Chapter 3 - Expectation

Q1: Start with c dollar $p(\text{double}) = p(\text{half}) = 0.5$
expected fortune after n trials?

$$E(x_n) = 0.5 \times 2x_{n-1} + 0.5\,\frac{x_{n-1}}{2} = 1.25\,x_{n-1}$$

$$= 1.25^2\,x_{n-2} = 1.25^n\,x_0 = 1.25^n c$$

Q2: $V(X) = 0 \iff \exists c : P(X=c) = 1$

if $P(X=c) = 1 \Rightarrow V(x) = E(x^2) - E(x)^2 = 1 - 1 = 0$

if $V(X) = 0 \Rightarrow \int_{-\infty}^{\infty} (x-\mu)^2\, dF_{(x)} = 0$

F_x non-decreasing $\rightarrow dF_X \geqslant 0$

assume $x \neq \mu \rightarrow dF_X = 0 \rightarrow P(X \neq \mu) = 0$

$\Rightarrow x = \mu$ always.

Q3: $X_1, \dots, X_n \sim \text{Unif}(0,1)$ $Y_n = \max\{X_1, \dots, X_n\}$
$E(Y_n) = ?$

find CDF for Y_n dist

$P(Y < y) = P(X_i < y) = y^n \quad \forall i$

$$\Rightarrow f_y = \frac{dF_Y}{dy} = n y^{n-1}$$

$$\Rightarrow E(Y) = \int_0^1 y \, n y^{n-1} = \frac{n}{n+1} y^{n+1} \Big|_0^1 = \frac{n}{n+1}$$

Q4: particle at 0, jumps of 1 unit

$P(\text{lef}) = p \qquad p(\text{right}) = 1-p$

$E(X_n)$, $V(X_n) = ?$ $\quad X_n$ position afte n jumps

$$X_n = \sum_{i=1}^{n} x_i \qquad E(X_i) = -1 \times p + 1(1-p) = 1-2p$$

$$\Rightarrow E(X_n) = n(1-2p)$$

$$V(x_n) = \sum_{i=1}^{n} V(x_i) = nV(x_i)$$

$$V(x_i) = E(x_i^2) - E(x_i)^2$$

$$= 1 - (1-2p)^2 = 4p(1-p)$$

$$\Rightarrow V(x_n) = 4np(1-p)$$

Q5: toss coin untill a head is obtained

expected number of tosses ?

$$P(n=1) = 1/2 \qquad \Rightarrow P(N=n) = 2^{-n}$$

$$P(n=2) = 1/2^2$$

$$E(N) = \sum_{n=1}^{\infty} n\left(\tfrac{1}{2}\right)^n = \sum_{n=1}^{\infty} n r^n \qquad r = \tfrac{1}{2}$$

we know $\sum_{x=1}^{\infty} r^x = \frac{r}{1-r}$

note: $\sum_{n\geq 1}^{\infty} n r^n = r \sum n r^{n-1} = r \sum \frac{d}{dr}(r^n)$

$$= r \frac{d}{dr} \sum r^n = r \frac{d}{dr}\left(\frac{r}{1-r}\right) = \frac{r}{(1-r)^2}$$

$$\Rightarrow E(N) = \sum_{n=1}^{\infty} n\left(\tfrac{1}{2}\right)^n = \frac{0.5}{0.5^2} = 2$$

Q6: prove $E(Y) = \sum_{i=1}^{n} r(x_i)\, P(x_i)$ $\qquad Y = r(X)$

X discrete r.v.

$A_j = \{x : r(x) = y_j\}$ A_j Partition on $\mathcal{X}$

$$E(Y) = \sum_j y_j\, P(Y = y_j) = \sum_j y_j\, P(x \in A_j)$$

$$= \sum_j \sum_{x \in A_j} r(x)\, p(x)$$

$$= \sum_{i=1}^{n} r(x_i)\, P(x_i)$$ since A_j is partition on $\mathcal{X}$

Q 7: assume $P(X \geq 0) = 1$ & $E(X)$ exists.

