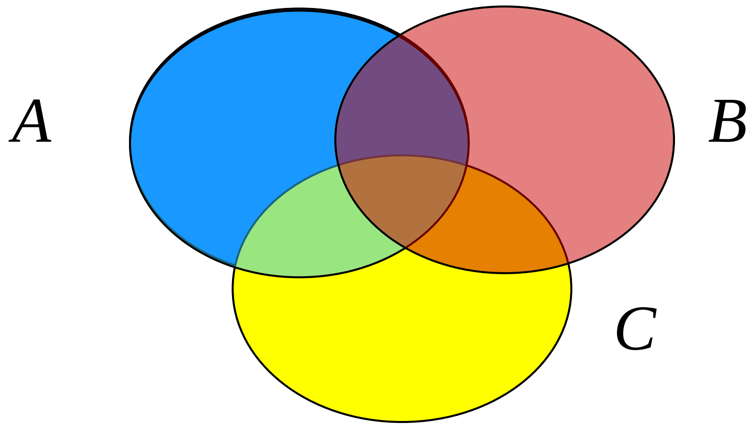


Discrete mathematics

Prof. Chintan Panchal

Set Theory

This Lecture

- We will first introduce set theory.
  - *Basic Definitions*
  - *Operations on Sets*
  - *Set Identities*

Sets

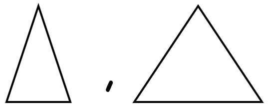
- *A set is an unordered collection of objects/data.*
- *Studying sets helps us categorize information.*
- *It allows us to make sense of a large amount of information by breaking it down into smaller groups.*

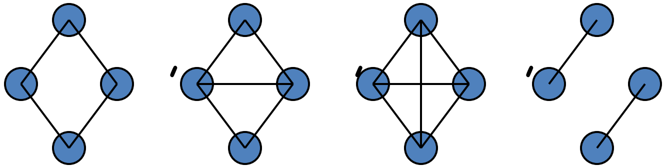
Examples of sets

- *Here are three sets:*
  - *The set of bits* $\{0, 1\}$
  - *The set of prime numbers* $\{2, 3, 5, 7, 11, \ldots\}$
  - *The set of basic arithmetic operators* $\{+, -, \cdot, /\}$

Other Examples of Sets

The set of all n-bit strings: {000…0, 000…1, …, 111…1}

The set of triangles with different angle: {  ,… }

The set of all graphs with four nodes: {  ,…}

Sets must be well-defined

- *A set is well-defined if we can tell whether a particular object is an element of that set.*

Defining a Set

- *A collection of well-defined unordered objects is called a set.*
- *For example,*
  - *'the set of former Nobel Prize winners' is a well-defined set.*
  - *'the set of Roll/En number of the students in class' is a well-defined set.*
  - *'the set of tall students in our university' is not a well-defined set.*

Key points

- ***Each object in a set is called an element or*** *a* ***member of the set.***.
  - ➢*Sets notation:* ***S, A, B, C, …*** ***(****single capital letter****)***
  - ➢*Elements Notation:* ***a, b, c, …***
  - ➢*Examples: S = {a, b, c, 2, 4}*
  - ➢$b \in S$, *'b is an element of set S' or 'b is in S'*
  - ➢$f \notin S$, *'f is not an element of set S' or 'f is not in S'*

Key points Cont…

- *Each element of the set is written only once.*
- *The order of elements in a set is not important.*

Denoting a set as an object

- *Particular symbols are reserved for the most important sets of numbers:*
  - *Ø – empty set*
  - *Z – integers*
  - *R – real numbers*
  - *Q – rational numbers*
  - *C – complex numbers*

Set cardinality/cardinal Numbers

- *The cardinality of a set S, denoted by |S| or n(S), is the* ***number of elements*** *in S.*

Set cardinality cont…

- The cardinality of the set of bits is ?

$$S = \{0,1\}$$

$$|S| = 2$$

- The cardinality of the set *S of prime numbers between 10 and 20 is ?*

$$S = \{\ 11, 13, 17, 19\}$$

$$|S| = 4$$

e.g. if S = {2, 3, 5, 7, 11, 13, 17, 19}, then |S|=8.

if S = {CSC1130, CSC2110, ERG2020, MAT2510}, then |S|=4.

if S = {{1,2}, {1,3}, {1,4}, {2,3}, {2,4}, {3,4}}, then |S|=6.

