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CS660 Homework 2

① Consider $\mathbb{R}^3$ with the inner product

$$\langle x, y\rangle = x^T \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} y$$

Furthermore, we define e_1, e_2 and e_3 as the standard/ Canonical bases in $\mathbb{R}^3$.

(a) Determine the orthogonal projection $\pi_U(e_2)$ of e_2 onto $U = \text{span}[e_1, e_3]$.

(b) Compute the distance $d(e_2, U)$.

⇒ Given,

$$\langle x, y\rangle = x^T \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} y$$

$$\therefore \langle e_1, e_3\rangle = (1,0,0) \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= (1\,0\,0) \begin{pmatrix} 2\times 0 + 1\times 0 + 0\times 1 \\ 1\times 0 + 2\times 0 + (-1)\times 1 \\ 0\times 0 + (-1)\times 0 + 2\times 1 \end{pmatrix}$$

$$= (1\,0\,0) \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

$$= 1\times 0 + 0\times(-1) + 0\times 2$$

$$= 0.$$

Therefore e_1 and e_2 are orthogonal to each other.

$\rightarrow \langle e_1, e_1 \rangle = (1,0,0)\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$= (1\ 0\ 0)\begin{pmatrix} 2\times1+1\times0+0\times0 \\ 1\times1+2\times0+(-1)\times0 \\ 0\times1+(-1)\times0+2\times0 \end{pmatrix}$

$= (1\ 0\ 0)\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

$= 1\times2+0\times1+0\times0$

$= 2$

$\rightarrow \langle e_3, e_3 \rangle = (0,0,1)\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$= (0\ 0\ 1)\begin{pmatrix} 2\times0+1\times0+0\times1 \\ 1\times0+2\times0+(-1)\times1 \\ 0\times0+(-1)\times0+2\times1 \end{pmatrix}$

$= (0\ 0\ 1)\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$

$= 0\times0+(-1)\times0+1\times2$

$= 2.$

Therefore $\left\{\frac{1}{\sqrt{2}}e_1, \frac{1}{\sqrt{2}}e_2\right\}$ are orthonormal

and $U = \text{Span}\{e_1, e_2\} = \text{Span}\left\{\frac{1}{\sqrt{2}}e_1, \frac{1}{\sqrt{2}}e_2\right\}$

a) Projection of e_2 on U:-

$$\pi_U(e_2) = \langle e_2, \tfrac{1}{\sqrt{2}} e_1 \rangle \tfrac{1}{\sqrt{2}} e_1 + \langle e_2, \tfrac{1}{\sqrt{2}} e_3 \rangle \tfrac{1}{\sqrt{2}} e_3$$

$$= \frac{1}{2} \langle e_2, e_1 \rangle e_1 + \frac{1}{2} \langle e_2, e_3 \rangle e_3$$

Now,

$$\langle e_2, e_1 \rangle = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$= 0 \times 2 + 1 \times 1 + 0 \times 0$$

$$= 1$$

$$\langle e_2, e_3 \rangle = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

$$= 0 \times 0 + 1 \times (-1) + 0 \times 2$$

$$= -1.$$

Therefore,

$$\pi_U(e_2) = \frac{1}{2} \left[1 e_1 - e_3 \right]$$

$$= \frac{1}{2} (1, 0, -1)$$

b) Required distance $d(e_2, u)$ = distance between e_2 and $\pi_u(e_2)$

$$= \|e_2 - \pi_u(e_2)\|$$

$$= \|(0,1,0) - \frac{1}{2}(1,0,-1)\|$$

$$= \|(-\frac{1}{2}, 1, \frac{1}{2})\|$$

$$= \sqrt{\langle(-1/2, 1, 1/2), (-1/2, 1, 1/2)\rangle}$$

Computing

$\langle(-1/2, 1, 1/2), (-1/2, 1, 1/2)\rangle$

$$= \begin{pmatrix} -1/2 & 1 & 1/2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} -1/2 \\ 1 \\ 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 & 1 & 1/2 \end{pmatrix} \begin{pmatrix} 2 \times -1/2 + 1 \times 1 + 0 \times 1/2 \\ 1 \times -1/2 + 2 \times 1 + (-1) \times 1/2 \\ 0 \times -1/2 + (-1) \times 1) + 2 \times 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 & 1 & 1/2 \end{pmatrix} \begin{pmatrix} -1 + 1 + 0 \\ -1/2 + 2 - 1/2 \\ 0 - 1 + 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 & 1 & 1/2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= 0 \times -1/2 + 1 \times 1 + (1/2) \times 0$$

$$= 1$$

Therefore the required distance is $= \sqrt{\langle(-1/2, 1, 1/2), (-1/2, 1, 1/2)\rangle}$
$= \sqrt{1}$
$= 1$

(2) Let W be the sequence of $\mathbb{R}^3$ spanned by $u_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 7 \\ -2 \\ 11 \end{bmatrix}$.

(i.e, $W = \text{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 7 \\ -2 \\ 11 \end{pmatrix}\right\}$).

Find the vector(s) than span(s) the orthogonal complement $W^{\perp}$ of W.