Show that $E(x) = \int_0^\infty P(X > x)\, dx$

Hint: consider integration by parts.

if $E(X)$ exists $\rightarrow \lim_{x \to \infty} x[1 - F(x)] = 0$

$$E(x) = \int_0^\infty \underbrace{x}_{u} \underbrace{f(x)\, dx}_{v'} = uv - \int_0^\infty v\, du$$

$$= x F_X \Big|_0^\infty - \int_0^\infty F_X\, dx$$

$$= \lim_{x \to \infty} x F_X - \int_0^\infty F_x\, dx$$

from the fact $\lim_{x \to \infty} x F_X = \lim_{x \to \infty} x$

$$\Rightarrow E(x) = \lim_{x \to \infty} x - \int_0^\infty F_X\, dx =$$

$$= \int_0^\infty dx - \int_0^\infty F_X\, dx = \int_0^\infty (1 - F_X)\, dx$$

$$= \int_0^\infty P(X > x)\, dx$$

Q8: $X_1, \ldots, X_n$ IID $\mu = E(x_i)$, $\sigma^2 = V(x_i)$

prove $E(\bar{X}_n) = \mu$ $\quad V(\bar{X}_n) = \frac{\sigma^2}{n}$ $\quad E(S_n^2) = \sigma^2$

$\bar{X}_n = 1/n \sum x_i \qquad S_n^2 = 1/{n-1} \sum (x_i - \bar{x}_n)^2$

$$E(\bar{x}_n) = \frac{1}{n}\sum E(x_i) = \frac{1}{n} n\mu = \mu$$

$$V(\bar{x}_n) = \sum \frac{1}{n^2} V(x_i) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

$$E(S_n^2) = \frac{1}{n-1} E\left[\sum x_i^2 + \sum \bar{x}_n^2 - \sum 2x_i \bar{x}_n\right]$$

note $\sum x_i \bar{x}_n = n\bar{x}_n^2$

$$= \frac{1}{n-1}\left[n E(x_i^2) + n E(\bar{x}_n^2) - 2n E(\bar{x}_n^2)\right]$$

$$= \frac{1}{n-1}\left[n E(x_i^2) - n E(\bar{x}_n^2)\right]$$

$$E(x_i^2) = \sigma^2 + \mu^2$$

$$E(\bar{x}^2) = \frac{\sigma^2}{n} + \mu^2 \left(\text{since } V(\bar{x}) = E(\bar{x}^2) - E(\bar{x})^2 = \frac{\sigma^2}{n}\right)$$

$\Rightarrow$

$$E(S_n^2) = \frac{n}{n-1}\left(\sigma^2 + \cancel{\mu^2} - \frac{\sigma^2}{n} - \cancel{\mu^2}\right)$$

$$= \frac{n}{n-1}\sigma^2\left(1 - \frac{1}{n}\right) = \frac{\cancel{n}}{\cancel{n-1}} \frac{\cancel{n-1}}{\cancel{n}} \sigma^2 = \sigma^2$$

Q9: $X_1, \ldots, X_n \sim$ Cauchy

$\bar{X}_n = \frac{1}{n}\sum x_i$ Plot $\bar{X}_n$ vs $n = 1, \ldots, 10^4$

for Cauchy $E(x), V(x)$ does not exist.

Q10: $X \sim N(0,1) \quad Y = e^X \quad E(Y) = ? \quad V(Y) = ?$

$$E(Y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{x} e^{-x^2/2} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{x(2-x)}{2}} \, dx$$

take $u = x - 1$

$$\Rightarrow E(Y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{1-u^2}{2}} \, du = e^{1/2} \underbrace{\int_{-\infty}^{\infty} \frac{e^{-u^2/2}}{\sqrt{2\pi}} \, du}_{1}$$

$$= e^{1/2}$$

$$E(Y^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{2x - x^2}{2}} \, dx \quad , \text{ take } u = x - 2$$

