

Chapter 1: Probability

Q1: Fill in details of proof Theorem 1.8

prove monotone decreasing case

part 1: $B_i \cap B_j = \emptyset \quad i \neq j \rightarrow \text{disjoint}$

$$A_1 \subset A_2 \subset \dots \subset A_n \Rightarrow A_n = \bigcup_{i=1}^n A_i$$

$$\bigcup_{i=1}^n B_i = \left\{ \omega \in \Omega : \omega \in A_1 \text{ } || (\omega \in A_2 \text{ } \& \omega \notin A_1) \right. \\ \left. \text{ } || \omega \in A_3 \text{ } \& \omega \notin A_2 \text{ } \& \omega \notin A_1), \dots \right\}$$

$$= \left\{ \omega \in \Omega : \omega \in A_1 \vee \omega \in A_2 \dots \vee \omega \in A_n \right\}$$

$$A_1 \subset A_2 \subset \dots \subset A_n \Rightarrow \bigcup_{i=1}^n B_i = \left\{ \omega \in \Omega : \omega \in A_n \right\} = A_n$$

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i = \lim_{n \rightarrow \infty} A_n = A$$

Part 2: monotone decreasing:

$$A_1 \supset A_2 \supset \dots \supset A_n$$

$$A = \lim_{n \rightarrow \infty} A_n = \bigcap_{i=1}^{\infty} A_i \quad \lim_{n \rightarrow \infty} P(A_n) = P(A)$$

$$P(A_n) = P\left(\bigcap_{i=1}^n A_i\right) \Rightarrow$$

$$\lim_{n \rightarrow \infty} P(A_n) = \lim_{n \rightarrow \infty} P\left(\bigcap_{i=1}^n A_i\right) = P(A)$$

Q2: prove statements in equation 1.1

1) $P(\emptyset) = 0$

using axioms 2 & 3:

$$P(\Omega) = 1 \Rightarrow P(\emptyset) = 0$$

2) $A \subset B \Rightarrow P(A) \leq P(B)$

define $C = B - A \Rightarrow C$ and A are disjoint

$$\text{Axiom 3} \Rightarrow P(B) = P(C) + P(A)$$

$$\text{Axiom 1} \Rightarrow P(C) \geq 0$$

$$\Rightarrow P(B) \geq P(A)$$

$$3) \quad 0 \leq P(A) \leq 1$$

$$\text{Axiom 1} \Rightarrow P(A) > 0$$

$$\text{Axiom 2} \Rightarrow P(\Omega) = 1$$

$$A \subset \Omega, \text{ using prev part} \Rightarrow P(A) \leq P(\Omega)$$

$$\Rightarrow P(A) \leq 1$$

$$4) \quad P(A^c) = 1 - P(A)$$

$$\text{Axiom 1: } P(\Omega) = 1 \Rightarrow P(A) + P(A^c) = 1$$

$$5) \quad A \cap B = \emptyset \Rightarrow P(A \cup B) = P(A) + P(B)$$

A and B are disjoint

$$\text{Axiom 3} \Rightarrow P(A \cup B) = P(A) + P(B)$$

$$\text{Q3: events } A_1, A_2, \dots \quad B_n = \bigcup_{i=n}^{\infty} A_i \quad C_n = \bigcap_{i=n}^{\infty} A_i$$

$$\text{a) show } B_1 \supset B_2 \supset \dots \quad C_1 \subset C_2 \subset \dots$$

$$B_{k+1} = \bigcup_{i=k+1}^{\infty} A_i \Rightarrow B_k = B_{k+1} \cup A_k \Rightarrow B_k \supset B_{k+1}$$

$$B_K = \bigcup_{i=K}^{\infty} A_i$$

$$C_{K+1} = \bigcap_{i=K+1}^{\infty} A_i$$

$$C_K = \bigcap_{i=K}^{\infty} A_i \Rightarrow C_K = C_{K+1} \cap A_K \Rightarrow C_K \subset C_{K+1}$$

b) $w \in \bigcap_{n=1}^{\infty} B_n \iff w \text{ belongs to infinite number of events } A_1, A_2, \dots$

if $w \in \bigcap_{n=1}^{\infty} B_n \Rightarrow w \in B_1 \wedge B_2 \wedge \dots \wedge B_n \wedge B_{\infty}$

$B_n = \bigcup_{i=n}^{\infty} A_i \Rightarrow$ if $w \in A_1, \dots, A_K$ i.e. finite number of A_i :

$\Rightarrow \exists j \text{ s.t. } w \notin B_j$

$\Rightarrow w \notin \bigcap_{n=1}^{\infty} B_n$ contradiction!