Representation of Sets

- *The List Method*
- *The Defining-Property Method*
- *The Venn Diagram Method*

The List Method

- For instance,

C = {I, You, He, She, We, They}

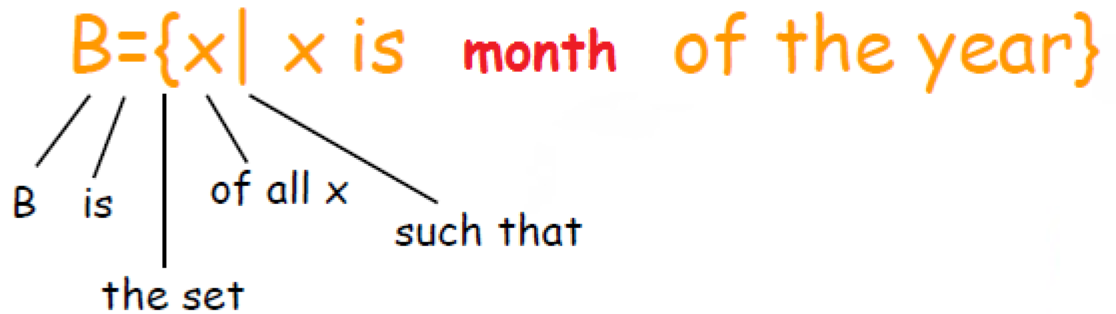
A = {+, -, %, $}

The List Method cont…

- *A = {Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday}*
- *B = {a, b, c, d, e}*
- *C = {m, a, t, h, e, i, c, s} Each element is written only once.*
- *D = {1, 2, 3, …, 1000} The first few elements of D are written to establish a pattern. The three dots (…), called an ellipsis, indicate that the last continues in the same way up to the last number of the set, which is 1000.*
- *W = {0, 1, 2, 3, …} If we don't put a number after the ellipsis*, this means that the list doesn't end.

The Defining-Property Method

- *For example,*

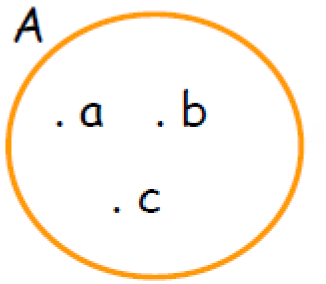


The Defining-Property Method cont…

- *A = {x| x is a day of the weekend}*
- *B = {x| x is a season of the year}*
- *C = {y| y is a whole number less than 25 & divisible by 3}*
- *D = {z| z is a blood type}*

The Venn Diagram Method

- *As an example,*



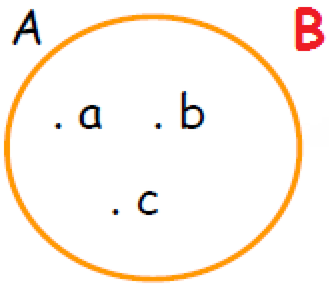
Each element of the set is represented by a point inside the closed shape

Equivalent Sets

- ***Two sets are equivalent if they have the same number of elements.***
- *If A & B are equivalent*
- $A \equiv B$
- *'set A is equivalent to set B'*
- *For example,*
  - $A = \{a, b, c, d\}$ & $B = \{1, 2, 3, 4\}$
  - $n(A) = n(B) = 4$ so $A \equiv B$

Equal Sets

- ***Two sets are equal if they have exactly the same elements.***
- *If A & B are equal*
- *A = B*
- *'set A is equal to set B'*



Note

- *If A is equal to B, then A is also equivalent to B.*

$$A = B \rightarrow A \equiv B$$

- *However,*
  - *If A is equivalent to B, then A might not be equal to B.*

$$A \equiv B \nrightarrow A = B$$

Types of Set

- *Empty (Null) Set*
- *The Universal Set*
- *Finite & Infinite Sets*

Empty (Null) Set

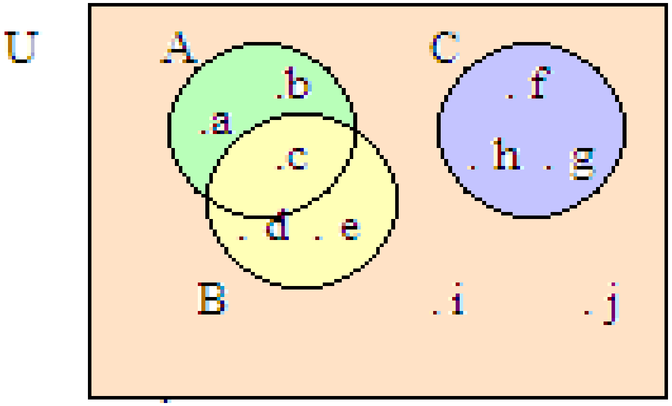
- ***The set that contains no element is called*** *the* ***empty set or the null set.***
- *Denotation: Ø or* **{ }**
- *As an example,*

$$A = \{x \mid x \text{ is a month containing 32 days}\}$$

$$\text{So } A = \varnothing \ \& \ n(\varnothing) = 0$$

The Universal Set

- ***The Universal set is the set of all elements under consideration in a given discussion.***
- *Denotation: U*



$A = \{a, b, c\}$
$B = \{c, d, e\}$
$C = \{f, g, h\}$

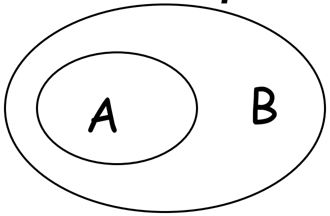
- *For instance,* $U = \{a, b, c, d, e, f, g, h, i, j\}$

Finite & Infinite Sets

- ***If the number of elements in a set is a whole number, the set is a finite set. If a set is not finite then it is an infinite set.***
- *As an example,*
  - *'the set of days of the week' is a finite set*
  - *'the set of all integer numbers' is an infinite set*

Subsets

- *Given two sets A and B, we say A is a subset of B, denoted by $A \subseteq B$, if every element of A is also an element of B.*

