GENERAL COMMENTS:

- Use the words "assume" and "let" properly. A lot of you defined a function $g(x) = f(x) - x$, for example, and then wrote, "Let $g(a) = f(a) - a$." This is not a proper use of "let." $g(a)$ does in fact just equal $f(a) - a$ because of how you defined $g(x)$. You are not "letting" it happen. Also, if you use "assume" or "let," you are setting up a particular scenario/case. If there are other cases to address, you need to be sure to do so.
- In order to use a theorem/lemma/postulate, you need to make sure that you establish that the associated hypotheses are satisfied (i.e. Does the function need to be continuous? Does the domain of the function need to be a closed interval or an open interval?)
- Make sure to proof read your solutions and eliminate unnecessary and redundant material.
- Writing is still getting better, and all of the following solutions are from student submissions.

1. (Chapter 3, exercise 41)
Find an interval of length 1 that contains a root of the equation $xe^x = 1$.

Student Solution:

Proof. Let $f(x) = xe^x - 1$. WTS $f(x)$ is continuous. $g(x) = x$ is continuous and $h(x) = e^x$ is continuous. Using theorem 3.2 point (ii), $\phi(x) = xe^x$ is continuous. Also, $\mu(x) = -1$ is continuous, then by using theorem 3.2 point (i), $f(x)$ is continuous.

Observe, $f(0) = -1$ {*which is less than zero*} and $f(1) = e - 1$ {*which is greater than zero*}. Hence, there is a $c \in (0, 1)$, s.t. $f(c) = 0$, by theorem 3.13 (Bolzano's Theorem). Therefore, the interval $(0, 1)$, which is of length 1, contains a root of the eq'n $xe^x = 1$, as required. ☐

2. (Chapter 3, exercise 43)
Suppose $f : [a, b] \to \mathbb{R}$ is continuous and $f(b) \leq y \leq f(a)$. Prove that there is $c \in [a, b]$ such that $f(c) = y$.

Student Solution:

Proof. Suppose $f : [a, b] \to \mathbb{R}$ is continuous and $f(b) \leq y \leq f(a)$. Equivalently, $f(b) - y \leq 0 \leq f(a) - y$. Let $g(x) = f(x) - y$. As $f(x)$ is continuous and $h(x) = -y$ is continuous, then by Theorem 3.2, $g(x)$ is continuous.

$g(a) = f(a) - y \leq 0$. $g(b) = f(b) - y \leq 0$. If $g(a) = 0$ or $g(b) = 0$, then there exists $c \in [a, b]$ such that $g(c) = 0$. If $g(a) \neq 0$ and $g(b) \neq 0$, then $g(a) > 0$ and $g(b) < 0$. Hence, there exists $c \in (a, b)$ such that $g(c) = 0$ by Theorem 3.13 (Bolzano's Theorem). Thus, there exists $c \in [a, b]$ such that $f(c) - y = 0$. Equivalently, there exists $c \in [a, b]$ such that $f(c) = y$. ☐

3. (Chapter 3, exercise 44)
Suppose that $f : [a, b] \to [a, b]$ is continuous. Prove that there is at least one fixed point in $[a, b]$, that is, x such that $f(x) = x$.

Student Solution:

Proof. If $f(a) = a$, $x = a \in [a, b]$ and $f(x) = x$, thus there is a fixed point. If $f(b) = b$, $x = b \in [a, b]$ and $f(x) = x$, thus there is a fixed point. Else, $f(a) > a$ and $f(b) < b$ {*because the range of f is $[a, b]$*}. Let $g(x) = f(x) - x$. Since $f(x)$ is continuous and x is continuous, $g(x)$ is continuous {*by arithmetic of continuous functions*}. Since $f(a) > a$ and $f(b) < b$, then $g(a) = f(a) - a > 0$ and $g(b) = f(b) - b < 0$. Since g is continuous and $g(a)$ and $g(b)$ have opposite signs, there is some $x \in (a, b)$ s.t. $g(x) = 0$. For this x, $g(x) = f(x) - x = 0$, thus $f(x) = x$. Thus there exists a fixed point $x \in [a, b]$. ☐