$$= 1/\sqrt{2\pi} \int_{-\infty}^{\infty} e^{2} e^{-u^2/2} \, du = e^2$$

$$V(Y) = E(Y^2) - E(Y)^2 = e^2 - e$$

Q11: $P(Y_i = 1) = P(Y_i = -1) = 1/2 \qquad X_n = \sum_{i=1}^{n} Y_i$

$E(X_n) = ? \quad V(X_n) = ?$

$$E(Y_i) = -\frac{1}{2} + \frac{1}{2} = 0 \Rightarrow E(X_n) = \sum E(Y_i) = 0$$

$$E(Y_i^2) = \frac{1}{2} + \frac{1}{2} = 1 \Rightarrow V(Y_i) = 1 - 0 = 1$$

$$\Rightarrow V(X_n) = \sum V(Y_i) = n$$

Q_{12}:

$X \sim \text{Bernoulli}(p)$ $E(X) = p$ $V(X) = p(1-p)$

$E(X) = p + 0(1-p) = p$

$E(X^2) = p \rightarrow V(X) = p - p^2 = p(1-p)$

Q_{12}

$X \sim \text{Poisson}(\lambda)$ $E(X) = \lambda$ $V(X) = \lambda$

Hint: $\sum_{x=0}^{\infty} \frac{a^x}{x!} = e^a$ for var first compute $E(X(X-1))$

$$f(x) = e^{-\lambda} \frac{\lambda^x}{x!} \quad x \geq 0$$

$$E(X) = \sum_{x=0}^{\infty} x e^{-\lambda} \frac{\lambda^x}{x!} = \sum_{x=1}^{\infty} \frac{e^{-\lambda} \lambda^x}{(x-1)!} = \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda} e^{\lambda} = \lambda$$

$$E(X(X-1)) = \sum_{x=0}^{\infty} x(x-1) e^{-\lambda} \frac{\lambda^x}{x!}$$

$$= \sum_{x=2}^{\infty} e^{-\lambda} \frac{\lambda^{x-2} \lambda^2}{(x-2)!} = \lambda^2 e^{-\lambda} e^{\lambda} = \lambda^2$$

$$\Rightarrow E(X^2) - E(X) = \lambda^2 \rightarrow E(X^2) = \lambda^2 + \lambda$$

$$\rightarrow V(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Q_{12}

$X \sim \text{uniform}(a, b) \qquad E(x) = \frac{a+b}{2} \qquad V(x) = \frac{(b-a)^2}{12}$

$$E(x) = \int_a^b \frac{x}{b-a}\,dx = \frac{1}{b-a}\,\frac{x^2}{2}\Big|_a^b = \frac{1}{2(b-a)} \times (b-a)(b+a) = \frac{b+a}{2}$$

$$E(x^2) = \int_a^b \frac{x^2}{b-a}\,dx = \frac{b^3-a^3}{3(b-a)} = \frac{a^2+b^2+ab}{3}$$

$$V(x) = E(x^2) - E(x)^2 = \frac{a^2+b^2+ab}{3} - \frac{a^2+b^2+2ab}{4}$$

$$= \frac{\cancel{4a^2}+\cancel{4b^2}+4ab-\cancel{3a^2}-\cancel{3b^2}-6ab}{12} =$$

$$= \frac{a^2+b^2-2ab}{12} = \frac{(b-a)^2}{12}$$

Q_{12}

$X \sim \text{Exp}(\beta) \qquad E(x) = \beta \qquad V(x) = \beta^2$

$f(x) = \frac{1}{\beta} e^{-x/\beta} \quad x > 0 \qquad\qquad \int u v' = uv - \int v\,du$

$$E(x) = \frac{1}{\beta}\int_0^\infty \underbrace{x}_{u}\, \underbrace{e^{-x/\beta}\,dx}_{v'} \qquad v' = e^{-x/\beta}dx \rightarrow v = -\beta e^{-x/\beta}$$

$$= \frac{1}{\beta}\left[-\beta x e^{-x/\beta}\Big|_0^\infty - \int_0^\infty -\beta e^{-x/\beta}\right]$$

$$= -xe^{-x/\beta}\Big|_0^\infty - \beta e^{-x/\beta}\Big|_0^\infty = -\lim_{x\to\infty} x e^{-x/\beta} - 0 + \beta = \beta$$

