16-811 Math Fundamentals for Robotics
Assignment 3

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October 10, 2019

1 Collaborations

1.1 Question 1

For Question 1.c, I used an online resource to solve for the answer. The URL used id : `https://math.stackexchange.com/questions/1736763/if-a-function-is-odd-even-then-its-` `fbclid=IwAR1HRzbhC_2B_3REtz_qVOQse0hMP3t1qyovkQpJR5bInRuJx_Lx_WDd9Jw`.

2 Instructions for Running code and Analysis

All codes are written in MATLAB.

2.1 Question 2

For question 1, the input values are taken in the variable y. I first use the data points to find the value of y, where its derivative becomes the highest ($y = 30$), and then fit two cubic polynomials from index $(0, 3)$ and $(31, end)$ respectively. The plot for both the input and output is attached below.

2.2 Question 4

The code for Question 4 is divided over two files. The files are namely, "q4.m" and "q4e.m". The first file contains blocks of code required to solve the first four parts of the question. The second file contains the code to solve the last part of the question.

The evaluator can comment/uncomment the respective parts in the file "q4.m" to run the codes for the first four parts. The ouputs corresponding to each part can be seen in the command line.

I generate the output for the first two parts using SVD, that is getting the plane equations which satisfy the given set of data points in the least square sense.

For the third part, I exploit the RANSAC algorithm and rely on it to generate planes which do not contain the cluttered data points. (We can assume RANSAC to work here since it is given that the number of correct data points is much more). The tolerance for a point to be considered as an inlier is set to the average distance between plane and a 'clean point' (calculated in part (a)).

The fourth part is also similar to the third part in its application od RANSAC, with the only difference being that RANSAC is applied four times to get points from each plane. Specifically, we apply RANSAC over the entire 'unvisited data' (initially, the entire data is unvisited). The points that are found by the RANSAC algorithm to be in a plane are then marked visited with a plane being constructed. Since we know that there are a total of four planes, this process is iterated over four times. A sanity check for this part is that each plane should contain approximately equal number of points.

For the last part, the code "q4e.m" should be run. This code will output the coefficient of the four planes corresponding to the hallway along with a roughness score for each plane. The more rough the plane, the less it is preferred for the locomotion. In this part, I first find two planes, which are easily found using RANSAC similar to the previous part.

Now, since the other two walls are noisy, our first aim is to separate them to avoid mixing and hence, the potential risk of decreasing RANSAC performance. We then apply RANSAC on one of the planes with more number of data points, asssuming that since it contains more points, it might be less noisy and there are more inliers that RANSAC can work with. This gives us the third plane.

Now we are left with points we are sure that belong to either the fourth plane or are noisy versions of it. Since RANSAC can not work here due to high noise, we shift to a makeshift version of majority vote model. We iterate over the remaining points, and for each point find the two closest points in the remaining points' set. These three points are used to construct an equation of a plane. I hypothesize that for majority combinations, the plane we will get will be a representative of the true surface plane. (The points closest to a given point are more likely to be on the surface plane than on the noisy plane on an average). Therefore, once we get the plane equations from each point, we take the most occurring value of each coefficient (this is not exactly appropriate. Ideally we need to choose the set of maximum occurring coefficients simultaneously within a threshold but that will add an extra hyperparameter and in practice even picking the coefficients individually seems to be working fine). This is the equation for the last plane.

