Aspects and Applications of Chern-Simons Theories

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Abstract

Chern-Simons theories are topological quantum field theories that have many applications in physics, ranging from gravity and string theory to condensed matter physics and mathematics. We study Chern-Simons theories in three dimensions. We present a brief review of many interesting properties of the theory, its differential geometry setting, subtleties in gauge invariance, the presence of anyonic excitations, its ability to calculate topological invariants. We then quantize the theory using the framework of geometric quantization. We take two different approaches to the quantization of the theory, obtaining the classical phase space first and then quantizing using the geometric quantization and quantizing the spacetime manifold first and then constraining the Hilbert space so obtained using the equations of motion. We discuss the Hilbert space of the pure Chern-Simons theory on a sphere, disc, and torus. We also illustrate how the partition function of the CS theory on two different manifolds can be glued to obtained the partition function on the glued manifold. The relationship between Chern-Simons theories and Wess-Zumino-Witten models is also discussed. We motivate the need for CS matter theories as theories with dynamical degrees of freedom. We discuss several aspects of these theories. In particular, we compute the beta function at two-loop order of CS theory coupled to real scalar fields in the fundamental representation. We also calculate thermal free energy of the Chern-Simons theory coupled to complex scalars and Dirac fermions in the fundamental representation. Apart from these we discuss the conjectured dualities among Chern-Simons matter theories and discuss its applications in condensed matter physics. The applications that we discuss are in describing the spectrum of non-Abelian anyons in a harmonic trap and as effective field theories of fractional quantum Hall effect.

Acknowledgements

This dissertation is the culmination of five years of dreaming of being a physicist. I do not claim that I have become one (or will ever become one), but I am closer (in some sense) to being one than I ever was. As a freshman at NISER, inspired by biographies of the physicists of the particle physics revolution, I wanted to do string theory (even though I had no idea what it entailed, I still don't). But during the first four years, I worked on problems in condensed matter physics and quantum information theory. With experience in these fields, working on something closely related to the dissertation was natural. But the romance of string theory hasn't waned yet. So I asked Prof. Shamik Banerjee for a dissertation problem. He was kind enough to suggest a topic to me that has the flavour of the kind of physics string theory is and has tremendous applications in condensed matter physics and quantum information theory. For this, I would like to extend my utmost gratitude to him. I also want to thank him for being extremely generous with his time and for being very kind during our meetings, which were always very productive. Above all, I am very grateful for the freedom he entrusted his students with to work on our projects independently.

I always knew research in physics would entail more than solving for trajectories of particles under the influence of fancy fields. But what I didn't realize was that there would not even be a particle to talk about. Initially, I struggled with abstract ideas and concepts even though I had anticipated and prepared for them. I reached the summit of confusion when I learned that 'particles are irreducible representations of the Lorentz group'. My first impression was that this was some sort of a joke on the use of mathematics in physics (it was not!). Now I am more at peace with all this, primarily because of the courses by Prof. Chethan Gowdigere, Prof. Yogesh Srivastava, Prof. Sayantani Bhattacharya, and Prof. Shamik Banerjee. Through these courses, I developed some perspective on abstract ideas in theoretical physics and learned to do long, tedious computations that I absolutely despised early on.

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Contents

1

Introduction

We are going to discuss the most famous topological quantum field theory in three dimensions, the Chern-Simons theory (CST). In 1988, Sir Michael Atiyah asked physicists to come up with a three-dimensional interpretation for the Jones polynomial of knots. The Jones polynomial is a topological invariant of a knot. Although knots live in three dimensions, their Jones polynomial is computed by looking at a two-dimensional projection of the knot. In July 1988, Edward Witten constructed a three-dimensional quantum field theory (the CST) where the vacuum expectation value of a Wilson loop along a knot is the Jones polynomial of that knot. CST is a topological quantum field theory, so any correlator in this theory remains invariant under a homeomorphism, and hence, they are topological invariants of the manifold $\mathcal{M}$ on which the theory lives. If we can compute these correlators, we have a factory that produces topological invariants of $\mathcal{M}$ and knots embedded in it. However, CST is an interacting quantum field theory, and the usual perturbative method for computing correlators will not result in topological invariants. Witten, rather remarkably, solved the CST on an arbitrary manifold $\mathcal{M}$. This led to a lot of activity in trying to understand CST better and its applications in various areas of both mathematics and physics.

The Chern-Simons action is,

$$S_{CS}[A] = \frac{k}{4\pi} \int_{\mathcal{M}} \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \tag{1.1}$$

where, A is the (differential one-form) gauge field. The equations of motion are,

$$F = dA + A \wedge A = 0. \tag{1.2}$$

In the $A_0 = 0$ gauge, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0$, implies,

$$\partial_t A_i = 0, \tag{1.3}$$

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j] = 0,\ i \in \{1, 2\}. \tag{1.4}$$

So the gauge fields are time-independent and satisfy a *Gauss law* constraint. The CS action (3.4) in this gauge,

$$S_{CS} = \frac{k}{4\pi} \int dt \int_\Sigma \varepsilon^{ij} \mathrm{Tr} \left(A_i \frac{\partial}{\partial t} A_j \right). \tag{1.5}$$

This is a simple quadratic action of the form, $S = \int p \partial_t q$. The gauge field A_i is canonically conjugate to itself which means the Hamiltonian of the CS theory vanishes!

$$H_{CS} = 0 \tag{1.6}$$

As the Hamiltonian of the theory is zero, the Heisenberg equations of motion tell as that the time evolution of any operator $\mathcal{O}$ is zero.

$$\frac{\partial \mathcal{O}}{\partial t} = 0 \tag{1.7}$$

This means the theory has no dynamical degrees of freedom. Everything in a CS theory is static with time. Even though there are topological excitations in the theory they do not have any dynamics. Such gluon-less gauge theory is boring for physical purposes. Even though CS theory has many applications in computing topological invariants, it has limited applications in describing real physics systems. The vanishing of the Hamiltonian is typical of a topological quantum field theory. To ensure dynamics we need to break topological invariance of the CS theory.

Recently, CS matter theories have found great popularity. These are quantum field theories in three dimensions where the CS field is coupled to matter fields in various representations of the CS gauge group. The topological invariance is immediately broken once we add matter fields. These CS matter theories are very interesting, and they are capable of modeling various physical phenomena as well. In particular, these theories are interesting because they have fixed points along their renormalization group flows which allows one to construct conformal field theories in three dimensions, they form the dual of higher spin theories of gravity via the AdS/CFT correspondence, their excitations are non-Abelian anyons which are relevant for topological quantum computation, they form effective field theories of fractional quantum Hall effect and they admit level-rank like dualities among themselves.

In this thesis we study various aspects and applications of both pure Chern-Simons theories and Chern-Simons matter theories. The thesis is organized as follows:

1. In chapter 2, we discuss the framework of geometric quantization which we will use to quantize pure Chern-Simons theory. In this chapter, we discuss the basics of differential geometry and introduce classical mechanics in the language of differential geometry. We also provide two explicit examples of geometric quantization. This chapter is adapted from [Nai05] and [BM94].

2. In chapter 3, we introduce the Chern-Simons action and its differential geometry setting. We discuss various aspects of the action and quantize the theory on different manifolds. We also state the CS/WZW correspondence. This chapter is adapted from various sources [Dun98], [EMSS89], [Wit89] and [BM94].

3. In chapter 4, we study a Chern-Simons matter theory. In particular, we compute the beta function of a Chern-Simons theory coupled to scalars in the fundamental representation of the $O(N)$ gauge group. We also review the ideas behind the renormalization group and large N expansion of quantum field theories. This review of renormalization group is adapted from [PS90] and of large N expansion is adapted from [Ton]. While the computation of the beta function was done in [AGAY12].

4. In chapter 5, we compute the thermal free energy at large N of Chern-Simons matter theories coupled to scalars and fermions in the fundamental representation. This chapter is adapted from [AGGA$^+$13].

5. In chapter 6, we discuss the dualities that Chern-Simons matter theories admit, and a couple of applications in computing the spectrum of non-Abelian anyons and as an effective field theory of fractional quantum Hall effect. The dualities are conjectured in [Aha16], the conformal spectrum of non-Abelian anyons and a model for effective field theory of FQHE are given in [DTT18] and [TT15], respectively.

6. Finally in chapter 7, we conclude with the summary of what we have learned and some future directions.

2

Geometric Quantization

2.1 Differential Geometry

In this section we introduce basic ideas of differential geometry. The language developed in this section will be used extensively in the quantization of the Chern-Simons theory.

1. *Topological manifold.* An n-dimensional topological manifold is a topological space such that the neighborhood of every point on the manifold is homeomorphic to an open disc in $\mathcal{R}^n$.

   A *chart* is a mapping from the open set in the manifold to an n-dimensional open disc. On open sets that have a non-empty intersection we have two choices of coordinates, *transition* functions take us from one coordinate to another.

2. *Vector fields.* Consider curves on the manifold, $U_t : t \in [0,1] \longrightarrow \mathcal{M}$. The points on the curve are $U_t(p)$ with $U_0(p) = p$, as the initial point. Consider $\mathcal{F}(\mathcal{M})$, the set of all differentiable functions on $\mathcal{M}$. If we choose $f \in \mathcal{F}$, we can define a vector X associated to the curve $U_t(p)$ at the point p as,

   $$(X \cdot f)(p) = \frac{d}{dt} f(U_t(p))|_{t=0}. \tag{2.1}$$

   X is the tangent vector at point p. If we consider all the curves starting at p, we get a vector space called the tangent vector space at p. A vector field is obtained by choosing an element of the tangent space for all $p \in \mathcal{M}$.

3. *Differential one-forms.* The differential one-form is obtained by choosing an element of the dual vector space of the tangent space at each point $p \in \mathcal{M}$. The basis of the dual vector space dx^i in local coordinates is defined by,
$$\left(\frac{\partial}{\partial x^i}, dx^j\right) = \delta_i^j. \tag{2.2}$$
The components of a differential one-form are contravariant tensors, $\omega = \omega_i dx^i$.

4. *Differential k-forms.* A differential k-form is obtained by choosing an element from $\wedge_k(T_p^*\mathcal{M})$, this is the antisymmetric product of the tangent vector space at the point p, for each point $p \in \mathcal{M}$. In local coordinates it is given by,
$$\omega = \frac{1}{k!}\omega_{i_1 i_2 \cdots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}. \tag{2.3}$$

5. *Pullback of a k-form.* Consider a map φ from a manifold $\mathcal{M}$ to a manifold $\mathcal{N}$. Let ω be a differential k-form defined on $\mathcal{N}$,
$$\omega = \frac{1}{k!}\omega_{i_1 i_2 \cdots i_k}(x) dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}. \tag{2.4}$$
Then the pullback of ω on $\mathcal{M}$ is given by,
$$\varphi^*\omega = \frac{1}{k!}\,\omega_{i_1 i_2 \cdots i_k}\Big(\varphi(y)\Big)\, d\varphi(y)^{i_1} \wedge d\varphi(y)^{i_2} \wedge \cdots \wedge d\varphi(y)^{i_k}, \tag{2.5}$$
$$= \frac{1}{k!}\,\omega_{i_1 i_2 \cdots i_k}\Big(\varphi(y)\Big)\left(\frac{\partial \varphi^{i_1}}{\partial y^{j_1}}\right)\cdots\left(\frac{\partial \varphi^{i_k}}{\partial y^{j_k}}\right) dy^{j_1} \wedge \cdots \wedge dy^{j_k}. \tag{2.6}$$

6. *Exterior derivative.* This is an analog of differentiation of differential forms. This takes k-forms to $k+1$ forms. It is denoted by d. In local coordinates we have,
$$d\omega = \frac{1}{k!}\frac{\partial \omega_{i_1 i_2 \cdots i_k}}{\partial x^j}\, dx^j \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}, \tag{2.7}$$
$$= \frac{1}{(k+1)!}\omega_{i_1 i_2 \cdots i_k i_{k+1}}\, dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} \wedge dx^{i_{k+1}}, \tag{2.8}$$
$$\omega_{i_1 i_2 \cdots i_k i_{k+1}} = \frac{\partial}{\partial x^{i_1}}\omega_{i_2 i_3 \cdots i_{k+1}} - \frac{\partial}{\partial x^{i_2}}\omega_{i_1 i_3 \cdots i_{k+1}} + \cdots + (-1)^k \frac{\partial}{\partial x^{i_{k+1}}}\omega_{i_1 i_2 \cdots i_k}. \tag{2.9}$$

7. *Exact and closed forms.* If the exterior derivative of a differential form

vanishes, it is called closed, ω is closed if $d\omega = 0$. While if a differential k-form can be written as an exterior derivative of a differential $k-1$ form, then it is exact. So ω is exact if $\omega = d\alpha$ for some differential form α.

8. *Lie derivative.* The Lie derivative of a function f on $\mathcal{M}$ with respect to a vector field ξ is the change in the function due to a coordinate transformation generated by the vector field. The Lie derivative L_ξ is given by,

$$L_\xi = i_\xi d + d i_\xi, \tag{2.10}$$

where d is the exterior derivative and,

$$i_\xi \omega = \xi^i \omega_i, \tag{2.11}$$

is the interior contraction operator.

2.2 Classical Mechanics

In this section we discuss various aspects of classical mechanics in the language of differential geometry.

1. The classical phase space is a smooth even-dimensional manifold $\mathcal{M}$ with a symplectic structure Ω. Ω is a differential two-form on $\mathcal{M}$ which is closed and non-degenerate. In local coordinates we have,

$$\Omega = \frac{1}{2}\Omega_{\mu\nu} dq^\mu \wedge dq^\nu. \tag{2.12}$$

2. The closeness of Ω allows us to write at least locally,

$$\Omega = d\mathcal{A} \tag{2.13}$$

here $\mathcal{A}$ is called the symplectic potential. Since, $\mathcal{A}$ and $\mathcal{A} + d\Lambda$ would lead to the same symplectic form, there is an ambiguity in the definition of $\mathcal{A}$. This ambiguity results in the freedom of canonical transformations.

3. Canonical transformations are diffeomorphisms of the classical phase space that preserve the symplectic form. Infinitesimally, canonical transformations

are generated by vector fields ξ such that $L_\xi\Omega = 0$. This implies,

$$d(i_\xi\Omega) = 0. \tag{2.14}$$

4. As $i_\xi\Omega$ is closed, we have,

$$i_\xi\Omega = -df, \tag{2.15}$$

for some function f on $\mathcal{M}$. So to every infinitesimal canonical transformation we can associate a function on the classical phase space.

5. Consider vector fields ξ and η along which the Lie derivative of the symplectic form vanishes. Consider also functions f and g on the phase space which correspond to the infinitesimal transformations generated by these vector fields. Then the Lie bracket of the vector fields defined as,

$$[\xi, \eta]^{mu} = \xi^\nu\partial_\nu\eta^\mu - \eta^\nu\partial_\nu\xi^\mu, \tag{2.16}$$

also generates an infinitesimal transformation and the function corresponding to this infinitesimal transformation is the Poisson bracket of f and g.

