16-811 Math Fundamentals for Robotics
Assignment 1

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1 Collaborations

1.1 Question 5

For Question 5, I discussed the solution approach I found online at https://igl.ethz.ch/projects/ARAP/svd$_{r}$*ot.pdf* with Mayank Gupta and Shamit Lal.
I also used this approach in my solution to Question 5. I had a discussion with Shamit Lal and Mayank Gupta about ways to verify the correctness of the codes written, at the end of which we decided to use random matrices and angles for Question 5, to verify the code correctness of the code.

2 Instructions for Running code

All codes are written in MATLAB.

2.1 Question 1

For question 1, at the start of the code, I declare a variable N, which is a randomly generated integer. This is used to create a random matrix A of size $N \times N$. I then call the function *LDUDecompose* which return permutation matrix P, Lower Triangular matrix L, Upper Triangular Matrix U and diagonal matrix D. I additionally, generate an error matrix, showing the difference in values of $P \times A$ and $L \times D \times U$. A scalar value which is the sum of all values of the above mentioned matrix is also calculated to show the code correctness.

The evaluator can change the matrix A as per his/her convenience to test the code.

2.2 Question 2

For question 2, I have taken three different matrices mentioned in the question and calculated the SVD for each matrix. The evaluator can change the matrices accordingly.

2.3 Question 3

For question 1, at the start of the code, I declare two variables $A1$ and $b1$, which can be changed by the evaluator. I then call the function *findSolution* which returns the solution vector($ansA1$) for the given equation $A \times ansA1 = b1$. The ouput of the function is the solution while the nature of the solution (Unique, SVD solution, etc.) is printed on the command window. The command window also prints the error between the true value $b1$ and the value obtained after the operation $A \times ansA1$.

2.4 Question 5

For question 1, at the start of the code, I declare three variables generated randomly corresponding to the rotation and N points generated randomly. These points are rotated and further translated by a randomly generated vector $translationGiven$, which can be changed by the evaluator. The code then verifies that $translationGiven$ and rotation matrix is retrieved using the algorithm explained in the write-up. The error between values of the rotation matrix and the translation vector is printed on the command line.

Q1 .

A =

0.1273	0.9500	0.5615	0.7891	0.9958	0.0967	0.3766
0.2493	0.1750	0.0357	0.7519	0.7196	0.2910	0.8627
0.9769	0.6529	0.9475	0.5272	0.3418	0.1596	0.2831
0.6080	0.9292	0.7037	0.5902	0.6902	0.5527	0.3726
0.2765	0.9323	0.9619	0.1631	0.3234	0.8952	0.8467
0.2057	0.9384	0.8746	0.7512	0.6974	0.8942	0.4109
0.2472	0.1251	0.8377	0.7673	0.8640	0.2085	0.0915

Result :

P =

1	0	0	0	0	0	0
0	1	0	0	0	0	0
0	0	1	0	0	0	0
0	0	0	1	0	0	0
0	0	0	0	1	0	0
0	0	0	0	0	1	0
0	0	0	0	0	0	1

L =

1.0000	0	0	0	0	0	0
1.9589	1.0000	0	0	0	0	0
7.6758	3.9381	1.0000	0	0	0	0
4.7767	2.1405	0.3615	1.0000	0	0	0
2.1727	0.6713	0.5509	-0.5006	1.0000	0	0
1.6159	0.3540	0.4153	-1.2366	-1.9764	1.0000	0
1.9421	1.0202	1.0055	-4.0294	7.0015	-1.8127	1.0000

D =

0.1273	0	0	0	0	0	0
0	-1.6858	0	0	0	0	0
0	0	0.8283	0	0	0	0
0	0	0	-0.6109	0	0	0
0	0	0	0	0.0657	0	0
0	0	0	0	0	3.9084	0
0	0	0	0	0	0	-4.0181

U =

$$
\begin{matrix}
1.0000 & 7.4641 & 4.4118 & 6.1996 & 7.8243 & 0.7595 & 2.9591 \\
0 & 1.0000 & 0.6312 & 0.4708 & 0.7302 & -0.0603 & -0.0742 \\
0 & 0 & 1.0000 & -2.9020 & -2.9626 & -1.1865 & -3.7425 \\
0 & 0.0000 & 0 & 1.0000 & 0.8910 & -0.3743 & 0.9385 \\
0 & 0 & 0.0000 & 0 & 1.0000 & 19.3657 & 20.7734 \\
0 & 0 & 0 & 0 & 0 & 1.0000 & 0.7765 \\
0 & 0 & 0 & 0 & 0 & 0 & 1.0000
\end{matrix}
$$

Q2 (a) $A_1 = \begin{bmatrix} 10 & 9 & 2 \\ 5 & 3 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

Initialize L with $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Pivot 1 : $A_1(1,1)$

Step 1 : Subtract $(1/2) R_1$ from R_2 i.e.