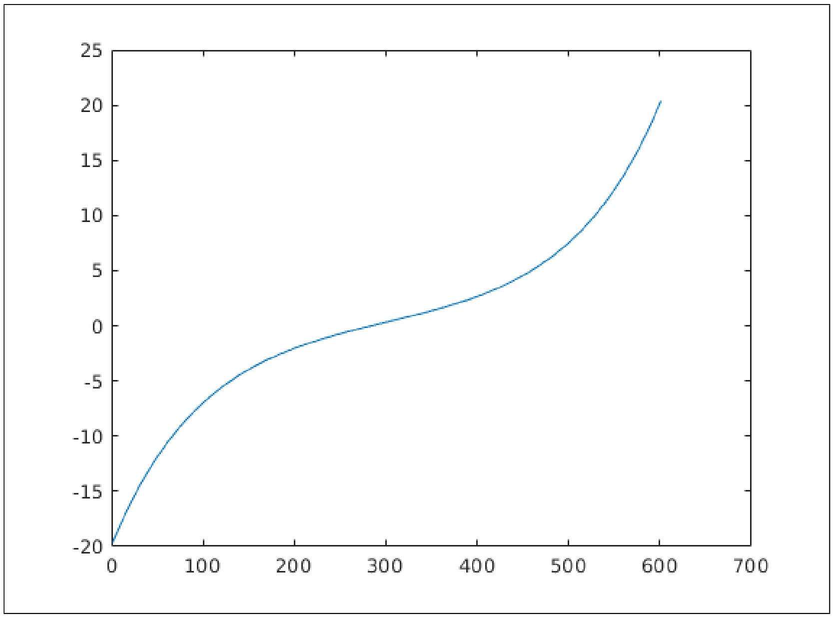
I use the following logic for smoothness/roughness calculation. For each plane, we can safely assume that the plane is smooth if the variance of the distance of the inlier points in the plane is less. However, it might be possible that the variance of inliers is less because there are less inliers found. We need to penalize that. Therefore, I generate a roughness score of the plane by the equation :

$$roughness = \frac{(Var(dist(inliers)))}{(num(inliers))} \tag{1}$$

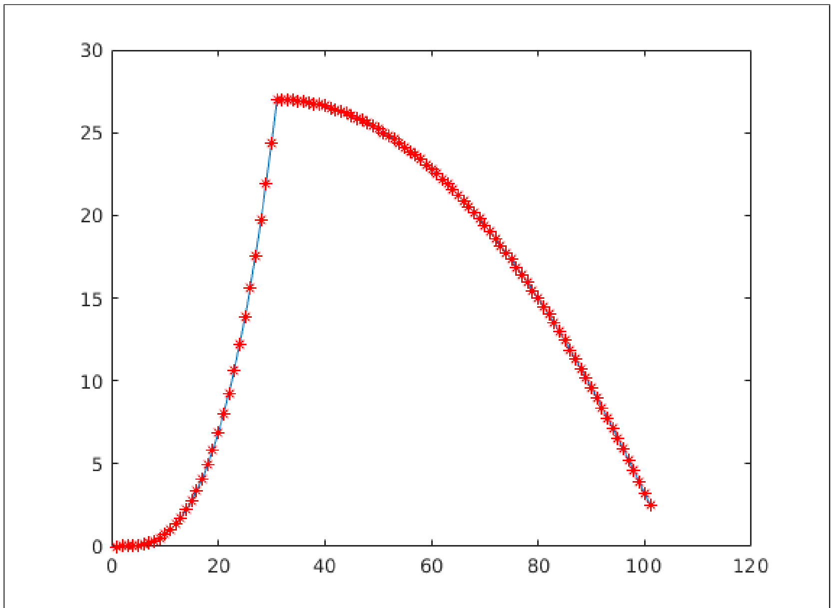
The more rough the surface, the less smooth it is, and more dangerous it is for the robot to perfom locomotion on that surface.