note: $\lim_{x\to\infty} \frac{x}{e^{x/\beta}} = \lim_{x\to\infty} \frac{1}{\frac{1}{\beta}e^{x/\beta}} = 0 \implies E(x) = \beta$

$$E(X^2) = \frac{1}{\beta}\int_0^\infty \underbrace{x^2}_{u} \underbrace{e^{-x/\beta}}_{v'} dx$$

$$= \frac{-\beta}{\beta} x^2 e^{-x/\beta}\Big|_0^\infty + \int_0^\infty 2x e^{-x/\beta} dx$$

from prev part: $\int_0^\infty x e^{-x/\beta} dx = \beta^2$

$$\Rightarrow E(X^2) = -\lim_{x\to\infty} x^2 e^{-x/\beta} + 2\beta^2 = 2\beta^2$$

$$\Rightarrow V(X) = 2\beta^2 - \beta^2 = \beta^2$$

Q12

$X \sim Gamma(\alpha, \beta)$ $\quad E(X) = \alpha\beta \quad V(X) = \alpha\beta^2$

Hint: multiply & divide by $\Gamma(\alpha+1)/\beta^{\alpha+1}$

use the fact that density integrates to 1.

$$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} \qquad x > 0$$

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy$$

$$E(X) = \int_0^\infty x \frac{\Gamma(\alpha+1)\beta^{\alpha+1}}{\Gamma(\alpha+1)\beta^{\alpha+1}} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} dx$$

$$= \frac{\Gamma(\alpha+1)\beta^{\alpha+1}}{\Gamma(\alpha)\beta^\alpha} \underbrace{\int_0^\infty \frac{1}{\beta^{\alpha+1}\Gamma(\alpha+1)} x^\alpha e^{-x/\beta} dx}_{Gamma(\alpha+1,\beta) \Rightarrow \text{ equal to } 1.}$$

$$\Rightarrow E(x) = \frac{\Gamma(\alpha+1)\,\beta}{\Gamma(\alpha)} = \alpha\beta$$

prove $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$?

$$\Gamma(\alpha+1) = \int_0^\infty \underbrace{y^\alpha}_{u} \underbrace{e^{-y}\,dy}_{v'} = -y^\alpha e^{-y}\Big]_0^\infty - \int -e^{-y}\alpha y^{\alpha-1}\,dy$$

$$= 0 + \alpha\int e^{-y} y^{\alpha-1}\,dy = \alpha\Gamma(\alpha)$$

also note : $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1) = (\alpha-1)!\,\Gamma(1)$

$$\Gamma(1) = \int_0^\infty y^0 e^{-y}\,dy = 1 \Longrightarrow \Gamma(\alpha) = (\alpha-1)!$$

Special case $\alpha = 1, 2, \ldots, n$

$$E(x^2) = \int_0^\infty x^2 \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)} \times \frac{\beta^{\alpha+2}\Gamma(\alpha+2)}{\beta^{\alpha+2}\Gamma(\alpha+2)}\,dx$$

$$= \frac{\beta^{\alpha+2}\Gamma(\alpha+2)}{\beta^\alpha \Gamma(\alpha)} \underbrace{\int_0^\infty \frac{x^{\alpha+1} e^{-x/\beta}}{\beta^{\alpha+2}\Gamma(\alpha+2)}\,dx}_{\text{Gamma}(\alpha+2,\beta)} = 1$$

$$\Rightarrow E(x^2) = \beta^2(\alpha+1)\alpha$$

$$\Rightarrow V(x) = \beta^2(\alpha^2+\alpha) - \alpha^2\beta^2 = \alpha\beta^2$$

Q12

$X \sim Beta(\alpha, \beta)$ $E(X) = \frac{\alpha}{\alpha+\beta}$ $V(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Hint: multiply & divide by $\Gamma(\alpha+1)\Gamma(\beta) / \Gamma(\alpha+\beta+1)$