6. We have,

$$-d\{g, f\} = i_{[\xi,\eta]}\Omega, \tag{2.17}$$

where,

$$\{f, g\} = \Omega^{\mu\nu}\partial_\mu f\partial_\nu g, \tag{2.18}$$

is the Poisson bracket of the functions on the classical phase space. The Poisson bracket is anti-symmetric and satisfies the Jacobi identity.

7. The symplectic form also defines a volume form on the phase space,

$$d\sigma(\mathcal{M}) = c\frac{\Omega \wedge \Omega \wedge \cdots \wedge \Omega}{(2\pi)^n}, \tag{2.19}$$

we take an n-fold product of Ω for a $2n$-dimensional phase space.

8. *Darboux's theorem.* In the neighborhood of a piont on the phase space it is possible to choose coordinates p_i and x_i such that the symplectic two-form is,

$$\Omega = dp_i \wedge dx_i. \tag{2.20}$$

2.3 The Goal

The goal of geometric quantization is as follows:

1. We want a mapping from the classical phase space to a Hilbert space. Constructing a Hilbert space involves stating the vectors and defining their inner products.

2. This mapping between the classical phase space and the Hilbert space is actually a mapping between canonical transformations on the phase space and unitary transformations on the Hilbert space.

3. Since functions on phase space generate canonical transformations they must correspond to Hermitian operators on the Hilbert space which generate unitary transformations.

4. We need to ensure that the representation of unitary transformations on the Hilbert space is irreducible otherwise there would be redundant states in the Hilbert space.

5. On the classical phase space we have the Poisson bracket algebra for the functions on phase space. This algebra is replaced by the commutation relation algebra for Hermitian matrices on the Hilbert space. We want this commutation relation algebra to have an irreducible representation on the Hilbert space.

2.4 The Recipe

Now that we understand what we hope to achieve using geometric quantization following is a recipe to achieve the goal.

1. The prequantum Hilbert space is the sections of the complex line bundle on the classical phase space with symplectic two-form Ω. Under canonical transformations of the form $\mathcal{A} \longrightarrow \mathcal{A}' = \mathcal{A} + d\Lambda$, where $\Omega = d\mathcal{A}$, we have,

$$\Psi \longrightarrow \Psi' = e^{i\Lambda}\Psi. \tag{2.21}$$

2. The prequantum inner product is defined as,

$$\langle 1|2\rangle = \int d\sigma(\mathcal{M})\Psi_1^*\Psi_2 \tag{2.22}$$

where $d\sigma(\mathcal{M})$ is the Liouville measure on the classical phase space $\mathcal{M}$ defined by the symplectic two-form.

3. We can define prequantum operators corresponding to functions on the classical phase space. A function f generates a canonical transformation with $\Lambda = i_\xi\mathcal{A} - f$. The prequantum operator corresponding to f is given by,

$$\mathcal{P}(f) = -i\xi \cdot \mathcal{D} + f, \tag{2.23}$$

where $\mathcal{D}_\mu = \partial_\mu - i\mathcal{A}_\mu$.

4. It is easy to check that the prequantum operators form a representation of the Poisson bracket algebra of the functions on phase space.

$$[\mathcal{P}(f), \mathcal{P}(g)] = i\mathcal{P}(\{f, g\}). \tag{2.24}$$

5. We have called everything 'prequantum' because the wavefunction that we have defined depends on all the phase space variables. The representation of the Poisson bracket algebra is reducible in such cases.

6. To ensure that the representation is irreducible, we need to eliminate the dependence of the wavefunctions on half the phase space variables. We do this by choosing a polarization direction and then projecting the wavefunctions along these directions.

7. To implement the polarization condition we choose vector fields P_i^μ so that,

$$\Omega_{\mu\nu}P_i^\mu P_j^\nu = 0. \tag{2.25}$$

And we impose,

$$P_i^\mu \mathcal{D}_\mu \Psi = 0. \tag{2.26}$$

8. Once we have imposed the polarization condition we can only define the inner product in a natural way if the classical phase space was Kahler. In this case we can use local complex coordinates a, $\bar{a}$, and write,

$$\Omega = \Omega_{a\bar{a}}dx^a \wedge dx^{\bar{a}}. \tag{2.27}$$

The covariant derivatives are,

$$\mathcal{D}_a = \partial_a - i\mathcal{A}_a, \tag{2.28}$$

$$\mathcal{D}_{\bar{a}} = \partial_{\bar{a}} - i\mathcal{A}_{\bar{a}}. \tag{2.29}$$

$$\tag{2.30}$$

For Kähler manifolds, we have a Kähler potential K which can be used to write the symplectic potential as follows,

$$\mathcal{A}_a = -\frac{i}{2}\partial_a K, \tag{2.31}$$

$$\mathcal{A}_{\bar{a}} = \frac{i}{2}\partial_{\bar{a}} K. \tag{2.32}$$

The holomorphic polarization is imposed as,

$$\mathcal{D}_{\bar{a}}\Psi = \left(\partial_{\bar{a}} + \frac{1}{2}\partial_{\bar{a}}K\right)\Psi = 0 \tag{2.33}$$

whose solution is of the form,

$$\Psi = e^{\left(-\frac{1}{2}K\right)}F \tag{2.34}$$

where F is some holomorphic function. The inner product is,

$$\langle 1|2\rangle = \int d\sigma(\mathcal{M})e^{-K}F_1^*F_2. \tag{2.35}$$

2.5 Topological Considerations

In this section we discuss the effects on quantization of the topological features of the classical phase space. In particular we discuss the effect of non-trivial first and second cohomology groups on quantization.

1. *Non-trivial first cohomology group.* A non-trivial first cohomology group implies there are closed loops that are not contractible to a point. Consider the action for a path C from a point a to b, parametrized as $q^\mu(t)$,

$$S = \int dt\left(\mathcal{A}_\mu\frac{dq^\mu}{dt} - H\right) + \int_a^b A_\mu dq^\mu. \tag{2.36}$$

If the first cohomology group had been trivial then changing the path slightly from C to C' would not change the integral of A.

$$\int_C A - \int_{C'} A = \oint_{C-C'} A \tag{2.37}$$

$$= \int_\Sigma dA \tag{2.38}$$

$$= 0 \tag{2.39}$$

However, in the case of non-trivial cohomology this is not going to be the case. Consider an example where the first cohomology group is the group is the group of integers. This corresponds to the case where there is only one topologically distinct noncontractible loop apart from multiple traversals of the same. In this case, $A = \theta\alpha$ where θ is a constant and α is normalized to unity along the noncontractible loop for going around once. Then the integral of A along closed paths is of the form θn. This has the effect of addition of a θ-term to the action.

2. *Non-trivial second cohomology group.* A non-trivial second cohomology implies there are closed two-surfaces that are not the boundary of any three-dimensional region. If the symplectic form is a non-trivial element of the second cohomology group, we do not have a globally defined symplectic potential. We need to represent the symplectic potential with different functions in different coordinate patches. In the overlap regions we will have transition functions relating different representations of the symplectic potential. Consider the example of a closed noncontractible two-sphere. We can cover it with two coordinate patches corresponding to the two hemispheres, we denote them by N and S and the symplectic potentials on them by $\mathcal{A}_N$ and $\mathcal{A}_S$. On the equator we have,

$$\mathcal{A}_N = \mathcal{A}_S + d\Lambda \tag{2.40}$$

where Λ is a function on the overlap region which gives the canonical transformation between the two $\mathcal{A}$'s. Similarly the wavefunctions on the equator must be related by canonical transformations,

$$\Psi_N = e^{i\Lambda}\Psi_S. \tag{2.41}$$

Lets consider the integral of $d\Lambda$ over the equator,

$$\oint_E d\Lambda = \Delta\Lambda = \int_E \mathcal{A}_N - \int_E \mathcal{A}_S \tag{2.42}$$

$$= \int_{\partial N} \mathcal{A}_N - \int_{\partial S} \mathcal{A}_S \tag{2.43}$$

$$= \int_N \Omega - \int_S \Omega \tag{2.44}$$

$$= \int_\Sigma \Omega \tag{2.45}$$

If $\Delta\Lambda$ is not zero or one, the wavefunction at the equator would be a single-valued function. So we must have,

$$\int_\Sigma \Omega = 2\pi n, \tag{2.46}$$

where n is an integer.

2.6 Examples

In this section we do two explicit examples of geometric quantization. We will write the symplectic two-form and solve for the wavefunction from the polarization condition. We will then define the inner product for these wavefunctions by retaining the prequantum inner product, we can do this because in the following examples the classical phase space would be a Kähler manifold. We will not discuss the action of operators on the wavefunctions in this section.

2.6.1 Coherent States

We consider a one-dimensional quantum system. The symplectic two-form on the phase space can be taken as,

$$\Omega = dp \wedge dx, \tag{2.47}$$

using the Darboux's theorem. Further by defining $z = (p + ix)/\sqrt{2}$ we can write the symplectic two-form as,

$$\Omega = idz \wedge d\bar{z}. \tag{2.48}$$

We choose the symplectic potential $\mathcal{A}$ such that $\Omega = d\mathcal{A}$.

$$\mathcal{A} = \frac{i}{2}(zd\bar{z} - \bar{z}dz) \tag{2.49}$$

The covariant derivatives $\mathcal{D}_a = \partial_a - i\mathcal{A}_a$ are,

$$\mathcal{D}_z = \partial_z - \frac{1}{2}\bar{z}, \tag{2.50}$$

$$\mathcal{D}_{\bar{z}} = \partial_{\bar{z}} - \frac{1}{2}z. \tag{2.51}$$

For holomorphic polarization we choose the polarization vector $\partial_{\bar{z}}$ which gives us the following polarization condition,

$$\left(\partial_{\bar{z}} + \frac{1}{2}z\right)\Psi = 0. \tag{2.52}$$

The solutions to this equation are of the form,

$$\Psi = \exp\left\{\left(-\frac{1}{2}z\bar{z}\right)\right\}\phi(z), \tag{2.53}$$

where $\phi(z)$ is a holomorphic function of z. The inner product on the Hilbert space of $\phi(z)$'s can now be defined as follows,

$$\langle 1|2\rangle = \int i\frac{dz\wedge d\bar{z}}{2\pi}\Psi_1^*\Psi_2, \tag{2.54}$$

$$= \int i\frac{dz\wedge d\bar{z}}{2\pi}e^{-z\bar{z}}\phi_1^*\phi_2. \tag{2.55}$$

This is the coherent state realization of the Heisenberg algebra.

2.6.2 Sphere

We consider the classical phase space to be a two-sphere. In coordinates, $z = x+iy$ the standard Kähler form on the manifold is given by,

$$\omega = i\frac{dz\wedge d\bar{z}}{(1+z\bar{z})^2}. \tag{2.56}$$

Note that the two-sphere is a compact manifold. Since it does not have a boundary, there are no one-forms α such that $d\alpha = \omega$, so the second cohomology class is non-zero. So the symplectic two-form must belong to an integral cohomology class of the two-sphere for geometric quantization.

$$\Omega = n\omega \tag{2.57}$$

Where $n \in \mathrm{Z}$. We have the quantization condition,

$$\int_M \Omega = 2\pi n. \tag{2.58}$$

The symplectic potential can be taken to be as in the previous example,

$$\mathcal{A} = \frac{in}{2}\left(\frac{z d\bar{z} - \bar{z} dz}{1 + z\bar{z}}\right), \tag{2.59}$$

the corresponding covariant derivatives can be obtained as in the previous example. This time the holomorphic polarization condition corresponds to,

$$\left(\partial_{\bar{z}} + \frac{n}{2}\frac{z}{1 + z\bar{z}}\right)\Psi = 0. \tag{2.60}$$

The solution of this equation is of the form,

$$\Psi = \exp\left\{\left(-\frac{n}{2}\log\left(1 + z\bar{z}\right)\right)\right\}\phi(z), \tag{2.61}$$

where $\phi(z)$ is once again a holomorphic function of z. The inner product on the $\phi(z)$ HIlbert space is,

$$\langle 1|2\rangle = \int i\frac{dz \wedge d\bar{z}}{2\pi}\frac{1}{(1 + z\bar{z})^{n+2}}\phi_1^*\phi_2. \tag{2.62}$$

3

The Chern-Simons Theory

3.1 The Chern-Simons Action

Consider an oriented three-manifold $\mathcal{M}$ (the spacetime manifold) and a compact, connected, simple Lie group G (the gauge group) with Lie algebra $\mathcal{G}$. A manifold is orientable when on any non-empty intersection of two open sets, the Jacobian of the transformation from one set of local coordinates to another is positive. As G is compact, $\mathcal{G}$ has a finite number of generators. If these generators cannot be separated into two mutually commuting sets, G is simple.

Consider a $\mathcal{G}$-valued one-form $A = A_\mu dx^\mu$, where $A_\mu = A^a_\mu(x)T^a$, T^a are the generators of $\mathcal{G}$. We want to construct a Lagrangian, which should be a 3-form because it will be integrated over a 3-manifold to give the action, which only involves A. Such a Lagrangian can only be a linear combination of $A \wedge A \wedge A$[1], $A \wedge dA$[2], and $dA \wedge A$. For gauge invariance we should also take a trace over Lie algebra indices. So the Lagrangian may be written as

$$L = \mathrm{Tr}(aA \wedge dA + cA \wedge A \wedge A), \tag{3.1}$$

[1] $\wedge$ denotes the 'wedge product'. For two 1-forms, A and B,

$$A \wedge B = \frac{1}{2}(A_i B_j - B_j A_i)(dx^i \otimes dx^j - dx^j \otimes dx^i),$$

the wedge product of a p-form and a q-form is a $p+q$-form.

[2]

$$dA = \frac{\partial A_\nu}{\partial x^\mu} dx^\mu \wedge dx^\nu,$$

is the exterior derivative of A, the exterior derivative of a k-form is a $k+1$-form.

where a and b are some constants[3]. We choose these constants such that,

$$dL = \mathrm{Tr}(F \wedge F), \tag{3.2}$$

where $F = dA + A \wedge A$. The quantity, $\mathrm{Tr}(F \wedge F)$, is called the second Chern form, and it remains invariant under gauge transformations of A, and its integral over a compact and oriented manifold is a constant. The reason behind defining the Lagrangian as in (3.2) will be clear once we discuss the gauge invariance of the CS Lagrangian. We get the desired Lagrangian for $a = 1$ and $b = 2/3$,

$$L = \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right). \tag{3.3}$$

We define the Chern-Simons action as,

$$S_{CS}[A] = \frac{k}{4\pi} \int_{\mathcal{M}} \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right). \tag{3.4}$$

where k is a constant called *level*. Below we discuss a few interesting and important aspects of this action.

