

2023-2

1. Let $f : [1, \infty) \rightarrow \mathbb{R}$ be a differentiable function such that $f(1) = \frac{1}{3}$ and $3 \int_1^x f(t) dt = xf(x) - \frac{x^2}{3}, x \in [1, \infty)$. Let e denote the base of the natural logarithm. Then the value of $f(e)$ is
 - A. $\frac{e^2+4}{3}$
 - B. $\frac{\log_e 4+e}{3}$
 - C. $\frac{4e^2}{3}$
 - D. $\frac{e^2-4}{3}$
2. Consider an experiment of tossing a coin repeatedly until the outcomes of two consecutive tosses are the same. If the probability of a random toss resulting in heads is $\frac{1}{3}$, then the probability that the experiment stops with head is
 - (a) $\frac{1}{3}$
 - (b) $\frac{5}{21}$
 - (c) $\frac{4}{21}$
 - (d) $\frac{2}{7}$
3. For any $y \in \mathbb{R}$, let $\cot^{-1}(y) \in (0, \pi)$ and $\tan^{-1}(y) \in ((-\frac{\pi}{2}, \frac{\pi}{2}))$. Then the sum of all the solutions of the equation $\tan^{-1}(\frac{6y}{9-y^2}) + \cot^{-1}(\frac{9-y^2}{6y}) = \frac{2\pi}{3}$ for $0 < |y| < 3$ is equal to
 - (a) $2\sqrt{3} - 3$
 - (b) $3 - 2\sqrt{3}$
 - (c) $4\sqrt{3} - 6$
 - (d) $6 - 4\sqrt{3}$

4. Let the position vectors of the points P, Q, R , and S be $\vec{a} = \hat{i} + 2\hat{j} - 5\hat{k}$, $\vec{b} = 3\hat{i} + 6\hat{j} + 3\hat{k}$, $\vec{c} = \frac{17}{5}\hat{i} + \frac{16}{5}\hat{j} + \frac{7}{5}\hat{k}$, and $\vec{d} = 2\hat{i} + \hat{j} + \hat{k}$, respectively. Then which of the following statements is true?
- The points P, Q, R , and S are NOT coplanar
 - $\frac{\vec{b} + 2\vec{d}}{3}$ is the position vector of a point which divides PR internally in the ratio $5 : 4$
 - $\frac{\vec{b} + 2\vec{d}}{3}$ is the position vector of a point which divides PR externally in the ratio $5 : 4$
 - The square of the magnitude of the vector $\vec{b} \times \vec{d}$ is 95
5. Let $M = (a_{ij})$, $a_{ij} \in \{1, 2, 3\}$ be the 3×3 matrix such that $a_{ij} = 1$ if $j + 1$ is divisible by i , otherwise $a_{ij} = 0$. Then which of the following statements is(are) true?
- M is invertible
 - There exists a nonzero column matrix $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ such that $M \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix}$
 - The set $\{X \in \mathbb{R}^3 : MX = 0\} \neq \{0\}$, where $0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
 - The matrix $(M - 2I)$ is invertible, where I is the 3×3 identity matrix
6. Let $f : (0, 1) \rightarrow \mathbb{R}$ be the function defined as $f(x) = [4x] \left(x - \frac{1}{4}\right)^2 \left(x - \frac{1}{2}\right)$, where $[x]$ denotes the greatest integer less than or equal to x . Then, which of the following statements is(are) true?
- The function f is discontinuous exactly at one point in $(0, 1)$.
 - There is exactly one point in $(0, 1)$ at which the function f is continuous but NOT differentiable.
 - The function f is NOT differentiable at more than three points in $(0, 1)$.
 - The minimum value of the function f is $\frac{1}{512}$.

7. Let S be the set of all twice differentiable functions f from \mathbb{R} to \mathbb{R} such that $\frac{d^2f}{dx^2}(x) > 0$ for all $x \in (-1, 1)$. For $f \in S$, let X_f be the number of points $x \in (-1, 1)$ where $f(x) = x$. Then, which of the following statements is(are) true?
- There exists a function $f \in S$ such that $X_f = 0$.
 - For every function $f \in S$, we have $X_f \leq 2$.
 - There exists a function $f \in S$ such that $X_f = 2$.
 - There does NOT exist any function f in S such that $X_f = 1$.
8. For $x \in \mathbb{R}$, let $\tan^{-1}(x) \in ((-\frac{\pi}{2}, \frac{\pi}{2}))$. Then the minimum value of the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \int_0^{x \tan^{-1} x} \frac{e^{t - \cos t}}{1 + t^{2023}} dt$
9. For $x \in \mathbb{R}$, let $y(x)$ be a solution of the differential equation $(x^2 - 5)\frac{dy}{dx} - 2xy = 2x(x^2 - 5)^2$ such that $y(2) = 7$. Then the maximum value of the function $y(x)$ is
10. Let X be the set of all five-digit numbers formed using the digits 1, 2, 2, 4, 4, 0. For example, 22240 is in X , while 02244 and 44422 are not in X . Suppose that each element of X has an equal chance of being chosen. Let p be the conditional probability that an element chosen at random is a multiple of 20 given that it is a multiple of 5. Then the value of $38p$ is equal to
11. Let $A_1, A_2, A_3, \dots, A_8$ be the vertices of a regular octagon that lie on a circle of radius 2. Let P be a point on the circle, and let PA_i denote the distance between the points P and A_i for $i = 1, 2, \dots, 8$. If P varies over the circle, then the maximum value of the product $PA_1 \cdot PA_2 \cdot PA_3 \cdots PA_8$ is
12. Let $R = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \{0, 3, 5, 7, 11, 13, 17, 19\} \right)$. Then the number of invertible matrices in R is
13. Let C_1 be the circle of radius 1 with center at the origin. Let C_2 be the circle of radius r with center at the point $A = (4, 1)$, where $1 < r < 3$. Two distinct common tangents PQ and ST of C_1 and C_2 are drawn. The tangent PQ touches C_1 at P and C_2 at Q . The tangent ST touches C_1 at S and C_2 at T . Midpoints of the line segments PQ and ST are joined to form a line which meets the x -axis at a point B . If $AB = \sqrt{5}$, then the value of r^2 is