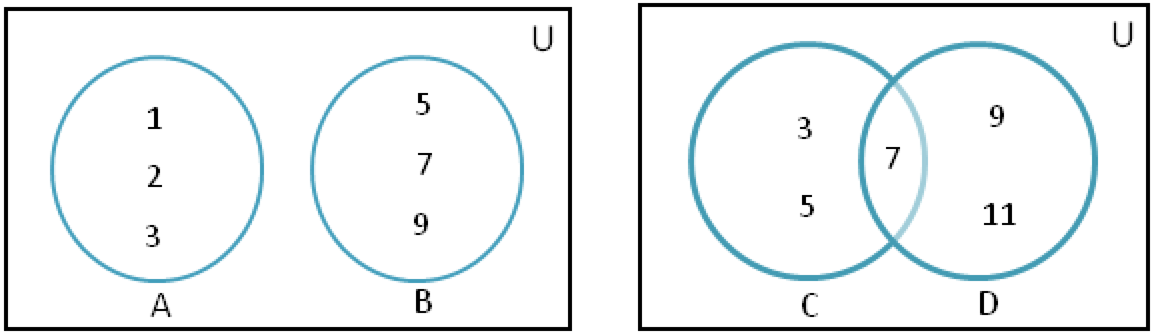


- *For example,*
  - *if B={1,2,3,4} and A ={1,2} then*
  - *A is a **subset** of B it is represented by $A \subseteq B$*
  - *B is a **superset** of A it is represented by $B \supseteq A$.*

*Note: if **every element** of A is also an element of B*

Disjoint set

- *Two sets are said to be disjoint sets if they have no element in common.*



Disjoint Sets

Not Disjoint Sets

Power Sets

- *The power set of any set S is the set of all subsets of S, including the empty.*
- *Example-*

pow({a,b}) = {∅, {a}, {b}, {a,b}}

pow({a,b,c}) = {∅, {a}, {b}, {c}, {a,b}, {a,c}, {b,c}, {a,b,c}}

pow({a,b,c,d}) = {∅, {a}, {b}, {c}, {d},
{a,b}, {a,c}, {b,c}, {a,d}, {b,d}, {c,d},
{a,b,c}, {a,b,d}, {a,c,d}, {b,c,d}, {a,b,c,d}}

Fact (to be explained later): If A has n elements, then pow(A) has 2^n elements.

Operation Of Sets

- *Union of sets*
- *Intersection of sets*
- *Compliments of sets*
- *Cartesian Product*

Union

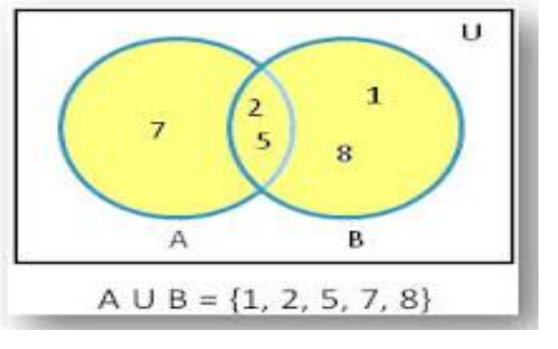
- *The union of two sets would be wrote as A U B, which is the set of elements that are members of A or B, or both too.*

$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$$

Example-

In the examples A = {2,5,7} and B = {1,2,5,8}

$A \cup B = \{1,2,5,7,8\}$



Intersection

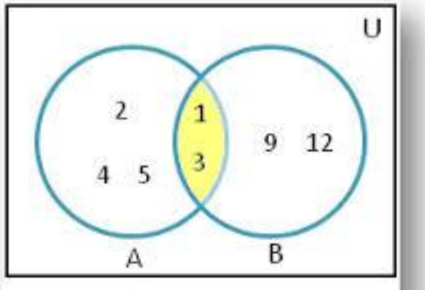
- *Intersection are written as $A \cap B$, is the set of elements that are in A and B.*

$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$$

Example-

In the examples $A = \{1,2,3,4,5\}$ and $B = \{1,3,9,12\}$

$A \cap B = \{1,3\}$



A ∩ B = {1, 3}

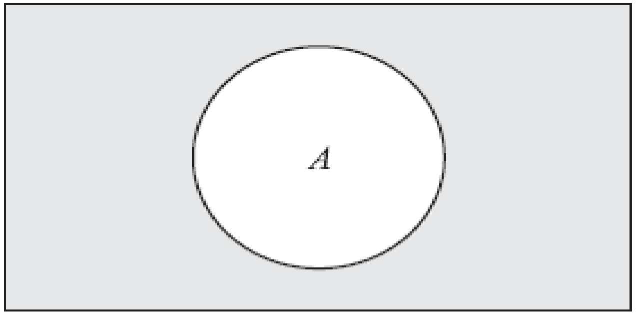
Example

- Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7\}$, $C = \{2, 3, 8, 9\}$.
- *Then*
- $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$,
- $A \cup C = \{1, 2, 3, 4, 8, 9\}$,
- $B \cup C = \{2, 3, 4, 5, 6, 7, 8, 9\}$,
- $A \cap B = \{3, 4\}$,
- $A \cap C = \{2, 3\}$,
- $B \cap C = \{3\}$.