1. *Geometry.* Let us understand more formally the geometrical setting of the CS theory. This will help us quantize the theory. On our spacetime manifold $\mathcal{M}$, we pick a principal G-bundle E. E is a manifold with $E = U_i \times G$ for each open set U_i of $\mathcal{M}$. A principal G-bundle also comes with a right G-action $\alpha : E \times G \longrightarrow E$. Sections of E are maps $g : U_i \longrightarrow G$, which assigns each point x on $\mathcal{M}$ to a gauge group element $g(x)$. We also place a principal connection, A, on E. This connection can be viewed as a $\mathcal{G}$-valued 1-form on $\mathcal{M}$. In local coordinates on $\mathcal{M}$ we can write $A = A^a_\mu(x) T^a(x) dx^\mu$, where the index a runs from 1 to the dimension of the gauge group. The covariant derivative of a $\mathcal{G}$-valued form ω is given by $D_\mu \omega = \partial_\mu \omega + [A_\mu, \omega]$, where $[\cdot, \cdot]$ is the Lie bracket. The local derivative ∂_μ is twisted by the connection, the measure of this twist is given by the curvature F of the connection defined as $F_{\mu\nu} \equiv [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. When $F = 0$, we call the connections flat. The connection A is the gauge field. Under gauge transformations,

$$A \longrightarrow A' = g^{-1} A g + g^{-1} dg \tag{3.5}$$

where g is a section of E.

[3]We do not have both $A \wedge dA$ and $dA \wedge A$ because $\mathrm{Tr}(A \wedge dA) = -\mathrm{Tr}(dA \wedge A)$.

2. *Boundary of a 4D theory.* Consider the CS action on a 3-manifold $\partial\mathcal{M}$, using Stokes' theorem and (3.2) we have,

$$S_{CS} = \frac{k}{4\pi}\int_{\partial\mathcal{M}} \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) = \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{Tr}(F\wedge F)$$
$$= \frac{k}{2\pi}\int_{\mathcal{M}} d^4x\, \mathrm{Tr}(F_{\mu\nu}\star F^{\mu\nu}), \tag{3.6}$$

where $\star F^{\alpha\beta} = (1/2)\varepsilon^{\alpha\beta\rho\sigma F_{\rho\sigma}}$ is the Hodge dual of $F^{\alpha\beta}$. This action is equal to the theta-term of non-abelian gauge theories in 4-dimensions up to an overall normalization. So we can think of CS as the boundary theory of topological Yang-Mills theory, whose action is,

$$S^{\theta}_{YM} = \frac{\theta}{16\pi^2}\int_{\mathcal{M}} d^4x\, \mathrm{Tr}(F_{\mu\nu}\star F^{\mu\nu}). \tag{3.7}$$

I find this correspondence interesting because we will later see that a $3D$ CS theory on a manifold with a boundary will be related to a conformal field theory on this boundary.

3. *Gauge invariance.* We should investigate the gauge invariance of the CS action,

$$S_{CS} = \frac{k}{4\pi}\int_{\mathcal{M}} d^3x\varepsilon^{\rho\mu\nu}\mathrm{Tr}\left(A_\rho(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{2}{3}A_\rho[A_\mu, A_\nu]\right). \tag{3.8}$$

Under a non-abelian gauge transformation of A (3.5), the change in the CS action is,

$$\delta S_{CS} = \frac{k}{4\pi}\int_{\mathcal{M}} d^3x\varepsilon^{\rho\mu\nu}\left(\partial_\rho \mathrm{Tr}[(\partial_\mu g)(g^{-1}A_\nu)]\right) + 2\pi k w(g). \tag{3.9}$$

The first term is a total derivative which vanishes on compact manifolds or when suitable boundary conditions are chosen. The second term where,

$$w(g) = \frac{1}{24\pi^2}\int_{\mathcal{M}} d^3x\varepsilon^{\rho\mu\nu}\mathrm{Tr}[(g^{-1}\partial_\rho g)(g^{-1}\partial_\mu g)(g^{-1}\partial_\nu g)], \tag{3.10}$$

is the winding number of the map g, this is a topological invariant of g on $\mathcal{M}$. For local gauge transformations $w(g) = 0$. For large gauge transformations, $w(g) \in \mathrm{Z}$. So the classical CS action is gauge-invariant only for local gauge transformations. However, for the quantum theory we only need the invariance of $e^{iS_{CS}}$, which happens only when $k \in \mathrm{Z}$. To maintain gauge-invariance of the quantum theory, the CS level must be an integer.

4. *Topological invariants and observables.* As the CS action does not have a metric, the partition function,

$$Z = \int D\mathcal{A} e^{iS_{CS}} \tag{3.11}$$

is a topological invariant of the spacetime manifold $\mathcal{M}$. For different gauge groups, we obtain different invariants. The gauge invariant observables of the theory are Wilson loops. These are the traces of the path-ordered exponential of the integral of the gauge field around a closed curve C,

$$W_R(C) = \mathrm{Tr}_R \left\{ \mathcal{P}exp \left(i \int_C A \right) \right\}, \tag{3.12}$$

where R is an irreducible representation of the gauge group. The closed curve C is diffeomorphic to S^1, we call them knots and,

$$\langle W_R(C) \rangle = \int D\mathcal{A} e^{iS_{CS}} W_R(C) \tag{3.13}$$

the vacuum expectation value of the Wilson loop is a knot invariant. On a manifold $\mathcal{M}$, different representations and gauge groups lead to different knot invariants. On $\mathcal{M} = S^3$ and $G = SU(2)$ in the fundamental representation, these invariants are the Jones polynomials.

5. *Anomalies and framing.* Consider the partition function of the CS theory (3.11), $D\mathcal{A}$ represents integration over all equivalence classes of connections modulo gauge transformations. It may not be possible to define $D\mathcal{A}$ without choosing some metric or a complex structure on the manifold, which would ruin topological invariance. When this happens, we call the theory anomalous. It can be shown that the CS theory is anomaly-free; we do need to pick a complex structure on the spacetime manifold, but the partition function is independent of this complex structure. We also need to provide additional data for the spacetime manifolds and knots embedded in them; this is called framing. This additional data regulates the divergences of the theory.

6. *Anyons.* Consider the abelian ($G = U(1)$) CS action on R^3.

$$\mathcal{S}_{CS} = \frac{k}{4\pi} \int_{\mathrm{R}^3} \mathrm{Tr}(A \wedge dA) \tag{3.14}$$

The $A \wedge A \wedge A$ term vanishes because of the anti-symmetry of the wedge product. Let's couple a matter current $J = \bar{\Psi}\gamma^\mu\Psi = (\rho, \mathbf{J})$ to the CS gauge field, the action becomes,

$$S_{CS} = \frac{k}{4\pi}\int d^3x \left(\varepsilon^{\rho\mu\nu}A_\rho\partial_\mu A_\nu + \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi + A_\mu J^\mu\right). \tag{3.15}$$

The equations of motion for A are,

$$\frac{k}{2\pi}\varepsilon^{\rho\mu\nu}\partial_\mu A_\nu = J^\rho. \tag{3.16}$$

We solve these equations for N charges placed at $\mathbf{x}_a(t)$, the resulting charge density is, $\rho = \sum_a \delta^{(2)}(\mathbf{x} - \mathbf{x}_a)$ and $\mathbf{J} = 0$. The gauge freedom is fixed by taking $A_0 = 0$ and imposing $\nabla \cdot \mathbf{A} = 0$. We obtain,

$$A_i(\mathbf{x}, t) = \frac{1}{k}\varepsilon^{ij}\frac{x^j - x_a^j}{|\mathbf{x} - \mathbf{x}_a|} \equiv -\partial_i\left(\frac{1}{k}\theta(\mathbf{x} - \mathbf{x}_a)\right). \tag{3.17}$$

Note that A_i is a total derivative that can be gauged away with an appropriate gauge transformation. Under this gauge transformation, the matter fields transform as follows,

$$\Psi(x) \longrightarrow \Psi'(x) = \exp\left(i\frac{\theta}{k}\right)\Psi(x) = \exp\left(\frac{i}{k}\int_C A_i dx^i\right)\Psi(x). \tag{3.18}$$

Suppose we move one of these particles around the other. This can be viewed as a double exchange; the particle that moves trades position with the stationary particle twice, once for every π rotation. For bosonic or fermionic particles, this would mean that Ψ returns to itself. However, in our case, Ψ carries an extra phase of $exp(2\pi i/k)$. These particles are called *anyons*. Alexei Kitaev, in 1997, demonstrated that these particles and their braiding statistics can be exploited to do fault-tolerant quantum computation.

3.2 Quantization of Chern-Simons Theory

The CS theory is an interacting quantum field theory. Such theories are typically intractable. It is not possible to find the Hilbert space of states, or correlation functions exactly. So we resort to perturbative calculations using Feynman diagrams. However, the CS theory, even though it looks like an intractable non-linear

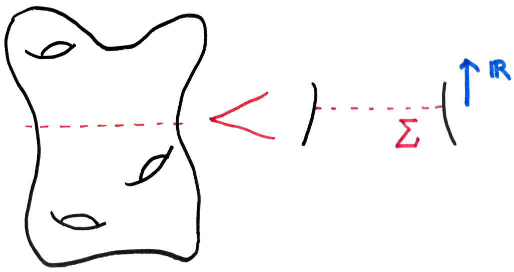


Figure 3.1: Cutting the spacetime manifold along a Riemann surface Σ, near the cut the manifold looks like, $\Sigma \times \mathcal{R}$.

quantum field theory, is exactly soluble. In this section, we will demonstrate how to quantize the CS action on various interesting spacetime manifolds.

The key to quantizing CS theory on a three-manifold $\mathcal{M}$ is to understand the quantization of the theory on a manifold $\Sigma \times \mathcal{R}$, where Σ is a two-dimensional Riemann surface. The manifold $\Sigma \times \mathcal{R}$ results from chopping the spacetime manifold along a Riemann surface Σ. Once we quantize CS theories on these surfaces, we can glue everything back together to obtain the quantum theory on $\mathcal{M}$.

To set up the quantization problem, we first study the classical CS on $\Sigma \times \mathcal{R}$. The variation of the CS action is,

$$
\begin{aligned}
\delta S_{CS} &= \frac{k}{4\pi}\int_{\Sigma\times\mathcal{R}} \mathrm{Tr}(\delta A \wedge dA + A \wedge d\delta A + 2A \wedge A \wedge \delta A) \\
&= \frac{k}{4\pi}\int_{\Sigma\times\mathcal{R}} \mathrm{Tr}(A \wedge d\delta A + dA \wedge \delta A + 2\delta A \wedge dA + 2A \wedge A \wedge \delta A) \\
&= \frac{k}{4\pi}\int_{\Sigma\times\mathcal{R}} \mathrm{Tr}(d(A \wedge \delta A) + 2\delta A \wedge (dA + A \wedge A)) \\
&= \frac{k}{4\pi}\int_{\partial\Sigma\times\mathcal{R}} \mathrm{Tr}(A \wedge \delta A) + \frac{k}{2\pi}\int_{\Sigma\times\mathcal{R}} \mathrm{Tr}(\delta A \wedge F)
\end{aligned}
$$

The first term in the above expression vanishes when Σ does not have a boundary. When Σ does have a boundary, we should choose the boundary conditions such that this term is zero. In the following, we will achieve this by working in the $A_0 = 0$ gauge, where A_0 is the component of A in the $\mathcal{R}$ direction (we interpret this as the time direction). The classical equations of motion are solutions of $\delta S_{CS} = 0$, as the variation in A can be arbitrary. This implies the equations of motion are,

$$F = dA + A \wedge A = 0. \tag{3.19}$$

In the $A_0 = 0$ gauge, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0$, implies,

$$\partial_t A_i = 0, \tag{3.20}$$

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j] = 0,\ i \in \{1, 2\}. \tag{3.21}$$

So the gauge fields are time-independent and satisfy a *Gauss law* constraint. Let's look at the CS action (3.4) in this gauge,

$$S_{CS} = \frac{k}{4\pi}\int dt \int_\Sigma \varepsilon^{ij}\mathrm{Tr}\left(A_i \frac{\partial}{\partial t} A_j\right). \tag{3.22}$$

This is a simple quadratic action of the form, $S = \int p\partial_t q$. The gauge field A_i is canonically conjugate to itself. We can extract the canonical commutation relations from this action,

$$\{A_i^a(x), A_j^b(y)\} = \frac{4\pi}{k}\delta^{ab}\varepsilon_{ij}\delta^{(2)}(x-y). \tag{3.23}$$

Before we proceed with quantization, notice an interesting implication of the action (3.22), the Hamiltonian of the CS theory vanishes! So, CS theories have no dynamical degrees of freedom. All the states of the quantum theory are degenerate with zero energy.

The goal of the quantization procedure is to find connections A satisfying both (3.19) and the commutator version of (3.23) simultaneously. Typically, in canonical quantization of quantum field theory, we quantize the commutators first and then impose the constraints on the states, for example, in the Gupta-Bleuler quantization of quantum electrodynamics. However, one could also impose the constraints first to obtain a phase space of the classical theory and then quantize this phase space. This is called constrained quantization. In subsequent sections, we demonstrate both ways of quantization.

3.2.1 Quantize, Constrain

We will consider the $SU(N)$ gauge group with the trace normalized as $\mathrm{Tr}(t^a t^b) = (1/2)\delta^{ab}$. The $\mathcal{G}$-valued gauge field A is $A = -it^a A_\mu^a$. On Σ we work in complex coordinates z and $\bar{z}$, the equations of motion in these coordinates become,

$$\partial_t A_z = \partial_t \left(\frac{1}{2}(A_1 + iA_2)\right) = 0 \tag{3.24}$$

$$\partial_t A_{\bar{z}} = \partial_t \left(\frac{1}{2}(A_1 - iA_2)\right) = 0 \tag{3.25}$$

$$F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z + [A_z, A_{\bar{z}}] = 0. \tag{3.26}$$

From the action of our theory,

$$S = \frac{ik}{\pi} \int dt \int_\Sigma \mathrm{Tr}(A_{\bar{z}} \partial_t A_z), \tag{3.27}$$

we obtain the symplectic two-form Ω on $\mathcal{A}$, the space of gauge fields on Σ, which is the phase space of the theory before the on-shell condition is imposed. Ω is derivative of the boundary term obtained upon the variation of the action,

$$\Omega = \frac{ik}{\pi} \int_\Sigma \mathrm{Tr}(\delta A_{\bar{z}} \delta A_z). \tag{3.28}$$

We have a symplectic manifold $(\mathcal{A}, \Omega)$. In taking the first steps towards quantization, we extract the Poisson bracket on this manifold by inverting Ω and promote it to commutators. The canonical commutation relations for the quantum theory are,

$$[A_z^a(z), A_{\bar{w}}^b(w)] = i\frac{2\pi}{k}\delta^{ab}\delta^{(2)}(z-w). \tag{3.29}$$

Now we need to associate a Hilbert space to $(\mathcal{A}, \Omega)$. In geometric quantization, we first describe the *prequantum* Hilbert space. Wavefunctions in this Hilbert space are the sections of the complex line bundle on $\mathcal{A}$; these are maps $\psi : \mathcal{A} \longrightarrow \mathcal{C}$, $\mathcal{C}$ is the one-dimensional complex vector space. The inner product on this space is defined as,

$$\langle \psi_1 | \psi_2 \rangle = \int \frac{\Omega}{2\pi} \psi_1^* \psi_2. \tag{3.30}$$

The wavefunctions, since they are sections of line bundles on $\mathcal{A}$, depend on all the phase space coordinates. It turns out that when this happens, the states in this Hilbert space do not form an irreducible representation of the Poisson bracket algebra. One needs to reduce the number of phase space variables the wavefunctions depend on by half. This is done by choosing a suitable *polarization* condition. We basically project these sections on a submanifold which has half the dimension of the phase space. We choose the *Bargmann* polarization condition,

$$\left(\frac{\delta}{\delta A_z} - \frac{k}{4\pi}A_{\bar{z}}\right)\psi[A_z, A_{\bar{z}}] = 0. \tag{3.31}$$

The wavefunctions in the quantum Hilbert space are,

$$\psi = e^{\frac{k}{4\pi}\int A^a_{\bar{z}}A^a_z}\phi[A^a_{\bar{z}}]. \tag{3.32}$$

The prequantum inner product now gives the inner product on the quantum Hilbert space,

$$\langle\phi_1|\phi_2\rangle = \int \frac{\Omega}{2\pi}e^{\frac{k}{2\pi}\int A^a_{\bar{z}}A^a_z}\phi_1^*\phi_2. \tag{3.33}$$

On these wavefunctionals, holomorphic in $A^a_{\bar{z}}$, A^a_z acts as,

$$A^a_z\psi[A^a_{\bar{z}}] = \frac{2\pi}{k}\psi[A^a_{\bar{z}}]. \tag{3.34}$$

We impose the Gauss law constraint on the wavefunctionals,

$$F^a_{z\bar{z}}\psi[A^a_{\bar{z}}] = 0 \tag{3.35}$$

$$\frac{k}{2\pi}\partial_z A^a_{\bar{z}}\psi[A^a_{\bar{z}}] = \left(\partial_{\bar{z}}\frac{\delta}{\delta A_z} + \left[A^a_{\bar{z}}, \frac{\delta}{\delta A_z}\right]\right)\psi[A^a_{\bar{z}}]. \tag{3.36}$$

The wavefunctionals that satisfy this are physical. To summarize, we have provided the canonical commutation relations for the fields and a Hilbert space of states which form an irreducible representation of the classical Poisson algebra. We have successfully quantized the theory. However, this is rather an opaque way of quantization; solving the Gauss law constraint is typically not going to be easy. Also, in quantum field theory, we rarely talk about wavefunctionals. We are interested in the states of the Hilbert space.

