Lesson: Gram-Schmidt Orthogonalisation Process

Simplifying Linear Algebra Through Orthogonal Bases

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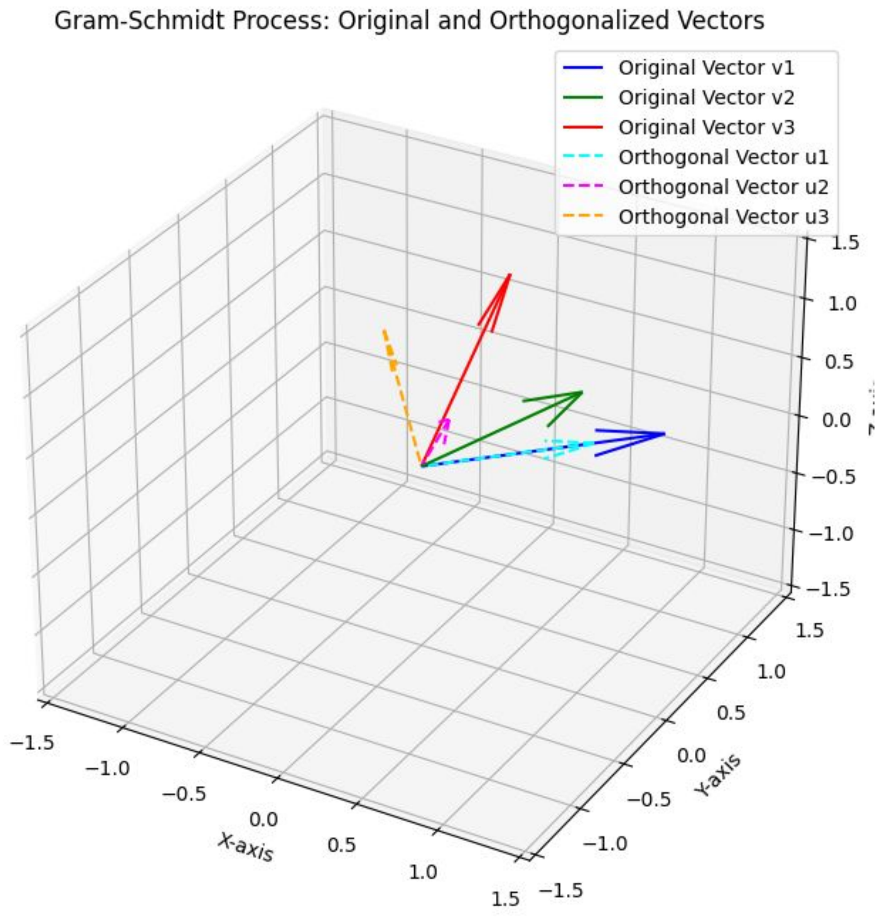
Introduction

What is the Gram-Schmidt Orthogonalisation Process?

The **Gram-Schmidt Orthogonalisation Process** is a method in linear algebra used to transform a set of linearly independent vectors into an orthogonal or orthonormal set of vectors while maintaining the span of the original set.

- A method to convert a given basis into an orthonormal basis.
- Used in various mathematical, engineering, and scientific applications.



Imagine you have a bunch of arrows (vectors) on a flat surface or in 3D space. These arrows are not necessarily at right angles to each other (they're not "orthogonal").

The Gram-Schmidt Process is a step-by-step method to straighten them out so that:

- All arrows point in unique directions but don't overlap or lean toward each other (they become "perpendicular" or orthogonal).
- You can also make each arrow a specific length, like 1 unit (this makes them "orthonormal").

Key Definitions

1. **Basis**: A set of vectors that span a vector space.
2. **Orthogonal Vectors:** A set of vectors is orthogonal if every pair of vectors in the set satisfies:

$$\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0 \quad \text{for } i \neq j$$

3. **Orthonormal Vectors:** Orthogonal vectors that are also normalized to have unit length

$$\|\mathbf{u}_i\| = 1$$

4. **Inner Product**: The "dot product" operation that helps in determining orthogonality.
5. **Orthonormal Basis**: A basis where vectors are orthogonal and of unit length.
6. **Projections:** The projection of a vector v onto another vector u is:

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}$$

Steps of Gram-Schmidt Process

Start with a basis $\{\beta 1,\beta 2,\ldots,\beta n\}\backslash\{\beta 1,\beta 2,\ldots,\beta n\}$.

