Propositional Logic

Introduction

- *Many algorithms and proofs use logical expressions such as:*

  "IF p THEN q" or "If p1 AND p2, THEN q1 OR q2"

  Therefore it is necessary to know the cases in which these expressions are TRUE or FALSE, that is, to know the "truth value" of such expressions.
- *In this chapter, we will explain what makes up a correct mathematical argument and introduce tools to construct these arguments.*

Proposition (statement)

- *A proposition (or statement) is a declarative statement which is true or false, but not both.*

Investigate using truth value

- *1+1=2*
- *9<6*
- *Surat is in India*
- *Where are you going?*
- *Do your homework*

true

false

true

Not proposition, Its question

Not proposition, Its command

Law of Proposition (statement)

- *Excluded middle*
  - ❑ *Not both true and false*
  - ❑ *Example: I am 20 years old.*
- *Contradiction*
  - ❑ *May true but sure that not false or vice versa*
  - ❑ *Example: 2 + 2 > red*

Compound Propositions

- *It is composed of sub propositions.*
- *Example:*

Rahul is smart or he studies every night.

Roses are red and violets are blue.

Atomic Propositions

- *A proposition is said to be atomic/primitive if it cannot be broken down into simpler propositions.*

- *Example:*

Rahul is smart

Roses are red

Basic logical operations

- *Conjunction, $P \wedge Q$, (and)*
- *Disjunction, $P \vee Q$, (or)*
- *Negation, $\neg P$, $\sim P$, (not)*
- *NAND*
- *NOR*
- *EX-OR*
- *EX-NOR*

Negation (¬, ~)

P	~P
T	F
F	T

If P is true, then ¬P is false;
and
if P is false, then ¬P is true

Ice floats in water ¬ *Ice does not float in water*

China is in Europe ¬ *China is not in Europe*

Example

- *Find the negation of the proposition*

  "Michael's PC runs Linux"

Solution: The negation is
"It is not the case that Michael's PC runs Linux."

This negation can be more simply expressed as
"Michael's PC does not run Linux."

Example

- *Find the negation of the proposition*

  “Rahul’s Smartphone has at least 32GB of memory”
- *Also express this in simple English.*

- *Solution: The negation is*

“It is not the case that Rahul’s Smartphone has at least 32GB of memory.”

- *This negation can also be expressed as*

  “Rahul’s Smartphone does not have at least 32GB of memory”

  or

  “Rahul’s Smartphone has less than 32GB of memory.”

Disjunction (∨)

P	Q	(P ∨ Q)
T	T	T
T	F	T
F	T	T
F	F	F

If P and Q are false, then $P \vee Q$ is false; otherwise $P \vee Q$ is true.

Consider the following four statements:
(i) Ice floats in water or 2 + 2 = 4.
(ii) Ice floats in water or 2 + 2 = 5.
(iii) China is in Europe or 2 + 2 = 4.
(iv) China is in Europe or 2 + 2 = 5.

False ?

Only the last statement (iv) is false. Each of the others is true since at least one of its sub-statements is true.

Conjunction (∧)

P	Q	(P ∧ Q)
T	T	T
T	F	F
F	T	F
F	F	F

If P and Q are true, then $P \wedge Q$ is true; otherwise $P \wedge Q$ is false.

Consider the following four statements:

(i) Ice floats in water and 2 + 2 = 4. (iii) China is in Europe and 2 + 2 = 4.

(ii) Ice floats in water and 2 + 2 = 5. (iv) China is in Europe and 2 + 2 = 5.

True?

Only the first statement is true. Each of the others is false since at least one of its sub statements is false.