3.2.2 Constrain, Quantize

In the case of CS theory, it is better to first apply the Gauss law constraint to reduce the phase space and then quantize this phase space. The phase space of the classical CS theory is the space of flat connections modulo gauge transformations on Σ, i.e. connections for which $F = 0$ and two connections that are equivalent under a gauge transformation are identified. The phase space is completely characterized by Wilson loops (3.12) on Σ. As Wilson loops are topological invariants, the phase space is the space of inequivalent Wilson loops on Σ. These are loops which cannot be contracted to each other under diffeomorphisms. So the classical phase space is characterized by non-contractible loops on Σ, and hence on the first cohomology group $\mathcal{H}^1(\Sigma, \mathcal{R})$. In the following sections, we study the classical phase space and the resulting Hilbert space of the CS theory on different Σ. Quantizing the phase space actually results in modification of the Poisson bracket equation (3.23). However, we will only focus on the Hilbert space structure and states of the theory. The setting in this subsection is going to be as follows, we will decompose $d = dt\partial_t + \tilde{d}$ and $A = A_0 + \tilde{A}$ into space and time components, the action is,

$$S = -\frac{k}{4\pi}\int_{\Sigma\times\mathcal{R}} \mathrm{Tr}\left(\tilde{A}\frac{\partial}{\partial t}\tilde{A}\right). \tag{3.37}$$

We will derive an effective action for those $\tilde{A}$ that satisfy the Gauss law constraint, $\tilde{F} = 0$.

$\mathbf{\Sigma = S^2}$

S^2 is the spherical shell. The first cohomology group of S^2 is trivial. Every loop can be continuously deformed to a point on a sphere. This means the classical phase space of inequivalent Wilson loops is a point. Consequently, the resulting Hilbert space is one-dimensional. This rather trivial theory will play an important role in the prescription of gluing manifolds of the form $\Sigma\times\mathcal{R}$ back to the spacetime manifold $\mathcal{M}$.

$\mathbf{\Sigma = D}$

D is a disc. This is a manifold with a boundary. In this case, we have to specify the gauge transformations carefully. We are working in the $A_0 = 0$ gauge; any transformation that leaves this condition intact leaves the action invariant. Transformations that satisfy $\partial_t g = 0$ on the boundary respect our gauge condition. However, all of these transformations are not gauge transformations. Only the transformations that are constant at the boundary are gauge transformations, transformations which do not depend on t but are not constant are global symmetries of the theory.

Again since D is simply connected, the constraint is solved for $\tilde{A} = -\tilde{d}UU^{-1}$, where U are gauge transformations which are constant at the boundary S^1 of the disc. We can compute the effective action by substituting this $\tilde{A}$ in the action

(3.37). We work on polar coordinates on D, and choose $r = 0$ as the center,

$$\mathrm{Tr}\varepsilon^{ij}(A_i\partial_t A_j) = \mathrm{Tr}\varepsilon^{ij}(\partial_i UU^{-1}\partial_t\partial_j UU^{-1}) - \mathrm{Tr}\varepsilon^{ij}(\partial_i UU^{-1}\partial_j UU^{-1}\partial_t UU^{-1}), \quad (3.38)$$

the first term can be expanded as ($\varepsilon^{r\phi} = 1$),

$$\partial_r \mathrm{Tr}(\partial_\phi U^{-1}\partial_t U) - \partial_\phi \mathrm{Tr}(\partial_r U^{-1}\partial_t U), \quad (3.39)$$

the second term is simplified using,

$$\mathrm{Tr}\varepsilon^{ij}(\partial_i UU^{-1}\partial_j UU^{-1}\partial_t UU^{-1}) = \frac{1}{3}\mathrm{Tr}\varepsilon^{\mu\nu\rho}(U^{-1}\partial_\mu UU^{-1}\partial_\nu UU^{-1}\partial_\rho U), \quad (3.40)$$

putting all of this together, the effective action is,

$$S^{\mathrm{eff}} = \frac{k}{4\pi}\int_{S^1}\mathrm{Tr}(U^{-1}\partial_\phi UU^{-1}\partial_t U) + \frac{k}{12\pi}\int_{D\times R} Tr(U^{-1}dU)^3. \quad (3.41)$$

In the bulk, the value of U can be changed using gauge transformations, so the action only depends on the value of U on the boundary. The second term in the action is just the winding number of the map U, and as discussed in the first section, is an integer. The effective action is invariant under,

$$U \longrightarrow \tilde{V}(\phi)UV(t). \quad (3.42)$$

These are global symmetries of the theory. The spectrum of the quantum theory is in the representation of these symmetries. There are no local degrees of freedom in theory. The only degree of freedom is at the boundary of D. The states are in the representation of the loop group LG. This is the group of continuous maps from S^1 to the group G. Obtaining the states is an exercise in the representation theory of the loop group for our choice of the gauge group. The classical phase space is LG/G because U is only defined up to an element of G. We can extract the two-form on this phase space using the effective action and invert it to find the canonical commutation relations.

$\mathbf{\Sigma = T^2}$

T^2 is a torus. This is going to be the simplest example of a compact manifold where the Hilbert space of the CS theory is non-trivial. The classical phase space is characterized by the two inequivalent Wilson loops. The general solution to the

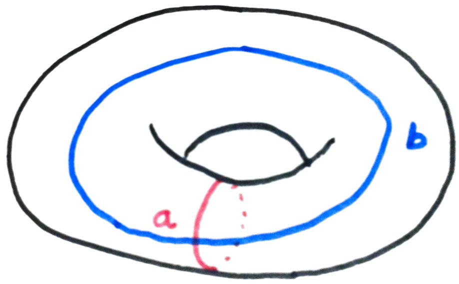


Figure 3.2: The two inequivalent, non-contractible loops on T^2.

constraint is,

$$\tilde{A} = -\tilde{d}UU^{-1} + U\theta(t)U^{-1}, \tag{3.43}$$

where $\theta(t)$ is a Lie algebra valued one-form that only depends on time. The components of $\theta(t)$ are the holonomies around the two inequivalent non-contractible cycles on the torus. We can obtain the effective action by substituting (3.43) in (3.37).

$$S^{\text{eff}} = -\frac{k}{4\pi}\int \text{Tr}(\theta \wedge \partial_t \theta) \tag{3.44}$$

We can extract the symplectic two-form using this effective action. Taking θ_1 and θ_2 to be the components of θ, we obtain the canonical commutation relations,

$$[\theta_1^i, \theta_2^j] = i\frac{2\pi}{k}\delta^{ij} \tag{3.45}$$

where $i\,j$ are the Lie algebra indices. The analysis of Hilbert space in the case of $SU(N)$ involves the representation theory of Lie groups. Let's consider the $U(1)$ gauge group. The effective action remains the same. The canonical commutation relations are also similar,

$$[\theta_1, \theta_2] = -i\frac{2\pi}{k}. \tag{3.46}$$

θ_i are periodic with $\theta_i = \theta_i + 2\pi$, due to large gauge transformations. In terms of the Wilson loops, $U_1 = e^{ia_1}$ and $U_2 = e^{ia_2}$, the commutator algebra is defined as,

$$U_1U_2 = e^{i2\pi/k}U_2U_1. \tag{3.47}$$

This is the algebra of clock and shift matrices in k-dimensions, so the Hilbert space of the CS theory on a torus is k-dimensional.

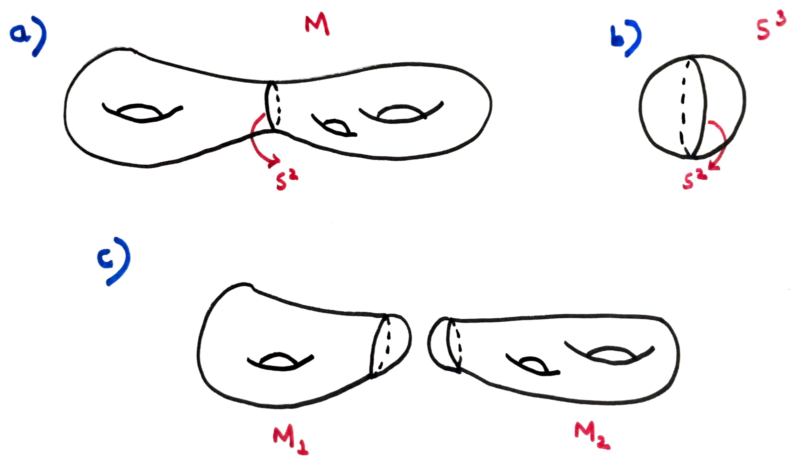


Figure 3.3: The gluing procedure. The manifold M is cut along S^2 into two parts, M_1 and M_2. On M_1 and M_2, we stick the pieces (cut along S^2) of S^3. (a) The spacetime manifold M. (b) Cutting S^3 along S^2. (c) Gluing the pieces of S^3 on M_1 and M_2.

3.2.3 Gluing

We have discussed the quantization of CS theory on manifolds of the form $\Sigma \times \mathcal{R}$. To obtain the quantum theory on spacetime manifold $\mathcal{M}$, upon quantization on $\Sigma \times \mathcal{R}$, we need a prescription of gluing everything back to $\mathcal{M}$.

Let $\mathcal{M}$ be the connected sum of M_1 and M_2 along a Riemann surface $\Sigma = S^2$. If we consider an action on a manifold with a boundary in quantum field theory, in the resulting path integral, the integration is over gauge fields taking specific values on the boundary. The path integral is a functional of this boundary data and, hence, is a vector in the Hilbert space of the theory associated with the boundary. In our example of $\mathcal{M} = M_1 \# M_2$ the partition function on $\mathcal{M}$ is,

$$\int_{\mathcal{A}_\Sigma/\mathcal{G}} DA_\Sigma e^{iS'_\Sigma[A_\Sigma]} \bar{\psi}_1[A_\Sigma]\psi_2[A_\Sigma] \tag{3.48}$$

where,

$$\psi_i[A_\Sigma] = \int_{\mathcal{A}_{M_i}/\mathcal{G},\, A_{M_i}|_\Sigma = A_\Sigma} DA_{M_i} e^{iS''_\Sigma[A_{M_i}]}, \tag{3.49}$$

are the wave functionals created by the path integrals on M_1 and M_2. As the boundaries of M_1 and M_2 are opposite in orientation, we obtain states that are dual to each other. The partition function on $\mathcal{M}$ can be thought of as the inner product of states created by the partition functions on M_1 and M_2,

$$Z(\mathcal{M}) = \langle \psi_2 | \psi_1 \rangle. \tag{3.50}$$

We can obtain a similar result for S^3, which can also be split along S^2,

$$Z(S^3) = \langle \phi_2 | \phi_1 \rangle. \tag{3.51}$$

The states $|\psi_1\rangle$, $|\psi_2\rangle$, $|\phi_1\rangle$ and $|\phi_2\rangle$ belong to the Hilbert space of CS on S^2. This Hilbert space is one-dimensional. So $|\psi_1\rangle = a\,|\phi_1\rangle$ and $\langle\psi_2| = b\,\langle\phi_2|$, where a and b are constants. Thus we have,

$$Z(M)Z(S^3) = Z(M_1)Z(M_2), \tag{3.52}$$

given the partition functions on M_1 and M_2, we can obtain the partition function on their connected sum $\mathcal{M} = M_1 \# M_2$.

3.2.4 CS/WZW Correspondence

We briefly describe how CS theory on $\Sigma \times R$ is related to Wess-Zumino-Witten (WZW) models. WZW models are two-dimensional conformal field theories. The conformal symmetry algebra in two dimensions is the Virasoro algebra. Two-dimensional conformal field theories are typically representations of this algebra. However, for some conformal field theories, there are fields that satisfy a symmetry algebra. These conformal field theories have a larger symmetry group with contains the Virasoro algebra. WZW conformal field theories are representations of the Kac-Moody algebras.

There are two ways in which these conformal field theories are related to CS theory on $\Sigma \times \mathcal{R}$. If Σ is a compact manifold, then the resulting Hilbert space is the space of conformal blocks of the WZW model on Σ. Conformal blocks are special functions that form a basis for correlation functions. Otherwise, if Σ is a manifold with a boundary, the Hilbert space of the CS theory is a representation of the chiral algebra of the WZW model.

We can demonstrate both of these from the analysis done in this section. For the case of CS on $D \times \mathcal{R}$, we obtained the effective action (3.41). This is exactly the action for the chiral WZW model, so trivially the Hilbert state is the representation of the chiral Kac-Moody algebra. When we did geometric quantization of the CS theory, upon imposing the Gauss law constraints on the states we obtained (3.36), it can be shown that the partition function of the WZW model on Σ satisfies a similar equation.

