1. $y" - 4y = \cos(2x)$
This is a second order non-homogeneous differential equation.
The general solution of a homogeneous function is $Y_h(x) = c_1e^{2x} + c_2e^{-2x}$ which was derived in our previous worksheets.
Suppose we guess that $Y_p(x) = A\cos(2x) + B\sin(2x)$ such that A and B are undetermined coefficients.
Then substituting in y" $= -4A\cos(2x) - 4B\sin(2x)$ and y $= A\cos(2x) + B\sin(2x)$ then we obtain the following

$$-4A\cos(2x)-4B\sin(2x)-4A\cos(2x)-4B\sin(2x) = -8A\cos(2x)-8B\sin(2x) = \cos(2x)$$

Then algebraically solving we see that A $= -\frac{1}{8}$ and B $= 0$. such that $Y_p(x) = -\frac{1}{8}\cos(2x)$.
Therefore, the particular solution is

$$Y_H(x) + Y_P(x) = c_1e^{2x} + c_2e^{-2x} + -\frac{1}{8}\cos(2x)$$

such that $c_1, c_2 \in \mathbb{R}$.

2. $y" - 4y = -3e^x$ with the initial condition $y(0) = 1$ and y'(0) = 5.
From the previous problem, the general solution to the homogeneous function is $Y_H(x) = c_1e^{2x} + c_2e^{-2x}$ such that $c_1, c_2 \in \mathbb{R}$
Suppose we guess that $Y_p(x) = Ae^x$ such that A is an undetermined coefficient. Plugging our guess to the differential equation $y" = Ae^x$ and y $= Ae^x$

$$Ae^x - 4(Ae^x) = -3Ae^x = -3e^x$$

Algebraically, we find that A = 1 then $Y_P(x) = e^x$.
Therefore, the general solution is

$$y(x) = c_1e^{2x} + c_2e^{-2x} + e^x$$

Given that y(0) = 1 and y'(0) = 5 implies that x = 0, y = 1, y' = 5. Then we have that

$$y(0) = 1 = c_1e^0 + c_2e^0 + e^0 = c_1 + c_2 + 1$$

and

$$y'(0) = 5 = 2c_1e^0 - 2c_2e^0 + e^0 = 2c_1 - 2c_2 + 1$$

Setting up the matrix from the system of linear equations and using Gaussian elimination.
To begin with multiply $\mathbf{R_2}$ with a factor of $\frac{1}{2}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & -2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$$

Next subtract row 2 with row 1 such that $\mathbf{R_2} - \mathbf{R_1}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 2 \end{bmatrix}$$

Then multiply $\mathbf{R_2}$ with a factor of $-\frac{1}{2}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

gives that $c_1 = 1$ and $c_2 = -1$.
Thus, the particular solution is

$$y(x) = e^{2x} - e^{-2x} + e^{x}$$

3. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = x$ with y(0) = 1 and y'(0) = 0
The general solution of a homogeneous of $y" + 4y' + 3y = 0$ is $Y_H(x) = c_1e^{-3x} + c_2e^{-x}$. Suppose we guess that $Y_P(x) = AX + B$. Then substituting our guess to the differential equation with $y" = 0$, $y' = A$, and$y = AX + B$. Then $0 + 4A + 3AX + 3B = 3AX + 4A + 3B = x$. Then algebraically, A = $\frac{1}{3}$ and B = $-\frac{4}{9}$. Therefore, $Y_p(H) = \frac{1}{3}x - \frac{4}{9}$. Therefore, the general solution of the differential equation is

$$y(x) = c_1e^{-3x} + c_2e^{-x} + \frac{1}{3}x - \frac{4}{9}$$

where $c_1, c_2 \in \mathbb{R}$.
Given that the initial conditions are y(0) = 1 and y'(0) = 0 implies that when x = 0 then y = 1 and y' = 0. Then we have the following

$$y(0) = c_1e^0 + c_2e^0 + \frac{1}{3}(0) - \frac{4}{9} = c_1 + c_2 - \frac{4}{9} = 1$$

and

$$y'(0) = -3c_1e^0 - c_2e^0 + \frac{1}{3} = -3c_1 - c_2 + \frac{1}{3} = 0$$

Using Gaussian elimination, pivot the 1st column by $\mathbf{R_2} + 3\mathbf{R_1}$

$$\begin{bmatrix} 1 & 1 & \frac{13}{9} \\ -3 & -1 & -\frac{1}{3} \end{bmatrix} \to \begin{bmatrix} 1 & 1 & \frac{13}{9} \\ 0 & 2 & 4 \end{bmatrix}$$