Truth table

P	Q	~Q	(P ∧ ~Q)	~(P ∧ ~Q)
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

Know as logically equivalent

Alternate Method for Constructing a Truth Table

P	Q	~	(P	∧	~	Q)
T	T	T	T	F	F	T
T	F	F	T	T	T	F
F	T	T	F	F	F	T
F	F	T	F	F	T	F
		4	1	3	2	1

Tautologies and contradiction

Some propositions contain only T in the last column of their truth tables

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

Some propositions contain only F in the last column of their truth tables

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

DeMorgan's laws:

$\sim(P \lor Q) \equiv \sim P \land \sim Q$ $\longrightarrow$ ***Know as logically equivalent***

P	Q	(P ∨ Q)	~(P ∨ Q)	~P	~Q	~P ∧ ~Q
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

$\sim(P \land Q) \equiv \sim P \lor \sim Q$

Example

- *p : “Roses are red*
- *q : “Violets are blue*

- *Solution : $(p \land q)$: “Roses are red and violets are blue”*

- $\neg(p \land q)$ *: “It is not true that roses are red and violets are blue.”*

- $\neg(p \land q) \equiv \neg p \lor \neg q$
- *Same meaning statement:*
- *“Roses are not red, or violets are not blue.”*

Algebra of propositions

Idempotent laws:	(1a) $p \vee p \equiv p$	(1b) $p \wedge p \equiv p$
Associative laws:	(2a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$	(2b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws:	(3a) $p \vee q \equiv q \vee p$	(3b) $p \wedge q \equiv q \wedge p$
Distributive laws:	(4a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	(4b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws:	(5a) $p \vee F \equiv p$ (6a) $p \vee T \equiv T$	(5b) $p \wedge T \equiv p$ (6b) $p \wedge F \equiv F$
Involution law:	(7) $\neg\neg p \equiv p$	
Complement laws:	(8a) $p \vee \neg p \equiv T$ (9a) $\neg T \equiv F$	(8b) $p \wedge \neg p \equiv F$ (9b) $\neg F \equiv T$
DeMorgan's laws:	(10a) $\neg(p \vee q) \equiv \neg p \wedge \neg q$	(10b) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Conditional Statement: Implication (>, $\longrightarrow$)

(if p then q) (p implies q) *if only if*

P is called the Hypothesis ; ***Q*** *is called the conclusion*

P	Q	(P→ Q)
T	T	*T*
T	F	*F*
F	F	*T*
F	T	*T*

"If you study, then you Pass."

"If your CPI is 6.0, then you don't need to pay tuition fee."

Example

- *Let p be the statement "Maria learns discrete mathematics" and q the statement "Maria will find a good job." Express the statement $p \rightarrow q$ as a statement in English.*

- *Solution :*
- *"If Maria learns discrete mathematics, then she will find a good job."*
- *"Maria will find a good job when she learns discrete mathematics."*

Example

- *What is the value of the variable x after the statement*

$$\textbf{if } 2 + 2 = 4 \textbf{ then } x := x + 1$$

- *if $x = 0$ before this statement is encountered?*

- *Solution: Because $2 + 2 = 4$ is true, the assignment statement $x := x + 1$ is executed. Hence, x has the value $0 + 1 = 1$ after this statement is encountered.*

Converse, Inverse, and Contrapositive

"If you do your homework, you will not be punished."

Inverse : $P \rightarrow Q$ is $\neg P \rightarrow \neg Q$

"If you do not do your homework, you will be punished."

Converse : $P \rightarrow Q$ is $Q \rightarrow P$

"If you will not be punished, you do your homework"

Contra-positive $P \rightarrow Q$ is $\neg Q \rightarrow \neg P$

"If you are punished, you did not do your homework".

Converse, Inverse, and Contrapositive cont…

					Inverse	Converse	Contra-positive
P	**Q**	**$\neg P$**	**$\neg Q$**	**$P \rightarrow Q$**	**$\neg P \rightarrow \neg Q$**	**$Q \rightarrow P$**	**$\neg Q \rightarrow \neg P$**
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Same

$$P \to Q = \neg P \vee Q$$

P	Q	P→Q	¬P	¬P ∨ Q
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Same

$$\neg(p \to q) \equiv p \wedge \neg q$$

Example

- *What are the contrapositive, the converse, and the inverse of the conditional statement*

  "The home team wins whenever it is raining?"
- ***Solution:***
- *the conditional statement $p \rightarrow q$, the original statement can be rewritten as **"If it is raining, then the home team wins."***
- *The **contrapositive** of this conditional statement is*

  "If the home team does not win, then it is not raining."
- *The **converse** is*

  "If the home team wins, then it is raining."
- *The **inverse** is*

  "If it is not raining, then the home team does not win."
- *Only the contrapositive is equivalent to the original statement.*

Biconditional statements

$\longleftrightarrow$ if and only if

In the first conditional, p is the hypothesis and q is the conclusion; in the second conditional, q is the hypothesis and p is the conclusion.