4

The Beta Function

In this chapter we will study our first Chern-Simons matter theory. In particular, we are going to study a CS theory coupled to scalar matter fields in the fundamental representation of the $O(N)$ group. We will show how this theory has a fixed point where it shows conformal invariance and hence by tuning the coupling constants of the theory we can construct conformal field theories in three dimensions. We also present brief overviews of the renormalization group and large N expansion of quantum field theories. Renormalization group studies in case of CS matter theories are important to identify the fixed points of the theories, these fixed points are important for two primary reasons, description of anyons and in arguments for why dualities between certain CS matter theories hold. Most of the CS matter theories are studied at large N because they simplify computations and render many exact results. In this chapter, however, we will compute the beta function both for infinite N and finite but large N.

4.1 Renormalization Group

We briefly review the idea of renormalization group and the beta function using the ϕ^4 theory as an example. The action of the theory is,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4. \tag{4.1}$$

To renormalize this theory, we express the bare quantities in terms of the renormalized ones,

$$\phi_0 = Z_\phi^{1/2}\phi, \quad m_0^2 = m^2 + \delta m^2, \quad \lambda_0 = \mu^\epsilon(\lambda + \delta\lambda).$$

Substituting into the bare Lagrangian gives,

$$\mathcal{L}_0 = \frac{1}{2}Z_\phi(\partial_\mu\phi)^2 + \frac{1}{2}Z_\phi(m^2 + \delta m^2)\phi^2 + \frac{Z_\phi^2}{4!}\mu^\epsilon(\lambda + \delta\lambda)\phi^4$$

We separate this into the renormalized Lagrangian and the counterterm Lagrangian,

$$\mathcal{L}_{\text{ren}} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4,$$

$$\mathcal{L}_{\text{ct}} = \frac{1}{2}\delta Z_\phi(\partial_\mu\phi)^2 + \frac{1}{2}\left(\delta Z_\phi m^2 + \delta m^2\right)\phi^2 + \frac{1}{4!}\left(2\delta Z_\phi\lambda + \delta\lambda\right)\mu^\epsilon\phi^4.$$

The renormalized lagrangian has coupling constants that depend on the energy scale μ. The renormalization group flow is the evolution of the coupling constants of the theory with the energy scale at which we look at the theory. The beta function tells us this dependence of the coupling constant on the energy scale of the theory,

$$\beta = \mu\frac{d\lambda}{d\mu}. \tag{4.2}$$

The renormalized coupling constant can be calculated using perturbation theory upto arbitrarily high order. For the ϕ^4 theory, the beta function at one-loop order is,

$$\beta(\lambda) = \frac{3}{16\pi^2}\lambda^2 + \mathcal{O}(\lambda^3). \tag{4.3}$$

Whenever the beta function is zero, the theory has conformal invariance, and that point is called the fixed point. For the ϕ^4 theory in $d = 4$ theory is only the $\lambda = 0$ fixed point which is trivial. Since the theory is non-interacting at $\lambda = 0$. There is a non-trivial fixed point in $d = 4 - \epsilon$ dimensions which is called the Wilson-Fischer fixed point. In section 3, we are going to renormalize the CS action coupled to real scalar fields in the fundamental representation of $O(N)$ and compute the beta function of the theory in the minimal subtraction scheme at two-loop order. We will then find three dimensional conformal field theories at the fixed points of that theory.

4.2 Large N

In this section we discuss the motivation for taking the large N limit of quantum field theories with the Yang-Mills theory as an example. The action for $SU(N)$ Yang-Mills is,

$$S_{\mathrm{YM}} = -\frac{1}{2g^2}\int d^4x \ \mathrm{Tr}\, F^{\mu\nu}F_{\mu\nu}. \tag{4.4}$$

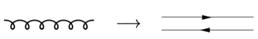
The large N limit is taken by keeping the 't Hooft coupling,

$$\lambda = g^2 N \tag{4.5}$$

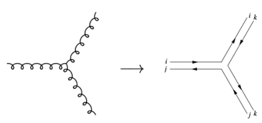
fixed while taking $N \longrightarrow \infty$. With these new coupling, the YM action is,

$$S_{YM} = -\frac{N}{2\lambda}\int d^4x \mathrm{Tr} F^{\mu\nu}F_{\mu\nu}. \tag{4.6}$$

What is going to happen in the 't Hooft limit ($N \longrightarrow \infty$, λ fixed) is that some of the diagrams that are leading order in coupling constant may become subdominant in the 't Hooft limit. So there is a rearrangement in the importance of Feynman diagrams of the theory. To see which diagrams dominate in the 't Hooft limit we use a different notation for the Feynman diagrams,



$$\sim \frac{\lambda}{N} \tag{4.7}$$

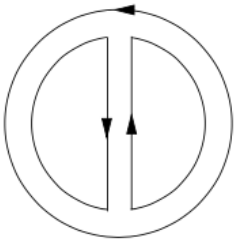


$$\sim \frac{N}{\lambda} \tag{4.8}$$

The general scaling for diagrams is going to be,

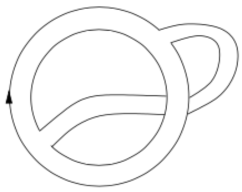
$$\text{diagram} \sim \left(\frac{\lambda}{N}\right)^{\#\text{propagators}} \left(\frac{N}{\lambda}\right)^{\#\text{vertices}} N^{\#\text{index contractions}}. \tag{4.9}$$

The idea is that if we can draw a diagram using this double line notation on a plane, it is going to be dominant in the large N limit, like the diagram as follows,



$$\sim \left(\frac{N}{\lambda}\right)^2 \left(\frac{\lambda}{N}\right)^3 N^3 \sim \lambda N^2 \tag{4.10}$$

while the diagrams that cannot be drawn on a plane is sub-dominant in the 't Hooft limit,



$$\sim \left(\frac{N}{\lambda}\right)^2 \left(\frac{\lambda}{N}\right)^3 N \sim \lambda. \tag{4.11}$$

So when working in the large N limit, we first need to identify the 't Hooft coupling and then rewrite the action in terms of it. Once we have done that to leading order in $1/N$ we only need to compute the planar diagrams.

4.3 Computation of the beta function

We consider $O(N)$ Chern-Simons gauge field A_μ coupled to bosons or Dirac fermions in the fundamental representation of $O(N)$, in three Euclidean dimensions. The CS action is,

$$S_{CS} = -\frac{ik}{8\pi}\int d^3x \epsilon^{\mu\nu\rho}\left(A^a_\mu \partial_\nu A^a_\rho + \frac{1}{3} f^{abc} A^a_\mu A^b_\nu A^c_\rho\right) \tag{4.12}$$

where $k \in \mathbb{Z}$ is called the CS level of the theory. In Euclidean coordinates the invariant tensors are $\delta_{33} = \delta_{+-} = 1$ and $\epsilon_{+-3} = 1$. We minimally couple this action to the following bosonic and fermionic action,

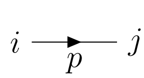
$$S_B = \int d^3x |D_\mu \phi|^2 + \frac{g_6}{3! \cdot 2^3}(\phi^6), \tag{4.13}$$

$$S_F = \int d^3x \bar{\psi}\gamma^\mu D_\mu \psi, \tag{4.14}$$

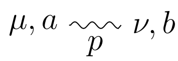
where $D_\mu = \partial_\mu + A_\mu$. The γ matrices in three dimensions are given by $\gamma^\mu = \sigma^\mu$ for $\mu = 1, 2, 3$, where σ are the Pauli matrices. So, the action that we will study is going to be,

$$S = S_{CS} + S_B/S_F. \tag{4.15}$$

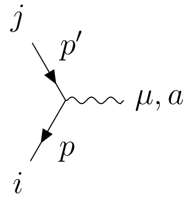
The Feynman rules for the bosonic theory can be read off from the action,



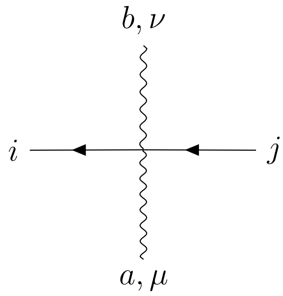
$$= \frac{\delta_{ij}}{p^2}$$



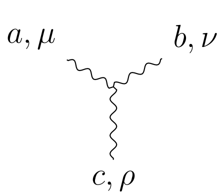
$$= G_{\nu\mu}(p)\delta_{ab}$$



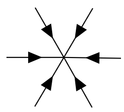
$$= (p' + p)^\mu T^a_{ij}$$



$$= \frac{1}{2}\{T^a, T^b\}_{ij}$$

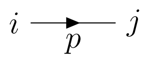


$$= \frac{k}{24\pi}\epsilon^{\mu\nu\rho} f^{abc}$$



$$= \frac{g_6}{3! \cdot 2^3}$$

where $G_{\nu\mu}$ is the Gluon propagator, which depends on the choice of our gauge. Similarly, the Feynman rule for the propagator fo the fermionic theory is,



$$= \frac{\delta_{ij} p_\mu \gamma^\mu}{p^2}.$$

We compute the beta function only for the bosonic case, we will work in the Lorentz gauge $\partial^\mu A_\mu = 0$. The regularized action is written as the sum of physical couplings and the counter terms.

$$S = S_{\rm CS}^{\rm phys.} + S_{\rm gh}^{\rm phys.} + S_{\rm b}^{\rm phys.} + S_{\rm CS}^{\rm c.t.} + S_{\rm gh}^{\rm c.t.} + S_{\rm b}^{\rm c.t.},$$

The physical coupling part of the action is as follows,

$$S_{\rm CS}^{\rm phys.} = \int d^dx \left\{ -\frac{i}{2}\epsilon_{\mu\nu\lambda}A_\mu^a \partial_\nu A_\lambda^a - \frac{i}{6}\mu^{\epsilon/2} g \epsilon_{\mu\nu\lambda} f^{abc} A_\mu^a A_\nu^b A_\lambda^c \right\},$$

$$S_{\rm gh}^{\rm phys.} = \int d^dx \left\{ -\frac{1}{2\alpha}(\partial_\mu A_\mu^a)^2 + \partial_\mu \bar{c}^a \partial^\mu c^a + \mu^{\epsilon/2} g f^{abc} \partial_\mu \bar{c}^a A_\mu^b c^c \right\},$$

this is the action of the CS ghost field as obtained by the Faddeev-Popov quantization of the CS theory.,

$$S_{\rm b}^{\rm phys.} = \int d^dx \bigg\{ \frac{1}{2}(\partial_\mu \phi_i)^2 + \mu^{\epsilon/2} g \partial_\mu \phi_i T_{ij}^a A_\mu^a \phi_j - \frac{1}{4}\mu^\epsilon g^2 \{T^a, T^b\}_{ij} \phi_i \phi_j A_\mu^a A_\mu^b \\ + \mu^{2\epsilon} \frac{g_6}{3! \cdot 2^3} (\phi_i \phi_i)^3 \bigg\}.$$

The counter terms may be written as,

$$S_{\rm CS}^{\rm c.t.} = \int d^dx \left\{ -\frac{i}{2}\delta Z_A \epsilon_{\mu\nu\lambda} A_\mu^a \partial_\nu A_\lambda^a - \frac{i}{6}\mu^{\epsilon/2} g \delta Z_g \epsilon_{\mu\nu\lambda} f^{abc} A_\mu^a A_\nu^b A_\lambda^c \right\},$$

$$S_{\rm gh}^{\rm c.t.} = \int d^dx \left\{ -\delta\alpha (\partial_\mu A_\mu^a)^2 + \delta Z_{\rm gh} \partial_\mu \bar{c}^a \partial^\mu c^a + \mu^{\epsilon/2} \delta \tilde{Z}_g g f^{abc} \partial_\mu \bar{c}^a A_\mu^b c^c \right\},$$

$$S_{\rm b}^{\rm c.t.} = \int d^dx \bigg\{ \frac{1}{2}\delta Z_\phi (\partial_\mu \phi_i)^2 + \mu^{\epsilon/2} \delta Z_g' g \partial_\mu \phi_i T_{ij}^a A_\mu^a \phi_j - \frac{1}{4}\mu^\epsilon \delta Z_g'' g^2 \{T^a, T^b\}_{ij} \phi_i \phi_j A_\mu^a A_\mu^b \\ + \mu^{2\epsilon} \delta Z_{g_6} \frac{g_6}{3! \cdot 2^3} (\phi_i \phi_i)^3 \bigg\}.$$

We define,

$$\delta Z_x = Z_x - 1, \;\; \delta\alpha = \frac{1}{\gamma_R} - \frac{1}{2\alpha} \tag{4.16}$$

to obtain the full action.

$$
\begin{aligned}
S = \int d^d x \Big\{ & -\frac{i}{2} Z_A \epsilon_{\mu\nu\lambda} A^a_\mu \partial_\nu A^a_\lambda - \frac{i}{6}\mu^{\epsilon/2} g Z_g \epsilon_{\mu\nu\lambda} f^{abc} A^a_\mu A^b_\nu A^c_\lambda \\
& - \frac{1}{2\gamma_R}(\partial_\mu A^a_\mu)^2 + Z_{\mathrm{gh}} \partial_\mu \bar{c}^a \partial^\mu c^a + \mu^{\epsilon/2} \tilde{Z}_g g f^{abc} \partial_\mu \bar{c}^a A^b_\mu c^c \\
& + \frac{1}{2} Z_\phi (\partial_\mu \phi_i)^2 + \mu^{\epsilon/2} Z'_g g \partial_\mu \phi_i T^a_{ij} A^a_\mu \phi_j - \frac{1}{4}\mu^\epsilon Z''_g g^2 \{T^a, T^b\}_{ij} \phi_i \phi_j A^a_\mu A^b_\mu \\
& + \mu^{2\epsilon} Z_{g_6} \frac{g_6}{3! \cdot 2^3} (\phi_i \phi_i)^3 \Big\}
\end{aligned}
\tag{4.17}
$$

1. We regulate the theory using dimensional reduction. This regularization procedure is Lorentz invariant and has been shown to be gauge invariant at least up to two loops (we do not do computations beyond two loops in this project). We compute the divergent integrals by first doing the ϵ contraction (or computing the γ traces in case of fermions) in $d = 3$ and then evaluate the resulting scalar integrals by analytic continuation to $d = 3 - \epsilon$.

2. We have the following relation between the CS level k and the coupling constant g,
$$k = \frac{4\pi}{g^2}. \tag{4.18}$$

3. In taking the large N limit, we fix the couplings $\lambda = g^2 N$ and $\lambda_6 = g_6 N^2$. Also note that the theory remains invariant under the combined transformation $\lambda \longrightarrow -\lambda$ ($k \longrightarrow -k$) and parity so the physical results should also remain invariant under this transformation.

4. We work in the Landau gauge in the next section, where $\alpha \longrightarrow 0$, in which the gluon propagator becomes,
$$G_{\nu\mu} = -\epsilon_{\mu\nu\lambda} \frac{p^\lambda}{p^2}. \tag{4.19}$$

5. The $O(N)$ generators in the fundamental representation are real and anti-symmetric. They satisfy,

   (a) $(T^a)^\dagger = (T^a)^T = -T^a$,

   (b) $\mathrm{Tr}(T^a T^b) = \delta^{ab} C_1$,

   (c) $f^{acd} f^{bcd} = \delta^{ab} C_2$,

   (d) $T^a_{ij} T^a_{kl} = I_{ij,kl} C_3$,

(e) $f^{abc}T^b_{ik}T^c_{kj} = \frac{1}{2}C_2 T^a_{ij}$,

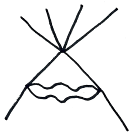
(f) $f^{abc} = \mathrm{Tr}\left(T^a[T^b, T^c]\right)$,

where, $C_1 = C_3 = 1$, $I_{ij,kl} = \frac{1}{2}\left(\delta_{il}\delta_{kj} - \delta_{ik}\delta_{jl}\right)$ and $C_2 = 2 - N$.

