Sep 14. Homework 1.

1) The matrix multiplication is possible if the number of columns in the first matrix equals the number of rows in the second matrix.

(a) $\begin{bmatrix} 1 & 5 \\ 2 & 3 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

matrix A (3×2) :- $\begin{bmatrix} 1 & 5 \\ 2 & 3 \\ 4 & 7 \end{bmatrix}$ & matrix B (3×3) :- $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

matrix A has dimensions 3×2 and matrix B has dimensions 3×3. The product is not possible because the number of columns in matrix A does not match the number of rows in matrix B.

Therefore matrix product cannot be performed.

(b) $\begin{bmatrix} 1 & 4 & 6 \\ 7 & 2 & 5 \\ 9 & 8 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

matrix A (3×3) :- $\begin{bmatrix} 1 & 4 & 6 \\ 7 & 2 & 5 \\ 9 & 8 & 3 \end{bmatrix}$ & matrix B (3×3) :- $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

→ Matrix multiplication :-

$$\begin{bmatrix} 1 & 4 & 6 \\ 7 & 2 & 5 \\ 9 & 8 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times1+4\times0+6\times1 & 1\times0+4\times1+6\times1 & 1\times1+4\times1+6\times0 \\ 7\times1+2\times0+5\times1 & 7\times0+2\times1+5\times1 & 7\times1+2\times1+5\times0 \\ 9\times1+8\times0+3\times1 & 9\times0+8\times1+3\times1 & 9\times1+8\times1+3\times0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+6 & 0+4+6 & 1+4+0 \\ 7+0+5 & 0+2+5 & 7+2+0 \\ 9+0+3 & 0+8+3 & 9+8+0 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 10 & 5 \\ 12 & 7 & 9 \\ 12 & 11 & 17 \end{bmatrix}$$

(c) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 & 6 \\ 7 & 2 & 5 \\ 9 & 8 & 3 \end{bmatrix}$

matrix A(3x3):- $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ & matrix B (3x3):- $\begin{bmatrix} 1 & 4 & 6 \\ 7 & 2 & 5 \\ 9 & 8 & 3 \end{bmatrix}$

⇒ Matrix Multiplication :-

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 4 & 6 \\ 7 & 2 & 5 \\ 9 & 8 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times1+0\times7+1\times9 & 1\times4+0\times2+1\times8 & 1\times6+0\times5+1\times3 \\ 0\times1+1\times7+1\times9 & 0\times4+1\times2+1\times8 & 0\times6+1\times5+1\times3 \\ 1\times1+1\times7+0\times9 & 1\times4+1\times2+0\times8 & 1\times6+1\times5+0\times3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+9 & 4+0+8 & 6+0+3 \\ 0+7+9 & 0+2+8 & 0+5+3 \\ 1+7+0 & 4+2+0 & 6+5+0 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 12 & 9 \\ 16 & 10 & 8 \\ 8 & 6 & 11 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 & 3 & 2 & 1 \\ 4 & 1 & -4 & -1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 2 & 2 \\ 4 & 2 \end{bmatrix}$$

matrix A (2×4):- $\begin{bmatrix} 1 & 3 & 2 & 1 \\ 4 & 1 & -4 & -1 \end{bmatrix}$ & matrix B (4×2):- $\begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 2 & 2 \\ 4 & 2 \end{bmatrix}$

$\Rightarrow$ Matrix Multiplication:-

$$\begin{bmatrix} 1 & 3 & 2 & 1 \\ 4 & 1 & -4 & -1 \end{bmatrix} \times \begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 2 & 2 \\ 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times0+3\times(-1)+2\times2+1\times4 & 1\times2+3\times1+2\times2+1\times2 \\ 4\times0+1\times(-1)+(-4)\times2+(-1)\times4 & 4\times2+1\times1+(-4)\times2+(-1)\times2 \end{bmatrix}$$

$$= \begin{bmatrix} 0+(-3)+4+4 & 2+3+4+2 \\ 0+(-1)+(-8)+(-4) & 8+1+(-8)+(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 0-3+4+4 & 2+3+4+2 \\ 0-1-8-4 & 8+1-8-2 \end{bmatrix}$$

$$= \begin{bmatrix} -3+8 & 11 \\ -9-4 & 9-10 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 11 \\ -13 & -1 \end{bmatrix}$$