3 Code generate images

3.1 Q.1

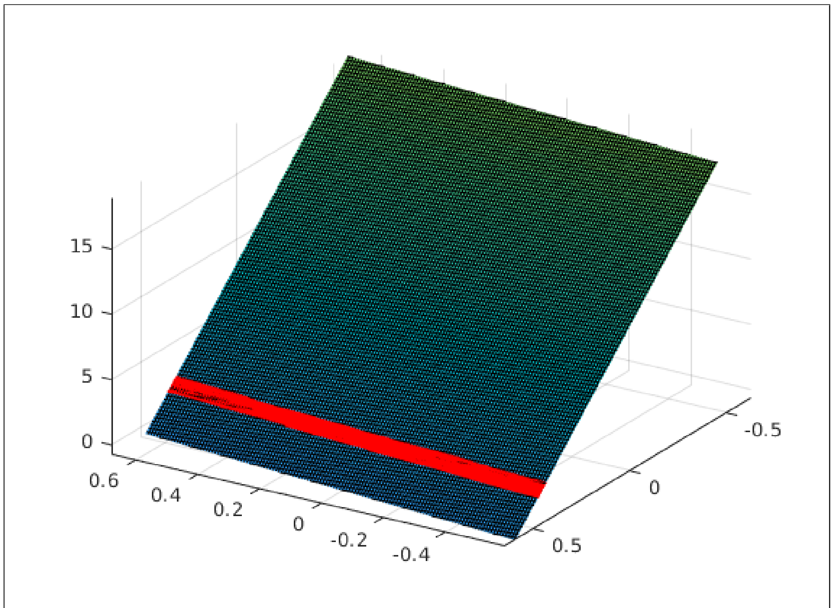


3.2 Q.2



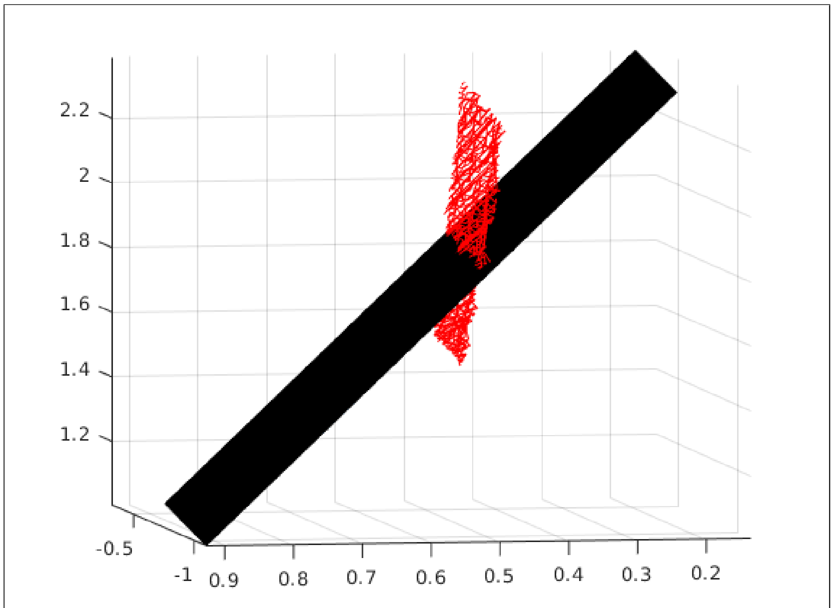

```
index =

    30


coeff1 =

   -0.0000
    0.0000
   -0.0000
    1.0000


coeff2 =

   22.5000
    3.0000
   -0.5000
   -0.0000

fx >>
```

3.3 Q4.1



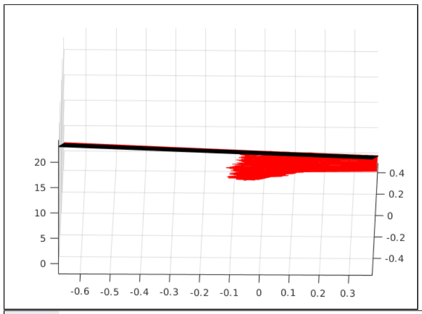

```
coeff_clean =

   -1.7931
  -18.7452
    9.4391


dist =

    0.0029
```

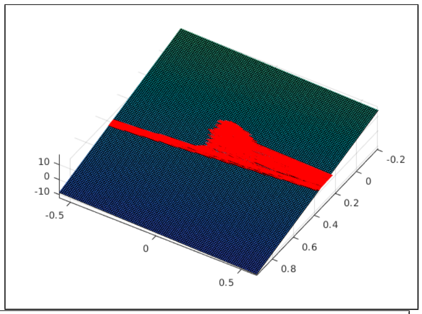
3.4 Q4.2




```
coeff_noisy =

   -0.2001
   -2.0224
    2.6567
```

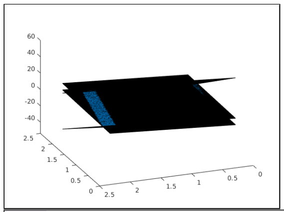
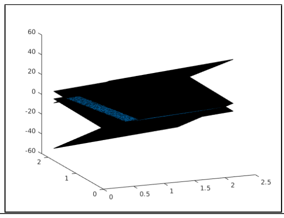
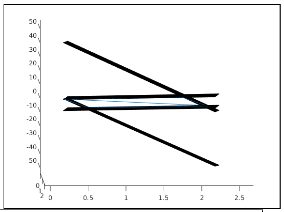
3.5 Q4.3