$\Rightarrow w \text{ belongs to infinite number of events } A_i$

if w belongs to infinite number of events A_i :

$\Rightarrow w \in B_1 \wedge B_2 \wedge \dots \wedge B_{\infty} \Rightarrow w \in \bigcap_{n=1}^{\infty} B_n$

c) $w \in \bigcup_{n=1}^{\infty} C_n \iff w \in A_1, A_2, \dots$
except finite number

$$w \in \bigcup_{n=1}^{\infty} C_n$$

$\iff \exists k \text{ where } w \in C_k$

$$\iff w \in \bigcap_{i=k}^{\infty} A_k$$

$$\iff w \in A_k, A_{k+1}, \dots, A_{\infty}$$

Q4: Show $(\bigcup_{i=1}^n A_i)^c = \bigcap_{i=1}^n A_i^c$

$$B = \bigcup_{i=1}^n A_i$$

$$B^c = \{w \in \Omega, w \notin B\}$$

$$\Rightarrow w \notin A_1 \text{, and } w \notin A_2 \dots \text{ and } w \notin A_n$$

$$\Rightarrow B^c = \bigcap_{i=1}^n A_i^c$$

Show $(\bigcap_{i=1}^n A_i)^c = \bigcup_{i=1}^n A_i^c$

$$\begin{aligned} (\bigcap_{i=1}^n A_i)^c &= \{w \in \Omega : w \notin A_1 \text{ or } w \notin A_2 \text{ or } \dots \text{ or } w \notin A_n\} \\ &= \bigcup_{i=1}^n A_i^c \end{aligned}$$

$$\text{Index} = I = \{i, j, k, \dots\}$$

define $B_j \ j=1, \dots, n$ where $n = \text{size}(\mathcal{I})$

$$= A_i : i \in \mathcal{I}$$

Q5: toss coin until get 2 heads

Sample space?

probability that K tosses are required?

$$\Omega = \{(H, H), (T, T, \dots, H, \dots, H), \dots, (\dots, H, \dots, H)\}$$

$$P(n=k) = P(\text{one } H \text{ between } k-1 \text{ tosses}) \times P(w_k = H)$$

$$P(w_k = H) = 0.5$$

$$P(H, \underbrace{T, T, \dots, T}_{k-2}) = 0.5 (0.5)^{k-2}$$

$$\Rightarrow P(\text{one } H \text{ in } k-1 \text{ tosses}) = (k-1) \times 0.5^{k-1}$$

$$\begin{aligned} \Rightarrow P(n=k) &= (k-1) 0.5^{k-1} \times 0.5 \\ &= (k-1) 0.5^k \end{aligned}$$

Q6: $\Omega = \{0, 1, \dots\}$

prove does not exist a unif. dist. on Ω

let $A_0 = \{0\}$, $A_1 = \{1\}$, ...

if uniform dist. $\Rightarrow P(A_i) = c \quad i = 0, 1, \dots, \infty$

$\Rightarrow P(\Omega) = c \times \infty = \infty \neq 1 \Rightarrow \text{contradiction}$

Q7: events A_1, A_2, \dots

show $P\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P(A_n)$

hints: define $B_n = A_n - \bigcup_{i=1}^{n-1} A_i$

then show $\{B_n\}$ are disjoint

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$$

$$B_n = A_n - \bigcup_{i=1}^{n-1} A_i = \{w \in \Omega : w \in A_n, w \notin A_1, \dots, A_{n-1}\}$$

$$B_i \cap B_j = \underbrace{(A_i - \bigcup_{k=1}^{i-1} A_k)}_{\substack{w \in A_i \\ \notin A_1, \dots, A_{i-1}}} \cap \underbrace{(A_j - \bigcup_{k=1}^{j-1} A_k)}_{\substack{w \in A_j \\ \notin A_1, \dots, A_{j-1}}}$$

$$\Rightarrow B_i \cap B_j = \emptyset$$

$\Rightarrow B_n$ are disjoint

$$\left\{ \begin{array}{l} B_1 = A_1 \\ B_2 = A_2 - A_1, \quad B_3 = A_3 - (A_1 \cup A_2) \\ B_1 \cup B_2 = A_1 \cup A_2 \\ B_1 \cup B_2 \cup B_3 = B_1 \cup B_2 \cup [A_3 - (B_1 \cup B_2)] \end{array} \right.$$

proved for $k=1, 2, 3$

assume it is true for k , prove for $k+1$

$$\text{assume } \bigcup_{i=1}^k B_i = \bigcup_{i=1}^k A_i$$

$$\begin{aligned} \Rightarrow \bigcup_{i=1}^{k+1} B_i &= \left(\bigcup_{i=1}^k B_i \right) \cup B_{k+1} \\ &= \left(\bigcup_{i=1}^k A_i \right) \cup (A_{k+1} - \bigcup_{i=1}^k A_i) \\ &= \bigcup_{i=1}^{k+1} A_i \end{aligned}$$