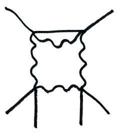
In this section we compute the β function for the bosonic theory coupled to the CS field. Our motivation is to check when this theory has conformal symmetry. The renormalization of the CS level leads to a shift in λ of the order $1/N$ and hence is ignored in the computations the we do in this section. We need to compute the β function for the λ_6 coupling constant. We do this at two loop level, which is first non-trivial order where the β function exists. This β function can be computed by calculating the divergent part of two-loop amplitudes where all external momenta are zero.

1. All external momenta are put to zero, because we only wish to extract the divergent part of the amplitudes.
2. We work at two-loop level because at one-loop all the diagrams are finite in the dimensional regularization scheme that we use. There can be three kinds of divergences
   (a) linear, these are rendered finite by the regularization scheme that we use.
   (b) quadratic and logarithmic, they require an odd power of loop momentum in the numerator, if this is the case the integral vanishes due to the $q \longrightarrow -q$ symmetry.
3. For renormalization we use the minimal subtraction scheme. Which means we subtract only the divergent part of loop integrals using the counter terms.

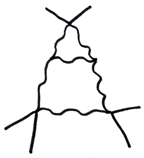
Even at two-loop order there are a large number of diagrams that one might need to compute even in the planar limit. The $\langle\phi^6\rangle$ correlator is superficially log-divergent but the diagrams contributing to its divergence is greatly reduced by an argument due to Aharony et. al., this only leaves eight diagrams that we need to compute. The argument goes as follows, consider a diagram with $\phi A^\mu \partial_\mu \phi$ vertex, with a gluon with loop momentum q and a bosonic external line with momentum p. The numerator will have $\epsilon_{\mu\nu\rho}q^\rho$ from the gluon propagator and $(q+2p)^\mu$ from the vertex. The leading high-energy term of the order q^2 cancels by antisymmetry reducing the degree of divergence and rendering the diagram finite. The following diagrams need to be computed. Here the first three diagrams are planar, whereas the next five diagrams are suppressed in $1/N$ and required if we want results at large but finite N.



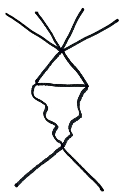
$$= -g^4 g_6 \left(\delta_{i_1 i_2}\delta_{i_3 i_4}\delta_{i_5 i_6} + 14 \text{ perms.}\right)\left(\frac{3}{2}N^2 + \frac{21}{2}N - 12\right)\frac{1}{64\pi^2}\frac{1}{\epsilon} \tag{4.20}$$



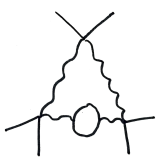
$$= g^8 \left(\delta_{i_1 i_2}\delta_{i_3 i_4}\delta_{i_5 i_6} + 14 \text{ perms.}\right)\left(N^2 + N - 2\right)\frac{3}{64\pi^2}\frac{1}{\epsilon} \tag{4.21}$$



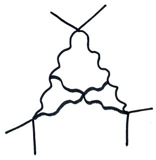
$$= \; = -g^8 \left(\delta_{i_1 i_2}\delta_{i_3 i_4}\delta_{i_5 i_6} + 14 \text{ perms.}\right)\left(N^2 - 3N + 2\right)\frac{3}{64\pi^2}\frac{1}{\epsilon} \tag{4.22}$$



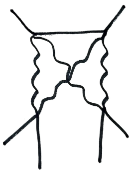
$$= -g^4 g_6 \left(\delta_{i_1 i_2}\delta_{i_3 i_4}\delta_{i_5 i_6} + 14 \text{ perms.}\right)\left(N - 1\right)\frac{9}{32\pi^2}\frac{1}{\epsilon} \tag{4.23}$$



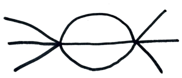
$$= g^8 \left(\delta_{i_1 i_2}\delta_{i_3 i_4}\delta_{i_5 i_6} + 14 \text{ perms.}\right)\left(N - 1\right)\frac{3}{64\pi^2}\frac{1}{\epsilon} \tag{4.24}$$



$$= 0 \tag{4.25}$$



$$= g^8 \left(\delta_{i_1 i_2}\delta_{i_3 i_4}\delta_{i_5 i_6} + 14 \text{ perms.}\right)(N-1)\frac{9}{32\pi^2}\frac{1}{\epsilon} \tag{4.26}$$

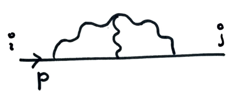


$$= g_6^2 \left(\delta_{i_1 i_2}\delta_{i_3 i_4}\delta_{i_5 i_6} + 14 \text{ perms.}\right)(3N+22)\frac{1}{32\pi^2}\frac{1}{\epsilon} \tag{4.27}$$

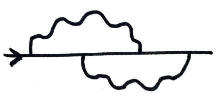
From the above diagrams we can compute the renormalization constant Z_{g_6},

$$g_6 Z_{g_6} \equiv g_6 + \sum_i (A_i) = g_6 + \frac{66g^8(N-1) + 4g_6^2(3N+22) - 3g^4 g_6(N^2+19N-20)}{128\pi^2}\frac{1}{\epsilon}. \tag{4.28}$$

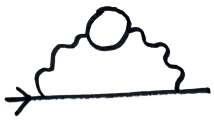
We also need the anomalous dimension of the scalar field at the two-loop order in order to compute the β function. This can be computed by the following diagrams.



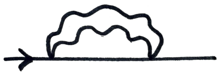
$$= -g^4 \delta_{ij} p^2 \left(N^2 - 3N + 2\right)\frac{1}{96\pi^2}\frac{1}{\epsilon} \tag{4.29}$$



$$= g^4 \delta_{ij} p^2 \left(N^2 - N\right)\frac{1}{384\pi^2}\frac{1}{\epsilon} \tag{4.30}$$



$$= g^4 \delta_{ij} p^2 \, (N-1) \, \frac{1}{48\pi^2} \frac{1}{\epsilon} \tag{4.31}$$



$$= g^4 \delta_{ij} p^2 \, (N-1) \, \frac{1}{96\pi^2} \frac{1}{\epsilon} \tag{4.32}$$

From the above diagram we can compute,

$$Z_\phi = 1 - \frac{g^4(3N^2 - 23N + 20)}{384\pi^2} \frac{1}{\epsilon}. \tag{4.33}$$

The bare coupling,

$$g_{6,0} = \frac{\mu^{2\epsilon g_6 Z_{g_6}}}{Z_\phi^3} = g_6 + b_1(g, g_6)/\epsilon + \text{(other terms)}, \tag{4.34}$$

where,

$$b_1(g, g_6) = \frac{33g^8(N-1) - 40g^4 g_6(N-1) + 2g_6^2(3N+22)}{64\pi^2}. \tag{4.35}$$

The β function for the λ_6 coupling is,

$$\beta_{\lambda_6}(\lambda, \lambda_6) = \frac{33(N-1)\lambda^4 - 40(N-1)\lambda^2\lambda_6 + 2(3N+22)\lambda_6^2}{32N^2\pi^2}. \tag{4.36}$$

1. In the limit of infinite N, $\beta_{\lambda_6} = 0$. So λ_6 is marginal in this limit.

2. At finite but large N, β_{λ_6} does not vanish.

   (a) When $\lambda = 0$, $\beta_{\lambda_6} > 0$, so the theory is IR-free.

   (b) When $\lambda \neq 0$ and for large N (not infinite), we have two lines of non-trivial fixed points of β_{λ_6},

   $$\lambda_6^{\pm}(\lambda) = \frac{\left(20N - 20 \pm \sqrt{1852 - 2054N + 202N^2}\right) \lambda^2}{44 + 6N}, \tag{4.37}$$

   the line λ_6^+ is IR-stable while the other line is UV-stable.

3. Since $\beta_\lambda = 0$, the renormalization group flow is always in the λ_6 direction.

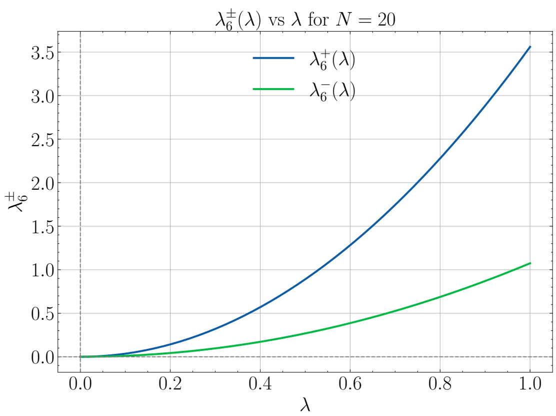


Figure 4.1: λ vs. λ_6 for both the IR-stable and UV-stable line of fixed points. The flow lines are towards the blue line and away from the green line.

5

The Thermal Free Energy

5.1 Setup

In this chapter we work with the $U(N)$ CS field coupled to complex scalars or Dirac fermions, in the fundamental representation. In the usual CS action, the large N limit is taken by keeping N/k fixed. We work in light-cone gauge in the x^1-x^2 directions, $A_- = 0$. The gauge group generators are anti-Hermitian and satisfy,

1. $\mathrm{Tr}(T^a T^b) = -1/2\delta^{ab}$,
2. and the properties satisfied by the $O(N)$ generators but with, $C_1 = C_3 = 1$, $I_{ij,kl} = \delta_{il}\delta_{kj}$ and $C_2 = -2N$.

The CS action reduces to,

$$S_{CS} = \frac{k}{4\pi} \int d^3x A_+^a \partial_- A_3^a. \tag{5.1}$$

We work on $\mathcal{R}^2 \times S^1$ where x^3 is compactified with length β. We take the holonomy along the thermal circle to be,

$$a = i\beta \mathcal{A}_3 \tag{5.2}$$

where $\mathcal{A}_3$ is the zero mode,

$$\mathcal{A}_3 = \frac{1}{V_2} \int_{\mathcal{R}^2} d^2x A_3 \tag{5.3}$$

where V_2 is the volume of $\mathcal{R}^2$. The eigenvalues of the holonomy are real and line on a circle of length 2π, due to identifications following from gauge symmetry. In the large N limit the eigenvalues of a are uniformly spread around 0 with width $2\pi|\lambda|$,

$$a_{ii} \longrightarrow a(u) = 2\pi|\lambda|u, \quad u \in [-1/2, 1/2], \tag{5.4}$$

and replace sums over eigenvalues by integrals over u,

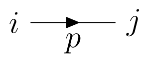
$$\sum_{i=1}^{N} f(a_{ii}) \longrightarrow N \int_{-1/2}^{1/2} du f(2\pi|\lambda|u). \tag{5.5}$$

5.2 Bosonic Theory

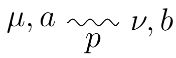
The effect of holonomy is to shift the scalar momentum,

$$\tilde{p}_\mu = p_\mu - i\mathcal{A}_\mu \tag{5.6}$$

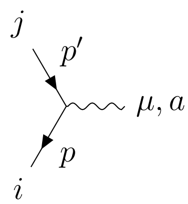
where, $\mathcal{A}_\ni$ is the zero-mode and $\mathcal{A}_\pm = 0$ .The Feynman rules in this case are,



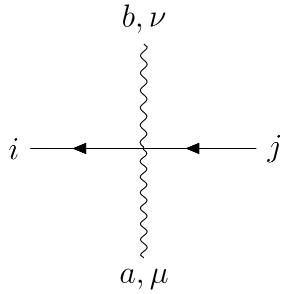
$= (\tilde{p}^{-2})_{ji}$



$= G_{\nu\mu}(p)\delta_{ab}$



$= i(T^a\tilde{p}'_\mu + \tilde{p}_\mu T^a)_{ij}$



$= \{T^a, T^b\}_{ij}\delta_{\mu 3}\delta_{\nu 3}$

The gluon propagator is,

$$G_{+3}(p) = -G_{3+}(p) = \frac{4\pi i}{k}\frac{1}{p^+}, \tag{5.7}$$

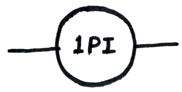
Since this does not depend on p_3 it is not affected by the holonomy.

To compute the thermal free energy we need the exact propagator for the theory, which is given by,

$$\langle \phi(p)^j \phi^\dagger(-q)_i \rangle = \left[\frac{1}{\tilde{p} - \Sigma_B}\right]_{ji} (2\pi)^3 \delta^3(p-q), \tag{5.8}$$

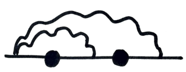
where $(\Sigma_B)_{ji}$ is the sum of 1PI diagrams.

$$\Sigma_B(p;\lambda)_{ji} = \tag{5.9}$$

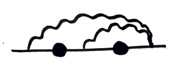


The 1PI diagrams involve the sum of the following four diagrams,

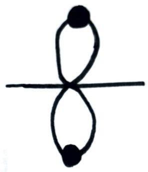
$$\begin{aligned} = (a) = &-\left(\frac{4\pi}{k}\right)^2 \int \mathcal{D}^3 l\, \mathcal{D}^3 q\, \frac{(l+p)^+\,(q+p)^+}{(l-p)^+\,(q-p)^+} \\ &\times \left[T^b \frac{1}{\tilde{l}^2 - \Sigma_B(l)} \{T^a, T^b\} \frac{1}{\tilde{q}^2 - \Sigma_B(q)} T^a\right]_{ji} \end{aligned} \tag{5.10}$$



$$\begin{aligned} = (b) = &\left(\frac{4\pi}{k}\right)^2 \int \mathcal{D}^3 l\, \mathcal{D}^3 q\, \frac{(l+p)^+\,(q+l)^+}{(l-p)^+\,(q-l)^+} \\ &\times \left[T^a \frac{1}{\tilde{l}^2 - \Sigma_B(l)} T^b \frac{1}{\tilde{q}^2 - \Sigma_B(q)} \{T^a, T^b\}\right]_{ji} \end{aligned} \tag{5.11}$$



$$\begin{aligned} = (c) = &\left(\frac{4\pi}{k}\right)^2 \int \mathcal{D}^3 l\, \mathcal{D}^3 q\, \frac{(l+p)^+\,(q+l)^+}{(l-p)^+\,(q-l)^+} \\ &\times \left[T^a \frac{1}{\tilde{l}^2 - \Sigma_B(l)} T^b \frac{1}{\tilde{q}^2 - \Sigma_B(q)} \{T^a, T^b\}\right]_{ji} \end{aligned} \tag{5.12}$$



$$= (d) = -\frac{1}{2}\frac{\lambda_6}{N^2}\int \mathcal{D}^3 l\, \mathcal{D}^3 q \left[\frac{1}{\tilde{q}^2 - \Sigma_B(q)}\right]_{j_1 i_1}\left[\frac{1}{\tilde{l}^2 - \Sigma_B(l)}\right]_{j_2 i_2}$$

$$\times \left[\delta_{ij}\delta_{i_1 j_1}\delta_{i_2 j_2} + (5 \text{ permutations})\right] \tag{5.13}$$