Complements

- *If A is any set which is the subset of a given universal set then its complement is the set which contains all the elements that are in but not in A.*

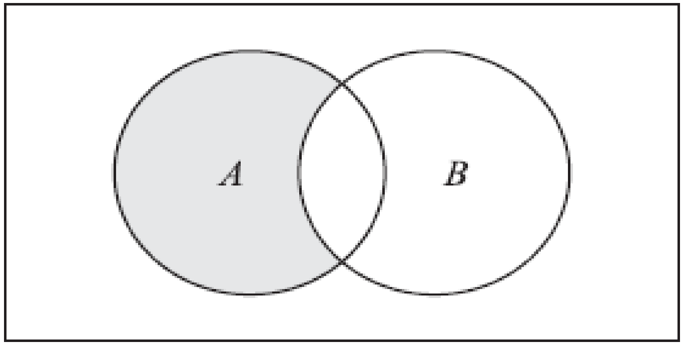
$$A^C = \{x \mid x \in U, x \notin A\}$$



Difference

$$A\text{-}B = A \backslash B = \{x \mid x \in A, x \notin B\}$$

Fact: $|A - B| = |A| - |A \cap B|$

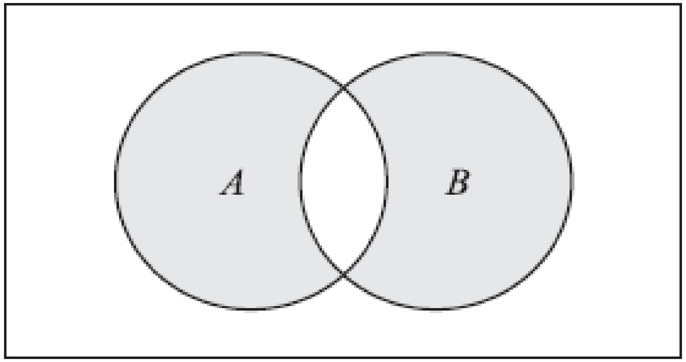


Symmetric difference

- The *symmetric difference of sets A and B, denoted by A ⊕ B, consists of those elements which belong to A* or *B but not to both.*

$$A \oplus B = (A \cup B) \setminus (A \cap B)$$

$$= (A \setminus B) \cup (B \setminus A)$$



Example

- **U = N = {1, *2, 3, . . .} is the universal set.***
- $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7\}$, $C = \{2, 3, 8, 9\}$, $E = \{2, 4, 6, . . .\}$ (Here *E is the set of even integers.)*
- *Find ?*
- A^C , B^C , E^c
- $A \backslash B$, $A \backslash C$, $B \backslash C$, $A \backslash E$
- $B \backslash A$, $C \backslash A$, $C \backslash B$, $E \backslash A$.
- $A \oplus B$, $B \oplus C$, $A \oplus C$, $A \oplus E$

Example cont…

- **U = N = {1, *2, 3, . . .} is the universal set.***
- $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7\}$, $C = \{2, 3, 8, 9\}$, $E = \{2, 4, 6, . . .\}$ (Here *E is the set of even integers.)*

- *Then:*
- $A^C = \{5, 6, 7, . . .\}$, $B^C = \{1, 2, 8, 9, 10, . . .\}$, $E^C = \{1, 3, 5, 7, . . .\}$

- $A \backslash B = \{1, 2\}$, $A \backslash C = \{1, 4\}$, $B \backslash C = \{4, 5, 6, 7\}$, $A \backslash E = \{1, 3\}$,
- $B \backslash A = \{5, 6, 7\}$, $C \backslash A = \{8, 9\}$, $C \backslash B = \{2, 8, 9\}$, $E \backslash A = \{6, 8, 10, 12, . . .\}$.

- $A \oplus B = (A \backslash B) \cup (B \backslash A) = \{1, 2, 5, 6, 7\}$, $B \oplus C = \{2, 4, 5, 6, 7, 8, 9\}$,
- $A \oplus C = (A \backslash C) \cup (B \backslash C) = \{1, 4, 8, 9\}$, $A \oplus E = \{1, 3, 6, 8, 10, . . .\}$.

Algebra of sets

- Commutative Laws:

$$A \cup B = B \cup A,\ A \cap B = B \cap A$$

- Associative Laws:

$$A \cup (B \cup C) = (A \cup B) \cup C,\ A \cap (B \cap C) = (A \cap B) \cap C$$

- Distributive Laws:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

- Identity Laws:

$$A \cup \emptyset = A,\ A \cap U = A$$

- Complement Laws:

$$A \cup A^c = U,\ A \cap A^c = \emptyset$$

- DeMorgan's Laws:

