Problem 1

Rewrite the differential equation as its auxiliary equation $r^3 - 5r - 22r - 56 = 0$. From the hint we are given that one of the roots is 2 so using synthetic division

$$\begin{array}{r|rrrr} 2 & 1 & -5 & -22 & 56 \\ \hline & & 2 & -6 & -56 \\ \hline & 1 & -3 & -28 & 0 \end{array}$$

Then it follows that the remaining roots can be found through $r^2 - 3r - 28$, therefore, from the quadratic formula

$$r = \frac{3 \pm \sqrt{9 - 4(-28)(1)}}{2} = -4, 7$$

Thus, the general solution for the differential equation is

$$y(x) = c_1 e^{2x} + c_2 e^{-4x} + c_3 e^{7x}$$

where $c_1, c_2, c_3 \in \mathbb{R}$

Given the initial condition of $y(0) = 1, y'(0) = -2, y"(0) = -4$ then the following implications are made

- $y(0) = c_1 + c_2 + c_3 = 1$
- $y'(0) = 2c_1 - 4c_2 + 7c_3 = -2$
- $y"(0) = 4c_1 + 16c_2 + 49c_3 = -4$

Then row reducing the matrix set up results to

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -4 & 7 & -2 \\ 4 & 16 & 49 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -\frac{16}{55} \\ 0 & 1 & 0 & \frac{14}{33} \\ 0 & 0 & 1 & \frac{13}{15} \end{bmatrix}$$

Thus, $c_1 = -\frac{16}{55}$, $c_2 = \frac{14}{33}$, $c_3 = \frac{13}{15}$.

Therefore, the solution that meets the initial condition is

$$y(x) = -\frac{16}{55} e^{2x} + \frac{14}{33} e^{-4x} + \frac{13}{15} e^{7x}$$

Problem 2

Similar to the previous problem, begin by doing synthetic division with the root being $\frac{1}{2}$ then

$$\begin{array}{r|rrrrr} \frac{1}{2} & 2 & 11 & 18 & 4 & -8 \\ \hline & & 1 & 6 & 12 & 8 \\ \hline & 2 & 12 & 24 & 16 & 0 \end{array}$$

results to $2r^3 + 12r^2 + 24r + 16 = 2(r^3 + 6r^2 + 12r + 8)$.
Suppose that one of the roots are -2. Then by plugging in r = -2 then

$$2(-2^3 + 6(-2)^2 + 12(-2) + 8) = 0$$

then the guess is correct so do another synthetic division

$$\begin{array}{r|rrrr} -2 & 1 & 6 & 12 & 8 \\ \hline & & -2 & -8 & -8 \\ \hline & 1 & 4 & 4 & 0 \end{array}$$

results to $r^2 + 4r + 4$ which has a repeated root of -2. Overall, the roots are

$$\text{r} = \tfrac{1}{2}, -2, -2, -2$$

Therefore the general solution for the given differential equation is

$$y(x) = c_1 e^{\frac{1}{2}x} + c_2 x^2 e^{-2x} + c_3 x e^{-2x} + c_4 e^{-2x}$$

such that $c_1, c_2, c_3, c_4 \in \mathbb{R}$

Problem 3

From the general solution the following assumption can be made that the roots are

$$\text{r} = 0,\ 1,\ \text{-}2,\ 3$$

or

$$(r)(r-1)(r+2)(r-3) = (r^4 - 2r^3 - 5r^2 + 6r)$$

Since the auxiliary equation is $r^4 - 2r^3 - 5r^2 + 6r$ then the differential equation is

$$y^{(4)} - 2y^{(3)} - 5y" + 6y = 0$$

Problem 4

On Problem 2, solving the problem have shown that $Y_h = c_1 e^{\frac{1}{2}t} + c_2 t^2 e^{-2t} + c_3 t e^{-2t} + c_4 e^{-2t}$.
Let D $= \frac{d}{dt}$ then the equation will be rewritten in the following

$$2D^4 + 11D^3 + 18D^2 + 4D - 8 = 12 - 32e^{-8t} + 2e^{-2t}$$

or the particular integral is

$$y_p = \frac{12 - 32e^{-8t} + 2e^{-2t}}{2D^4 + 11D^3 + 18D^2 + 4D - 8}$$

Solving for the particular integral results to

$$y_p(t) = -\frac{1}{15}t^3 e^{-2t} - \frac{4}{459}e^{-8t} - \frac{3}{2}$$

Therefore, the general solution is

$$y(t) = c_1 e^{\frac{1}{2}t} + c_2 t^2 e^{-2t} + c_3 t e^{-2t} + c_4 e^{-2t} - \frac{1}{15}t^3 e^{-2t} - \frac{4}{459}e^{-8t} - \frac{3}{2}$$

Problem 5

$y"' - y" = 2x + 1 + 3\sin(x)$
The solution for the homogeneous differential equation comes from the auxiliary equation $r^3 - r^2 = r^2(r-1)$ such that the roots are r = 0, 0, 1 then

$$Y_h = c_1e^{0x} + c_2xe^{0x} + c_3e^x = c_1 + c_2x + c_3e^x$$

where $c_1, c_2, c_3 \in \mathbb{R}$.
To solve for the particular solution break $g(x) = 2x + 1 + 3\sin(x)$ into two parts such that

- $g_1(x) = 2x + 1$
- $g_2(x) = 3\sin(x)$
- such that the particular solution has the form $y_p(x) = y_{p1} = Ax^3 + Bx^2$ and $y_{p2} = C\sin(x) + D\cos(x)$ where A,B,C, and D are undetermined coefficients

Begin with y_{p1} then finding the second and third derivative and

$$y^{'''} - y'' = 6A - 6AX - B = 2x + 1$$

Then algebraically solving

- $-6AX = 2x \to A = -\frac{1}{3}$
- $6A - 2B = 1 \to B = -\frac{3}{2}$

Then taking the guess for the second part subbing in the equation results to

$$(-C + D)\cos(x) + (C + D)\sin(x) = 3\sin(x)$$

results to the following that

- -C + D $= 0 \to C = \frac{3}{2}$
- $C + D = 3 \to D = \frac{3}{2}$

The the overall particular solution is

$$y_p(x) = -\frac{x^3}{3} - \frac{3}{2}x^2 + \frac{3}{2}\sin(x) + \frac{3}{2}\cos(x)$$

Then the solution for the differential equation is

$$y(x) = y_p(x) + y_h(x) = c_1 + c_2x + c_3e^x - \frac{x^3}{3} - \frac{3}{2}x^2 + \frac{3}{2}\sin(x) + \frac{3}{2}\cos(x)$$