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CS660 Homework 3.

① Find the singular value decomposition (SVD) of matrix A.

$$A = \begin{bmatrix} 4 & 2 \\ 2 & -7 \\ 4 & -2 \end{bmatrix}$$

→ $$A = \begin{bmatrix} 4 & 2 \\ 2 & -7 \\ 4 & -2 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 4 & 2 & 4 \\ 2 & -7 & -2 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & -7 \\ 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 4\times4+2\times2+4\times4 & 4\times2+2\times(-7)+4\times(-2) \\ 2\times4+(-7)\times2+(-2)\times4 & 2\times2+(-7)\times(-7)+(-2)\times(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 16+4+16 & 8-14-8 \\ 8-14-8 & 4+49+4 \end{bmatrix}$$

$$= \begin{bmatrix} 36 & -14 \\ -14 & 57 \end{bmatrix}$$

- Determining eigen values:-

$$A^TA - \lambda I = \begin{bmatrix} 36-\lambda & -14 \\ -14 & 57-\lambda \end{bmatrix}$$

$A^TA - \lambda I =$

$$(36-\lambda)(57-\lambda)-196 = 36(57-\lambda)-\lambda(57-\lambda)-196$$
$$= 2052-36\lambda-57\lambda+\lambda^2-196$$
$$= \lambda^2-93\lambda+1856$$
$$= \lambda^2-64\lambda-29\lambda+1856$$
$$= \lambda(\lambda-64)-29(\lambda-64)$$
$$= (\lambda-29)(\lambda-64) = 0$$
$$\boxed{\lambda_1 = 29,\ \lambda_2 = 64},\ \lambda = 0$$

(9) $\lambda = 64$

$$\begin{bmatrix} 36-\lambda & -14 \\ -14 & 57-\lambda \end{bmatrix}$$

$$= \begin{bmatrix} 36-64 & -14 \\ -14 & 57-64 \end{bmatrix}$$

$$= \begin{bmatrix} -28 & -14 \\ -14 & -7 \end{bmatrix}$$

$R_1 \to \frac{-R_1}{7}$ and $R_2 \to \frac{-R_2}{7}$

$$= \begin{bmatrix} \cancel{-}4 & \cancel{-}2 \\ \cancel{-}2 & \cancel{-}1 \end{bmatrix}$$

$R_1 \to R_1 - R_2$ ~~and R_2~~

$$= \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$= \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

$R_1 \to \frac{R_1}{2}$

$$= \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}$$

$\lambda_1 = \frac{1}{2}\lambda_2$

$\lambda_2 = \lambda_2$

$\therefore v_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

Finding magnitude of vector:-

$$|v| = \sqrt{\left(\frac{1}{2}\right)^2 + 1^2}$$

$$= \sqrt{\frac{1}{4} + 1}$$

$$= \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$$

$$\therefore \vec{V_1} = \frac{1}{\frac{\sqrt{5}}{2}} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2/2\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} \frac{2}{2\sqrt{5}} \times \frac{2\sqrt{5}}{2\sqrt{5}} \\ 2/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 4\sqrt{5}/20 \\ 2/\sqrt{5} \end{bmatrix}$$

$$\vec{v_1} = \begin{bmatrix} \sqrt{5}/4 \\ 2/\sqrt{5} \end{bmatrix}$$

(ii) $\lambda = 29$

$$\begin{bmatrix} 36-\lambda & -14 \\ -14 & 57-\lambda \end{bmatrix}$$

$$= \begin{bmatrix} 36-29 & -14 \\ -14 & 57-29 \end{bmatrix}$$

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$$= \begin{bmatrix} 7 & -14 \\ -14 & 28 \end{bmatrix}$$

$R_1 \rightarrow \frac{R_1}{7}$, $R_2 \rightarrow \frac{R_2}{7}$

$$= \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

$R_2 \rightarrow \frac{R_2}{2}$

$$= \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

$R_2 \rightarrow R_2 + R_1$

$$= \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$x_1 = -2x_2$

$x_2 = x_2$

$\therefore V_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Finding magnitude of vector :-

$|v| = \sqrt{(-2)^2 + (1)^2}$

$= \sqrt{4+1}$

$= \sqrt{5}$

$$\therefore \vec{v_2} = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\vec{v_2} = \begin{bmatrix} -2/\sqrt{5} \\ \sqrt{5} \end{bmatrix}$$