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(e) $\begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 2 & 2 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 & 1 \\ 4 & 1 & -4 & -1 \end{bmatrix}$

matrix A (4×2):- $\begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 2 & 2 \\ 4 & 2 \end{bmatrix}$ & matrix B (2×4):- $\begin{bmatrix} 1 & 3 & 2 & 1 \\ 4 & 1 & -4 & -1 \end{bmatrix}$

$\Rightarrow$ Matrix multiplication:-

$$\begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 2 & 2 \\ 4 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 2 & 1 \\ 4 & 1 & -4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0\times1+2\times4 & 0\times3+2\times1 & 0\times2+2\times(-4) & 0\times1+2\times(-1) \\ (-1)\times1+1\times4 & (-1)\times3+1\times1 & (-1)\times2+1\times(-4) & (-1)\times1+1\times(-1) \\ 2\times1+2\times4 & 2\times3+2\times1 & 2\times2+2\times(-4) & 2\times1+2\times(-1) \\ 4\times1+2\times4 & 4\times3+2\times1 & 4\times2+2\times(-4) & 4\times1+2\times(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 0+8 & 0+2 & 0+(-8) & 0-2 \\ -1+4 & -3+1 & -2-4 & -1-1 \\ 2+8 & 6+2 & 4-8 & 2-2 \\ 4+8 & 12+2 & 8-8 & 4-2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 2 & -8 & -2 \\ 3 & -2 & -6 & -2 \\ 10 & 8 & -4 & 0 \\ 12 & 14 & 0 & 2 \end{bmatrix}$$

2) Write $y = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$ as a linear combination of $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $x_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

→ To compute :-

$$\lambda_1 \vec{x}_1 + \lambda_2 \vec{x}_2 + \lambda_3 \vec{x}_3 = \vec{y} \, ?$$

$$\therefore \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 1 & 2 & -1 & 2 \\ 1 & 3 & 1 & 5 \end{array}\right]$$

The goal is to row reduce matrix into $\left[\begin{array}{ccc|c} 1 & 0 & 0 & \# \\ 0 & 1 & 0 & \# \\ 0 & 0 & 1 & \# \end{array}\right]$

$R_2 \to R_2 - R_1$, and
$R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & -3 & 3 \\ 0 & 2 & -1 & 6 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 5 & 0 \end{array}\right]$$

$R_3 \to \frac{1}{5} R_3$ and $R_1 \to R_1 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & -4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

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3) Consider two subspaces U_1 and U_2 where U_1 is the solution space of the homogeneous equations system $A_1x=0$ and U_2 is the solution space of the homogeneous equations system $A_2x=0$.

$$A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 2 \\ 3 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 2 & -2 & 0 \\ 1 & 2 & 2 \\ 6 & -4 & 2 \\ 2 & -1 & 3 \end{bmatrix}$$

(a) Determine the dimension of U_1 and U_2

⇒ To determine the dimension of subspaces U_1 and U_2, we need to find the nullity of the matrices A_1 and A_2. The dimension of the solution space (nullity) is given by the rank-nullity theorem :

nullity = n - rank(A) ; 'n' is number of columns of the matrix

$$A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 2 \\ 3 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

A_1 is (4×3) matrix,
If 'r' is the rank of $A_1 (4\times3)$;
then $r \leq 4$ or $r \leq 3$.
As 'r' will always ~~be the one~~ take the smaller value irrespective of rows or columns whichever is smaller.

$\therefore R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$

$$A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 4 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$R_2 \to -1R_2$
$R_4 \to R_4 - R_2$

$$A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix has 3 non-zero rows, so the rank of A_1 is 3.

$\therefore$ By rank-nullity theorem, the dimension of $V_1 = 3 - \text{rank}(A_1)$
$= 3 - 3$
$= 0$.