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \quad 0 < x < 1$$

$$E(X) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1)} \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x\, x^{\alpha-1}(1-x)^{\beta-1}\, dx$$

$$= \frac{\Gamma(\alpha+1)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+1)\Gamma(\alpha)} \underbrace{\int_0^1 \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta)} x^{\alpha}(1-x)^{\beta-1}\, dx}_{Beta(\alpha+1,\beta)}$$

$$= \frac{\alpha\cancel{\Gamma(\alpha)}\,\cancel{\Gamma(\alpha+\beta)}}{(\alpha+\beta)\cancel{\Gamma(\alpha+\beta)}\,\cancel{\Gamma(\alpha)}}$$

$$\Rightarrow E(X) = \frac{\alpha}{\alpha+\beta}$$

$$E(X^2) = \frac{\Gamma(\alpha+2)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\alpha+\beta+2)} \int_0^1 \underbrace{\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+2)\Gamma(\beta)} x^{\alpha+1}(1-x)^{\beta-1}\, dx}_{=1}$$

$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

$$V(X) = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - \frac{\alpha^2}{(\alpha+\beta)^2} = \frac{(\alpha^2+\alpha)(\alpha+\beta) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$= \frac{\cancel{\alpha^3} + \cancel{\alpha^2\beta} + \cancel{\alpha^2} + \alpha\beta - \cancel{\alpha^3} - \cancel{\alpha^2\beta} - \cancel{\alpha^2}}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Q13: flip fair Coin → head $X \sim Unif(0,1)$

tail $X \sim Unif(3,4)$

$E(X) = ?$ $Std(X) = ?$

$$E(X) = \int_0^1 \frac{x}{2}\,dx + \int_3^4 \frac{x}{2}\,dx = \frac{x^2}{4}\Big)_0^1 + \frac{x^2}{4}\Big)_3^4 = 2$$

$$E(X^2) = \int_0^1 \frac{x^2}{2}\,dx + \int_3^4 \frac{x^2}{2}\,dx = \frac{19}{3}$$

$$V(X) = \frac{19}{3} - 4 = \frac{7}{3} \longrightarrow Std(X) = \sqrt{\frac{7}{3}}$$

Q14: $Cov\left(\sum_{i=1}^{m} a_i X_i, \sum_{j=1}^{n} b_j Y_j\right) \overset{?}{=} \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j Cov(X_i, Y_j)$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$\Rightarrow Cov(\Sigma X_i, \Sigma Y_j) = E\left(\Sigma a_i X_i \Sigma b_j Y_j\right) - E(\Sigma a_i X_i)\,E(\Sigma b_j Y_j)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j E(X_i Y_j) - \Sigma a_i E(X_i)\,\Sigma b_j E(Y_j)$$

$$= \sum_i \sum_j a_i b_j \underbrace{\left[E(X_i Y_j) - E(X_i)E(Y_j)\right]}_{Cov(X_i, Y_j)}$$

Q_{15}: $f(x,y) = \begin{cases} \frac{1}{3}(x+y) & 0 \leq x \leq 1,\ 0 \leq y \leq 2 \\ 0 & o.w. \end{cases}$

$V(2X-3Y+8) = ?$

$f(x) = \int_0^2 \frac{1}{3}(x+y)\,dy = \frac{xy}{3} + \frac{y^2}{6}\Big)_0^2 = \frac{2}{3}(x+1)$

$f(y) = \int_0^1 \frac{1}{3}(x+y)\,dx = \frac{x^2}{6} + \frac{xy}{3}\Big)_0^1 = \frac{y}{3} + \frac{1}{6}$

$E(X) = \int_0^1 \frac{2}{3}(x^2+x)\,dx = \frac{2}{3}\left(\frac{x^3}{3} + \frac{x^2}{2}\right)\Big)_0^1 = \frac{2}{3}\left(\frac{1}{3} + \frac{1}{2}\right) = \frac{5}{9}$

$E(X^2) = \int_0^1 \frac{2}{3}(x^3+x^2)\,dx = \frac{7}{18}$

$E(Y) = \frac{11}{9}$ $\quad E(Y^2) = \frac{16}{9}$

$\Rightarrow V(X) = E(X^2) - E(X)^2 = \frac{7}{18} - \frac{25}{81} = \frac{13}{162}$

