CS502: Graph Theory and Applications **Fall 2019-20**

Lecture 1: Basics of Graph Theory

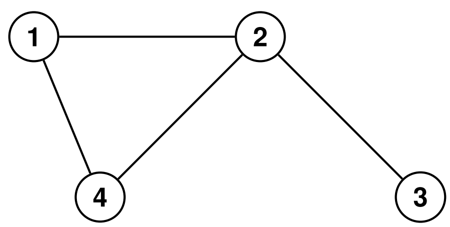
Instructor: Vinod Reddy *Scribe: GVN Sai Uday(11740320) & E Suyra Teja(11740310)*

In this lecture, we will be dealing with some basic terminology required to learn Graph Theory.

A graph G is a pair $G = (V, E)$, where V is a finite set and E is a set of 2-element subsets of V. The elements of V are called vertices of the graph G and the elements of E are called edges of G.

For a graph $G = (V, E)$, let $V(G)$ & $E(G)$ denote the vertex and edge set of G respectively.

Example:-



$$V = \{1, 2, 3, 4\}$$
$$E = \{\{1, 2\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$$

The number of vertices of a graph G is its order(or size), denoted as $|G|$ or $|V|$.

Note:-

- From this lecture onwards *n* denotes the number of vertices in a graph.
- Also *m* denotes the number of edges in a graph.
- The graphs we consider in this course are finite unless otherwise stated.

An edge x, y is usually written as xy (or yx). Two vertices x, y of G are adjacent or neighbours if xy is an edge of G. Two edges e, f are adjacent if they have an end in common.

Definition 1.1 (Degree(or Valency))**.** *The* degree(or valency) *of a vertex v in a graph G is the number of edges incident with v, with loops counted twice. It is denoted by deg$_G(v)$ or d(v).*

Lemma 1.2 (Handshaking Lemma)**.** *The sum of the degrees of the vertices of a graph equals twice the number of edges.*

Proof. Each edge contributes two to the sum of the degrees, one for each of its endpoints.

$$2|E| = \sum_{v \in V} d(v)$$

□

Corollary 1.3. *In any graph G, the number of vertices of odd degree is even.*

Proof. We know that

$$2|E| = \sum_{v \in V} d(v)$$

$$\underbrace{2|E|}_{\text{EVEN}} = \underbrace{\sum_{\substack{v \in V \\ d(v) is odd}} d(v)}_{\mathbf{x}} + \underbrace{\sum_{\substack{v \in V \\ d(v) is even}} d(v)}_{\text{EVEN}}$$

For the above equation to satisfy the value of **x** should be even(i.e. for the LHS to be even sum of terms of RHS should be even, which is only possible when value of **x** is even since the other term is even.) □

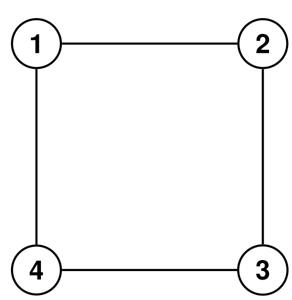
Definition 1.4 (Isolated vertex)**.** *A vertex of degree zero is called isolated vertex.*

Definition 1.5 (Minimum degree)**.** *The degree of the vertex with the least number of edges incident to it (i.e. $min\{d(v) : v \in V\}$) is the minimum degree of G. It is represented by $\delta(G)$*

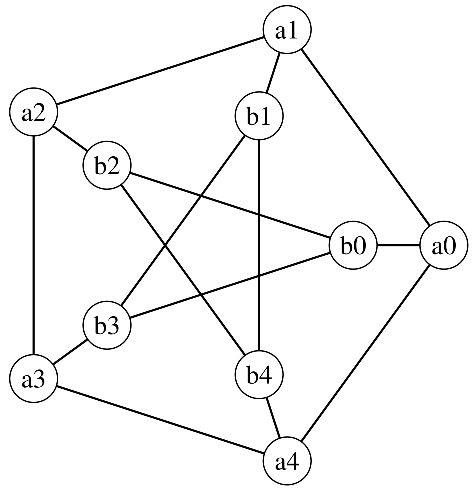
Definition 1.6 (Maximum degree)**.** *The degree of the vertex with the most number of edges incident to it (i.e. $max\{d(v) : v \in V\}$) is the maximum degree of G. It is represented by $\Delta(G)$*

Definition 1.7 (Regular Graph)**.** *If all the vertices of G have the same degree k then the graph G is called k-regular.*

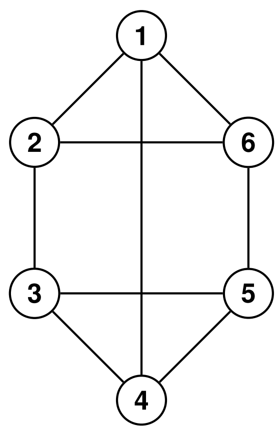
Example:-
(a) 2-Regular Graph



(b) Petersen's Graph(3-Regular Graph)