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Note: $P \longleftrightarrow Q$ is equivalent to $(P \longrightarrow Q) \wedge (Q \longrightarrow P)$

Note: $P \longleftrightarrow Q$ is equivalent to $(P \longrightarrow Q) \wedge (\neg P \longrightarrow \neg Q)$

"I am breathing if and only if I am alive."

Example

Let p be the statement "You can take the flight," and let q be the statement "You buy a ticket." Then $p \leftrightarrow q$ is the statement

"You can take the flight if and only if you buy a ticket."

Argument

An argument is a sequence of statements.

All statements except final one are called assumptions or hypothesis.

The final statement is called the conclusion.

An argument is valid if:

whenever all the assumptions are true, then the conclusion is true.

Valid: If today is Wednesday, then yesterday was Tuesday.
Today is Wednesday so Yesterday was Tuesday

InValid: If today is Wednesday, then yesterday was Monday.
Today is Wednesday and Yesterday was not Monday

Argument cont…

An argument is an assertion that a given set of propositions $P1, P2, \ldots, Pn$, called assumptions, yields another proposition Q, called the conclusion. Such an argument is denoted by

$$P1, P2, \ldots, Pn \vdash Q$$

$\vdash$ *turnstile symbol*

Argument is said to be valid if Q is true whenever all the assumptions $P1, P2, \ldots, Pn$ are true.

An argument which is not valid is called fallacy.

Argument cont…

$$P, Q \vdash P \vee Q$$ → *argument*

P	Q	(P ∨ Q)
T	T	T
T	F	T
F	T	T
F	F	F

(P ∨ Q)	T	assumptions
P	T	conclusion

Invalid argument

P	T	assumptions
(P ∨ Q)	T	conclusion

Valid argument

Disjunction (∨)

P	Q	(P ∨ Q)
T	T	T
T	F	T
F	T	T
F	F	F

assumptions	**P**	**~(P ∨ Q)**	**(P ∨ Q)**	**(P ∨ Q)** **P**	**(P ∨ Q)** **P**
conclusion	**(P ∨ Q)**	**P**	**~P**	**Q**	**~Q**
	valid	*Invalid*	*Invalid*	*Invalid*	*Invalid*

Conjunction (∧)

P	Q	(P ∧ Q)
T	T	T
T	F	F
F	T	F
F	F	F

assumptions	**P**	**~(P ∧ Q)**	**(P ∧ Q)**	**~(P ∧ Q)** **~P**	**~(P ∧ Q)** **P**
conclusion	**(P ∧ Q)**	**~P**	**P**	**Q**	**~Q**
	Invalid	*Invalid*	*valid*	*Invalid*	*valid*

Argument cont…

The argument P1, P2, . . . , Pn $\vdash$ *Q is valid if and only if the proposition (P1* $\wedge$ *P2 . . .* $\wedge$ *Pn)* $\rightarrow$ *Q is a tautology.*

"If P implies Q and Q implies R, then P implies R"

$$P \rightarrow Q, Q \rightarrow R \vdash P \rightarrow R = [(P \rightarrow Q) \wedge (Q \rightarrow R)] \rightarrow (P \rightarrow R)$$

Can toy Verify using truth table?