The bootstrap equation for the scalar propagator is,

$$\Sigma_B(p)\delta_{ij} = (a) + (b) + (c) + (d). \tag{5.14}$$

Once we take the derivative of this bootstrap equation with respect to p^- using,

$$\frac{\partial}{\partial p^-}\frac{1}{p^+} = 2\pi\delta^2(p), \qquad \frac{\partial p_s}{\partial p^-} = \frac{p^+}{p_s}. \tag{5.15}$$

The resulting equation,

$$\partial_{p^-}\Sigma_B(p) = 0 \tag{5.16}$$

tells us that we can evaluate Σ_B by setting $p_s = 0$. This renders $(b) = (c) = 0$ and,

$$\text{(a)} = -4\pi^2\lambda^2\delta_{ji}\left[\int_{-1/2}^{1/2} du \int D^3 q\, \frac{1}{\vec{q}^2 + \beta^{-2}\mu_B^2}\right]^2, \tag{5.17}$$

$$\text{(d)} = -\frac{\lambda_6}{2}\delta_{ji}\left[\int_{-1/2}^{1/2} du \int D^3 q\, \frac{1}{\vec{q}^2 + \beta^{-2}\mu_B^2}\right]^2, \tag{5.18}$$

where the integral over uniformly spread eigenvalues is substituted. The sum over q_3 momentum is,

$$F(q_s, u) \equiv \frac{1}{\beta}\sum_{n=-\infty}^{\infty} \frac{1}{\left(\frac{2\pi}{\beta}(n + |\lambda| u)\right)^2 + q_s^2 + \beta^{-2}\mu_B^2}. \tag{5.19}$$

The bootstrap equation for the self-energy can now be written as,

$$\Sigma_B = -\hat{\lambda}^2\left[\int_{-1/2}^{1/2} du \int dq_s q_s F(q_s, u)\right]^2 \tag{5.20}$$

where,

$$\hat{\lambda}^2 \equiv \lambda^2 + \frac{\lambda_6}{8\pi^2}. \tag{5.21}$$

And the bootstrap equation for the thermal mass μ_B is,

$$|\mu_B(\lambda,\lambda_6)| = \left|\frac{\hat{\lambda}}{2}\int_{-1/2}^{1/2} du \int_0^\infty dx \, \frac{x}{\sqrt{x^2+\mu_B^2}} \, \mathrm{Re}\left(\coth\left[\frac{\sqrt{x^2+\mu_B^2}+2\pi i|\lambda|u}{2}\right]\right)\right| \tag{5.22}$$

where $x=\beta q_s$. By regularizing the radial integral and subtracting the divergence using a mass counter-term the bootstrap equation becomes,

$$|\mu_B| = \left|\hat{\lambda}\int_{-1/2}^{1/2} du \, \log\left|2\sinh\left(\frac{\mu_B+2\pi i|\lambda|u}{2}\right)\right|\right| \tag{5.23}$$

which can be solved analytically,

$$\pm\mu_B = -\frac{|\hat{\lambda}\mu_B|}{2} - \frac{1}{2\pi i}\frac{|\hat{\lambda}|}{|\lambda|}\left[\mathrm{Li}_2\left(e^{-|\mu_B|-\pi i|\lambda|}\right) - \mathrm{Li}_2\left(e^{-|\mu_B|+\pi i|\lambda|}\right)\right]. \tag{5.24}$$

The exact scalar propagator is,

$$\langle\phi(p)^j\phi^\dagger(-q)_i\rangle = \left[\frac{1}{\tilde{p}-\beta^{-2}\mu_B(\lambda,\lambda_6)}\right]_{ji}(2\pi)^3\delta^3(p-q), \tag{5.25}$$

where μ_B is obtained by solving the previous equation.

To compute the free energy of the theory we need to evaluate the following expression in terms of the self-energy. The subtlety is that we include the integral over the holonomy in our computations. At large N, the thermal free energy is,

$$\beta F_B = NV_2\int_{-1/2}^{1/2} du \int \frac{d^2p}{(2\pi)^2}\sum_{n=-\infty}^{\infty}\left[\log\left(\tilde{p}^2-\Sigma_B\right)+\frac{2}{3}\frac{\Sigma_B}{\tilde{p}^2-\Sigma_B}\right], \tag{5.26}$$

where,

$$p_3 = \frac{2\pi n}{\beta} \;\; \tilde{p}_\mu = p_\mu - \frac{2\pi|\lambda|u}{\beta}\delta_{3,\mu}. \tag{5.27}$$

We regulate the divergence in sum over n in the first log term by introducing a negative contribution with a large mass $\beta^{-1}M$, so we need to compute,

$$\int \frac{d^2p}{(2\pi)^2} \sum_{n=-\infty}^{\infty} \log\left[\frac{p_s^2 + \beta^{-2}\mu_B^2}{p_s^2 + \beta^{-2}\mu_B^2 + \beta^{-2}M^2}\right] = \frac{1}{2\pi\beta^2} \int_0^{\Lambda'} dx\, x \log[2\cosh\left(\sqrt{x^2+\mu_B^2}\right) \\ - 2\cos(2\pi|\lambda|u)] - (\mu_B \to M)$$

where $x = \beta p_s$. The first term has UV divergence which can be regularised. The computation for second term is similar to the thermal mass bootstrap equation. From these computations one can compute that the free energy is,

$$\beta F_B = -\frac{NV_2}{2\pi\beta^2}\left\{\frac{\mu_B^3}{3}\left(1 \pm \frac{2}{|\hat{\lambda}|}\right) + \frac{1}{\pi i|\lambda|}\int_{\mu_B}^{\infty} dyy[\mathrm{Li}_2(e^{-y+\pi i|\lambda|}) - \mathrm{Li}_2(e^{-y-\pi i|\lambda|})]\right\}. \tag{5.28}$$

This free energy vanishes for fixed N in the strong coupling limit $|\lambda| \longrightarrow 1$.

5.3 Fermionic Theory

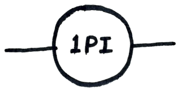
The fermionic part of the action is,

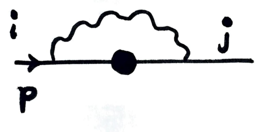
$$S_F = \int d^3x \bar{\psi}\gamma^\mu D_\mu \psi \tag{5.29}$$

Once again to compute the free energy we need the exact fermionic propagator. The exact fermionic propagator written as,

$$\langle \psi(p)_i \bar{\psi}(-q)_j \rangle = \left[\frac{1}{i\tilde{p}_\mu\gamma^\mu - \Sigma_F(p)}\right]_{ji} (2\pi)^3\delta^3(p-q), \tag{5.30}$$

where just as in the bosonic case Σ_F is the sum of 1PI diagrams. The bootstrap equation for the exact propagator is,

$\Sigma(p;\lambda)_{ji} =$ 

$=$ 

. (5.31)

We can expand $\Sigma_F = i\Sigma_-\gamma^- + i\Sigma_+\gamma^+ + i\Sigma_3\gamma^3 + \Sigma_I I$. The diagram in the bootstrap equation is given by,

$$\begin{aligned}
&= \frac{4\pi i}{k} T^a_{jm} T^a_{ki} \int \frac{D^3 q}{(2\pi)^3} \frac{1}{(q-p)^+} \gamma^{[3|} \left(\frac{1}{i\tilde{q}_\mu \gamma^\mu - \Sigma_F(q)} \right)_{mk} \gamma^{|+]} \\
&= -\frac{2\pi i}{k} \delta_{ji} \mathrm{Tr}_N \int \frac{D^3 q}{(2\pi)^3} \frac{1}{(q-p)^+} \gamma^{[3|} \frac{1}{i\tilde{q}_\mu \gamma^\mu - \Sigma_F(q)} \gamma^{|+]} \\
&= \frac{2\pi i}{k} \delta_{ji} \mathrm{Tr}_N \int \frac{D^3 q}{(2\pi)^3} \frac{1}{(q-p)^+} \frac{1}{(\tilde{q} - \Sigma_F)^2 + \Sigma_I^2} \gamma^{[3|} \left[i(\tilde{q} - \Sigma_F)_\mu \gamma^\mu + \Sigma_I I \right] \gamma^{|+]}
\end{aligned} \tag{5.32}$$

Using the properties of gamma matrices the bootstrap equation can be written as,

$$\begin{aligned}
\Sigma_F(p) &= \frac{4\pi i}{k} \mathrm{Tr}_N \int \frac{D^3 q}{(2\pi)^3} \frac{1}{(q-p)^+} \frac{\Sigma_I(q)\gamma^+ - i(\tilde{q} - \Sigma_F) + I}{(\tilde{q} - \Sigma_F)^2 + \Sigma_I^2} \\
&= 4\pi i \int_{-1/2}^{1/2} du \int \frac{d^2 q}{(2\pi)^2} \frac{1}{(q-p)^+} \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \frac{\Sigma_I(q)\gamma^+ - i(\tilde{q} - \Sigma_F) + I}{(\tilde{q} - \Sigma_F)^2 + \Sigma_I^2}
\end{aligned} \tag{5.33}$$

Now we compute the sum by setting $x = \beta q_s$ as in the bosonic case,

$$\begin{aligned}
\frac{1}{\beta} \sum_{n=-\infty}^{\infty} \frac{1}{(\tilde{q} - \Sigma_F)^2 + \Sigma_I^2(q)} &= \beta \sum_n \left[2\pi \left(n + \frac{1}{2} - |\lambda| u \right)^2 + x^2 (1 - g + f^2) \right]^{-1} \\
&= \frac{\beta}{2} \frac{x\sqrt{1 - g(x) + f^2(x)}}{x^2 + (1 - g + f^2)} \times \\
&\quad \mathrm{Re} \left[\tanh \left(\frac{x\sqrt{1 - g(x) + f^2(x)} + 2\pi i |\lambda| u}{2} \right) \right] \\
&\equiv \frac{\beta}{2} G(x)
\end{aligned} \tag{5.34}$$

Proceeding as in the bosonic case we differentiate the bootstrap equation and regularize and analytically evaluate the resulting integrals to obtain the thermal mass bootstrap equation,

$$\pm \mu_F(\lambda) = \lambda \mu_F + \frac{1}{\pi i} \left(\mathrm{Li}_2 \left(-e^{-\mu_F - \pi i \lambda} \right) - \text{c.c.} \right) \tag{5.35}$$

From this we compute the exact propagator,

$$\langle \psi(p)_i \bar{\psi}(-q)_j \rangle = \frac{-i\tilde{p}_\mu \gamma^\mu + i g p_+ \gamma^+ - f p_s I}{\tilde{p}^2 + T^2 \mu_F^2} (2\pi)^3 \delta^3(p - q). \tag{5.36}$$

The free energy in terms of the self-energy with the integration over thermal holonomy is,

$$\beta F_F = -NV_2 \int_{-1/2}^{1/2} du \int \frac{d^2q}{(2\pi)^2} \sum_{n=-\infty}^{\infty} \mathrm{Tr}_{\mathrm{fer}}\{\log\left(i\tilde{q}_\mu \gamma^\mu - \tilde{\Sigma}_F(q)\right) \tag{5.37}$$

$$+ \frac{1}{2}\Sigma_F(q)\frac{1}{i\tilde{q}_\mu\gamma^\mu - \Sigma_F(q)}\}. \tag{5.38}$$

We first compute the trace over the fermionic indices,

$$\begin{aligned} &\mathrm{Tr}_{\mathrm{fer}}\left\{\log\left(i\tilde{q}_\mu\gamma^\mu - \Sigma_F(q)\right) + \frac{1}{2}\Sigma_F(q)\frac{1}{i\tilde{q}_\mu\gamma^\mu - \Sigma_F(q)}\right\} \\ &= \log\left[(\tilde{q} - \Sigma_F)^2 + \Sigma_I^2\right] + \frac{1}{2}\frac{(g-2f^2)q_s^2}{(\tilde{q}-\Sigma_F)^2 + \Sigma_I^2} \end{aligned} \tag{5.39}$$

to finally write the free energy as,

$$\beta F_F = -NV_2 \int_{-1/2}^{1/2} du \int \frac{d^2q}{(2\pi)^2} \sum_{n=-\infty}^{\infty} \left[\log\left(q_3^2 + \beta^{-2}\mu_F^2\right) + \frac{1}{2}\frac{(g-2f^2)q_s^2}{q_3^2 + \beta^{-2}\mu_F^2}\right]. \tag{5.40}$$

Regulating the terms as in the bosonic case, we obtain the free energy of the fermionic theory as follows,

$$\beta F_F = \frac{NV_2}{2\pi\beta^2}\left\{\frac{\mu_F^3}{3}\left(1 \mp \frac{1}{\lambda}\right) + \frac{1}{\pi i \lambda}\int_{\mu_f}^{\infty} dy y[\mathrm{Li}_2(-e^{-y+\pi i\lambda}) - c.c.]\right\}. \tag{5.41}$$

6

Applications

6.1 Dualities?

Chern-Simons theories admit many dualities. The CS/WZW correspondence relates CS theory on a manifold to a WZW model on the boundary of that manifold. They also admit level/rank dualities. These are dualities among different CS theories unlike in the CS/WZW correspondence where the duality is between a CS theory and a WZW model. As the name suggests level/rank dualities relate the CS level and the rank of the gauge group of two different CS theories. In the Yang-Mills regularization procedure, the value of CS level k is quantized and can take any integer value. In this regularization when we have a theory with gauge group,

$$U(N) = \frac{SU(N) \times U(1)}{Z_N} \tag{6.1}$$

denoted by $U(N)_{k_1,k_2}$, where k_1 is the rank of the $SU(N)$ group and k_2 is the rank of the $U(1)$ group, we have the following level-rank dualities,

$$SU(N)_k \longleftrightarrow U(k)_{-N,-N}, \tag{6.2}$$

$$U(N)_{k,k+N} \longleftrightarrow U(k)_{-N,-k-N}. \tag{6.3}$$

When we couple matter fields to the CS action, CS/WZW correspondence is immediately ruined. However, as we have seem in section 3, we can still find a conformal field theory by tuning the coupling constants of the theory. The level-rank dualities of CS theories are actually motivated from the CS/WZW correspondence.

Such dualities were known to exist for WZW models and the CS/WZW correspondence extended them to CS theories. So once we add matter fields since there is no CS/WZW correspondence, there is no level-rank duality. However, it turns out that even CS matter theories admit level-rank like dualities among themselves. We discuss some of aspects of these dualities.

1. *What are these dualities?* The conjectured CS matter dualities are as follows,

   $SU(N)_k$ coupled to scalars $\longleftrightarrow$ $U(k)_{-N+N_f/2,\,-N+N_f/2}$ coupled to fermions,
   $U(N)_{k,k}$ coupled to scalars $\longleftrightarrow$ $SU(k)_{-N+N_f/2}$ coupled to fermions,
   $U(N)_{k,k+N}$ coupled to scalars $\longleftrightarrow$ $U(k)_{-N+N_f/2,\,-N-k+N_f/2}$ coupled to fermions.