$V(Y) = E(Y^2) - E(Y)^2 = \frac{16}{9} - \frac{121}{81} = \frac{23}{81}$

$E(XY) = \int_0^1 \int_0^2 xy\,\frac{x+y}{3}\,dx\,dy = \frac{1}{3}\int_0^1\int_0^2 (x^2y + xy^2)\,dx\,dy$

$= \frac{1}{3}\int_0^2 \left(\frac{x^3 y}{3} + \frac{x^2 y^2}{2}\right)_0^1 dy = \frac{1}{3}\int_0^2 \left(\frac{y}{3} + \frac{y^2}{2}\right) dy$

$= \frac{1}{3}\left(\frac{y^2}{6} + \frac{y^3}{6}\right)_0^2 = \frac{1}{18}(4+8) = \frac{2}{3}$

$\Rightarrow Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{2}{3} - \frac{5}{9}\frac{11}{9} = \frac{-1}{81}$

$V(2X-3Y+8) = V(2X-3Y) = 4V(X) + 9V(Y) - 2\times2\times3\,Cov(X,Y)$

$= 4\times\frac{13}{162} + 9\times\frac{23}{81} + 12\,\frac{1}{81} = \frac{245}{81}$

Q_{16}: $r(x)$ function of x, $S(y)$ function of y

$$E(r(X)\,s(Y)\mid X) \overset{?}{=} r(X)\,E(s(Y)\mid X)$$

$$E(r(X)\mid X) \overset{?}{=} r(X)$$

$$E(r(X)\,S(Y)\mid X) = \int r(x)\,S(y)\,f(y\mid x)\,dy$$

$$= r(x)\int S(y)\,f(y\mid x)\,dy$$

$$= r(X)\;E(S(Y)\mid X)$$

$$E(r(X)\mid X) = r(X)\,\underbrace{P(X=x\mid x)}_{1} = r(X)$$

Q_{17}: $V(Y) = EV(Y\mid X) + VE(Y\mid X)$

hints $E(Y)=m$ $\quad E(Y\mid X) = b(x)$

$$E(b(x)) = EE(Y\mid X) = E(Y) = m$$

$$V(Y) = E(Y-m)^2 = E((Y-b)+(b-m))^2$$

$$E(\text{Stuff}) = EE(\text{Stuff}\mid X)$$

$$V(Y) = E\left[(y-b)^2 + (b-m)^2 + 2(y-b)(b-m)\right]$$

$$E(y-b)^2 = EE((y-b)^2\mid X) = EE(y^2+b^2-2yb\mid x)$$

$$= E\left[E(y^2\mid X) + \underbrace{E(b^2\mid X)}_{b^2} - 2b\,\underbrace{E(y\mid x)}_{b}\right]$$

$$\Rightarrow E(y-b)^2 = E\left[E(y^2|x) - E(y|x)^2\right]$$
$$= E\,V(y|x)$$

$$E(b-m)^2 = E(b-E(b))^2 = V(b) = V E(y|x)$$

$$E[(y-b)(b-m)] = E\left[E((y-b)(b-m)|x)\right]$$
$$= E\left[(b-m)\,\cancel{E(y-b|x)}^{\,0}\right] = 0$$