$$(A \cup B)^c = A^c \cap B^c,\ (A \cap B)^c = A^c \cup B^c$$

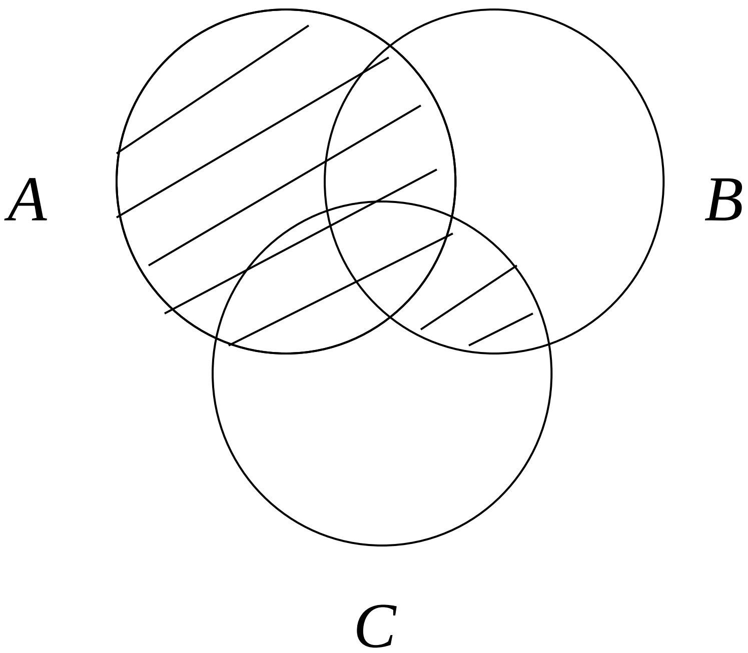
- Involution Law:

$$(A^c)^c = A$$

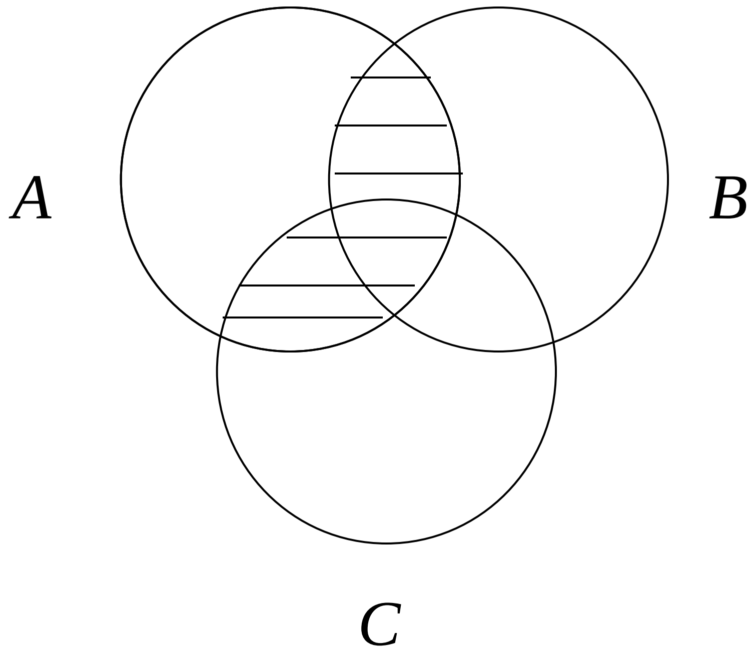
Set Identities

Distributive Law: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (1)

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (2)



(1)

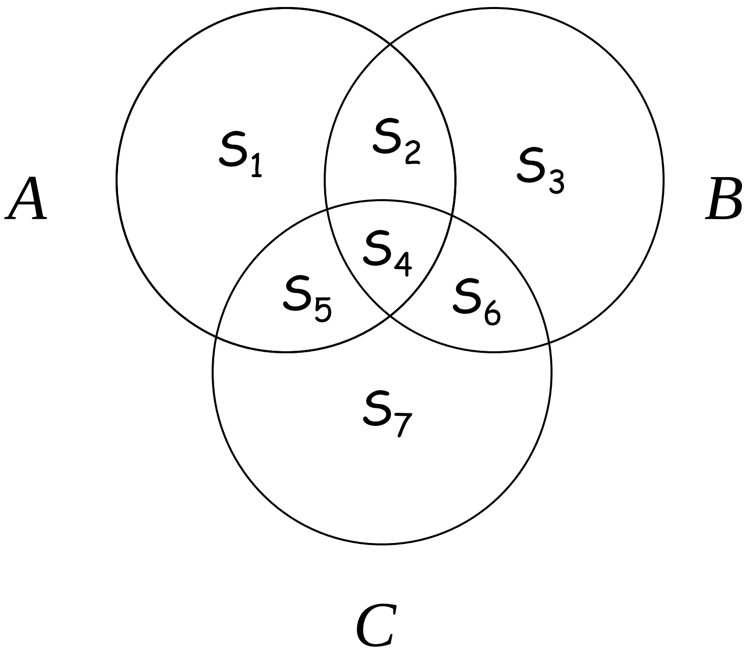


(2)

Set Identities

Distributive Law: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

We can also verify this law more carefully



L.H.S

$A = S_1 \cup S_2 \cup S_4 \cup S_5$

$B \cap C = S_4 \cup S_6$

$A \cup (B \cap C) = S_1 \cup S_2 \cup S_4 \cup S_5 \cup S_6$

R.H.S.