\Rightarrow proved for case $k+1 \Rightarrow \bigcup_{i=1}^k B_i = \bigcup_{i=1}^k A_i$

$$\Rightarrow P\left(\bigcup_{n=1}^{\infty} A_n\right) = P\left(\bigcup_{n=1}^{\infty} B_n\right)$$

$$\begin{aligned} B_i \text{ disjoint} &\Rightarrow P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(B_n) \\ B_n = A_n - \bigcup_{i=1}^{n-1} A_i &\Rightarrow P(B_n) \leq P(A_n) \end{aligned} \quad \Rightarrow$$

$$\Rightarrow P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(B_n) \leq \sum_{n=1}^{\infty} P(A_n)$$

Q8: if $P(A_i) = 1$ $\forall i$ prove $P(\bigcap_{i=1}^{\infty} A_i) = 1$

$$\left. \begin{array}{l} A_i \subset \Omega \\ |A_i| = |\Omega| \end{array} \right\} \Rightarrow A_i = \Omega$$

$$\Rightarrow P(\bigcap_{i=1}^{\infty} A_i) = P(\Omega) = 1$$

Q9: $P(B > 0)$

Show $P(\cdot | B)$ satisfies prob. axioms

Axiom 1: $P(\cdot | B) = \frac{P(\cdot B)}{P(B)} \geq 0$

Axiom 2: $P(\Omega | B) = \frac{P(\Omega B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

Axiom 3: if A_1, A_2, \dots disjoint $\Rightarrow P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

$$P(\bigcup_i A_i | B) \stackrel{?}{=} \sum_i P(A_i | B)$$

$$P\left(\bigcup_i A_i \mid B\right) = P\left[\left(\bigcup_i A_i\right) \cap B\right] / P(B)$$

$$= P\left[\bigcup_i (A_i \cap B)\right] / P(B)$$

$$\stackrel{A_i \text{ disjoint}}{=} \sum_{i=1}^{\infty} P(A_i \cap B) / P(B)$$

$$= \sum_{i=1}^{\infty} P(A_i \mid B)$$

Q10: Monty Hall problem

you pick door 1 \rightarrow monty shows 2 or 3 is empty.

prove that you should switch after Monty

$$\Omega = \{(w_1, w_2) : w_i \in \{1, 2, 3\}\}$$

w_1 prize
 w_2 Monty

$$\Omega = \{(1, 2), (1, 3), (2, 3), (3, 2)\}$$

assumption: we always choose door 1 first

if $\omega_1 = 1 \Rightarrow$ Monty chooses 2 or 3 with same prob

$$\Rightarrow P(1,2) = P(1,3)$$

if $\omega_1 = 2$ or $\omega_1 = 3 \Rightarrow$ Monty chooses other door

$$\Rightarrow P(3,2) = P(2,3)$$

lets compare $P(1,2)$ and $P(3,2)$

if $P(3,2) > P(1,2) \Rightarrow$ we should switch

$$P(3,2) = P(\omega_1 = 3 \mid \omega_2 = 2) = \frac{P(\omega_2 = 2 \mid \omega_1 = 3) P(\omega_1 = 3)}{P(\omega_2 = 2)}$$

$$= \frac{1 \times \frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

$$P(1,2) = P(\omega_1 = 1 \mid \omega_2 = 2) = \frac{P(\omega_2 = 2 \mid \omega_1 = 1) P(\omega_1 = 1)}{P(\omega_2 = 2)}$$

$$= \frac{\frac{1}{2} \times \frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}$$

$$\Rightarrow \text{if } w_2 = 2 \quad \begin{cases} P(w_1=3 | w_2=2) = \frac{2}{3} \\ P(w_1=1 | w_2=2) = \frac{1}{3} \end{cases}$$

$$\text{if } w_2 = 3 \quad \begin{cases} P(w_1=2 | w_2=3) = \frac{2}{3} \\ P(w_1=1 | w_2=3) = \frac{1}{3} \end{cases}$$

