Section 1.4 Exercise 34

Sol.

If

$$a_n := (-1)^n \left(1 - \frac{1}{n}\right) \qquad (n \in \mathbb{N}),$$

then $(a_{2k})_{k\in\mathbb{N}}$ is a convergent subsequence. Indeed,

$$a_{2k} = (-1)^{2k} \left(1 - \frac{1}{2k}\right) = 1 - \frac{1}{2k} \to 0 \qquad (k \to \infty).$$

Section 1.4 Exercise 35

Sol.

Let x be an accumulation point of the set $\{a_n\}_{n\in\mathbb{N}}$. Consider the interval $I_1 := (x-1, x+1)$. This interval contains x and thus has to contain infinitely many elements of the set $\{a_n\}_{n\in\mathbb{N}}$. In particular, there exists $n_1 \in \mathbb{N}$ such that $a_{n_1} \in I_1$.
Now, consider $I_2 := (x - 1/2, x + 1/2)$. Again, the interval I_2 must contain infinitely many elements of the set $\{a_n\}_{n\in\mathbb{N}}$, so there exists $n_2 > n_1$ such that $a_{n_2} \in I_2$.
Continuing in this way, we obtain a sequence of positive integers $n_1 < n_2 < \dots$ such that $a_{n_k} \in I_k$ for all k. In particular, we have

$$|a_{n_k} - x| < \frac{1}{k} \qquad (k \in \mathbb{N})$$

which implies that the subsequence $(a_{n_k})_{k\in\mathbb{N}}$ converges to x.

Section 1.4 Exercise 38

Sol.

Let $c > 1$ and consider the sequence

$$a_n := c^{1/n} \qquad (n \in \mathbb{N}).$$

Then

$$a_n - a_{n+1} = c^{\frac{1}{n+1}} \left(c^{\frac{1}{n(n+1)}} - 1\right) > 0$$

since $c > 1$, so that the sequence (a_n) is decreasing. Moreover, we have that $a_n \geq 1$ for all n, and the sequence (a_n) is bounded from below.

It follows that $a_n \to a$ for some real number a. Thus the subsequence $(a_{2n})_{n\in\mathbb{N}}$ also converges to a. But $a_{2n} = \sqrt{a_n} \to \sqrt{a}$, and we must have $a = \sqrt{a}$, so that $a = 0$ or $a = 1$. The first case is impossible since $a_n \geq 1$ for all n.
Therefore $a_n \to 1$.

Section 1.4 Exercise 40

Sol.

Consider the sequence defined by $a_1 := 6$ and $a_n := \sqrt{6 + a_{n-1}}$ for $n > 1$. We prove by induction that $a_{n+1} \leq a_n$ and $a_n \geq \sqrt{6}$.

First, we have
$$a_2 = \sqrt{6+6} = \sqrt{12} \leq \sqrt{36} = a_1$$
and clearly $a_1 \geq \sqrt{6}$. This proves the base induction case.
Suppose now that $a_{n+1} \leq a_n$ and $a_n \geq \sqrt{6}$ for some $n \geq 1$. We have
$$a_{n+2} = \sqrt{6 + a_{n+1}} \leq \sqrt{6 + a_n} = a_{n+1}$$
and
$$a_{n+1} = \sqrt{6 + a_n} \geq \sqrt{6 + \sqrt{6}} \geq \sqrt{6}.$$
This completes the proof by induction.

It follows that $a_n \to a$ for some number a. But then $a_{n+1} \to a$ as well. But $a_{n+1} = \sqrt{6 + a_n} \to \sqrt{6 + a}$. Thus a satisfies the equation
$$a = \sqrt{6 + a}$$
and
$$a^2 - a - 6 = 0.$$
The solutions of this equation are $a = 3$ and $a = -2$. The second solution is impossible since we must have $a \geq \sqrt{6}$. Hence $a_n \to 3$ as $n \to \infty$.

Section 1.4 Exercise 47

Sol.

Suppose that $a_n \to a$ and that b is an accumulation point of the set $\{a_n\}_{n\in\mathbb{N}}$. By Exercise 35, there is a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ that converges to b. Since the whole sequence $(a_n)_{n\in\mathbb{N}}$ converges to a, we must have that $(a_{n_k})_{k\in\mathbb{N}}$ converges to a as well, by a theorem proved in class. It follows from the uniqueness of the limit that $a = b$, as required.

Section 2.1 Exercise 2

Sol.

Let $f : (-2, 0) \to \mathbb{R}$ defined by

$$f(x) := \frac{2x^2 + 3x - 2}{x + 2}.$$

Note that for $x \in (-2, 0)$, we have $f(x) = 2x - 1$. We claim that

$$\lim_{x \to -2} f(x) = -5.$$

Indeed, let $\epsilon > 0$. Let $\delta := \epsilon/2$. Then if $x \in (-2, 0)$ and $|x - (-2)| = |x + 2| < \delta$, we have

$$|f(x) - (-5)| = |2x - 1 + 5| = |2x + 4| = 2|x + 2| < 2\delta = \epsilon.$$

This shows that

$$\lim_{x \to -2} f(x) = -5$$

as claimed.

Section 2.1 Exercise 7

Sol.

Let $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = x \cos(1/x)$. We claim that

$$\lim_{x \to 0} f(x) = 0.$$

By a theorem proved in class, it suffices to show that if (x_n) is a sequence of numbers in $(0, 1)$ with $x_n \to 0$, then $f(x_n) \to 0$.
Let (x_n) be such a sequence. Then

$$f(x_n) = x_n \cos(1/x_n)$$

is the product of the bounded sequence $\cos(1/x_n)$ and the sequence (x_n) which converges to 0. It follows from a theorem proved in class that $f(x_n) \to 0$, as required.