a)|, |f(b)|\}$. Also note, a, b are accumulation points of the domain $[a, b]$. We'll use Theorem 2.1 to show that f has a limit at b; the proof that f has a limit at a is the same and will be omitted. We'll also suppose that f is increasing; the argument for f decreasing works the same.

First consider the monotonically increasing sequence $\left\{b - \frac{1}{k}\right\}_{k=1}^{\infty}$ that converges to b. (Technically, we should start the sequence at $k = M$, where M is chosen large enough so that the first term is in the interval $[a, b]$; I've omitted that technicality for simplicity.) Since f is increasing, it follows that $\left\{f(b - \frac{1}{k})\right\}_{k=1}^{\infty}$ is increasing and bounded. Hence, by Theorem 1.16, we have that $\left\{f(b - \frac{1}{k})\right\}_{n=1}^{\infty}$ converges to $L = \sup_k f(b - \frac{1}{k})$. Let $\varepsilon > 0$. From the definition of convergence, there exists $N_1 \in \mathbb{N}$ such that $L - \epsilon < f(b - \frac{1}{k})$ for $k \geq N_1$.

Now let $\left\{x_n\right\}_{n=1}^{\infty}$ be any sequence in $[a, b)$ converging to b. Then there exists $N_k \in \mathbb{N}$ such that for all $n \geq N_k$ we have $|x_n - b| < \frac{1}{k}$. This yields $b - \frac{1}{k} < x_n$. Hence, since f is increasing, we have $L - \epsilon < f(b - \frac{1}{k}) \leq f(x_n)$ for all $n \geq N_2 = \max(N_1, N_k)$. Next we get an upper bound: Given $n \in \mathbb{N}$, since $x_n < b$ there is a k_n such that $x_n < b - \frac{1}{k_n}$. Hence $f(x_n) \leq f(b - \frac{1}{k_n}) \leq L$, since $L = \sup_k f(b - \frac{1}{k})$. Combined with the previously found lower bound, we have that $n \geq N_2$ implies $L - \epsilon < f(x_n) \leq L$; i.e. $\lim_{n\to\infty} f(x_n) = L$. Then by Theorem 2.1, it follows that $\lim_{x\to b} f(x)$ exists.

As mentioned above, the same argument works to show that f has a limit at a. $\square$