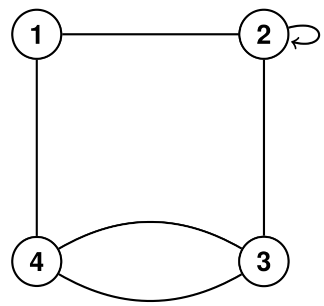
(c) 3-Regular Graph



Definition 1.8 (Multiple edges)**.** *Multiple edges are two or more edges that are incident to the same two vertices. Multiple edges are also known as parallel edges*

Definition 1.9 (Self loop)**.** *A self-loop is an edge that connects a vertex to itself.*

Example:-



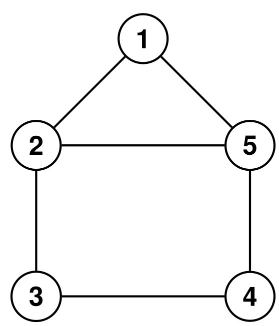
Definition 1.10 (Simple Graph)**.** *A graph with no self loops and no multiple edges is called a simple graph.*

Note:- In this course unless specified all graphs are simple graphs.

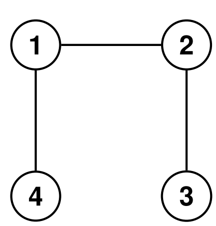
Definition 1.11 (Subgraphs)**.** *A graph H is called a subgraph of a graph G, if* $V(H) \subseteq V(G)$ *and* $E(H) \subseteq E(G)$.

Definition 1.12 (Induced Subgraph)**.** *A subgraph H of G is called induced subgraph of G, if for any two vertices* $u, v \in H$*, if* $uv \in E(G)$ *then* $uv \in E(H)$

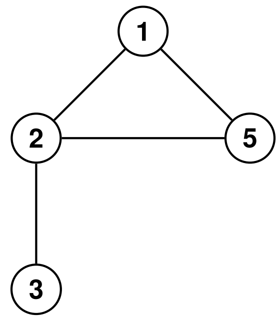
Example:-
(a) Let graph G be



Let the below graph be H_1



Let the below graph be H_2



Note:-

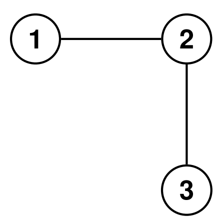
- H_1 is a subgraph of G, but not induced because $3, 4 \in V$ and edge $34 \in E(G)$ but $34 \notin E(H_1)$.
- H_2 is an induced subgraph of G because $\forall\ u, v \in V$ and edge $uv \in E(G)$ we have $uv \in E(H_2)$.

Representation of graphs:

1. **Adjacency matrix:** Adjacency matrix of a graph G is nxn matrix, where

$$a_{ij} = \begin{cases} 1 & \text{if } ij \in E(G); \\ 0 & \text{otherwise;} \end{cases}$$

Example:- Let graph G be,



For the given graph the adjacency matrix would look like,

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

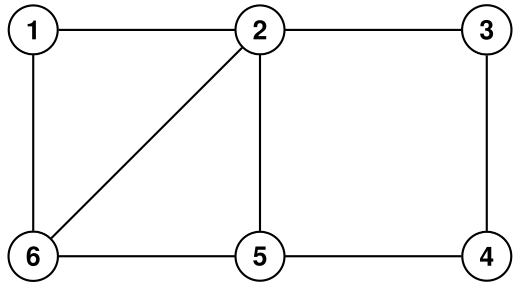
2. **Adjacency list:** Adjacency list is a collection of unordered lists used to represent a finite graph. Each list describes the set of neighbors of a vertex in the graph.

The adjacency list of the given graph G is,

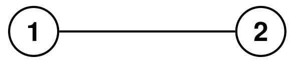
$$\boxed{1} \rightarrow \boxed{2} \rightarrow \boxed{3}$$