(iii) $\lambda = 0$

$$\begin{bmatrix} 36-\lambda & -14 \\ -14 & 57-\lambda \end{bmatrix}$$

$$= \begin{bmatrix} 36-0 & -14 \\ -14 & 57-0 \end{bmatrix}$$

$$\begin{bmatrix} 36 & -14 \\ -14 & 57 \end{bmatrix}$$

$R_1 \to R_1 + R_2$

$$= \begin{bmatrix} 22 & 43 \\ -14 & 57 \end{bmatrix}$$

$R_1 \to R_1 + R_2$

$$= \begin{bmatrix} 8 & 100 \\ -14 & 57 \end{bmatrix}$$

$R_1 \to \frac{R_1}{4}$

$$= \begin{bmatrix} 2 & 25 \\ -14 & 57 \end{bmatrix}$$

$R_2 \to R_2 + 7R_1$

$$\begin{bmatrix} 2 & 25 \\ 0 & 232 \end{bmatrix}$$

$R_1 \to \frac{R_1}{2}$, $R_2 \to \frac{R_2}{232}$

$$= \begin{bmatrix} 1 & 25/2 \\ 0 & 1 \end{bmatrix}$$

$\lambda_1 = \frac{25}{2} \lambda_2$ $\qquad$ $\vec{V_3} = \begin{bmatrix} 25/2 \\ 1 \end{bmatrix}$

$\lambda_2 = 1$

Magnitude of vector:-

$$|v| = \sqrt{\left(\frac{25}{2}\right)^2 + 1}$$

$$= \sqrt{\frac{625}{4} + 1}$$

$$= \frac{\sqrt{629}}{2}$$

$$\vec{V_3} = \frac{1}{\frac{\sqrt{629}}{2}} \begin{bmatrix} 25/2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{629}}{25} \\ 2/\sqrt{629} \end{bmatrix}$$

$\vec{V_1} = \begin{bmatrix} \sqrt{5}/4 \\ 2/5 \end{bmatrix}$, $\vec{V_2} = \begin{bmatrix} -2/\sqrt{5} \\ \sqrt{5} \end{bmatrix}$, $\vec{V_3} = \begin{bmatrix} \sqrt{629}/25 \\ 2/\sqrt{629} \end{bmatrix}$

$$V = \begin{bmatrix} \sqrt{5}/4 & -2\sqrt{5} & \sqrt{629}/25 \\ 2\sqrt{5} & \sqrt{5} & 2/\sqrt{629} \end{bmatrix}_{2\times3}$$

$$V^T = \begin{bmatrix} -\sqrt{5}/4 & 2\sqrt{5} \\ -2\sqrt{5} & \sqrt{5} \\ \sqrt{629}/25 & 2/\sqrt{629} \end{bmatrix}_{3\times2}$$

The singular values of AA^T in order from greatest to ~~largest~~ least are:

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{64} = 8$$
$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{29}$$

As U is an orthogonal matrix with colums equal to unit eigenvectors of ~~V~~ $A \cdot A^T$.

$$\cancel{A = \begin{bmatrix} 4 & 2 \\ 2 & -7 \\ 4 & \sqrt{2} \end{bmatrix} \text{ and } A^T = \begin{bmatrix} \ \end{bmatrix}}$$

$$\Sigma = \begin{bmatrix} 8 & 0 \\ 0 & \sqrt{29} \end{bmatrix}$$

$$\therefore A = U \Sigma V^T$$

$$= \begin{bmatrix} \sqrt{5}/4 & -2/\sqrt{5} & \sqrt{629}/25 \\ 2\sqrt{5} & \sqrt{5} & 2/\sqrt{629} \end{bmatrix} \times \begin{bmatrix} 8 & 0 \\ 0 & \sqrt{29} \end{bmatrix} \times \begin{bmatrix} -\sqrt{5}/4 & 2\sqrt{5} \\ -2/\sqrt{5} & \sqrt{5} \\ \sqrt{629}/25 & 2/\sqrt{629} \end{bmatrix}$$

② Decompose matrix.

$$B = \begin{bmatrix} 11 & -8 & 4 \\ -8 & -1 & -2 \\ 4 & -2 & -2 \end{bmatrix}$$

Find the characteristic polynomial, the eigenvalues and eigenvectors.

→ ⓘ To find characteristic polynomial, we calculate the determinant of $(B-\lambda I)$.

$$B = \begin{bmatrix} 11-\lambda & -8 & 4 \\ -8 & -1-\lambda & -2 \\ 4 & -2 & -2-\lambda \end{bmatrix}$$

$$= 11-\lambda\left[(-1-\lambda)(-2-\lambda) - (-2)(-2)\right] + 8\left[(-8)(-2-\lambda) + 8\right] + 4\left[+16 - (4)(-1-\lambda)\right]$$

$$= 11-\lambda\left[-1(-2-\lambda) - \lambda(-2-\lambda) - 4\right] + 8\left[+16 + 8\lambda + 8\right] + 4\left[+16 + 4 + 4\lambda\right]$$

$$= 11-\lambda\left[2+\lambda+2\lambda+\lambda^2-4\right] + 8\left[8\lambda+24\right] + 4\left[4\lambda+20\right]$$

$$= 11-\lambda\left[\lambda^2+3\lambda-2\right] + 64\lambda + 192 + 16\lambda + 80$$

$$= 11\left[\lambda^2+3\lambda-2\right] - \lambda\left[\lambda^2+3\lambda-2\right] + 80\lambda + 272$$

$$= 11\lambda^2 + 33\lambda - 22 - \lambda^3 - 3\lambda^2 + 2\lambda + 80\lambda + 272$$

$$\boxed{\begin{aligned} &\cancel{= 8\lambda^2 + 115\lambda - 134 - \lambda^3} \\ &\cancel{= -\lambda^3 + 8\lambda^2 + 115\lambda + 90} \\ &\cancel{= -\lambda(8\lambda^2 + 115 + 134) = 0} \end{aligned}}$$

$$= 8\lambda^2 + 115\lambda + 250 - \lambda^3$$

$$= -\lambda^3 + 8\lambda^2 + 115\lambda + 250 = 0$$

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$= \cancel{-\lambda(\lambda^2 + 8\lambda + 115) + 250 = 0}$