\Rightarrow better to switch

Q11: Show if $A \& B$ indep. $\rightarrow A^c \& B^c$ are indep.

$$P(A^c \cup B^c) = P(A^c) + P(B^c) - P(A^c \cap B^c)$$

also using Venn diagram one can prove that:

$$P(A^c \cup B^c) = 1 - P(A \cap B) = 1 - P(A) \cdot P(B)$$

(since $A \& B$ indep)

$$\Rightarrow P(A^c) + P(B^c) - P(A^c \cap B^c) = 1 - P(A)P(B)$$

$$\begin{aligned} \Rightarrow P(A^c \cap B^c) &= 1 - P(A) + 1 - P(B) - 1 + P(A)P(B) \\ &= (1 - P(A))(1 - P(B)) \\ &= P(A^c)P(B^c) \end{aligned}$$

$\Rightarrow A^c \& B^c$ are independent.

Q₁₂: 3 Cards : red-red , green-green, red-green

- chooses a card at random

see one side at random

$$P(\underbrace{\text{second side} = \text{green}}_{\omega_2} \mid \underbrace{\text{first side green}}_{\omega_1}) = ?$$

$$\Omega = \{(\omega_1, \omega_2) : \omega_1, \omega_2 \in \{r, g\}\}$$

$$\Omega = \{(r, r), (r, r), (g, g), (g, g), (r, g), (g, r)\}$$

$$P(\omega_2 = g \mid \omega_1 = g) = \frac{P(\omega_2 = g, \omega_1 = g)}{P(\omega_1 = g)} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

Q₁₃: toss fair coin until one Head one Tail

a) sample space?

b) probability that 3 tosses required?

$$a) \Omega = \{(\omega_1, \dots, \omega_n) : \omega_i \in \{H, T\}, n = 2, 3, \dots, \infty\}$$

$$\omega_n \neq \omega_i \quad i = 1, 2, \dots, n-1$$

$$b) P(n=K) = P\left(\omega_{i=1, \dots, K-1} = H, \omega_K = T\right) + P\left(\omega_{i=1, \dots, K-1} = T, \omega_K = H\right)$$

$$= 2 \times \frac{1}{2} \times \left(\frac{1}{2}\right)^{K-1} = \frac{1}{2^{K-1}}$$

$$\Rightarrow P(n=3) = \frac{1}{4}$$

Q14: show if $P(A)=0$ or $1 \rightarrow A$ indep of all events

show if A indep. of itself $\rightarrow P(A)=0$ or 1

- if $P(A)=0$ or $1 \Rightarrow$

assume general event B

$$P(A \cap B) = \begin{cases} P(B) & \text{if } P(A)=1 \\ 0 & \text{if } P(A)=0 \end{cases}$$

$$\Rightarrow P(A \cap B) = P(A) P(B) \Rightarrow A \text{ & } B \text{ indep.}$$

- A indep of $A \Rightarrow P(A) = P(A \cap A) = P(A) \cdot P(A)$

$$\Rightarrow P(A) = P(A)^2 \Rightarrow P(A) = 1 \text{ or } 0$$

Q₁₅: prob blue eye = 1/4

family with 3 children

assume indep between children

a) if at least one child has blue eye

prob of at least 2 child blue eyes?

b) if youngest has blue eyes

prob of at least 2 child blue eyes?

$$\Omega = \left\{ (\omega_1, \omega_2, \omega_3) : \omega_i \in \{1, 0\} \right\}$$

1: blue
0: not blue

$$n = \omega_1 + \omega_2 + \omega_3, \quad 0 \leq n \leq 3$$

$$P(\omega_i = 1) = 1/4 = p$$

$$P(\omega_i = 0) = 3/4 = q$$

$$a) P(n \geq 2 | n \geq 1) = \frac{P(n \geq 2)}{P(n \geq 1)} = \frac{1 - P(n=0) - P(n=1)}{1 - P(n=0)}$$

$$P(n=0) = q^3 = 27/64$$

$$P(n=1) = 3pq^2 = 27/64$$

$$\Rightarrow P(n \geq 2 | n \geq 1) = \frac{1 - \frac{27}{64} - \frac{27}{64}}{1 - \frac{27}{64}} = \frac{10}{37}$$

b) $P(n \geq 2 | \omega_1=1) = \frac{P(\omega_1=1, \omega_2+\omega_3 \geq 1)}{P(\omega_1=1)}$