$\Rightarrow$ We need to find vectors that are orthogonal to both u_1 and u_2.

Let $v = (x, y, z)$ be a vector in $W^{\perp}$.
To be orthonogonal component, 'v' must satisfy
$\langle u_1, v\rangle = 0$ and $\langle u_2, v\rangle = 0$.
$\therefore$ $x + 2y + 3z = 0$ — (i)
$7x - 2y + 11z = 0$ — (ii)

$\langle u, v\rangle = ax + by + cz$ in $\mathbb{R}^3$,
where $u = \{a, b, c\}$ and $v = \{x, y, z\}$.

Now,
Solving equations (i) and (ii)

$$\begin{aligned} x + \cancel{2y} + 3z &= 0 \\ 7x - \cancel{2y} + 11z &= 0 \\ \hline 8x + 14z &= 0. \end{aligned}$$

$$z = -\frac{4}{7}x$$

Simplifying ~~x~~ in eq (i)

$$x + 2y + 3\left(\frac{-4}{7}\right)x = 0.$$

$$x - \frac{12x}{7} + 2y = 0.$$

$$-\frac{5x}{7} + 2y = 0.$$

$$y = \frac{5}{14}x.$$

So, any vector of the form $v = \{x, \frac{5}{14}x, -\frac{4}{7}x\}$ will be orthogonal to both u_1 and u_2 and thus span the orthogonal complement $W^\perp$ of W.

We can choose any nonzero value for x to obtain a vector in $W^\perp$.

If $x = 14$, then $v = [14, 5, -8]$ span $W^\perp$.

$\therefore$ One vector that spans $W^\perp$ is $[14, 5, -8]$

(3) Let W be the subspace of R^5 spanned by $u = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \\ 2 \end{bmatrix}$,

$v = \begin{bmatrix} 2 \\ 4 \\ 7 \\ 2 \\ -1 \end{bmatrix}$.

Find a basis of the orthogonal complement $W^{\perp}$ of W.

$\Rightarrow$ To discover $W^{\perp}$, 'x' should be orthogonal to both u and v, i.e ($u \cdot x = 0$) and ($v \cdot x = 0$),

where $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$

From ($u \cdot x = 0$):-

$[1x_1 + 2x_2 + 3x_3 - 1x_4 + 2x_5 = 0]$

$\therefore x_1 + 2x_2 + 3x_3 - x_4 + 2x_5 = 0$ — (i)

From condition ($v \cdot x = 0$):-

$2x_1 + 4x_2 + 7x_3 + 2x_4 - x_5 = 0$ — (ii)

from eq (i) : $x_1 = -2x_2 - 3x_3 + x_4 - 2x_5$

putting it in eq (ii)

$2[-2x_2 - 3x_3 + x_4 - 2x_5] + 4x_2 + 7x_3 + 2x_4 - x_5 = 0.$

$-4x_2 - 6x_3 + 2x_4 - 4x_5 + 4x_2 + 7x_3 + 2x_4 - x_5 = 0.$

$[(4x_2 - 4x_2) + (-6x_3 + 7x_3) + (2x_4 + 2x_4) + (-4x_5 - x_5) = 0]$

$\therefore x_3 + 4x_4 - 5x_5 = 0$ — (iii)

So, $x_3 = 5x_5 - 4x_4$

Substituting x_3 in x_1:

$$x_1 = -2x_2 - 3(5x_5 - 4x_4) + x_4 - 2x_5$$

$$x_1 = -2x_2 - 15x_5 + 12x_4 + x_4 - 2x_5$$

$$x_1 = -2x_2 + 13x_4 - 17x_5$$

$\therefore$ The arrangement of x is:

$$x = \begin{pmatrix} -2x_2 + 13x_4 - 17x_5 \\ x_2 \\ 5x_5 - 4x_4 \\ x_4 \\ x_5 \end{pmatrix}$$

$\rightarrow$ The pecemise vectors are :-

For $(x_2 = 1)$, $(x_4 = 0)$, $(x_5 = 0)$: $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

For $(x_2 = 0)$, $(x_4 = 1)$, $(x_5 = 0)$ = $\begin{pmatrix} 13 \\ 0 \\ -4 \\ 1 \\ 0 \end{pmatrix}$

For $(x_2 = 0)$, $(x_4 = 0)$, $(x_5 = 1)$ = $\begin{pmatrix} -17 \\ 0 \\ 5 \\ 0 \\ 1 \end{pmatrix}$

Hence, the ~~[illegible]~~ orthogonal component $W^{\perp}$ is:

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 13 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -17 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(4) Find the Orthonormal basis for the subspace U of $\mathbb{R}^4$ spanned by the vectors.