1. **Step 1**: Orthogonalize the vectors by subtracting projections:
2. **Step 2**: Normalize the orthogonal vectors:

$$\alpha_k = \beta_k - \sum_{j=1}^{k-1} \frac{\langle \beta_k, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} \alpha_j$$

3. Repeat for all vectors in the set.

$$\tilde{\alpha_k} = \frac{\alpha_k}{\|\alpha_k\|}$$

Given Basis:

$$\beta_1=(1,1,0),\ \beta_2=(1,0,1),\ \beta_3=(0,1,1)$$

Step 1: Find $\alpha_1=\beta_1$.

Step 2: Orthogonalize β_2 to get α_2:

$$\alpha_2 = \beta_2 - \frac{\langle \beta_2, \alpha_1 \rangle}{\langle \alpha_1, \alpha_1 \rangle} \alpha_1$$

Step 3: Orthogonalize β_3 to get α_3:

$$\alpha_3 = \beta_3 - \frac{\langle \beta_3, \alpha_1 \rangle}{\langle \alpha_1, \alpha_1 \rangle} \alpha_1 - \frac{\langle \beta_3, \alpha_2 \rangle}{\langle \alpha_2, \alpha_2 \rangle} \alpha_2$$

Step 4: Normalize all $\boldsymbol{\alpha}$ vectors.

WHY IT IS USED?

The **Gram-Schmidt Orthogonalisation Process** is used because of its critical role in transforming a set of vectors into a more structured form that facilitates computations and analysis.

There are many main reasons why Gram-Schmidt Orthogonalisation Process is widely applied:-

The Gram-Schmidt Orthogonalisation Process is invaluable in simplifying vector operations, ensuring stability in computations, and serving as a cornerstone for advanced applications in mathematics, physics, and engineering.

1. To Create Orthogonal or Orthonormal Bases

- Orthogonal vectors are simpler to work with because the inner product (dot product) of any two distinct orthogonal vectors is zero.
- Orthonormal bases (orthogonal and of unit length) simplify many mathematical operations, such as projections and transformations.

2. Improves Numerical Stability

- Orthogonal or orthonormal vectors help avoid issues with rounding errors in numerical computations.
- This is particularly important in algorithms like QR factorization, which rely on orthogonal matrices for solving linear systems efficiently.

3. Applications in Linear Algebra

- **Projections**: The process simplifies projecting vectors onto subspaces.
- **QR Decomposition**: Decomposing a matrix into a product of an orthogonal matrix (QQQ) and an upper triangular matrix (RRR).
- **Least Squares Method**: Used to find the best-fit solution to overdetermined systems.

4. Dimensionality Reduction

- In applications like Principal Component Analysis (PCA), the Gram-Schmidt process helps in creating orthonormal bases to reduce data dimensions while preserving variance.

5. Simplifies Computations

- When working with orthonormal bases:
  - Length of a vector is computed easily as the square root of the sum of squares of its coefficients.
  - Projection of one vector onto another becomes straightforward.

6. Foundations in Physics and Engineering

- In quantum mechanics, orthonormal wave functions are used to describe quantum states.
- In signal processing, orthogonal signals minimize interference, which is essential for encoding and decoding information.

7. Theoretical Importance

- It provides a systematic way to understand and construct vector spaces.
- Serves as a building block for advanced linear algebra concepts like eigenvalues, eigenvectors, and singular value decomposition (SVD).

Example 1

Example : Apply Gram-Schmidt process to the vectors $\beta_1 = (1, 0, 1)$, $\beta_2 = (1, 0, -1)$, $\beta_3 = (0, 3, 4)$ to obtain an orthonormal basis for $R^3(R)$ with standard inner product space.

We are given the vectors:

β1=(1,0,1), β2=(1,0,−1), β3=(0,3,4)

in , $\mathbb{R}^3$ and we aim to find an orthonormal basis using the **standard inner product**.

Step 1: Set the First Vector as $\alpha 1$

$$\alpha 1 = \beta 1 = (1,0,1)$$

Normalize $\alpha 1$:

$$\|\alpha_1\| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$$

$$\tilde{\alpha}_1 = \frac{\alpha_1}{\|\alpha_1\|} = \frac{1}{\sqrt{2}}(1,0,1)$$

$$= \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

Step 2: Orthogonalize *β2*

Remove the projection of *β2*:

$$\text{Projection of } \beta_2 \text{ onto } \alpha_1: \quad \mathrm{proj}_{\alpha_1}(\beta_2) = \frac{\langle \beta_2, \alpha_1 \rangle}{\langle \alpha_1, \alpha_1 \rangle} \alpha_1$$