$(A \cup B) = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$

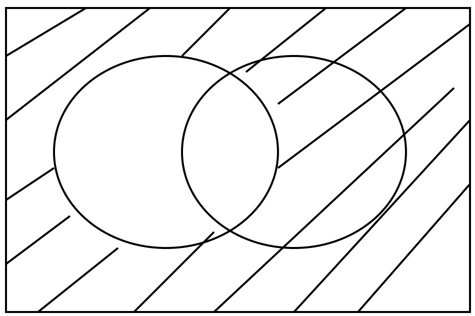
$(A \cup C) = S_1 \cup S_2 \cup S_4 \cup S_5 \cup S_6 \cup S_7$

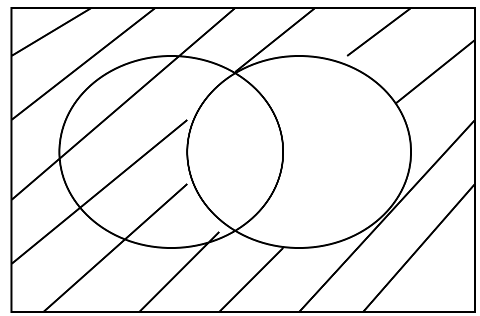
$(A \cup B) \cap (A \cup C) = S_1 \cup S_2 \cup S_4 \cup S_5 \cup S_6$

Set Identities

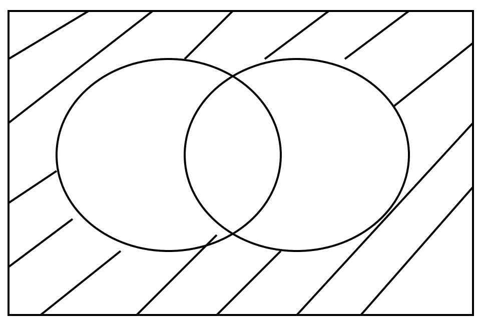
De Morgan's Law: $\overline{A \cup B} = \overline{A} \cap \overline{B}$

$\overline{A}$





$\overline{B}$

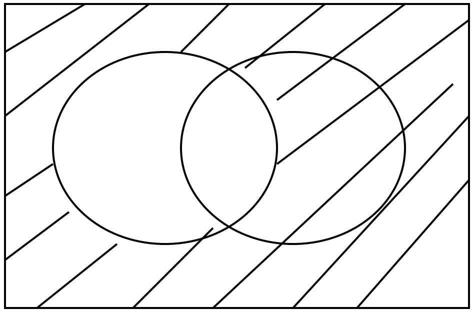


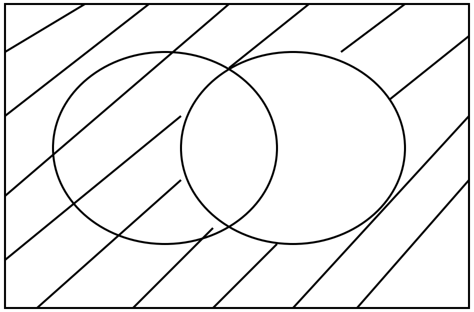
$\overline{A \cup B} = \overline{A} \cap \overline{B}$

Set Identities

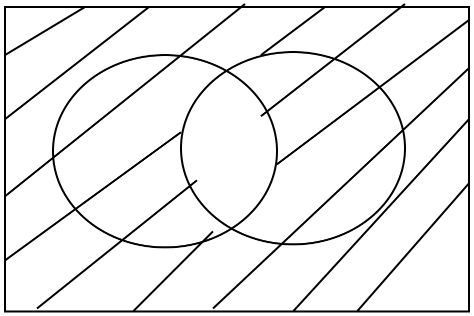
De Morgan's Law: $\overline{A \cap B} = \overline{A} \cup \overline{B}$

