First Order ODE

Separable ODE

Definition An ordinary differential equation is considered to be separable if it can be algebraically manipulated so that all the independent and dependent variables are each on one side. The technique is called separation of variables.

First Order Linear

Definition an equation is considered to be first order

$$y'(x) + p(x)y(x) = q(x)$$

1. Mulitiply both sides by the integrating factor obtained by $m(x) = e^{\int p(x)dx}$ as a result

$$\frac{d}{dx}[m(x)y(x)] = m(x)q(x)$$

2. integrate the left side using the Fundamental Theorem of Calculus and use an appropriate method to integrate the left

$$y(x) = [m(x)]^{-1}[\int m(x)q(x)dx + C]$$

   where $C \in \mathbb{R}$

Exact Equations

Definition Suppose that the first order differential equation is written as

$$M(x, y)dx + N(x, y)dy = 0$$

Then it is exact iff $M_y = N_x$.

Integrating Factors

- If the equation is x dependent then the integrating factor to make the differential exact can be obtained by

$$e^{\int g(x)dx}$$

  where

$$g(x) = \frac{M_y - N_x}{N}$$

- otherwise if dependent on y

$$e^{\int h(y)dy}$$

  where

$$h(y) = \frac{M_y - N_x}{M}$$

Homogeneous Equations

Definition the case where the differential equation of

$$M(x,y)dx + N(x,y)dy = 0$$

where M and N have the same degree of homogeneity then it is possible to perform a change of variable $z = \frac{y}{x}$ and make the equation separable.

Exact and Uniqueness

- **Prove Existence**
  a solution exists if $y' = f(t,y)$ is continuous in the neighborhood of $t_0, y_0)$
- **Definition** Neighborhood
  a set of $\mathbb{R}$ has a neighborhood Q filled with real number x iff Q contains a positive length interval centered at x
- **Prove Uniqueness**
  if f_y is continuous in the neighborhood of (t_0, y_0)

Euler's Method

$$y_{n+1} = y_n + f(t_n, y_n)$$

where $\frac{dy}{dt} = f(t,y)$

Second Order ODE

$$ay" + by' + c = d$$

Homogeneous Second ODE

where d = 0.
First find roots $r_1 \& r_2$ of the associated auxiliary equation: $ar^2 + br + c = 0$

- **2 real roots**
  The general solution: $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2(x)}$ where $c_1 \quad c_2 \in \mathbb{R}$
- **1 real roots**
  $y(x) = c_1 e^{rx} + c_2 x e^{rx}$ where $c_1, c_2 \in \mathbb{R}$
- **Imaginary Roots**
  where $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ then $y(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$

Methods of Undetermined Coefficients

Consider an equation with the form

$$ay" + by' + cy = f(x)$$

1. First solve the associated homogeneous equation

$$ay" + by' + cy = 0$$

2. Second, solve the for the particular equation by making a guess based on f(x)

f(x)	$Y_p(x)$
$ae^{\beta x}$	$Ae^{\beta x}$
$a\cos(\beta x)$	$A\cos(\beta x) + B\sin(\beta x)$
$b\sin(\beta t)$	$A\cos(\beta x) + B\sin(\beta x)$
$a\cos(\beta t) + b\sin(\beta t)$	$A\cos(\beta x) + B\sin(\beta x)$
n-th term polynomial	$A_n t^n + ... + A_0$

Short Cut Method

a more efficient way for solving

$$u_1'(t) = -\frac{y_2(t)g(t)}{W(t)}$$

$$u_2(t) = \frac{y_1(t)g(t)}{W(t)}$$

where the following are

- W(t): Wronskian
- y_1, y_2 come from the homogeneous solution

Higher Order ODE

consider the equation with the form

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_0 y = f$$

- **Distinct Real Roots**
  for a given associated polynomial, if the roots $r_1, r_2, ..., r_n$ are real and distinct then it follows that

$$y(x) = c_1 e^{r_1 x} + ...c_n e^{r_n x}$$

- **Repeated Roots** the solution will take a form of

$$e^{rx}, xe^{rx}, ...x^k e^{rx}$$

  where the roots are repeated k times

- **Complex Roots**
  in the case of complex roots are repeated to order k then

$$e^{\alpha x}\cos(bx), xe^{\alpha x}\cos(bx), ...x^{k-1}e^{\alpha x}\cos(bx)$$

and

$$e^{\alpha x}\sin(bx), xe^{\alpha x}\sin(bx), ...x^{k-1}e^{\alpha x}\sin(bx)$$