2023-2

1. Let $f:[1,\infty)\to\mathbb{R}$ be a differentiable function such that $f(1)=\frac{1}{3}$ and $3\int_1^x f(t)dt = xf(x)-\frac{x^2}{3}, x\in[1,\infty)$. Let e denote the base of the natural logarithm.Then the value of $f(e)$ is

   A. $\frac{e^2+4}{3}$
   B. $\frac{\log_e 4+e}{3}$
   C. $\frac{4e^2}{3}$
   D. $\frac{e^2-4}{3}$

2. Consider an experiment of tossing a coin repeatedly until the outcomes of two consecutive tosses are the same. If the probability of a random toss resulting in heads is $\frac{1}{3}$, then the probability that the experiment stops with head is

   (a) $\frac{1}{3}$
   (b) $\frac{5}{21}$
   (c) $\frac{4}{21}$
   (d) $\frac{2}{7}$

3. For any $y\in\mathbb{R}$, let $\cot^{-1}(y)\in(0,\pi)$and $\tan^{-1}(y)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ .Then the sum of all the solutions of the equation $\tan^{-1}(\frac{6y}{9-y^2})+\cot^{-1}(\frac{9-y^2}{6y})=\frac{2\pi}{3}$ for $0<|y|<3$ is equal to

   (a) $2\sqrt{3}-3$
   (b) $3-2\sqrt{3}$
   (c) $4\sqrt{3}-6$
   (d) $6-4\sqrt{3}$

4. Let the position vectors of the points P, Q, R, and S be $\vec{a} = \hat{i}+2\hat{j}-5\hat{k}$, $\vec{b} = 3\hat{i}+6\hat{j}+3\hat{k}$, $\vec{c} = \frac{17}{5}\hat{i}+\frac{16}{5}\hat{j}+\frac{7}{5}\hat{k}$, and $\vec{d} = 2\hat{i}+\hat{j}+\hat{k}$, respectively. Then which of the following statements is true?

   (a) The points P, Q, R, and S are NOT coplanar

   (b) $\frac{\vec{b}+2\vec{d}}{3}$ is the position vector of a point which divides PR internally in the ratio $5:4$

   (c) $\frac{\vec{b}+2\vec{d}}{3}$ is the position vector of a point which divides PR externally in the ratio $5:4$

   (d) The square of the magnitude of the vector $\vec{b} \times \vec{d}$ is 95

5. Let $M = (a_{ij})$, $a_{ij} \in \{1,2,3\}$ be the 3×3 matrix such that $a_{ij} = 1$ if $j+1$ is divisible by i, otherwise $a_{ij} = 0$. Then which of the following statements is(are) true?

   (a) M is invertible

   (b) There exists a nonzero column matrix $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ such that $M \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix}$

   (c) The set $\{X \in \mathbb{R}^3 : MX = 0\} \neq \{0\}$, where $0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

   (d) The matrix $(M - 2I)$ is invertible, where I is the 3×3 identity matrix

6. Let $f : (0,1) \to \mathbb{R}$ be the function defined as $f(x) = [4x]\left(x - \frac{1}{4}\right)^2\left(x - \frac{1}{2}\right)$, where $[x]$ denotes the greatest integer less than or equal to$x$.Then, which of the following statements is(are) true?

   (a) The function f is discontinuous exactly at one point in $(0,1)$.

   (b) There is exactly one point in $(0,1)$ at which the function f is continuous but NOT differentiable.

   (c) The function f is NOT differentiable at more than three points in $(0,1)$.

   (d) The minimum value of the function f is $\frac{1}{512}$.

7. Let S be the set of all twice differentiable functions $f from \mathbb{R} to \mathbb{R}$ such that $\frac{d^2f}{dx^2}(x) > 0$for all$x \in (-1,1)$. For $f \in S$, let X_f be the number of points $x \in (-1,1)$ where $f(x) = x$. Then, which of the following statements is(are) true?

   (a) There exists a function $f \in S$ such that $X_f = 0$.

   (b) For every function $f \in S$, we have $X_f \leq 2$.

   (c) There exists a function $f \in S$ such that $X_f = 2$.

   (d) There does NOT exist any function $f \quad in \quad S$ such that $X_f = 1$.

8. For $x \in \mathbb{R}$, let $\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the minimum value of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \int_0^{x \tan^{-1} x} \frac{e^{t-\cos t}}{1+t^{2023}}\, dt$

9. For $x \in \mathbb{R}$, let $y(x)$ be a solution of the differential equation $(x^2-5)\frac{dy}{dx} - 2xy = 2x(x^2-5)^2$ such that $y(2) = 7$. Then the maximum value of thefunction $y(x)$ is

10. Let X be the set of all five-digit numbers formed using the digits 1, 2, 2, 2, 4, 4, 0. For example, 22240 is in X, while 02244 and 44422 are not in X. Suppose that each element of$X$ has an equal chance of being chosen. Let $p$be the conditional probability that an element chosen at random is a multiple of 20 given that it is a multiple of 5. Then the value of $38p$ isequal to

11. Let $A_1, A_2, A_3, \ldots, A_8$ be the vertices of a regular octagon that lie on a circle of radius 2. Let P be a point on the circle, and let PA_i denote the distance between the points P and A_i for $i = 1, 2, \ldots, 8$. If P varies over the circle, then the maximum value of the product $PA_1 \cdot PA_2 \cdot PA_3 \cdots PA_8$is

12. Let $R = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \{0, 3, 5, 7, 11, 13, 17, 19\} \right)$. Then the number of invertible matrices in R is

13. Let C_1 be the circle of radius 1 with center at the origin. Let C_2 be the circle of radius r with center at the point $A = (4,1)$,where $1 < r < 3$. Two distinct common tangents PQ and ST of C_1 and C_2 are drawn. The tangent PQ touches C_1 at P and C_2 at Q. The tangent ST touches C_1 at S and C_2 at T. Midpoints of the line segments PQ and ST are joined to form a line which meets the x-axis at a point B. If $AB = \sqrt{5}$, then the value of r^2 is

Consider an obtuse angled triangle ABC in which the difference between the largest and the smallest angle is $\frac{\pi}{2}$ and whose sides are in arithmetic progression. Suppose that the vertices of this triangle lie on a circle of radius 1.

14. Let a be the area of the triangle ABC. Then the value of $(64a)^2$ is

15. Then the in radius of the triangle ABC is

Consider the 6×6 square in the figure. Let $A_1, A_2, \ldots, A_{49}$ be the points of intersections (dots in the picture) insome order. We say that A_i and A_j are friends if they are adjacent along a row or along a column. Assume that each point A_i has an equal chance of being chosen

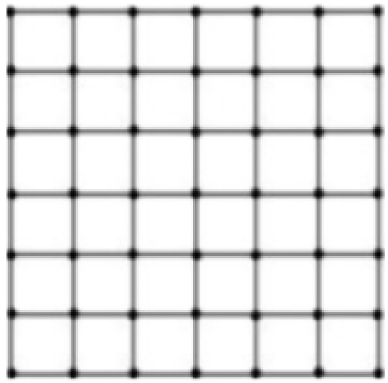

Figure 1: A 6x6 Square

16. Let p_i be the probability that a randomly chosen point has i many friends, $i = 0, 1, 2, 3, 4$. Let X be a random variable such that for $i = 0, 1, 2, 3, 4$, the probability $P(X = i) = p_i$. Then the value of $7E(X)$ is

17. Two distinct points are chosen randomly out of the points $A_1, A_2, A_3, \ldots, A_{49}$. Let p be the probability that they arefriends.Then the value of $7p$ is