$\overline{A}$





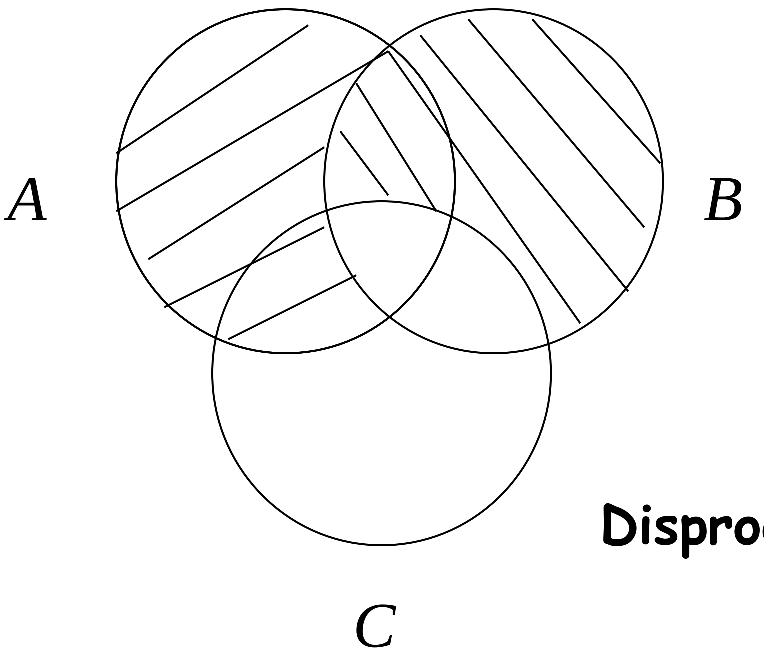
$\overline{B}$

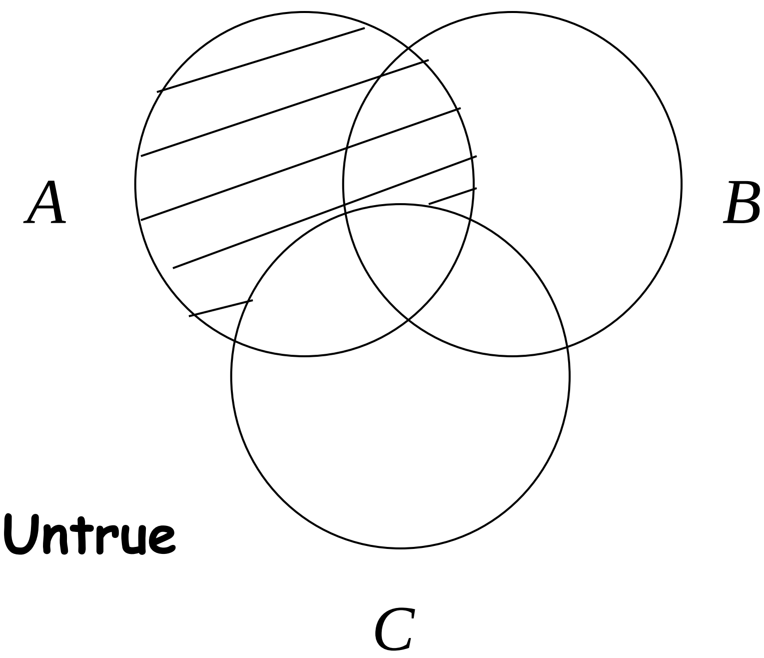


$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Proof

$$(A - B) \cup (B - C) = A - C?$$





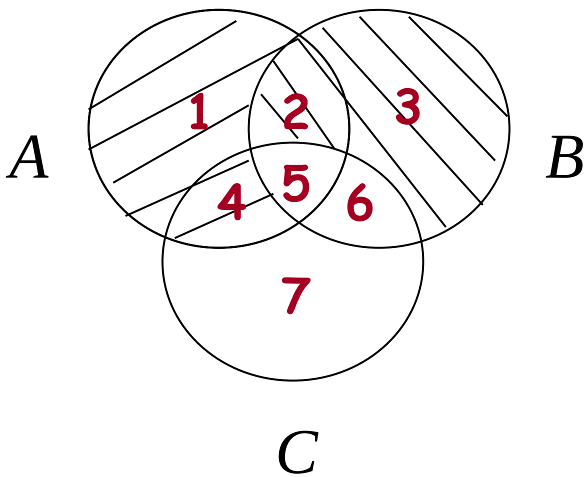
Disproof / Untrue

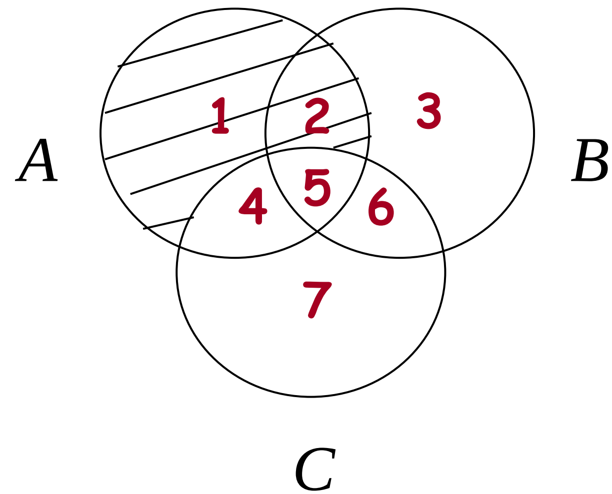
L.H.S

R.H.S

Disproof

$$(A - B) \cup (B - C) = A - C?$$





We can easily construct a **counterexample** to the equality, by putting a number in each region in the figure.

Let A = {1,2,4,5}, B = {2,3,5,6}, C = {4,5,6,7}.

Then we see that L.H.S = {1,2,3,4} and R.H.S = {1,2}.

Cartesian Product

- $A = \{1,2\}$ & $B = \{3,4\}$
- $A \times B = \{(1,3), (1,4), (2,3), (2,4)\}$

e.g.

▪Let A be the set of letters, i.e. {a,b,c,…,x,y,z}.

▪Let B be the set of digits, i.e. {0,1,…,9}.

▪AxA is just the set of strings with two letters.

▪BxB is just the set of strings with two digits.

▪AxB is the set of strings where the first character is a letter and the second character is a digit.

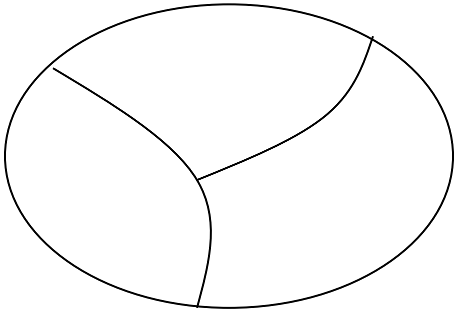
Partitions of Sets

Two sets are disjoint if their intersection is empty.

A collection of nonempty sets $\{A_1, A_2, \ldots, A_n\}$ is a partition of a set A if and only if

$$A = A_1 \cup A_2 \cup \cdots \cup A_n$$

$A_1, A_2, \ldots, A_n$ are mutually disjoint (or pairwise disjoint).



e.g. Let A be the set of integers.

Let A_1 be the set of negative integers.

Let A_2 be the set of positive integers.

