1. Given that the free damped system when m = 1, $\gamma = 2$, and $k = 1$ then setting up the second order differential equation will

$$u"(t) + 2u' + u = 0$$

since there is no extreral force then F(t) = 0.
The associated polynomial to the differential is

$$r^2 + 2r + 1 = (r+1)^2 = 0$$

Which has a single root r = -1 therefore, the general solution to the differential equation is

$$u(t) = c_1 e^{-t} + c_2 t e^{-t}$$

where $c_1, c_2 \in \mathbb{R}$.

2. Based on the given, w = 4 and since w = mg then to find m

$$m = \frac{w}{g} = \frac{4}{32} = \frac{1}{8}$$

Since mg - kl = w - kl = 0 then to find k,

$$k = \frac{w}{l} = \frac{4}{0.5} = 8$$

0.5 because 6 in. to feet is half a foot.
Finally to find γ use the relation such that

$$\gamma = 0$$

Then recall the model of

$$F(t) = mu" + \gamma u' + ku$$

then substituting the values known then

$$F(t) = \frac{1}{8}u" + 8u = 0$$

or

$$0 = u" + 64u$$

then the auxiliary equation will be

$$r^2 + 64 = (r+8i)(r-8i)$$

then the general solution will be in the form of

$$u(t) = c_1 \cos(8t) + c_2 \sin(8t)$$

Given the initial conditions of

- $u(0) = 8(\frac{1}{12}) = \frac{2}{3}$
- $u'(0) = \frac{4}{3}$

Then algebraically solving for the unknown constants then

- $u(0) = c_1 + 0 = \frac{2}{3}$
- $u'(0) = 8c_2 = -\frac{4}{3} \to c_2 = -\frac{1}{6}$

Therefore, the solution to the differential equation is

$$u(t) = \frac{2}{3}\cos(8t) - \frac{1}{6}\sin(8t)$$

3. Given the following values of m = 1, k = 25, and $\gamma = 8$. Therefore, using the model we have that

$$F(t) = u"(t) + 8u'(t) + 25u(t)$$

While in the problem we are given that there is an external force of F(t) = $73e^{4t} - 18e^{-4t}$, to begin solving assume that F(t) = 0 then the auxiliary equation to the differential equation is

$$r^2 + 8r + 25 = 0$$

which has the roots of $-4 \pm 3i$ then the homogeneous solution will be

$$u_h(t) = e^{-4t}[c_1\cos(3t) + c_2\sin(3t)]$$

To find the particular solution, guess that

$$u_p(t) = Ae^{4t} + Be^{-4t}$$

where A and B are undetermined coefficients.
Then, $u' = 4Ae^{4t} - 4Be^{-4t}$ and $u" = 16Ae^{4t} + 16Be^{-4t}$ and then subbing in and algebraically solving for A and B results to

A = 1 and B = -2

then the particular solution is $u_p(t) = e^{4t} - 2e^{-4t}$. Then the general solution will be the form of u(t) = $u_h(t) + u_p(t)$

$$u(t) = e^{-4t}[c_1\cos(3t) + c_2\sin(3t)] + e^{4t} - 2e^{-4t}$$

Furthermore, the problem gives an initial condition of the following: u(0) = 0 and u'(0) = 3.
Then the following is made: $u(0) = 0 = e^0[c_1 + 0] + 1 - 2 = c_1 - 1$.
Then algebraically solving shows that $c_1 = 1$.
Then taking the derivative then

$$u'(t) = -4e^{(-4t)}[\cos(3t) + c_2\sin(3t)] + e^{-4t}[-3\sin(3t) + 3c_2\cos(3t)] + 4e^{4t} + 8e^{-4t}$$

Then subbing in with t = 0 then

$$u'(0) = -4(1) + (3c_2) + 4 + 8 = 3$$

and algebraically solving for c_2 gives $c_2 = -\frac{5}{3}$.
Therefore,

$$u(t) = e^{(-4t)}[\cos(3t) - \frac{5}{3}\sin(3t] + e^{4t} - 2e^{-4t}$$

4. With the given information then the following equation can be constructed

$$25y" - 175y = -300x^2$$

or

$$y" - 7y = -12x^2$$

Then the auxiliary equation for the homogeneous case is

$$r^2 - 7 = 0$$

has the roots of $\pm\sqrt{7}$ therefore the solution for the homogeneous case is

$$y_h(x) = c_1 e^{\sqrt{7}x} + c_2 e^{-\sqrt{7}x}$$

Suppose that the particular solution has the form of $y_p(x) = Ax^2 + Bx + C$ where $A, B, C \in \mathbb{R}$.
Then the following implication can be made

