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CS 66 Homework 4

① Compute the derivative $f'(x)$ for:

$$\cancel{f(x) = e - \frac{(x-\mu)^2}{2\sigma^2}}$$

$$f(x) = e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where μ and σ can be treated like constants

→

$$f(x) = e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Using chain rule,

$$f'(x) = e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot \left(-\frac{1}{2\sigma^2}\right) \cdot 2(x-\mu)$$

$$\therefore f'(x) = \left(-\frac{x-\mu}{\sigma^2}\right) \cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

② Compute the derivatives df/dx of the following functions. Describe your steps in detail.

(a)

$$f(z) = e^{-1/2 z}$$

$$z = g(y) - y^T S^{-1} y$$

$$y = h(x) = x - \mu$$

where $x, y, \mu \in \mathbb{R}^D$, $S \in \mathbb{R}^{D \times D}$

Use the chain rule. Provide the dimensions of every single partial derivate

→

$$f(z) = e^{-\frac{1}{2}z}$$

$$z = g(y) = y^T S^{-1} y$$

$$y = h(x) = x - \mu$$

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we need to find $\frac{df}{dx}$

$$\frac{df}{dz} = \frac{d}{dx}\left(e^{-\frac{1}{2}z}\right)$$
$$= \frac{1}{2} e^{\left(-\frac{1}{2}z\right)}$$

$$\frac{dz}{dy} = \frac{d}{dy}\left(y^T S^{-1} y\right)$$
$$= 2S^{-1}y$$

$\frac{dy}{dx} = \frac{d}{dx}(x-\mu) = I$, which is identity matrix.

∴ Using the chain rule:-

$$\frac{df}{dx} = \frac{df}{dz} \times \frac{dz}{dy} \times \frac{dy}{dx}$$

$$\therefore \frac{df}{dx} = \frac{1}{2} e^{-\frac{1}{2}z} \cdot 2S^{-1}y \cdot I$$

(b) $f(x) = tr\left(xx^T + \sigma^2 I\right)$, $x \in \mathbb{R}^D$. (Hint: xx^T is the outer product, so you perform the outer product operation ~~quickly~~ explicitly first to make it easier.)

→ $f(x) = tr\left(x \cdot x^T + \sigma^2 I\right)$

we need to find $\frac{df}{dx}$

$$\frac{df}{dx} = \frac{d}{dx}\left[tr(x \circ x^T + \sigma^2 I)\right]$$

$$= \frac{d}{dx}\left[tr(xx^T) + tr(\sigma^2 I)\right]$$

$$\therefore \frac{df}{dx} = 2x + 0 = 2x$$

(c) Use the chain rule. Provide the dimensions of every single partial derivative. You do not need to compute the product of the partial derivatives explicitly.

$$f = \sin(z) \in \mathbb{R}^M$$

$$z = Ax + b,\ x \in \mathbb{R}^N,\ A \in \mathbb{R}^{M \times N},\ b \in \mathbb{R}^M$$

Here 'sin' is applied to every component of z.

$\rightarrow$ $f = \sin(z)$

$z = Ax + b$

(x) is a vector with dimensions (N).

Using chain rule:

$$\frac{df}{dz} = \sin(z)$$

$$= \cos(z)$$

$$\frac{dz}{dx} = \frac{d}{dx}(z = Ax + b).$$

$$\frac{dz}{dx} = A$$

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$$\therefore \frac{df}{dx} = \frac{df}{dz} \times \frac{dz}{dx}$$

$$\frac{df}{dx} = \cos(z) \cdot A.$$

The dimension of $\left(\frac{df}{dx}\right)$ is $(M \times N)$, which matches the dimensions of (A).

(3) Consider the following functions:

$f(x) = \sin(x) \cos(x)$, $x \in R^2$ 112

$f_2(x, y) = x^T y$, $x, y \in R^n$

$f_3(x) = x x^T$, $x \in R^n$

(a) What are the dimensions of $\frac{df}{dx}$?

(b) Compute the Jacobians

→

(a) - For the function $f(x) = \sin(x) \cos(x)$, the derivate $\frac{df}{dx}$ will have the same dimensions as x, ~~which is~~ which is n.

- The function $f_2(x, y) = x^T y$, the derivate $\frac{df}{dx}$ will have dimensions $n \times n$ because it takes x and y as inputs and has an inner product operation.

- The function $f_3(x) = x x^T$, the derivative $\frac{df}{dx}$ will have

(a) dimension $n \times n$ because it involves an outer product of x with itself.

(b)

i) $f(x) = \sin(x)\cos(x)$

Using chain rule:

$$\frac{df}{dx} = \frac{d}{dx}[\sin(x)\cos(x)]$$

$$= \cos(x)\cos(x) - \sin(x)\sin(x)$$

$$\frac{df}{dx} = \cos^2(x) - \sin^2(x)$$

(ii) $f_2(x, y) = x^T y$

$$\frac{df_2}{dx} = \frac{d}{dx}(x^T y)$$

$$\frac{df_2}{dx} = y$$

(iii) $f_3(x) = xx^T$

$$\frac{df}{dx} = \frac{d}{dx}(xx^T)$$

$$\frac{df_3}{dx} = 2x$$

(4) Let $f(x) = e^x$.
(a) Find the maclaurin series for f.
(b) Use the result from (a) to find the Maclaurin series for $g(x) = xe^x$.

→

(a) $f(x) = e^x$

Using taylor series expansion for e^x centered at $x=0$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad -(i)$$

(b) Multiplying each term in eq (i) by x.

$$g(x) = x \cdot e^x.$$

$$= x \cdot \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\therefore g(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$