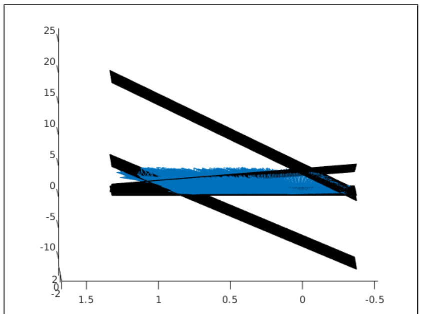

```
finalCoeff =

   -2.1149
  -20.8633
   10.3169
```

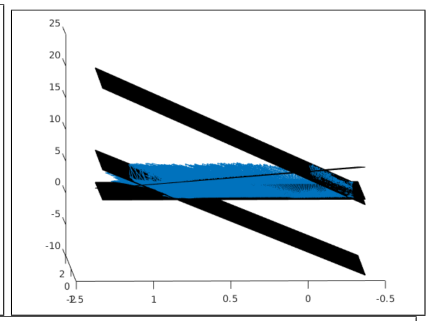
3.6 Q4.4

```
coeffPlane1 =

    4.4266
  -25.8214
    5.5886


coeffPlane2 =

    5.3291
  -30.4858
   55.6860


coeffPlane3 =

    5.0282
    0.9014
   -1.2310


coeffPlane4 =

    4.5516
    0.8132
   -8.4752
```

3.7 Q4.5






```
coeffPlane1 =

    0.1217
    9.9088
   -5.8302


roughness =

  527.6455


coeffPlane2 =

    0.1804
   11.1212
    5.8827


roughness =

  494.4130


coeffPlane3 =

   20.1196
   -0.4091
  -20.8519


roughness =

  542.6698


coeffPlane4 =

   -1.0000   -1.8953    4.6530


roughness =

   2.7023e+04
```

Q 1 (a) $f(x) = 1/3 + 2\sinh(x)$

Taylor Series expansion ~~at~~ around $x = a$ becomes:

$$\frac{1}{3} + 2\sinh(a) + \frac{2\cosh(a)}{1!}(x-a) + \frac{2\sinh}{2!}(x-a)^2$$

$$+ \frac{2\cosh(a)}{3!}(x-a)^3 + \cdots$$

At $x=0$, $\sinh(0) = 0$ and $\cosh(0) = 1$

$\therefore$ the expansion becomes.

$$1/3 + 2x + \frac{2}{3!}x^3 + \frac{2}{5!}x^5 + \cdots$$

$$= 1/3 + 2\left[x + x^3/3! + x^5/5! + \cdots\right] \quad \underline{\text{Ans}}.$$

(b) Plotted through code [PLEASE SEE ~~ON~~ THE THIRD SECTION OF THE PDF].

(c) $f(x) = 1/3 + 2\sinh(x)$

let the approximating quadratic polynomial be $p(x) = ax^2 + bx + c$.

Now, since $f(x)$ is an odd function, the approximating polynomial also has to be odd.

$\Rightarrow$ a in $p(x) = ax^2 + bx + c$ has the value 0.

This reduces our problem in the linear approximation domain

However, we still need to treat it as a quadratic problem ($\because$ $f''(x)$ changes sign at end points, $f'''(x)$ does not).

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Now, $\because$ $f'''(x)$ does not change sign ~~end points at~~ ~~between~~ -3 and 3, two points that can be taken to approximate the quadratic are -3 and 3

$x_0 = -3$ $\quad x_1 = ?$ $\quad x_2 = ? \; (= -x_1)$ $\quad x_3 = 3$

(since f is odd)

Let the error func^n be $e(x) = f(x) - p(x)$

$e(x_0) = f(x_0) - p(x_0)$ $\quad e(x_2) = f(x_2) - p(x_2)$

$e(x_1) = f(x_1) - p(x_1)$ $\quad e(x_3) = f(x_3) - p(x_3)$

$e(x_0) = -e(x_1)$ — (1) and $e(x_0) = -e(x_3)$ — (1.1)