Now;

$$A_2 = \begin{bmatrix} 2 & -2 & 0 \\ 1 & 2 & 2 \\ 6 & -4 & 2 \\ 2 & -1 & 3 \end{bmatrix}$$

$R_2 \to R_2 - \frac{1}{2} R_1$
$R_3 \to R_3 - 3R_1$
$R_4 \to R_4 - R_1$

$$\begin{bmatrix} 2 & -2 & 0 \\ 0 & 3 & 2 \\ 0 & 2 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

$R_3 \to R_3 - \frac{2}{3} R_2$
$R_4 \to R_4 - \frac{1}{3} R_2$

$$A_2 = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 2/3 \\ 0 & 0 & 1/3 \end{bmatrix}$$

Since all rows are non-zero, the rank of A_2 is 3.

$\therefore$ dimension of $V_2 = 3 - \text{rank}(A_2)$

$= 3 - 3$

$= 0$

The dimension of $V_1 = 0$, and
The dimension of $V_2 = 0$.

(b) Determine bases of V_1 and V_2.

$\Rightarrow$ As both matrices A_1 and A_2 have a rank of 3 and each matrix has 3 columns, their nullities are 0. This implies that the only solution to the homogeneous systems is the trivial solution $\vec{x} = 0$.

- Null Space and Basis for V_1.

$$A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 2 \\ 3 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

The matrix got row-reduced to the form: $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

rank of A_1 is 3 and there are no free variables as each column contains a leading 1, the null space of A_1 contains only the trivial solution. Therefore, the basis of V_1 is empty.

$\therefore$ Basis of V_1: None (trivial subspace)

- Null space and Basis for V_2.

$$A_2 = \begin{bmatrix} 2 & -2 & 0 \\ 1 & 2 & 2 \\ 6 & -4 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

The matrix got row-reduced to the form: $\begin{bmatrix} 2 & -2 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 2/3 \\ 0 & 0 & 1/3 \end{bmatrix}$

Similarly as matrix A_1, rank of A_2 is 3 and there are no free variables, meaning the null space of A_2 also contains only the trivial solution.

$\therefore$ Basis of U_2 : None (trivial subspace)

(c) Determine a basis for $U_1 \cap U_2$

$\Rightarrow$ Need to find:- looking for vectors $\vec{x}$ that satisfy both $A_1\vec{x} = 0$ and $A_2\vec{x} = 0$.

- Solving the system $A_1 x = 0$.

$$A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 2 \\ 3 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

The matrix got row-reduced to: $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

This system has the trivial solution $x = 0$, so the null space of A_1 consists only of the zero vector.

$$\therefore U_1 = \{0\}$$

- Solving the system $A_2 x = 0$

$$A_2 = \begin{bmatrix} 2 & -2 & 0 \\ 1 & 2 & 2 \\ 6 & -4 & 2 \\ 2 & -1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -2 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 2/3 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$R_1 \to \frac{1}{2}R_1, \; R_3 \to \frac{3}{2}R_3, \; R_4 \Rightarrow 3R_4$$

The matrix got row-reduced to: $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

This system also has the trivial solution $x=0$,

$$\therefore V_2 = \{0\}.$$

- Intersection of $V_1 \cap V_2$
As both V_1 and V_2 are trivial subspaces containing only the zero vector, their intersection is also the trivial subspace

$$V_1 \cap V_2 = \{0\}$$

- Determining basis for $V_1 \cap V_2$:-
The intersection of $V_1 \cap V_2$ is the trivial subspace, so its dimension is 0 and it has no non-trivial base vectors.

$\boxed{\therefore \text{ The basis for } V_1 \cap V_2 \text{ is } \underline{\text{No basis}} \text{ (only the zero vector)}}$

4) Suppose $S = \{v_1, v_2, \ldots, v_m\}$ spans a vector space V.
Prove:
(a) If $w \in V$, then $\{w, v_1, v_2, \ldots, v_m\}$ is linearly dependent and spans V.
(b) If v_i is a linear combination of $\{v_1, v_2, v_{i-1}\}$, then S without v_i spans V.

⇒ (a) $S = \{v_1, v_2, \ldots, v_m\}$ spans the vector space V. This means any vector $v \in V$ can be written as a linear combination of the vectors in S.