$$= P(\omega_2 + \omega_3 \geq 1) = 1 - P(\omega_2 + \omega_3 = 0)$$

$$= 1 - q^2 = 7/16$$

Q16: prove if $A \text{ } \& \text{ } B \text{ indep} \rightarrow P(A|B) = P(A)$

prove $\forall A, B: P(AB) = P(A|B)P(B) = P(B|A)P(A)$

- if $A \text{ } \& \text{ } B \text{ indep}$

$$\Rightarrow P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

$$P(A|B) = \frac{P(AB)}{P(B)} \Rightarrow P(AB) = P(A|B)P(B)$$

$$P(B|A) = \frac{P(AB)}{P(A)} \Rightarrow P(AB) = P(B|A)P(A)$$

Q17: Show $P(ABC) = P(A|BC)P(B|C)P(C)$

$$P(ABC) = P(A|BC)P(BC)$$

$$P(BC) = P(B|C)P(C)$$

$$\Rightarrow P(ABC) = P(A|BC)P(B|C)P(C)$$

Q18: A_i are disjoint, $\bigcup_{i=1}^k A_i = \Omega$

prove if $P(A_1|B) < P(A_1)$ $P(B) > 0$

$\Rightarrow P(A_i|B) > P(A_i)$ for some $i = 2, \dots, k$

$$\sum_{i=1}^k P(A_i) = \sum_{i=1}^k P(A_i|B) = 1$$

$$\Rightarrow P(A_1) + \sum_{i=2}^k P(A_i) = P(A_1|B) + \sum_{i=2}^k P(A_i|B)$$

$$P(A_1|B) < P(A_1) \Rightarrow \sum_{i=2}^k P(A_i|B) > \sum_{i=2}^k P(A_i)$$

$$\Rightarrow \exists i \in \{2, \dots, k\} \text{ s.t. } P(A_i|B) > P(A_i)$$

Q19: $P(\text{Mac}) = 0.3 \quad P(\text{Win}) = 0.5 \quad P(\text{Lin}) = 0.2$

$$P(V|\text{Mac}) = 0.65 \quad P(V|\text{Win}) = 0.82 \quad P(V|\text{Lin}) = 0.5$$

$$P(\text{Win}|V) = ?$$

$$P(\text{Win}|V) = \frac{P(V|\text{Win}) P(\text{Win})}{P(V)} = \frac{0.82 \times 0.5}{\sum_x P(V|x) P(x)}$$

$$x \in \{\text{Mac}, \text{Win}, \text{Lin}\} = \frac{0.82 \times 0.5}{0.65 \times 0.3 + 0.82 \times 0.5 + 0.5 \times 0.2} = 0.58$$

Q20: p_1, \dots, p_5 prob of head for each coin

$$p_1 = 0 \quad p_2 = \frac{1}{4} \quad p_3 = \frac{1}{2} \quad p_4 = \frac{3}{4} \quad p_5 = 1$$

C_i event that coin i is selected

$$a) P(C_i | H) = ? \quad i=1, \dots, 5$$

$$b) P(H_2 | H_1) = ? \quad H_j = \text{heads on toss } j$$

$$c) P(C_i | B_4) = ? \quad B_4 = \text{first head on toss 4}$$

$$P(H | C_i) = P_i \quad , \quad \sum_i P_i = \frac{5}{2} \quad , \quad P(C_i) = \frac{1}{5}$$

$$a) P(C_i | H) = \frac{P(H | C_i) P(C_i)}{P(H)} = \frac{P_i P(\cancel{C_i})}{\sum P_i P(\cancel{C_i})} = \frac{P_i}{\sum P_i}$$

$$= \frac{2}{5} P_i$$

$$b) P(H_2 | H_1) = \frac{P(H_2 \cap H_1)}{P(H_1)} = \frac{\sum P_i^2 P(\cancel{C_i})}{\sum P_i P(\cancel{C_i})}$$

$$= \frac{2}{5} \sum_i P_i^2$$

$$c) P(C_i | B_4) = \frac{P(B_4 | C_i) P(\cancel{C_i})}{\sum P(B_4 | C_i) P(\cancel{C_i})}$$

$$P(B_4 | C_i) = (1 - P_i)^3 P_i$$

$$\Rightarrow P(C_i | B_4) = \frac{(1-p_i)^3 p_i}{\sum_j (1-p_j)^3 p(j)}$$

Q21: Computer experiment

find solution in Jupyter notebook.

Q22: Computer experiment

find solution in Jupyter notebook.

Q23: Computer experiment

find solution in Jupyter notebook.