Then multiply $\mathbf{R_2}$ with a factor of $\frac{1}{2}$

$$\begin{bmatrix} 1 & 1 & \frac{13}{9} \\ 0 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & \frac{13}{9} \\ 0 & 1 & 2 \end{bmatrix}$$

Then subtract $\mathbf{R_2} - \mathbf{R_1}$ then

$$\begin{bmatrix} 1 & 1 & \frac{13}{9} \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{5}{9} \\ 0 & 1 & 2 \end{bmatrix}$$

gives that $c_1 = -\frac{5}{9}$ and $c_2 = 2$

4. $x^2y" - 3xy' + 4y = x^2\ln(x) \qquad , y(1) = 0, y'(1) = 1$
To begin, find the solution to the homogeneous differential equations by letting y = x^n, $y' = nx^{n-1}$, $y" = (n^2 - n)x^{n-2}$ then the differential equation can be rewritten into

$$(n^2 - 4n + 4)(x^n)$$
$$(n-2)^2(x^n)$$

Then the solutions of the homogeneous equation are $y_1 = x^2$ and $y_2 = x^2\ln(x)$.
Then taking the wronskian of the homogeneous solutions

$$W(y_1, y_2) = \det(\begin{bmatrix} x^2 & x^2\ln(x) \\ 2x & 2x\ln(x) + x \end{bmatrix})$$
$$= 2x^3\ln(x) + x^3 - 2x^3\ln(x) = x^3$$

Let g(x) = $\ln(x)$.
Let A = - $\int \frac{y_2 g}{W}$ = $\int \frac{x^2(\ln(x))^2}{x^3}dx$. To solve the integration apply the u substitution method u = $\ln(x)$ and $du = \frac{1}{x}$ then

$$A = -\int u^2 du = \frac{u^3}{3} = -\frac{(\ln(x))^3}{3}$$

Let B = $\int \frac{y_2 g}{W}$ = $\int \frac{x^2\ln(x)}{x^3}dx$. Use u substitution to solve by setting u = $\ln(x)$ and $du = \frac{dx}{x}$ then

$$B = \int \frac{x^2\ln(x)}{x^3}dx = \int udu = \frac{u^2}{2} = \frac{(\ln(x))^2}{2}$$

Therefore the particular solution of the differential equation is

$$y_p(x) = y_1 A + y_2 B =$$
$$= -x^2(\frac{(\ln(x))^3}{3} + x^2\ln(x)(\frac{\ln(x)^2}{2} = \frac{x^2\ln(x)^3}{6}$$

The solution to the homogeneous equation

$$y_h(x) = c_1x^2 + c_2x^2\ln(x)$$

where $c_1, c_2 \in \mathbb{R}$.
Then the solution for the differential equation is

$$y(t) = y_h + y_p = c_1x^2 + c_2x^2\ln(x) + \frac{x^2\ln(x)^3}{6}$$

5. $e^{-t}Q" + t^2Q' + e^tQ = 3\sin(t)$.
Let a = $e^(-t)$, b = t^2, and c = $e^(t)$. For the following functions, the three are continuous on the following intervals

- $e^{-t} : (-\infty, \infty)$ where $e^(- t) \neq 0$
- $t^2 : (-\infty, \infty)$
- $e^(t) : (-\infty, \infty), e^t \neq= 0$

Let f(t) = $\sin(3t)$ then f(t) is continuous on $(-\infty, \infty)$.
Based on the given let $t_0 = 0$ and since $0 \in (-\infty, \infty)$ then there exists a unique solution.