$$P \rightarrow Q, Q \rightarrow R \vdash P \rightarrow R = [(P \rightarrow Q) \land (Q \rightarrow R)] \rightarrow (P \rightarrow R)$$

p	q	r	$[(p$	$\rightarrow$	$q)$	$\land$	$(q$	$\rightarrow$	$r)]$	$\rightarrow$	$(p$	$\rightarrow$	$r)$
T	T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T	F	F	T	T	F	F
T	F	T	T	F	F	F	F	T	T	T	T	T	T
T	F	F	T	F	F	F	F	T	F	T	T	F	F
F	T	T	F	T	T	T	T	T	T	T	F	T	T
F	T	F	F	T	T	F	T	F	F	T	F	T	F
F	F	T	F	T	F	T	F	T	T	T	F	T	T
F	F	F	F	T	F	T	F	T	F	T	F	T	F
Step			1	2	1	3	1	2	1	4	1	2	1

Example

Consider the following argument:

S1 : If a man is a bachelor, he is unhappy.
S2 : If a man is unhappy, he is playing game.

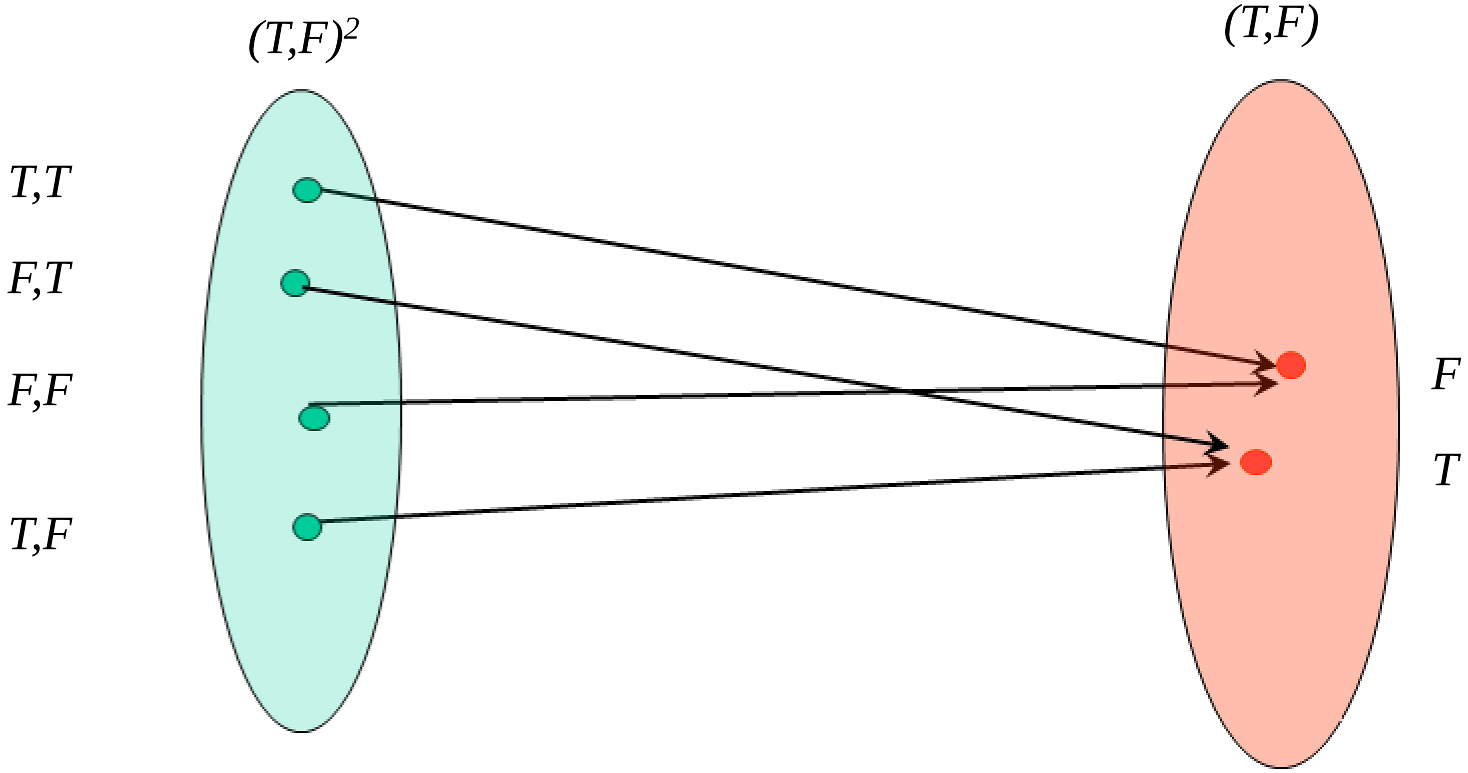
S : man is playing game

Valid or invalid?