Compute the inner product:

$$\langle \beta 2, \alpha 1 \rangle = (1)(1) + (0)(0) + (-1)(1) = 0$$

$$\langle \alpha_1, \alpha_1 \rangle = 1^2 + 0^2 + 1^2 = 2$$

$$\mathrm{proj}_{\alpha_1}(\beta_2) = \frac{0}{2} \alpha_1$$

$$= (0,0,0)$$

Thus:

$$\alpha_2 = \beta_2 - \mathrm{proj}_{\alpha_1}(\beta_2) = (1, 0, -1) - (0, 0, 0) = (1, 0, -1)$$

Normalize *α2*:

$$\|\alpha_2\| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$$

$$\tilde{\alpha}_2 = \frac{\alpha_2}{\|\alpha_2\|} = \frac{1}{\sqrt{2}}(1, 0, -1)$$

$$= \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$$

Step 3: Orthogonalize ***β3***

Remove the projections of *β3* onto *α1* and *α2*:

$$\alpha_3 = \beta_3 - \mathbf{proj}_{\alpha_1}(\beta_3) - \mathbf{proj}_{\alpha_2}(\beta_3)$$

Compute $\text{proj}_{\alpha_1}(\beta_3)$**:**

$$\langle \beta_3, \alpha_1 \rangle = (0)(1) + (3)(0) + (4)(1) = 4$$

$$\text{proj}_{\alpha_1}(\beta_3) = \frac{4}{2}\alpha_1 = 2(1, 0, 1) = (2, 0, 2)$$

Compute $\text{proj}_{\alpha_2}(\beta_3)$:

$$\langle \beta 3, \alpha 2 \rangle = (0)(1)+(3)(0)+(4)(-1) = -4$$

$$\text{proj}_{\alpha_2}(\beta_3) = \frac{-4}{2}\alpha_2 = -2(1,0,-1) = (-2,0,2)$$

Compute *α3*:

$$\alpha_3 = \beta_3 - \text{proj}_{\alpha_1}(\beta_3) - \text{proj}_{\alpha_2}(\beta_3)$$

$$\alpha 3 = (0,3,4)-(2,0,2)-(-2,0,2) = (0-2+2, 3-0+0, 4-2-2)$$

$$\alpha 3 = (0,3,0)$$

Normalize *α3*:

$$\|\alpha_3\| = \sqrt{0^2+3^2+0^2} = 3$$

$$\tilde{\alpha}_3 = \frac{\alpha_3}{\|\alpha_3\|} = \frac{1}{3}(0,3,0)$$

$$= (0,1,0)$$

The orthonormal basis for $\mathbb{R}^3$ is:

$$\tilde{\alpha}_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \quad \tilde{\alpha}_2 = \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right), \quad \tilde{\alpha}_3 = (0, 1, 0)$$

Example 2

Apply the Gram-Schmidt orthogonalization process to obtain an orthonormal basis from the basis $B = \{\beta_1, \beta_2, \beta_3\}$ of $R^3(R)$.

Where $\beta_1 = (1, 0, 1)$, $\beta_2 = (1, 2, -2)$, $\beta_3 = (2, -1, 1)$.

Given: $\beta 1=(1,0,1)$, $\beta 2=(1,2,-2)$, $\beta 3=(2,-1,1)$

Applying the **Gram-Schmidt Orthogonalization Process** to this set of vectors step by step to obtain an **orthonormal basis**.

Step 1: Start with $\alpha_1=\beta_1$

$$\alpha_1=\beta_1=(1,0,1)$$

Normalize α_1:

$$\|\alpha_1\| = \sqrt{1^2+0^2+1^2} = \sqrt{2}$$

$$\tilde{\alpha}_1 = \frac{\alpha_1}{\|\alpha_1\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

Step 2: Orthogonalize β_2

Remove the projection of β_2 onto α_1:

$$\mathbf{proj}_{\alpha_1}(\beta_2) = \frac{\langle \beta_2, \alpha_1\rangle}{\langle \alpha_1, \alpha_1\rangle}\alpha_1$$

Compute Inner Product:

$$\langle\beta_2,\alpha_1\rangle=(1)(1)+(2)(0)+(-2)(1)=1-2=-1$$

$$\langle \alpha_1, \alpha_1\rangle = 1^2+0^2+1^2 = 2$$

Compute Projection:

$$\text{proj}_{\alpha_1}(\beta_2) = \frac{-1}{2}\alpha_1 = \frac{-1}{2}(1,0,1) = \left(-\frac{1}{2}, 0, -\frac{1}{2}\right)$$