$\Rightarrow$

$$V(y) = E(y-b)^2 + E(b-m)^2 = E\,V(y|x) + V E(y|x)$$

Q_{18} if $E(X|Y=y) = c$ constant c

$\Rightarrow$ X & Y are uncorrelated

$$E(XY) = E E(XY|Y) = E\left[Y E(X|Y)\right] = E(Yc)$$
$$= cE(Y) = E(X)E(Y)$$

why $c \stackrel{?}{=} E(X)$

$$E(X) = E E(X|y) = E(c) = c \quad \text{constant}$$

$$\Rightarrow \mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = 0$$

Q_{19}: $X_1, \dots, X_n \sim \text{Unif}(0,1)$

$\bar{X} = \frac{1}{n} \sum X_i \rightarrow E(\bar{X}) = ? \quad V(\bar{X}) = ?$

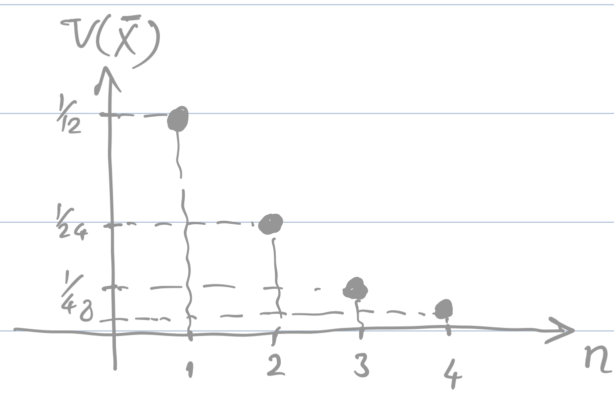
plot them as fun of n

$$E(\bar{X}) = E(X_i) = \int_0^1 x \, dx = \frac{1}{2}$$

$$V(\bar{X}) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

$$V(X_i) = \sigma^2 = E(X_i^2) - E(X_i)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\Rightarrow V(\bar{X}) = \frac{1}{12n}$$



Q_{20}: X random vec mean "μ" var "Σ"

vector "a" $\Rightarrow E(a^T X) = a^T \mu \quad V(a^T X) = a^T \Sigma a$

matrix "A" $\Rightarrow E(AX) = A\mu \quad V(AX) = A \Sigma A^T$

$$E(a^T X) = E\left(\sum_{i=1}^n a_i x_i\right) = \sum_{i=1}^n a_i E(x_i) = \sum a_i \mu_i$$

$$= a^T \mu$$

$$V(a^T x) = E\left((a^T x - a^T \mu)(a^T x - a^T \mu)^T\right)$$
$$= E\left(a^T (x-\mu)(x-\mu)^T a\right)$$
$$= a^T \Sigma a$$

$$E(Ax) = E\left(\sum_{j=1}^{n} A_{i,j} x_j\right) = \sum_{j=1}^{n} A_{i,j} \mu_j$$
$$= A\mu$$

$$V(Ax) = E\left((Ax - A\mu)(Ax - A\mu)^T\right)$$
$$= E\left(A(x-\mu)(x-\mu)^T A^T\right)$$
$$= A \Sigma A^T$$

Q21: if $E(Y|X) = x \overset{?}{\Longrightarrow} Cov(X,Y) = V(X)$

$$E(y) = EE(y|x) = E(x)$$
$$E(xy) = EE(xy|x) = E(x E(y|x)) = E(x^2)$$
$$Cov(x,y) = E(xy) - E(x)E(y) = E(x^2) - E(x)E(x)$$
$$= V(x)$$

Q_{22}: $X \sim unif(0,1)$ $\quad 0<a<b<1$

$$Y = \begin{cases} 1 & 0<x<b \\ 0 & o.w. \end{cases} \qquad Z = \begin{cases} 1 & a<x<1 \\ 0 & o.w. \end{cases}$$

a) Y & Z are independent?

b) $E(Y|Z)=?$

a) $P(Y=1) = P(x<b) = b$

$P(Z=1) = P(x>a) = 1-a$

$P(Y=1, Z=1) = P(a<x<b) = b-a$

$P(Y=1)P(Z=1) \neq P(Y=1, Z=1) \rightarrow$ not indep.

b) $E(Y|Z)=?$

if $Z=0 \rightarrow x<a \rightarrow x<b \rightarrow Y=1 \rightarrow E(Y)=1$

if $Z=1 \rightarrow x>a \rightarrow P(Y=1) = P(a<x<b \mid a<x<1)$

$$= \frac{b-a}{1-a}$$

$$\Rightarrow E(Y|Z=1) = \frac{b-a}{1-a}$$

$$\Rightarrow E(Y|Z) = \begin{cases} 1 & Z=0 \\ \frac{b-a}{1-a} & Z=1 \end{cases}$$

Q23:

moment generating function Poisson dist.

$$f(x) = e^{-\lambda} \frac{\lambda^x}{x!} \qquad x = 0, 1, \ldots, \infty$$

$$MGF = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} e^{-\lambda} \frac{\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}$$

Q23:

MGF Normal dist.