$R_2 \leftarrow R_2 - (1/2) R_1$

$$A_1 = \begin{bmatrix} 10 & 9 & 2 \\ 5 - \frac{10}{2} & 3 - \frac{9}{2} & 1 - \frac{2}{2} \\ 2 & 2 & 2 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 10 & 9 & 2 \\ 0 & -3/2 & 0 \\ 2 & 2 & 2 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 2 : Subtract $(1/5) R_1$ from R_3 i.e.

$R_3 \leftarrow R_3 - (1/5) R_1$

$$A_1 = \begin{bmatrix} 10 & 9 & 2 \\ 0 & -3/2 & 0 \\ 2 - \frac{10}{5} & 2 - \frac{9}{5} & 2 - \frac{2}{5} \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 10 & 9 & 2 \\ 0 & -3/2 & 0 \\ 0 & 1/5 & 8/5 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/5 & 0 & 1 \end{bmatrix}$$

Pivot 2 : $A_1(2,2)$

Step 1 : Add $(2/15) R_2$ to R_3

$R_3 \leftarrow R_3 + (2/15) R_2$

$$A_1 = \begin{bmatrix} 10 & 9 & 2 \\ 0 & -3/2 & 0 \\ 0 & 0 & 8/5 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/5 & -2/15 & 1 \end{bmatrix}$$

Thus, $U \cancel{A} = \begin{bmatrix} 10 & 9 & 2 \\ 0 & -3/2 & 0 \\ 0 & 0 & 8/5 \end{bmatrix}$

Now, we decompose U into DU

$$D = \begin{bmatrix} 10 & 0 & 0 \\ 0 & -3/2 & 0 \\ 0 & 0 & 8/5 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 9/10 & 2/10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the LDU decomposition becomes

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/5 & -2/15 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 10 & 0 & 0 \\ 0 & -3/2 & 0 \\ 0 & 0 & 8/5 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 9/10 & 2/10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

Q.2 (b) $A_2 = \begin{bmatrix} 16 & 16 & 0 & 0 \\ 4 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

Initialize L as $L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Pivot 1 : $\boxed{A_2(1,1)}$

Step 1 : Subtract (1/4) R_1 from R_2, i.e.

$R_2 \leftarrow R_2 - (1/4) R_1$

$$A_2 = \begin{bmatrix} 16 & 16 & 0 & 0 \\ 4-\frac{16}{4} & 0-\frac{16}{4} & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 16 & 0 & 0 \\ 0 & -4 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Pivot 2 : $A_2(2,2)$

Step 1 : ~~Subtract~~ Add (1/4) R_2 to R_3, i.e.

$R_3 \leftarrow R_3 + R_2/4$

$$A_2 = \begin{bmatrix} 16 & 16 & 0 & 0 \\ 0 & -4 & -2 & 0 \\ 0 & 1+\frac{-4}{4} & -1+\frac{-2}{4} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 16 & 0 & 0 \\ 0 & -4 & -2 & 0 \\ 0 & 0 & -3/2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 & 0 \\ 0 & -1/4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Pivot 3 $A_2(3,3)$

Step 1 : Add $(2/3)\, R_3$ to R_5, i.e.

$$R_5 \leftarrow R_5 + (2/3)\, R_3$$

$$A_2 = \begin{bmatrix} 16 & 16 & 0 & 0 \\ 0 & -4 & -2 & 0 \\ 0 & 0 & -3/2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 & 0 \\ 0 & -1/4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2/3 & 0 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 16 & 16 & 0 & 0 \\ 0 & -4 & -2 & 0 \\ 0 & 0 & -3/2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 & 0 \\ 0 & -1/4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2/3 & 0 & 1 \end{bmatrix}$$