Compute *α2*:

$$\alpha_2 = \beta_2 - \text{proj}_{\alpha_1}(\beta_2)$$

$$\alpha_2 = (1,2,-2) - \left(-\frac{1}{2}, 0, -\frac{1}{2}\right) = \left(1 + \frac{1}{2}, 2+0, -2+\frac{1}{2}\right)$$

$$\alpha_2 = \left(\frac{3}{2}, 2, -\frac{3}{2}\right)$$

Normalize *α2*:

$$\|\alpha_2\| = \sqrt{\left(\frac{3}{2}\right)^2 + 2^2 + \left(-\frac{3}{2}\right)^2}$$

$$= \sqrt{\frac{9}{4} + 4 + \frac{9}{4}} = \sqrt{\frac{25}{4} + 4} = \sqrt{\frac{25}{4}} = \frac{5}{2}$$

$$\tilde{\alpha}_2 = \frac{\alpha_2}{\|\alpha_2\|} = \frac{1}{\frac{5}{2}}\left(\frac{3}{2}, 2, -\frac{3}{2}\right) = \left(\frac{3}{5}, \frac{4}{5}, -\frac{3}{5}\right)$$

Step 3: Orthogonalize *β3*

Remove the projections of *β3* onto *α1* and *α2*:

$$\alpha_3 = \beta_3 - \mathbf{proj}_{\alpha_1}(\beta_3) - \mathbf{proj}_{\alpha_2}(\beta_3)$$

Compute $\mathbf{proj}_{\alpha_1}(\beta_3)$**:**

$$\langle \beta 3, \alpha 1 \rangle = (2)(1) + (-1)(0) + (1)(1) = 2 + 1 = 3$$

$$\mathbf{proj}_{\alpha_1}(\beta_3) = \frac{3}{2}\alpha_1 = \frac{3}{2}(1, 0, 1) = \left(\frac{3}{2}, 0, \frac{3}{2}\right)$$

Compute $\mathbf{proj}_{\alpha_2}(\beta_3)$**:**

$$\langle \beta_3, \alpha_2 \rangle = \left(2 \cdot \frac{3}{5}\right) + \left(-1 \cdot \frac{4}{5}\right) + \left(1 \cdot \frac{-3}{5}\right) = \frac{6}{5} - \frac{4}{5} - \frac{3}{5} = \frac{-1}{5}$$

$$\mathbf{proj}_{\alpha_2}(\beta_3) = \frac{-1}{5}\alpha_2 = \frac{-1}{5}\left(\frac{3}{2}, 2, -\frac{3}{2}\right)$$

$$= \left(\frac{-3}{10}, \frac{-2}{5}, \frac{3}{10}\right)$$

Compute $\alpha 3$**:**

$$\alpha_3 = \beta_3 - \mathbf{proj}_{\alpha_1}(\beta_3) - \mathbf{proj}_{\alpha_2}(\beta_3)$$

$$\alpha_3 = (2, -1, 1) - \left(\frac{3}{2}, 0, \frac{3}{2}\right) - \left(\frac{-3}{10}, \frac{-2}{5}, \frac{3}{10}\right)$$

$$\alpha_3 = \left(2 - \frac{3}{2} + \frac{3}{10}, -1 - 0 + \frac{2}{5}, 1 - \frac{3}{2} - \frac{3}{10}\right)$$

$$\alpha_3 = \left(\frac{20}{10} - \frac{15}{10} + \frac{3}{10}, -1 + \frac{4}{10}, \frac{10}{10} - \frac{15}{10} - \frac{3}{10}\right)$$

$$\alpha_3 = \left(\frac{8}{10}, -\frac{6}{10}, -\frac{8}{10}\right) = \left(\frac{4}{5}, -\frac{3}{5}, -\frac{4}{5}\right)$$

Normalize $\alpha 3$**:**

$$\|\alpha_3\| = \sqrt{\left(\frac{4}{5}\right)^2 + \left(-\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2} = \sqrt{\frac{16}{25} + \frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{41}{25}} = \frac{\sqrt{41}}{5}$$

$$\tilde{\alpha}_3 = \frac{\alpha_3}{\|\alpha_3\|} = \frac{1}{\frac{\sqrt{41}}{5}} \left(\frac{4}{5}, -\frac{3}{5}, -\frac{4}{5}\right)$$

$$= \left(\frac{4}{\sqrt{41}}, -\frac{3}{\sqrt{41}}, -\frac{4}{\sqrt{41}}\right)$$

The orthonormal basis for $\mathbb{R}^3$ is:

$$\tilde{\alpha}_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \quad \tilde{\alpha}_2 = \left(\frac{3}{5}, \frac{4}{5}, -\frac{3}{5}\right), \quad \tilde{\alpha}_3 = \left(\frac{4}{\sqrt{41}}, -\frac{3}{\sqrt{41}}, -\frac{4}{\sqrt{41}}\right)$$