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 2 \\ -4 \\ -3 \end{bmatrix}$$

$\Rightarrow$ Let $u_1 = v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

then $u_2 = v_2 - \dfrac{\langle v_2 \cdot u_1 \rangle}{\langle u_1 \cdot u_1 \rangle} u_1$

$$= \begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} \times \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$\cancel{1\times1+1\times1+2\times1+4\times1}$

$$= \begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix} - \frac{8}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

$$u_3 = v_3 - \frac{\langle v_3 \cdot u_1 \rangle}{\langle u_1 \cdot u_1 \rangle} u_1 - \frac{\langle v_3 \cdot u_2 \rangle}{\langle u_2 \cdot u_2 \rangle} u_2$$

$$= \begin{bmatrix} 1 \\ 2 \\ -4 \\ -3 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 2 \\ -4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} * \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 2 \\ -4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ 0 \\ 2 \end{bmatrix}}{\begin{bmatrix} -1 \\ -1 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ 0 \\ 2 \end{bmatrix}} \times \begin{bmatrix} -1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ -4 \\ -3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} -1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 \\ 3/2 \\ -3 \\ 1 \end{bmatrix}$$

Finding unit vector $u_1 = \frac{u_1}{||u_1||}$ where $u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

and $||u_1|| = \sqrt{1^2 + 1^2 + 1^2 + 1^2} = 2$

Thus $u_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$

Finding unit vector $U_2 = \dfrac{u_2}{||u_2||}$ where

$$u_2 = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 2 \end{bmatrix} \text{ and } ||u_2|| = \sqrt{(-1)^2 + (-1)^2 + 0^2 + 2^2}$$
$$= \sqrt{6}$$

$$\therefore U_2 = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 0 \\ 2/\sqrt{6} \end{bmatrix}$$

Finding unit vector $U_3 = \dfrac{u_3}{||u_3||}$ where

$$u_3 = \begin{bmatrix} 1/2 \\ 3/2 \\ -3 \\ 1 \end{bmatrix} \text{ and } ||u_3|| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + (-3)^2 + 1^2}$$
$$= \frac{5}{\sqrt{2}}$$

$$\therefore U_3 = \begin{bmatrix} \sqrt{2}/10 \\ \frac{3\sqrt{2}}{10} \\ \frac{-3\sqrt{2}}{5} \\ \sqrt{2}/5 \end{bmatrix}$$

Therefore, orthonormal basis for U is

$$B = \{U_1, U_2, U_3\} = \left\{ \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 0 \\ 2/\sqrt{6} \end{bmatrix}, \begin{bmatrix} \sqrt{2}/10 \\ 3\sqrt{2}/10 \\ -3\sqrt{2}/5 \\ \sqrt{2}/5 \end{bmatrix} \right\}$$

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(5) Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{bmatrix}$

(a) Are the rows of A orthogonal? YES
(b) Is A an orthogonal matrix? NO
(c) Are the columns of A orthogonal? NO
Provide support for your answers.
$\Rightarrow$

(a) Formula for dot product:-
$(x,y,z) \cdot (m,n,p) = xm + yn + zp$
$A_1 = (1,1,-1)$, $A_2 = (1,3,4)$, $A_3 = (7,-5,2)$

$A_1 \cdot A_2 = (1,1,-1) \cdot (1,3,4)$
$= 1+3-4$
$= 0$

$A_2 \cdot A_3 = (1,3,4) \cdot (7,-5,2)$
$= 7-15+8$
$= 0$

$A_1 \cdot A_3 = (1,1,-1) \cdot (7,-5,2)$
$= 7-5-2$
$= 0$

$\therefore$ All the rows are orthogonal

(b) $A^T = \begin{bmatrix} 1 & 1 & 7 \\ 1 & 3 & -5 \\ -1 & 4 & 2 \end{bmatrix}$

finding product of A and A^T

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 7 \\ 1 & 3 & -5 \\ -1 & 4 & 2 \end{bmatrix} =$$

$$= \begin{bmatrix} 1\times1+1\times1+(-1)\times(-1) & 1\times1+1\times3+(-1)\times4 & 1\times7+1\times(-5)+(-1)\times2 \\ 1\times1+3\times1+4\times(-1) & 1\times1+3\times3+4\times4 & 1\times7+3\times(-5)+4\times2 \\ 7\times1+(-5)\times3+2\times4 & 7\times1+(-5)\times3+2\times4 & 7\times7+(-5)\times(-5)+2\times2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+1 & 1+3-4 & 7-5-2 \\ 1+3-4 & 1+9+16 & 7-15+8 \\ 7-15+8 & 7-15+8 & 49+25+4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 26 & 0 \\ 0 & 0 & 78 \end{bmatrix}$$

The matrix A is not an orthogonal matrix as it is not equal to identity matrix

(c) The column vectors are $A_{C1} = (1,1,7)$, $A_{C2} = (1,3,-5)$, $A_{C3} = (-1,4,2)$

$$A_{C1} \cdot A_{C2} = (1,1,7) \cdot (1,3,-5)$$
$$= 1+3-35$$
$$= -31$$

$$A_{C3} \cdot A_{C2} = (-1,4,2) \cdot (1,3,-5)$$
$$= -1+12-10$$
$$= 1$$

$$A_{C1} \cdot A_{C3} = (1,1,7) \cdot (-1,4,2)$$
$$= 1+4+14$$
$$= 17$$

$\therefore$ The columns are not orthogonal to each other