Now, U can be further decomposed into a diagonal component with D as, where we assign elements of leading diagonal of U to D.

$$D = \begin{bmatrix} 16 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & -3/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The new U, therefore comes out to be:

$$U = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The decomposition thus becomes,

$$\begin{bmatrix} 16 & 16 & 0 & 0 \\ 4 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 & 0 \\ 0 & -1/4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2/3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 16 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & -3/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times$$

$$A \quad = \quad L \quad . \quad D$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$U$$

Q.2 (c) $A_3 = \begin{bmatrix} 10 & 6 & 4 \\ 5 & 3 & 2 \\ 1 & 1 & 0 \end{bmatrix}$ $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Pivot 1 $A_3(1,1)$

Step 1 : Subtract $(1/2) R_1$ from R_2, i.e.

$$R_2 \leftarrow R_2 - (1/2) R_1$$

$$A = \begin{bmatrix} 10 & 6 & 4 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since, the second row is now [0, 0, 0] we need to interchange the 2^{nd} and 3^{rd} row. Thus, the permutation matrix becomes

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 10 & 6 & 4 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}$$

Step 2 : Remove $(1/10) R_1$ from R_2, i.e.

$$R_2 \leftarrow R_2 - (1/10) R_1$$

$$A = \begin{bmatrix} 10 & 6 & 4 \\ 0 & 1-\frac{6}{10} & 0-4/10 \\ 0 & 0 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 1/10 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 10 & 6 & 4 \\ 0 & 4/10 & -4/10 \\ 0 & 0 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 1/10 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 10 & 6 & 4 \\ 0 & 4/10 & -4/10 \\ 0 & 0 & 0 \end{bmatrix} = D \begin{bmatrix} 1 & 6/10 & 4/10 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

where

$$D = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 4/10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus,

$$PA = LDU$$

$$\underset{P}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 10 & 6 & 4 \\ 5 & 3 & 2 \\ 1 & 1 & 0 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 1/10 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 10 & 0 & 0 \\ 0 & 4/10 & 0 \\ 0 & 0 & 0 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & .6 & .4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}}$$

Q2. (SVD) calculations:

(a)

A =

10	9	2
5	3	1
2	2	2

U =

-0.8991	0.1788	0.3997
-0.3861	0.1066	-0.9163
-0.2064	-0.9781	-0.0268

S =

15.1186	0	0
0	1.5362	0
0	0	1.0334

V =

-0.7497	0.2376	-0.6177
-0.6391	-0.0176	0.7689
-0.1718	-0.9712	-0.1651

(b)

A =

16	16	0	0
4	0	-2	0
0	1	-1	0
0	0	0	1
0	0	1	1

U =

-0.9914	0.1250	-0.0030	0.0268	-0.0265
-0.1269	-0.9728	0.1589	-0.0326	0.1060
-0.0311	-0.0136	-0.5399	-0.7264	0.4239
0.0000	0.0173	0.3936	-0.6638	-0.6358
0.0005	0.1937	0.7269	-0.1729	0.6358

S =

22.8186	0	0	0
0	3.4932	0	0
0	0	1.6873	0
0	0	0	1.1227
0	0	0	0

V =

-0.7174	-0.5414	0.3482	0.2664
-0.6965	0.5687	-0.3485	-0.2645
0.0125	0.6163	0.5624	0.5511
0.0000	0.0604	0.6641	-0.7452

(c)

A =

10	6	4
5	3	2
1	1	0

U =

-0.8905	-0.0840	-0.4472
-0.4452	-0.0420	0.8944
-0.0939	0.9956	0.0000

S =

13.8451	0	0
0	0.5595	0
0	0	0.0000

V =

-0.8107	-0.0967	0.5774
-0.4892	0.6538	-0.5774
-0.3216	-0.7505	-0.5774

Q4 $u \to$ n dimensional column vector in $\mathbb{R}^n$ $\|u\| = 1$

$A = I - uu^T$

(a) To look at geometric interpretaⁿ, we look at two specific values for special vectors

i) We look at the product Au

$$Au = (I - uu^T)u$$
$$= Iu - u\underbrace{u^Tu}_{1}$$
$$= Iu - u$$
$$= Iu - Iu$$
$$= 0$$

ii) We look at product Av, where $v \perp^r u$, i.e. $u^Tv = 0$

$$Av = (I - uu^T)v$$
$$= Iv - u\underbrace{u^Tv}_{0}$$
$$= Iv - 0 = v$$

We see that the product is 0 for u and 1 for v.