- $y_p'(x) = 2Ax + B$
- $y"_p(x) = 2A$

Then substituting them to the differential equation solves that

$$(2A) - 7(Ax^2 + Bx + C) = -12x^2$$

Then algebraically solving results

- $-7A = -12 \to A = \frac{12}{7}$
- $-7B = 0 \to B = 0$
- $2A - 7C = 0 \to C = \frac{24}{49}$

Thus the particular solution is

$$y_p(x) = \frac{12}{7}x^2 - \frac{24}{49}$$

Therefore the general solution of the equation is

$$y(x) = y_h(x) + y_p(x) = c_1 e^{\sqrt{7}x} + c_2 e^{-\sqrt{7}x} + \frac{12}{7}x^2 + \frac{24}{49}$$

Given the following initial conditions of y(0) = 0 and y'(10) = 0 then then

$$y(0) = 0 = c_1 + c_2 - \frac{24}{29}$$

and

$$y'(10) = \sqrt{7}e^{10\sqrt{7}} + \sqrt{7}e^{-10\sqrt{7}} + \frac{240}{7} = 0$$

Then algebraically solving for the constants then $c_2 \approx -0.4898$ and $c_1 \approx -4.2 \times 10^{-11}$ therefore the solution is

$$y(x) = -0.4898e^{-\sqrt{7}x} + -4.2 \times 10^{-11}e^{\sqrt{7}x} + \frac{12}{7}x^2 + \frac{24}{49}$$

5. Given with a m = 2000, a spring constant k = 2×10^5, with an external force of $2000\sin(10t)$ such that the equation

$$2000u"(t) + 2 \times 10^5 u(t) = 2000\sin(10t)$$

which can simplified as

$$u"(t) + 100u(t) = \sin(10t)$$

To begin with, find the homogeneous solution by setting the auxiliary equation as

$$r^2 + 100 = (r \pm 10i)$$

then the homogeneous general solution is

$$u_h(t) = c_1\cos(10t) + c_2\sin(10t)$$

For the particular solution, take the guess that the particular solution has the form of $y_p = At\sin(10t) + Bt\cos(10t)$ where A and B are undetermined coefficients.
Then subbing the particular solution of the following

- $u_p' = A\sin(10t) + 10At\cos(10t) + B\cos(10t) - 10Bt\sin(10t)$
- $u_p'' = 10A\cos(10t) + 10A\cos(10t) - 100At\sin(10t) - 10B\sin(10t) - 10B\sin(10t) - 100Bt\cos(10t)$

Then

$$20A\cos(10t) - 20B\sin(10t) - 100At\sin(10t) - 100Bt\cos(10t) + 100At\sin(10t) + 100Bt\cos(10t) = \sin(10$$

or

$$20A\cos(10t) - 20B\sin(10t) = \sin(10t)$$

Then algebraically solving it follows that

- 20A = 0 $\rightarrow$ A = 0
- -20B = 1 $\rightarrow$ B = $-\frac{1}{20}$

Thus,

$$u_p(t) = \frac{-t\cos(10t)}{20}$$

Then the general solution will be

$$u(t) = u_h(t) + u_p(t) = c_1\cos(10t) + c_2\sin(10t) - \frac{t\cos(10t)}{20}$$

Since the problem has given us of an initial conditions of u(0) = 0.1 then since sin(0) = 0 and from the multiplication rule of 0 then algebraically solving for c_1 is

$$c_1 = 0.1$$

Furthermore, at t = 0 since there is no velocity then the following initial condition is implied $u'(0) = 0$, then algebraically solving for c_2 results as

$$u'(0) = 0 = 10c_2\cos(0) + 0 - \frac{1}{20}$$

results $c_2 = 0.005$.
Therefore,

$$u(t) = 0.1\cos(10t) + 0.005\sin(10t) - \frac{t\cos(10t)}{20}$$

Since t is in seconds then the following conversions will need to be made

- 1 minutes $\rightarrow$ 60 seconds
- 10 minutes $\rightarrow$ 600 seconds

Then

- u(60) $= 0.1\cos(10 * 60) + 0.005\sin(10 * 60) - \frac{60\cos(10\times 60)}{20} \approx 2.897$
- u(600) $= 0.1\cos(10\times 600) + 0.005\sin(10\times 600) - \frac{600\cos(10\times 600)}{20} \approx -270291$

Therefore,

- $u(60) \approx 2.89$ m
- $u(600) \approx -27.091$ m