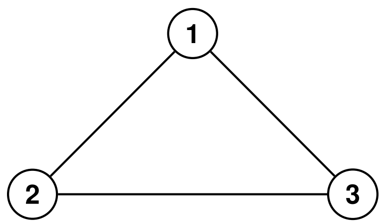
$$\boxed{2} \rightarrow \boxed{1}$$

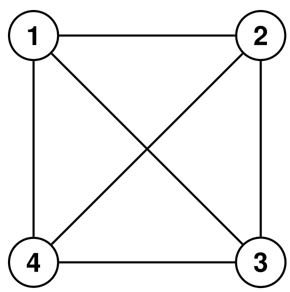
$$\boxed{3} \rightarrow \boxed{1}$$

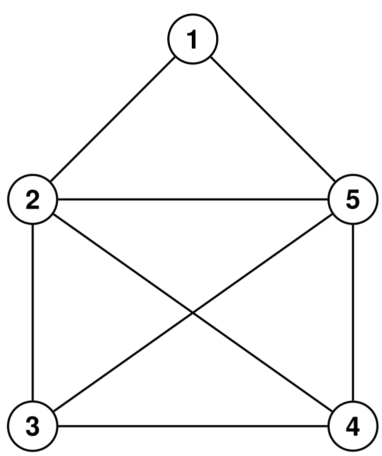

1
2
3
6
5
4

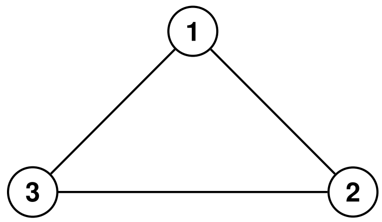

1

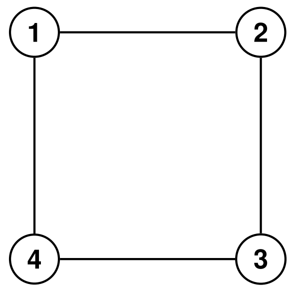

1
2

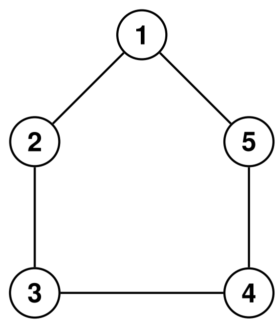

1
2
3

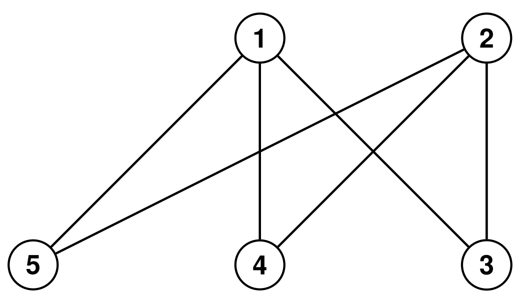

1
2
4
3

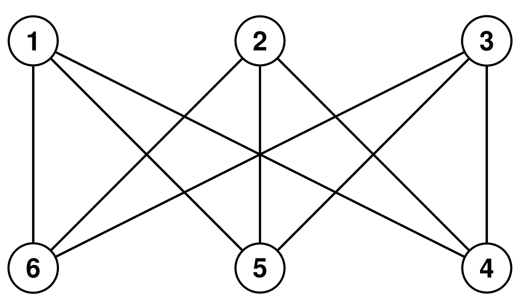

1
2
5
3
4


1
3
2


1
2
4
3


1
2
5
3
4


1
2
5
4
3


1
2
3
6
5
4

1.1 Some theorems and stuff

We now delve right into the proof.

Lemma 1.13. *This is the first lemma of the lecture.*

Proof. The proof is by induction on .... We also throw in a figure (which you might want to make larger).

This is the end of the proof, which is marked with a little box. □

1.1.1 A few items of note

Here is an itemized list:

- this is the first item;
- this is the second item.

Here is an enumerated list:

1. this is the first item;
2. this is the second item.

Here is an exercise:

Exercise: Find an efficient algorithm for triangulation.

Here is how to define things in the proper mathematical style. Let f_k be the $AND - OR$ function, defined by

$$f_k(x_1, x_2, \ldots, x_{2^k}) = \begin{cases} x_1 & \text{if } k = 0; \\ \text{AND}(f_{k-1}(x_1, \ldots, x_{2^{k-1}}), f_{k-1}(x_{2^{k-1}+1}, \ldots, x_{2^k})) & \text{if } k \text{ is even;} \\ \text{OR}(f_{k-1}(x_1, \ldots, x_{2^{k-1}}), f_{k-1}(x_{2^{k-1}+1}, \ldots, x_{2^k})) & \text{otherwise.} \end{cases}$$

Here is another equation that uses one of the AMS commands, align

$$\begin{aligned} p(x_{1:5}) &= \sum_{x_{2:5}} p(x_1)p(x_2|x_1)p(x_3|x_1)p(x_4|x_2)p(x_5|x_1)p(x_6|x_2, x_5) \\ &= p(x_1) \sum_{x_2} p(x_2|x_1) \sum_{x_3} p(x_3|x_1) \sum_{x_4} p(x_4|x_2) \sum_{x_5} p(x_5|x_1)p(x_6|x_2, x_5) \\ &= p(x_1) \sum_{x_2} p(x_2|x_1) \sum_{x_3} p(x_3|x_1) \sum_{x_4} p(x_4|x_2) \sum_{x_5} p(x_5|x_1)\phi_{X_6}(x_2, x_5) \end{aligned}$$

which assumes that $X_4 \perp\!\!\!\perp \{X_1, X_3\}|X_2$.

Theorem 1.14. *This is the first theorem.*

Proof. This is the proof of the first theorem. We show how to write pseudo-code now.

Consider a comparison between x and y:

if x or y or both are in S **then**
answer accordingly
else
Make the element with the larger score (say x) win the comparison
if $F(x) + F(y) < \frac{n}{t-1}$ **then**
$F(x) \leftarrow F(x) + F(y)$
$F(y) \leftarrow 0$
else
$S \leftarrow S \cup \{x\}$
$r \leftarrow r + 1$
endif
endif

This concludes the proof. $\square$

1.2 Next topic

Here is some citations [JB00] and [L2000]

References

[JB00] M.I. JORDAN and C. BISHOP, "An Introduction to Graphical Models," *To be published*, 2000

[L2000] S.L. LAURITZEN "Graphical Models," *Oxford Science Publications*, 1996