Now, for any generic vector(α) in the column space,

$$\alpha = a_1\bar{u} + a_2\bar{v} + a_3\bar{z} + a_4\bar{b} + a_5\bar{c} + \cdots$$

Where $\bar{b}, \bar{c}, \bar{u}, \bar{v}, \ldots$ are the orthonormal biases which span the column space $\mathbb{R}^n$.

$$A\alpha = A(a_1\bar{u} + a_2\bar{v} + a_3\bar{z} + a_4\bar{b} + a_5\bar{c} + \cdots)$$
$$= a_1A\bar{u} + a_2A\bar{v} + a_3A\bar{z} + a_4A\bar{b} + a_5A\bar{c} + \cdots$$
$$= 0 + a_2\bar{v} + a_3\bar{z} + a_4\bar{b} + a_5\bar{c} + \cdots$$

Thus, geometrically, Au will give us the components of a vector α in column space $\mathbb{R}^n$ which are not $\bar{u}$, i.e. all except the u^{th} component in α

(b) Using information in the previous part, we can see that for all vectors $\bar{v}$, such that $\bar{u}.\bar{v} = 0$, $\boxed{(I - uu^T) v = v}$

$\Rightarrow$ We will have $(n-1)$ Eigen values as 1 {for the $(n-1)$ biases in $\mathbb{R}^n$, except u}

For the remaining Eigen value, we know that $(I - uu^T) x = \lambda x$ for $x \neq 0$ is 0 at $x = u$

$$\Rightarrow (I - uu^T) x = \underline{0} x$$

Thus, the remaining Eigen value is 0.

The set of Eigen values becomes $= \{0, \underbrace{1, 1, 1, \ldots 1}_{N-1 \text{ times.}}\}$

(c) Using the calculation in part (a) it has already been established that:

$$Au = 0$$

$\Rightarrow$ the null space of A is $\boxed{\underset{\downarrow}{c}u}$.

scalar constant

(d) $A^2 = (I - uu^T)(I - uu^T)$

$\because$ matrices are distributive

$$A^2 = (I - uu^T) I - (I - uu^T) uu^T$$
$$= I (I - uu^T) - uu^T (I - uu^T)$$
$$= I - Iuu^T - uu^T I + uu^T uu^T$$
$$= I - uu^T - uu^T I + uu^T$$
$$= I - uu^T - \cancel{uu^T I} + \cancel{uu^T I}$$
$$= I - uu^T$$
$$= A$$

$\therefore \boxed{A^2 = A}$

Q5 Given: Points $P = \{p_1, \dots p_n\}$ rigidly transformed to points $Q = \{q_1, \dots q_n\}$.

Calculate: The translaⁿ vector and rotation matrix used to achieve the transformation from $\{p_1, \dots, p_n\}$ to $\{q_1, \dots, q_n\}$

Solnⁿ: For an individual point, this problem is stated as:

$$\cancel{p_i = R}$$

$$q_i = Rp_i + \bar{t}$$

$$\Rightarrow \bar{t} = q_i - Rp_i \quad -(1)$$

$\because$ we know that there is no noise in the data (Given), the translation can be calculated using Eqn. (1) with the centroids of $p_i \; \forall i = \{1 \dots n\}$ and $q_i \; \forall i \in \{1, \dots, n\}$ in place of individual point.

$$\Rightarrow \boxed{\bar{t} = \frac{\sum_{i=1}^{n} q_i}{n} - \frac{\sum_{i=1}^{n} p_i}{n} \quad -(2)}$$

Now, we need to calculate R, in order to calculate $\bar{t}$.

For rotaⁿ, we can shift the centroids of both sets of points $\{P$ and $Q\}$ to the origin without any effect on rotation.