Then $\{A_1, A_2\}$ is not a partition of A, because $A \neq A_1 \cup A_2$

as 0 is contained in A but not contained in $A_1 \cup A_2$

Exercises

$$A - (A \cap B) = A - B?$$

$$(A \cup B) - C = (A - C) \cup (B - C)?$$

$$\overline{(A \cup B \cup C)} = \overline{A} \cap \overline{B} \cap \overline{C}?$$

Example

List the elements of each set

(a) $A = \{x \in \mathbf{N} \mid \boldsymbol{3 < x < 9}\}$

(b) $B = \{x \in \mathbf{N} \mid \textbf{\textit{x is even, x < 11}}\}$

(c) $C = \{x \in \mathbf{N} \mid \boldsymbol{4 + x = 3}\}$

(a) A consists of the positive integers between 3 and 9; hence $A = \{4, 5, 6, 7, 8\}$.

(b) B consists of the even positive integers less than 11; hence $B = \{2, 4, 6, 8, 10\}$.

(c) No positive integer satisfies $4 + x = 3$; hence $C = \emptyset$, the empty set.

where **N = { N*atural numbers or positive integers: 1, 2, 3, . . .*}**

Example

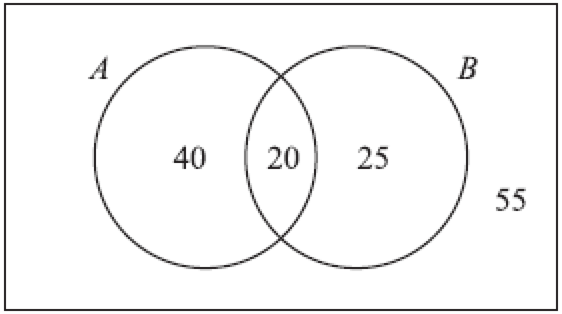
Each student in college has a mathematics requirement *A and a science requirement B.*
A poll of 140 students shows that:
60 completed *A, 45 completed B, 20 completed both A and B.*

Find the number of students who have completed:
(*a) At least one of A and B; (b) exactly one of A or B; (c) neither A nor B.*

(*a*) $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 60 + 45 - 20 = 85$

(*b*) $n(A \oplus B) = 65.$

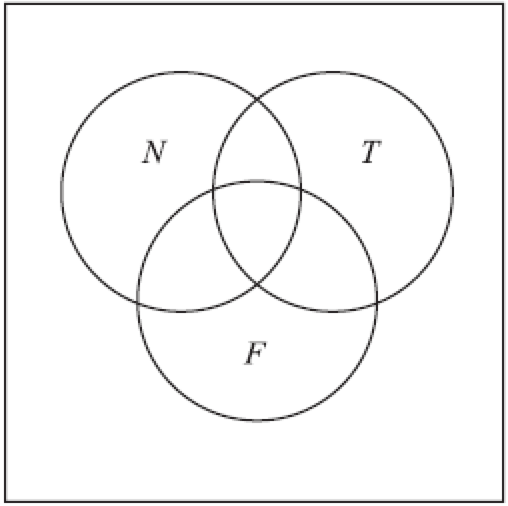
(*c*) $n[(A \cup B)^C] = 140 - 85 = 55.$



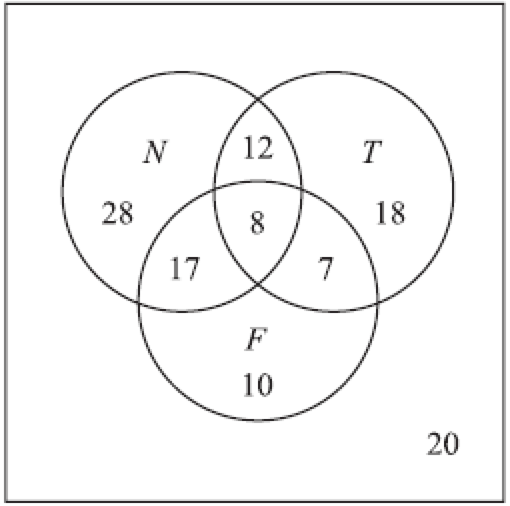
Example

In a survey of 120 people, it was found that:
65 read *Newsweek magazine, 20 read both Newsweek and Time*, 45 read *Time, 25 read both Newsweek and Fortune*, 42 read *Fortune, 15 read both Time and Fortune*, 8 read all three magazines.

(*a) Find the number of people who read at least one of the three magazines.*
(*b) Fill in the correct number of people in each of the eight regions of the Venn diagram in Fig where N, T , and F denote the set of people who read Newsweek, Time, and Fortune, respectively.*



(a) 100



Example

Let $S = \{a, b, c, d, e, f, g\}$. *Determine which of the following are partitions of S:*

(*a*) $P1 = [\{a, c, e\}, \{b\}, \{d, g\}]$,

(*b*) $P2 = [\{a, e, g\}, \{c, d\}, \{b, e, f\}]$,

(*c*) $P3 = [\{a, b, e, g\}, \{c\}, \{d, f\}]$,

(*d*) $P4 = [\{a, b, c, d, e, f, g\}]$.

(*a*) *P1 is not a partition of S since* $f \in S$ *does not belong to any of the cells.*

(*b*) *P2 is not a partition of S since* $e \in S$ *belongs to two of the cells.*

(*c*) *P3 is a partition of S since each element in S belongs to exactly one cell.*

(*d*) *P4 is a partition of S into one cell, S itself.*

?