$$P \rightarrow Q, Q \rightarrow R \vdash P \rightarrow R \quad \textbf{\textit{means}} \quad \textbf{\textit{valid}}$$

Propositional function

$$Map \quad f(T,F)^n \rightarrow (T,F)$$



Propositional function cont…

P(x) denotes the value of propositional function P at x.

The domain is often denoted by U (the universe)

Example:

- *Let P(x) denote "x > 5" and U be the integers. Then*

  P(8) is true.

  P(5) is false.

- *Let P(x, y, z) denote that x + y = z and U be the integers for all three variables.*

  P(−4, 6, 2) is true.

  P(5, 2, 10) is false.

  P(5, x, 7) is not a proposition.

Quantifiers

- *We need quantifiers to formally express the meaning of the words/sentence for "all" and "some".*

- *The two most important quantifiers are:*

  Universal quantifier, "For all". Symbol: $\forall$

  Existential quantifier, "There exists". Symbol: $\exists$

- *$\forall x\, P(x)$ asserts that $P(x)$ is true for every x in the domain.*
- *$\exists x\, P(x)$ asserts that $P(x)$ is true for some x in the domain.*

- *Variables in the scope of some quantifier are called bound variables.*
- *All other variables in the expression are called free variables.*

Universal Quantifier

$\forall x$ P(x) is read as "For all x, P(x)" or "For every x, P(x)".

The truth value depends not only on P, but also on the domain U.

Example:

Let P(x) denote $x > 0$.

If U is the integers then $\forall x$ P(x) is false.
If U is the positive integers then $\forall x$ P(x) is true.

Existential Quantifier

$\exists x\, P(x)$ is read as "For some x, P(x)" or "There is an x such that, P(x)", or "For at least one x, P(x)".

The truth value depends not only on P, but also on the domain U.

Example:

Let P(x) denote $x < 0$.

If U is the integers then $\exists x\, P(x)$ is true.
If U is the positive integers then $\exists x\, P(x)$ is false

Negation of Quantified statements

Consider the statement: "All majors are male."

Its negation reads:
"It is not the case that all majors are male" or, equivalently, "There exists at least one major who is a female (not male)"

Symbolically, using M to denote the set of majors, the above can be written as

$$\neg(\forall x \in M)(x \text{ is male}) \equiv (\exists x \in M)\,(x \text{ is not male})$$

or, when p(x) denotes "x is male,"

$$\neg(\forall x \in M)p(x) \equiv (\exists x \in M)\neg p(x)$$

or

$$\neg\forall x p(x) \equiv \exists x \neg p(x)$$

De Morgan's Law for Quantifiers

The rules for negating quantifiers are:

• $\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$

In other words, the following two statements are equivalent:

(1) It is not true that, for all $a \in A$, $p(a)$ is true. (2) There exists an $a \in A$ such that $p(a)$ is false.

• $\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$

That is, the following two statements are equivalent:

(1) It is not true that for some $a \in A$, $p(a)$ is true. (2) For all $a \in A$, $p(a)$ is false.