To accomplish this, we subtract each set of points with their centroids, i.e.

$$x_i = p_i - \frac{\sum_{i=1}^{n} p_i}{n} \qquad y_i = q_i - \frac{\sum_{i=1}^{n} q_i}{n}$$

The rotation equation, now becomes

$$y_i = Rx_i \quad \forall i \in \{1, 2, \ldots, n\}$$

which can be re-stated as a least-squares soln. Mathematically, we need to find R such that

$$R = \underset{R}{\operatorname{argmin}} \left(\sum_{i=1}^{n} \| Rx_i - y_i \| \right) \quad - (3)$$

We simplify the expression in (3)

$$\cancel{\sum \| Rx_i - y_i \|}$$

$$\| Rx_i - y_i \|^2 = (Rx_i - y_i)^T (Rx_i - y_i)$$

$$= ((Rx_i)^T - y_i^T)(Rx_i - y_i)$$

$$= (x_i^T R^T - y_i^T)(Rx_i - y_i)$$

$$= x_i^T \underbrace{R^T R}_{I} x_i - x_i^T R^T y_i - y_i^T R x_i + y_i^T y_i$$

$$\| Rx_i - y_i \|^2 = x_i^T x_i - \underbrace{\underset{(1\times n)}{x_i^T} \underset{(n\times n)}{R^T} \underset{(n\times 1)}{y_i}}_{(1\times 1)\ \{\text{Scalar}\}} - y_i^T R x_i + y_i^T y_i \quad - (4)$$

As seen in Eqn. (4) $x_i^T R^T y_i$ is a scalar

$$\Rightarrow (x_i^T R^T y_i)^T = x_i^T R^T y_i$$

$$\Rightarrow y_i^T R x_i = x_i^T R^T y_i \quad - (5)$$

Substituting Eqn. (5) in Eqn. (4) we get

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$$\|Rx_i - y_i\|^2 = x_i^T x_i - 2y_i^T Rx_i + y_i^T y_i \quad -(6)$$

As can be seen from Eqn. (6), the only term dependent on R is "$-2y_i^T R x_i$".

$\therefore$ Eqn. (3) can be written as

$$R = \underset{R}{\operatorname{argmin}} \left(-2\sum_{i=1}^{n} y_i^T R x_i\right)$$

$$R = \underset{R}{\operatorname{argmax}} \left(\sum_{i=1}^{n} y_i^T R x_i\right) \quad -(7)$$

Eqn. (7) shows we need R such that $\sum_{i=1}^{n} y_i^T R x_i$ is maximum.

Writing it in matrix form we get

$$\sum y_i^T R x_i = \underbrace{tr}_{\text{Trace}}(Y^T R X) \text{ where}$$

$$Y^T = \begin{bmatrix} - & y_1^T & - \\ - & y_2^T & - \\ & \vdots & \\ - & y_n^T & - \end{bmatrix} \qquad X = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix}$$

Now, $tr(AB) = tr(BA)$

$$\Rightarrow tr(\underbrace{Y^T}_{A} \underbrace{RX}_{B}) = tr(RXY^T) \quad -(8)$$

let $S = XY^T$. The SVD of S is given by $S = U\Sigma V^T$

Substituting SVD of S in (8), we get

$$tr(RXY^T) = tr(\underbrace{RU}_{A}\underbrace{\Sigma V^T}_{B}) = tr(\Sigma V^T R U) \quad -(9)$$

Let $V^T R U = M$. $\because V^T$ and U are the decomposition matrices of S, V^T and U are orthonormal.

$\because$ R is a rotation matrix, R is also orthonormal

$\Rightarrow$ M is an orthonormal matrix.

Also, using Eq ⑨ and Eqn ⑦ we can state that we need to ~~minimize~~ maximize $tr(\Sigma M)$ to get R.

$$tr(\Sigma M) = \sum_{i=1}^{3} \sigma_i m_{ii} \quad — (10)$$

If we need to maximize the expression in Eqn. (10) under the constraint that M is orthonormal,

$$m_{ii} = 1 \qquad m_{ij} = 0 \ \forall j \neq i.$$

$\Rightarrow$ M is Identity matrix

$$\Rightarrow M = I = V^T R U$$
$$\Rightarrow I = V^T R U$$
$$\Rightarrow V = V V^T R U$$
$$\Rightarrow V = R U$$
$$\Rightarrow V U^T = R U U^T$$
$$\Rightarrow \boxed{R = V U^T} \quad \text{Ans}$$

$V^T = V^{-1}$ & $U^T = U^{-1}$ due to SVD.

With rotation matrix available, translation can be calculated using Eqn. ①