Case 1: $\mu = 0$, $\sigma^2 = 1$ $\qquad f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$

$$\Rightarrow E(e^{tx}) = \int_{-\infty}^{\infty} \frac{e^{tx}}{\sqrt{2\pi}} e^{-x^2/2} \, dx$$

$$tx - x^2/2 = -\tfrac{1}{2}(x-t)^2 + t^2/2$$

$$\Rightarrow E(e^{tx}) = e^{t^2/2} \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2}}_{N(t,1)} dx = e^{t^2/2}$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{tx} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = e^{t^2/2} \qquad (1)$$

general case $N(\mu, \sigma^2)$

$$E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} \frac{e^{\frac{-(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} dx$$

$$Z = \frac{x-\mu}{\sigma} \rightarrow x = \sigma Z + \mu \quad , \quad dx = \sigma dz$$

$$\Rightarrow$$

$$E(e^{tx}) = e^{\mu t} \underbrace{\int_{-\infty}^{\infty} \frac{e^{t\sigma Z}}{\sqrt{2\pi}} e^{-Z^2/2} dz}_{\text{from (1)} = e^{\frac{t^2\sigma^2}{2}}}$$

$$\Rightarrow E(e^{tx}) = e^{\mu t + \frac{t^2\sigma^2}{2}}$$

Q23:

MGF Gamma (α, β) dist.

take $\lambda = 1/\beta \rightarrow E(e^{tx}) = \int_0^\infty e^{tx} \frac{1}{\Gamma(\alpha)} \lambda^\alpha x^{\alpha-1} e^{-\lambda x} dx$

$$\Rightarrow E(e^{tx}) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\lambda-t)x} dx$$

note:

$$\int x^a e^{-bx} dx = \frac{\Gamma(a+1)}{b^{a+1}} \Rightarrow \int_0^\infty x^{\alpha-1} e^{-(\lambda-t)x} dx = \frac{\Gamma(\alpha)}{(\lambda-t)^\alpha}$$

$$\Rightarrow E(e^{tx}) = \frac{\lambda^\alpha}{\cancel{\Gamma(\alpha)}} \frac{\cancel{\Gamma(\alpha)}}{(\lambda-t)^\alpha} = \left(\frac{\lambda}{\lambda-t}\right)^\alpha = \left(\frac{1}{1-\beta t}\right)^\alpha$$

for the valid integral $b = \lambda - t > 0 \rightarrow t < \lambda \rightarrow t < 1/\beta$

$\Rightarrow t < 1/\beta$ valid range.

Q24: $X_1, \dots, X_n \sim Exp(\beta)$

- find MGF of X_i
- prove $\sum_{i=1}^{n} X_i \sim Gamma(n, \beta)$

$Exp(\beta) \sim \frac{1}{\beta} e^{-x/\beta} \quad x > 0$

$$E(e^{tx}) = \int_0^\infty e^{tx} \frac{e^{-x/\beta}}{\beta} dx = \frac{1}{\beta} \int_0^\infty e^{x(t - 1/\beta)} dx$$

$$= \frac{1}{\beta(t - 1/\beta)} e^{x(t - 1/\beta)} \Big)_0^\infty$$

if $t < 1/\beta \rightarrow \lim_{x \to \infty} e^{x(t - 1/\beta)} = 0$

$$\Rightarrow E(e^{tx}) = \frac{-1}{\beta t - 1} = \frac{1}{1 - \beta t}$$

using lemma 3.31:

$$Y = \sum X_i \Rightarrow \psi_y(t) = \prod_i \psi_i(t) = \left(\frac{1}{1 - \beta t}\right)^n$$

$\Rightarrow \psi_Y(t) = \text{MGF Gamma}(n, \beta)$

Using Theorem 3.33:

$$Y \overset{d}{=} \text{Gamma}(n, \beta).$$