(i) Linear Dependence:-
Since $w \in V$ and $\{v_1, v_2, \ldots, v_m\}$ spans V, 'w' can be written as a linear combination of the vectors $v_1, v_2, \ldots, v_m$.
That is, there exist scalars $c_1, c_2, \ldots c_m$ such that:
$$w = c_1 v_1 + c_2 v_2 + \ldots + c_m v_m.$$

The equation shows that 'w' is dependent on the vectors $v_1, v_2, \ldots, v_m$. Now, consider the set $\{w, v_1, v_2, \ldots, v_m\}$.
If we take the following linear combination:
$$w - (c_1 v_1 + c_2 v_2 + \ldots + c_m v_m) = 0$$

We can see that there is a non-trivial linear combination of the vectors $\{w, v_1, v_2, \ldots, v_m\}$ that equals 0. Therefore, the set is linearly dependent

~~(ii) Spanning set~~
(ii) Spanning:-
Since $\{v_1, v_2, \ldots, v_m\}$ already spans V (assumptions), adding 'w' to the set does not change the span. The vector 'w' can already be expressed as a linear combination of the vectors $v_1, v_2, \ldots, v_m$. Thus, the set $\{w, v_1, v_2, \ldots, v_m\}$ also spans V, because any

vector in V can be written as a linear combination of the vectors in $\{v_1, v_2, \dots, v_m\}$ and hence also in $\{w, v_1, v_2, \dots, v_m\}$.
Hence, The set $\{w, v_1, v_2, \dots, v_m\}$ is linearly dependent and spans V.

(b) To prove: The set $S' = \{v_1, v_2, \dots, v_m\}$ spans V.
Since v_i is a linear combination of $\{v_1, v_2, \dots, v_{i-1}\}$, there exist scalars $c_1, c_2, \dots, c_{i-1}$ such that:

$$v_i = c_1 v_1 + c_2 v_2 + \dots + c_{i-1} v_{i-1}$$

Consider the set $S' = \{v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots v_m\}$. We need to show that S' spans V.
Since S spans V, any vector $v \in V$ can be written as a linear combination of the vector in S, i.e

$$v = a_1 v_1 + a_2 v_2 + \dots + a_m v_m$$

As v_i can be written as a linear combination of $\{v_1, v_2, \dots, v_{i-1}\}$, we can substitute this expression for v_i into the linear combination:

$$v = a_1 v_1 + a_2 v_2 + \dots + a_{i-1} v_{i-1} + a_i(c_1 v_1 + c_2 v_2 + \dots + c_{i-1} v_{i-1}) + a_{i+1} v_{i+1} + \dots + a_m v_m$$

Thus, v can be written as a linear combination of the vectors $\{v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_m\}$
Therefore, the set S' spans V.

5) Consider the basis

$$B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

of $\mathbb{R}^3$. Find the change of bases matrix P from the standard basis $\{e_1, e_2, e_3\}$ to the basis B.

$\Rightarrow$

$$P = [e_1, e_2, e_3] = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

finding P^{-1}:

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

$$\det(P) = 1\left(\begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix}\right) - 1\left(\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\right) + 0\left(\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}\right)$$

$$\det(P) = 1(9-2) - 1(6-0) + 0$$

$$\therefore \det(P) = 1$$

Now, finding the matrix of minors.

$$\text{Minors} = \begin{bmatrix} \det\begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix} & \det\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} & \det\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} \\ \det\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} & \det\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} & \det\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \\ \det\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} & \det\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} & \det\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \end{bmatrix}$$

$\therefore$ cofactors = $\begin{bmatrix} 7 & 6 & 4 \\ 3 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

Transpose the cofactor matrix (adjugate)

$$\text{Adj}(P) = \begin{bmatrix} 7 & 3 & 1 \\ 6 & 3 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{\det(P)} \times \text{Adj}(P)$$

$$\therefore P^{-1} = \begin{bmatrix} 7 & 3 & 1 \\ 6 & 3 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

$\therefore$ Change of basis matrix P from standard basis $\{e_1, e_2, e_3\}$ to basis B is

$$B = \left\{ \begin{bmatrix} 7 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$