$e(x) = (1/3) + 2\sinh(x) - (bx + c)$

$e(x_0) = (1/3) + 2\sinh(x_0) - (bx_0 + c)$

$x_0 = -3$

$e(x_0) = (1/3) + 2\sinh(-3) - (-3b + c)$

$|||^{ly}$

$e(x_3) = (1/3) + 2\sinh(3) - (3b + c)$

Putting $e(x_0)$ and $e(x_3)$ in (1.1)

$\cancel{1/3 + 2\sinh(-3) + 3b - c = -1/3 - 2\sinh(3) + 3b + c}$

$(1/3) + 2\sinh(-3) + \cancel{3b} - c = -1/3 - 2\sinh(3) + \cancel{3b} + c$

$2/3 + \underbrace{\cancel{2\sinh(-3) + 2\sinh(3)}}_{\text{(odd function)}} = 2c$

$$\boxed{c = 1/3}$$

$e(x_1) = f(x_1) - p(x_1)$

$= 1/3 + 2\sinh(x_1) - [bx_1 + 1/3]$

$e(x_1) = 2\sinh(x_1) - bx_1$

$\because$ x_1 is an extremum point

$\cancel{\frac{d\, e(x_1)}{dx_1}}$

$$\frac{d}{dx} e(x) \Big|_{x=x_1} = 0$$

$$\Rightarrow 2\cosh(x) - b \Big|_{x=x_1} = 0$$

$$\Rightarrow 2\cosh(x) = b \quad - (2)$$

And

$$e(x_0) = -e(x_1) \quad [\text{Eqn. } 1]$$

$$\Rightarrow 1/3 + 2\sinh(-3) - [-3b + 1/3] = 2\sinh(x_1) - bx_1$$

$$\cancel{1/3} + 2\sinh(-3) + 3b - \cancel{1/3} = 2\sinh(x) - bx_1$$

$$2\sinh(-3) + 3b - 2\sinh(x_1) + bx_1 = 0$$

$$2\sinh(-3) + 6\cosh(x_1) - 2\sinh(x_1) + 2x_1\cosh(x_1) = 0$$

Finding the roots of this equation, we get two possible values,

$$x_1 = -1.64 \text{ and } x_1 = 3$$

However, we need the error to be max.

$$\Rightarrow \frac{d^2 e(x)}{dx}\Big|_{x=x_1} < 0 \Rightarrow x_1 = -1.64$$

Substituting this in Eqn. (2) we get

$$b = 2\cos(-1.64) = 5.35$$

Thus, the polynomial we get is

$$\boxed{p(x) = 5.35x + 1/3}$$

$$\|e(x)\|_\infty = \max_{a \le x \le b} |e(x)|$$

$\therefore$ by definition, L_∞ has to be at one of the end points, we select $x = 3$

$$L_\infty(e(x)) = |e(3)| = |f(3) - p(3)|$$

$$= \left| \tfrac{1}{3} + 2\sinh(3) - 5.35 \times 3 - \tfrac{1}{3} \right|$$

$$= 3.9857$$

For L_2 norm

$$\|e(x)\|_2 = \sqrt{\int_{-3}^{3} |e(x)|^2 \, dx}$$

$$= \sqrt{\int_{-3}^{3} \left| \tfrac{1}{3} + 2\sinh(x) - 5.35x - \tfrac{1}{3} \right|^2 dx}$$

$$= \sqrt{42.7086} = 6.53 \text{ Ans}$$

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1) (d) We know that we need to project the func^n on orthonormal bases. Orthonormal bases are found using the recurrence:

$p_0(x) = 1 \qquad p_{(-1)}(x) = 0$

and $p_{i+1}(x) = \left[x - \frac{\langle x p_i, p_i \rangle}{\langle p_i, p_i \rangle} \right] p_i(x) - \frac{\langle p_i, p_i \rangle}{\langle p_{i-1}, p_{i-1} \rangle} p_{i-1}(x)$

$$p_1(x) = \left[x - \frac{\langle x p_0, p_0 \rangle}{\langle p_0, p_0 \rangle} \right] p_0(x)$$

$$= \left[x - \frac{\int_{-3}^{3} x \, dx}{\int_{-3}^{3} 1 \, dx} \right] 1 = x$$

$$p_2(x) = \left[x - \frac{\langle x p_1, p_1 \rangle}{\langle p_1, p_1 \rangle} \right] p_1(x) - \frac{\langle p_1, p_1 \rangle}{\langle p_0, p_0 \rangle} p_0(x)$$

$$= \left[x - \frac{\int_{-3}^{3} x^3 \, dx}{\int_{-3}^{3} x^2 \, dx} \right] x - \frac{\int_{-3}^{3} x^2 \, dx}{\int_{-3}^{3} dx} \cdot 1$$

$$= [x - 0] \, x - \frac{18}{6} \cdot 1 = x^2 - 3$$

Now, $1/3 + 2\sinh(x)$

$$= \frac{1}{3} + 2\left(\frac{e^x - e^{-x}}{2} \right)$$

$$= 1/3 + e^x - e^{-x}$$

$$\langle p_0, f \rangle = \int_{-3}^{3} \left(\frac{1}{3} + e^x - e^{-x} \right) 1 \, dx$$

$$= 2$$

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$$\langle P_1, f \rangle = \int_{-3}^{3} \frac{x}{3} + xe^{+x} - xe^{-x} \, dx$$

$$= 80.74$$

$$\langle P_2, f \rangle = \int_{-3}^{3} (x^2-3)\left(1/3 + e^{x} - e^{-x}\right) dx = 0.$$

The polynomial, therefore becomes,

$$\frac{2}{6} + \frac{80.74}{18} x + 0x^2$$

$$\boxed{1/3 + \frac{80.74}{18} x}$$ Ans

Q. 3

Given: $T_{n+1}(x) = 2x T_n - T_{n-1}(x)$

$T_n(\cos\theta) = \cos(n\theta)$ — 2

It can be seen that $T_0 = 1$ and $T_1 = x$ [from eqn. 2]

Now, we proceed on deriving T_3 and T_4.

First, we derive the expression for T_3.

~~$T_2 = 2 \times 2 \times T_1$~~

$$T_2(x) = 2x . T_1(x) - T_0(x)$$
$$= 2x^2 - 1$$
$$T_3(x) = 2x T_2(x) - T_1(x)$$
$$= 2x[2x^2 - 1] - x$$
$$= 4x^3 - 3x$$

Substituting $x = \cos\theta$

$$T_3(\cos\theta) = 4\cos^3\theta - 3\cos\theta$$
$$= 2\cos\theta[2\cos^2 - 1] - \cos\theta$$
$$= 2\cos\theta \cos 2\theta - \cos\theta$$
$$= \cos\theta\cos 2\theta + \cos\theta\cos 2\theta - \cos\theta$$
$$= \cos\theta\cos 2\theta + \cos\theta(\cancel{1 - \cos 2\theta}\ \cos 2\theta - 1)$$
$$= \cos\theta\cos 2\theta + \cos\theta(\cancel{\cos^2\theta} - \sin^2\theta - \sin^2\theta - \cancel{\cos^2\theta})$$
$$= \cos\theta\cos 2\theta + (-1)2\cos\theta\sin^2\theta$$
$$= \cos\theta\cos 2\theta - 2\cos\theta . \cos\theta\sin\theta$$
$$= \cos\theta\cos 2\theta - \cos\theta \sin 2\theta$$
$$= \cos 3\theta.$$

Proved.

Now, we calculate $T_4(x)$

$$T_4(x) = 2x T_3(x) - T_2(x)$$
$$= 2x(4x^3 - 3x) - (2x^2 - 1)$$
$$= 8x^4 - 8x^2 + 1$$

Again substituting $x = \cos\theta$, we get

$$T_4 = 8\cos^4\theta - 8\cos^2\theta + 1$$
$$= 8\cos^2\theta(\cos^2\theta - 1) + 1$$
$$= 8\cos^2\theta(\cancel{\cos^2\theta} - \sin^2\theta - \cancel{\cos^2\theta}) + 1$$
$$= 1 - 8\cos^2\theta\sin^2\theta$$
$$= 1 - 4\cos\theta\sin\theta . 2\cos\theta\sin\theta$$
$$= 1 - 4\cos\theta\sin\theta\sin 2\theta$$
$$= 1 - 2.\cos\theta\sin\theta . 2\sin 2\theta$$
$$= 1 - 2\sin^2 2\theta$$
$$= \cos^2 2\theta + \sin^2 2\theta - 2\sin^2 2\theta$$
$$= \cos^2 2\theta - \sin^2 2\theta$$
$$= \cos 4\theta.$$