Consider an obtuse angled triangle ABC in which the difference between the largest and the smallest angle is $\frac{\pi}{2}$ and whose sides are in arithmetic progression. Suppose that the vertices of this triangle lie on a circle of radius 1.

14. Let a be the area of the triangle ABC . Then the value of $(64a)^2$ is

15. Then the in radius of the triangle ABC is

Consider the 6×6 square in the figure. Let A_1, A_2, \dots, A_{49} be the points of intersections (dots in the picture) in some order. We say that A_i and A_j are friends if they are adjacent along a row or along a column. Assume that each point A_i has an equal chance of being chosen

16. Let p_i be the probability that a randomly chosen point has i many friends, $i = 0, 1, 2, 3, 4$. Let X be a random variable such that for $i = 0, 1, 2, 3, 4$, the probability $P(X = i) = p_i$. Then the value of $7E(X)$ is

17. Two distinct points are chosen randomly out of the points $A_1, A_2, A_3, \dots, A_{49}$. Let p be the probability that they are friends. Then the value of $7p$ is

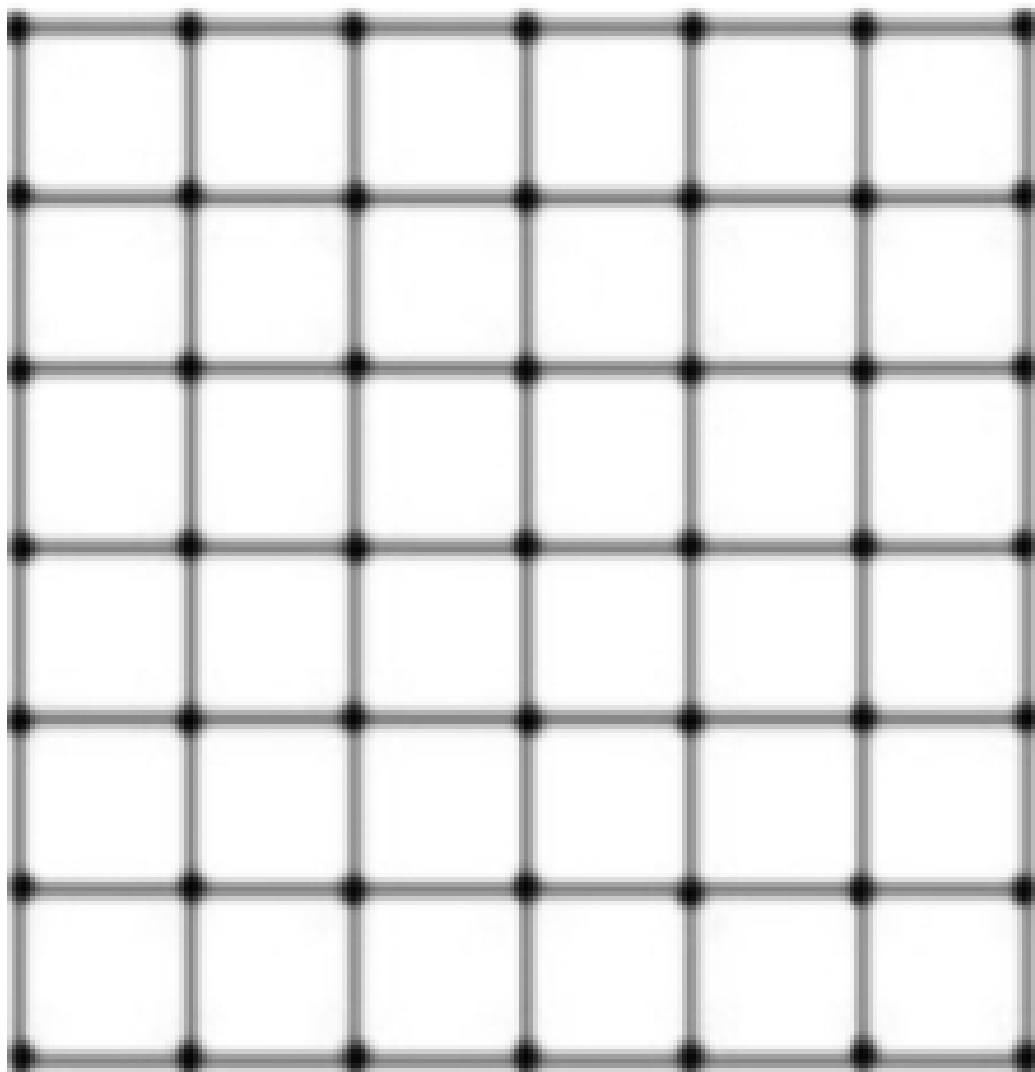


Figure 1: A 6x6 Square