CS502: Graph Theory and Applications **Fall 2019-20**

Lecture 1: Basics of Graph Theory

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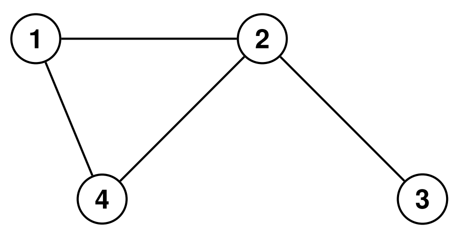
In this lecture, we will be dealing with some basic terminology required to learn Graph Theory.

A graph G is a pair $G = (V, E)$, where V is a finite set and E is a set of 2-element subsets of V. The elements of V are called vertices of the graph G and the elements of E are called edges of G.

For a graph $G = (V, E)$, let $V(G)$ & $E(G)$ denote the vertex and edge set of G respectively.

Example:-



$$V = \{1, 2, 3, 4\}$$
$$E = \{\{1, 2\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$$

The number of vertices of a graph G is its order(or size), denoted as $|G|$ or $|V|$.

Note:-

- From this lecture onwards *n* denotes the number of vertices in a graph.
- Also *m* denotes the number of edges in a graph.
- The graphs we consider in this course are finite unless otherwise stated.

An edge x, y is usually written as xy (or yx). Two vertices x, y of G are adjacent or neighbours if xy is an edge of G. Two edges e, f are adjacent if they have an end in common.

Definition 1.1 (Degree(or Valency))**.** *The* degree(or valency) *of a vertex v in a graph G is the number of edges incident with v, with loops counted twice. It is denoted by deg*$_G(v)$ *or d*(v)*.*

Lemma 1.2 (Handshaking Lemma)**.** *The sum of the degrees of the vertices of a graph equals twice the number of edges.*

Proof. Each edge contributes two to the sum of the degrees, one for each of its endpoints.

$$2|E| = \sum_{v \epsilon V} d(v)$$

□

Corollary 1.3. *In any graph G, the number of vertices of odd degree is even.*

Proof. We know that

$$2|E| = \sum_{v \epsilon V} d(v)$$

$$\underbrace{2|E|}_{\text{EVEN}} = \underbrace{\sum_{\substack{v \epsilon V \\ d(v) isodd}} d(v)}_{\mathbf{1}} + \underbrace{\sum_{\substack{v \epsilon V \\ d(v) iseven}} d(v)}_{\text{EVEN}}$$

For the above equation to satisfy the value of **1** should be even(i.e. for the LHS to be even sum of terms of RHS should be even, which is only possible when value of **1** is even since the other term is even.) □

Definition 1.4 (Isolated vertex)**.** *A vertex of degree zero is called isolated vertex.*

Definition 1.5 (Minimum degree)**.** *The degree of the vertex with the least number of edges incident to it (i.e.* $min\{d(v) : v\epsilon V\}$*) is the minimum degree of G. It is represented by* $\delta(G)$

Definition 1.6 (Maximum degree)**.** *The degree of the vertex with the most number of edges incident to it (i.e.* $max\{d(v) : v\epsilon V\}$*) is the maximum degree of G. It is represented by* $\Delta(G)$

1.1 Some theorems and stuff

We now delve right into the proof.

Lemma 1.7. *This is the first lemma of the lecture.*

Proof. The proof is by induction on .... We also throw in a figure (which you might want to make larger).

This is the end of the proof, which is marked with a little box. □

1.1.1 A few items of note

Here is an itemized list:

- this is the first item;
- this is the second item.

Here is an enumerated list:

1. this is the first item;
2. this is the second item.

Here is an exercise:

Exercise: Find an efficient algorithm for triangulation.

Here is how to define things in the proper mathematical style. Let f_k be the $AND - OR$ function, defined by

$$f_k(x_1, x_2, \ldots, x_{2^k}) = \begin{cases} x_1 & \text{if } k = 0; \\ \text{AND}(f_{k-1}(x_1, \ldots, x_{2^{k-1}}), f_{k-1}(x_{2^{k-1}+1}, \ldots, x_{2^k})) & \text{if } k \text{ is even;} \\ \text{OR}(f_{k-1}(x_1, \ldots, x_{2^{k-1}}), f_{k-1}(x_{2^{k-1}+1}, \ldots, x_{2^k})) & \text{otherwise.} \end{cases}$$

Here is another equation that uses one of the AMS commands, align

$$\begin{aligned} p(x_{1:5}) &= \sum_{x_{2:5}} p(x_1)p(x_2|x_1)p(x_3|x_1)p(x_4|x_2)p(x_5|x_1)p(x_6|x_2, x_5) \\ &= p(x_1) \sum_{x_2} p(x_2|x_1) \sum_{x_3} p(x_3|x_1) \sum_{x_4} p(x_4|x_2) \sum_{x_5} p(x_5|x_1)p(x_6|x_2, x_5) \\ &= p(x_1) \sum_{x_2} p(x_2|x_1) \sum_{x_3} p(x_3|x_1) \sum_{x_4} p(x_4|x_2) \sum_{x_5} p(x_5|x_1)\phi_{X_6}(x_2, x_5) \end{aligned}$$

which assumes that $X_4 \perp\!\!\!\perp \{X_1, X_3\} | X_2$.

Theorem 1.8. *This is the first theorem.*

Proof. This is the proof of the first theorem. We show how to write pseudo-code now.

Consider a comparison between x and y:

if x or y or both are in S **then**
  answer accordingly
else
  Make the element with the larger score (say x) win the comparison
  if $F(x) + F(y) < \frac{n}{t-1}$ **then**
    $F(x) \leftarrow F(x) + F(y)$
    $F(y) \leftarrow 0$
  else
    $S \leftarrow S \cup \{x\}$
    $r \leftarrow r + 1$
  endif
endif

This concludes the proof. □

1.2 Next topic

Here is some citations [JB00] and [L2000]

References

[JB00] M.I. JORDAN and C. BISHOP, "An Introduction to Graphical Models," *To be published*, 2000

[L2000] S.L. LAURITZEN "Graphical Models," *Oxford Science Publications*, 1996