(b) $\langle g, h \rangle = \int_{-1}^{1} (1-x)^{-1/2} g(x)\, h(x)\, dx$

We substitute $g(x) = T_3(x)$ and $h(x) = T_4(x)$

$\langle g, h \rangle = \int_{-1}^{1} (1-x)^{-1/2} T_3(x)\, T_4(x)\, dx$

$\because x \in [-1, 1]$ in the integral, we substitute $x = \cos\theta$.

$$\langle g, h \rangle = \int_{-\pi}^{0} \frac{1}{\sqrt{1-\cos^2\theta}} T_3(\cos\theta) \cdot T_4(\cos\theta)\, d(\cos\theta)$$

(The limits are $[-\pi, 0]$ and not $[2\pi, \pi]$ because $\sqrt{1-\cos^2\theta}$ needs to be +ve).

$$\langle g, h \rangle = \int_{\pi}^{0} \frac{1}{\sin\theta} \cos 3\theta \cos 4\theta \sin\theta\, d\theta$$

$$= \int_{\pi}^{0} \cos 3\theta \cos 4\theta\, d\theta.$$

Let us have a closer look at a more general form of this expression

$$I = \int_{\pi}^{0} \cos(m\theta) \cos(n\theta)\, d\theta$$

We know that $\cos A \cos B = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right)$

Using this identity in the expression we get

$$I = \frac{1}{2}\int_{-\pi}^{0} \cos(m+n)\theta\, d\theta + \frac{1}{2}\int_{-\pi}^{0} \cos(m-n)\theta\, d\theta$$

* Now, $(m+n)$ and $(m-n)$ can be either even, or odd.

If $\int_{-\pi}^{0} \cos A\alpha\, d\alpha$ is calculated when A is even, the result is 0 [multiple of period]

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III^{rly}, if $\int_{-\pi}^{\pi} \cos A\alpha \, d\alpha$ is calculated if A is odd, we will have (A-1) full periods, where the I value will cancel each other and one half-period remaining, which is in effect equal to

$$I = \int_{-\pi}^{0} \cos\theta \, d\theta = 0.$$

$\therefore$ if $m \neq n$ $\int_{-\pi}^{\pi} \cos m\theta \cos n\theta \, d\theta = 0.$

$\Rightarrow$ $\langle g, h \rangle$ when $g = T_3(x)$ and $h = T_4(x)$ when $x \in [-1, 1]$ $= 0.$ Ans.

) As mentioned in the previous part $\langle g, g \rangle$ where $g = T_n(x)$ with $\langle g, h \rangle = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} g(x) h(x)$ is equal to

$$\int_{-\pi}^{0} \cos^2 n\theta \, d\theta = \frac{1}{2}\int_{-\pi}^{0} \cos 2n\theta \, d\theta + \frac{1}{2}\int_{-\pi}^{0} \cos 0\theta \, d\theta$$

$$= 0 + \frac{1}{2}\int_{-\pi}^{0} d\theta$$

$$= 0 + \pi/2.$$

$$= \pi/2$$ Ans.

(d) As mentioned in part (a), (b) and (c) if $\langle g, h \rangle$ is defined as

$$\langle g, h \rangle = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} g(x)\, h(x)\, dx$$

Then in the range of integral limits $[-1, 1]$

~~$\langle g, h \rangle = \int \cos$~~

and $g = T_n(x)$ and $h = T_m(x)$

$$\langle g, h \rangle = \int_{\pi}^{0} \cos m\theta \cos n\theta \, d\theta .$$

Also, $\int_{\pi}^{0} \cos m\theta \cos n\theta = 0$ if $m \neq n$ and $\pi/2$ if $m = n$.

$\Rightarrow \langle T_i, T_j \rangle = 0 \quad \forall\, i, j$ if $i \neq j$. Ans.