Comparing it with $ax^3 + bx^2 + cx + d = 0$, we get

$\lambda_1 = -5$, $\lambda_2 = \frac{13}{2} - \frac{3\sqrt{41}}{2}$, $\lambda_3 = \frac{13}{2} + \frac{3\sqrt{41}}{2}$

$$\therefore V_1 = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}$$

$$V_2 = \begin{pmatrix} 17/10 - 3\sqrt{41}/10 \\ -17/20 - 3\sqrt{41}/20 \\ 1 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} \frac{17 + 3\sqrt{41}}{10} \\ -3\sqrt{41} + 17/20 \\ 1 \end{pmatrix}$$

(3) Find the determinant of following matrix using Laplace expansion

→

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 15 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\det(A) = \sum_{j=1}^{n} A_{ij} C_{ij}$$

$$= 1(135-48) - 2(36-42) + 3(32-105)$$

$$= 87 + 12 - 219$$

$$|A| = -120$$

(4) Find the rank -1 approximation of

$$D = \begin{bmatrix} 2 & 4 & 3 \\ -2 & 3 & 2 \end{bmatrix}$$

→ The SVD of matrix is:-

$$A = U\Sigma V^t$$

$$D = \begin{bmatrix} 2 & 4 & 3 \\ -2 & 3 & 2 \end{bmatrix}$$

$$R_2 \to R_2 + R_1$$

$$= \begin{bmatrix} 2 & 4 & 3 \\ 0 & 7 & 5 \end{bmatrix}$$

The rank of matrix is number of nonzeroes rows in the reduced matrix, so the rank is 2.

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(5) Find the Cholesky decomposition of

$$E = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 10 & 7 \\ 6 & 7 & 21 \end{bmatrix}$$

$\rightarrow$ $E = [L][L]^T$

$$= \underbrace{\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}}_{L^T}$$

$l_{11}^2 = 4 \therefore l_{11} = 2$

$l_{11} * l_{21} + 0 \times l_{22} + 0 \times 0 = 2$

$2 l_2 + 0 = 2$

$\therefore l_{21} = 1$

$l_{11} \times l_{31} + 0 \times l_{32} + 0 \times l_{33} = 6$

$\therefore l_{31} = 3.$

$l_{21} \times l_{11} + l_{22} \times 0 + 0 = 2$

$\therefore l_{21} = 1$

As we get the same value of l_{21}, we know that it is a ~~[illegible]~~ symmetric matrix, so move to the next column.

$l_{21} \times l_{21} + l_{22} \times l_{22} = 10$

$l_{21}^2 + l_{22}^2 = 10$

$1 + l_{22}^2 = 10$

$l_{22}^2 = 9$

$\therefore l_{22} = 3$

$l_{31} \times l_{11} = 6$

$l_{31} \times 2 = 6$

$\therefore l_{31} = 3$

$l_{31} \times l_{21} + l_{32} \times l_{22} = 7$

$3 \times 1 + 3 l_{32} = 7$

$3 l_{32} = 4$

$l_{32} = 4/3$

$l_{31} \times l_{31} + l_{32} \times l_{32} + l_{33} \times l_{33} = 21$

$3 \times 3 + \frac{16}{9} + l_{33}^2 = 21 \quad \Big| \quad \frac{97}{9} + l_{33}^2 = 21$

$$l_{33} = \frac{\sqrt{92}}{3}$$

$$L = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 3 & 4/3 & \sqrt{92}/3 \end{pmatrix}, \quad L^T = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 3 & 4/3 \\ 0 & 0 & \sqrt{92}/3 \end{pmatrix}$$

$$L \times L^T = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 3 & 4/3 & \sqrt{92}/3 \end{bmatrix} \times \begin{bmatrix} 2 & 1 & 3 \\ 0 & 3 & 4/3 \\ 0 & 0 & \sqrt{92}/3 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times 2 & 2\times 1 & 2\times 3 \\ 1\times 2 + 3\times 0 & 1\times 1 + 3\times 3 & 1\times 3 + \frac{4}{3}\times 3 \\ 3\times 2 & 3\times 1 + \frac{4}{3}\times 3 & 3\times 3 + \frac{4}{3}\times\frac{4}{3} + \frac{\sqrt{92}}{3}\times\frac{\sqrt{92}}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 6 \\ 2 & 10 & 7 \\ 6 & 7 & 21 \end{bmatrix} = A$$