2. *Why do we think they exist?* The intuition for dualities like the ones listed above comes from the behavior of CS matter theories along RG flows. Consider a fermionic CS matter theory and deform it by a fermion mass term such that depending on the sign of the mass term the theory flows to pure CS theory at low energies. When we map this deformation to the scalar side we obtain a different pure CS theory at low energies. If these two CS theories are related by a level rank duality. The CS matter theories that we start with should also be related by some duality. Using the precise level-rank dualities that exist for CS theories and the choice of gauge group for the CS matter theories, we can restrict the possible dualities in CS matter theories to the above list.

3. *What evidence do we have for their existence?* There are several computations that hint that the proposed dualities are correct. These computations are typically all done at leading order in large N limit. Quantities such as the thermal free energy and large N correlation functions have been computed for both sides of the duality and matched at various orders. The baryonic and monopole operators on both sides of the duality have also been shown to be consistent.

4. *How do we look for them!* While the dualities are conjectured to hold at any N and at all orders in the perturbation theory. The current evidence for the duality only comes from large N computations at finite order in the perturbation theory. The most straightforward way of gathering evidence for these dualities is to compute physical quantities like the thermal free energy and correlation functions at higher orders and for finite N. However,

unless we have some non-perturbative arguments or computations that can help prove these dualities, they will remain a conjecture.

6.2 Spectrum of Non-Abelian Anyons

We have already seen how excitations in CS theory are non-abelian anyons. This remains true even in the case of Chern-Simons matter theories. In this example we will show how a CS theory coupled to either bosonic or fermionic non-relativistic matter field can describe multiple non-Abelian anyons in a harmonic trap. Let us first describe the bosonic and fermionic actions, the bosonic action is,

$$S_B = \int dt d^2x \left[i\phi_a^\dagger \mathcal{D}_0 \phi_a - \frac{1}{2m} \vec{\mathcal{D}}\phi_a^\dagger \vec{\mathcal{D}}\phi_a - \lambda(\phi_a^\dagger t^\alpha[R_a]\phi_a)(\phi_b^\dagger t^\alpha[R_b]\phi_b) \right] \tag{6.4}$$

there are N_f scalar fields ϕ_a, each of them transform in some representation R_a of $SU(N)$. The generators of the gauge group are $t^\alpha[R_a]$ where $\alpha = 1, \cdots, N^2 - 1$. All the anyons have the same mass m and the quartic term gives rise to a delta-function interaction among the anyons. For the fermionic theory the action is,

$$S_F = \int dt d^2x \left[i\psi_a^\dagger \mathcal{D}_0 \psi_a - \frac{1}{2m} \vec{\mathcal{D}}\psi_a^\dagger \vec{\mathcal{D}}\psi_a - \frac{1}{2m} \psi_a^\dagger F_{12}^\alpha t^\alpha[R_a]\psi_a \right] \tag{6.5}$$

where rest of the symbols have the same meaning but the matter fields are now N_f complex, Grassman-valued fields ψ_a. The role of quartic interactions is played by the coupling to the non-Abelian magnetic field F_{12}. For both these theories we have fixed points where they demonstrate $SO(2,1)$ conformal invariance. The choice of the fixed point governs the kind of interaction we will have among anyons and the resulting spectrum. The problem of non-Abelian anyons in a harmonic trap is reduced to computing the spectrum of the dilatation operator of the conformal field theory at the chosen fixed point. In the rest of the discussion we will talk about bosons, the case of fermions is slightly different but the key ideas remain the same. Due to the state-operator correspondence of conformal field theories the state that describes a single anyon sitting in a harmonic trap transforming in representation R_a is dual to the following Wilson line,

$$\Phi_a(x) = \mathcal{P} \exp\left\{ \left(i \int_\infty^x A^\alpha t^\alpha[R_a] \right) \right\} \phi_a(x) \tag{6.6}$$

for n-particles in the harmonic trap we need to look for operators

$$\mathcal{O} \sim \Pi_{i=1}^{n}(\partial^{l_i}\bar{\partial}^{m_i}\Phi_{a_i}^{\dagger}) \tag{6.7}$$

from these operators we need to choose the primary operators of the theory. There are two ways in which this spectrum can be computed. One is the brute force method which involves perturbation theory. Just like in section 3, we can compute the quantum corrections to the dimensions of the operators upto some order in perturbation theory and find out all the primary operators. The perturbation theory for the current action is not as tedious because there are no anti-particles which reduces the number of diagrams that we need to compute by a lot. The second method focuses on a special class of states,

$$\mathcal{O} \sim \Pi_{i=1}^{n}(\partial^{m_i}\Phi_{a_i}^{\dagger}) \tag{6.8}$$

the scaling dimension of these operators are fixed by their angular momentum. And the angular momentum of an operator $\mathcal{O}$ sitting in the representation R is

$$J = -\frac{C_2(R) - \sum_i C_2(R_{a_i})}{2k} \tag{6.9}$$

where C_2 is the quadratic Casimir of the representation. Using these methods, the spectrum of two anyons in $SU(2)_k$ CS theory coupled to scalars in the fundamental representation is computed and it indicates that there is no level-crossing for the ground state.

6.3 Effective theory of FQHE

We described how the quantization of the CS theory on a disk results in an edge mode with remarkable resemblence with the edge mode in quantum Hall effect. Here, we will discuss how even CS matter theories acts as an effective field theory of low energy excitations in fractional quantum Hall effect (FQHE).

We discuss a non-relativistic, supersymmetric CS matter theory coupled to a scalar ϕ and a complex fermion ψ. The low energy physics of this theory will turn out be the theory of fractional quantum Hall effect. Fractional quantum Hall effect refers to the phenomena where the hall conductance of electrons confined in two spatial dimensions has plateaus at quantized values of applied magnetic field, these quantized values are fractions of e^2/h. The excitations of fractional quantum Hall states have fractional charge and spin. One can show that the vortices in the following models have a wavefunction in the same universality class as the Laughlin state. Laughlin states are many-body wavefunctions that explain the fractional quantum Hall effect at specific filling fractions. By being in the same universality class we mean both the states have the same topological order, ground state degeneracy, the same edge theory and the same quasiparticle statistics.

$$
\begin{aligned}
S = \int dt\, d^2x \Bigg\{ & i\phi^\dagger \mathcal{D}_0 \phi + i\psi^\dagger \mathcal{D}_0 \psi - \frac{1}{2m} \mathcal{D}_\alpha \phi^\dagger \mathcal{D}_\alpha \phi - \frac{1}{2m} \mathcal{D}_\alpha \psi^\dagger \mathcal{D}_\alpha \psi \\
& - \frac{k}{4\pi} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - \mu A_0 + \frac{1}{2m} \psi^\dagger B \psi \\
& - \frac{\pi}{mk} \left(|\phi|^4 - \mu |\phi|^2 + 3|\phi|^2 |\psi|^2 \right) \Bigg\}
\end{aligned} \tag{6.10}
$$

The fermion carries no spinor index and $|\psi|^2 = -\psi\psi^\dagger = \psi^\dagger\psi$. The other parameters in the action are k the CS level, m the mass of bosons and fermions and μ the chemical potential, which is thought of as the background magnetic field for the vortices. Since we called the action supersymmetric, let us see the supersymmetries it has. In particular this action has kinematical and dynamical supersymmetries. The kinematical supersymmetry is given by the transformation,

$$
\delta_1 \phi = \epsilon_1^\dagger \psi, \quad \delta_1 \psi = -\epsilon_1 \phi, \quad \delta_1 A_z = 0, \quad \delta_1 A_0 = \frac{\pi}{mk} \left(\epsilon_1 \phi \psi^\dagger - \epsilon_1^\dagger \psi \phi^\dagger \right)
$$

while the dynamical supersymmetry is given by,

$$
\delta_2 \phi = \epsilon_2^\dagger \mathcal{D}_{\bar{z}} \psi, \quad \delta_2 \psi = \epsilon_2 \mathcal{D}_z \phi, \quad \delta_2 A_z = -\frac{i\pi}{k} \epsilon_2^\dagger \psi \phi^\dagger, \quad \delta_2 A_0 = \frac{i\pi}{mk} \left(\epsilon_2^\dagger \phi^\dagger \mathcal{D}_{\bar{z}} \psi - \epsilon_2 \phi \mathcal{D}_z \psi^\dagger \right).
$$

These supersymmstries render this model exactly solvable. The quantum state describing m quasi-holes at complex coordinates η_i, $i = 1, \cdots m$, is,

$$
|\eta_1, \ldots, \eta_m\rangle_k \propto \prod_{i=1}^{m} \det\left(Z^\dagger - \eta_i^\dagger \right) |\text{ground}\rangle_k
$$

where Z is $n \times n$ complex matrix which parametrises the moduli space of vortices. This is indeed the Laughlin wavefunction,

$$
|\eta_1, \ldots, \eta_m\rangle_k \quad \to \quad \prod_a (z_a - \eta) \,|\text{Laughlin}\rangle_k
$$

where z_a are the eigenvalues of Z.

7

Conclusions

Chern-Simons theories are interesting both because of the many applications they have in diverse areas of physics and because of its rich mathematical implications. In this dissertation we have touched upon both these aspects.

We studied geometric quantization as a systematic way of quantizing a classical system. To do this we got acquainted with the basics of differential geometry and studied classical mechanics in the the laguage of differential geometry. The geometric quantization method gives us a recipe of associating a Hilbert space to the classical phase space such that observables on the phase space correspond to Hermitian operators on the corresponding Hilbert space. It also ensures that the Poisson bracket algebra of the observables on the classical phase space has an appropriate representation in terms of commutators on the associated Hilbert space. We discuss a couple of subtleties in the procedure arising from special topologies of the classical phase space. Finally we explicitly carry out the geometric quantization procedure for coherent states and for a spherical classical phase space.

Once acquainted with the tools of differential geometry and geometric quantization we move on to study pure Chern-Simons theory. First of all we discuss the differential geometry setting of the classical action, this is very important as we later quantize the theory using geometric quantization. We also discuss various aspects of the action. We learned that the classical Chern-Simons action is not gauge invariant but the quantum theory is gauge invariant. We looked at the three-dimensional CS theory as the boundary theory of four-dimensional topological Yang-Mills theory. When CS theory is coupled to a matter current anyons emerge, we discussed how this happens and demonstrate the Aharanov-Bohm phase that these anyons accumulate when braided around each other. When knots are included in the spacetime manifold, the CS action admits a framing anomaly, we briefly discuss the implications of this anomaly and also discuss how knot invariants can be computed using CS theory. Computation of knot invariants is one of the most interesting applications of CS theories in mathematics. In fact the interpretation of the Jones polynomial as the Wilson loop expectation value of the knots in CS spacetime manifold is what made the theory popular in the first place.

Once we understood the CS action and its properties well we move on to quantize the theory on various manifolds. The theory is quantized on a manifold by chopping the manifold in many pieces such that each piece locally looks like $\Sigma \times \mathcal{R}$ where Σ is a Riemannian manifold and then gluing these pieces together. We discuss how this gluing procedure works and how chopping the manifold on such pieces makes the theory easy to quantize. During quantization we demonstrate two ways of approaching the problem. We could either constrain the classical theory first by finding its phase space using equations of motion of the CS theory and then quantize this phase space using the geometric quantization recipe or we could apply the recipe to the spacetime manifold to quantize it and then constrain the Hilbert space using equations of motion. The second method is similar to the Gupta-Bleuler method of quantization where the theory is quantized first and then only certain states satisfying a criterion are chosen to live in the Hilbert space.

Finally we comment on the Chern-Simons/Wess-Zumino-Witten correspondence. This correspondence related CS theories on some spacetime manifold to a WZW conformal field theory on the boundary of that spacetime manifold. While this correspondence does not say that the two theories are equivalent (their partition function is the same), it does provide enough data to build one theory from the other. It should also be noted that CS theory itself may be viewed as the boundary theory of a four-dimensional topological Yang-Mills theory and so the CS/WZW correspondence hints at a correspondence between a four-dimensional theory and a two-dimensional theory.

While studying CS matter theories we started with a review of renormalization group and large N expansion in quantum field theories. Both these concepts are central in the study of CS matter theories. By studying the RG flows of the theory we come up with recipes for constructing three dimensional conformal field theories from CS matter theories. While large N expansion renders most computation doable in these theories. At finite N there are terms of the order $1/N$ that complicates most computations. Also additional non-planar diagrams have to be computed.

We computed the β function corresponding to the ϕ^6 self-interaction coupling for the CS theory coupled to real scalars in the fundamental representation of the gauge group. We found that the coupling is exactly marginal for infinite N but has two non-trivial lines of fixed points for large but finite N. One of these lines is IR-stable while the other line is UV-stable. We also computed the thermal free energy of CS matter theories coupled to complex scalars and Dirac fermions in the fundamental representation of the gauge group. We discussed the subtlety in computing the thermal free energy for such theories arising from the assumption that the eigenvalues of the thermal gauge holonomy is localized. The eigenvalues are instead distributed uniformly over the unit circle, and once we take care of this, we get correct results consistent with the proposed dualities between CS matter theories.

All of our computations have been done at the leading order in the large N limit. Also the dimensional regularization scheme that we use is only known to be gauge invariant upto two-loop level. Fortunately, that is the level at which our computations are done. But it would be interesting to do these computations with a better regularization scheme or we should check the consistency of the dimensional regularization scheme beyond two-loop level. Also, we should do these computations at higher orders in the perturbation theory to gain a better understanding of the behavior of these theories. We have chosen to work in the large N limit to avoid cumbersome calculations and so that we have to compute only a manageable number of diagrams. However, computations with small N are necessary to understand the dualities.

We have also briefly discussed the proposed CS matter dualities and the motivation for their existence. The evidence for these dualities come from calculations that are similar to the ones done in this report. While we have not explicitly done a computation to match a certain quantity on both sides of the duality, the thermal free energy computation for the regular fermion theory that we have done can be readily extended to compute the thermal free energy for critical fermion theory which is dual to the regular boson theory. It can be shown that the thermal free energy of these theories match and provide an evidence for the duality. Topological quantum computation in enabled by non-abelian anyons. We discuss a paper that describes non-abelian anyons in a harmonic trap using non-relativistic CS (bosonic and fermionic) matter theory. The low energy physics of the fractional quantum Hall effect is effectively described by a CS theory. We also discuss a paper where this effective theory is a supersymmetric CS matter theory instead of a pure CS theory.

Now that we have been acquainted to computations in CS matter theories there are several problems that we can work on. One can extend the present calculation of physical quantities to higher orders and lower N, we can look at CS matter theories with matter fields in different representations, for example, bifundamental CS matter theories have several interesting properties and their dualities are not well understood. But the most interesting application of these theories are in understanding non-abelian anyons and fractional quantum Hall effect. Some CS matter theories are exactly solved in large N limit, if we can map these theories using CS matter dualities to the CS matter theory which describes non-Abelian anyons, we could have the first exactly solvable model of interacting non-Abelian anyons.

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