Propositional Functions with more than One Variable

A propositional function preceded by a quantifier for each variable

$$\forall x \exists y, p(x, y)$$

denotes a statement and has a truth value

$\forall x \exists y, p(x, y)$, *that is, "For every x, there exists a y such that p(x,Y) is true*

Example

Let $S = \{1, 2, 3, \ldots\ 9\}$ *and* $p(x, y)$ *denote "*$x + y = 10$*."*

The following is a statement

$\forall x \exists y, p(x, y)$*, that is,*
"For every x*, there exists a* y *such that* $x + y = 10$*"*

This statement is true.
For example, if $x = 1$*, let* $y = 9$*; if* $x = 2$*, let* $y = 8$*, and so on.*

Quick Summary

- *Conditional Statements*
  - *The meaning of IF and its logical forms*
  - *Contrapositive*
  - *If, only if, if and only if*
- *Arguments*
  - *definition of a valid argument*

Key points:

(1) Make sure you understand conditional statements and contrapositive.

(1) Make sure you can check whether an argument is valid.

Example

Let *p be "It is cold" and let q be "It is raining". Give a simple verbal sentence which describes each of the* following statements:
(*a*) $\neg p$; (*b*) $p \wedge q$; (*c*) $p \vee q$; (*d*) $q \vee \neg p$.

(*a*) *It is not cold.*
(*b*) *It is cold and raining.*
(c) It is cold or it is raining.
(d) It is raining or it is not cold.

Example

Verify that the proposition $p \vee \neg(p \wedge q)$ *is a tautology*

Construct the truth table of $p \vee \neg(p \wedge q)$

p	q	$p \wedge q$	$\neg(p \wedge q)$	$p \vee \neg(p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

T for all values of *p and q, the proposition is a tautology.*

Example

Rewrite the following statements without using the conditional:

(*a*) *If it is cold, he wears a hat.*

(*b*) *If productivity increases, then wages rise.*

$$p \rightarrow q \equiv \neg p \lor q$$

(*a*) *It is not cold or he wears a hat.*

(*b*) *Productivity does not increase or wages rise.*

Example

Determine the contrapositive of each statement:

(*a*) *If Erik is a poet, then he is poor.*

(*b*) *Only if Marc studies will he pass the test.*

$$p \to q \equiv \neg q \to \neg p$$

(*a*) If Erik is not poor, then he is not a poet..

(*b*) If Marc does not study, then he will not pass the test.

Example

Verify that

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q \equiv \neg p \leftrightarrow q$

Example

Write the negation of each statement as simply as possible:

(*a*) *If she works, she will earn money.*

(*b*) *He swims if and only if the water is warm.*

$$\neg(p \rightarrow q) \equiv p \wedge \neg q$$

$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q \equiv \neg p \leftrightarrow q$$

(*a*) She works or she will not earn money.

(*b*)

He swims if and only if the water is not warm.

He does not swim if and only if the water is warm.

ARGUMENTS Examples

Show that the following argument is a fallacy:
$p \to q, \neg p \vdash \neg q$

Construct the truth table for $[(p \to q) \land \neg p] \to \neg q$ as in Fig. 4-12. Since the proposition $[(p \to q) \land \neg p] \to \neg q$ is not a tautology, the argument is a fallacy. Equivalently, the argument is a fallacy since in the third line of the truth table $p \to q$ and $\neg p$ are true but $\neg q$ is false.

p	q	$p \to q$	$\neg p$	$(p \to q) \land \neg p$	$\neg q$	$[(p \to q) \land \neg p] \to \neg p$
T	T	T	F	F	F	T
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	F	T	T	T	T	T

ARGUMENTS Examples

Prove the following argument is valid:
$p \rightarrow \neg q, r \rightarrow q, r \vdash \neg p$

Construct the truth table of the premises and conclusions as in Fig. 4-14(*a*). *Now,* $p \rightarrow \neg q$, $r \rightarrow q$, *and* r *are*
true simultaneously only in the fifth row of the table, where $\neg p$ *is also true. Hence the argument is valid.*

	p	q	r	$p \rightarrow \neg q$	$r \rightarrow q$	$\neg q$
1	T	T	T	F	T	F
2	T	T	F	F	T	F
3	T	F	T	T	F	F
4	T	F	F	T	T	F
5	F	T	T	T	T	T
6	F	T	F	T	T	T
7	F	F	T	T	F	T
8	F	F	F	T	T	T

ARGUMENTS Examples

Determine the validity of the following argument:
If 7 is less than 4, then 7 is not a prime number.
7 is not less than 4.

7 is a prime number.

First translate the argument into symbolic form. Let *p be "7 is less than 4" and q be "7 is a prime number." Then* the argument is of the form

$$p \to \neg q,\ \neg q \vdash q$$

construct a truth table

p	q	$\neg q$	$p \to \neg q$	$\neg p$
T	T	F	F	F
T	F	T	T	F
F	T	F	T	T
F	F	T	T	T

Argument is fallacy since, in the fourth line of the truth table, the premises *$p \to \neg q$ and $\neg p$ are true, but the conclusion q is false.*

ARGUMENTS Examples

Test the validity of the following argument:
If two sides of a triangle are equal, then the opposite angles are equal.
Two sides of a triangle are not equal.
The opposite angles are not equal.

$p \rightarrow q, \neg p \vdash \neg q$, where p is "Two sides of a triangle are equal" and *q is "The opposite angles are equal."*

p	q	$p \rightarrow q$	$\neg p$	$(p \rightarrow q) \wedge \neg p$	$\neg q$	$[(p \rightarrow q) \wedge \neg p] \rightarrow \neg p$
T	T	T	F	F	F	T
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	F	T	T	T	T	T

This argument is a fallacy

Examples

Let $A = \{1, 2, 3, 4, 5\}$. Determine the truth value of each of the following statements:

(a) $(\exists x \in A)(x + 3 = 10)$ (c) $(\exists x \in A)(x + 3 < 5)$

(b) $(\forall x \in A)(x + 3 < 10)$ (d) $(\forall x \in A)(x + 3 \leq 7)$

(a) False. For no number in A is a solution to $x + 3 = 10$.

(b) True. For every number in A satisfies $x + 3 < 10$.

(c) True. For if $x_0 = 1$, then $x_0 + 3 < 5$, i.e., 1 is a solution.

(d) False. For if $x_0 = 5$, then $x_0 + 3$ is not less than or equal 7. In other words, 5 is not a solution to the given condition.

Examples

Determine the truth value of each of the following statements where U = {1, *2, 3}* ***is the universal set:***

(a) $\exists x \forall y,\ x^2 < y + 1$;

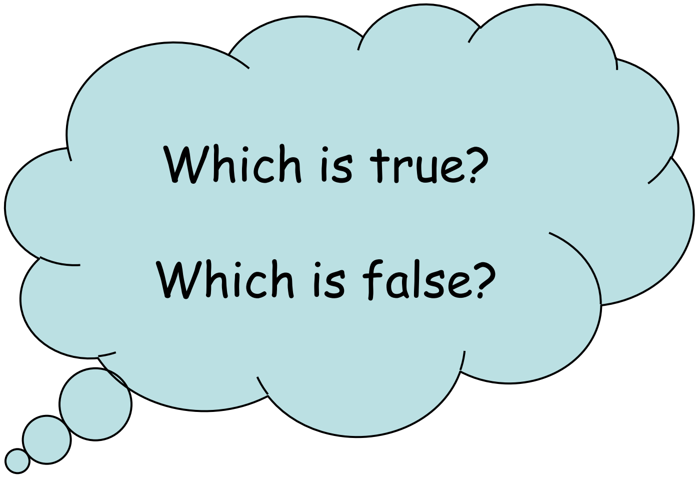
(b) $\forall x \exists y,\ x^2 + y^2 < 12$;

(c) $\forall x \forall y,\ x^2 + y^2 < 12$.

(*a*) *True. For if* $x = 1$, *then 1, 2, and 3 are all solutions to* $1 < y + 1$.

(*b*) *True. For each* x_0, *let* $y = 1$; *then* $x^2_0 + 1 < 12$ *is a true statement.*

(*c*) *False. For if* $x_0 = 2$ *and* $y_0 = 3$, *then* $x^2_0 + y^2_0 < 12$ *is not a true statement.*



"The sentence below is false."

"The sentence above is true."